Received: 16 November 2021 | Revised: 6 April 2022 | Accepted: 21 June 2022

DOI: 10.1002/mma.8542

RESEARCH ARTICLE

WILEY

Depth of almost strictly sign regular matrices

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Communicated by: E. Venturino

Funding information
Ministerio de Asuntos Económicos y Transformación Digital, Gobierno de España, Grant/Award Number: TIN2017-87600-P; Ministerio de Ciencia e Innovación, Grant/Award Number: PGC2018-096321-B-I00

The concept of depth of an almost strictly sign regular matrix is introduced and used to simplify some algorithmic characterizations of these matrices.

KEYWORDS

almost strictly sign regular matrices, depth, characterization, Neville elimination

MSC CLASSIFICATION

94D05, 68T37, 68U10

1 | INTRODUCTION

Matrices with all their minors nonnegative are called totally positive (TP), and matrices whose minors of the same order have the same sign are called sign regular (SR). They arise in approximation theory, differential equations, statistics, combinatorics, mechanics, computer-aided geometric design and economics, among other subjects (see previous studies[1–3] and[4]). An important subclass of TP matrices are the almost strictly totally positive (ASTP) matrices, matrices whose minors are positive if and only if all their diagonal entries are positive (see Gasca and Peña[5]). ASTP matrices contain Hurwitz matrices and B-splines collocation matrices. In Huang et al,[6] the class of almost strictly sign regular (ASSR) matrices was introduced and characterized by a reduced number of minors. ASSR matrices form a subclass of SR matrices including all ASTP matrices.

In Alonso et al,[7] an algorithmic characterization of ASSR matrices is provided. It used the Neville elimination (NE) of a matrix, which is an elimination procedure alternative to Gaussian elimination. Roughly speaking, NE makes zeros in a column of a matrix by adding to each row an adequate multiple of the previous one (see Gasca and Peña[8] for more details), instead of using just a row with a fixed pivot as in Gaussian elimination. So, NE transforms a nonsingular matrix into an upper triangular matrix. More results about NE and SR matrices can be seen in previous studies[9–14] and Alonso et al.[15]

In this paper, we introduce the concept of depth of an ASSR matrix. At the end of Section 4, we see that strictly M-banded ASSR matrices (see Alonso et al.[16]) are included in the class of ASSR matrices with depth $n-M$. We also see in Section 4 that the use of the depth of an ASSR matrix allows us to simplify the algorithms of Alonso et al[7] and to reduce their computational costs. If the matrix has a large depth, then this reduction is considerable.

The paper is organized as follows. Section 2 contains basic notations and definitions concerning the zero pattern of a matrix. Section 3 recalls definitions and some fundamental results on ASSR matrices. In Section 4, we introduce the concept of depth of an ASSR matrix and prove that the depth of an ASSR matrix determines its initial signature. We also provide the mentioned simplified characterization and the corresponding algorithms. Section 5 includes some numerical results and applications. Finally, Section 6 summarizes the main conclusions of this work.

2 | BASIC NOTATIONS AND DEFINITIONS

For $k, n \in \mathbb{N}$, with $1 \le k \le n$, $Q_{k,n}$ denotes the set of all increasing sequences of k natural numbers not greater than n. If A is a real $n \times n$ matrix and $\alpha = (\alpha_1, \dots, \alpha_k)$, $\beta = (\beta_1, \dots, \beta_k) \in Q_{k,n}$, then $A[\alpha|\beta]$ is by definition the $k \times k$ submatrix of A containing rows $\alpha_1, \dots, \alpha_k$ and columns $\beta_1, \dots, \beta_k$ of A. In particular, when $\alpha = \beta$, $A[\alpha] := A[\alpha|\alpha]$ is the corresponding principal submatrix. Besides, $Q^0_{k,n}$ denotes the set of increasing sequences of k consecutive natural numbers not greater than n and if $\alpha \in Q^0_{k,n}$, then $\det A[\alpha]$ is a principal minor.

From now on, it will be frequently used the backward identity matrix $n \times n$, P_n, whose element (i,j) is defined as

$$\begin{cases} 1 & \text{if } i+j = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Throughout this document, we will work with matrices whose zero and nonzero entries are grouped in certain positions. Then we introduce the type-I and type-II staircase matrices.

Definition 2.1. A real matrix $A = (a_{ij})_{1 \le i,j, \le n}$ is called type-I staircase matrix if it satisfies simultaneously the following conditions:

(a) $a_{ii} \neq 0, \forall i \in \{1, \dots, n\}$;
(b) $a_{ij} = 0, i > j \Rightarrow a_{kl} = 0$, if $l \le j$, $i \le k$;
(c) $a_{ij} = 0, i < j \Rightarrow a_{kl} = 0$, if $k \le i$, $j \le l$.

Definition 2.2. Matrix A is called type-II staircase if $P_n A$ is a type-I staircase matrix.

Conditions introduced in definitions 2.1 and 2.2 produce a staircase structure for the zero pattern, which can be defined through the following indices (see earlier research[5,7]):

Definition 2.3. For a matrix $A = \left(a_{ij}\right)_{1 \le i,j \le n}$, type-I staircase, we define

$$i_0 = 1, \qquad j_0 = 1, \tag{1}$$

and for $k = 1, \dots, \ell$:

$$i_k = \max\left\{ i \,/\, a_{ij_{k-1}} \neq 0 \right\} + 1 \; (\le n+1), \tag{2}$$

$$j_k = \max\left\{ j \le i_k \,/\, a_{i_k j} = 0 \right\} + 1 \; (\le n+1), \tag{3}$$

where ℓ is given in this recurrent definition by $j_\ell = n+1$.

Analogously, we define

$$\hat{j}_0 = 1, \qquad \hat{i}_0 = 1 \tag{4}$$

and for $k = 1, \dots, r$:

$$\hat{j}_k = \max\left\{ j \,/\, a_{\hat{i}_{k-1} j} \neq 0 \right\} + 1 (\le n+1), \tag{5}$$

$$\hat{i}_k = \max\left\{ i \le \hat{j}_k \,/\, a_{i \hat{j}_k} = 0 \right\} + 1 (\le n+1), \tag{6}$$

where $\hat{i}_r = n+1$.

Finally, we denote by I, J, $\hat{I}$, and $\hat{J}$ the following sets of indices

$$I = \{i_0, i_1, \dots, i_\ell\}, \quad J = \{j_0, j_1, \dots, j_\ell\},$$
$$\hat{I} = \left\{\hat{i}_0, \hat{i}_1, \dots, \hat{i}_r\right\}, \quad \hat{J} = \left\{\hat{j}_0, \hat{j}_1, \dots, \hat{j}_r\right\},$$

thereby defining the zero pattern in the matrix A.

Note 2.4. *Note that if card(I) = 2, then $a_{ij} \neq 0$ when $1 \le j < i \le n$, where card(I) denotes the number of elements that the set I has. In the same way, if card($\hat{I}$) = 2, then $a_{ij} \neq 0$ when $1 \le i < j \le n$.*

So, if A is a type-II staircase matrix, the zero pattern of A is the zero pattern of $P_n A$.

To describe the zero pattern of a type-I staircase matrix, we define the following indices.

Definition 2.5. Let A be a real $n \times n$ matrix, type-I staircase, with zero pattern $I, J, \hat{I}$, and $\hat{J}$. Let $1 \leq i, j \leq n$. If $j \leq i$, we define

$$j_t = \max \{ j_s / 0 \leq s \leq k-1,\, j - j_s \leq i - i_s \}, \tag{7}$$

being k the unique index satisfying that $j_{k-1} \leq j < j_k$, and if $i < j$,

$$\hat{i}_t = \max \left\{ \hat{i}_s / 0 \leq s \leq k' - 1,\, i - \hat{i}_s \leq j - \hat{j}_s \right\}, \tag{8}$$

being k' the only index satisfying that $\hat{i}_{k'-1} \leq i < \hat{i}_{k'}$.

3 | ASSR MATRICES

Next, we define ASSR matrices, which are matrices whose nontrivial minors of the same order have all the same strict sign. To store the sign, we are going to define the vector of signatures.

Definition 3.1. Given a vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \mathbb{R}^n$, we say that ε is a signature sequence or, simply, is a signature, if $\varepsilon_i \in \{+1, -1\}$ for all $1 \leq i \leq n$.

ASSR matrices form a subclass of SR matrices. A matrix is SR if all its minors of the same order have the same sign. That is as follows:

Definition 3.2. A real matrix $A = (a_{ij})_{1 \leq i,j, \leq n}$ is said to be SR, with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, if all its minors satisfy that

$$\varepsilon_m \det A[\alpha|\beta] \geq 0, \;\; \alpha, \beta \in Q_{m,n}, \qquad 1 \leq m \leq n. \tag{9}$$

In a staircase matrix, there are some minors which are trivially zero due to the position of their zero entries. We are going to distinguish those minors from those that do not verify that condition.

Definition 3.3. Let $A = \left(a_{ij}\right)_{1 \leq i,j \leq n}$ be a type-I (type-II) staircase matrix. A submatrix $A[\alpha|\beta]$, with $\alpha, \beta \in Q_{m,n}$, is said to be nontrivial if all its main diagonal (secondary diagonal) elements are nonzero.

The secondary diagonal (sometimes called antidiagonal) of a square matrix $A = \left(a_{ij}\right)_{1 \leq i,j \leq n}$ is the collection of entries a_{ij} such that $i + j = n + 1$ for all $1 \leq i, j \leq n$.
The minor associated to a nontrivial submatrix ($A[\alpha|\beta]$) is called nontrivial minor ($\det A[\alpha|\beta]$).
The nontrivial minors play an important role in ASSR matrices.

Definition 3.4. A real matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ with type-I or type-II staircase is said to be ASSR, with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, if all its nontrivial minors $\det A[\alpha|\beta]$ satisfy that

$$\varepsilon_m \det A[\alpha|\beta] > 0, \;\; \alpha, \beta \in Q_{m,n}, \qquad 1 \leq m \leq n. \tag{10}$$

Note 3.5. *Observe that an ASSR matrix is SR, since the trivial minors are zero and the nontrivial minors satisfy the strict inequality (9). Observe also that an ASSR matrix is nonsingular.*

Next, we present the characterization given in Huang et al.,[6] Tehorem 10 for ASSR matrices.

Theorem 3.6. *Let A be a real $n \times n$ matrix and $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ be a signature. Then A is ASSR with signature ε if and only if A is a type-I or type-II staircase matrix, and all its nontrivial minors with $\alpha, \beta \in Q^0_{m,n}$, $m \leq n$, satisfy*

$$\varepsilon_m \det A[\alpha|\beta] > 0. \tag{11}$$

A characterization of the ASSR matrices using the pivots of NE is given in Alonso et al.[7] This characterization uses the following results:

Theorem 3.7. *Let $B = (b_{ij})_{1 \leq i,j \leq n}$ be a nonsingular type-I staircase matrix, with zero pattern defined by $I, J, \hat{I}$, and $\hat{J}$. If B is ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, then the NE of B can be performed without row exchanges and the pivots*

p_{ij} *satisfy, for* $1 \le j \le i \le n$,

$$p_{ij} = 0 \Leftrightarrow b_{ij} = 0, \tag{12}$$

$$\varepsilon_{j-j_t}\varepsilon_{j-j_t+1}p_{ij} > 0 \Leftrightarrow b_{ij} \neq 0, \tag{13}$$

where $\varepsilon_0 = 1$ *and* j_t *is defined in (7).*

Theorem 3.8. *Let* $B = (b_{ij})_{1\le i,j\le n}$ *be a nonsingular type-II staircase matrix, with zero pattern defined by* I, J, $\hat{I}$, *and* $\hat{J}$. *If* B *is ASSR with signature* $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, *then the NE of* B^T *can be performed without row exchanges and the pivots* q_{ij} *satisfy, for* $1 \le i < j \le n$,

$$q_{ij} = 0 \Leftrightarrow b_{ij} = 0, \tag{14}$$

$$\varepsilon_{i-\hat{i}_t}\varepsilon_{i-\hat{i}_t+1}q_{ij} > 0 \Leftrightarrow b_{ij} \neq 0, \tag{15}$$

where $\varepsilon_0 = 1$ *and* $\hat{i}_t$ *is defined in (8).*

From now on, we will denote by $A_h = A[h, \dots, n]$. Note that $A_1 = A$ and $A_n = (a_{nn})$.

Note 3.9. *Let* A *be a type-I (type-II) staircase matrix. Then, in relation to* A_h (A_h^T) *matrices, the next results are verified:*

1. A_h *and* A_h^T *are also type-I (type-II) staircase matrix for all* $h \in \{1, 2, \dots, n\}$.
2. *If the zero pattern of* A *is given by* $I = \{i_0, \dots, i_\ell\}$, $J = \{j_0, \dots, j_\ell\}$, $\hat{I} = \{\hat{i}_0, \dots, \hat{i}_r\}$, $\hat{J} = \{\hat{j}_0, \dots, \hat{j}_r\}$, *then the zero pattern of* A_h *matrices is given by* $I^h = \{1, i_a - h + 1, \dots, i_\ell - h + 1\}$, $J^h = \{1, j_a - h + 1, \dots, j_\ell - h + 1\}$, $\hat{I}^h = \{1, \hat{i}_b - h + 1, \dots, \hat{i}_r - h + 1\}$, $\hat{J}^h = \{1, \hat{j}_b - h + 1, \dots, \hat{j}_r - h + 1\}$,
   where $a = \min\{s \,/\, j_s - h + 1 \ge 2\}$ *and* $b = \min\{s \,/\, \hat{i}_s - h + 1 \ge 2\}$.
3. *If* A *is ASSR matrix with signature* $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ *then* A_h *and* A_h^T *are ASSR matrices with signature* $\varepsilon' = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-h+1})$.

Finally, in the following result (Theorem 5 of Alonso et al.[7]), a characterization of ASSR matrices, with $\varepsilon_2 = 1$, is presented:

Theorem 3.10. *A nonsingular matrix* $A = \left(a_{ij}\right)_{1\le i,j\le n}$ *is ASSR with signature* $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, *with* $\varepsilon_2 = 1$ *if and only if for every* $h = 1, \dots, n-1$, *the following properties hold simultaneously:*

(i) A *is type-I staircase;*
(ii) the NE of the matrices $A_h = A[h, \dots, n]$ *and* A_h^T *can be performed without row exchanges;*
(iii) the pivots p_{ij}^h *of the NE of* A_h *satisfy conditions corresponding to (12), (13), and the pivots* q_{ij}^h *of the NE* A_h^T *satisfy (14) and (15);*
(iv) for the positions (i^h, j^h) *of matrix* A_h:

- *if* $i^h \ge j^h$ *and* $i^h - j^h = i_t^h - j_t^h$ *then* $\varepsilon_{j^h - j_t^h}\varepsilon_{j^h - j_t^h + 1} = \varepsilon_{j^h-1}\varepsilon_{j^h}$,
- *if* $i^h < j^h$ *and* $i^h - j^h = \hat{i}_t^h - \hat{j}_t^h$ *then* $\varepsilon_{i^h - \hat{i}_t^h}\varepsilon_{i^h - \hat{i}_t^h + 1} = \varepsilon_{i^h-1}\varepsilon_{i^h}$,

where indices $i_t^h, j_t^h, \hat{i}_t^h, \hat{j}_t^h$ *are given by conditions corresponding to (7) and (8).*

4 | DEPTH AND CHARACTERIZATION OF ASSR MATRICES

In this section, the characterization of ASSR matrices obtained in Theorem 3.10 is simplified. To that end, following the Definition 2.3, we denote by $r = \text{card}(I) - 1$ and by $s = \text{card}(\hat{I}) - 1$. Observe that if $r = 1$ and $s = 1$, then A has not zero entries.

For further results, it is convenient to introduce the depth of an ASSR matrix, that is, the length of the longest diagonal with nonzero elements which is close to a zero entry.

Definition 4.1. Let A an $n \times n$ matrix, type-I staircase with zero pattern give by $I, J, \hat{I}, \hat{J}$. We define θ_L (θ_U) as the length of the shortest diagonal below (above) the main diagonal without zero entries, that is,

$$\theta_L := \begin{cases} 1 & \text{if} \quad r = 1, \\ \max\limits_{k\in\{1,\dots,r-1\}} \{n - (i_k - j_k)\} & \text{if} \quad r > 1 \end{cases}$$

and

$$\theta_U := \begin{cases} 1 & \text{if } s = 1, \\ \max\limits_{k\in\{1,\ldots,s-1\}} \{n - (\hat{j}_k - \hat{i}_k)\} & \text{if } s > 1. \end{cases}$$

To illustrate Definition 4.1, let us look at the following example.

Example 4.2. Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 10^{-4} & 0 & 0 & 0 \\ 2 & 6 & 6 & 8 & 0 & 0 \\ 0 & 6 & 21 & 30 & 9 & 0 \\ 0 & 8 & 30 & 48 & 42 & 28 \\ 0 & 10 & 39 & 82 & 172 & 176 \\ 0 & 0 & 0 & 28 & 176 & 259 \end{pmatrix},$$

the zero pattern bellow the main diagonal is $I = \{1, 3, 6, 7\}$ and $J = \{1, 2, 4, 7\}$. We observe that $\theta_L = \max\{6 - (3 - 2), 6 - (6 - 4)\} = 5$ and 5 is the length of the shortest diagonal below the main diagonal without zero entries.

The zero pattern above the main diagonal is $\hat{I} = \{1, 2, 3, 4, 7\}$ and $\hat{J} = \{1, 4, 5, 6, 7\}$. We observe that $\theta_U = \max\{6 - (4 - 2), 6 - (5 - 3), 6 - (6 - 4)\} = 4$ and 4 is the length of the shortest diagonal above the main diagonal without zero entries.

Considering the previous definition, the depth θ of A is defined.

Definition 4.3. Let $A = (a_{ij})_{1\le i,j\le n}$ be a real matrix. We define the depth θ of A as follows:

- if the matrix A is type-I staircase, then $\theta = \max\{\theta_L, \theta_U\}$,
- if the matrix A is type-II staircase, then θ is the depth of $P_n A$.

Example 4.4. Let A be the matrix given in Example 4.2. Then $\theta = \max\{5, 4\} = 5$ and 5 is the length of the longest diagonal close to a zero.

The next result shows that the depth θ of an ASSR matrix determines the first θ components of its signature.

Theorem 4.5. *Let $A = (a_{ij})_{1\le i,j\le n}$ be an ASSR type-I staircase matrix with depth θ. If A is nonnegative, then its signature is $\varepsilon = (1, 1, \ldots, 1, \varepsilon_{\theta+1}, \ldots, \varepsilon_n)$. If A is nonpositive, then its signature is $\varepsilon = (-1, 1, \ldots, (-1)^{\theta-1}, (-1)^{\theta}, \varepsilon_{\theta+1}, \ldots, \varepsilon_n)$.*

Proof. Observe that it is enough to see that

$$\varepsilon_k = (\varepsilon_1)^k \ \forall k \in \{1, 2, \ldots, \theta\}.$$

Firstly, we suppose that $\varepsilon_1 = 1$. If $\theta = 1$, the result holds.

We can suppose that $\theta = \theta_L$ because, otherwise, we can apply the same reasoning to A^t. Let m be such that $\theta = \theta_L = n - (i_m - j_m)$. The proof is performed by induction on k.

If $k = 1$, then $\varepsilon_1 = 1$ by assumption.

We suppose that $\varepsilon_k = 1$ for all $k \le \ell - 1$ with $\ell \le \theta$, and we will prove that $\varepsilon_\ell = 1$.

If $\ell \le n + 2 - i_m$, we consider the square matrix of order m

$$B = A[i_m - 1, \ldots, i_m + \ell - 2 | j_m - 1, \ldots, j_m + \ell - 2]$$

then

$$\det B = a_{i_m-1, j_m-1} \det A[i_m, \ldots, i_m + \ell - 2 | j_m, \ldots, j_m + \ell - 2].$$

Thus, $\varepsilon_\ell = \varepsilon_1 \varepsilon_{\ell-1}$. By hypothesis $\varepsilon_1 = 1$ and by the induction hypothesis, $\varepsilon_{\ell-1} = 1$; thus, $\varepsilon_\ell = 1$.

If $\ell > n + 2 - i_m$, we consider the square matrix of order m

$$B = A[n - \ell + 1, \ldots, n | n - (i_m - j_m) - \ell + 1, \ldots, n - (i_m - j_m)]$$

then

$$\det B = \det A[n-\ell+1,\ \dots\ ,i_m-1|n-(i_m-j_m)-\ell+1,\ \dots\ ,j_m-1)]\cdot\det A[i_m,\ \dots\ ,n|j_m,\ \dots\ ,n-(i_m-j_m)].$$

Thus, $\varepsilon_\ell=\varepsilon_{\ell+i_m-n-1}\varepsilon_{n-i_m+1}$. By the induction hypothesis $\varepsilon_{\ell+i_m-n-1}=1$ and $\varepsilon_{n-i_m+1}=1$, thus, $\varepsilon_\ell=1$.

Therefore, in any case, $\varepsilon_\ell=1$, and the result holds; that is, $\varepsilon=(1,1,\ \dots\ ,1,\varepsilon_{\theta+1},\ \dots\ ,\varepsilon_n)$.

Suppose now that $\varepsilon_1=-1$, then we define $B=\varepsilon_1A=-A$. B is an ASSR matrix with signature ε' with $\varepsilon'_k=(-1)^k\varepsilon_k$, and $\theta'=\theta$. Applying the previous reasoning to B, $\varepsilon'_k=1$ for all $k\in\{1,\ \dots\ ,\theta\}$ and

$$\varepsilon_k=(-1)^k\varepsilon'_k=(\varepsilon_1)^k.$$

So, the result holds; that is, $\varepsilon=(-1,1,\ \dots\ ,(-1)^{\theta-1},(-1)^\theta,\varepsilon_{\theta+1},\ \dots\ ,\varepsilon_n)$. □

Example 4.6. Given the matrix

$$A=\begin{pmatrix} -4 & -16 & -2 & 0 & 0 & 0\\ -8 & -40 & -44 & -64 & 0 & 0\\ -2 & -32 & -133 & -240 & -60 & 0\\ 0 & -32 & -184 & -368 & -248 & -144\\ 0 & 0 & -36 & -240 & -972 & -1344\\ 0 & 0 & 0 & -144 & -1752 & -13300 \end{pmatrix}$$

ASSR with signature $\varepsilon=(-1,1,-1,1,-1,1)$ and $\theta=4$. Then $\varepsilon_1=-1$, $\varepsilon_2=(-1)^2$, $\varepsilon_3=(-1)^3$, $\varepsilon_4=(-1)^4$.

In a previous study,[7] the authors establish the relationship between the signatures of A and P_nA that we are going to use.

Proposition 4.7. *A matrix A is ASSR if and only if P_nA is also ASSR. Furthermore, if the signature of A is $\varepsilon=(\varepsilon_1,\varepsilon_2,\ \dots\ ,\varepsilon_n)$, then the signature of P_nA is $\varepsilon'=(\varepsilon'_1,\varepsilon'_2,\ \dots\ ,\varepsilon'_n)$, with $\varepsilon'_m=(-1)^{\frac{m(m-1)}{2}}\varepsilon_m$, $m=1,\ \dots\ ,n$.*

Taking into account the previous result, it is possible to determine the first θ components of the signature vector for type-II staircase ASSR matrices.

Corollary 4.8. *Let A be an $n\times n$ ASSR type-II staircase matrix with signature ε. It is verified that $\varepsilon_k=(-1)^{\frac{k(k+1)}{2}}(\varepsilon_1)^k$ for all $k\in\{1,\ \dots\ ,\theta\}$ where θ is the depth of the matrix A.*

Proof. If we define $B=\varepsilon_1P_nA$ and we call ε' its signature, then we have that $\varepsilon'_1=1$ and B is type-I. By Theorem 4.5, $\varepsilon'_k=1$ for all $k\in\{1,\ \dots\ ,\theta\}$. Finally, using Proposition 4.7, we have $\varepsilon_k=(-1)^{\frac{k(k+1)}{2}}(\varepsilon_1)^k\varepsilon'_k=(-1)^{\frac{k(k+1)}{2}}(\varepsilon_1)^k$. □

In order to simplify the relationship between the elements of the signature vector collected in Equations (13) and (15), the following auxiliary results are obtained.

Lemma 4.9. *Let $B=(b_{ij})_{1\le i,j\le n}$ be a type-I staircase nonsingular matrix, with zero pattern I, J, $\hat{I}$, and $\hat{J}$ and depth θ. Let $\varepsilon=(\varepsilon_1,\varepsilon_2,\ \dots\ ,\varepsilon_n)$ be a signature vector with $\varepsilon_2=1$ and $\varepsilon_k=(\varepsilon_1)^k$ for all $k\in\{1,\ \dots\ ,\theta\}$. Then for all the pairs (i,j) such that $1\le j\le i\le n$, it holds that*

$$\varepsilon_{j-j_t}\varepsilon_{j-j_t+1}=\varepsilon_{j-1}\varepsilon_j.$$

Proof.

- If $j_t=1$ then it is direct, $\varepsilon_{j-j_t}\varepsilon_{j-j_t+1}=\varepsilon_{j-1}\varepsilon_j$.
- If $j_t>1$ then by (7), there exists s such that $j-j_s\le i-i_s$. So,

$$j\le i-i_s+j_s=i-(i_s-j_s)\le n-(i_s-j_s)\le\max_{k\in\{1,\dots,r-1\}}\{n-(i_k-j_k)\}=\theta_L\le\theta.$$

Thus,

$$\varepsilon_{j-j_t}\varepsilon_{j-j_t+1}=(\varepsilon_1)^{j-j_t}(\varepsilon_1)^{j-j_t+1}=(\varepsilon_1)^{2(j_t-1)}(\varepsilon_1)^{j-j_t}(\varepsilon_1)^{j-j_t+1}=$$

$$(\varepsilon_1)^{j_t-1}(\varepsilon_1)^{j-j_t}(\varepsilon_1)^{j_t-1}(\varepsilon_1)^{j-j_t+1}=(\varepsilon_1)^{j-1}(\varepsilon_1)^j=\varepsilon_{j-1}\varepsilon_j.$$

□

By applying Lemma 4.9 to matrix B^T, the next result is obtained.

Lemma 4.10. *Let $B = (b_{ij})_{1\leq i,j\leq n}$ be a type-I staircase nonsingular matrix, with zero pattern I, J, $\hat{I}$, and $\hat{J}$ and depth θ. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ be a signature vector with $\varepsilon_2 = 1$ and $\varepsilon_k = (\varepsilon_1)^k$ for all $k \in \{1, \ldots, \theta\}$. Then for all the pairs (i, j) such that $1 \leq i < j \leq n$, it is verified that*

$$\varepsilon_{i-\hat{i}_t}\varepsilon_{i-\hat{i}_t+1} = \varepsilon_{i-1}\varepsilon_i.$$

The following result provides a simplification of the conditions given in Theorem 3.7.

Proposition 4.11. *Let $B = (b_{ij})_{1\leq i,j\leq n}$ be a nonsingular matrix ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ and $\varepsilon_2 = 1$. Then the NE of B can be performed without row exchanges and the pivots p_{ij} satisfy, for any $1 \leq j \leq i \leq n$,*

$$p_{ij} = 0 \Leftrightarrow b_{ij} = 0, \tag{16}$$

$$\varepsilon_{j-1}\varepsilon_j p_{ij} > 0 \Leftrightarrow b_{ij} \neq 0, \tag{17}$$

where $\varepsilon_0 = 1$.

In addition, for $j \in \{1, \ldots, \theta\}$, condition (17) can be expressed as follows:

$$\varepsilon_1 p_{ij} > 0 \Leftrightarrow b_{ij} \neq 0.$$

Proof. As B is an ASSR matrix with $\varepsilon_2 = 1$, by Theorem 2 in Alonso et al,[7] the matrix B is type-I staircase. We denote by $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ the signature, the zero pattern I, J, $\hat{I}$, $\hat{J}$, and θ the depth of B.

By Theorem 3.7, we know that the NE of B can be performed without row exchanges and the pivots p_{ij} satisfy (12) and (13), for any $1 \leq j \leq i \leq n$, that is,

$$\begin{aligned} p_{ij} = 0 \Leftrightarrow &\quad b_{ij} = 0, \\ \varepsilon_{j-j_t}\varepsilon_{j-j_t+1} p_{ij} > 0 \Leftrightarrow &\quad b_{ij} \neq 0, \end{aligned}$$

where $\varepsilon_0 = 1$ and j_t is defined in (7).

So, (16) holds and we only have to prove that (13) and (17) are equivalent.

By Theorem 4.5, $\varepsilon_k = (\varepsilon_1)^k$ for $k \leq \theta$ and so the hypothesis of Lemma 4.9 holds. Therefore, we have

$$\varepsilon_{j-j_t}\varepsilon_{j-j_t+1} = \varepsilon_{j-1}\varepsilon_j.$$

Then, $\varepsilon_{j-1}\varepsilon_j p_{ij} = \varepsilon_{j-j_t}\varepsilon_{j-j_t+1} p_{ij} > 0$, and (17) holds.

Finally, if $1 \leq j \leq \theta$, then $j - 1 < \theta$ and $\varepsilon_j = (\varepsilon_1)^j$ and $\varepsilon_{j-1} = (\varepsilon_1)^{j-1}$

$$\varepsilon_{j-1}\varepsilon_j = (\varepsilon_1)^{j-1}(\varepsilon_1)^j = (\varepsilon_1)^{2j-1} = \varepsilon_1.$$

Thus, the condition $\varepsilon_{j-1}\varepsilon_j p_{ij} > 0$ is simplified to $\varepsilon_1 p_{ij} > 0$. □

Analogously, by using Theorem 3.10, conditions (14) and (15) can be simplified.

Proposition 4.12. *Let $B = (b_{ij})_{1\leq i,j\leq n}$ be a nonsingular matrix ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ where $\varepsilon_2 = 1$. Then, the NE of B^T can be performed without row exchanges and the pivots q_{ij} satisfy, for any $1 \leq i < j \leq n$,*

$$q_{ij} = 0 \Leftrightarrow b_{ij} = 0, \tag{18}$$

$$\varepsilon_{i-1}\varepsilon_i q_{ij} > 0 \Leftrightarrow b_{ij} \neq 0, \tag{19}$$

where $\varepsilon_0 = 1$.

In addition, for indices $i \in \{1, \ldots, \theta\}$, condition (19) can be expressed as follows:

$$\varepsilon_1 q_{ij} > 0 \Leftrightarrow b_{ij} \neq 0.$$

Note 4.13. *Let A be an $n \times n$ matrix, type-I staircase with depth θ. Then, the depth of A_h, which is denoted by θ^h, verifies that $\theta^h \leq \theta - h + 1$.*

The following theorem is a simplification of Theorem 3.10 and the depth θ of the matrix will play a key role.

Theorem 4.14. *A nonsingular matrix $A = (a_{ij})_{1\leq i,j\leq n}$ is ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ and $\varepsilon_2 = 1$ if and only if A is type-I staircase with depth θ, $\varepsilon_k = (\varepsilon_1)^k$ for all $k \in \{1, 2, \dots, \theta\}$, and for every $h = 1, \dots, n - \theta + 1$, the following properties hold simultaneously:*

(i) the NE of the matrices A_h and A_h^T can be performed without row exchanges;
(ii) the pivots of the NE of A_h, denoted as p_{ij}^h, satisfy conditions corresponding to (16) and (17), and the pivots of the NE of A_h^T, denoted as q_{ij}^h, satisfy (18) and (19).

Proof. Let us start by assuming that the matrix A is ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ with $\varepsilon_2 = 1$.

By Theorem 2 in Alonso et al,[7] as $\varepsilon_2 = 1$, A is type-I staircase and using Theorem 4.5, $\varepsilon_k = (\varepsilon_1)^k$ for all $k \in \{1, 2, \dots, \theta\}$ holds.

For each $h \in \{1, \dots, \theta\}$, we apply Theorems 4.11 and 4.12 to $B = A_h$, and conditions (i) and (ii) hold.

For the converse, we assume that A is type-I staircase with depth θ, $\varepsilon_k = (\varepsilon_1)^k$ for all $k \in \{1, 2, \dots, \theta\}$, and for every $h = 1, \dots, n - \theta + 1$, (i) and (ii) hold and we shall prove that the matrix is ASSR.

The proof is based on Theorem 3.10, so we have to prove that the conditions (i)–(iv) of this theorem are fulfilled.

- Condition (i) and (ii) of Theorem 3.10 are trivial by considering that A is type-I staircase and (i).
- Using the hypothesis (ii), $\varepsilon_k = (\varepsilon_1)^k$ if $1 \leq k \leq \theta$, and Lemma 4.9, condition (iii) of Theorem 3.10 is fulfilled.
- Finally, let us check condition (iv) of the Theorem 3.10.
  Given the matrix A_h, its depth θ^h, and the signature vector $\varepsilon^h = (\varepsilon_1^h, \dots, \varepsilon_{n-h+1}^h) = (\varepsilon_1, \dots, \varepsilon_{n-h+1})$. It is clear that $\varepsilon_k^h = \varepsilon_k = (\varepsilon_1)^k = (\varepsilon_1^h)^k$, so we are under the hypothesis of Lemma 4.9 and Lemma 4.10 and it is verified, for the positions (i^h, j^h) of matrix A_h, that:
  - if $i^h \geq j^h$ and $i^h - j^h = i_t^h - j_t^h$ then $\varepsilon_{j^h - j_t^h} \varepsilon_{j^h - j_t^h + 1} = \varepsilon_{j^h - 1} \varepsilon_{j^h}$,
  - if $i^h < j^h$ and $i^h - j^h = \hat{i}_t^h - \hat{j}_t^h$ then $\varepsilon_{i^h - \hat{i}_t^h} \varepsilon_{i^h - \hat{i}_t^h + 1} = \varepsilon_{i^h - 1} \varepsilon_{i^h}$,

  where indices $i_t^h, j_t^h, \hat{i}_t^h, \hat{j}_t^h$ are given by conditions corresponding to (7) and (8).

The last part of the proof is that the conditions must hold for the matrices A_h with $h \in \{1, \dots, \theta\}$. To prove this, we take the matrix $B = \varepsilon_1 A$, whose signature verifies that $\varepsilon_k = 1$ for all $k \in \{1, \dots, \theta\}$. So the matrix $B_{n-\theta+1}$ with the signature vector $\varepsilon'^h = (1, \dots, 1)$ verifies the hypothesis of Theorem 3.3 in[5] and the matrix $B_{n-\theta+1}$ it is ASTP. Thus, the matrix $A_{n-\theta+1} = \varepsilon_1 B_{n-\theta+1}$ is ASSR and conditions (i)–(iv) are fulfilled for it and all its submatrices A_k with $k \in \{n - \theta + 1, \dots, n\}$. Thus, conditions (i)–(iv) of Theorem 3.10 are fulfilled and the matrix A is ASSR with signature ε. □

Note 4.15. *Notice that the previous result is a generalization of Theorem 6 of Alonso et al.[16] That result is applied to strictly M-banded matrices, while the new proposal allows us to characterize any ASSR matrices of type-I staircase, taking into account their depth. In fact, one can observe that a strictly M-banded type-I staircase matrix has depth $n - M$. On the other hand, we should take advantage of this moment to correct the last condition of Theorem 6 of Alonso et al,[16] which should be written as $A_{M+1} = A[M + 1, \dots, n]$.*

When the matrix A is type-II staircase, a result similar to Theorem 4.14 is presented. For that, we consider the matrix $B = P_n A$ and the submatrices $B_h = B[h, \dots, n]$ and B_h^T with $h \in \{1, \dots, n-\theta+1\}$ being θ the depth of A. So, the following result is a consequence of Corollary 4.8 and applying Theorem 4.14 to B.

Theorem 4.16. *A nonsingular matrix $A = (a_{ij})_{1\leq i,j\leq n}$ is ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ and $\varepsilon_2 = -1$ if and only if $B = P_n A$ is type-I staircase with depth θ, $\varepsilon_k = (-1)^{\frac{k(k-1)}{2}} (\varepsilon_1)^k$ for all $k \in \{1, 2, \dots, \theta\}$, and for every $h = 1, \dots, n - \theta + 1$, the following properties hold simultaneously:*

(i) the NE of the matrices B_h and B_h^T can be performed without row exchanges;

(ii) the pivots p_{ij}^h of the NE of B_h satisfy conditions corresponding to (16) and (17), and the pivots q_{ij}^h of the NE B_h^T satisfy (18) and (19) for the signature vector $\bar{\varepsilon} = (\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_n)$ with $\bar{\varepsilon}_k = (-1)^{\frac{k(k-1)}{2}} \varepsilon_k$.

Theorems 4.14 and 4.16 allow us to implement a fast algorithm to check if matrix A is ASSR by performing the NE algorithm to the submatrices A_h.

Algorithm 1 NE characterization

Input: A: matrix of order n and a signature vector ε
Output: If matrix A is ASSR with signature ε, TRUE else FALSE

```
1: Check that A is type-I staircase, calculate its depth θ and check that ε_k = (ε_1)^k for k ∈ {2, ..., θ}
2: for h = 1 to n − θ + 1 do
3:     Apply NE to matrix A_h, checking that no row exchanges are needed and pivots p^h_ij satisfy (16) and (17)
4:     Apply NE to matrix A_h^T, checking that no row exchanges are needed and pivots q^h_ij satisfy (18) and (19)
5: end for
```

Algorithm 1 allows us to test if a type-I staircase matrix is ASSR using the simplification obtained in Theorem 4.14. If we consider a type-II staircase matrix, then we apply Theorem 4.16 and the following changes in Algorithm 1 should be considered:

1. Check that $B = P_n A$ is type-I staircase, calculate its depth θ, and check that $\varepsilon_k = (-1)^{\frac{k(k-1)}{2}} (\varepsilon_1)^k$ for $k \in \{2, \dots, \theta\}$
3. Apply NE to matrix B_h, checking that no row exchanges are needed and pivots p_{ij}^h satisfy (16) and (17) for the signature vector $\bar{\varepsilon} = (\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_n)$ with $\bar{\varepsilon}_k = (-1)^{\frac{k(k-1)}{2}} \varepsilon_k$
4. Apply NE to matrix B_h^T, checking that no row exchanges are needed and pivots q_{ij}^h satisfy (18) and (19) for the signature vector $\bar{\varepsilon} = (\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_n)$ with $\bar{\varepsilon}_k = (-1)^{\frac{k(k-1)}{2}} \varepsilon_k$

It is possible to implement the algorithm so that the input argument is only the matrix and the output arguments are the signature, the zero pattern, and the depth in case the matrix is ASSR.

Notice that the computational complexity of Algorithm 1 depends on the number of times NE is applied to the matrices A_h and A_h^T, with $h = 1, \dots, n - \theta + 1$. If we consider that computational cost of NE is $2n^3/3t_c$, where t_c is the time spent to carry out one operation in floating point, it is evident that the characterization presented in Theorem 4.14 has a lower computational cost than the characterization achieved in Theorem 3.10, where $h = 1, \dots, n-1$.

We have done it in these terms, and we have applied it to the following examples.

Example 4.17. Given the matrix

$$A = \begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ -2 & -6 & -6 & -8 & 0 & 0 \\ 0 & 0 & -21 & -30 & -9 & 0 \\ 0 & 0 & -30 & -48 & -42 & -28 \\ 0 & 0 & -9 & -42 & -172 & -176 \\ 0 & 0 & 0 & -28 & -176 & -259 \end{pmatrix},$$

the algorithm returns the following results

$$\theta = 6, I = \{1,3,6,7\}, J = \{1,3,4,7\}, \hat{I} = \{1,2,3,4,7\}, \hat{J} = \{1,3,5,6,7\},$$
$$\varepsilon = (-1,1,-1,1,-1,1).$$

It should be noted that since the depth θ is 6, NE elimination must only be applied to the matrices $A_1 = A$ and $A_1^T = A^T$, while with the previous algorithms, it was necessary to apply the NE elimination to the matrices

$$\{A_1, A_1^T, A_2, A_2^T, A_3, A_3^T, A_4, A_4^T, A_5, A_5^T\}.$$

In this example, the computational cost of Algorithm 1, considering the number of floating point operations resulting from the application of the NE, would be $2 * (2 * (6^3)/3) = 288$, while if we use the characterization obtained in Theorem 4.14, it would be necessary to perform approximately 587 floating point operations.

Example 4.18. Given the matrix

$$B = \begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ -2 & -6 & -6 & -8 & 0 & 0 \\ 0 & 0 & -21 & -30 & -9 & 0 \\ 0 & 0 & -30 & -48 & -42 & -45 \\ 0 & 0 & -9 & -42 & -172 & -176 \\ 0 & 0 & 0 & -28 & -176 & -259 \end{pmatrix},$$

the algorithm returns the following results

$$\theta = 6, I = \{1,3,6,7\}, J = \{1,3,4,7\}, \hat{I} = \{1,2,3,4,7\}, \hat{J} = \{1,3,5,6,7\},$$
$$\varepsilon = \text{“false”},$$

because the matrix is not ASSR, $\det(B[1,2|1,2]) = \det\begin{pmatrix} -1 & -2 \\ -2 & -6 \end{pmatrix} = 2$ and $\det(B[4,5|5,6]) = \det\begin{pmatrix} -42 & -45 \\ -172 & -176 \end{pmatrix} = -348$. In this example, the ASSR conditions fail when we apply the NE elimination to B^T. At the beginning of the fifth step, we get the matrix

$$(B^T)^{(5)} = \begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -21 & -30 & -9 & 0 \\ 0 & 0 & 0 & -5.1429 & -29.1430 & -28 \\ 0 & 0 & 0 & 0 & -3 & -17.3330 \\ 0 & 0 & 0 & 0 & 101.5000 & 116 \end{pmatrix};$$

it can be seen that the pivots of the fifth column must all be negative; however, the pivot $q_{56} = (B^T)^{(5)}[6|5] = 101.5$ is positive.

5 | NUMERICAL EXPERIMENTS AND APPLICATIONS

In this section, we present a numerical experiment associated with a problem modeled through a differential equation. In this sense, it should be noted that classic problems as:

- an elastic cord held at both ends with unitary tension and subjected to a transverse load of intensity f(x)
- an elastic bar held at both ends and subjected to axial load of intensity f(x) or
- the conduction of heat in a bar subjected to a distributed heat source f(x) with constant temperature at the ends

can be established from a differential equation in one dimension and of the second order of the type:

$$u''(t) + g(t)u(t) = f(t), \quad t \in [a,b],$$
$$u(a) = 0,$$
$$u(b) = 0.$$

The finite element method consists of looking for the solution in a finite dimensional vector space and reducing the problem to calculating the coordinates of the solution with respect to a given base. This reduces the problem to a system of linear equations in which the matrix of coefficients, called the stiffness matrix of the system, is a structured matrix. When applying the method to certain cases, an ASSR matrix results. In these cases, high precision methods specifically designed for this type of matrices can be used and a highly accurate solution is obtained.

We performed several experiments, and here, we show one. We solve the next differential equation

$$-u''(t) + 3000\,u(t) = t\sin(t + 2\pi/3), \quad t \in [1,3],$$
$$u(1) = 0,$$
$$u(3) = 0.$$

Using 51 nodes in the equation, we obtain a tridiagonal linear system, $A \cdot X = b$, where the matrix A (stiffness matrix) is of order 49

$$A = \begin{pmatrix} 8050 & 1975 & 0 & \dots & 0 \\ 1975 & 8050 & 1975 & \dots & 0 \\ 0 & 1975 & 8050 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 8050 \end{pmatrix},$$

and the right-hand side vector is obtained as $b = (b_1, b_2, \dots, b_{49}) = (2.8871e - 04, -0.0014, \dots, -0.1115)$, with $b_k = \frac{1}{25}\int_0^1 \left(x * f\left(\frac{x}{25} + t_{k-1}\right) + (1-x) * f\left(\frac{x}{25} + t_k\right)\right) dx$, where $f(x) = x \sin\left(x + \frac{2\pi}{3}\right)$ and $t_k = 1 + \frac{k(3-1)}{50}$, with $k = 1, 2, \dots, 49$.

Thus, applying Algorithm 1, it is possible to verify that the obtained matrix (stiffness matrix) is an ASSR matrix with signature sequence $\varepsilon = (1, 1, 1, \dots, 1, 1)$. Next, we will use Neville's method to solve the resulting system, and we will analyze the error made with respect to the exact solution of that system. It should be noted that the exact solution of the system has been obtained using symbolic computation in Matlab.

To carry out the Neville process (see Gasca and Peña[8]), the following algorithm has been implemented using Matlab (see Alonso et al.[15]):

```
function [A Piv Ak] = Neville(A,n)
% Function to apply the Neville Elimination method to a square matrix.
% Input arguments:
%     A  ........ a square matrix
%     n ......... the order of the matrix A (optional)
% Output arguments:
%     A ......... the matrix we obtain at the end of the process
%     Piv ....... Matrix with the pivots of the NE algorithm
%     Ak ........ A 3-dimensional array with all the matrices A_k
%                 of the procedure A_1=Ak(:,:,1), A_2=Ak(:,:,2),...
if nargin < 2; n = size(A,2); end
Piv = zeros(n,n);
for j = 1:n
    Ak(:,:,j) = A;
    [I,bnd] = Pivoting(A(j:end,j),j,n);
    A = A(I,:);
    Piv(j:end,j) = A(j:end,j);
    for i = bnd:-1:(j+1)
        A(i,:) = A(i,:)-A(i,j)/A(i-1,j)*A(i-1,:);
    end
end

function [I bnd] = Pivoting(v,j,n)
% Function to apply pivoting strategy
% Input arguments:
%     v ......... a column vector with the elements (a_jj,a_{j+1j},...,a_nj)
%     j  ........ index of the column we are applying the method
%     n ......... order of the matrix
% Output arguments:
%     I ......... the new order to put the zero entries down
%     bnd ....... the position of the last element different from zero
vv = ones(n,1);
vv(j:n) = abs(v)/max(abs(v)) ~= 0;%We can use (>10^k*eps) instead of (~=0)
[vv I] = sort(vv,'descend');
bnd = find(vv,1,'last');
```

It should be noted that other efficient tools that use NE to work with SR matrices can be found, for instance, in[15] and.[17–19] In the first one, several algorithms (functions) have been implemented using Matlab to work with ASSR matrices. The second includes a software package that can perform virtually all matrix computations with nonsingular TP matrices to

TABLE 1 Absolute and relative errors: ML, NE, and TNS

	$\|\|ML-SY\|\|$	$\|\|NE-SY\|\|$	$\|\|TNS-SY\|\|$	$\|\|ML-SY\|\|/\|\|SY\|\|$	$\|\|NE-SY\|\|/\|\|SY\|\|$	$\|\|TNS-SY\|\|/\|\|SY\|\|$
1-norm	3.0626e-20	2.2327e-20	2.3757e-20	1.2002e-16	8.7502e-17	9.3104e-17
2-norm	6.4365e-21	5.4095e-21	5.5032e-21	1.5055e-16	1.2653e-16	1.2872e-16
∞-norm	1.6941e-21	1.6941e-21	1.6941e-21	1.4442e-16	1.4442e-16	1.4442e-16

high relative accuracy (HRA), under certain conditions. HRA means that the relative errors of the computations are of the order of machine precision, independently of the size of the condition number. If the stiffness matrix A is a TP matrix, it is possible to use the function TNBD, to compute the bidiagonal decomposition of the matrix A by performing NE, next, using the function TNSolve, we can solve the triangular linear system $AX = b$ using backward substitution.

In Table 1, we compare the solution obtained by using the MatLab command $X = A\backslash b$ (ML), the Neville algorithm (NE), the functions TNBD and TNSolve (TNS), and symbolic computation (SY), with the usual norms:

It can be seen that the combination between the algorithm characterization and the Neville algorithm is efficient for this type of applications.

6 | CONCLUSIONS

It is shown that the new concept of depth (θ) of an $n \times n$ staircase matrix A is a very useful tool to deal with ASSR matrices. On the one hand, it determines the initial θ components of the signature of A. In particular, if the depth is maximal, that is, $\theta = n$, then the signature is completely determined. On the other hand, the depth can be used to simplify and reduce the computational cost of the algorithm to check if a given matrix is ASSR with a given signature. This reduction increases with the depth of the matrix.

ACKNOWLEDGEMENTS

This work has been partially supported by the Spanish Research Grants PGC2018-096321-B-I00, TIN2017-87600-P, and MTM2017-90682-REDT and by the Gobierno de Aragón (E41_20R).

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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How to cite this article: Alonso P, Jiménez J, Peña JM, Serrano ML. Depth of almost strictly sign regular matrices. *Math Meth Appl Sci*. 2022;1-13. doi:10.1002/mma.8542