Jacobi Multipliers and Hamel's formalism

José F. Cariñena[a] and Patricia Santos[b]

[a]Department of Theoretical Physics, Univ. of Zaragoza, Spain

[b]Polytechnic Institute of Coimbra, ISEC, and CMUC, University of Coimbra, Portugal

Abstract

In this work we establish the relation between the Jacobi last multiplier, which is a geometrical tool in the solution of problems in mechanics and that provides Lagrangian descriptions and constants of motion for second-order ordinary differential equations, and nonholonomic Lagrangian mechanics where the dynamics is determined by Hamel's equations.

Keywords: Jacobi multiplier, Nonholonomic systems, Hamel's equations, Quasivelocities

Running title:

MSC Classification: 70F25, 70G45, 70H03

PACS numbers: 02.30.Hq, 02.40.Yy, 45.20.Jj

1 Introduction

The study of the integrability by quadratures of a given system of first-order differential equations is not an easy task and a very relevant advance is due to Jacobi, who introduced the concept of Jacobi multiplier [27] and proved that the knowledge of one such Jacobi multipliers and some first integrals is sufficient to carry out the integrability by quadratures [22, 28, 41]. Jacobi multipliers have been attracting some attention all over the last years (see e.g. [8] and references therein) and have several applications in the analysis of systems of first-order ordinary differential equations [2, 12, 23, 35, 36, 42].

In the geometric approach autonomous systems of first-order differential equations are replaced by vector fields on a manifold Q and the solutions of the corresponding system in a chart provide us the integral curves of the vector field. Systems of second-order differential equations can also be seen as systems with a double number of first-order differential equations and the associated vector field is then a vector field of a special kind on the tangent bundle $\tau : TQ \to Q$.

In many problems in nonholonomic classical mechanics, where the systems of second-order differential equations appear, it is useful to consider quasi-velocities, and then the dynamics is determined by Hamel's equations [25, 34, 37]. For instance for a system with linear nonholonomic constraints one can define quasi-velocities in such a way that some of them coincide with the constraints, obtaining as a result less equations to solve.

Our aim in this work is to establish some relations between Jacobi multipliers for such nonholonomic systems [14, 21, 28] and Hamel's formalism. The structure of this article is as follows: In Section 2 in order to the paper be more self-contained we recall the geometric theory behind Jacobi multipliers. Section 3 is devoted to study Jacobi multipliers in the framework of regular Lagrangian systems, and we prove that the determinant of the Hessian matrix of the Lagrangian in the velocities is a Jacobi multiplier for the system with respect to the natural volume element on the velocity phase space. A more general result in the particular case of generalised forces only depending on the positions is explicitly proved. The theory is illustrated in Section 4 with some simple problems: starting with that of spherical geometry, then we analyse the case of Liénard's equation and Chiellini's condition, and finally the dynamics of a position-dependent mass particle is analysed from this perspective. In Section 5, we consider a system determined by linear nonholonomic constraints, and the use of quasi-coordinates and the Boltzmann-Hamel equations are reviewed, and then we write the dynamics and determine Jacobi multipliers with respect to several proportional volume forms in the Lagrangian formulation in terms of quasi-coordinates. Jacobi multipliers for a nonholonomic system is studied in Section 6. The main result of Jacobi on multipliers is given in Section 7 in geometric terms: it is shown how a Jacobi multiplier of a nonholonomic system can be used with other constants of motion, to obtain a reduced system endowed with a Jacobi multiplier. Then, when such reduced system is two-dimensional, this Jacobi last multiplier allows us to solve the reduced system by quadratures.

2 Jacobi multipliers and volume forms

Let (M, Ω) be an oriented n-dimensional manifold, where Ω stands for a volume form on the manifold, i.e. a never vanishing n-form on the manifold. Given a vector field X on M we define the *divergence* of X as the unique function $\operatorname{div}(X) : M \to \mathbb{R}$ satisfying (see [8])

$$\mathcal{L}_X \Omega = \operatorname{div}(X)\, \Omega. \tag{2.1}$$

A *Jacobi multiplier* for X is a non-vanishing function $\mu : M \to \mathbb{R}$ satisfying

$$\mathcal{L}_{\mu\, X}(\Omega) = 0, \tag{2.2}$$

that is, $\operatorname{div}(\mu\, X) = 0$. Notice that as Ω is closed, Cartan's magic formula implies that $\mathcal{L}_{\mu\, X}(\Omega) = d(\mu\, i(X)\Omega)$ and then μ is a Jacobi multiplier for X if and only if $\mu\, i(X)\Omega$ is a closed form, and equivalently, if and only if $\mathcal{L}_X(\mu\, \Omega) = 0$. Furthermore, using that $\mathcal{L}_X$ is a derivation of degree zero, we see that $\mathcal{L}_X(\mu\, \Omega) = (X\mu + \mu \operatorname{div}(X))\Omega$, which proves that (2.2) is equivalent to

$$X\mu + \mu \operatorname{div}(X) = 0, \tag{2.3}$$

sometimes written as (see e.g. [21, 36, 42])

$$X(\ln |\mu|) + \operatorname{div}(X) = 0. \tag{2.4}$$

This means that along the integral curves of the vector field X

$$\frac{d}{dt} \ln |\mu| + \operatorname{div}(X) = 0, \tag{2.5}$$

which is called the *generalised Liouville equation.*

Remark that, as indicated above,

$$\mathcal{L}_X(\mu\, \Omega) = d(\mu\, i(X)\Omega) = d(i(\mu\, X)\Omega) = \mathcal{L}_{\mu\, X}(\Omega),$$

and we see that μ is a Jacobi multiplier for X if and only if the vector field X is divergence-free with respect to $\mu\, \Omega$. In this sense μ plays the role of the component of the X-invariant n-form $\mu\, \Omega$, which means that the behaviour of the Jacobi multiplier under changes of coordinates is not that of a function but that of a pseudo-function.

This property is very relevant because it shows the equivalence of searching for Jacobi multipliers and for invariant volume-forms: a volume form $\Omega' = \mu\,\Omega$ is X-invariant, i.e. $\mathcal{L}_X\Omega' = 0$, if and only if μ is a Jacobi multiplier for the vector field X on the oriented manifold (M, Ω). Moreover this property also shows that μ is a Jacobi multiplier for X in the oriented manifold (M, Ω) if, and only if, the function $f\,\mu$ is a Jacobi multiplier for X on the oriented manifold $(M, f^{-1}\,\Omega)$, for each positive function f.

The theory of Jacobi multipliers which was introduced to integrate the system is also particularly useful to find first-integrals (i.e. constants of motion): the knowledge of two Jacobi multipliers μ_1 and μ_2 for X implies that $I = \mu_1/\mu_2$ is a first-integral of X (see e.g. [8, 22, 41]). In the case of a two-dimensional system, a Jacobi multiplier μ determines a (only locally defined) first-integral of the system, I, and vice versa, by means of $\mu\, i(X)\Omega = dI$. Recall that if a first-integral I of a vector field X on a two-dimensional manifold is known, then using the condition $I(x_1, x_2) = k \in \mathbb{R}$, if, for instance, $\partial I/\partial x_2 \neq 0$, the implicit function theorem shows that we can locally express the variable x_2 as a function of x_1 for each value of k, what provides us the general solution for the integral curves of X. Conversely, if a Jacobi multiplier μ for a vector field on a two-dimensional manifold is known, there exists a, at least locally defined, function I such that $\mu\, i(X)\omega = dI$, and consequently the function I is a first-integral which can be found by a quadrature.

When considering another volume form Ω' on M, there exists a non-vanishing function $\eta : M \to \mathbb{R}$, such that, $\Omega' = \eta\,\Omega$, and there is a relation between two Jacobi multipliers of X for both volume forms, $\mu' = \mu/\eta$. This particular relation will be crucial to deduce the integrating factor of a system with nonholonomic constraints.

The simplest case is when M is an open set of the n-dimensional Euclidean space. There exist global (Euclidean) coordinates $(x^1, \dots, x^n)$ and we can choose the volume form $\Omega = dx^1 \wedge \cdots \wedge dx^n$. If the vector field X is given by

$$X = \sum_{i=1}^{n} X^i(x) \frac{\partial}{\partial x^i}\,, \tag{2.6}$$

then the divergence of X with respect to such volume form takes the usual form

$$\mathrm{div}(X) = \sum_{i=1}^{n} \frac{\partial X^i}{\partial x^i}. \tag{2.7}$$

As a first instance of the usefulness of Jacobi multipliers, it has been proved [2, 22, 41] that in the case of a three-dimensional system with a Jacobi multiplier μ and a first-integral I_1 satisfying the condition $\partial I_1/\partial x_1 \neq 0$, which allows us to express x_1 as a function of x_2 and x_3, we get a reduced system in the coordinates (x^2, x^3) that has an integrating factor $\mu'(x_2, x_3)$ given by $\mu' = \mu/\frac{\partial I_1}{\partial x_1}$, where in the functions of the right hand side x_1 is expressed as a function of x_2 and x_3, and this integrating factor determines a second first-integral I_2, depending only on the coordinates (x^2, x^3), by $\mu'\, i(X)(dx^2 \wedge dx^3) = dI_2$, and then, as indicated above, the general solution can be found by quadratures. This is but a particular example of the more general case which motivated Jacobi for the introduction of Jacobi multipliers: If a Jacobi multiplier μ and $(n-2)$ functionally independent first-integrals, $\{I_1, \dots, I_{n-2}\}$, for a given vector field X on an oriented manifold (M, Ω) are known, the determination of the integral curves of the vector field X is reduced to quadratures.

3 Jacobi multipliers for regular Lagrangian systems

Even if the concept of Jacobi multiplier is defined for any vector field X on an oriented manifold (M, Ω), in the search for integral curves of X the existence of geometric structures compatible with X is very useful and in particular in the very important case of second-order differential equations appearing in many physical applications and deserving therefore an special attention. We will see that in the particular case in which such systems of second-order differential equations admit a Lagrangian formulation it is possible to find a Jacobi multiplier directly from such a Lagrangian.

It is a well-known fact that *autonomous* systems of second-order differential equations

$$\ddot{x}^i = f(x^j, \dot{x}^j), \qquad i, j = 1, \dots, n, \tag{3.1}$$

on an open set U of $\mathbb{R}^n$, can also be seen as systems with a double number of first-order differential equations

$$\begin{cases} \dot{x}^i = v^i \\ \dot{v}^i = f^i(x, v) \end{cases} . \tag{3.2}$$

In the simplest case of the n-dimensional Euclidean space, the Euclidean coordinates $(x^1, \dots, x^n)$ mentioned above induce global coordinates in its tangent bundle, denoted $(x^1, \dots, x^n, v^1, \dots, v^n)$, and there is a natural volume element (tangent and cotangent bundles are identified by the Euclidean metric)

$$\Omega = dx^1 \wedge \cdots \wedge dx^n \wedge dv^1 \wedge \cdots \wedge dv^n. \tag{3.3}$$

The autonomous system of second-order differential equations (3.1) and its corresponding system (3.2) can be described by a special kind of vector field Γ, to be called SODE vector field, on the open set $\tau^{-1}(U)$ of $T\mathbb{R}^n$, with $\tau : T\mathbb{R}^n \to \mathbb{R}^n$ and coordinate expression

$$\Gamma = \sum_{i=1}^{n} \left(v^i \frac{\partial}{\partial x^i} + f^i(x,v) \frac{\partial}{\partial v^i} \right), \tag{3.4}$$

whose integral curves are the tangent lift of solutions of the given system (3.1), i.e. solutions of (3.2). More specifically, such curves are the solutions of the associated system of first-order differential equations (3.2). Observe that in this case $\operatorname{div}(\Gamma) = \sum_{i=1}^{n} \frac{\partial f^i}{\partial v^i}$.

The geometric framework for the study of Lagrangian mechanics is that of tangent bundles [7, 15, 16]. The tangent bundle $\tau : TQ \to Q$ is characterised by two geometric tensors, the vertical endomorphism S, a (1,1)-tensor field on TQ, also called tangent structure, which satisfies $\operatorname{Im} S = \ker S$ and an integrability condition and the Liouville vector field Δ generating dilations along fibres in TQ [17]. If (U, φ) is a local chart on Q and $\pi^i : \mathbb{R}^n \to \mathbb{R}$ are the natural projections on the i-th-factor and $q^i = \pi^i \circ \varphi$ we define the coordinate system $(U, q^1, \ldots, q^n)$ on Q, and the corresponding chart in $\mathscr{U} = \tau^{-1}(U)$, given by $(\mathscr{U}, \varphi, \varphi_*)$ defines a coordinate system $(U, q^1, \ldots, q^n, v^1, \ldots, v^n)$ on TQ. Correspondingly, we consider the coordinate basis of $\mathfrak{X}(U)$ usually denoted $\{\partial/\partial q^j \mid j = 1, \ldots, n\}$ and its dual basis for $\Omega^1(U)$, $\{dq^j \mid j = 1, \ldots, n\}$. Then a vector v in a point $q \in U$ is $v = v^j \, (\partial/\partial q^j)_q$, i.e. $v^i = dq^i(v)$. With this notation the coordinate expressions of the vertical endomorphism S and the Liouville vector field Δ, are [15, 16]:

$$S(x,v) = \frac{\partial}{\partial v^i} \otimes dq^i, \qquad \Delta(x,v) = v^i \frac{\partial}{\partial v^i}. \tag{3.5}$$

Recall also that given a Lagrangian $L \in C^\infty(TQ)$ we can define a 1-form $\theta_L = dL \circ S$ and the exact 2-form $\omega_L = -d\theta_L$. When ω_L is regular the Lagrangian L is said to be regular and then ω_L is a symplectic form and the dynamics is given by the uniquely defined SODE vector field Γ such that

$$i(\Gamma)\omega_L = dE_L \Longleftrightarrow \mathscr{L}_\Gamma \theta_L - dL = 0, \tag{3.6}$$

where the energy function E_L is defined by $E_L = \Delta L - L$.

It is also a well-known property that if a second-order differential equation $\ddot{x} = f(x, \dot{x})$ admits a Lagrangian formulation in terms of a Lagrangian function L, then the function

$$\mu = \frac{\partial^2 L}{\partial v^2} \tag{3.7}$$

is a Jacobi last multiplier for the volume 2-form $dx \wedge dv$ [41]. Actually, if the system (3.1) admits a Lagrangian formulation, then the function f is given by

$$f(x,v) = \frac{1}{\mu}\left(\frac{\partial L}{\partial x} - v\frac{\partial^2 L}{\partial x \partial v}\right) . \tag{3.8}$$

We can now see that the function function μ defined by (3.7) satisfies the condition for being a Jacobi last multiplier, because if Γ is given by

$$\Gamma = \left(v\frac{\partial}{\partial x} + f(x,v)\frac{\partial}{\partial v}\right) , \tag{3.9}$$

the condition

$$\operatorname{div}(\mu\,\Gamma) = 0 \Longleftrightarrow \frac{\partial(\mu\, v)}{\partial x} + \frac{\partial(\mu\, f)}{\partial v} = 0,$$

when μf is given by (3.8), becomes

$$v\frac{\partial\mu}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial L}{\partial x} - v\frac{\partial^2 L}{\partial x\partial v}\right) = v\frac{\partial^3 L}{\partial v^2\partial x} + \frac{\partial^2 L}{\partial x\partial v} - \frac{\partial^2 L}{\partial x\partial v} - v\frac{\partial^3 L}{\partial v^2 \partial x} = 0\,,$$

and then μ is a Jacobi last multiplier.

The corresponding result for n-dimensional systems is given in the following theorem:

Theorem 1 *If a regular Lagrangian $L \in C^\infty(T\mathbb{R}^n)$ determines the autonomous system (3.2), i.e. the vector field (3.4), then the determinant of the Hessian matrix in the velocities, $W = \left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right)$, is a Jacobi multiplier for the system (3.2), i.e. for the vector field (3.4).*

Proof.- The vector field determined by the regular Lagrangian is the solution of the equation $i(\Gamma)\omega_L = dE_L$, where, as defined above, E_L is the energy of the system, ω_L is the Cartan 2-form $\omega_L = -d\theta_L$ and $\theta_L = S^*(dL)$ is the Cartan 1-form determined by the Lagrangian, with S denoting the vertical endomorphism (see [15, 16]). Then the solution of the equation (3.6) is a Hamiltonian vector field w.r.t. ω_L, and then $\mathcal{L}_\Gamma\omega_L = 0$. Hence, if $\Omega' = \omega_L^{\wedge n}$, then $\mathcal{L}_\Gamma\Omega' = 0$, and therefore $\mu' = 1$ is a Jacobi multiplier for (Γ, Ω'). The relation between Ω given by (3.3) and Ω' is

$$\begin{aligned}
\Omega' &= (-1)^n\, d\left(\sum_{i_1=1}^n \frac{\partial L}{\partial v^{i_1}}\, dx^{i_1}\right) \wedge \cdots \wedge d\left(\sum_{i_n=1}^n \frac{\partial L}{\partial v^{i_n}}\, dx^{i_n}\right) \\
&= \sum_{i_1,\dots,i_n=1}^n dx^{i_1} \wedge d\left(\frac{\partial L}{\partial v^{i_1}}\right) \wedge \cdots \wedge dx^{i_n} \wedge d\left(\frac{\partial L}{\partial v^{i_n}}\right) \\
&= (-1)^{n(n-1)/2}\, n!\; dx^1 \wedge \cdots \wedge dx^n \wedge d\left(\frac{\partial L}{\partial v^1}\right) \wedge \cdots \wedge d\left(\frac{\partial L}{\partial v^n}\right) \\
&= (-1)^{n(n-1)/2}\, n!\; \det(W)\,\Omega
\end{aligned} \tag{3.10}$$

and as $\det(W)$ is the proportionality factor between both volume forms the results follows. $\square$

Remark that, as a corollary, if we have two equivalent regular Lagrangians L_1 and L_2, the quotient $\det(W_2)/\det(W_1)$ is a constant of motion. This result has been given in [18] for the onedimensional case and in [26] as a particular case of other more general result, which is that the characteristic polynomial of the recursion operator $R = \widehat{\omega}_{L_1}^{-1} \circ \widehat{\omega}_{L_2}$, is a constant of motion, or in other words, the traces of powers of the recursion operator R are constants of the motion (see e.g. [6] and references therein).

The theory can be extended to more general cases where the configuration space is not the Euclidean space. If a given second-order differential equation vector field Γ admits a Lagrangian description by the regular Lagrange function L in the tangent bundle $\tau : TQ \to Q$, i.e. $i(\Gamma)\omega_L = dE_L$, then $\mathcal{L}_\Gamma \omega_L = 0$ implies that $\mathcal{L}_\Gamma \omega_L^{\wedge n} = 0$, and therefore the vector field Γ is divergence-free with respect to the volume form $\omega_L^{\wedge n}$. In other words, we are able to find a Γ-invariant volume form.

If a chart in the open set $U \subset Q$ is chosen, i.e. we consider a coordinate system $(U, q^1, \ldots, q^n)$ on the open set U of Q and the corresponding chart in $\mathscr{U} = \tau^{-1}(U)$, then this coordinate system determines a basis $((\partial/\partial q^1)_{|m}, \ldots, (\partial/\partial q^n)_{|m})$ of the tangent bundle at each point m of $\mathscr{U}$, and the local basis of the $C^\infty(U)$-module of vector fields $\mathfrak{X}(U)$ given by $\{\partial/\partial q^i \mid i = 1, \ldots, n\}$ defines a local basis of the $C^\infty(T\mathscr{U})$-module of vector fields $\mathfrak{X}(T\mathscr{U})$ given by $\{\partial/\partial q^i, \partial/\partial v^i \mid i = 1, \ldots, n\}$. When restricted to such a tangent chart on the open set $\mathscr{U}$ of TQ, i.e. in terms of the associated local coordinate system $(q^1, \ldots, q^n, v^1, \ldots, v^n)$ on the tangent bundle TQ, we can consider the volume form locally given as follows

$$\Omega = dq^1 \wedge \cdots \wedge dq^n \wedge dv^1 \wedge \cdots \wedge dv^n, \tag{3.11}$$

and then the volume form $\omega_L^{\wedge n}$ on $\mathscr{U} = \tau^{-1}(U)$ is proportional to $\Omega = dq^1 \wedge \cdots \wedge dq^n \wedge dv^1 \wedge \cdots \wedge dv^n$ with proportionality factor $\det(W)$, and this shows that $\det(W)$ is a Jacobi multiplier for the restriction of Γ onto the oriented manifold $(\mathscr{U}, \Omega)$.

Let (Q, g) be a n-dimensional manifold endowed with a Riemann metric g, and consider a coordinate system $(U, q^1, \ldots, q^n)$ on the open set U of Q. The metric g is represented on U by the matrix G with elements $g_{ij} = g(\partial/\partial q^i, \partial/\partial q^j)$. In the associated local coordinate system $(q^1, \ldots, q^n, v^1, \ldots, v^n)$ of the tangent bundle TQ, the volume form is locally given by (3.11). The remarkable point is that if the local coordinates are orthonormal, i.e. $g_{ij} = \delta_{ij}$, Ω is well defined no matter of the choice of the orthonormal system of coordinates, because for an orthonormal change of coordinates the determinant of the Jacobian matrix with elements $J_{ij} = \partial \bar{q}^i / \partial q^j$ is 1, and therefore $dq^1 \wedge \cdots \wedge dq^n \wedge dv^1 \wedge \cdots \wedge dv^n = d\bar{q}^1 \wedge \cdots \wedge d\bar{q}^n \wedge d\bar{v}^1 \wedge \cdots \wedge d\bar{v}^n$.

Suppose we have an *autonomous* second-order vector field Γ in a neighbourhood of a point in the velocity phase space TQ, with coordinate expression (3.4) whose integral curves are the solutions of the associated system of first-order differential equations (3.2). Then, if we choose in such a neighbourhood the volume form (3.11), we can conclude, in full similarity with the preceding result for the Euclidean case that:

Theorem 2 *If the vector field Γ in (3.4) is determined by a regular Lagrangian $L \in C^\infty(TQ)$, where (Q, g) is a n-dimensional Riemann manifold, and the local coordinates are orthonormal, then the determinant of the Hessian matrix in the velocities, with coordinate expression $W = \left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right)$, is a Jacobi multiplier of the system with respect to the volume form (3.11).*

As a consequence of Theorem 2 for second-order differential equations (i.e. one degree of freedom) we recover the above mentioned well-known result on the theory of Jacobi multipliers, that is, the function (3.7) is a Jacobi multiplier of the system (see e.g. [8]).

Nucci *et al.* extended in [35] this result for systems with one-degree of freedom to systems with n-degrees of freedom if the *generalised force is independent of velocities.* The authors stated that, for each pair of indices, the partial derivative $\mu_{ij} = \partial^2 L / \partial v^i \partial v^j$ is a Jacobi multiplier of the system and proved the result for the particular case of systems of two second-order differential equations. The extended result given in [35], valid for SODE vector fields Γ of the form

$$\Gamma = \sum_{i=1}^{n} \left(v^i \frac{\partial}{\partial q^i} + f^i(q) \frac{\partial}{\partial v^i} \right) . \tag{3.12}$$

We first remark that if the system (3.2) is such that the generalised forces $f^i(q, v)$ do not depend on the velocities, then as the divergence of Γ is zero, μ is a Jacobi multiplier of the system iff it is a constant of motion, because since the generalised force of the system is independent of the velocities, $\operatorname{div}(\Gamma) = 0$ and then $\operatorname{div}(\mu\, \Gamma) = \Gamma(\mu)$.

The extended result is summarised in the following theorem:

Theorem 3 *If the vector field Γ in (3.12) is determined by a regular Lagrangian $L \in C^\infty(TQ)$, where (Q, g) is a n-dimensional Riemann manifold, and the local coordinates are orthonormal, then, for each pair of indices the partial derivative $\mu_{ij} = \partial^2 L/\partial v^i \partial v^j$ is a Jacobi multiplier of the system with respect to the volume form (3.11).*

Proof.- If a chart of orthonormal local coordinates in the open set $U \subset Q$ is chosen and we consider the corresponding chart on $\mathscr{U} = \tau^{-1}(U)$, and the corresponding local bases of the $C^\infty(U)$-module $\mathfrak{X}(U)$ and the $C^\infty(\mathscr{U})$-module $\mathfrak{X}(\mathscr{U})$, then, having in mind the explicit form of the 1-form θ_L,

$$\theta_L = \sum_{i=1}^{n} \frac{\partial L}{\partial v^i}\, dq^i, \tag{3.13}$$

we see that for each SODE vector field Γ on $\tau^{-1}(U)$, as

$$\mathscr{L}_\Gamma \theta_L = \mathscr{L}_\Gamma \left(\frac{\partial L}{\partial v^i}\right) dq^i + \frac{\partial L}{\partial v^i}\, dv^i,$$

we have that

$$\left\{ \begin{array}{rcl} i\left(\dfrac{\partial}{\partial q^i}\right) \mathscr{L}_\Gamma \theta_L & = & \mathscr{L}_\Gamma \left(\dfrac{\partial L}{\partial v^i}\right) \\ i\left(\dfrac{\partial}{\partial v^i}\right) \mathscr{L}_\Gamma \theta_L & = & \dfrac{\partial L}{\partial v^i} \end{array} \right. , \tag{3.14}$$

and consequently Γ is a solution of the dynamical equation (3.6) if and only if

$$i\left(\frac{\partial}{\partial q^i}\right) dL = \frac{\partial L}{\partial q^i} = \mathscr{L}_\Gamma \left(\frac{\partial L}{\partial v^i}\right), \tag{3.15}$$

because if $\mathscr{L}_\Gamma \theta_L - dL = 0$, the second equation in (3.14) reduces to an identity and the first one to (3.15), and conversely, if (3.15) holds, the 1-forms dL and $\mathscr{L}_\Gamma \theta_L$ on TQ take the same value on a local basis of vector fields, and then both coincide.

We can make use of these results to prove that if the SODE vector field Γ is solution of the dynamical equation (3.6) for a Lagrangian L such that the f^j components $f^j = i(\partial/\partial v^j)\Gamma$ are basic functions, namely, the vector field Γ is given by (3.12), then the Lagrangian L is such that

$$\frac{\partial^2 L}{\partial q^i \partial v^j} = \frac{1}{2} \frac{\partial^2 (\Gamma L)}{\partial v^i \partial v^j}.$$

In fact, taking into account the commutation relation

$$\left[\Gamma, \frac{\partial}{\partial v^i}\right] = -\frac{\partial}{\partial q^i}, \tag{3.16}$$

we obtain the mentioned result, because

$$\frac{\partial^2 L}{\partial q^i \partial v^j} = \frac{\partial}{\partial v^j}\left(\frac{\partial L}{\partial q^i}\right) = \frac{\partial}{\partial v^j}\left(\mathscr{L}_\Gamma\left(\frac{\partial L}{\partial v^i}\right)\right) = \frac{\partial}{\partial v^j}\left(\frac{\partial(\Gamma L)}{\partial v^i} - \frac{\partial L}{\partial q^i}\right),$$

where use has been made of $\mathscr{L}_\Gamma \circ i(\partial/\partial v^i) - i(\partial/\partial v^i)\circ \mathscr{L}_\Gamma = i([\Gamma, \partial/\partial v^i]) = -i(\partial/\partial q^i)$. Therefore, the regular Lagrangian regular L defining a vector field Γ as in (3.12) satisfies

$$\frac{\partial^2 L}{\partial q^i \partial v^j} = \frac{\partial^2 L}{\partial q^j \partial v^i}. \tag{3.17}$$

Hence, the functions $\mu_{ij} = \partial^2 L/\partial v^i \partial v^j$ are constants of the motion, because using the commutation relation (3.16) of Γ with $\partial/\partial v^j$, we see that

$$\Gamma\left(\frac{\partial}{\partial v^j}\left(\frac{\partial L}{\partial v^i}\right)\right) = \frac{\partial}{\partial v^j}\left(\Gamma\left(\frac{\partial L}{\partial v^i}\right)\right) - \frac{\partial}{\partial q^j}\left(\frac{\partial L}{\partial v^i}\right),$$

which can be rewritten as

$$\Gamma\left(\frac{\partial}{\partial v^j}\left(\frac{\partial L}{\partial v^i}\right)\right) = \frac{\partial}{\partial v^j}\left(\frac{\partial L}{\partial q^i}\right) - \frac{\partial}{\partial q^j}\left(\frac{\partial L}{\partial v^i}\right),$$

as a consequence of the Euler–Lagrange equations, and then (3.17) show that the functions $\partial^2 L/\partial v^i \partial v^j$ are constants of the motion. Therefore, $d\mu_{ij}/dt = 0$ and μ_{ij} is a Jacobi multiplier for (Γ_L, Ω). $\square$.

Using this result in the example of a 2-dimensional coupled oscillator it is possible to find a general Lagrangian for such system [35].

4 Examples

In this section, some examples are given to illustrate several applications of the theory of Jacobi multipliers in the integration of dynamical systems, or as a way to find new constants of motion. A Jacobi multiplier can also be used to solve the inverse problem for a second-order differential equation, allowing us to find a (may be non-standard) Lagrangian description for a given second-order differential equation when a Jacobi multiplier is known [5, 28, 31, 36], or also constants of motion when two inequivalent Jacobi multipliers are known, leading to the result, usually attributed to Currie and Saletan [18], that if two regular Lagrangians L_1 and L_2 are known for a second-order differential equation, the quotient function f defined by

$$f\,\frac{\partial^2 L_1}{\partial v^2} = \frac{\partial^2 L_2}{\partial v^2} \tag{4.1}$$

is a constant of the motion.

4.1 Spherical geometry

Consider the motion of a unity mass point on a sphere of radius $R = 1/\sqrt{\lambda}$ centred at the origin. In the usual spherical polar coordinates [3], a chart is fixed by two coordinates (θ, ϕ) such that $0 < \theta < \pi$, $0 < \phi < 2\pi$, and

$$\mathbf{x}(\theta, \phi) = (R \sin\theta \, \cos\phi, R \sin\theta \, \sin\phi, R \cos\theta).$$

The sphere can be seen as a Riemann manifold with the metric induced by the Euclidean metric in $\mathbb{R}^3$ and then

$$g_{\theta\theta} = R^2, \quad g_{\theta\phi} = 0, \quad g_{\phi\phi} = R^2 \sin^2\theta,$$

i.e. the arc-length is $ds^2 = R^2(d\theta^2 + \sin^2\theta \, d\phi^2)$. The geodesics are given by maximal length arcs. In particular, if we choose *as origin* in the sphere the North pole N, the distance of a generic point with coordinates (θ, ϕ) to N is $R\,\theta$. If we consider the free motion on the sphere the Lagrangian reduces to the kinetic energy

$$T(\theta, \phi, v_\theta, v_\phi) = \frac{1}{2} R^2 (v_\theta^2 + \sin^2\theta \, v_\phi^2).$$

Suppose a central motion, i.e. under the action described by a potential function $V(\theta)$ that does not depend on ϕ but only on θ, a parameter proportional to the distance to the North pole. The Lagrangian of such mechanical system is (see e.g. [3])

$$L(\theta, \phi, v_\theta, v_\phi) = \frac{1}{2} R^2 (v_\theta^2 + \sin^2\theta \, v_\phi^2) - V(\theta).$$

The dynamics is given by the integral curves of the vector field

$$\Gamma_L = v_\theta \frac{\partial}{\partial\theta} + v_\phi \frac{\partial}{\partial\phi} + \left(\sin\theta \, \cos\theta \, v_\phi^2 - \frac{1}{R^2}\frac{\partial V}{\partial\theta}\right) \frac{\partial}{\partial v_\theta} - 2\cot\theta \, v_\theta v_\phi \frac{\partial}{\partial v_\phi}.$$

Then, by Theorem 1, we conclude that, as

$$W_{\theta\theta} = R^2, \quad W_{\theta\phi} = 0, \quad W_{\phi\phi} = R^2 \sin^2\theta,$$

$\mu = \det(W) = R^4 \sin^2\theta$ is a Jacobi multiplier for Γ_L, with volume form $\Omega = d\theta \wedge d\phi \wedge dv_\theta \wedge dv_\phi$. In fact, one can check that

$$\Gamma_L \mu = 2R^4 \, v_\theta \, \sin\theta \, \cos\theta, \qquad \operatorname{div}\Gamma_L = -2\cot\theta \, v_\theta,$$

and therefore $\Gamma_L \mu + \mu \operatorname{div}\Gamma_L = 0$.

4.2 Liénard's equation and Chiellini's condition

Consider the classical Liénard equation [33]

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{4.2}$$

This is a class of second-order differential equations (SODE) in which the damping term is proportional to the velocity, $f(x)\dot{x}$, where f is a non-vanishing function. The Liénard equation defines a SODE vector field in $T\mathbb{R} \approx \mathbb{R}^2$ given by

$$\Gamma = v\frac{\partial}{\partial x} - (f(x)v + g(x))\frac{\partial}{\partial v}. \tag{4.3}$$

We can consider the 2-form $\omega = dx \wedge dv$ on $T\mathbb{R}$ and look for (see [4]) a Jacobi last multiplier for Γ. Since $\operatorname{div}\Gamma = -f$, the function μ is a Jacobi multiplier of the system iff $\Gamma\mu - \mu f = 0$. This is partial differential equation whose most general solution is not easy to find but we can look for a particular solution of the form $\mu(x, v) = (v - W(x))^{1/s}$, wherein s is a non-zero real constant [4]. This function μ is a Jacobi multiplier of the system iff

$$\begin{aligned}
0 &= v\frac{\partial\mu}{\partial x} - (f(x)v + g(x))\frac{\partial\mu}{\partial v} - \mu f(x) \\
&= -\frac{\mu^{1-s}}{s}W'(x)\,v - \frac{\mu^{1-s}}{s}(f(x)v + g(x)) - \mu f(x) \\
&= -\frac{\mu^{1-s}}{s}\big(W'(x)\,v + f(x)v + g(x) + s\mu^s f(x)\big) \\
&= -\frac{\mu^{1-s}}{s}\big(\big[W'(x) + (1+s)f(x)\big]\,v + g(x) - sf(x)W(x)\big).
\end{aligned}$$

So μ is a Jacobi multiplier iff $sW(x) = g(x)/f(x)$ and $W'(x) = -(1+s)f(x)$. Combining the two equations we obtain the following compatibility condition between f and g for the existence of a Jacobi last multiplier μ of the previously chosen form

$$\frac{d}{dx}\left(\frac{g}{f}\right) = kf, \quad \text{with } k = -s(1+s). \tag{4.4}$$

The above condition (4.4) is known in the literature as Chiellini condition [13], and appears in this example as a consequence of the Jacobi multiplier theory. More details can be found in [4, 31].

The Chiellini condition (4.4) is used to integrate the first kind Abel equations [24], and has been applied for obtaining exact solutions of second-order differential equations that can be reduced to an Abel equation. It was shown in [30] the relationship of this condition to its linearisability under an appropriate Sundman transformation [39], and then it was shown in [31] that the corresponding Lagrangian can be easily derived from that of the damped harmonic oscillator.

Using (3.7), the Jacobi multiplier we have found can be used to derive a non-standard Lagrangian for the classical Liénard equation [4].

4.3 Dynamics of a position-dependent mass particle

In a recent paper [11] Casetta studied a particular example of a position-dependent mass one-dimensional system whose dynamical evolution is given by a Meshchersky's equation,

$$m(q)\,\ddot{q} + \frac{dV}{dq} - \alpha\,\dot{q}^2\,\frac{dm}{dq} = 0, \quad \alpha \in \mathbb{R}, \tag{4.5}$$

where $m = m(q)$ is the position-dependent mass function and the real function V is the potential energy. This kind of position-dependent mass systems were studied in [40] in the context of the quadratic Liénard type equation, and it was proved to have an autonomous first integral and, consequently, a Jacobi last multiplier, and therefore to be derived from a Lagrangian.

The associated vector field is

$$\Gamma_\alpha = v\,\frac{\partial}{\partial q} + F_\alpha(q, v)\,\frac{\partial}{\partial v}, \tag{4.6}$$

where

$$F_\alpha(q, v) = \frac{1}{m(q)}\left(-\frac{dV}{dq} + \alpha\,v^2\,\frac{dm}{dq}\right). \tag{4.7}$$

Note that

$$\operatorname{div}\Gamma_\alpha = \frac{\partial F_\alpha}{\partial v} = 2\alpha\,\frac{1}{m(q)}\,\frac{dm}{dq}\,v,$$

and we can look for a Jacobi multiplier μ, for the vector field Γ_α that depends only on the position, i.e. $\partial\mu/\partial v = 0$, and then a solution for the differential equation for the Jacobi last multipliers:

$$\Gamma_\alpha(\mu) + \mu\operatorname{div}\Gamma_\alpha = v\,\frac{d\mu}{dq} + 2\alpha\,\frac{\mu}{m(q)}\,\frac{dm}{dq}\,v = 0,$$

which admits as a particular solution $\mu = (m(q))^{-2\alpha}$.

Therefore, the dynamical system can be defined by a Lagrangian L, uniquely defined, up to addition of a gauge term, that verifies $\mu = \partial^2 L/\partial v^2$. The Lagrangian is then given, up to a gauge term, by

$$L_\alpha(q, v) = \frac{1}{2}(m(q))^{-2\alpha}\,v^2 - \widetilde{V}_\alpha(q), \tag{4.8}$$

for an appropriated function $\widetilde{V}_\alpha$, with an associated energy function

$$E_{L_\alpha}(q,v) = \frac{1}{2}(m(q))^{-2\alpha}\, v^2 + \widetilde{V}_\alpha(q), \tag{4.9}$$

and then, in order to the dynamical vector field to be (4.6) with the function F_α given by (4.7), the function $\widetilde{V}_\alpha$ must be such that

$$-(m(q))^{-2\alpha}\, F_\alpha(q,v) = \frac{\partial E_{L_\alpha}}{\partial q} = -\alpha\,(m(q))^{-2\alpha-1} m'(q)\, v^2 + \widetilde{V}'_\alpha(q),$$

which determines the function $\widetilde{V}_\alpha$ as follows:

$$\widetilde{V}_\alpha(q) = \int^q (m(\zeta))^{-2\alpha-1} V'(\zeta)\, d\zeta, \tag{4.10}$$

as established in [11]. This means that (4.9) with this value of the function $\widetilde{V}_\alpha$ is the only functionally independent constant of motion.

A particularly interesting case is when $\alpha = -1/2$, for which (4.5) reduces to

$$m(q)\, \ddot{q} + \frac{dV}{dq} + \frac{1}{2}\, \dot{q}^2\, \frac{dm}{dq} = 0, \tag{4.11}$$

which, as $\widetilde{V}_{-1/2}(q) = V(q)$, is described by the Lagrangian

$$L_{-1/2}(q,v) = \frac{1}{2} m(q)\, v^2 - V(q). \tag{4.12}$$

Casetta deduced a conservation law along curves in the same level E of the energy function E_L by

$$I(q,v) = f_E \left(\frac{1}{2} m v^2 + V \right),$$

where the following function f_E,

$$f_E = \exp\left(-\left(\frac{1}{2} + \alpha\right) \int^q \frac{2m' m^{2\alpha}(E - \widetilde{V})}{m^{2\alpha+1}(E - \widetilde{V}) + V} d\zeta \right),$$

depends only on the position. Note that, there is a different function for each value of the parameter E. We can prove that $dI = -f_E\, m\, \eta$, with $\eta = F_{-1/2}(q,\, v)dq - v\ dv$, so, as expected, the new constant of motion I and the generalised energy E_L are functionally dependent. In fact,

$$dI = \left[f'_E \left(\frac{1}{2} m v^2 + V \right) + f_E \left(\frac{1}{2} m' v^2 + V' \right) \right] dq + f_E m v\, dv$$

and since $f'_E = -f_E(\frac{1}{2}+\alpha)m'v^2/(\frac{1}{2}mv^2+V)$ we obtain

$$\begin{aligned} dI &= \left[-f_E\left(\tfrac{1}{2}+\alpha\right)m'v^2 + f_E\left(\tfrac{1}{2}m'v^2+V'\right)\right]dq + f_E mv\,dv \\ &= f_E m\left(\left[-\tfrac{1}{m}\left(\tfrac{1}{2}+\alpha\right)m'v^2 + \tfrac{1}{m}\left(\tfrac{1}{2}m'v^2+V'\right)\right]dq + vdv\right) \\ &= f_E m\left(\left[-\tfrac{1}{m}\alpha m'v^2 + \tfrac{1}{m}V'\right]dq + v\,dv\right) = f_E m\left(-F_{-1/2}(q,v)\,dq + v\,dv\right) = -f_E m\,\eta. \end{aligned}$$

However, if the potential energy function is identically zero we obtain $f_E = m^{-1-2\alpha}$ thus $mf_E = \mu$ and $I = E_L$, and then if $\alpha = -\frac{1}{2}$ we have $f_E = 1$ and $I = E_L$.

5 Jacobi multipliers for Boltzmann-Hamel equations

The use of quasi-coordinates or nonholonomic coordinates (see e.g. [9] and [25, 29, 37] for a geometric interpretation of quasi-coordinates) has been shown to be very efficient to deal with many problems in Mechanics. In this section we recall the concept of quasi-coordinates and the implications they have on the equations of motion. We will see that the choice of quasi-coordinates will have a direct effect on the Jacobi multipliers of a Lagrangian system, the associated Lagrangian being the counterpart of the one proved by Ghori [21] for the Hamiltonian formalism.

5.1 Quasi-coordinates overview and Boltzmann-Hamel equations

Consider a configuration manifold Q where a mechanical system is evolving. Recall that usual coordinates on a tangent bundle TQ are induced from a chart on its base manifold Q. As indicated above, given a coordinate chart (U,φ) of Q, we can induce a chart on $\mathscr{U} = \tau^{-1}(U)$ by the tangent map $\phi = T\varphi$, i.e. $\phi(m,v) = (\varphi(m), \varphi_{*m}(v))$, and then, if $\varphi = (q^1,\dots,q^n)$ then the coordinates of a vector v are given by $dq^i(v) = v(q^i)$. However, we can substitute the 1-forms $dq^1,\dots,dq^n$, by n linearly independent non-exact 1-forms $\alpha^1,\dots,\alpha^n$,

$$\alpha^i = \sum_{j=1}^n \alpha^i{}_j(q)\,dq^j\ , \qquad i=1,\dots,n, \qquad \text{with} \quad \det(\alpha^i{}_j(q)) \neq 0\,,$$

and the inverse relation

$$dq^i = \sum_{j=1}^n \beta^i{}_j(q)\ \alpha^j\,, \quad i=1,\dots,n\,, \quad \text{with} \quad \sum_{j=1}^n \alpha^i{}_j(q)\,\beta^j{}_k(q) = \delta^i_k.$$

Then the 1-forms α^i are such that $\alpha^1\wedge\cdots\wedge\alpha^n = \det(\mathscr{A})\,dq^1,\dots,dq^n$, with $\mathscr{A}$ the matrix with elements $\alpha^i{}_j$, and then $\alpha^1\wedge\cdots\wedge\alpha^n$ is a volume form on Q. Consider the basis $\{X_1,\dots,X_n\}$ of the

$C^\infty(Q)$-module of vector fields on Q dual to the basis of 1-forms $\{\alpha^1,\ldots,\alpha^n\}$, i.e. $\langle\alpha^i, X_j\rangle = \delta^i_j$. The expressions of such vector fields are

$$X_j = \sum_{i=1}^n \beta^i{}_j\, \frac{\partial}{\partial q^i}\,, \qquad \frac{\partial}{\partial q^j} = \sum_{i=1}^n \alpha^i{}_j\, X_i. \tag{5.1}$$

They provide a local trivialization of the tangent bundle TQ on $\mathscr{U}$ and the quasi-velocities $\bar v^i$ at a point $m\in Q$ (also called nonintegrable velocities) are the coordinates of the vector with respect to the basis $\{X_{1m},\ldots,X_{nm}\}$ of T_mU determined by such basis of vector fields at $m\in U$.

The base coordinates $\bar q^i$ on TQ are the original ones but the fibre coordinates are related by

$$\bar v^i = \sum_{j=1}^n \alpha^i{}_j\, v^j\,, \qquad v^i = \sum_{j=1}^n \beta^i{}_j\, \bar v^j\ .$$

The coordinate expression of the Lagrange function in terms of these quasi-coordinates is

$$\bar L(\bar q^i, \bar v^i) = L\left(q^i, \sum_{j=1}^n \alpha^i{}_j\, v^j\right).$$

Remark that the notation $\bar q^i$ is convenient because $\bar q^i = q^i$, but

$$\frac{\partial}{\partial q^i} = \frac{\partial}{\partial \bar q^i} + \sum_{k,l=1}^n v^l \frac{\partial \alpha^k{}_l}{\partial \bar q^i}\frac{\partial}{\partial \bar v^k} = \sum_{k,l,r=1}^n \frac{\partial}{\partial \bar q^i} + \sum_{k,l,r=1}^n \beta^l{}_r \bar v^r \frac{\partial \alpha^k{}_l}{\partial \bar q^i}\frac{\partial}{\partial \bar v^k} = \frac{\partial}{\partial \bar q^i} - \sum_{k,l,r=1}^n \bar v^r \alpha^k{}_l \frac{\partial \beta^l{}_r}{\partial \bar q^i}\frac{\partial}{\partial \bar v^k},$$

while

$$\frac{\partial}{\partial \bar q^i} = \frac{\partial}{\partial q^i} + \sum_{k,l=1}^n \bar v^l \frac{\partial \beta^k{}_l}{\partial \bar q^i}\frac{\partial}{\partial v^k} = \frac{\partial}{\partial q^i} + \sum_{k,l,r=1}^n \alpha^l{}_r v^r \frac{\partial \beta^k{}_l}{\partial q^i}\frac{\partial}{\partial v^k} = \frac{\partial}{\partial q^i} - \sum_{k,l,r=1}^n v^r \beta^k{}_l \frac{\partial \alpha^l{}_r}{\partial q^i}\frac{\partial}{\partial v^k}.$$

On the other side,

$$\frac{\partial}{\partial v^i} = \sum_{j=1}^n \alpha^j{}_i \frac{\partial}{\partial \bar v^j}, \qquad \frac{\partial}{\partial \bar v^i} = \sum_{k=1}^n \beta^k{}_i \frac{\partial}{\partial v^k}. \tag{5.2}$$

As a consequence, we also have the following relation

$$\frac{\partial}{\partial v^i}\frac{\partial}{\partial v^j} = \alpha^l{}_i \frac{\partial}{\partial \bar v^l}\left(\alpha^k{}_j \frac{\partial}{\partial \bar v^j}\right) = \alpha^l{}_i \alpha^k{}_j\, \frac{\partial}{\partial \bar v^l}\frac{\partial}{\partial \bar v^j}. \tag{5.3}$$

The Euler–Lagrange equations for the system defined by the Lagrangian function $L\in C^\infty(TQ)$ are (see [9] and [25] for a geometric approach)

$$\frac{d}{dt}\left(\frac{\partial \bar L}{\partial \bar v^l}\right) = \sum_{j=1}^n \beta^j{}_l\, \frac{\partial \bar L}{\partial \bar q^j} + \sum_{j,k=1}^n \bar v^j \gamma^k_{jl} \frac{\partial \bar L}{\partial \bar v^k}\ , \quad l = 1,\ldots,n, \tag{5.4}$$

where the Hamel symbols γ^k_{ml} are the functions given by

$$\gamma^k_{ml} = \sum_{i,j=1}^n \beta^j{}_m \beta^i{}_l \left(\frac{\partial \alpha^k{}_j}{\partial q^i} - \frac{\partial \alpha^k{}_i}{\partial q^j} \right) . \tag{5.5}$$

The geometric interpretation in terms of the Lie algebroid structure was given in [9], where it was shown that the use of quasi-velocities is equivalent to use the Lie algebroid structure of TQ but deliberately forgetting the tangent structure, and using a local basis of sections given by the above mentioned vector fields X_j. Note that the structure functions in such a basis, given by the relation $[X_i, X_j] = C_{ij}{}^k X_k$, coincide with the corresponding Hamel symbols, because

$$\begin{aligned} \left[\sum_{l=1}^n \beta^l{}_i \frac{\partial}{\partial q^l}, \sum_{m=1}^n \beta^m{}_j \frac{\partial}{\partial q^m}\right] &= \sum_{l,m=1}^n \left(\beta^l{}_i \frac{\partial \beta^m{}_j}{\partial q^l} \frac{\partial}{\partial q^m} - \beta^m{}_j \frac{\partial \beta^l{}_i}{\partial q^m} \frac{\partial}{\partial q^l} \right) \\ &= \sum_{klm} \left(\beta^m{}_i \frac{\partial \beta^l{}_j}{\partial q^m} - \beta^m{}_j \frac{\partial \beta^l{}_i}{\partial q^m} \right) \alpha^k{}_l X_k \end{aligned}$$

and if we take into account that, as $\sum_{j=1}^n \alpha^i{}_j(q)\, \beta^j{}_k(q) = \delta^i_k$,

$$\sum_{l=1}^n \frac{\partial \beta^l{}_j}{\partial q^m} \alpha^k{}_l = -\sum_{l=1}^n \beta^l{}_j \frac{\partial \alpha^k{}_l}{\partial q^m}, \qquad \sum_{l=1}^n \frac{\partial \beta^l{}_i}{\partial q^m} \alpha^k{}_l = -\sum_{l=1}^n \beta^l{}_i \frac{\partial \alpha^k{}_l}{\partial q^m},$$

we obtain

$$[X_i, X_j] = \sum_{k,l,m=1}^n \left(\beta^m{}_j \beta^l{}_i \frac{\partial \alpha^k{}_l}{\partial q^m} - \beta^m{}_i \beta^l{}_j \frac{\partial \alpha^k{}_l}{\partial q^m} \right) X_k = \sum_{k,l,m=1}^n \beta^m{}_j \beta^l{}_i \left(\frac{\partial \alpha^k{}_l}{\partial q^m} - \frac{\partial \alpha^k{}_m}{\partial q^l} \right) X_k,$$

from where we see that $C_{ij}{}^k$ coincides with the Hamel symbol γ^k_{ij} given by (5.5).

The Boltzmann-Hamel equations [34, 37] determine the projections on the configuration manifold of the integral curves of the vector field with local expression

$$\Gamma = \sum_{i,j=1}^n \beta^i{}_j \bar{v}^j \frac{\partial}{\partial \bar{q}^i} + \sum_{i=1}^n \bar{f}^i(\bar{q}, \bar{v}) \frac{\partial}{\partial \bar{v}^i}, \tag{5.6}$$

where (see later on for a proof)

$$\bar{f}^i(\bar{q}, \bar{v}) = \sum_{l=1}^n \bar{W}^{il} \left(\sum_{k=1}^n \beta^k{}_l \frac{\partial \bar{L}}{\partial \bar{q}^k} + \sum_{j,k=1}^n \bar{v}^j \gamma^k_{jl} \frac{\partial \bar{L}}{\partial \bar{v}^k} - \sum_{j,k=1}^n \bar{v}^j \beta^k{}_j \frac{\partial^2 \bar{L}}{\partial \bar{q}^k \partial \bar{v}^l} \right), \tag{5.7}$$

and $\bar{W}^{ik}$ represents the matrix element of the inverse matrix of $\bar{W} = \left(\frac{\partial^2 \bar{L}}{\partial \bar{v}^i \partial \bar{v}^j}\right)$. More explicitly, such integral curves are solution of

$$\begin{cases} \dot{\bar{q}}^i = \sum_{j=1}^{n} \beta^i{}_j \, \bar{v}^j \\ \dot{\bar{v}}^i = \bar{f}^i(\bar{q}, \bar{v}) \end{cases} . \tag{5.8}$$

In such quasi-coordinates on the tangent bundle we can consider the following two alternative volume forms

$$\bar{\Omega} = dq^1 \wedge \cdots \wedge dq^n \wedge d\bar{v}^1 \wedge \cdots \wedge d\bar{v}^n, \quad \bar{\bar{\Omega}} = \alpha^1 \wedge \cdots \wedge \alpha^n \wedge d\bar{v}^1 \wedge \cdots \wedge d\bar{v}^n,$$

which are related to Ω by

$$\bar{\Omega} = \det(\mathscr{A})\ \Omega, \qquad \bar{\bar{\Omega}} = (\det(\mathscr{A}))^2\, \Omega. \tag{5.9}$$

Similarly a covector $\zeta \in T_q^*Q$ can be expressed as $\zeta = \sum_{j=1}^{n} \pi_j \alpha^j(q)$, and then $(\pi_1, \ldots, \pi_n)$ are called the quasi-momenta of ζ in the given basis and (q^i, π_k) are called the quasi-coordinates of $\zeta \in T^*Q$ (see [9]). The relation between standard momenta and quasi-momenta is given by the well-known basis change formulas, $\pi_j = \sum_{i=1}^{n} p_i\, \beta^i{}_j(q)$. The Hamel symbols are such that $d\alpha^k = -\sum_{l,m=1}^{n} \frac{1}{2}\gamma^k_{ml} \alpha^m \wedge \alpha^l$. Note that when the 1-forms α^k are all exact the quasi-velocities $\bar{v}^j$'s and the quasi-momenta π_j's would be the velocities v^j's and momenta p_j's w.r.t. a new local coordinate system on Q.

5.2 Dynamical equations in terms of quasi-velocities

Let us first remark that in terms of quasi-coordinates the two geometric objects characterising the tangent bundle structure of TQ, the Liouville vector field Δ, infinitesimal generator of dilations along fibres, and the vertical endomorphism S are respectively given by

$$\Delta = \sum_{i=1}^{n} \bar{v}^i \frac{\partial}{\partial \bar{v}^i}, \quad S = \sum_{i,k=1}^{n} \alpha^k{}_i \frac{\partial}{\partial \bar{v}^k} \otimes d\bar{q}^i,$$

because

$$\frac{\partial}{\partial \bar{v}^i} = \sum_{k=1}^{n} \beta^k{}_i \frac{\partial}{\partial v^k}, \quad \frac{\partial}{\partial v^i} = \sum_{k=1}^{n} \alpha^k{}_i \frac{\partial}{\partial \bar{v}^k}.$$

Consequently, the expression in terms of quasi-coordinates of the energy is given by

$$E_{\bar{L}} = \sum_{j=1}^{n} \bar{v}^j \, \frac{\partial \bar{L}}{\partial \bar{v}^j} - \bar{L},$$

because

$$\frac{\partial \bar{L}}{\partial \bar{v}^j} = \sum_{k=1}^{n} \frac{\partial L}{\partial v^k} \beta^k{}_j,$$

and therefore,

$$\sum_{j=1}^{n} \bar{v}^j \, \frac{\partial \bar{L}}{\partial \bar{v}^j} = \sum_{j,k=1}^{n} \frac{\partial L}{\partial v^k} \beta^k{}_j \bar{v}^j = \sum_{k=1}^{n} \frac{\partial L}{\partial v^k} v^k.$$

Hence,

$$dE_{\bar{L}} = \sum_{j=1}^{n} \frac{\partial \bar{L}}{\partial \bar{v}^j} d\bar{v}^j + \sum_{j,k=1}^{n} \bar{v}^j \, \frac{\partial^2 \bar{L}}{\partial \bar{q}^k \partial \bar{v}^j} d\bar{q}^k + \sum_{j,k=1}^{n} \bar{v}^j \, \frac{\partial^2 \bar{L}}{\partial \bar{v}^j \partial \bar{v}^k} d\bar{v}^k - \sum_{j=1}^{n} \frac{\partial \bar{L}}{\partial \bar{q}^j} d\bar{q}^j - \sum_{j=1}^{n} \frac{\partial \bar{L}}{\partial \bar{v}^j} d\bar{v}^j,$$

i.e.

$$dE_{\bar{L}} = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \bar{v}^j \, \frac{\partial^2 \bar{L}}{\partial \bar{q}^k \partial \bar{v}^j} - \frac{\partial \bar{L}}{\partial \bar{q}^k} \right) d\bar{q}^k + \sum_{j,k=1}^{n} \bar{v}^j \, \frac{\partial^2 \bar{L}}{\partial \bar{v}^j \partial \bar{v}^k} d\bar{v}^k. \tag{5.10}$$

Moreover, the expression in quasi-coordinates of the Liouville 1-form θ_L is

$$\theta_L = dL \circ S = \sum_{i=1}^{n} \frac{\partial L}{\partial v^i} \, dq^i = \sum_{i,k=1}^{n} \frac{\partial \bar{L}}{\partial \bar{v}^k} \, \frac{\partial \bar{v}^k}{\partial v^i} \, dq^i = \sum_{i,k=1}^{n} \frac{\partial \ \bar{L}}{\partial \bar{v}^k} \, \alpha^k{}_i \, d\bar{q}^i = \sum_{k=1}^{n} \frac{\partial \bar{L}}{\partial \bar{v}^k} \, \alpha^k,$$

and therefore,

$$d\theta_L = \sum_{k=1}^{n} d \left(\frac{\partial \bar{L}}{\partial \bar{v}^k} \right) \wedge \alpha^k + \sum_{k=1}^{n} \frac{\partial \bar{L}}{\partial \bar{v}^k} \, d\alpha^k,$$

and developing these expressions

$$\omega_L = - \sum_{j,k=1}^{n} \frac{\partial^2 \bar{L}}{\partial \bar{q}^j \partial \bar{v}^k} d\bar{q}^j \wedge \alpha^k - \sum_{j,k=1}^{n} \frac{\partial^2 \bar{L}}{\partial \bar{v}^k \partial \bar{v}^j} d\bar{v}^j \wedge \alpha^k - \sum_{j,k,l=1}^{n} \frac{\partial \bar{L}}{\partial \bar{v}^k} \frac{\partial \alpha^k{}_j}{\partial \bar{q}^l} d\bar{q}^l \wedge d\bar{q}^j.$$

and therefore,

$$i \left(\frac{\partial}{\partial \bar{q}^j} \right) \omega_L = - \sum_{k,l=1}^{n} \frac{\partial^2 \bar{L}}{\partial \bar{q}^j \partial \bar{v}^k} \alpha^k{}_l \, d\bar{q}^l + \sum_{i,k=1}^{n} \left(\frac{\partial^2 \bar{L}}{\partial \bar{q}^i \partial \bar{v}^k} \alpha^k{}_j \, d\bar{q}^i + \frac{\partial^2 \bar{L}}{\partial \bar{v}^i \partial \bar{v}^k} \alpha^k{}_j d\bar{v}^i - \frac{\partial \bar{L}}{\partial \bar{v}^k} \left(\frac{\partial \alpha^k{}_i}{\partial \bar{q}^j} - \frac{\partial \alpha^k{}_j}{\partial \bar{q}^i} \right) \, d\bar{q}^i \right),$$

and

$$i \left(\frac{\partial}{\partial \bar{v}^j} \right) \omega_L = - \sum_{k=1}^{n} \frac{\partial^2 \bar{L}}{\partial \bar{v}^j \partial \bar{v}^k} \alpha^k = - \sum_{i,k=1}^{n} \frac{\partial^2 \bar{L}}{\partial \bar{v}^j \partial \bar{v}^k} \alpha^k{}_i \, d\bar{q}^i.$$

In other words, the matrix representing $\widehat{\omega}_L$ is

$$\widehat{\omega}_L = \begin{pmatrix} C & -\mathscr{A}^T\bar{W} \\ \bar{W}\mathscr{A} & 0 \end{pmatrix}. \tag{5.11}$$

Here $\widehat{\omega}_L$ denotes the map $\widehat{\omega}_L : \mathfrak{X}(TQ) \to \bigwedge^1(TQ)$ defined by $\widehat{\omega}_L(X) = i(X)\omega_L$. In the usual bases given by $\{\partial/\partial\bar{q}^1, \ldots, \partial/\partial\bar{q}^n, \partial/\partial\bar{v}^1, \ldots, \partial/\partial\bar{v}^n\}$ and its dual it is represented by a matrix

$$\widehat{\omega}_L = \begin{pmatrix} C & M \\ N & 0 \end{pmatrix}$$

where

$$C_{ij} = i\left(\frac{\partial}{\partial\bar{q}^i}\right) i\left(\frac{\partial}{\partial\bar{q}^j}\right)\omega_L, \quad M_{ij} = i\left(\frac{\partial}{\partial\bar{q}^i}\right) i\left(\frac{\partial}{\partial\bar{v}^j}\right)\omega_L, \quad N_{ij} = i\left(\frac{\partial}{\partial\bar{v}^i}\right) i\left(\frac{\partial}{\partial\bar{q}^j}\right)\omega_L,$$

Then we have

$$N_{ij} = i\left(\frac{\partial}{\partial\bar{v}^i}\right) i\left(\frac{\partial}{\partial\bar{q}^j}\right)\omega_L = \sum_{k=1}^{n} \bar{W}_{ik}\alpha^k{}_j = (\bar{W}\mathscr{A})_{ij},$$

and similarly,

$$M_{ij} = i\left(\frac{\partial}{\partial\bar{q}^i}\right) i\left(\frac{\partial}{\partial\bar{v}^j}\right)\omega_L = -\sum_{k=1}^{n} \bar{W}_{jk}\alpha^k{}_i = -(\mathscr{A}^T\bar{W})_{ij},$$

and

$$C_{ij} = \sum_{k=1}^{n}\left(-\alpha^k{}_i\frac{\partial^2\bar{L}}{\partial\bar{q}^j\partial\bar{v}^k} + \frac{\partial^2\bar{L}}{\partial\bar{q}^i\partial\bar{v}^k}\alpha^k{}_j + \frac{\partial\bar{L}}{\partial\bar{v}^k}\left(\frac{\partial\alpha^k{}_j}{\partial\bar{q}^i} - \frac{\partial\alpha^k{}_i}{\partial\bar{q}^j}\right)\right),$$

i.e the matrix representation is the above matrix where $\mathscr{A}^T$ denotes the transpose matrix of $\mathscr{A}$.

Therefore, taking into account (5.10) we see that a vector field $\Gamma = \sum_{i=1}^{n}(\xi^i\partial/\partial\bar{q}^i + \eta^i\partial/\partial\bar{v}^i)$ is a solution of the dynamical equation (3.6) if and only if

$$\sum_{j,k=1}^{n} \bar{W}_{ik}\alpha^k{}_j\xi^j = \sum_{j=1}^{n} \bar{W}_{ij}\bar{v}^j \Longleftrightarrow \xi^i = \sum_{k=1}^{n} \beta^i{}_k\,\bar{v}^k,$$

and (summation on repeated indices is understood)

$$\left[-\alpha^k{}_i\frac{\partial^2\bar{L}}{\partial\bar{q}^j\partial\bar{v}^k} + \frac{\partial^2\bar{L}}{\partial\bar{v}^k\partial\bar{q}^i}\alpha^k{}_j + \frac{\partial\bar{L}}{\partial\bar{v}^k}\left(\frac{\partial\alpha^k{}_j}{\partial\bar{q}^i} - \frac{\partial\alpha^k{}_i}{\partial\bar{q}^j}\right)\right]\xi^j - \frac{\partial^2\bar{L}}{\partial\bar{v}^j\partial\bar{v}^k}\alpha^k{}_i\,\eta^j = \bar{v}^j\frac{\partial^2\bar{L}}{\partial\bar{q}^i\partial\bar{v}^j} - \frac{\partial\bar{L}}{\partial\bar{q}^i}.$$

Recall that according to the definition (5.5) of Hamel symbols,

$$\sum_{i,j=1}^{n}\left(\frac{\partial\alpha^k{}_j}{\partial\bar{q}^i} - \frac{\partial\alpha^k{}_i}{\partial\bar{q}^j}\right)\beta^i{}_l\beta^j{}_m = \gamma^k_{ml} \iff \frac{\partial\alpha^k{}_j}{\partial\bar{q}^i} - \frac{\partial\alpha^k{}_i}{\partial\bar{q}^j} = \sum_{l,m=1}^{n}\alpha^m{}_j\,\alpha^l{}_i\,\gamma^k_{ml},$$

and then using this together with $\xi^i = \sum_{k=1}^{n} \beta^i{}_k \, \bar{v}^k$, the preceding equation becomes (summation on repeated indices is understood)

$$\left(-\alpha^k{}_i \frac{\partial^2 \bar{L}}{\partial \bar{q}^j \partial \bar{v}^k} + \frac{\partial^2 \bar{L}}{\partial \bar{v}^k \partial \bar{q}^i} \alpha^k{}_j + \frac{\partial \bar{L}}{\partial \bar{v}^k} \, \alpha^m{}_j \, \alpha^l{}_i \, \gamma^k_{ml}\right) \beta^j{}_m \, \bar{v}^m - \bar{W}_{jk} \alpha^k{}_i \, \eta^j = \bar{v}^k \frac{\partial^2 \bar{L}}{\partial \bar{q}^i \partial \bar{v}^k} - \frac{\partial \bar{L}}{\partial \bar{q}^i},$$

i.e.,

$$-\alpha^k{}_i \frac{\partial^2 \bar{L}}{\partial \bar{q}^j \partial \bar{v}^k} \beta^j{}_m \, \bar{v}^m + \frac{\partial \bar{L}}{\partial \bar{v}^k} \, \alpha^l{}_i \, \gamma^k_{ml} \, \bar{v}^m - \bar{W}_{jk} \alpha^k{}_i \, \eta^j = -\frac{\partial \bar{L}}{\partial \bar{q}^i},$$

and we can obtain then the second component of the dynamical vector field

$$\eta^r = \sum_{i,s=1}^{n} \beta^i{}_s \bar{W}^{rs} \left(\frac{\partial \bar{L}}{\partial \bar{q}^i} - \sum_{j,k,m=1}^{n} \alpha^k{}_i \frac{\partial^2 \bar{L}}{\partial \bar{q}^j \partial \bar{v}^k} \beta^j{}_m \, \bar{v}^m + \sum_{k,l,m=1}^{n} \frac{\partial \bar{L}}{\partial \bar{v}^k} \, \alpha^l{}_i \, \gamma^k_{ml} \, \bar{v}^m \right),$$

and then, using that $\sum_{i=1}^{n} \alpha^k{}_i \beta^i{}_l = \delta^k_l$,

$$\eta^r = \sum_{s=1}^{n} \bar{W}^{rs} \left(\sum_{i=1}^{n} \beta^i{}_s \frac{\partial \bar{L}}{\partial \bar{q}^i} - \sum_{j,m=1}^{n} \frac{\partial^2 \bar{L}}{\partial \bar{q}^j \partial \bar{v}^s} \beta^j{}_m \, \bar{v}^m + \sum_{k,m=1}^{n} \frac{\partial \bar{L}}{\partial \bar{v}^k} \, \gamma^k_{ms} \, \bar{v}^m \right)$$

which shows (5.7) with a different notation.

5.3 Jacobi multipliers for systems defined by Lagrangians in quasi-coordinates

As far as Jacobi multipliers for the dynamics of systems defined by Lagrangians in quasi-coordinates is concerned, we have the following result:

Theorem 4 *If the vector field Γ whose expression in quasi-coordinates is (5.6) is defined by a regular Lagrangian with quasi-coordinate expression $\bar{L}(\bar{q}, \bar{v})$, then the determinant of the product $\mathscr{A}\bar{W}$ is a Jacobi multiplier of Γ with respect to the volume form $\bar{\Omega}$, and therefore the determinant of the product $\mathscr{A}^2\bar{W}$ is a Jacobi multiplier of Γ with respect to the volume form Ω.*

Proof.- Computing the n-exterior power of the Cartan 2-form, we obtain in quasi-coordinates

$$\begin{aligned}
\Omega_L = \omega_L^{\wedge n} &= \sum_{i_1,\ldots,i_n=1}^{n} \alpha^{i_1} \wedge d\left(\frac{\partial \bar{L}}{\partial \bar{v}^{i_1}}\right) \wedge \cdots \wedge \alpha^{i_n} \wedge d\left(\frac{\partial \bar{L}}{\partial \bar{v}^{i_n}}\right) \\
&= \sum_{i_1,\ldots,i_n=1}^{n} (-1)^{n(n-1)/2} \, n! \, \det(\bar{W}) \, \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_n} \wedge d\bar{v}^1 \wedge \cdots \wedge d\bar{v}^n \\
&= \sum_{i_1,\ldots,i_n=1}^{n} (-1)^{n(n-1)/2} \, n! \, \det(\bar{W}) \det(\mathscr{A}) \, dq^1 \wedge \cdots \wedge dq^n \wedge d\bar{v}^1 \wedge \cdots \wedge d\bar{v}^n, \quad (5.12)
\end{aligned}$$

i.e. $\Omega_L = (-1)^{n(n-1)/2}\, n!\, \det(\bar{W})\, \bar{\bar{\Omega}} = (-1)^{n(n-1)/2}\, n!\, \det(\bar{W})\, \det(\mathscr{A})\bar{\Omega}$. The solution of the dynamical system defined by a Lagrangian is a Hamiltonian vector field w.r.t. the symplectic form ω_L, then $\mu' = 1$ is a Jacobi multiplier for $(\Gamma, \omega_L^{\wedge n})$. Therefore, $\bar{\mu} = \det(\mathscr{A}\bar{W})$ is a Jacobi multiplier of Γ w.r.t. the volume form $\bar{\Omega}$, and $\mu = \det(\mathscr{A}^2\bar{W})$ is a Jacobi multiplier of Γ w.r.t. the volume form Ω. $\Box$

The above result is the Lagrangian equivalent of the one proved by Ghori [21] for the Hamiltonian formalism in quasi-coordinates. In [21] Ghori proved that the determinant of $\mathscr{A}$ is a Jacobi multiplier for Hamel's equations.

Example 1 *(Kepler problem [3, 9]) Consider a particle P of mass m moving in a plane under the action of a central force $F(r) = -kmm'/r^2$ on the direction of a fixed point O of mass $m' \gg m$, where $k > 0$ and $r = \operatorname{dist}(O, P)$. The configuration space of the system is $Q = \mathbb{R}^2 - \{O\}$. Let θ be the angle that the line OP makes with a fixed direction on the plane. The dynamics is determined by the regular Lagrangian in the chart of polar coordinates given by*

$$L = \frac{m}{2}(v_r^2 + r^2 v_\theta^2) + \frac{kmm'}{r}. \tag{5.13}$$

Therefore, as

$$\frac{\partial^2 L}{\partial v_r^2} = m, \qquad \frac{\partial^2 L}{\partial v_r \partial v_\theta} = 0, \qquad \frac{\partial^2 L}{\partial v_\theta^2} = m\, r^2,$$

Theorem 1 gives us a Jacobi multiplier for the system in polar coordinates with respect to the volume form $\Omega = dr \wedge d\theta \wedge dv_r \wedge dv_\theta$: $\mu = \det(W) = m^2\, r^2$.

As the coordinate θ is cyclic, $\partial L/\partial\dot{\theta} = mr^2\,\dot{\theta}$ is a constant of motion. This suggests us to consider the following set of quasi-velocities on the tangent bundle TQ: $\bar{v}_r = \dot{r}$ and $\bar{v}_\theta = r^2\dot{\theta}$, determined by the 1-forms $\alpha_1 = dr$ and $\alpha_2 = r^2\, d\theta$, with associated vector fields $X_1 = \partial/\partial r$ and $X_2 = r^{-2}\partial/\partial\theta$ and Hamel symbols $\gamma_{12}^1 = 0$ and $\gamma_{12}^2 = -2/r$. The dynamics in quasi-coordinates is determined by the Lagrangian

$$\bar{L}(r, \theta, \bar{v}_r, \bar{v}_\theta) = \frac{m}{2}\left(\bar{v}_r^2 + \frac{\bar{v}_\theta^2}{r^2}\right) + \frac{kmm'}{r}, \tag{5.14}$$

and the matrices $\mathscr{A} = (\alpha^i{}_j)$ and $\mathscr{B} = (\beta^i{}_j)$ associated to this set of quasi-velocities are given by

$$\mathscr{A} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \qquad \mathscr{B} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}.$$

Therefore, $\bar{\mu} = \det(\alpha \bar{W}) = m^2$, i.e. any constant, is a Jacobi multiplier for the system in quasi-coordinates and the volume form $\bar{\Omega} = dr \wedge d\theta \wedge d\bar{v}_r \wedge d\bar{v}_\theta$. Note that as $\gamma^1_{12} = 0$, $\gamma^2_{12} = -2/r$ and

$$\frac{\partial \bar{L}}{\partial r} = -\frac{m}{r^3}\bar{v}_\theta^2 - \frac{kmm'}{r^2}, \quad \frac{\partial \bar{L}}{\partial \theta} = 0,$$

the vector field determined by the system of second-order differential equations is

$$\Gamma = \bar{v}_r \frac{\partial}{\partial r} + \frac{\bar{v}_\theta}{r^2}\frac{\partial}{\partial \theta} + \left(-\frac{\gamma m'}{r^2} + \frac{\bar{v}_\theta^2}{r^3}\right)\frac{\partial}{\partial \bar{v}_r}.$$

6 Jacobi multipliers and nonholonomic systems

The use of quasi-coordinates is particularly useful when we have linear nonholonomic constraints, i.e. constraints on the velocities of the type $\sum_{i=1}^n \alpha_i(q) v^i = 0$, that are not derivable from holonomic constraints [1, 9, 29, 34]. If ζ is a 1-form on Q, with coordinate expression $\zeta = \sum_{i=1}^n \zeta_i(q)\, dq^i$, then we denote $\widehat{\zeta}$ the function $\widehat{\zeta} \in C^\infty(TQ)$ given by $\widehat{\zeta}(v) = \zeta_{\tau(v)}(v)$. In the usual coordinates $\widehat{\zeta}$ is given by $\widehat{\zeta}(q,v) = \sum_{i=1}^n \zeta_i(q) v^i$, and therefore the linear nonholonomic constraints are given by the 1-forms $\alpha = \sum_{i=1}^n \alpha_i(q)\, dq^i$, and those that are not derivable from holonomic constraints correspond to non-exact 1-forms.

In the usual formulation [10, 32, 38] for a n degrees of freedom system with $r < n$ functionally independent linear nonholonomic constraints $\widehat{\zeta}^A = 0$, for $A = n-r+1, \ldots, n$, we can set the last r quasi-velocities equal to the constraints and solve the remaining $2n - r$ Hamel's equations: such a set of r linear constraints defines a rank r vector subbundle $\mathscr{D}$ of $\tau : TQ \to Q$ called the *constraint submanifold* S_0. The admissible velocities are the elements of $\mathscr{D}$, and a curve in Q is said to be admissible if its velocities take values in $\mathscr{D}$. From the annihilator $\mathscr{D}^\circ \subset T^*Q$ of $\mathscr{D}$, i.e. the set of 1-forms on TQ vanishing on the elements of $\mathscr{D}$, we can construct the vector subbundle $\widetilde{\mathscr{D}^\circ}$ of $T(TQ)$ defined by $\widetilde{\mathscr{D}^\circ} = \{\, \zeta \circ T\tau \in T^*(TQ) \mid \zeta \in \mathscr{D}^\circ \,\}$.

Given a regular Lagrangian function $L \in C^\infty(TQ)$, consider the nonholonomic system defined by the Lagrangian L and the linear constraints given by $\mathscr{D}$. The evolution of the nonholonomic system is determined by the Lagrange–d'Alembert principle, which states that the dynamics of the system is given by the integral curves of the vector fields Γ tangent to $\mathscr{D}$ satisfying the Lagrange–d'Alembert equation

$$(i(\Gamma)\omega_L - dE_L)|_{\mathscr{D}} \in \widetilde{\mathscr{D}^\circ}. \tag{6.1}$$

If $\mathscr{D}^\circ$ is generated by the set $\{\zeta^A \mid A = n'+1,\dots,n\}$ of 1-forms in Q, with $n' = n - r$, then the vector bundle $\widetilde{\mathscr{D}^\circ}$ is generated by the set of basic 1-forms $\{\tau^*\zeta^A \mid A = n'+1,\dots,n\}$.

More explicitly, the dynamics is given by a vector field Γ determined by the equation (see e.g. [10])

$$i(\Gamma)\omega_L - dE_L = \sum_{A=n'+1}^{n} \lambda_A\, \tau^*\zeta^A, \tag{6.2}$$

where the functions $\lambda_A \in C^\infty(TQ)$ are the Lagrangian multipliers of the system, to be determined by the tangency conditions $\mathcal{L}_\Gamma \widehat{\zeta}^A = 0$, for all $A = n'+1,\dots,n$. This vector field can be written as a sum $\Gamma = \Gamma_L + \sum_{A=n'+1}^{n} \lambda_A\, Z^A$, where Γ_L is given by the right hand side of (5.6) and (5.7), and Z^A is the vertical vector field corresponding by $\widehat{\omega}_L$ to $\tau^*\zeta^A$, i.e. $i(Z^A)\omega_L = \tau^*\zeta^A$. Each vector fields Z^A is vertical, because it corresponds to a basic 1-form $\tau^*\zeta^A$, and therefore $Z^A = \sum_{i=1}^{n} z^{Ai}\partial/\partial v^i$ with $z^{Ai} = -\sum_{j=1}^{n} W^{ij}\zeta^A_j$, i.e. the local expression in quasi-coordinates of Z^A is

$$Z^A = -\sum_{i,j=1}^{n} W^{ij}\zeta^A_j \frac{\partial}{\partial v^i} = -\sum_{i,j,k,l,r=1}^{n} \beta^i{}_j \bar{W}^{jk}\beta^l{}_k \zeta^A_l \alpha^r{}_i \frac{\partial}{\partial \bar{v}^r} = -\sum_{j,k,l=1}^{n} \bar{W}^{jk}\beta^l{}_k\zeta^A_l \frac{\partial}{\partial \bar{v}^j},$$

where use has been made of (5.3), which implies $W = (\mathscr{A})^T\bar{W}\mathscr{A}$. The same result can be obtained directly in quasi-coordinates using the form of $\widehat{\omega}_L$ given by (5.11) and so, if $Z^A = \sum_{i=1}^{n} \bar{z}^{Ai}\partial/\partial\bar{v}^i$, then $\bar{z}^{Ai} = -\sum_{j,k=1}^{n} \bar{W}^{ij}\beta^k{}_j\zeta^A_k$.

Consider appropriate quasi-coordinates as mentioned before, $(\bar{q}^i, \bar{v}^j) = (\bar{q}^i, \bar{v}^a, \bar{v}^A)$ on TQ, with $i = 1,\dots,n$, $a = 1,\dots n'$ and $A = n'+1,\dots,n$, for which the equations defining the constraint manifold $\mathscr{D}$ are simply $\bar{v}^A = \widehat{\zeta}^A = 0$, with $A = n'+1,\dots,n$. In other words, $(\bar{q}^i, \bar{v}^a)$, with $i = 1,\dots,n$ and $a = 1,\dots n'$, are the coordinates for $\mathscr{D}$. Recall that the tangency conditions are $\mathcal{L}_\Gamma \bar{v}^A = 0$, for all $A = n'+1,\dots,n$.

If the matrix of functions in TQ $M^{AB} = Z^A\widehat{\zeta}^B$ is regular we say that the constrained system is regular, and then the tangency conditions uniquely determine the coefficients λ_A as solutions of

$\Gamma_L\widehat{\zeta}^B + \displaystyle\sum_{A=n'+1}^{n} M^{AB}\lambda_A = 0$, and the equation (6.1) has a unique Γ solution of (6.2) and tangent to $\mathscr{D}$.

When using such quasi-coordinates is convenient to choose as a basis for vector fields in TQ, $\{X_i, \partial/\partial\bar{v}^i\}$, where the vector fielsds X_i are given by (5.1). In particular, a SODE vector field on TQ tangent to the constraint manifold is of the form

$$\Gamma = \sum_{i=1}^{n} \bar{v}^i X_i + \sum_{a=1}^{n'} \bar{f}^a(\bar{q}^i, \bar{v}^i)\frac{\partial}{\partial \bar{v}^a}. \tag{6.3}$$

The constraints $\bar{v}^A = \widehat{\zeta}^A = 0$ define a $(2n-r)$-dimensional submanifold S_0 of TQ,

$$j : U \to TQ, \quad j(\bar{q}_1, \dots, \bar{q}_n, \bar{v}^1, \dots, \bar{v}^{n-r}) = (\bar{q}_1, \dots, \bar{q}_n, \bar{v}^1, \dots, \bar{v}^{n-r}, 0, \dots, 0),$$

where U is an open set of $\mathbb{R}^{n-r}$, and there is a vector field $\bar{\Gamma}_0$ on U that is j-related to Γ.

Hence, the system of differential equations for the integral curves of vector field obtained by restricting Γ to the constraint submanifold, which has the same coordinate expression as in (6.3), are

$$\left\{\begin{array}{rcl} \dot{\bar{q}}^i &=& \displaystyle\sum_{a=1}^{n'} \beta^i{}_a(\bar{q})\bar{v}^a \\ \dot{\bar{v}}^a &=& \bar{f}^a(\bar{q}^i, \bar{v}^b) \end{array}\right. , \tag{6.4}$$

with

$$f^a(\bar{q}^i, \bar{v}^b) = \sum_{b=1}^{n'} \bar{W}^{ab} \sum_{k=1}^{n} \left(\beta^k{}_b \frac{\partial \bar{L}}{\partial \bar{q}^k} + \sum_{c=1}^{n'} \bar{v}^c \gamma^k_{cb} \frac{\partial \bar{L}}{\partial \bar{v}^k} - \sum_{c=1}^{n'} \bar{v}^c \beta^k{}_c \frac{\partial^2 \bar{L}}{\partial \bar{q}^k \partial \bar{v}^b} \right).$$

We can consider the volume form in TQ

$$\bar{\Omega} = d\bar{q}^1 \wedge \ldots \wedge d\bar{q}^n \wedge d\bar{v}^1 \wedge \ldots \wedge d\bar{v}^n, \tag{6.5}$$

and in the constraint manifold S_0 the following volume form

$$\bar{\Omega}_0 = d\bar{q}^1 \wedge \ldots \wedge d\bar{q}^n \wedge d\bar{v}^1 \wedge \ldots \wedge d\bar{v}^{n'}, \tag{6.6}$$

and then the divergence of $\bar{\Gamma}_0$ with respect to such a volume form is given by

$$\operatorname{div}(\bar{\Gamma}_0) = \sum_{a=1}^{n'} \left(\sum_{i=1}^{n} \bar{v}^a \frac{\partial \beta^i{}_a}{\partial q^i} + \frac{\partial \bar{f}^a}{\partial \bar{v}^a} \right). \tag{6.7}$$

Notice that, according to (6.2),

$$\mathcal{L}_\Gamma \omega_L = \sum_{a=1}^{n'} d(\lambda_A \tau^* \alpha^A).$$

So, in general, $\mathcal{L}_\Gamma \Omega_L \neq 0$, with $\Omega_L = \omega_L^{\wedge n}$. If μ is a Jacobi multiplier for (Γ, Ω_L) then, as $\Omega_L = \delta \det(\mathscr{A}\bar{W})\,\bar{\Omega}$, with δ a constant, $\bar{\mu} = \mu \det(\mathscr{A}\bar{W})$ is a Jacobi multiplier for $(\Gamma, \bar{\Omega})$. On the other hand, if a Jacobi multiplier $\bar{\mu}$ for $(\Gamma, \bar{\Omega})$ is known, then $\mu = \bar{\mu}/\det(\mathscr{A}\bar{W})$ is a Jacobi multiplier for (Γ, Ω_L).

It is to be remarked that the true tangency condition is the less restrictive condition $\left(\mathcal{L}_\Gamma \widehat{\zeta}^A\right)_{|\widehat{\zeta}^A=0} = 0$, and therefore the vector field Γ is not fully determined by the tangency conditions but only its restriction $\bar{\Gamma}_0$ to the constraint submanifold S_0.

Example 2 *Consider the motion of a free particle of unitary mass in the configuration space* $\mathbb{R}^3$, *under the action of the linear constraint* $\bar{v}_z = v_z - y v_x$ *(see [1, 9]). Let* $(x, y, z, \bar{v}_x, \bar{v}_y, \bar{v}_z)$ *be a system of quasi-coordinates on the tangent bundle* $T\mathbb{R}^3$, *where*

$$\bar{v}_x = v_x, \quad \bar{v}_y = v_y \quad \textit{and} \quad \bar{v}_z = v_z - y v_x,$$

i.e.

$$\zeta^1 = dx, \quad \zeta^2 = dy, \quad \zeta^3 = dz - y\,dx.$$

Hence, the transformation matrix is given by

$$\mathscr{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}, \qquad \mathscr{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & 0 & 1 \end{pmatrix}.$$

The motion of the free particle is defined by the regular Lagrangian function

$$\bar{L}(x, y, z, \bar{v}_x, \bar{v}_y, \bar{v}_z) = \frac{1}{2}\left(\bar{v}_x^2 + \bar{v}_y^2 + (\bar{v}_z + y\bar{v}_x)^2\right),$$

and the corresponding Hessian matrix in the quasi-velocities is given by

$$\bar{W} = \begin{pmatrix} 1+y^2 & 0 & y \\ 0 & 1 & 0 \\ y & 0 & 1 \end{pmatrix}.$$

Then, $\det(\mathscr{A}\bar{W}) = 1$ *is a Jacobi multiplier for the free system whose solution is given by the vector field*

$$\Gamma_L = \bar{v}_x\,\partial/\partial x + \bar{v}_y\,\partial/\partial y + (\bar{v}_z + y\bar{v}_x)\,\partial/\partial z - \bar{v}_x\bar{v}_y\,\partial/\partial\bar{v}_z.$$

The nonholonomic system is given by the vector field $\Gamma = \Gamma_L + \lambda\,Z$ *tangent to* $\mathscr{D}$*, where* Z *is a vertical vector field determined by the equation* $i_Z\omega_L = \tau^*\zeta^3$*, namely:*

$$Z = y\,\frac{\partial}{\partial v_x} - \frac{\partial}{\partial v_z} = y\,\frac{\partial}{\partial\bar{v}_x} - (1+y^2)\,\frac{\partial}{\partial\bar{v}_z},$$

λ *is the Lagrange multiplier associated to the constraint* $\bar{v}_z = 0$ *determined by the tangency condition,*

$$\lambda = -\frac{\bar{v}_x\bar{v}_y}{1+y^2}\,,$$

and then

$$\Gamma = \bar{v}_x\,\frac{\partial}{\partial x} + \bar{v}_y\,\frac{\partial}{\partial y} + (\bar{v}_z + y\bar{v}_x)\frac{\partial}{\partial z} - \frac{y\,\bar{v}_x\bar{v}_y}{1+y^2}\,\frac{\partial}{\partial\bar{v}_x},$$

whose restriction to the constraint submanifold coincides with

$$\Gamma_0 = \bar{v}_x\,\frac{\partial}{\partial x} + \bar{v}_y\,\frac{\partial}{\partial y} + y\bar{v}_x\,\frac{\partial}{\partial z} - \frac{y\,\bar{v}_x\bar{v}_y}{1+y^2}\,\frac{\partial}{\partial\bar{v}_x}.$$

The divergence of the solution Γ *with respect to the volume form* $\bar{\Omega}$ *is non-zero,*

$$\operatorname{div}(\Gamma) = -\frac{y\bar{v}_y}{1+y^2}, \tag{6.8}$$

and similarly for the divergence of the vector field $\bar{\Gamma}_0$ *with respect to the volume form* $\bar{\Omega}_0$*.*

However, the equation (6.8) implies that

$$\frac{d}{dt}\ln(\sqrt{1+y^2}) + \operatorname{div}(\Gamma) = 0, \tag{6.9}$$

and similarly for $\bar{\Gamma}_0$ *Therefore,* $\mu = \sqrt{1+y^2}$ *is a Jacobi multiplier for* $(\Gamma,\bar{\Omega})$ *and also for* (Γ,Ω_L)*, because* $\det(A\bar{W}) = 1$*. Additionally, we can prove that* $\eta = (1+y^2)\bar{v}_x$ *is also a Jacobi multiplier of the system, and then* $I = \mu/\eta = \sqrt{1+y^2}\,\bar{v}_x$ *is a constant of motion for the nonholonomic system.*

7 A reduction theorem of a nonholonomic system in quasi-coordinates

Consider the nonholonomic system defined in the previous section, given by the vector field Γ. Remark that $i(\Gamma)dE_L = 0$, that is, the energy of the free system is a constant of motion of the nonholonomic system. Indeed, by the formula (6.2),

$$i(\Gamma)dE_L = i(\Gamma)i(\Gamma)\omega_L - \sum_{A=n'+1}^{n} \lambda_A i(\Gamma)\,\tau^*\alpha^A = -\sum_{A=n'+1}^{n} \lambda_A i(\Gamma)\,\tau^*\alpha^A. \tag{7.1}$$

Since the tangency condition $i(\Gamma)\,\tau^*\alpha^A = 0$, for all $A = n'+1,\ldots,n$, then we obtain $i(\Gamma)dE_L = 0$.

Recall now the main property leading to the introduction of the concept of Jacobi last multiplier for a given vector field.

Theorem 5 *Let μ be a Jacobi multiplier for a vector field $X \in \mathfrak{X}(M)$ on an oriented n-dimensional manifold (M,Ω). If $\{I_1,\ldots,I_k\}$ is a family of k functionally independent first-integrals of X, then for any k real numbers, $a_1,\ldots,a_k$, the vector field X is tangent to the $(n-k)$-dimensional manifold $S_{(a_1,\ldots,a_k)}$ given by the level set*

$$S_{(a_1,\ldots,a_k)} = \mathbf{I}^{-1}(a_1,\ldots,a_k) = \bigcap_{l=1}^{k} I_l^{-1}(a_l)\,,$$

where $\mathbf{I}: M \to \mathbb{R}^k$ is the map $\mathbf{I}(p) = (I_1(p),\ldots,I_k(p))$, $p \in M$. Moreover, $j^*\mu \in C^\infty(S_{(a_1,\ldots,a_k)})$ is a Jacobi multiplier for the restriction of X to such level set $S_{(a_1,\ldots,a_k)}$ endowed with the $(n-k)$-form $j^*\tau$, where $j: S_{(a_1,\ldots,a_k)} \to M$ defines the $(n-k)$-dimensional submanifold $S_{(a_1,\ldots,a_k)}$ and τ is the $(n-k)$-form $\tau = *(dI_1\wedge\cdots\wedge dI_k)$, with $*$ denoting the Hodge operator.

Proof.- The functions I_j are first-integrals, $XI_j = 0$, and therefore X is tangent to each one of the mentioned level sets, i.e. for any k real numbers, $a_1,\ldots,a_k$, there exists a vector field $\bar{X}$ in the manifold defining $S_{(a_1,\ldots,a_k)}$ such that X is j-related to $\bar{X}$.

Note that as the k-first integrals are assumed to be functionally independent, $dI_1\wedge\cdots\wedge dI_k \neq 0$, and the set of k 1-forms $\{dI_1,\ldots,dI_k\}$ span a $(n-k)$-dimensional integrable distribution, which can also be defined by the k-form $dI_1\wedge\cdots\wedge dI_k$. The leaves of such foliation are the above mentioned level sets of $\mathbf{I}$ and the vector field X belongs to such distribution. There are adapted local coordinates systems with coordinates $(I_1,\ldots,I_k,y^1,\ldots,y^{n-k})$ for which the expression of the vector field X is $X = Y_\alpha(I,y)\,\partial/\partial y^\alpha$. The leaf $S_{(a_1,\ldots,a_k)}$ is defined by a map $j: U \to M$, where U is an open set of $\mathbb{R}^{n-k}$, given by

$$j(y^1,\ldots,y^{n-k}) = (a_1,\ldots,a_k,y^1,\ldots,y^{n-k}).$$

Now note that the n-form $\Omega_1 = dI_1\wedge\cdots\wedge dI_k\wedge dy^1\wedge\cdots\wedge dy^{n-k}$ is proportional to the volume form Ω, i.e. there exists a function f such that $\Omega_1 = f\,\Omega$. The fact that μ is a Jacobi multiplier for X with respect to the volume form Ω means that $\mathscr{L}_{\mu\,X}(\Omega)=0$, i.e $\mathscr{L}_{\mu\,X}(\Omega_1)=(\mu\,Xf/f)\Omega_1$. On the other side, as $\mathscr{L}_{\mu\,X}(dI_1\wedge\cdots\wedge dI_k)=0$, we have $\mathscr{L}_{\mu\,X}\Omega_1 = dI_1\wedge\cdots\wedge dI_k\wedge\mathscr{L}_{\mu\,X}(dy^1\wedge\cdots\wedge dy^{n-k})$, and therefore, comparing both values of $\mathscr{L}_{\mu\,X}(\Omega_1)$ we arrive to $\mathscr{L}_{\mu\,X}(dy^1\wedge\cdots\wedge dy^{n-k}) = \mu\,(Xf/f)dy^1\wedge\cdots\wedge dy^{n-k}$. Now, using the Hodge operator defined on (M,Ω) we obtain that $\tau = *(dI_1\wedge\cdots\wedge dI_k) = (1/f)dy^1\wedge\cdots\wedge dy^{n-k}$, and then

$$\mathscr{L}_{\mu\,X}\tau = \mathscr{L}_{\mu\,X}\left(\frac{1}{f}dy^1\wedge\cdots\wedge dy^{n-k}\right) = -\frac{\mu\,Xf}{f^2}dy^1\wedge\cdots\wedge dy^{n-k}+\frac{1}{f}\mathscr{L}_{\mu\,X}(dy^1\wedge\cdots\wedge dy^{n-k}),$$

and therefore,

$$\mathscr{L}_{\mu\,X}\tau = \left(-\frac{\mu\,Xf}{f^2}+\frac{\mu\,Xf}{f^2}\right)dy^1\wedge\cdots\wedge dy^{n-k}=0.$$

Recall that $(j^*\mu)(y^1,\ldots,y^{n-k}) = \mu(a_1,\ldots,a_k,y^1,\ldots,y^{n-k})$ and that the vector field $j^*\mu\,\bar{X}$ is j-related to $\mu\,X$. Moreover, as

$$\mathscr{L}_{(j^*\mu)\,\bar{X}}(j^*\tau) = d\left(i((j^*\mu)\bar{X})(j^*\tau)\right) = d\left(j^*(i(\mu\,X)\tau)\right) = j^*\left(d(i(\mu\,X)\tau)\right) = j^*\left(\mathscr{L}_{\mu\,X}\tau\right)\,,$$

and μ is a Jacobi multiplier for X, then we see that $\mathscr{L}_{(j^*\mu)\,\bar{X}}(j^*\tau)=0$, i.e. $j^*\mu$ is a Jacobi multiplier for $\bar{X}$ with respect to the volume form $j^*\tau$.□

Corollary 6 *If a Jacobi multiplier μ and $(n-2)$ functionally independent first-integrals, $\{I_1,\ldots,I_{n-2}\}$, for a given vector field X on an oriented manifold (M,Ω) are known, the determination of the integral curves of the vector field X is reduced to quadratures.*

Proof.- It is a particular case of the previous result for $k=n-2$. Once that $n-2$ values $a_1,\ldots,a_{n-2}$, of the first integrals $I_1,\ldots,I_{n-2}$ have been fixed, we consider the 2-dimensional manifold $j: S_{(a_1,\ldots,a_{n-2})}\to M$ endowed with the 2-form $j^*\tau$, defined as above, $\tau = *(dI_1\wedge\cdots\wedge dI_{n-2})$. Now the pull back $j^*\mu$ is a Jacobi last multiplier for the restriction of the vector field, i.e. $j^*\mu\;[i(\bar{X})j^*\tau]$ is closed, and the problem is reduced to quadratures.□

The preceding result is due to Jacobi (see e.g. [28]) and this is the reason for the adjective 'last' for this Jacobi multiplier.

The case of a nonholonomic system described by a regular Lagrangian L defined in TQ, for a n-dimensional manifold Q, and under the action of r nonholonomic linear constraints $\widehat{\zeta}^A = 0$, for $A = n - r + \dots, n$, is a particular case of a $N = 2n$ dimensional oriented manifold $(TQ, \omega_L^{\wedge n})$ and the constants of motion are given by the r linear constraints $\widehat{\zeta}^A = 0$ and the leaf we are interested in is given by the zero values set.

The functions $\widehat{\zeta}^A$ are first-integrals, because the vector Γ has been chosen to be tangent to each one of the mentioned level sets, and in particular for the zero level, and then there exists a vector field $\bar{\Gamma}_0$ in the manifold defining S_0 such that Γ is j-related to $\bar{\Gamma}_0$. Moreover, if μ is a Jacobi multiplier for the vector field Γ with respect to $\bar{\Omega}$, then $j^*\mu \in C^\infty(S_0)$ is a Jacobi multiplier for the restriction of Γ to such level S_0 endowed with the $(2n-r)$-form $j^*\tau$, where $j : S_0 \to TQ$ defines the $(2n-r)$-dimensional submanifold S_0 and τ is the $(2n-r)$-form $\tau = *(d\bar{v}_{n-r+1} \wedge \cdots \wedge d\bar{v}_n)$, with $*$ being the Hodge operator. If, furthermore, we know $s = 2n - r - 2$ functionally independent constants of motion $I_1, I_2, \dots, I_s$, the dynamics reduces to a two-dimensional system endowed with a symplectic form that admits a Jacobi multiplier, and consequently the reduced two-dimensional system is integrable by quadratures.

8 Summary and outlook

In this paper we have reviewed from a geometric point of view the theory of Jacobi multipliers and some of its applications, and in particular in the case of second order dynamics described by a Lagrangian. As our aim was to show how the Jacobi multiplier theory can be used to solve nonholonomic systems that are described by Hamel's formalism we have presented the concept of quasi-velocity in a geometric approach and the Boltzmann-Hamel equations of the dynamics are rederived. The theory of Jacobi multipliers in the particular case of systems with linear nonholonomic constraints have been developed as well as an updated geometric rederivation in geometric terms of the Jacobi's main result concerning the role played by the Jacobi 'last' multiplier for the integrability by quadratures. The particular case of systems with linear nonholonomic constraints and its reduction has been analysed.

There remain many applications to be developed from this new perspective of Jacobi multipliers and the use of appropriate quasi-coordinates. The study of Chaplygin systems [19, 20] deserves a special attention and will be developed in a future paper. Moreover, as the problem of a Jacobi multiplier for a given vector field in an oriented manifold can be seen as the search for a conformally related vector field leaving invariant the considered volume form, this suggests the study of geometric structures, given by tensor fields, that are invariant under a conformally related vector field, a particular case being the process of Hamiltonisation of a vector field.

9 Acknowledgments

The work of Patrícia Santos was partially supported by the Centre for Mathematics of the University of Coimbra UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES. José F. Cariñena acknowledges the financial support from the Spanish Ministerio de Ciencia, Innovación y Universidades project PGC2018-098265-B-C33.

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