High Rise vs Low Rise: does Building Typology Matter in High Density Property Investments?

HIGH RISE VS LOW RISE: DOES BUILDING TYPOLOGY MATTER IN HIGH DENSITY PROPERTY INVESTMENTS?

Chiara D'Alpaos[1]*, Valentina Antoniucci[1] and Giuliano Marella[1]

1: Department of Civil Environmental and Architectural Engineering
School of Engineering
University of Padova
Via Venezia, 1 - 35131
chiara.dalpaos@unipd.it, http://www.dicea.unipd.it
valentina.antoniucci@dicea.unipd.it, giuliano.marella@unipd.it

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Abstract *It is argued in the literature that skyscrapers are the most profitable building typology that guarantees an efficient use of land when both land values and population density are high. Nonetheless evidence from on-going projects in Europe shows that many investments in skyscrapers construction fail and are abandoned in favour of low-rise high density buildings. When construction has started, tall building projects must be completed in order to gain profits and returns on investment. Due to the specific construction typology and phasing, the developer has no option to defer completion or proceed by sequential investments. On the contrary, low-rise high density buildings are characterized by high operational flexibilities as they can be expanded by sequential investments and easily modified over time in order to adapt to changes in the state variables (e.g. demand, construction costs, market prices, etc.). Investments in low-rise construction generate multiple interacting options or compounded options whose values may increase the project Net Present Value. This paper investigates the role of flexibility in high density building construction projects. We model the value of flexibility to proceed by sequential investments according to the Real Option Approach. Sequential investments can actually be seen as a series of compound and growth options, where an earlier investment cost represents the exercise price required to acquire the subsequent option to continue operating the project until the next stage comes due.*

1. INTRODUCTION

In urban economic theory, densification processes in cities are caused by the allocative preferences of firms and households: cities guarantee more resources and greater potential in terms of economic and human capital, labour, and infrastructure than small towns. These processes are even more significant in contemporary megalopolis. The entire American population is concentrated in the three per cent of the Country territory [1] and the fifty per cent of the world's population lives in cities [2]. If compared to areas of sprawl, high urban density calls for specific building typologies: high-rise buildings (e.g. tall buildings and skyscrapers) and low-rise settlements (e.g. block buildings).

The rise and fortune of Skyscraper as symbols of 20th Century modern cities is widely discussed in literature. Starting from 1930, skyscrapers were recognized as the most efficient land use in the presence of high market land values, continuous population growth and firms and capital concentration [3]. The challenge in the 20th Century to build the highest tower in the world proved that the capability to construct the highest building has an economic value per se, greater than the returns on the underlying real estate investment, which turned often to be a failure [4]. It was and still is a matter of status: the company value increases worldwide through its identification in a symbolic building. Typical examples are the Chrysler Building or the Trump Tower in New York City [5].

Nonetheless, evidence from on-going projects in Europe and especially in Southern Europe shows that many investments in skyscrapers construction fail and are abandoned in favour of more profitable building typologies such as low-rise high density buildings, same buildable volume [6]. Significant initial capital outlays and uncertainty over future revenues make de facto investments in tall buildings irreversible and somehow unacceptably risky especially in times of financial crisis and down real estate markets. These constructions are fragile in terms of returns on investments for the different risk components related to tall building's development. Although skyscrapers generate the higher value per square meter compared to other building typologies, they are the most expensive and the greatest time-consuming construction process.

The specificity of investments in tall buildings make hard to properly predict the developer's profits and returns on capital. High risks are mainly determined by long lasting and out of sequence cycle in construction that characterize tall buildings [4] [6]. The average duration for a skyscraper's development is about ten years, from ideation to construction. If we consider that the business model is structured and defined at the process beginning, it can be easily shown that there is an increasing market risk that affects predicted rent values, due to volatile demand and unpredictable market events [7] [8]: there is higher probability to occur in a negative real estate cycle in long-lasting development than in other short-lasting constructions such as low-rise high density blocks.

Although high density low-rise blocks are less profitable investments than tall buildings in terms of value per square meter, the entire development process is less costly (-25% on average compared to tall buildings) and might better perform in terms of risk, and specifically construction risks. When construction has started, tall building projects must be completed in order to gain profit and returns on investments. Due to the specific construction typology and phasing, the developer has no option to defer completion or proceed by sequential investments. On the contrary, low rise high density buildings are characterized by high operational flexibilities as they can be expanded by sequential investment and easily modified over time in order to adapt to changes in the state variables (e.g. demand, construction costs, market prices, etc.). Investments in low rise construction generate multiple interacting options or compounded options whose values may increase the project Net Present Value.

This paper investigates the role of flexibility in high density construction projects. We model the value of flexibility to proceed by sequential investments. Sequential investments can actually be seen as a collection of compound and growth options, where an earlier investment cost represents the exercise price required to acquire the subsequent option to continue operating the project until the next stage comes due. Compoundness within the same multi-stage project (i.e. intraproject interaction) generates a series of point in time (i.e. decision nodes) when the project might be better discontinued if it turns out not to perform satisfactorily. The possibility to proceed by sequential investments, i.e. block by block, that are less affected by adverse economic realizations reduces investments irreversibility and mitigate losses [9]. As new information arrives and uncertainty about future cash flows is gradually resolved, the developer may have valuable flexibility to alter his initial operating strategy in order to capitalize on favourable future opportunities [10] [11]. Indeed, the importance of operating options becomes of crucial importance when the environment is volatile and construction technology is flexible, thus permitting managerial intervention at limited cost. This flexibility gives developers the option to strategically decide the optimal construction scheme and can significantly contribute to limiting losses and hedging of investment risk. If optimally exercised, operational flexibility may be economically relevant and its value is strongly related to the developer's ability to decide his planned course of action in the future, given then-available information.

The remainder of the paper is organized as follows. In section 2 the model is presented, Section 3 provides a numerical example to test the model's theoretical predictions and Section 4 concludes.

2. SEQUENTIAL INVESTMENTS: THE MODEL

Sequential investments can be considered as a portfolio of growth options. In this section we provide a basic model to determine the optimal investment strategy of an investor who has the opportunity to proceed with sequential investments in the construction and operation of real estate investments (i.e. the construction of high density residential buildings). The developer has the possibility of choosing between two alternative projects A and B of different scales (e.g. gross lease area). When future payoff are volatile and market conditions are uncertain, the real estate developer can decide firstly to invest in the smaller scale project A and wait to invest in the larger scale one until new information arrives and uncertainty about future cash flows is gradually resolved in order to capitalize on favourable future opportunities.

Starting from McDonald and Siegel [12] and D'Alpaos and Moretto [13], we extend the benchmark case of a single indivisible project A assuming that the investor has the possibility of choosing between two alternative projects A and B of different scales.

We introduce the following simplifying assumptions:

a) Investment A and investment B are large-scale projects, but project B is larger in scale, and once installed they generate respectively $\Pi_t^A(X^A)$ and $\Pi_t^B(X^B)$ with $X^B>X^A$ where X is the project's dimension (i.e. the gross lease area in m^2).

b) The projects' net revenues can be simplified into a linear function:

$$\Pi^B(X^B) = \pi_t X^B \qquad \Pi^A(X^A) = \pi_t X^A$$

where π_t is the instantaneous profit per square meter (m^2) equal for both the projects. The unit profit can be described by the following geometric Brownian motion with instantaneous expected return $\mu \geq 0$ and instantaneous volatility $\sigma>0$:

$$d\pi_t = \mu\pi_t dt + \sigma\pi_t dz_t \qquad \pi_0 = \pi \tag{1}$$

where dz_t is the increment of a standard Brownian process with mean zero and variance dt (i.e. $E(dz_t) = 0$ and $E(dz_t^2) = dt$). If we assume that in equilibrium the investment value is equal to its discounted expected cash flows, π_t is the difference between the lease (€/m^2) and the operating, managerial and maintenance costs.

c) Investment in projects A and B (i.e. blocks construction) entails sunk capital costs I^A and I^B respectively, where $I^B>I^A$. Investment costs include capital expenditure, development costs and administrative costs.

d) The developer can operate one project at a time and the investment is sequential, where investment A occurs before B. In other words, the developer can always invest in the smaller scale project and subsequently invest in the bigger scale one, incorporating the former into the latter. Alternatively, the developer can invest in project B simultaneously incorporating A (i.e. invest in both the projects simultaneously).

e) Finally, for the sake of simplicity, we assume that for both the projects the opportunity to invest is not subject to time constraints and at the end of their lifetime T_u the projects value is equal to zero.

Since investments A and B are not traded assets, their expected rate of return μ falls below the equilibrium total expected rate of return $\hat{\mu}$ required by investors in the market from an risk-equivalent traded financial security. The resulting rate of return shortfall $\delta \equiv \hat{\mu} - \mu > 0$ represents the opportunity cost (in annual terms) to invest at time zero and it is analogous to a constant dividend yield [14] [15].

In equilibrium, according to the risk-neutral valuation approach [16] [17], the actual growth rate μ can be replaced by the risk-neutral equivalent drift r-δ.

We can therefore discount certainty-equivalent cash flows at the risk-free rate r.

Consequently we can therefore rewrite (1) as follows:

$$d\pi_t = (r-\delta)\pi_t dt + \sigma\pi_t dz_t \qquad \pi_0 = \pi \,. \tag{2}$$

Given the above assumptions, the market value of the project B is:

$$V^B(\pi) = E\left\{\int_0^{T_u} e^{-rt}\Pi_t^B dt\right\} = \frac{\pi X^B}{\delta}(1-e^{-\delta T_u}) \tag{3}$$

where T_u is the project useful life.

On the contrary, while determining the market value of project A we must take into account that once A is installed, it generates a growth option, i.e. the opportunity to invest in project B. It is optimal to switch to project B whenever the instantaneous profit π_t becomes large enough. In particular, we can express $V^A(\pi)$ as:

$$V^A(\pi) = \max_{\tau^*} E \int_0^{\tau^*} e^{-rt}\Pi_t^A dt + e^{-r\tau}\left(V^B(\pi_{\tau^*}) - I^B\right) \tag{4}$$

where τ^* is the optimal switching time from A to B.

The solution to problem (2) is to switch from A to B as soon as π_t exceeds the critical threshold π^*_{AB}:

$$\pi_{AB}^{\ *} = \frac{\alpha}{\alpha-1}\delta\frac{I^B}{(X^B - X^A)(1-e^{-\delta T_u})} \tag{5}$$

where $\alpha = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1$.

The market value of project A turns out to be:

$$V^A(\pi) = \begin{cases} \frac{\pi X^A}{\delta}(1-e^{-\delta T_u}) + \left(\frac{\pi}{\pi_{AB}^{\ *}}\right)^{\alpha}\frac{I^B}{\alpha-1} & \text{if} \quad \pi \le \pi_{AB}^{\ *} \\ \frac{\pi X^B}{\delta}(1-e^{-\delta T_u}) - I^B & \text{if} \quad \pi > \pi_{AB}^{\ *} \end{cases} \tag{6}$$

It deserves to mention that (5) is positive if and only if X^B-X^A>0. Some comments on (6) are necessary. For any $\pi \in (0, \pi^*_{AB})$, $V^A(\pi) \le V^B(\pi)$. The current values of projects A and B coincide when π=0, while for any $\pi \in [\pi^*_{AB}, \infty)$ we observe that $V^A(\pi) - V^B(\pi) = -I^B$. In other words the current value of project B is always greater than the value of project A, which includes the option value to switch, eventually, from A to B at cost I^B. It is worth incurring a sunk cost I^B whenever the net revenues generated by project B are greater than the net revenues generated by project A.

In order to determine under which conditions it is optimal to proceed with sequential investments, let's consider the opportunity to invest in project A, which embeds the option to switch at a future date to project B. For each $t \le \infty$, this is equivalent to solve the following problem:

$$F^A(\pi_t) = \max_{\tau^*} E_t \left\{ e^{-r(\tau^* - t)} \left(V^A(\pi_{\tau^*}) - I^A \right) \right\}. \tag{7}$$

According to (6), due to the non-linearity in π, there is a discontinuity in the threshold π_A^*, exceeded which it is optimal to invest in project A:

$$\pi_A^* = \begin{cases} \frac{\alpha}{\alpha-1}\delta \frac{I^A}{X^A(1-e^{-\delta T_u})} & \text{if} \quad \frac{X^B}{X^A} - 1 < \frac{I^B}{I^A} \\ \frac{\alpha}{\alpha-1}\delta \frac{I^B + I^A}{X^B(1-e^{-\delta T_u})} & \text{if} \quad \frac{X^B}{X^A} - 1 \ge \frac{I^B}{I^A} \end{cases} \tag{8}$$

In the first case $\pi_A^* < \pi^*_{AB}$ therefore it is optimal to invest firstly in project A and then wait until the instantaneous profit π_t exceeds π_{AB}^* to invest in project B incorporating A. On the contrary when $\pi_A^* \ge \pi_{AB}^*$ it is optimal to invest in both the projects simultaneously and, therefore, proceed directly with the implementation of B incorporating A.

3. NUMERICAL EXAMPLE

In this section we present a numerical example to clarify applications of the above model and test the model's theoretical predictions on a stylized case study. We compare the decision to invest in a tall building to the decision to invest in low-rise buildings (i.e. blocks) and proceed by sequential investments.

We analyse the case of a developer that wants to supply a certain gross building area but can decide to invest and construct a tall building (Alternative T) or to construct low-rise blocks (Alternative LR). As far as the LR alternative is concerned, the entrepreneur has the discretion of choosing between two alternative projects A and B of different scale (i.e. gross building area) and has the opportunity to proceed with sequential investments. The value of the investment in the tall building can be properly determined according to the Net Present Value rule under the hypothesis that markets are complete and the tower value is equal to its discounted future net revenues (i.e. revenues from lease minus operating, managerial and maintenance costs). Whereas, as previously said, traditional capital budgeting techniques can be misleading in the presence of operational flexibility. The value of the flexibility to implement sequential investments is therefore determined as described in the previous section.

Summary information on technical and economic data relative to the various investment alternatives are displayed in Table 1.

r [%]	4
σ [%]	10; 20; 30; 40
δ [%]	1; 2; 3; 4
Tall building	
Number of floors	14
Total gross lease area [m^2]	6
Construction and development costs, I^T [€]	5,390,000
Current net annual lease, π^T [€/m^2 year]	77
Useful life, T_u [years]	50
Low rise buildings	
Number of blocks	3
Total gross lease area, X [m^2]	6
Construction and development costs per 1 block, $I^{LR,1}$ [€]	1,924,200
Construction and development costs per 2 blocks, $I^{LR,2}$ [€]	3,451,400
Construction and development costs per 3 blocks $I^{LR,3}$ [€]	4,581,500
Current net annual lease, π^{LR} [€/m^2 year]	69
Useful life, T_t [years]	50

Table 1. Technical and economic data relative to alternative T and alternative LR

3.1. The value of the T alternative

We determine the value of the T alternative according to D'Alpaos and Moretto (2005) and we introduce the following simplifying assumptions.

a) Once installed, project T generates net revenues cash flows that can be simplified into a linear function:

$$\Pi^T(X^T) = \pi_t{}^T X^T$$

where $\pi^T{}_t$ is the instantaneous profit per square meter (m^2) and X^T is the project's dimension (i.e. gross building area in m^2). Analogously to the assumptions introduced in Section 2, the unit profit can be described by the following geometric Brownian motion with mean zero and variance dt:

$$d\pi^T{}_t = (r-\delta)\pi^T{}_t dt + \sigma\pi^T{}_t dz_t \qquad \pi^T{}_0 = \pi^T$$

b) Investment cost is I^T and it is considered to be completely irreversible.

c) Project T useful life T_u is equal to project A and B useful life.

d) At the end of its lifetime the investment value is equal to zero.

The project value is :

$$V^T(\Pi^T) = E\left\{\int_0^{T_u} e^{-rt}\Pi_t^T dt\right\} \equiv \frac{\Pi^T(X^T)}{\delta}(1-e^{-\delta T_u})$$

The investment expected Net Present Value, NPV^T, is therefore:

$$NPV^T = V^T(\Pi^T) - I^T.$$

The results of simulations are shown in Table 2.

	NPV^T [€]	**$uNPV^T$ [€/m²]**
δ=1%	10,778,426	1,960
δ=2%	7,500,152	1,367
δ=3%	5,081,812	923
δ=4%	3,269,637	594

Table 2. Alternative T Net Present Value [€] and unit Net Present Value [€/m²] for different values of δ

The tall building represents a positive net present value project for any δ. For increasing values of δ, the Net Present Value decreases: the smaller the cost of carry, the smaller the investment Net Present Value. The Net Present Value does not vary according to σ.

3.2. The value of the LR alternative

By (3) and (6) we obtain the value of projects A and B respectively, while by (5) and (8) we derive the thresholds that trigger the investments and we determine the profitability of whether or not it is optimal to proceed with sequential investments. We performed simulations assuming different values for the parameters. In particular we assumed σ=10%, 20%, 30%, 40% and δ=1%, 2%, 3%, 4%. The triggers obtained for σ=10% and σ=40% are shown in Table 3.

r=4%			Gross lease area (m²)		Gross lease area (m²)	
			3670	5500	3670	5500
δ=1%	Gross lease area (m²)	1830	π^*_{AB}=222	π^*_{AB}=147	π^*_{AB}=607	π^*_{AB}=265
			π^*_{A}=62	π^*_{A}=41	π^*_{A}=169	π^*_{A}=74
			$uENPV^A$=3,210	$uENPV^A$=5,122	$uENPV^A$=3,741	$uENPV^A$=2,983
		3670		π^*_{AB}=295		π^*_{AB}=805
				π^*_{A}=74		π^*_{A}=202
				$uENPV^A$=4,976		$uENPV^A$=5,573
δ=2%	Gross lease area (m²)	1830	π^*_{AB}=145	π^*_{AB}=96	π^*_{AB}=399	π^*_{AB}=96
			π^*_{A}=40	π^*_{A}=27	π^*_{A}=111	π^*_{A}=27
			$uENPV^A$=1,894	$uENPV^A$=3,166	$uENPV^A$=2,495	$uENPV^A$=2,031
		3670		π^*_{AB}=192		π^*_{AB}=530
				π^*_{AB}=48		π^*_{AB}=133
				$uENPV^A$=3,104		$uENPV^A$=3,776
δ=3%	Gross lease area (m²)	1830	π^*_{AB}=126	π^*_{AB}=83	π^*_{AB}=343	π^*_{AB}=228
			π^*_{A}=35	π^*_{A}=23	π^*_{A}=96	π^*_{A}=64
			$uENPV^A$=1,068	$uENPV^A$=1,898	$uENPV^A$=1,652	$uENPV^A$=1,390
		3670		π^*_{AB}=167		π^*_{AB}=455
				π^*_{A}=42		π^*_{A}=114
				$uENPV^A$=1,915		$uENPV^A$=2,538
δ=4%	Gross lease area (m²)	1830	π^*_{AB}=124	π^*_{AB}=82	π^*_{AB}=325	π^*_{AB}=216
			π^*_{A}=34	π^*_{A}=22	π^*_{A}=91	π^*_{A}=60
			$uENPV^A$=552	$uENPV^A$=1,026	$uENPV^A$=1,061	$uENPV^A$=940
		3670		π^*_{AB}=164		π^*_{AB}=431
				π^*_{A}=41		π^*_{A}=108
				$uENPV^A$=1,556		$uENPV^A$=1,658

Table 3. Alternative LR optimal triggers [€] and uENPV [€/m²] for different values of δ and σ=10%; 40%

The results displayed in Table 3 show that the optimal investment strategy for any σ and δ is to invest in the smaller scale project and wait to invest in the bigger scale one until the instantaneous profit π_t becomes greater than the threshold π^*_{AB}. In other words it is always optimal to invest in sequential investment and build one block at a time. Economies of scale are small and do not affect the optimal investment strategy. For increasing values of δ, ceteris paribus, the thresholds π^*_{AB} and π^*_{A} decreases (Tab. 3). For increasing values of σ, ceteris paribus, the threshold increases (Tab. 3). The uncertainty over future net payoffs induces the developer to build one block at the time and pay indeed the additional cost to switch to the subsequent investment. For increasing values of σ, the difference between π^*_{AB} and π^*_{A} increases and as a consequence the expected time to switch to the bigger scale project increases. The greater the uncertainty the greater the option value to invest sequentially; the greater the uncertainty, the longer the switching times. Table 3 illustrates the unit Net Present Value of the smaller scale investment that includes the option value to switch to further investments, uENPV henceforth. By comparison of Table 2 and Table 3, it emerges that it is always preferable to proceed by sequential investments instead of construct the tall building, regardless the tall building is a positive net present value project. In order to maximize the investment value, the developer should invest in low-rise buildings: the developer should start implementation of the smaller scale project and wait to invest in the bigger scale one as soon as the current net unit revenue exceeds the threshold that triggers the investment.

4. CONCLUSIONS

The paper investigates the role of flexibility in sequential property investments. The tall building unit Net Present Value is positive and greater than the unit Net Present value of the low-rise alternative (e.g. $uNPV^T$=1,960 €/m^2 *vs* $uNPV^{LR}$=1,665 €/m^2 when δ=1%). Nonetheless it is widely recognized that the Net Present Value rule assumes management's passive commitment to a "certain static operating strategy" and fails because it is not able to capture managerial flexibility to adapt and revise later decisions in response to unexpected market events. In particular traditional capital budgeting techniques fail to capture the value of flexibility that characterizes sequential investments. In fact the model results show that the unit Net Present Value of the tall building is always lower than the unit Net present Value of the low-rise alternative that includes the growth option value ($uNPV^{T=}$964 €/m^2 *vs* $uENPV^{LR}$=1,068 €/m^2 when δ=3% and σ=10%). It is therefore profitable for the developer to proceed by sequential investments and construct low-rise buildings.

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