GENERALIZED BIVARIATE HERMITE FRACTAL INTERPOLATION FUNCTION

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Abstract. Fractal interpolation provides an efficient way to describe smooth or non-smooth structure associated with nature and scientific data. The aim of this paper is to introduce a bivariate Hermite fractal interpolation formula, which generalizes the classical Hermite interpolation formula for two variables. It is shown here that the proposed Hermite fractal interpolation function and its derivatives of all order are good approximation of the original function even if the partial derivative of the original functions are non-smooth in nature.

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1. Introduction

The classical Hermite interpolation problem deals with the construction of suitable approximately function based on the functional values and its derivative values at prescribed node points. The Hermite's interpolation formula [20] provided an expression for a polynomial which passes through given points with fixed derivative values at the grid points. Spitzbart [19] proposed a generalization of Hermite's interpolation formula in one variable, and obtained a polynomial $p(x)$ of degree $n + \sum\limits_{j=0}^{n} r_j$ in x which interpolates to the values of a function and its derivatives up to order r_j at $x_j, j = 0, 1, \dots, n$. The derivatives of classical Hermite function are either smooth or piece-wise smooth and hence they are not ideal to capture varying non-smoothness associated with the derivatives of original function. Fractal interpolation technique defined via iterated function system (IFS) provides a efficient way to capture varying non-differentiability associated in a given domain. This method constitute an advance in the fact that all classical methods can be obtained as a particular case of fractal function. This method is introduced first by Barnsley [2]. After that fractal interpolation functions (FIFs) have been used widely in various field like computer graphics, image processing, modelling of natural surfaces and so on. Barnsley and Harrington [4] proved the existence of differentiable FIFs. Later on Navascués [16] constructed a family of fractal function f^α and depending on the scaling vector α, f^α is smooth or non-smooth in nature. The function f^α retains some properties such as continuity and integrability of f. The process of obtaining α-fractal function by using a continuous function determines an operator $\mathcal{F}^\alpha : \mathcal{C}(I) \mapsto \mathcal{C}(I),\ f \mapsto f^\alpha$. This map links the theory of classical approximation and FIF

([15], [21], [22]). Assuming some conditions on the scaling factors, convergence of f^α towards f is obtained [14].
Interpolation of surface data nowadays plays crucial role in science and technology. To cover the more complex geometry hidden within the construction, fractal surface technique is required. On the basis of construction of FIFs, fractal surfaces constructed via IFS were first introduced by Massopust [12]. He considered the case when the domain is triangular and the interpolation points are coplanar. A more general construction was studied by Geronimo and Hardin [11]. Zhao [24] gave even more general construction of both affine and non-affine fractal interpolation surfaces (FISs) using arbitrary contraction factors on triangular domains. After that Xie and Sun [23] constructed FISs on rectangular grids with the help of scaling factors and without using boundary condition. But this leads to attractor which are not graph of continuous functions. Dalla ([5], [9]) used co-linear data and proved that the attractor is a continuous surface. All the construction mentioned above leads to self-similar attractors. However, with the existing methods we cannot obtain fractal surfaces if there is a set of functional values at all the grid points along with derivative values of various orders in both directions. Thus our method will be useful to construct the fractal surfaces from a set of bivariate Hermite data. Ahlin [1] considered a bivariate generalization of Hermite's interpolation formula. He developed a bivariate osculatory interpolation polynomial which agrees with $f(x, y)$ and its partial and mixed partial derivatives up to a specified order at each of the nodes of a Cartesian grid. A bivariate generalization of Spitzbart's formula is given in [7], and it is useful for the cases where only functional values are given but no partial derivation along x or y direction is given. In this paper we introduce a new construction of fractal interpolation surfaces using the bivariate Hermite interpolation that gives rise to smooth surfaces.

The paper is organized as follows: In Section 2 we first give a brief introduction of fractal interpolation function and the classical bivariate Hermite interpolation formula. Then we extend this interpolation formula using fractal procedures and obtained error bounds of bivariate Hermite FIF functions in Section 3. Finally we give some example and graph of the interpolated function and its derivatives in both x and y direction in Section 4. Conclusion is given in Section 5.

2. Backgrounds and preliminaries

In this section we shall review some relevance general material on fractal functions that can be found in details ([2], [3], [6], [18]).

2.1. **Basic of FIF theory.** Let $\Delta := \{x_0, x_1, \ldots, x_N\}$ be a partition of a real compact interval $I = [a, b]$, satisfying $a = x_0 < x_1 < \cdots < x_N = b$. Let a set of data points $\{(x_n, y_n), n \in \mathbb{N}_N \cup \{0\}\}$ be given, where $\mathbb{N}_k$ is the first k natural numbers, and $I_n = [x_{n-1}, x_n]$. Let $L_n : I \to I_n, n \in \mathbb{N}_N$ be contractive homeomorphisms such that

$$L_n(x_0) = x_{n-1},\ L_n(x_N) = x_n. \tag{2.1}$$

Let $K = I \times \mathbb{R}$ and N continuous mappings, $F_n : K \to \mathbb{R}$ satisfying

$$F_n(x_0, y_0) = y_{n-1}, F_n(x_N, y_N) = y_n, \ |F_n(x, y) - F_n(x, y')| \le |c_n||y - y'|, \tag{2.2}$$

where $(x, y), (x, y') \in K, \ c_n \in (-1, 1), \ n \in \mathbb{N}_N$. Let X be a complete metric space and $\{w_n : X \to X; i = 1, 2, \ldots, N\}$ be a collection of continuous functions on X. Then $\{X; w_n : i = 1, 2, ..., N\}$ is called an Iterated Function System (IFS) on X. Now define functions $w_n : K \to K$ as $w_n(x, y) = (L_n(x), F_n(x, y)) \ \forall \ n \in \mathbb{N}_N$. The following is a fundamental theorem between the IFS and fractal functions.

Theorem 2.1. *(Barnsley [2]) Let $\mathcal{C}(I)$, the space of all real-valued continuous functions on a compact interval I, be endowed with the Chebyshev norm $\|g\|_\infty := \max\{|g(x)| : x \in I\}$ and consider the closed metric subspace*

$$\mathcal{C}_{y_0,y_N}(I) := \{g \in \mathcal{C}(I) : \ g(x_0) = y_0, \ g(x_N) = y_N\}.$$

The following hold.

(1) The IFS $\{K; w_n, n = 1, 2, \ldots, N\}$ has a unique attractor G which is the graph of a continuous function $f^* : I \to \mathbb{R}$ satisfying $f^*(x_n) = y_n$ for $n = 0, 1, \ldots, N$.
(2) The function f^* is the fixed point of the Read-Bajraktarević (RB) operator T on $\mathcal{C}_{y_0,y_N}(I)$ as

$$(Tg)(x) = F_n\big(L_n^{-1}(x), g \circ L_n^{-1}(x)\big), \ x \in I_n, \ n \in \mathbb{N}_N.$$

The function f^* appearing in the foregoing theorem is called a fractal interpolation function (FIF) corresponding to $\{(x_n, y_n), n \in \mathbb{N}_N \cup \{0\}\}$, and it is unique satisfying the functional equation

$$f^*(x) = F_n(L_n^{-1}(x), f^* \circ L_n^{-1}(x)) \ \ \forall x \in [x_{n-1}, x_n], \ \ n \in \mathbb{N}_N. \tag{2.3}$$

The most frequently used fractal functions are defined by the IFS

$$L_n(x) = a_n x + d_n, \ F_n(x, y) = \alpha_n y + q_n(x), \tag{2.4}$$

where $\alpha_n \in (-1, 1)$ is called the vertical scaling factor of the transformation w_n, $q_n : I \to \mathbb{R}$ are continuous functions satisfying

$$q_n(x_0) = y_{n-1} - \alpha_n y_0, \quad q_n(x_N) = y_n - \alpha_n y_N$$

due to conditions in (2.1) and (2.2). The factor $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in (-1, 1)^N$ is called the scale vector of a FIF and depending on the magnitude of scaling vector, it is possible to get a wide variety of interpolants.

2.2. **α-fractal functions.** Let $f \in \mathcal{C}(I)$ be a continuous function. For the partition Δ of I, consider the case $q_n(x) = f \circ L_n(x) - \alpha_n b(x)$, $n \in \mathbb{N}_N$, where b is defined through the linear map $L : \mathcal{C}(I) \to \mathcal{C}(I)$, $b = Lf$, such that L is bounded with respect to the sup-norm and satisfy $Lf(x_0) = f(x_0)$ and $Lf(x_N) = f(x_N)$.

Definition 2.2. [16] Let f^α be the continuous function defined by the IFS (2.3)-(2.4). f^α is the α-fractal function associated with f with respect to b, the partition Δ and scale vector α.

According to (2.3) and (2.4), f^α satisfies the fixed point equation

$$f^\alpha(x) = f(x) + \alpha_n (f^\alpha - b) \circ L_n^{-1}(x),\ x \in I_n,\ n \in \mathbb{N}_N. \tag{2.5}$$

The uniform distance between f^α and f is bounded as (see for instance [16])

$$\|f^\alpha - f\|_\infty \le \frac{|\alpha|_\infty}{1-|\alpha|_\infty}\|f-b\|_\infty, \tag{2.6}$$

where $|\alpha|_\infty = \max\{|\alpha_n|; n \in \mathbb{N}_N\}$.
According to (2.6), if $\alpha = 0$ or $f = b$, then $f^\alpha = f$. The existence of differentiable FIF is guaranteed by the following proposition.

Proposition 2.3. *[4] Let $\{(x_n, y_n) | n = 0, 1, 2, \dots, N\}$ be the interpolation data with $x_0 < x_1 < \dots < x_N$. Let $L_n(x) = a_n x + b_n$, $F_n(x, y) = \alpha_n y + q_n(x)$, for $n = 1, 2, \dots, N$. Suppose for some integer $s > 0$, $|\alpha_n| < a_n^s$ and $q_n \in \mathcal{C}^s[a, b], n = 1, 2, \dots, N$. Let $F_{n,k}(x, y) = \frac{\alpha_n y + q_n^k}{a_n^k}$, $y_{0,k} = \frac{q_1^k(x_0)}{a_1^k - \alpha_1}$, $y_{N,k} = \frac{q_1^k(x_N)}{a_N^k - \alpha_N}$, $k = 1, 2, \dots, s$. If $F_{n-1}(x_N, y_{N,k}) = F_n(x_0, y_{0,k})$ for $n = 2, 3, \dots, N$ and $k = 1, 2, \dots, s$, then $\{L_n(x), F_{n,k}(x, y)\}_{n \in \mathbb{N}_N}$ determines a FIF $f \in \mathcal{C}^s[x_0, x_N]$.*

2.3. **The bivariate Hermite interpolation formula.** Recall the existence of the generalized bivariate Hermite interpolation formula by Chawla et al. [7].

Theorem 2.4. *Suppose a set of values $f_{i,j}^{(k,l)}$, $i = 0, 1, \dots, m;\ j = 0, 1, \dots, n;\ k = 0, 1, \dots, r_i; l = 0, 1, \dots, s_j$ is generated from a function Φ, where $f_{i,j}^{(k,l)} = \frac{\partial^{k+l}\phi(x_i, y_j)}{\partial x^k \partial y^l}$. Then the unique polynomial $H_{M,N}(x, y)$ of degree $M = m + \sum_{i=0}^{m} r_i$ in x and of degree $N = n + \sum_{j=0}^{n} s_j$ in y such that*

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l} H_{M,N}(x_i, y_j) = f_{i,j}^{(k,l)},\ i = 0, 1, \dots, m;\ j = 0, 1, \dots, n;\ k = 0, 1, \dots, r_i; l = 0, 1, \dots, s_j \tag{2.7}$$

is given by

$$H_{M,N}(x, y) = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r_i}\sum_{l=0}^{s_j} A_{ik}(x) B_{jl}(y) f_{i,j}^{(k,l)}, \tag{2.8}$$

where $A_{ik}(x)$ and $B_{jl}(y)$ are given as

$$A_{ik}(x) = \frac{p_i(x)(x-x_i)^k}{k!}\sum_{t=0}^{r_i-k}\frac{g_i^{(t)}(x_i)(x-x_i)^t}{t!},$$

$$B_{jl}(y) = \frac{q_j(y)(y-y_j)^l}{l!}\sum_{t=0}^{s_j-l}\frac{h_j^{(t)}(y_j)(y-y_j)^t}{t!},$$

$$p_i(x) = (x-x_0)^{r_0+1}\dots(x-x_{i-1})^{r_{i-1}+1}(x-x_{i+1})^{r_{i+1}+1}\dots(x-x_m)^{r_m+1},$$
$$q_j(y) = (y-y_0)^{s_0+1}\dots(y-y_{j-1})^{s_{j-1}+1}(y-y_{j+1})^{s_{j+1}+1}\dots(y-y_n)^{s_n+1},$$

and

$$g_i(x) = [p_i(x)]^{-1},\ h_j(y) = [q_j(y)]^{-1}.$$

If $\frac{\partial^{r_i+s_j}\Phi}{\partial x^{r_i}\partial y^{s_j}}$, $(i=0,1,\dots,m;\ j=0,1,\dots,n)$ are non-smooth in nature, then $H_{M,N}$ is not an ideal approximating surface for Φ. Thus we have constructed fractal version of $H_{M,N}$ in the following using the α-fractal technique.

3. Generalization of Hermite function of two variables by fractal interpolation

Navascués and Sebastián [17] constructed Hermite fractal function for one variable. In this paper we shall be interested in the generalization of Theorem 2.4.

Theorem 3.1. *Let a finite set of equidistant data :* $x_0 < x_1 < \dots < x_m\,; y_0 < y_1 < \dots < y_n$ *and* $\{f^{(k,l)}_{i,j},\ i=0,1,\dots,m;\ j=0,1,\dots,n;\ k=0,1,\dots,r_i; l=0,1,\dots,s_j\}$ *be given. Let the fixed vertical scaling factors* $\alpha_U,\ U=1,2,\dots,m$ *and* $\beta_V,\ V=1,2,\dots,n$ *be chosen such that*

$$|\alpha_U| < \frac{1}{m^p},\ p = \max\{r_i; i=0,1,\dots,m\}$$

and

$$|\beta_V| < \frac{1}{n^q},\ q = \max\{s_j; j=0,1,\dots,n\}.$$

Then for fixed i,j *and any* $k=0,1\dots,r_i,\ l=0,1,\dots,s_j$, *there exist fractal functions* $A^{\alpha}_{ik}(x)$ *and* $B^{\beta}_{jl}(y)$ *such that*

$$(A^{\alpha}_{ik})^{(\xi)}(x_i) = (A_{ik})^{(\xi)}(x_i),\ i=0,1,\dots,m;\ \xi=0,1,\dots,p \tag{3.1}$$

and

$$(B^{\beta}_{jl})^{(\eta)}(y_j) = (B_{jl})^{(\eta)}(y_j),\ j=0,1,\dots,n;\ \eta=0,1,\dots,q. \tag{3.2}$$

Proof. We will give here the construction of $A^{\alpha}_{ik}(x)$, the fractal perturbation of $A_{i,k}(x)$. For the equidistant data in x direction, consider $a_U = \frac{x_U - x_{U-1}}{x_m - x_0} = \frac{1}{m},\ U=1,2,\dots,m$. We will define a suitable FIF $A^{\alpha}_{ik}(x)$ satisfying Proposition (2.3). Consider the IFS $\{(L_U, F^{ik}_U); U = 1,2,\dots,m\}$, where $L_U(x) = \frac{x}{U} + b_U$ satisfying (2.1) and $F^{ik}_U(x,y) = \alpha_U y + q^{ik}_U(x)$ such that

$$q^{ik}_U(x) = A_{ik}\circ L_U(x) - \alpha_U b_{ik}(x). \tag{3.3}$$

We will choose $b_{ik}(x)$ such that $b_{ik}\in\mathcal{C}^p(I)$ and (3.1) is satisfied. From (3.3),

$$f^{ik}_{m\xi} = \frac{(q^{ik}_m)^{(\xi)}(x_m)}{a^{\xi}_m - \alpha_m} = \frac{A^{(\xi)}_{ik}(x_m) - m^{\xi}\alpha_m b^{(\xi)}_{ik}(x_m)}{1 - m^{\xi}\alpha_m},$$

$$f^{ik}_{0\xi} = \frac{(q^{ik}_1)^{(\xi)}(x_0)}{a^{\xi}_1 - \alpha_1} = \frac{A^{(\xi)}_{ik}(x_0) - m^{\xi}\alpha_1 b^{(\xi)}_{ik}(x_0)}{1 - m^{\xi}\alpha_1}.$$

Using these end points in the join-up conditions $F^{ik}_{U-1,\xi}(x_m, f^{ik}_{m,\xi}) = F^{ik}_{U,\xi}(x_0, f^{ik}_{0,\xi})$, we have

$$\alpha_{U-1}\left[\frac{A^{(\xi)}_{ik}(x_m) - m^{\xi}\alpha_m b^{(\xi)}_{ik}(x_m)}{1-m^{\xi}\alpha_m} - b^{(\xi)}_{ik}(x_U)\right] = \alpha_U\left[\frac{A^{(\xi)}_{ik}(x_0) - m^{\xi}\alpha_1 b^{(\xi)}_{ik}(x_0)}{1-m^{\xi}\alpha_1} - b^{(\xi)}_{ik}(x_0)\right].$$

If all $\alpha_U = \alpha^*$, the above condition gives

$$A^{(\xi)}_{ik}(x_m) - b^{(\xi)}_{ik}(x_m) = A^{(\xi)}_{ik}(x_0) - b^{(\xi)}_{ik}(x_0),$$

which is true if we consider b_{ik} as Hermite interpolated function with respect to x_0, x_m such that

$$\left.\begin{aligned} b^{(\xi)}_{ik}(x_0) &= A^{(\xi)}_{ik}(x_0)\\ b^{(\xi)}_{ik}(x_m) &= A^{(\xi)}_{ik}(x_m)\end{aligned}\right\},\ \xi = 0,1,2,\dots,p. \tag{3.4}$$

The IFS associated with $(A^{\alpha}_{ik})^{(\xi)}(x)$ is $\{\mathbb{R}^2;\ (L_U(x), F^{ik}_{U,\xi}(x,y)),\ U = 1,2,\dots,m\}$, where

$$L_U(x) = \frac{x}{U} + b_U, F^{ik}_{U\xi}(x,y) = m^{\xi}\alpha^* x + m^{\xi}(q^{ik}_U)^{(\xi)}(x) = m^{\xi}\alpha^* x + A^{(\xi)}_{ik}(L_U(x)) - m^{\xi}\alpha^* b^{(\xi)}_{ik}(x). \tag{3.5}$$

Then from (3.3) and (3.4)

$$\begin{aligned}(A^{\alpha}_{ik})^{(\xi)}(x_0) = f^{ik}_{0\xi} &= \frac{(q^{ik}_1)^{(\xi)}(x_0)}{a^{\xi}_1 - \alpha_1}\\ &= \frac{1}{a^{\xi}_1 - \alpha_1}\left(\frac{A^{(\xi)}_{ik}(L_1(x_0))}{m^{\xi}} - \alpha^* b^{(\xi)}_{ik}(x_0)\right)\\ &= \frac{1}{1-\alpha^* m^{\xi}}\left(A^{(\xi)}_{ik}(x_0) - \alpha^* m^{\xi} b^{(\xi)}_{ik}(x_0)\right) = A^{(\xi)}_{ik}(x_0).\end{aligned}$$

Similarly, we have $(A^{\alpha}_{ik})^{(\xi)}(x_m) = A^{(\xi)}_{ik}(x_m)$. For all other partition points $U = 1,2,\dots,m-1$,

$$\begin{aligned}(A^{\alpha}_{ik})^{(\xi)}(x_U) &= F^{ik}_{U\xi}\left(L^{-1}_U(x_U), (A^{\alpha}_{ik})^{(\xi)}\circ L^{-1}_U(x_U)\right)\\ &= m^{\xi}\alpha^*(A^{\alpha}_{ik})^{(\xi)}\circ L^{-1}_U(x_m) + A^{(\xi)}_{ik}(x_U) - m^{\xi}\alpha^* b^{(\xi)}_{ik}\circ L^{-1}_U(x_U)\\ &= m^{\xi}\alpha^*(A^{\alpha}_{ik})^{(\xi)}(x_m) + A^{(\xi)}_{ik}(x_U) - m^{\xi}\alpha^* b^{(\xi)}_{ik}(x_m) = A^{(\xi)}_{ik}(x_U)\end{aligned}$$

A^{α}_{ik} is the required Hermite fractal function in the x direction. Similarly the IFS for B^{β}_{jl} is given by $\{\mathbb{R}^2;\ (L^*_V(y), F^{*jl}_{V,\eta}(y,z)),\ V = 1,2,\dots,n\}$, where

$$L^*_V(y) = \frac{y}{V} + b_V,\ F^{*jl}_{V,\eta}(y,z) = n^{\eta}\beta^* z + n^{\eta}(q^{*jl}_V)^{(\eta)}(y) = n^{\eta}\beta^* z + B^{(\eta)}_{jl}(L^*_V(y)) - n^{\eta}\beta^* b^{*(\eta)}_{jl}(y). \tag{3.6}$$

□

Definition 3.2. The generalized bivariate Hermite fractal interpolation function is defined with the help of the Hermite functions $A^{\alpha}_{ik}(x)$ and $B^{\beta}_{jl}(y)$ as

$$H^{\alpha\beta}_{M,N}(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r_i}\sum_{l=0}^{s_j} A^{\alpha}_{ik}(x) B^{\beta}_{jl}(y) f^{(k,l)}_{i,j}. \tag{3.7}$$

Remark 3.3. If we consider the scaling vectors $\alpha = \mathbf{0}$ and $\beta = \mathbf{0}$, then $A_{ik}^{\alpha} = A_{ik}$ and $B_{jl}^{\beta} = B_{jl}$ and hence $H_{M,N}^{\alpha\beta} = H_{M,N}$, and we obtain the classical bivariate Hermite function as a particular case.

Let I, J be two compact intervals in $\mathbb{R}$. For a fixed partition Δ, scaling vectors α, β we can define an operator $\mathcal{H}$ on $\mathcal{C}^r(I \times J)$ such that $\mathcal{H}(f)$ is the generalized bivariate Hermite FIF for a fixed $f \in \mathcal{C}^r(I \times J)$, where $r = \min(p, q)$.

Proposition 3.4. *$\mathcal{H}$ is a bounded linear operator on $\mathcal{C}^r(I \times J)$.*

Proof. For given $f, g \in \mathcal{C}^r(I \times J)$ and a real scalar c, we have

$$\mathcal{H}(f+g) = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r_i}\sum_{l=0}^{s_j} A_{ik}^{\alpha}(x) B_{jl}^{\beta}(y)(f+g)_{i,j}^{(k,l)} = \mathcal{H}(f) + \mathcal{H}(g),$$

$$\mathcal{H}(cf) = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r_i}\sum_{l=0}^{s_j} A_{ik}^{\alpha}(x) B_{jl}^{\beta}(y)(cf)_{i,j}^{(k,l)} = c\mathcal{H}(f).$$

For a function f of two variables, consider the norm $\|f\|_{\mathcal{C}^r(I\times J)} = \max\limits_{i+j\leq r} \sup\limits_{x\in I, y\in J} \left|\frac{\partial^{i+j} f}{\partial^i x \partial^j y}\right|$.
Now,

$$\begin{aligned} \|\frac{\partial^{u+v} H}{\partial^u x \partial^v y}\|_{\infty} &= \sup_{x,y\in I}\left|\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r_i}\sum_{l=0}^{s_j} (A_{ik}^{\alpha}(x))^{(u)}(B_{jl}^{\beta}(y))^{(v)} f_{i,j}^{(k,l)}\right| \qquad (3.8)\\ &\leq \|f\|_{\mathcal{C}^r(I\times J)} \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r_i}\sum_{l=0}^{s_j} \|(A_{ik}^{\alpha})^{(u)}\|_{\infty}\|(B_{jl}^{\beta})^{(v)}\|_{\infty}. \end{aligned}$$

Using IFS (3.5), we get the vertical scaling factor as $m^{\xi}\alpha^*$. Now using Proposition (2.3), for $\xi = 0, 1, 2, \ldots, r$,

$$\begin{aligned} \|(A_{ik}^{\alpha})^{(\xi)} - A_{ik}^{(\xi)}\|_{\infty} &\leq \frac{m^{\xi}|\alpha^*|}{1 - m^{\xi}|\alpha^*|}\|A_{ik}^{(\xi)} - b_{ik}^{(\xi)}\|_{\infty}\\ &\leq \frac{m^{r}|\alpha^*|}{1 - m^{r}|\alpha^*|}\|A_{ik} - b_{ik}\|_{\mathcal{C}^r(I)}. \end{aligned}$$

Thus

$$\|(A_{ik}^{\alpha})^{(\xi)}\|_{\infty} \leq \frac{m^{r}|\alpha^*|}{1 - m^{r}|\alpha^*|}\|A_{ik} - b_{ik}\|_{\mathcal{C}^r(I)} + \|A_{ik}\|_{\mathcal{C}^r(I)}. \qquad (3.9)$$

Similarly, for the Hermite function in y direction

$$\|(B_{jl}^{\beta})^{(\eta)}\|_{\infty} \leq \frac{n^{r}|\beta^*|}{1 - n^{r}|\beta^*|}\|B_{jl} - b_{jl}^*\|_{\mathcal{C}^r(J)} + \|B_{jl}\|_{\mathcal{C}^r(J)}. \qquad (3.10)$$

Using (3.8), (3.9) and (3.10), we get the upper bound for the operator $\mathcal{H}$ as

$$\|\mathcal{H}\| \leq \left(\frac{m^r|\alpha^*|}{1-m^r|\alpha^*|}\sum_{i=0}^{n}\sum_{k=0}^{r_i}\|A_{ik}-b_{ik}\|_{\mathcal{C}^r(I)}+\|A_{ik}\|_{\mathcal{C}^r(I)}\right)$$
$$\left(\frac{n^r|\beta^*|}{1-n^r|\beta^*|}\sum_{j=0}^{m}\sum_{l=0}^{s_l}\|B_{jl}-b^*_{jl}\|_{\mathcal{C}^r(J)}+\|B_{jl}\|_{\mathcal{C}^r(J)}\right).$$

□

3.1. **Upper bound of the error.** We will use the following result to prove the error bound.

Theorem 3.5. *[7] Let $x_0, x_1, \dots, x_m$ be $m+1$ and $y_0, y_1, \dots, y_n$ be $n+1$ distinct points in $[a,b]\times[c,d]$. Let $f(x)\in\mathcal{C}([a,b]\times[c,d])$ and suppose that all of its partial derivatives exist. If we keep y fixed, then we can write $f(x,y)$ as*

$$f(x,y)=\sum_{i=0}^{m}\sum_{k=0}^{r_i}A_{ik}(x)\frac{\partial^k}{\partial x^k}f(x_i,y)+\frac{\lambda(x)}{(M+1)!}\frac{\partial^{M+1}}{\partial x^{M+1}}f(\xi,y),$$

where $\lambda(x)=(x-x_0)^{r_0+1}\dots(x-x_m)^{r_m+1}$, $\min(x,x_0,\dots,x_m)\leq\xi\leq\max(x,x_0,\dots,x_m)$. Similarly, if we keep x fixed, then

$$f(x,y)=\sum_{j=0}^{n}\sum_{l=0}^{s_j}B_{jl}(y)\frac{\partial^l}{\partial y^l}f(x,y_j)+\frac{\mu(y)}{(N+1)!}\frac{\partial^{N+1}}{\partial y^{N+1}}f(x,\eta),$$

where $\mu(y)=(y-y_0)^{s_0+1}\dots(y-y_n)^{s_n+1}$, $\min(y,y_0,\dots,y_n)\leq\eta\leq\max(y,y_0,\dots,y_n)$. Thus

$$f(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r_i}\sum_{l=0}^{s_j}A_{ik}(x)B_{jl}(y)\frac{\partial^{k+l}f(x_i,y_j)}{\partial x^k\partial y^l}+\frac{\lambda(x)}{(M+1)!}\sum_{j=0}^{n}\sum_{l=0}^{s_j}B_{jl}(y)\frac{\partial^{M+l+1}f(\xi,y_j)}{\partial x^{M+1}\partial y^l}$$
$$+\frac{\mu(y)}{(N+1)!}\sum_{i=0}^{m}\sum_{k=0}^{r_i}A_{ik}(x)\frac{\partial^{k+N+1}f(x_i,\eta)}{\partial x^k\partial y^{N+1}}+\frac{\lambda(x)\mu(y)}{(M+1)!(N+1)!}\frac{\partial^{M+N+2}}{\partial x^{M+1}\partial y^{N+1}},$$

which can be expressed as

$$f(x,y)=H_{M,N}(x,y)+R_{M,N}(x,y),$$

where $H_{M,N}(x,y)$ is the Hermite interpolated function of f and $R_{M,N}(x,y)$ is the error function.

Theorem 3.6. *Let $\Phi\in\mathcal{C}^r(I\times J)$ be the original function approximated by the generalized Hermite FIF $H^{\alpha\beta}_{M,N}$ such that $|\alpha^*|<\frac{1}{m^p}$ for $p=\max\{r_i; i=0,1,\dots,m\}$ and $|\beta^*|<\frac{1}{n^q}$ for $q=\max\{s_j; j=0,1,\dots,n\}$. Then*

$$\|\Phi-H^{\alpha\beta}_{M,N}\|_{\mathcal{C}^r(I\times J)}\leq\|\Phi-H_{M,N}\|_{\mathcal{C}^r(I\times J)}$$
$$+\|\Phi\|_{\mathcal{C}^r(I\times J)}\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r_i}\sum_{l=0}^{s_j}\Bigg[\|A^{(u)}\|_\infty\frac{n^r|\beta^*|}{1-n^r|\beta^*|}\|B_{jl}-b^*_{jl}\|_{\mathcal{C}^r(J)}$$
$$+\frac{m^r|\alpha^*|}{1-m^r|\alpha^*|}\|A_{ik}-b_{ik}\|_{\mathcal{C}^r(I)}\Big(\frac{n^r|\beta^*|}{1-n^r|\beta^*|}\|B_{jl}-b^*_{jl}\|_{\mathcal{C}^r(J)}+\|B_{jl}\|_{\mathcal{C}^r(J)}\Big)\Bigg]$$

Proof. Now for $u+v=0,1,\dots,r$,

$$
\begin{aligned}
&\|\frac{\partial^{u+v}H^{\alpha\beta}_{M,N}}{\partial^u x\partial^v y}-\frac{\partial^{u+v}H_{M,N}}{\partial^u x\partial^v y}\|_\infty\\
&\leq \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r_i}\sum_{l=0}^{s_j}\|f^{(k,l)}\|_\infty\|(A^\alpha_{ik})^{(u)}(B^\beta_{jl})^{(v)}-(A_{ik})^{(u)}(B_{jl})^{(v)}\|_\infty\\
&\leq \|f\|_{\mathcal{C}^r(I\times J)}\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r_i}\sum_{l=0}^{s_j}\big(\|(A^\alpha_{ik})^{(u)}-(A_{ik})^{(u)}\|_\infty\|(B^\beta_{jl})^{(v)}\|_\infty\\
&\qquad\qquad + \|(A_{ik})^{(u)}\|_\infty\|(B^\beta_{jl})^{(v)}-(B_{jl})^{(v)}\|_\infty\big)\\
&\leq \|f\|_{\mathcal{C}^r(I\times J)}\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r_i}\sum_{l=0}^{s_j}\Big(\frac{m^r|\alpha^*|}{1-m^r|\alpha^*|}\|A_{ik}-b_{ik}\|_{\mathcal{C}^r(I)}\|(B^\beta_{ik})^{(v)}\|_{\mathcal{C}^r(J)}\\
&\qquad\qquad + \|A^{(u)}_{ik}\|_\infty\frac{n^r|\beta^*|}{1-n^r|\beta^*|}\|B_{jl}-b^*_{jl}\|_{\mathcal{C}^r(J)}\Big)\\
&\leq \|f\|_{\mathcal{C}^r(I\times J)}\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r_i}\sum_{l=0}^{s_j}\Big[\Big(\|A^{(u)}_{ik}\|_\infty\frac{n^r|\beta^*|}{1-n^r|\beta^*|}\|B_{jl}-b_{jl}\|_{\mathcal{C}^r(J)}\Big)\\
&+\frac{m^r|\alpha^*|\|A_{ik}-b_{ik}\|_{\mathcal{C}^r(I)}}{1-m^r|\alpha^*|}\Big(\frac{n^r|\beta^*|}{1-n^r|\beta^*|}\|B_{jl}-b^*_{jl}\|_{\mathcal{C}^r(J)}+\|B_{jl}\|_{\mathcal{C}^r(J)}\Big)\Big].
\end{aligned}
$$

The upper bound of error follows from the triangle inequality

$$\|\Phi-H^{\alpha\beta}_{M,N}\|_{\mathcal{C}^r(I\times J)}\leq\|\Phi-H_{M,N}\|_{\mathcal{C}^r(I\times J)}+\|H_{M,N}-H^{\alpha\beta}_{M,N}\|_{\mathcal{C}^r(I\times J)}.$$

□

The following proposition is useful to compute $\|A_{ik}-b_{ik}\|_\infty$, $\|B_{jl}-b^*_{jl}\|_\infty$.

Proposition 3.7. *[8] Let $h(t)\in\mathcal{C}^s[t_0,t_M]$ with $s\geq 2p+2$. Let Δ be any partition of $[t_0,t_m]$, $\Delta: t_0<t_1<\cdots<t_m$, and let $\Phi(t)$ be the unique Hermite interpolation of $h(t)$ such that $h^{(\xi)}(t_j)=\phi^{(\xi)}(t_j)$ for all $0\leq j\leq m,\ 0\leq\xi\leq p$. Then for all k with $0\leq k\leq p$*

$$\|h^{(k)}-\phi^{(k)}\|_\infty\leq\frac{\|\Delta\|^{2p+2-k}}{2^{2p+2-2k}p!(2p+2-2k)!}\|h^{(2p+2)}\|_\infty. \tag{3.11}$$

Note: Taking $h(t)=A_{ik}(t),\ \Phi(t)=b_{ik}(t)$ and $k=\xi$, we can find the bound $\|A^{(\xi)}_{ik}-b^{(\xi)}_{ik}\|_\infty$ using (3.11).

4. Examples and graphs

Example 4.1. Consider $m=2, n=2, r_i=1, s_j=1$, $I=[-2,2]$, $J=[-1,1]$. Then $H_{5,5}$ is a polynomial of degree 5 in x and degree 5 in y. The set of data points are given in Table 1. Here a quadrilateral represents the values of (f,f_x,f_y,f_{xy}) at (x_i,y_j) for $i,j=0,1,2$. Using

Table 1. Data values

$x \downarrow \mid y \rightarrow$	-1	0	1
-2	$(4,1,2,5)$	$(-1,0,5,8)$	$(0,5,-6,-3)$
0	$(18,6,0,4)$	$(6,-3,1,-7)$	$(-5,-9,4,2)$
2	$(22,-10,10,0)$	$(11,4,-4,1)$	$(10,8,3,-4)$

Table 1, we can construct the basis functions as

$$A_{00}(x) = \frac{(3x+8)x^2(x-2)^2}{128},\ A_{01}(x) = \frac{x^2(x-2)^2(x+2)}{64}, A_{10}(x) = \frac{(x+2)^2(x-2)^2}{16},$$
$$A_{11}(x) = \frac{x(x-2)^2(x+2)^2}{16}, A_{20}(x) = \frac{x^2(x+2)^2(8-3x)}{128}, A_{21}(x) = \frac{x^2(x-2)(x+2)^2}{64},$$
$$B_{00}(y) = \frac{y^2(4+3y)(y-1)^2}{4},\ B_{01}(y) = \frac{y^2(y-1)^2(y+1)}{4},\ B_{10}(y) = (y+1)^2(y-1)^2,$$
$$B_{11}(y) = y(y+1)^2(y-1)^2, B_{20}(y) = \frac{y^2(y+1)^2(4-3y)}{4}, B_{21}(y) = \frac{y^2(y+1)^2(y-1)}{4}.$$

For the construction of fractal Hermite functions consider the uniform partition of I as $\{-2,0,2\}$ and hence $a_U = \frac{1}{2}, U = 1,2$ in the x direction and $\{-1,0,1\}$ in the y direction with $a^*_V = \frac{1}{2}, V = 1,2$. According to Theorem 3.1, we have to consider $|\alpha_U| < \frac{1}{2^1}$ and $|\beta_V| < \frac{1}{2^1}$. Also, consider base functions using (3.4) as

$$b_{00}(x) = \frac{(4+x)(x-2)^2}{32},\ b_{01}(x) = \frac{(2+x)(x-2)^2}{16},\ b_{10}(x) = (\sin x - \sin 2)^2(\sin x + \sin 2)^2,$$
$$b_{11}(x) = (e^x - e^2)^2(e^x - e^{-2})^2, b_{20}(x) = \frac{(4-x)(x+2)^2}{32},\ b_{21}(x) = \frac{(x-2)(x+2)^2}{16},$$
$$b^*_{00}(y) = \frac{(y+2)(y-1)^2}{4},\ b^*_{01}(y) = \frac{(y+1)(y-1)^2}{4},\ b^*_{10}(y) = (e^y - e)^2(e^y - e^{-1})^2,$$
$$b^*_{11}(y) = (\sin y - \sin 1)^2(\sin y + \sin 1)^2, b^*_{20}(y) = \frac{(y+1)^2(2-y)}{4},\ b^*_{21}(y) = \frac{(y-1)(y+1)^2}{4}.$$

In view of (3.5), the IFS for generalized Hermite fractal functions are

$$\left.\begin{array}{c} L_U(x) = \frac{x}{2} + b_U \\ F_U^{ik}(x,y) = \alpha_U y + A_{ik} \circ L_U(x) - \alpha_U b_{ik}(x) \end{array}\right\},\ U = 1,2, \tag{4.1}$$

and

$$\left.\begin{array}{c} L^*_V(y) = \frac{y}{2} + b^*_V \\ F_V^{*jl}(y,z) = \beta_V z + B_{jl} \circ L^*_V(y) - \beta_V b^*_{jl}(y) \end{array}\right\},\ J = 1,2. \tag{4.2}$$

Here we have chosen $\alpha = 0.2$ and $\beta = -0.3$. With the above choice of scaling vectors and the functions A_{ik}, b_{ik}, B_{jl}, b^*_{jl} and using IFS (4.1),(4.2), the fractal functions $A^{\alpha}_{ik}(x)$ and $B^{\beta}_{jl}(y)$ are constructed. Using these $A^{\alpha}_{ik}(x)$ and $B^{\beta}_{jl}(y)$ and Definition (3.7) we have plotted the generalized bivariate Hermite fractal functions (See Figure 1). Also we have chosen $\alpha^* = 0 = \beta^*$ and plotted the second graph in Figure 1 which agrees with classical Hermite bivariate functions. Next we have plotted the graph of both partial derivative function of the proposed function in x and y direction (See Figure 2 and Figure 3). It is clear from Figure 2 and Figure 3 that, for the value of $\alpha = 0 = \beta$, fractal derivative functions agrees with the classical derivative function.

Remark 4.2. Several authors (see for instance Massopust [12, 13] and Drakopoulos et al. [10]) have proposed the construction of fractal surfaces through oriented simplices or parallelepipeds. This is a very versatile and interesting model for the definition of non-smooth surfaces, having fractal dimensions. However, in the case where there are prescribed derivative values at the nodes, the polygonal model involves major technical difficulties that complicate excessively the formulation. The approach described in this paper presents a simple definition of a surface matching value conditions on the nodes, with the additional advantage of being a generalization of the classical Hermite functions.

FIGURE 1. Bivariate Hermite fractal function for different values of α, β.

FIGURE 2. Partial derivative with respect to x of bivariate Hermite fractal function for different values of α, β.

FIGURE 3. Partial derivative with respect to y of bivariate Hermite fractal function for different values of α, β.

5. Conclusion

The present paper described a method to construct bivariate Hermite fractal function. The main advantage of the presented interpolation method is the constructibility, that is the possibility of implementing the method for approximating non-smooth derivatives of the original functions. Numerical examples are given to illustrate the feasibility of our method for the best possible choice of the scaling factors. The roughness of fractal interpolated surface can be adjusted with the scaling vectors. Using the same interpolation data we can obtain various shape of bivariate FIF by varying values of scaling vectors. The proposed bivariate Hermite FIF may be useful for surface modeling problem in computer graphics, CAGD and data visualization when there is fractality hidden in the partial derivatives of the original function.

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