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On the uniqueness conjecture for the maximum Stirling numbers of the second kind

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Abstract The Stirling numbers of the second kind $S(n,k)$ satisfy

$$S(n,0) < \cdots < S(n,k_n) \geq S(n,k_n+1) > \cdots > S(n,n).$$

A long standing conjecture asserts that there exists no $n \geq 3$ such that $S(n,k_n) = S(n,k_n+1)$. In this note, we give a characterization of this conjecture in terms of multinomial probabilities, as well as sufficient conditions on n ensuring that $S(n,k_n) > S(n,k_n+1)$.

Keywords Stirling number of the second kind · uniqueness conjecture · multinomial law

1 Introduction

For a fixed positive integer $n \geq 3$, it is well known that the sequence $(S(n,k))_{k=0}^{n}$ of Stirling numbers of the second kind is unimodal in k, namely,

$$0 = S(n,0) < \cdots < S(n,k_n) \geq S(n,k_n+1) > \cdots > S(n,n) = 1.$$

In 1973, Wegner [12] conjectured that there exists no $n \geq 3$ such that $S(n,k_n) = S(n,k_n+1)$. For the Stirling numbers of the first kind, such a property was shown by

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Erdős [5] in 1953. Up to our knowledge, the only specific result concerning Wegner's conjecture is the following (cf. Dobson [4], Mullin [8], and Canfield and Pomerance [3])

Theorem A. *Let $n \geq 3$. Then, k_{n+1} equals either k_n or $k_n + 1$. If $k_n = k_{n+1}$, then $S(n,k_n) > S(n,k_n+1)$.*

On the other hand, denote by

$$E(x) = \#\{n \leq x : S(n,k_n) = S(n,k_n+1)\}.$$

Canfield and Pomerance [3] showed that for any $\varepsilon > 0$

$$E(x) = O\left(x^{3/5+\varepsilon}\right).$$

The same authors also verified that Wegner's conjecture is true for $3 \leq n \leq 10^6$.

Our contribution to the problem is twofold. In first place, we give a characterization of such a conjecture in terms of multinomial probabilities (see Theorem 2 in Section 3). In second place, we provide in Theorem 1 below a sufficient condition on n guaranteeing that $S(n,k_n) > S(n,k_n+1)$.

To this end, let $n \geq 3$. We consider the strictly increasing function

$$g_n(x) = (x+1)\left(1 - \frac{1}{x}\right)^n, \qquad x > 1, \tag{1}$$

and denote by ν_n the unique solution to the equation $g_n(\nu_n) = 1$. We also consider the auxiliary strictly increasing function

$$f(x) = \frac{\log(x+1)}{\log(1+1/(x-1))}, \qquad x > 1. \tag{2}$$

Let $k \geq 2$ and denote by $\lfloor x \rfloor$ the integer part of x. Since $g_m(x) > g_{m+1}(x)$, $x > 1$, we note that

$$\begin{aligned}\{m \geq 3 : \lfloor \nu_m \rfloor = k\} &= \{m \geq 3 : g_m(k) < 1 < g_m(k+1)\} \\ &= \{m \geq 3 : f(k) < m < f(k+1)\}.\end{aligned} \tag{3}$$

Together with the equivalence $\log(1+y) \sim y$ as $y \to 0$, this implies that

$$\#\{m \geq 3 : \lfloor \nu_m \rfloor = k\} \sim f(k+1) - f(k) \sim 1 + \log(k+1), \qquad k \to \infty.$$

With these notations, we state the first result of this note.

Theorem 1 *Let $n \geq 3$. Then, $k_n \leq \lfloor \nu_n \rfloor$. If $k_n = \lfloor \nu_n \rfloor$, then*

$$k_m = \lfloor \nu_n \rfloor, \qquad n \leq m \leq \lfloor f(\lfloor \nu_n \rfloor + 1) \rfloor, \tag{4}$$

and

$$S(m,k_m) > S(m,k_m+1), \qquad n \leq m \leq \lfloor f(\lfloor \nu_n \rfloor + 1) \rfloor - 1. \tag{5}$$

The assumption $k_n = \lfloor \nu_n \rfloor$ is meaningful and leads us to the problem of location of k_n. A first result in this direction goes back to Harper [6], who showed that $k_n \sim n/\log n$, $n \to \infty$. More precise asymptotic results are due to Rennie and Dobson [10], Menon [7], and Canfield [2], among many others. More recently, non-asymptotic bounds for k_n have been obtained. In this regard, Yu [14] gave

$$\lfloor e^{W(n)} \rfloor - 2 \leq k_n \leq \lfloor e^{W(n)} \rfloor + 1, \qquad n \geq 2, \tag{6}$$

where $W(\cdot)$ stands for the Lambert-W function, that is, the solution to the equation $W(n)e^{W(n)} = n$. Finally, Wegner [13] showed that

$$\left\lfloor r_n - \frac{0.2}{\log(r_n+1)} \right\rfloor \leq k_n \leq \left\lfloor r_n + \frac{0.4}{\log(r_n+1)} \right\rfloor, \qquad n \geq 1,$$

where r_n is the unique solution to the equation

$$\left(1 - \frac{1}{2r_n}\right)(r_n+1)\log(r_n+1) = n.$$

Denote by $h_n(x) = xe^{-n/x}$, $x > 1$. Since $g_n(x) \sim h_n(x)$, as $x \to \infty$, we see that $g_n\left(e^{W(n)}\right) \sim h_n\left(e^{W(n)}\right) = 1$, as follows from the definition of $W(n)$. This means that $\nu_n \sim e^{W(n)}$, as $n \to \infty$. A more detailed analysis actually shows that

$$e^{W(n)} - 1 < \nu_n < e^{W(n)} + 1, \qquad n \geq 3.$$

In accordance with (6), this means that the assumption $k_n = \lfloor \nu_n \rfloor$ makes sense (see also Table 1 and the concluding remarks at the end of this note).

n	ν_n	k_n	$k_n = \lfloor \nu_n \rfloor$
1000	190.75187	189	no
1500	268.51842	267	no
2000	342.93881	342	yes
2500	415.04015	414	no
3000	485.38741	484	no
3500	554.33514	553	no
4000	622.12516	621	no
4500	688.93213	688	yes
5000	754.88769	754	yes
6000	884.63384	883	no
7000	1011.9673	1011	yes
8000	1137.3010	1136	no
9000	1260.9336	1260	yes
10000	1383.0907	1382	no
20000	2550.1253	2549	no
30000	3656.9585	3656	yes
40000	4727.8314	4726	no

Table 1 Numerical values of ν_n rounded to 8 significant digits, and values of k_n, obtained by using the Newton-Raphson method and the software Mathematica©. The cases when $k_n = \lfloor \nu_n \rfloor$ are highlighted.

To show Theorem 1, we use a probabilistic representation of $S(n,k)$ by means of a multinomial law, which is close to the classical representation in terms of occupancy problems (see Pitman [9] for more details). Indeed, we give in Theorem 2 (the main result in this note) a closed form expression for the difference $S(n,k)-S(n,k-1)$ in terms of multinomial probabilites and the function $g_n(x)$ defined in (1). This allows us to characterize Wegner's conjecture on the one hand and to give a short proof of Theorem 1, on the other.

2 Stirling numbers and multinomial laws

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Unless otherwise stated, we assume from now on that $n,k \in \mathbb{N}$. Let $(U_j)_{j\geq 1}$ be a sequence of independent identically distributed random variables having the uniform distribution on $[0,1]$. Sun [11] (see also [1]) gave the following probabilistic representation

$$S(n,k) = \binom{n}{k} \mathbb{E}(U_1 + \cdots + U_k)^{n-k}, \qquad 1 \leq k \leq n, \tag{7}$$

where $\mathbb{E}$ stands for the mathematical expectation. We will always consider Borel sets $B \subseteq [0,1]$, whose Lebesgue measure is denoted by $\lambda(B)$. Define the random variable

$$S_n(B) = \sum_{j=1}^{n} 1_B(U_j),$$

where 1_B means the indicator function of the set B. Clearly, $S_n(B)$ has the binomial law with parameters n and $p = \lambda(B)$, i. e.,

$$P(S_n(B) = l) = \binom{n}{l} p^l (1-p)^{n-l}, \qquad l = 0,1,\ldots,n.$$

Let $\mathscr{P}_k$ be the family of partitions of $[0,1]$ into k Borel sets $B_1,\ldots, B_k$, with $\lambda(B_j) = p_j$, $j = 1,2,\ldots,k$. Finally, we consider the integer simplex

$$\Delta_{n,k} = \{(\ell_1,\ldots,\ell_k) \in \mathbb{N}_0^k : \ell_1 + \cdots + \ell_k = n\}.$$

If $(B_1,\ldots,B_k) \in \mathscr{P}_k$, then the random vector $(S_n(B_1),\ldots,S_n(B_k))$ has the multinomial law with parameters n, $p_1,\ldots,p_k$, namely,

$$P(S_n(B_1) = \ell_1,\ldots,S_n(B_k) = \ell_k) = \frac{n!}{\ell_1!\cdots\ell_k!} p_1^{\ell_1} \cdots p_k^{\ell_k}, \quad (\ell_1,\ldots,\ell_k) \in \Delta_{n,k}. \tag{8}$$

The random vectors $(S_n(B_1),\ldots,S_n(B_k))$, with $n,k \in \mathbb{N}$ and $(B_1,\ldots,B_k) \in \mathscr{P}_k$, are defined on the same probability space. This coupling construction is the key tool for giving closed form expressions for $S(n,k)-S(n,k-1)$, since the probability law of $(S_n(B_1),\ldots,S_n(B_k))$ depends on $(B_1,\ldots,B_k)$ only through the corresponding Lebesgue measures $\lambda(B_j) = p_j$. With this condition being fulfilled, we will be free to choose the partition $(B_1,\ldots,B_k) \in \mathscr{P}_k$.

Denote

$$\mathscr{P}_k^* = \left\{(B_1,\ldots,B_k) \in \mathscr{P}_k : \lambda(B_j) = 1/k,\ j = 1,2,\ldots,k\right\},$$

as well as

$$D_{n,k} = \{(\ell_1,\dots,\ell_k) \in \Delta_{n,k} : \ell_1 \ge 1,\dots,\ell_k \ge 1\}.$$

Note that for $(B_1,\dots,B_k) \in \mathscr{P}_k^*$, we have from (8)

$$P(n,k) := P\left(\bigcap_{j=1}^{k}\{S_n(B_j) \ge 1\}\right) = \frac{1}{k^n}\sum_{D_{n,k}}\frac{n!}{\ell_1!\cdots\ell_k!}. \tag{9}$$

With these ingredients, we give the following probabilistic representation.

Lemma 1 *If $1 \le k \le n$ and $(B_1,\dots,B_k) \in \mathscr{P}_k^*$, then*

$$S(n,k) = \frac{k^n}{k!}P(n,k).$$

Proof Since $\mathbb{E}U_j^{\ell} = 1/(\ell+1)$, $\ell \in \mathbb{N}_0$, we have from (7) and (9)

$$\begin{aligned} S(n,k)) &= \binom{n}{k}\sum_{\Delta_{n-k,k}}\frac{(n-k)!}{\ell_1!\cdots\ell_k!}\mathbb{E}U_1^{\ell_1}\cdots\mathbb{E}U_k^{\ell_k} \\ &= \frac{1}{k!}\sum_{\Delta_{n-k,k}}\frac{n!}{(\ell_1+1)!\cdots(\ell_k+1)!} = \frac{1}{k!}\sum_{D_{n,k}}\frac{n!}{\widehat{\ell_1}!\cdots\widehat{\ell_k}!} = \frac{k^n}{k!}P(n,k), \end{aligned}$$

where $\widehat{\ell_j} = \ell_j + 1$, $j = 1,\dots,k$. The proof is complete. □

Let $2 \le k \le n$. In order to compare $P(n,k)$ and $P(n,k-1)$, we consider the following coupling construction. Given $(B_1,\dots,B_k) \in \mathscr{P}_k^*$, we define $(A_1,\dots,A_{k-1}) \in \mathscr{P}_{k-1}^*$ by decomposing B_k as a disjoint union of $k-1$ Borel sets C_j with

$$\lambda(C_j) = \frac{1}{k(k-1)}, \qquad j = 1,\dots,k-1,$$

and set

$$A_j = B_j \cup C_j, \qquad j = 1,\dots,k-1.$$

Observe that $B_j \cap C_j = \emptyset$, $j = 1,\dots,k-1$, and thus

$$\lambda(A_j) = \lambda(B_j) + \lambda(C_j) = \frac{1}{k-1}, \qquad j = 1,\dots,k-1.$$

Finally, denote

$$E = \bigcap_{j=1}^{k-1}\{S_n(A_j) \ge 1\}, \qquad F = \bigcap_{j=1}^{k}\{S_n(B_j) \ge 1\}. \tag{10}$$

In this setting, we give the following result.

Lemma 2 *Let* $2 \le k \le n$. *Then*

$$P(n,k) = \left(1 - \frac{g_n(k)}{k+1}\right) P(n,k-1) - Q(n,k),$$

where $g_n(\cdot)$ *is defined in (1) and*

$$Q(n,k) = P\left(E \cap \left(\bigcup_{j=1}^{k-1} \{S_n(B_j) = 0\}\right) \cap \{S_n(B_k) \ge 1\}\right).$$

Proof Denote the complement of F by

$$F^c = \bigcup_{j=1}^{k} \{S_n(B_j) = 0\}.$$

By construction, $F \subseteq E$. We thus have from (9) and (10)

$$\begin{aligned}
&P(n,k-1) = P(E) = P(E \cap F) + P(E \cap F^c) \\
&= P(F) + P\big(E \cap F^c \cap \{S_n(B_k) = 0\}\big) + P\big(E \cap F^c \cap \{S_n(B_k) \ge 1\}\big) \\
&= P(n,k) + P\big(E \cap \{S_n(B_k) = 0\}\big) \\
&+ P\left(E \cap \left(\bigcup_{j=1}^{k-1} \{S_n(B_j) = 0\}\right) \cap \{S_n(B_k) \ge 1\}\right).
\end{aligned} \tag{11}$$

Again by construction, (1), and (9), we have

$$\begin{aligned}
P\big(E \cap \{S_n(B_k) = 0\}\big) &= P\left(\bigcap_{j=1}^{k-1} \{S_n(B_j) \ge 1\} \cap \{S_n(B_k) = 0\}\right) \\
&= \frac{1}{k^n} \sum_{D_{n,k-1}} \frac{n!}{\ell_1! \cdots \ell_{k-1}!} = \frac{g_n(k)}{k+1} P(n,k-1).
\end{aligned}$$

This, in conjunction with (11), shows the result. □

3 The main result

Keeping the notations of the preceding sections, we state our main result.

Theorem 2 *Let* $n \ge 3$ *and* $2 \le k \le n$. *Then*

$$S(n,k) - S(n,k-1) = \frac{k^n}{k!} \left((1 - g_n(k)) P(n,k-1) - Q(n,k)\right). \tag{12}$$

As a consecuence, Wegner's conjecture is true if and only if

$$(1 - g_n(k_n + 1)) P(n,k_n) < Q(n,k_n + 1). \tag{13}$$

Proof Identity (12) follows from Lemmas 1 and 2 and some simple computations. Characterization (13) follows by choosing $k = k_n + 1$ in (12). The proof is over. □

Proof of Theorem 1

If $S(n,k) > S(n,k-1)$, identity (12) implies that

$$(1-g_n(k))P(n,k-1) > Q(n,k) > 0.$$

By (1), this means that $k < \nu_n$ and, a fortiori, $k_n \le \lfloor \nu_n \rfloor$.

Suppose that $k_n = \lfloor \nu_n \rfloor$. Again by (1), $1-g_n(k_n+1) < 0$. Thus, condition (13) is fulfilled and $S(n,k_n) > S(n,k_n+1)$. Finally, assume that $n \le m \le \lfloor f(\lfloor \nu_n \rfloor + 1) \rfloor$. Using Theorem A, (3), and the first statement of Theorem 1, we get

$$k_n \le k_m \le \lfloor \nu_m \rfloor = \lfloor \nu_n \rfloor = k_n,$$

which shows (4). Statement (5) readily follows from Theorem A. This completes the proof. □

Concluding remarks

Numerical computations suggest that k_n equals either $\lfloor \nu_n \rfloor$ or $\lfloor \nu_n \rfloor - 1$. If $k_n = \lfloor \nu_n \rfloor - 1$, characterization (13) becomes

$$(1-g_n(\lfloor \nu_n \rfloor))P(n,\lfloor \nu_n \rfloor - 1) < Q(n,\lfloor \nu_n \rfloor). \tag{14}$$

The difficulty in proving this inequality stems from the fact that both sides in (14) are positive and, apparently, have the same order of magnitude. This implies that a very precise estimate of $Q(n,\lfloor \nu_n \rfloor)$ is needed in order to prove (or disprove) Wegner's conjecture.

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