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Accurate computations with collocation and Wronskian matrices of Jacobi polynomials

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Abstract In this paper an accurate method to construct the bidiagonal factorization of collocation and Wronskian matrices of Jacobi polynomials is obtained and used to compute with high relative accuracy their eigenvalues, singular values and inverses. The particular cases of collocation and Wronskian matrices of Legendre polynomials, Gegenbauer polynomials, Chebyshev polynomials of the first and second kind and rational Jacobi polynomials are considered. Numerical examples are included.

Keywords High relative accuracy · Bidiagonal decompositions · Jacobi polynomials · Totally positive matrices

1 Introduction

Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ (see Section 3) form a class of classical orthogonal polynomials, which includes many important families of orthogonal polynomials such as Legendre and Chebyshev polynomials (see Section 5). In fact, Jacobi polynomials are orthogonal with respect to the weight $(1-x)^\alpha(1+x)^\beta$ on the interval $[-1,1]$ and present many useful applications. For instance, to approximation theory, to Gaussian quadrature to numerically compute integrals, to differential equations or to physical applications (cf. [2], [13]).

Let us recall that, given a system of functions $(u_0,\dots,u_n)$, its *collocation matrix* at points $x_1<\cdots<x_{n+1}$ is given by $(u_{j-1}(x_i))_{1\le i,j\le n+1}$. This paper deals with the accurate computa-

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tion when using collocation and Wronskian matrices (see Section 3) of Jacobi polynomials on $(1,\infty)$. As shown in this paper, for these matrices many algebraic computations (such as the computation of the inverse, of all the eigenvalues and singular values, or the solutions of some linear systems) can be performed with high relative accuracy (HRA, see Section 2). Up to now, this has been obtained only for a few classes of structured matrices. Among them we can mention the collocation matrices of Bernstein polynomials [16], of Laguerre polynomials [3] and of Bessel functions [4] as well as the Wronskian matrices of the monomials and of exponential polynomials [15]. In fact, this last paper was the unique paper guaranteeing HRA for some Wronskian matrices.

Crucial facts to derive our results have been to prove the strict total positivity (see Section 2) of the collocation matrices of Jacobi polynomials on $(1,\infty)$ and the total positivity of their Wronskian matrices. Then the bidiagonal factorization with HRA has been obtained for these matrices and the algorithms presented in [12] can be used for the algebraic computations mentioned above with HRA.

As mentioned before, accurate computations with collocation matrices of other interesting bases of orthogonal polynomials, such as Laguerre polynomials or Bessel polynomials, have been already achieved (see [3] and [4]). The analysis of the domain where the corresponding collocation or Wroskian matrices, or closely related matrices, are totally positive helps to obtain their bidiagonal factorization and the solution of algebraic problems with HRA for the parameters in this domain. We shall see that for the collocation or Wronskian matrices of Jacobi bases, this domain lies outside the interval where the polynomials are orthogonal and have their zeros.

The paper is organized as follows. Section 2 presents some basic concepts and results related to the bidiagonal factorization of totally positive matrices and with HRA. In Section 3, the strict total positivity and bidiagonal factorization of the collocation matrices of Jacobi polynomials on $(1,\infty)$ are obtained. In Section 4, the total positivity and bidiagonal factorization of the corresponding Wronskian matrices are derived. Section 5 particularizes the results for some well known families of Jacobi polynomials: Legendre polynomials, Gegenbauer polynomials, Chebyshev polynomials of the first and second kind and rational Jacobi polynomials. Section 6 presents numerical examples confirming the theoretical results for the computation of eigenvalues, singular values, inverses, and the solution of linear systems with some matrices used in this paper.

2 Notations and auxiliary results

As usual, given an n-times continuously differentiable function f and x in its parameter domain, $f'(x)$ denotes the first derivative of f at x and, for any $i \le n$, $f^{(i)}(x)$ denotes the i-th derivative of f at x. Let us recall that for a given basis $(u_0,\dots,u_n)$ of a space of n-times continuously differentiable functions, defined on a real interval I and $x \in I$, the *Wronskian matrix* at x is defined by

$$W(u_0,\dots,u_n)(x) := (u_{j-1}^{(i-1)}(x))_{i,j=1,\dots,n+1}.$$

A matrix is totally positive: TP (respectively, strictly totally positive: STP) if all its minors are nonnegative (respectively, positive). Two recent books on these matrices are [6, 18], where many applications of these matrices are presented, as well as in [1].

Neville elimination is an alternative procedure to Gaussian elimination and has been used to characterize TP and STP matrices. Given a nonsingular matrix $A = (a_{i,j})_{1\le i,j\le n+1}$, Neville elimination computes a matricial sequence

$$A^{(1)} := A \to A^{(2)} \to \cdots \to A^{(n+1)}, \tag{1}$$

such that, for $1 \le k \le n$, $A^{(k+1)} = (a_{i,j}^{(k+1)})_{1\le i,j\le n+1}$ has zeros below its main diagonal in the k first columns and is computed from $A^{(k)} = (a_{i,j}^{(k)})_{1\le i,j\le n+1}$ by:

$$a_{i,j}^{(k+1)} := \begin{cases} a_{i,j}^{(k)}, & \text{if } 1 \le i \le k, \\ a_{i,j}^{(k)} - \frac{a_{i,k}^{(k)}}{a_{i-1,k}^{(k)}} a_{i-1,j}^{(k)}, & \text{if } k+1 \le i, j \le n+1 \text{ and } a_{i-1,k}^{(k)} \ne 0, \\ a_{i,j}^{(k)}, & \text{if } \quad k+1 \le i \le n+1 \text{ and } a_{i-1,k}^{(k)} = 0. \end{cases} \tag{2}$$

At the end of the Neville elimination, an upper triangular matrix

$$U := A^{(n+1)} \tag{3}$$

is obtained. In this process, the element

$$p_{i,j} := a_{i,j}^{(j)}, \quad 1 \le j \le i \le n+1, \tag{4}$$

is called the (i,j) pivot and, in particular, $p_{i,i}$ is a diagonal pivot of the Neville elimination of A. If all the pivots are nonzero then Neville elimination can be carried out without row exchanges. In this case, by Lemma 2.6 of [7],

$$\begin{aligned} &p_{i,1} = a_{i,1}, \quad 1 \le i \le n+1, \\ &p_{i,j} = \frac{\det A[i-j+1,\ldots,i|1,\ldots,j]}{\det A[i-j+1,\ldots,i-1|1,\ldots,j-1]}, \quad 1 < j \le i \le n+1, \end{aligned} \tag{5}$$

where, given increasing sequences of integers α and β, $A[\alpha|\beta]$ denotes the submatrix of A containing rows of places α and columns of places β.

Moreover,

$$m_{i,j} := \begin{cases} a_{i,j}^{(j)}/a_{i-1,j}^{(j)} = p_{i,j}/p_{i-1,j}, & \text{if } a_{i-1,j}^{(j)} \ne 0, \\ 0, & \text{if } a_{i-1,j}^{(j)} = 0, \end{cases}, \quad 1 \le j < i \le n+1, \tag{6}$$

is called the (i,j) multiplier of the Neville elimination of A.

Neville elimination has been used to characterize TP and STP matrices (see [7–9]). The following characterization can be derived from Corollary 5.5 of [7].

Theorem 1 Let A be a nonsingular matrix. Then A is TP if and only if the Neville elimination of A and U^T, where U is the upper triangular matrix in (3), can be performed without row exchanges and all the pivots of both Neville eliminations are nonnegative.

By Theorem 4.2 and the arguments of p.116 of [9], a nonsingular TP matrix $A = (a_{i,j})_{1\le i,j\le n+1}$ admits a factorization of the form

$$A = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n, \tag{7}$$

where F_i and G_i are the TP, lower and upper triangular bidiagonal matrices given by

$$F_i = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & m_{i+1,1} & 1 & & & \\ & & & & m_{i+2,2} & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & m_{n+1,n+1-i} & 1 \end{pmatrix}, \quad G_i^T = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & \widetilde{m}_{i+1,1} & 1 & & & \\ & & & & \widetilde{m}_{i+2,2} & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \widetilde{m}_{n+1,n+1-i} & 1 \end{pmatrix}, \tag{8}$$

and $D = \operatorname{diag}(p_{1,1}, p_{2,2}, \dots, p_{n+1,n+1})$ has positive diagonal entries. The diagonal entries $p_{i,i}$ of D are the diagonal pivots of the Neville elimination of A and the elements $m_{i,j}$ and $\tilde{m}_{i,j}$ are nonnegative and coincide with the multipliers of the Neville elimination of A and A^T, respectively. If, in addition, the entries m_{ij}, $\widetilde{m}_{ij}$ satisfy

$$m_{ij} = 0 \quad \Rightarrow \quad m_{hj} = 0, \quad \forall h > i$$

and

$$\widetilde{m}_{ij} = 0 \quad \Rightarrow \quad \widetilde{m}_{ik} = 0, \quad \forall k > j,$$

then the decomposition (7) is unique. We shall denote the bidiagonal decomposition (7) of a TP matrix A as $BD(A)$ (see [11]).

Given $BD(A)$, using the results in [7–9], a bidiagonal decomposition of A^{-1} can be computed as

$$A^{-1} = \widetilde{G}_1 \widetilde{G}_2 \cdots \widetilde{G}_n D^{-1} \widetilde{F}_n \cdots \widetilde{F}_2 \widetilde{F}_1, \tag{9}$$

where $\widetilde{F}_i$ and $\widetilde{G}_i$, $i = 1, \dots, n$, are the lower and upper triangular bidiagonal matrices of the form of F_i and G_i, respectively, but replacing the off-diagonal entries $\{m_{i+1,1}, \dots, m_{n+1,n+1-i}\}$ and $\{\tilde{m}_{i+1,1}, \dots, \tilde{m}_{n+1,n+1-i}\}$ by $\{-m_{i+1,i}, \dots, -m_{n+1,i}\}$ and $\{-\tilde{m}_{i+1,i}, \dots, -\tilde{m}_{n+1,i}\}$, respectively.

Let us observe that if A is a nonsingular and TP matrix, then A^T is also nonsingular and TP. Moreover, the bidiagonal decomposition of A^T can be computed as

$$A^T = G_n^T G_{n-1}^T \cdots G_1^T D F_1^T \cdots F_{n-1}^T F_n^T, \tag{10}$$

where F_i and G_i, $i = 1, \dots, n$, are the lower and upper triangular bidiagonal matrices given in the bidiagonal factorization $BD(A)$, that is,

$$BD(A^T) = BD(A)^T.$$

Finally, let us recall that a real value x is obtained with high relative accuracy (HRA) if the relative error of the computed value $\tilde{x}$ satisfies

$$\frac{\|x - \tilde{x}\|}{\|x\|} < Ku,$$

where K is a positive constant independent of the arithmetic precision and u is the unit round-off. HRA implies that the relative errors of the computations are of the order of the machine precision. An algorithm can be computed with HRA when it only uses products, quotients, sums of numbers of the same sign, subtractions of numbers of opposite sign or subtraction of initial data (cf. [5], [10]).

In [11] it was shown that if $BD(A)$, the bidiagonal factorization (7) of a nonsingular TP matrix A, is computed with HRA then we can also compute with HRA its eigenvalues and singular values, the matrix A^{-1} and even the solution of $Ax = b$ for vectors b with alternating signs.

3 Total positivity and factorizations of collocation matrices of Jacobi polynomials

Given $\alpha,\beta\in\mathbb{R}$, the basis of Jacobi polynomials of the space $\mathbf{P}^n$ of polynomials of degree less than or equal to n is $(J_0^{(\alpha,\beta)},\dots,J_n^{(\alpha,\beta)})$ with

$$J_i^{(\alpha,\beta)}(x):=\frac{\Gamma(\alpha+i+1)}{i!\Gamma(\alpha+\beta+i+1)}\sum_{k=0}^{i}\binom{i}{k}\frac{\Gamma(\alpha+\beta+i+k+1)}{\Gamma(\alpha+k+1)}\left(\frac{x-1}{2}\right)^k,\ \ i=0,\dots,n. \tag{11}$$

Let us recall that Jacobi polynomials are orthogonal on the interval $[-1,1]$ with respect to the weight $(1-x)^\alpha(1+x)^\beta$.

Let us consider the lower triangular matrix $A=(a_{ij})_{1\le i,j\le n+1}$ given by

$$a_{i,j}:=\begin{cases}\frac{1}{(j-1)!(i-j)!}\prod_{k=j}^{i-1}(\alpha+k)\prod_{k=1}^{j-1}(\alpha+\beta+i+k-1), & \text{if}\quad i\ge j,\\ 0, & \text{if}\quad i<j.\end{cases} \tag{12}$$

It can be checked that

$$(J_0^{(\alpha,\beta)},\dots,J_n^{(\alpha,\beta)})^T=A(v_0,\dots,v_n)^T, \tag{13}$$

where $(v_0,\dots,v_n)$ is the basis of $\mathbf{P}^n$ such that

$$v_i(x):=\left(\frac{x-1}{2}\right)^i,\quad i=0,\dots,n. \tag{14}$$

The following result provides the multipliers and the diagonal pivots of the Neville elimination of the change of basis matrix A described in (12) and proves that this matrix is nonsingular and TP.

Theorem 2 *Let $A=(a_{ij})_{1\le i,j\le n+1}$ be the lower triangular matrix defined in* (12). *Then the multipliers $m_{i,j}$ and diagonal pivots $p_{i,i}$ of the Neville elimination of A are given by*

$$m_{i,1}:=\frac{\alpha+i-1}{i-1},\quad m_{i,j}:=\frac{\alpha+\beta+2i-j}{\alpha+\beta+2i-j-2}m_{i,j-1},\quad 1<j<i\le n+1,\ 1<i\le n+1, \tag{15}$$

$$p_{i,i}:=\prod_{r=1}^{i-1}\frac{(\alpha+\beta+2i-r-1)}{(i-r)},\quad 1\le i\le n+1.$$

Moreover, for any $\alpha,\beta>-1$, A is nonsingular and TP.

Proof Let $A^{(k)}:=(a_{ij}^{(k)})_{1\ge i,j\ge n+1}$, $k=2,\dots,n+1$, be the matrices obtained after $k-1$ steps of the Neville elimination of A. First, let us see by induction on k that

$$a_{i,j}^{(k)}=\frac{1}{(j-k)!(i-j)!}\prod_{r=1}^{k-1}\frac{(\alpha+\beta+2i-r-1)}{(i-r)}\prod_{r=j}^{i-1}(\alpha+r)\prod_{r=1}^{j-k}(\alpha+\beta+i+r-1), \tag{16}$$

for $1 \le j < i \le n+1$. For $k=2$, taking into account that $a_{i,j}^{(2)} = a_{i,j} - \frac{a_{i,1}}{a_{i-1,1}} a_{i-1,j}$, we have

$$
\begin{aligned}
a_{i,j}^{(2)} &= a_{i,j} - \frac{\alpha+i-1}{i-1} a_{i-1,j} \\
&= \frac{1}{(j-1)!(i-j-1)!} \prod_{r=j}^{i-1}(\alpha+r) \prod_{r=1}^{j-2}(\alpha+\beta+i+r-1) \left(\frac{\alpha+\beta+i+j-2}{i-j} - \frac{\alpha+\beta+i-1}{i-1} \right) \\
&= \frac{1}{(j-1)!(i-j-1)!} \prod_{r=j}^{i-1}(\alpha+r) \prod_{r=1}^{j-2}(\alpha+\beta+i+r-1) \left(\frac{(j-1)(\alpha+\beta+2i-2)}{(i-j)(i-1)} \right) \\
&= \frac{1}{(j-2)!(i-j)!} \frac{(\alpha+\beta+2i-2)}{(i-1)} \prod_{r=j}^{i-1}(\alpha+r) \prod_{r=1}^{j-2}(\alpha+\beta+i+r-1), \quad 1 \le j < i \le n+1.
\end{aligned}
$$

Therefore formula (16) holds for $k=2$. Let us now suppose that (16) holds for some $k \in \{2,\dots,n\}$. Taking into account that $a_{i,j}^{(k+1)} = a_{i,j}^{(k)} - \frac{a_{i,k}^{(k)}}{a_{i-1,k}^{(k)}} a_{i-1,j}^{(k)}$, we have

$$
a_{i,j}^{(k+1)} = a_{i,j}^{(k)} - \frac{1}{i-1} \prod_{r=1}^{k-1} \frac{(\alpha+\beta+2i-r-1)}{(\alpha+\beta+2i-r-3)} (\alpha+i-1) a_{i-1,j}^{(k)}.
$$

Then, by defining

$$
C_1 := \frac{\alpha+\beta+i+j-k-1}{i-j} - \frac{\alpha+\beta+i-1}{i-k} = \frac{(j-k)(\alpha+\beta+2i-k-1)}{(i-j)(i-k)},
$$

we can write

$$
\begin{aligned}
a_{i,j}^{(k+1)} &= \frac{1}{(j-k)!(i-j-1)!} \prod_{r=1}^{k-1} \frac{(\alpha+\beta+2i-r-1)}{(i-r)} \prod_{r=j}^{i-1}(\alpha+r) \prod_{r=1}^{j-k-1} (\alpha+\beta+i+r-1) C_1 \\
&= \frac{1}{(j-k-1)!(i-j)!} \prod_{r=1}^{k} \frac{(\alpha+\beta+2i-r-1)}{(i-r)} \prod_{r=j}^{i-1}(\alpha+r) \prod_{r=1}^{j-k-1} (\alpha+\beta+i+r-1),
\end{aligned}
$$

and formula (16) also holds for $k+1$.

Now, by (4) and (16), we can easily deduce that the pivots $p_{i,j}$ of the Neville elimination of A satisfy

$$
p_{i,j} = \frac{1}{(i-j)!} \prod_{r=1}^{j-1} \frac{(\alpha+\beta+2i-r-1)}{(i-r)} \prod_{r=j}^{i-1}(\alpha+r), \quad 1 \le j < i \le n+1, \tag{17}
$$

and, for the particular case $i=j$,

$$
p_{i,i} = \prod_{r=1}^{i-1} \frac{(\alpha+\beta+2i-r-1)}{(i-r)}, \quad 1 \le i \le n+1. \tag{18}
$$

Let us observe that, by formula (17), the pivots of the Neville elimination of A are nonzero and so, this elimination can be performed without row exchanges. Besides, since A is lower triangular with nonzero diagonal entries, A is nonsingular and the obtained matrix U (see (3)) is diagonal and so, the Neville elimination of U^T does not perform any operation. Then, by Theorem 1, we can conclude that A is nonsingular and TP for any $\alpha, \beta > -1$.

Finally, using (6) and (18), the multipliers $m_{i,j}$ can be written as

$$m_{i,j} = \frac{(i-1-j)!(\alpha+i-1)}{(i-j)!} \prod_{r=1}^{j-1} \frac{(\alpha+\beta+2i-r-1)(i-r-1)}{(\alpha+\beta+2i-r-3)(i-r)},$$

and we can deduce that

$$m_{i,1} = \frac{\alpha+i-1}{i-1}, \quad 1 < i \le n+1, \tag{19}$$

$$m_{i,j} = \frac{\alpha+\beta+2i-j}{\alpha+\beta+2i-j-2} m_{i,j-1}, \quad 1 < j < i \le n+1. \qquad \square$$

Corollary 1 *Let $A = (a_{ij})_{1\le i,j\le n+1}$ be the lower triangular matrix defined by* (12)*. Then, for any $\alpha, \beta > -1$, the matrix A admits a factorization of the form*

$$A = F_n F_{n-1} \cdots F_1 D, \tag{20}$$

where F_i, $i = 1,\dots,n$, is the lower triangular, bidiagonal matrix given by (8) *and* $D = \mathrm{diag}(p_{1,1}, p_{2,2}, \dots, p_{n+1,n+1})$*. The entries $m_{i,j}$ and $p_{i,i}$ can be obtained from* (15)*.*

Let us observe that the factorization (20) corresponds to $BD(A)$, the bidiagonal factorization (7) of A. Furthermore, for any $\alpha, \beta > -1$, $BD(A)$ can be computed with HRA, since it does not require subtractions (except of the initial data).

Remark 1 It is well known that the monomial basis $(1, t, \dots, t^n)$ of $\mathbf{P}^n$ is STP on $(0, \infty)$. Moreover, given a sequence of positive parameters $0 < t_0 < \cdots < t_n$, the bidiagonal factorization (7) of the corresponding STP collocation matrix can be described by

$$m_{i,j} = \frac{\prod_{k=1}^{j-1}(t_i - t_{i-k})}{\prod_{k=2}^{j}(t_{i-1} - t_{i-k})}, \quad \widehat{m}_{i,j} = t_j, \quad 1 \le j < i \le n+1,$$

$$p_{i,i} = \prod_{k=1}^{i-1}(t_i - t_k), \quad 1 \le i \le n+1 \tag{21}$$

(see [10] or Theorem 3 of [14]). Consequently, the basis $(v_0, \dots, v_n)$ defined in (14) is also STP on $(1, \infty)$. Furthermore, given $1 < x_1 < \cdots < x_{n+1}$, by considering $t_i := (x_i - 1)/2$, $i = 1, \dots, n+1$, and using the bidiagonal factorization (21) for the collocation matrix of the monomial basis at $0 < t_1 < \cdots < t_{n+1}$, it can be easily deduced that the bidiagonal decomposition (7) of the collocation matrix of $(v_0, \dots, v_n)$ at $x_1 < \cdots < x_{n+1}$ is given by:

$$m_{i,j} = \frac{\prod_{k=1}^{j-1}(x_i - x_{i-k})}{\prod_{k=2}^{j}(x_{i-1} - x_{i-k})}, \quad \widehat{m}_{i,j} = (x_j - 1)/2, \quad 1 \le j < i \le n+1,$$

$$p_{i,i} = \frac{1}{2^{i-1}} \prod_{k=1}^{i-1}(x_i - x_k), \quad 1 \le i \le n+1. \tag{22}$$

The following result proves that, for any $\alpha, \beta > -1$, the collocation matrix of the basis (11) of Jacobi polynomials at $1 < x_1 < \cdots < x_{n+1}$,

$$M_J := \left(J_{j-1}^{(\alpha,\beta)}(x_i) \right)_{1\le i,j\le n+1}, \tag{23}$$

is STP.

Theorem 3 *Given $\alpha, \beta > -1$, the corresponding basis of Jacobi polynomials defined in* (11) *is STP on* $(1, \infty)$.

Proof Given a sequence of parameters $1 < x_1 < \cdots < x_{n+1}$, by formula (13), the collocation matrix (23) of the Jacobi polynomial basis satisfies

$$M_J = MA^T, \tag{24}$$

where M is the collocation matrix at $1 < x_1 < \cdots < x_{n+1}$ of the basis $(v_0, \ldots, v_n)$ defined in (14) and A is the lower triangular matrix defined by (12).

Clearly, by Remark 1, M is a STP matrix. On the other hand, by Theorem 2, given $\alpha, \beta > -1$, the lower triangular matrix A defined by (12) is nonsingular and TP. So, A^T is also a nonsingular and TP matrix. As a direct consequence of these facts and taking into account that, by Theorem 3.1 of [1], the product of a STP matrix and a nonsingular TP matrix is a STP matrix, we can conclude that the collocation matrix (23) is STP. □

Remark 2 By Section 4 of [10], we can transpose the bidiagonal decomposition (20) of the lower triangular and TP matrix A to obtain the corresponding bidiagonal decompositon of A^T (see (10)). Clearly, since $BD(A)$ can be computed with HRA, $BD(A^T)$ can be also computed with HRA. Moreover, the collocation matrix of the basis $(v_0, \ldots, v_n)$ defined in (14) at nodes $1 < x_1 < \ldots < x_{n+1}$ is STP and its corresponding bidiagonal decomposition can be obtained with HRA (see (22)). If the bidiagonal decompositions of two nonsingular, TP matrices can be computed with HRA, using Algorithm 5.1 of [11], we can also obtain with HRA the bidiagonal decomposition of the nonsingular and TP product matrix. Consequently, we can derive with HRA the bidiagonal matrices (8) of the bidiagonal factorization (7) of the collocation matrices of Jacobi polynomials and thus, we can also compute with HRA its inverse matrix, its eigenvalues and singular values as well as the solutions of some linear systems.

In Section 6, Algorithm 2 provides the bidiagonal decomposition of the collocation matrix (23) of the basis of Jacobi polynomials. Moreover, Section 6 illustrates accurate results obtained when computing algebraic problems using this algorithm and the algorithms presented in [11] and [12].

4 Total positivity and factorizations of Wronskian matrices of Jacobi polynomials

Given $x \in \mathbb{R}$, let $W(J_0^{(\alpha,\beta)}, \ldots, J_n^{(\alpha,\beta)})(x)$ be the Wronskian matrix at x of the basis (11) of Jacobi polynomials. Using formula (13), it can be checked that

$$W(J_0^{(\alpha,\beta)}, \ldots, J_n^{(\alpha,\beta)})(x) = W(v_0, \ldots, v_n)(x)A^T, \tag{25}$$

where $W(v_0, \ldots, v_n)(x)$ is the Wronskian matrix of the basis $(v_0, \ldots, v_n)$ given in (14) and A is the lower triangular matrix defined by (12).

In Corollary 1 of [15] it was proved that the Wronskian matrix at any positive real value of the monomial basis $(1, x, \ldots, x^n)$ of the space of polynomials $\mathbf{P}^n$ is TP on $(0, \infty)$. It was also shown that this Wronskian matrix and its inverse can be computed with HRA. Now we are going to extend these results to the basis $(\ell_0, \ldots, \ell_n)$ given by

$$\ell_i(x) = (ax + b)^i, \quad x \in \mathbb{R}, \quad i = 0, \ldots, n, \tag{26}$$

where $a, b \in \mathbb{R}$ with $a > 0$. First let us prove the following auxiliary result.

Lemma 1 *The basis* $(\ell_0,\dots,\ell_n)$ *defined in* (26) *satisfies*

$$\frac{1}{a^i i!}\ell_j^{(i)}(x)=\frac{1}{a^{i-1}(i-1)!}\ell_{j-1}^{(i-1)}(x)+\frac{ax+b}{a^i i!}\ell_{j-1}^{(i)}(x),\quad 1\le i,j\le n. \tag{27}$$

Proof We prove the result by induction on i. Since $\ell_j(x)=(ax+b)\ell_{j-1}(x)$, we have

$$\ell_j'(x)=a\ell_{j-1}(x)+(ax+b)\ell_{j-1}'(x),\quad x\in\mathbb{R},$$

and so, formula (27) holds for $i=1$ and $1\le j\le n$. If (27) holds for $i>1$, we can write

$$\frac{1}{a^i i!}\ell_j^{(i+1)}(x)=\frac{a(i+1)}{a^i i!}\ell_{j-1}^{(i)}(x)+\frac{ax+b}{a^i i!}\ell_{j-1}^{(i+1)}(x),$$

and deduce that

$$\frac{1}{a^{i+1}(i+1)!}\ell_j^{(i+1)}(x)=\frac{1}{a^i i!}\ell_{j-1}^{(i)}(x)+\frac{ax+b}{a^{i+1}(i+1)!}\ell_{j-1}^{(i+1)}(x). \qquad \square$$

Now, for a given $x\in\mathbb{R}$, $k,n\in\mathbb{N}$ with $k\le n$, let $U_{k,n}=(u_{i,j}^{(k)})_{1\le i,j\le n+1}$ be the upper triangular, bidiagonal matrix with unit diagonal entries, such that

$$u_{i,i+1}^{(k)}:=0,\quad i=1,\dots,k-1,\quad u_{i,i+1}^{(k)}:=ax+b,\quad i=k,\dots,n. \tag{28}$$

The following result shows that the product matrix $U_{1,n}\cdots U_{n,n}$ coincides, up to a positive scaling, with the Wronskian matrix of $(\ell_0,\ell_1,\dots,\ell_n)$ at x.

Proposition 1 *For a given* $x\in\mathbb{R}$ *and* $n\in\mathbb{N}$, *let*

$$U_n:=U_{1,n}\cdots U_{n,n},$$

where $U_{k,n}$, $k=1,\dots,n$, *are the upper triangular, bidiagonal matrices with unit diagonal entries satisfying* (28). *Then* $U_n=(u_{i,j})_{1\le i,j\le n+1}$ *is an upper triangular matrix and*

$$u_{i,j}=\frac{1}{a^{i-1}(i-1)!}\ell_{j-1}^{(i-1)}(x),\quad 1\le i,j\le n+1. \tag{29}$$

Proof First, let us observe that U_n is the product of upper triangular, bidiagonal matrices and so, it is an upper triangular matrix. Now, we prove (29) by induction on n. For $n=1$,

$$U_1=U_{1,1}=\begin{pmatrix}1 & ax+b\\ 0 & 1\end{pmatrix}$$

and (29) clearly holds. Let us observe that

$$U_{n+1}:=U_{1,n+1}\cdots U_{n+1,n+1}=U_{1,n+1}\tilde U_{n+1},$$

where $\tilde U_{n+1}:=U_{2,n+1}\cdots U_{n+1,n+1}$ satisfies $\tilde U_{n+1}=(\tilde u_{i,j})_{1\le i,j\le n+2}$ with $\tilde u_{i,1}=\tilde u_{1,i}=\delta_{1,i}$, that is, $\delta_{1,1}=1$ and $\delta_{1,i}=0$ for $i=2,\dots,n+2$, and $\tilde U_{n+1}[2,\dots,n+2|2,\dots,n+2]=U_{1,n}\cdots U_{n,n}$. Let us now suppose that (29) holds for $n\ge 1$. Then we have that

$$\tilde u_{i,j}=\frac{1}{a^{i-2}(i-2)!}\ell_{j-2}^{(i-2)}(x),\quad 2\le i,j\le n+2.$$

Taking into account that $U_{n+1} = U_{1,n+1}\tilde{U}_{n+1}$ and using Lemma 1, we deduce that $U_{n+1} = (u_{i,j})_{1\le i,j\le n+2}$ satisfies

$$\begin{aligned} u_{i,j} &= \tilde{u}_{i,j} + (ax+b)\tilde{u}_{i+1,j} = \frac{1}{a^{i-2}(i-2)!}\ell_{j-2}^{(i-2)}(x) + \frac{ax+b}{a^{i-1}(i-1)!}\ell_{j-2}^{(i-1)}(x) \\ &= \frac{1}{a^{i-1}(i-1)!}\ell_{j-1}^{(i-1)}(x), \quad 1 \le i,j \le n+2. \end{aligned} \qquad \square$$

As a direct consequence of the previous result, we can provide the bidiagonal factorization (7) of the Wronskian matrix of $(\ell_0,\dots,\ell_n)$.

Proposition 2 *Let $n \in \mathbb{N}$ and $(\ell_0,\dots,\ell_n)$ be the basis given in* (26)*. Then, for any $x > -b/a$, the Wronskian matrix $W(\ell_0,\dots,\ell_n)(x)$ is TP and*

$$W(\ell_0,\dots,\ell_n)(x) = \begin{pmatrix} 0! & & & \\ & a^1\, 1! & & \\ & & \ddots & \\ & & & a^n\, n! \end{pmatrix} U_{1,n}\cdots U_{n,n}, \tag{30}$$

where $U_{k,n}$, $k = 1,\dots,n$, are the upper triangular, bidiagonal matrices with unit diagonal entries satisfying (28)*.*

Let us observe that the bidiagonal factorization (7) of $W(\ell_0,\dots,\ell_n)(x)$ is given by (30). Clearly, this factorization can be computed with HRA for any $x > -b/a$ and, consequently, using (9), its inverse matrix can also be computed with HRA as stated in the following result.

Proposition 3 *Let W be the Wronskian matrix at $x > -b/a$ of the basis $(\ell_0,\dots,\ell_n)$ given in* (26)*. Then W^{-1} can be computed with HRA.*

Now, using Proposition 2, we can immediately deduce the following factorization of the Wronskian matrix at $x \in \mathbb{R}$ of the basis $(v_0,\dots,v_n)$ in (14),

$$W(v_0,\dots,v_n)(x) := \begin{pmatrix} \frac{1}{2^0}0! & & & \\ & \frac{1}{2^1}1! & & \\ & & \ddots & \\ & & & \frac{1}{2^n}n! \end{pmatrix} U_{1,n}\cdots U_{n,n}, \tag{31}$$

where $U_{k,n} = (u_{i,j}^{(k)})_{1\le i,j\le n+1}$, $k = 1,\dots,n$, is the upper triangular, bidiagonal matrix with unit diagonal entries satisfying

$$u_{i,i+1}^{(k)} := 0, \quad i = 1,\dots,k-1, \quad u_{i,i+1}^{(k)} := (x-1)/2, \quad i = k,\dots,n. \tag{32}$$

Moreover, if $x > 1$, $W(v_0,\dots,v_n)(x)$ is a nonsingular and TP matrix. Then, taking into account (25), the fact that A^T is a nonsingular and TP matrix (see Theorem 2) and that the product of nonsingular TP matrices is a nonsingular and TP matrix (Theorem 3.1 of [1]), we deduce the following result on the total positivity of the Wronskian matrices of Jacobi polynomials.

Theorem 4 *Let $n \in N$ and $(J_0^{(\alpha,\beta)},\dots,J_n^{(\alpha,\beta)})$ be the Jacobi polynomial basis given in* (11). *For any $\alpha,\beta > -1$, the Wronskian matrix $W(J_0^{(\alpha,\beta)},\dots,J_n^{(\alpha,\beta)})(x)$ at $x > 1$ is nonsingular and TP.*

Remark 3 Taking into account (10), we can obtain the bidiagonal decomposition (20) of the matrix A^T in (25). Clearly, since $BD(A)$ can be computed with HRA, $BD(A^T)$ can be also computed with HRA. On the other hand, the Wronskian matrix of the basis $(v_0,\dots,v_n)$ defined in (14) is nonsingular and TP at any $x > 1$. Moreover, its corresponding bidiagonal decomposition (22) can be obtained with HRA. By Algorithm 5.1 of [11], if the bidiagonal decompositions of two nonsingular and TP matrices can be computed with HRA, then the bidiagonal decomposition of the product matrix can be also obtained with HRA. Consequently, the Wronskian matrix of the basis (11) of Jacobi polynomials can be computed with HRA and thus, we can compute with HRA its inverse matrix, its eigenvalues and singular values and the solutions of some linear systems.

In Section 6, Algorithm 3 provides the bidiagonal decomposition (7) of the Wronskian matrix (25) of the basis of Jacobi polynomials. Section 6 shows accurate results obtained when computing the mentioned algebraic problems using this algorithm and the algorithms presented in [11] and [12].

5 Collocation and Wronskian matrices of well known orthogonal bases

In this section we are going to see that the results on properties and factorizations of collocation and Wronskian matrices of Jacobi polynomials obtained in the previous sections can be used to derive properties of collocation and Wronskian matrices of other well known orthogonal bases.

The following auxiliary results can be easily checked and will be useful to derive the bidiagonal decomposition of matrices obtained by scaling with a diagonal matrix a nonsingular and TP matrix.

Lemma 2 *Let F_i and G_i, $i = 1,\dots,n$, be the lower and upper, respectively, triangular bidiagonal matrices described in* (8) *and $\Delta = \operatorname{diag}(d_1,d_2,\dots,d_{n+1})$ a nonsingular diagonal matrix. Then*

$$\Delta F_i = \widehat{F}_i\Delta \quad \text{and} \quad G_i\Delta = \Delta\widehat{G}_i, \quad i = 1,\dots,n, \tag{33}$$

where

$$\widehat{F}_i = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & r_{i+1,1} & 1 & & & \\ & & & & r_{i+2,2} & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & r_{n+1,n+1-i} & 1 \end{pmatrix}, \quad \widehat{G}_i^T = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & \widetilde{r}_{i+1,1} & 1 & & & \\ & & & & \widetilde{r}_{i+2,2} & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \widetilde{r}_{n+1,n+1-i} & 1 \end{pmatrix}, \tag{34}$$

with

$$r_{i,j} = \frac{d_i}{d_{i-1}} m_{i,j}, \quad \widetilde{r}_{i,j} = \frac{d_i}{d_{i-1}} \widetilde{m}_{i,j}, \quad 1 \le j < i \le n+1.$$

As a consequence, we have the following result.

Lemma 3 *Let $A = F_nF_{n-1}\cdots F_1DG_1\cdots G_{n-1}G_n$ be the bidiagonal decomposition* (7) *of a nonsingular and TP matrix A. Then, given a nonsingular matrix $\Delta = \operatorname{diag}(d_1,d_2,\ldots,d_{n+1})$, the bidiagonal decomposition* (7) *of ΔA and $A\Delta$ are given by*

$$\Delta A = \widehat{F}_n\widehat{F}_{n-1}\cdots\widehat{F}_1\widehat{D}G_1\cdots G_{n-1}G_n, \tag{35}$$

$$A\Delta = F_nF_{n-1}\cdots F_1\widehat{D}\widehat{G}_1\cdots\widehat{G}_{n-1}\widehat{G}_n, \tag{36}$$

where $\widehat{F}_i$ and $\widehat{G}_i$, $i = 1,\ldots,n$, are the lower and upper, respectively, triangular matrices described in (34) *and $\widehat{D} = \Delta D = D\Delta$.*

Let us start by considering the basis $(L_0,\ldots,L_n)$ of Legendre polynomials defined by

$$L_i(x) := J_i^{(0,0)}(x), \quad i = 0,\ldots,n, \tag{37}$$

where $(J_0^{(0,0)},\ldots,J_n^{(0,0)})$ is the basis of Jacobi polynomials given in (11) with $\alpha = \beta = 0$. From Theorem 3, Remark 2, Theorem 4 and Remark 3, we can deduce the following result.

Theorem 5 *The basis $(L_0,\ldots,L_n)$ of Legendre polynomials, defined by* (37)*, is STP on $(1,\infty)$. Given $x_1 < \cdots < x_{n+1}$, with $x_1 > 1$, the bidiagonal decomposition* (7) *of the corresponding collocation matrix can be obtained with HRA. Moreover, for any $x > 1$, the Wronskian matrix $W(L_0,\ldots,L_n)(x)$ is nonsingular and TP and its bidiagonal decomposition* (7) *can be obtained with HRA.*

Given $\lambda \in \mathbb{R}$, the basis of Gegenbauer polynomials of $\mathbf{P}^n$ is $(G_0,\ldots,G_n)$ with

$$G_i^{\lambda}(x) := \frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)}\frac{\Gamma(i+2\lambda)}{\Gamma(i+\lambda+1/2)}J_i^{(\lambda-1/2,\lambda-1/2)}(x), \quad i = 0,\ldots,n, \tag{38}$$

where $(J_0^{(\lambda-1/2,\lambda-1/2)},\ldots,J_n^{(\lambda-1/2,\lambda-1/2)})$ is the basis of Jacobi polynomials given in (11) with $\alpha = \beta = \lambda - 1/2$. By Theorem 3 and Remark 2, Lemma 3 and Remark 3, we can deduce the following result.

Theorem 6 *For any $\lambda > -1/2$, the basis $(G_0,\ldots,G_n)$ of Gegenbauer polynomials, defined by* (38)*, is STP on $(1,\infty)$. Given $x_1 < \cdots < x_{n+1}$, with $x_1 > 1$, the bidiagonal decomposition* (7) *of the corresponding collocation matrix can be obtained with HRA. Moreover, for any $x > 1$, the Wronskian matrix $W(G_0,\ldots,G_n)(x)$ is nonsingular and TP and its bidiagonal decomposition* (7) *can be obtained with HRA.*

The basis $(T_0,\ldots,T_n)$ of Chebyshev polynomials of the first kind is defined by

$$T_i(x) := \frac{J_i^{(-1/2,-1/2)}(x)}{J_i^{(-1/2,-1/2)}(1)}, \quad i = 0,\ldots,n, \tag{39}$$

where $(J_0^{(-1/2,-1/2)},\ldots,J_n^{(-1/2,-1/2)})$ is the basis of Jacobi polynomials given in (11) with $\alpha = \beta = -1/2$. Using again Theorem 3, Remark 2, Lemma 3 and Remark 3, we can deduce the following result.

Theorem 7 *The basis $(T_0,\ldots,T_n)$ of Chebyshev polynomials of the first kind, defined by* (39)*, is STP on $(1,\infty)$. Given $x_1 < \cdots < x_{n+1}$, with $x_1 > 1$, the bidiagonal decomposition* (7) *of the corresponding collocation matrix can be obtained with HRA. Moreover, for any $x > 1$, the Wronskian matrix $W(T_0,\ldots,T_n)(x)$ is nonsingular and TP and its bidiagonal decomposition* (7) *can be obtained with HRA.*

The basis $(U_0,\dots,U_n)$ of second kind Chebyshev polynomials is defined by

$$U_i(x) := (i+1)\frac{J_i^{(-1/2,-1/2)}(x)}{J_i^{(1/2,1/2)}(1)}, \quad i=0,\dots,n, \tag{40}$$

where $(J_0^{(1/2,1/2)},\dots,J_n^{(1/2,1/2)})$ is the basis of Jacobi polynomials given in (11) with $\alpha = \beta = 1/2$. Using again Theorem 3, Remark 2, Lemma 3 and Remark 3, we can deduce the following result.

Theorem 8 *The basis $(U_0,\dots,U_n)$ of Chebyshev polynomials of second kind, defined by* (40)*, is STP on $(1,\infty)$. Given $x_1 < \cdots < x_{n+1}$, with $x_1 > 1$, the bidiagonal decomposition* (7) *of the corresponding collocation matrix can be obtained with HRA. Moreover, for any $x > 1$, the Wronskian matrix $W(U_0,\dots,U_n)(x)$ is nonsingular and TP and its bidiagonal decomposition* (7) *can be obtained with HRA.*

In [19], induced by Jacobi polynomials, a new orthogonal system of rational functions was introduced. For given $\alpha,\beta \in \mathbb{R}$, the system $(R_0^{(\alpha,\beta)},\dots,R_n^{(\alpha,\beta)})$ of rational Jacobi functions is defined by

$$R_i^{(\alpha,\beta)}(x) := J_i^{(\alpha,\beta)}\left(\frac{x-1}{x+1}\right), \quad i=0,\dots,n, \tag{41}$$

where $(J_0^{(\alpha,\beta)},\dots,J_n^{(\alpha,\beta)})$ is the basis (11) of Jacobi polynomials. Using again Theorem 3, Remark 2, Lemma 3 and Remark 3, we can deduce the following result.

Theorem 9 *For any $\alpha,\beta > -1$, the basis $(R_0^{(\alpha,\beta)},\dots,R_n^{(\alpha,\beta)})$ of rational Jacobi functions given in* (41) *is STP on $(-\infty,-1)$. Given $x_1 < \cdots < x_{n+1}$, with $x_{n+1} < -1$, the bidiagonal decomposition* (7) *of the corresponding collocation matrix can be obtained with HRA.*

Similar results can be deduced by considering the rational counterparts of the basis of Legendre, Gegenbauer and the first and second kind Chebyshev polynomials.

Section 6 will show accurate results obtained when computing the eigenvalues, singular values, or the solutions of some linear systems associated with the collocation and Wronskian matrices of all the mentioned orthogonal bases, using their corresponding bidiagonal decompositions and the algorithms presented in [11] and [12].

6 Numerical experiments

Given a nonsingular and TP matrix whose bidiagonal factorization (7) can be computed with HRA, the functions `TNEigenvalues`, `TNSingularValues`, `TNInverseExpand` and `TNSolve`, available in the library TNTool of [12], can be used to compute with HRA its eigenvalues, its singular values, its inverse matrix and the solution of some linear systems, respectively. The function `TNProduct` is also avaliable in the mentioned library. If the bidiagonal decomposition (7) of two nonsingular and TP matrices A and B can be computed with HRA, `TNProduct` computes with HRA the bidiagonal decomposition (7) of AB.

Using Theorem 2, we have implemented the Matlab function `TNBDA` (see Algorithm 1) providing the bidiagonal decomposition (20) of the lower triangular matrix A given in (12), for given $\alpha,\beta > -1$ and $n \in \mathbb{N}$. Using `TNBDA` and taking into account Remark 1 and Theorem 3, we have also implemented the Matlab function `TNBDJ` (see Algorithm 2) for the computation of the bidiagonal decomposition (7) of the collocation matrix at $x = (x_i)_{i=1}^{n+1}$, with $1 < x_1 < \cdots < x_{n+1}$, of the Jacobi polynomial basis corresponding to given $\alpha, \beta > -1$. Furthermore, using TNBDA and taking into account Proposition 2, Theorem 4 and Remark 3, we have implemented the Matlab function TNBDWJ (see Algorithm 3), which provides the bidiagonal decomposition (7) of the Wronskian matrix at $x > 1$ of the Jacobi polynomial basis corresponding to given $\alpha, \beta > -1$.

Moreover, using TNBDJ (TNBDWJ, respectively) and taking into account Lemma 3, we have also implemented the Matlab functions TNBDG, TNBDT1 and TNBDT2 (TNBDWG, TNBDWT1, TNBDWT2, respectively) for the computation of the bidiagonal decomposition (7) of the collocation matrix at $x_1, \ldots, x_{n+1}$ (of the Wronskian matrix at $x > 1$, respectively) of the bases (38) of Gegenbauer polynomials at a given $\lambda > -1/2$, the basis (39) of Chebyshev polynomials of the first kind and the basis (40) of Chebyshev polynomials of the second kind, respectively.

Algorithm 1: Computation of the bidiagonal decomposition (20) of the matrix A in (12)

function $BDA = \mathtt{TNBDA}(\alpha, \beta, n+1)$
BDA=zeros(n+1,n+1)
for i :=2 to n+1
  $aux := \frac{\alpha+i-1}{i-1}$
  $BDA(i,1) := aux$
  for j :=2 to i-1
    $aux := aux \cdot \frac{\alpha+\beta+2i-j}{\alpha+\beta+2i-j-2}$
    $BDA(i,j) := aux$
  end j
end i
$BDA(1,1) = 1$
for i :=2 to n+1
  $aux := 1$
  for k :=1 to i-1
    $aux := aux \cdot \frac{\alpha+\beta+2i-k-1}{i-k}$
    $BDA(i,i) := aux$
  end k
end i

In order to check the accuracy of the solution of the above mentioned algebraic problems, obtained using the functions in [12] with the bidiagonal factorization (7), we have considered collocation matrices $\mathbf{M_n}$ at $x = (x_i)_{i=1}^{n+1}$ satisfying $1 < x_1 < \ldots < x_{n+1}$ and Wronskian matrices $\mathbf{W_n}$ at $x = 2$ or $x = 50$, for $(n+1)$-dimensional Jacobi, Legendre, Gegenbauer and Chebyshev of the first and second kind polynomial bases. Additionally, we have also considered collocation matrices at sequences $x = ((x_i-1)/(x_i+1))_{i=1}^{n+1}$ such that $x_1 < x_2 < \cdots < x_{n+1} < -1$ of their rational counterpart bases. For the considered collocation matrices, we have obtained the bidiagonal decomposition (7) by using TNBDJ, TNBDG, TNBDT1 and TNBDT2. For the considered Wronskian matrices, the factorization (7) has been obtained with the Matlab functions TNBDWJ, TNBDWG, TNBDWT1 and TNBDWT2. The software with the numerical experiments will be provided by the authors upon request.

Tables 1, 2, 3 and 4 illustrate the 2-norm condition number of the mentioned collocation and Wronskian matrices that have been obtained with the Mathematica command `Norm[A,2]· Norm[Inverse[A],2]`. Observe that the condition number of the matrices considerably increases with their dimension. Due to this ill conditioning, traditional methods do not achieve accurate solutions when solving the mentioned algebraic problems. The following numerical results confirm this fact and illustrate the high accuracy obtained when using the functions in [12] with the bidiagonal factorizations (7) obtained in this paper.

Algorithm 2: Computation of the bidiagonal decomposition of the collocation matrix of Jacobi polynomials

function $BDJ = \texttt{TNBDJ}(\alpha,\beta,x,n+1)$
$BDA = \texttt{TNBDA}(\alpha,\beta,n+1)$
$BDB = \texttt{zeros}(n+1,n+1)$
for i :=2 to n+1
$BDB(i,1) := 1$
$aux := 1$
  for j :=2 to i-1
    $aux := aux \cdot \dfrac{x_i - x_{i-j+1}}{x_{i-1} - x_{i-j}}$
    $BDB(i,j) := aux$
  end j
end i
for i=1 to n
$aux := (x_i - 1)/2$
  for j :=i+1 to n+1
    $BDB(i,j) := aux$
  end j
end i
$aux := 1$
$BDB(1,1) = 1$
for i :=2 to n+1
$aux := aux/2$
  for k :=1 to i-1
    $aux := aux \cdot (x_i - x_k)$
    $BDB(i,i) := aux$
  end k
end i
$BDJ = \texttt{TNProduct}(BDB,(BDA)^T)$

Algorithm 3: Computation of the bidiagonal decomposition of the Wronskian matrix of Jacobi polynomials

function $BDWJ = \texttt{TNBDWJ}(\alpha,\beta,x,n+1)$
$BDA = \texttt{TNBDA}(\alpha,\beta,n+1)$
$BDWB = \texttt{zeros}(n+1,n+1)$
for i=1 to n+1
  for j :=i+1 to n+1
    $BDWB(i,j) := (x-1)/2$
  end j
end i
$BDWB(1,1) := 1$
for i :=2 to n+1
  $BDWB(i,i) := (i-1) \cdot BDWB(i-1,i-1)/2$
end i
$BDWJ = \texttt{TNProduct}(BDWB,(BDA)^T)$

6.1 Eigenvalues and singular values

Let us recall that all considered matrices are STP and so, all their eigenvalues are positive and distinct (see Theorem 6.2 of [1]). On the other hand, the eigenvalues of the mentioned Wronskian matrices are integers and so, they can be exactly determined.

We have compared the eigenvalues and singular values obtained when using the Matlab commands `eig` and `svd`, respectively, and those computed using the bidiagonal decompositions (7) in this paper and the Matlab functions `TNEigenValues` and `TNSingularValues`, respectively. In order to determine the accuracy of the approximations, we have also calculated the eigenvalues and singular values of the matrices by using Mathematica with a precision of 100 digits and computed the relative errors corresponding to the approximations, considering the eigenvalues and singular values provided by Mathematica as exact.

Table 1 From left to right, condition number of collocation matrices at $x_i = 1 + i/(n+1)$, $i = 1,\dots,n+1$, of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind polynomial bases.

	Jacobi	**Legendre**	**Gegenbauer**	**Chebyshev 1st kind**	**Chebyshev 2nd kind**
n+1	$\kappa_2(\mathbf{M_n})$	$\kappa_2(\mathbf{M_n})$	$\kappa_2(\mathbf{M_n})$	$\kappa_2(\mathbf{M_n})$	$\kappa_2(\mathbf{M_n})$
10	4.2×10^{13}	5.3×10^{13}	1.2×10^{14}	1.4×10^{14}	1.8×10^{14}
15	5.1×10^{21}	9.2×10^{21}	2.1×10^{22}	3.2×10^{22}	2.1×10^{22}
20	8.2×10^{29}	1.9×10^{30}	4.4×10^{30}	7.7×10^{30}	4.4×10^{30}
25	1.5×10^{38}	4.4×10^{38}	1.0×10^{39}	2.0×10^{39}	1.0×10^{39}

Table 2 From left to right, condition number of collocation matrices at $x = ((x_i-1)/(x_i+1))_{i=1}^{n+1}$ such that $x_i = -3 + i/(n+1)$ of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind rational bases.

	Rational Jacobi	**Rational Legendre**	**Rational Gegenbauer**	**Rational Chebyshev 1st kind**	**Rational Chebyshev 2nd kind**
n+1	$\kappa_2(\mathbf{M_n})$	$\kappa_2(\mathbf{M_n})$	$\kappa_2(\mathbf{M_n})$	$\kappa_2(\mathbf{M_n})$	$\kappa_2(\mathbf{M_n})$
10	1.7×10^{17}	1.5×10^{17}	3.7×10^{17}	4.1×10^{17}	3.7×10^{17}
15	1.7×10^{27}	2.2×10^{27}	5.5×10^{27}	7.6×10^{27}	5.5×10^{27}
20	2.4×10^{37}	3.9×10^{37}	1.0×10^{38}	1.6×10^{38}	1.0×10^{38}
25	4.0×10^{47}	8.2×10^{47}	2.1×10^{48}	3.8×10^{48}	2.1×10^{48}

Table 3 From left to right, condition number of Wronskian matrices at $x_0 = 2$ of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind polynomial bases.

	Jacobi	**Legendre**	**Gegenbauer**	**Chebyshev 1st kind**	**Chebyshev 2nd kind**
n+1	$\kappa_2(\mathbf{W_n})$	$\kappa_2(\mathbf{W_n})$	$\kappa_2(\mathbf{W_n})$	$\kappa_2(\mathbf{W_n})$	$\kappa_2(\mathbf{W_n})$
10	2.1×10^{9}	5.8×10^{8}	2.4×10^{9}	1.6×10^{9}	2.4×10^{9}
15	1.4×10^{16}	3.6×10^{15}	1.9×10^{16}	1.2×10^{16}	1.9×10^{16}
20	5.9×10^{23}	1.4×10^{23}	8.1×10^{23}	5.6×10^{23}	8.4×10^{23}
25	$8{,}8\times10^{31}$	2.0×10^{31}	1.4×10^{32}	9.2×10^{31}	1.4×10^{32}

Table 4 From left to right, condition number of Wronskian matrices at $x_0 = 50$ of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind polynomial bases.

	Jacobi	**Legendre**	**Gegenbauer**	**Chebyshev 1st kind**	**Chebyshev 2nd kind**
n+1	$\kappa_2(\mathbf{W_n})$	$\kappa_2(\mathbf{W_n})$	$\kappa_2(\mathbf{W_n})$	$\kappa_2(\mathbf{W_n})$	$\kappa_2(\mathbf{W_n})$
10	5.8×10^{27}	1.0×10^{27}	5.6×10^{27}	2.8×10^{27}	5.6×10^{27}
15	7.1×10^{40}	1.2×10^{40}	7.7×10^{40}	4.0×10^{40}	7.7×10^{40}
20	1.5×10^{53}	2.6×10^{52}	1.9×10^{53}	1.0×10^{53}	1.9×10^{53}
25	9.4×10^{64}	1.6×10^{64}	1.3×10^{65}	7.2×10^{64}	1.3×10^{65}

We have computed the relative error of the approximations a of the exact eigenvalue and singular value $\tilde{a}$ by means of the formula $e = |a-\tilde{a}|/|a|$.

Tables 5, 6, 7, 8 and 9 show the relative errors of the approximations to the lowest eigenvalue and the lowest singular value obtained with both methods. Observe that the eigenvalues and singular values obtained using the factorization (7) are very accurate for all considered n, whereas the approximations of the eigenvalues and singular values obtained with the Matlab commands `eig` and `svd` are not very accurate when n increases.

6.2 Inverse matrix

We have also used the Matlab function `TNInverseExpand` (see Section 4 of [17]) with the factorization (7) proposed in this paper in order to compute the inverse of the con-

Table 5 From left to right, relative errors when computing the lowest eigenvalue of collocation matrices at $x_i = 1+i/(n+1)$, $i = 1,\ldots,n+1$, of Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind polynomial bases.

	Jacobi		**Legendre**		**Gegenbauer**		**Chebyshev 1st kind**		**Chebyshev 2nd kind**	
n+1	**eig**	`TNEV`	**eig**	`TNEV`	**eig**	`TNEV`	**eig**	`TNEV`	**eig**	`TNEV`
10	1.5×10^{-16}	5.4×10^{-16}	3.3×10^{-17}	6.9×10^{-16}	1.2×10^{-15}	4.9×10^{-16}	1.3×10^{-6}	1.0×10^{-15}	9.6×10^{-7}	3.0×10^{-16}
15	1.1×10^{-12}	2.2×10^{-16}	3.0×10^{-13}	3.5×10^{-16}	5.4×10^{-13}	2.3×10^{-15}	3.9	3.5×10^{-15}	1.3×10^{-1}	2.7×10^{-15}
20	1.2×10^{-11}	2.4×10^{-15}	7.7×10^{-12}	1.1×10^{-16}	4.3×10^{-12}	1.2×10^{-15}	5.9×10^{8}	5.3×10^{-15}	1.3×10^{5}	4.1×10^{-15}
25	1.3×10^{-10}	8.4×10^{-16}	1.9×10^{-10}	2.5×10^{-16}	1.8×10^{-10}	1.7×10^{-15}	1.5×10^{9}	9.9×10^{-15}	1.9×10^{9}	1.7×10^{-15}

Table 6 From left to right, relative errors when computing the lowest singular value of collocation matrices at $x_i = 1+i/(n+1)$, $i = 1,\ldots,n+1$, of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind polynomial bases.

	Jacobi		**Legendre**		**Gegenbauer**		**Chebyshev 1st kind**		**Chebyshev 2nd kind**	
n+1	**svd**	`TNSV`	**svd**	`TNSV`	**svd**	`TNSV`	**svd**	`TNSV`	**svd**	`TNSV`
10	2.1×10^{-16}	2.5×10^{-16}	5.9×10^{-17}	5.9×10^{-17}	1.6×10^{-15}	1.2×10^{-15}	1.3×10^{-6}	1.4×10^{-16}	9.5×10^{-7}	1.2×10^{-15}
15	1.9×10^{-12}	4.7×10^{-16}	3.7×10^{-13}	1.5×10^{-15}	8.2×10^{-13}	1.1×10^{-15}	4.6	3.8×10^{-15}	1.4×10^{-1}	3.4×10^{-15}
20	1.6×10^{-11}	2.5×10^{-15}	4.3×10^{-11}	1.4×10^{-15}	5.4×10^{-11}	7.8×10^{-16}	1.5×10^{5}	4.7×10^{-15}	1.5×10^{5}	1.2×10^{-15}
25	1.2×10^{-9}	7.4×10^{-16}	6.3×10^{-10}	6.9×10^{-16}	9.1×10^{-10}	1.8×10^{-15}	6.6×10^{9}	1.2×10^{-14}	8.5×10^{9}	1.5×10^{-15}

Table 7 From left to right, relative errors when computing the lowest eigenvalue of the collocation matrices at $x = ((x_i-1)/(x_i+1))_{i=1}^{n+1}$ such that $x_i = -3+i/(n+1)$ of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind rational bases.

	Rational Jacobi		**Rational Legendre**		**Rational Gegenbauer**		**Rational Chebyshev 1st kind**		**Rational Chebyshev 2nd kind**	
n+1	**eig**	`TNEV`	**eig**	`TNEV`	**eig**	`TNEV`	**eig**	`TNEV`	**eig**	`TNEV`
10	5.8×10^{-17}	8.8×10^{-17}	4.9×10^{-18}	7.3×10^{-16}	8.6×10^{-17}	8.6×10^{-17}	9.6×10^{-17}	4.9×10^{-16}	8.7×10^{-17}	2.6×10^{-16}
15	8.0×10^{-14}	4.4×10^{-16}	1.3×10^{-14}	4.2×10^{-16}	3.1×10^{-14}	1.7×10^{-15}	3.5×10^{-15}	4.8×10^{-15}	1.9×10^{-14}	2.4×10^{-15}
20	1.8×10^{-12}	8.2×10^{-16}	1.7×10^{-12}	1.6×10^{-16}	2.1×10^{-12}	1.1×10^{-16}	1.2×10^{-12}	2.1×10^{-15}	2.0×10^{-12}	1.3×10^{-15}
25	1.6×10^{-11}	8.3×10^{-16}	7.4×10^{-11}	2.0×10^{-16}	3.4×10^{-12}	1.8×10^{-16}	7.1×10^{-12}	6.8×10^{-15}	4.7×10^{-12}	4.1×10^{-15}

Table 8 From left to right, relative errors when computing the lowest singular value of the collocation matrices at $x = ((x_i-1)/(x_i+1))_{i=1}^{n+1}$ such that $x_i = -3+i/(n+1)$ of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind rational bases.

	Rational Jacobi		**Rational Legendre**		**Rational Gegenbauer**		**Rational Chebyshev 1st kind**		**Rational Chebyshev 2nd kind**	
n+1	**svd**	`TNSV`	**svd**	`TNSV`	**svd**	`TNSV`	**svd**	`TNSV`	**svd**	`TNSV`
10	2.8×10^{-17}	5.7×10^{-16}	9.4×10^{-17}	5.4×10^{-16}	1.8×10^{-16}	1.9×10^{-17}	2.8×10^{-17}	8.1×10^{-16}	2.0×10^{-17}	6.1×10^{-16}
15	7.0×10^{-14}	1.1×10^{-15}	9.9×10^{-15}	9.7×10^{-18}	3.2×10^{-14}	2.5×10^{-15}	4.4×10^{-15}	4.9×10^{-15}	1.4×10^{-14}	3.8×10^{-15}
20	3.9×10^{-12}	9.4×10^{-17}	2.2×10^{-12}	1.8×10^{-15}	3.3×10^{-12}	1.3×10^{-16}	1.5×10^{-12}	1.5×10^{-15}	3.0×10^{-12}	7.0×10^{-16}
25	3.9×10^{-12}	2.4×10^{-15}	2.3×10^{-11}	1.6×10^{-15}	2.5×10^{-11}	3.9×10^{-15}	2.7×10^{-11}	8.9×10^{-15}	2.2×10^{-11}	3.9×10^{-15}

sidered collocation and Wronskian matrices. We have also computed their approximations with the Matlab command `inv`. In order to determine the accuracy of the approximations, we have calculated the inverse of these matrices by using Mathematica with a precision of 100 digits and computed the relative errors corresponding to the approximations, considering the inverse matrix provided by Mathematica as exact. We have computed the relative error of each approximation $\tilde{A}^{-1}$ of the exact inverse matrix A^{-1} by means of the formula $e = \|A^{-1}-\tilde{A}^{-1}\|_2/\|A^{-1}\|_2$.

Tables 10, 11 and 12 show the relative errors of the approximations to the inverse of the collocation and Wronskian matrices obtained with both methods. For all considered cases, the approximation of the inverse matrix obtained by means of `TNInverseExpand`

Table 9 From left to right, relative errors when computing the lowest singular value of Wronskian matrices at $x_0 = 2$ of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind polynomial bases.

	Jacobi		**Legendre**		**Gegenbauer**		**Chebyshev 1st kind**		**Chebyshev 2nd kind**	
n+1	**svd**	`TNSV`	**svd**	`TNSV`	**svd**	`TNSV`	**svd**	`TNSV`	**svd**	`TNSV`
10	3.5×10^{-9}	6.1×10^{-17}	1.5×10^{-9}	7.9×10^{-17}	4.2×10^{-9}	6.6×10^{-17}	6.3×10^{-9}	9.5×10^{-16}	4.5×10^{-9}	6.6×10^{-17}
15	2.0×10^{-2}	1.7×10^{-16}	6.7×10^{-3}	2.3×10^{-17}	2.6×10^{-2}	2.0×10^{-16}	2.5×10^{-2}	8.9×10^{-16}	2.7×10^{-2}	2.4×10^{-16}
20	1.6	3.2×10^{-16}	1.3	3.6×10^{-16}	2.2	6.0×10^{-17}	3.7	1.6×10^{-16}	7.8×10^{-1}	6.0×10^{-17}
25	2.1	3.8×10^{-16}	4.1	6.3×10^{-16}	2.9	1.6×10^{-16}	4.2	6.0×10^{-16}	2.9	1.4×10^{-15}

and the factorization (7) is very accurate, providing much better results than those obtained by Matlab using the command `inv`.

Table 10 From left to right, relative errors when computing the inverse of collocation matrices at $x_i = 1 + i/(n+1)$, $i = 1,\dots,n+1$, of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind polynomial bases.

	Jacobi		**Legendre**		**Gegenbauer**		**Chebyshev 1st kind**		**Chebyshev 2nd kind**	
n+1	**inv**	`TNInvExp`	**inv**	`TNInvExp`	**inv**	`TNInvExp`	**inv**	`TNInvExp`	**inv**	`TNInvExp`
10	1.4×10^{-16}	3.8×10^{-16}	4.0×10^{-17}	8.2×10^{-16}	1.5×10^{-15}	5.4×10^{-16}	1.3×10^{-6}	4.8×10^{-16}	9.5×10^{-7}	1.6×10^{-15}
15	2.0×10^{-12}	7.5×10^{-16}	3.7×10^{-13}	3.6×10^{-16}	8.3×10^{-13}	1.9×10^{-15}	0.8	3.3×10^{-15}	1.6×10^{-1}	3.2×10^{-15}
20	3.3×10^{-11}	1.7×10^{-15}	4.6×10^{-11}	4.9×10^{-16}	5.8×10^{-11}	1.8×10^{-16}	1.0	4.6×10^{-15}	1.0	2.8×10^{-15}
25	1.4×10^{-9}	4.6×10^{-16}	6.3×10^{-10}	4.5×10^{-16}	9.2×10^{-10}	6.2×10^{-16}	1.0	9.9×10^{-15}	1.0	3.8×10^{-15}

Table 11 From left to right, relative errors when computing the inverse of the collocation matrices at $x = ((x_i-1)/(x_i+1))_{i=1}^{n+1}$ such that $x_i = -3 + i/(n+1)$ of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind polynomial bases.

	Rational Jacobi		**Rational Legendre**		**Rational Gegenbauer**		**Rational Chebyshev 1st kind**		**Rational Chebyshev 2nd kind**	
n+1	**inv**	`TNInvExp`	**inv**	`TNInvExp`	**inv**	`TNInvExp`	**inv**	`TNInvExp`	**inv**	`TNInvExp`
10	4.1×10^{-17}	8.5×10^{-17}	4.1×10^{-17}	1.9×10^{-16}	1.4×10^{-16}	2.9×10^{-16}	5.1×10^{-17}	3.9×10^{-16}	5.7×10^{-17}	3.0×10^{-16}
15	7.1×10^{-14}	2.2×10^{-16}	1.0×10^{-14}	3.0×10^{-16}	3.2×10^{-14}	2.5×10^{-15}	4.4×10^{-15}	3.7×10^{-15}	1.4×10^{-14}	2.6×10^{-15}
20	4.1×10^{-12}	7.0×10^{-16}	2.3×10^{-12}	3.2×10^{-16}	3.3×10^{-12}	5.3×10^{-16}	1.5×10^{-12}	2.9×10^{-15}	3.0×10^{-12}	8.2×10^{-16}
25	1.4×10^{-11}	5.9×10^{-16}	2.5×10^{-11}	1.1×10^{-15}	2.1×10^{-11}	1.0×10^{-15}	2.8×10^{-11}	7.8×10^{-15}	2.5×10^{-11}	2.4×10^{-15}

Table 12 From left to right, relative errors when computing the inverse of Wronskian matrices at $x_0 = 50$ of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind polynomial bases.

	Jacobi		**Legendre**		**Gegenbauer**		**Chebyshev of the first kind**		**Chebyshev of the second kind**	
n+1	**inv**	`TNInvExp`	**inv**	`TNInvExp`	**inv**	`TNInvExp`	**inv**	`TNInvExp`	**inv**	`TNInvExp`
10	1.6×10^{-13}	1.7×10^{-17}	1.4×10^{-13}	1.8×10^{-16}	1.1×10^{-13}	7.5×10^{-17}	1.3×10^{-14}	2.2×10^{-16}	8.0×10^{-14}	6.0×10^{-16}
15	7.8×10^{-11}	4.5×10^{-17}	2.2×10^{-11}	2.0×10^{-16}	2.2×10^{-11}	4.1×10^{-15}	5.9×10^{-13}	4.6×10^{-15}	4.6×10^{-11}	4.4×10^{-15}
20	7.3×10^{-9}	1.4×10^{-16}	8.3×10^{-9}	3.9×10^{-16}	1.1×10^{-8}	1.2×10^{-15}	6.6×10^{-10}	3.3×10^{-15}	6.7×10^{-10}	1.9×10^{-15}
25	2.4×10^{-6}	1.0×10^{-16}	5.3×10^{-6}	5.0×10^{-16}	8.0×10^{-7}	4.6×10^{-15}	1.4×10^{-7}	8.2×10^{-15}	3.7×10^{-7}	4.7×10^{-15}

6.3 Linear systems

We shall illustrate the accuracy of the solutions of linear systems computed by using the bidiagonal factorization (7). We have obtained the solution of the linear systems using Mathematica with a precision of 100 digits and considered this solution exact. Then we have also computed with Matlab two approximations, the first one using the previous functions and the second one using the Matlab command \. We have computed the relative error of every approximation $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_{n+1})$ of the solution c of the linear system by means of the formula $e = \|c - \tilde{c}\|_2 / \|c\|_2$.

Tables 13, 14 and 15 show the relative errors when solving the linear systems $\mathbf{M}_n c_n = \mathbf{d}_n$ and $\mathbf{W}_n c_n = \mathbf{d}_n$ where $\mathbf{d}_n = ((-1)^{i+1} d_i)_{1 \le i \le n+1}$ and d_i, $i = 1, \ldots, n+1$, random integer values. The computed results confirm the accuracy of the proposed method that, clearly, keeps the accuracy when the dimension of the problem increases. In contrast, when n increases the condition number of the considered matrices considerably increases and that explains the bad results obtained with the Matlab command \.

Table 13 From left to right, relative errors when solving $\mathbf{M_n c_n} = \mathbf{d_n}$ with collocation matrices at $x_i = 1 + i/(n+1)$, $i = 1, \ldots, n+1$ of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind polynomial bases.

	Jacobi		**Legendre**		**Gegenbauer**		**Chebyshev 1st kind**		**Chebyshev 2nd kind**	
n+1	$\mathbf{M_n} \backslash \mathbf{d_n}$	`TNsolve`	$\mathbf{M_n} \backslash \mathbf{d_n}$	`TNsolve`	$\mathbf{M_n} \backslash \mathbf{d_n}$	`TNsolve`	$\mathbf{M_n} \backslash \mathbf{d_n}$	`TNsolve`	$\mathbf{M_n} \backslash \mathbf{d_n}$	`TNsolve`
10	1.4×10^{-16}	4.1×10^{-16}	6.6×10^{-17}	9.4×10^{-16}	1.5×10^{-15}	5.4×10^{-16}	1.3×10^{-6}	4.8×10^{-16}	9.5×10^{-7}	6.1×10^{-16}
15	2.0×10^{-12}	6.5×10^{-16}	3.7×10^{-13}	1.4×10^{-16}	8.3×10^{-13}	1.9×10^{-15}	8.2×10^{-1}	3.3×10^{-15}	1.6×10^{-1}	3.0×10^{-15}
20	3.3×10^{-11}	1.2×10^{-15}	4.6×10^{-11}	3.9×10^{-16}	5.8×10^{-11}	1.8×10^{-16}	1.0	4.6×10^{-15}	1.0	2.7×10^{-15}
25	1.4×10^{-9}	5.4×10^{-16}	6.3×10^{-10}	2.3×10^{-16}	9.2×10^{-10}	6.2×10^{-16}	1.0	9.9×10^{-15}	1.0	1.1×10^{-15}

Table 14 From left to right, relative errors when solving $\mathbf{M_n c_n} = \mathbf{d_n}$ with collocation matrices at $x = ((x_i - 1)/(x_i + 1))_{i=1}^{n+1}$ such that $x_i = -3 + i/(n+1)$ of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind rational bases.

	Rational Jacobi		**Rational Legendre**		**Rational Gegenbauer**		**Rational Chebyshev 1st kind**		**Rational Chebyshev 2nd kind**	
n+1	$\mathbf{A_n} \backslash \mathbf{d_n}$	`TNsolve`	$\mathbf{A_n} \backslash \mathbf{d_n}$	`TNsolve`	$\mathbf{A_n} \backslash \mathbf{d_n}$	`TNsolve`	$\mathbf{A_n} \backslash \mathbf{d_n}$	`TNsolve`	$\mathbf{A_n} \backslash \mathbf{d_n}$	`TNsolve`
10	2.9×10^{-17}	1.1×10^{-16}	7.9×10^{-17}	3.8×10^{-16}	1.2×10^{-16}	2.1×10^{-16}	5.5×10^{-17}	8.48×10^{-16}	6.3×10^{-17}	2.1×10^{-16}
15	7.1×10^{-14}	1.9×10^{-16}	1.0×10^{-14}	1.9×10^{-16}	3.2×10^{-14}	2.4×10^{-15}	4.4×10^{-15}	3.6×10^{-15}	1.4×10^{-14}	2.8×10^{-15}
20	4.1×10^{-12}	4.3×10^{-16}	2.3×10^{-12}	3.0×10^{-16}	3.3×10^{-12}	3.7×10^{-16}	1.5×10^{-12}	2.5×10^{-15}	3.4×10^{-12}	5.6×10^{-16}
25	1.4×10^{-11}	7.3×10^{-16}	2.5×10^{-11}	1.2×10^{-15}	2.8×10^{-11}	1.7×10^{-15}	2.8×10^{-11}	7.3×10^{-15}	2.5×10^{-11}	2.7×10^{-15}

Table 15 From left to right, relative errors when solving $\mathbf{W_n c_n} = \mathbf{d_n}$ at $x_0 = 50$ of the Jacobi (with $\alpha = 1$, $\beta = 2$), Legendre, Gegenbauer (with $\lambda = 1$) and Chebyshev of the first and second kind polynomial bases.

	Jacobi		**Legendre**		**Gegenbauer**		**Chebyshev 1st kind**		**Chebyshev 2nd kind**	
n+1	$\mathbf{W_n} \backslash \mathbf{d_n}$	`TNsolve`	$\mathbf{W_n} \backslash \mathbf{d_n}$	`TNsolve`	$\mathbf{W_n} \backslash \mathbf{d_n}$	`TNsolve`	$\mathbf{W_n} \backslash \mathbf{d_n}$	`TNsolve`	$\mathbf{W_n} \backslash \mathbf{d_n}$	`TNsolve`
10	3.1×10^{-16}	1.1×10^{-16}	4.3×10^{-14}	2.3×10^{-16}	2.1×10^{-14}	4.5×10^{-17}	1.2×10^{-14}	2.6×10^{-16}	4.5×10^{-9}	6.6×10^{-17}
15	4.3×10^{-11}	9.5×10^{-17}	5.8×10^{-12}	2.8×10^{-16}	1.5×10^{-11}	3.2×10^{-15}	5.9×10^{-13}	4.6×10^{-15}	2.7×10^{-2}	2.4×10^{-16}
20	4.6×10^{-9}	1.5×10^{-16}	6.3×10^{-9}	6.1×10^{-16}	5.6×10^{-9}	1.3×10^{-15}	2.3×10^{-9}	2.6×10^{-15}	7.8×10^{-1}	6.0×10^{-17}
25	6.2×10^{-6}	1.4×10^{-16}	4.4×10^{-6}	4.2×10^{-16}	1.2×10^{-6}	7.8×10^{-16}	2.1×10^{-7}	4.5×10^{-15}	2.9	1.4×10^{-15}

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