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# Fractional Generalizations of Rodrigues-Type Formulas for Laguerre Functions in Function Spaces

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**Abstract:** Generalized Laguerre polynomials, $L_n^{(\alpha)}$, verify the well-known Rodrigues' formula. Using Weyl and Riemann–Liouville fractional calculi, we present several fractional generalizations of Rodrigues' formula for generalized Laguerre functions and polynomials. As a consequence, we give a new addition formula and an integral representation for these polynomials. Finally, we introduce a new family of fractional Lebesgue spaces and show that some of these special functions belong to them.

**Keywords:** Rodrigues' formula; Laguerre functions; function spaces; fractional calculus



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## 1. Introduction

In approximation theory, the classical orthogonal polynomials of Jacobi, Laguerre, and Hermite have many properties in common, namely, the Rodrigues formula, the differential equation, the derivative formula, and the three-term recurrence relation. Under some conditions, these common properties are equivalent and characterize these classical orthogonal polynomials. See more details, for example, in [1] [Chapter 12] and [2] [Chapter V].

Polynomial solutions in the differential equation

$$zw''(z) + (\alpha + 1 - z)w'(z) + nw(z) = 0,$$

with $n = 0, 1, 2\dots$ and $\alpha \in \mathbf{C}$ are called *generalized Laguerre polynomial, $L_n^{(\alpha)}$*. They verify Rodrigues' formula,

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(x^{n+\alpha}e^{-x}), \tag{1}$$

where we have

$$L_n^{(\alpha)}(x) := \sum_{m=0}^{n}(-1)^m\binom{n+\alpha}{n-m}\frac{x^m}{m!},$$

with $\binom{n+\alpha}{n-m} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+m+1)(n-m)!}$, see, for example, [2] [p. 241] and [1] [Chapter 12]. In particular, they are

$$\begin{aligned} L_0^{(\alpha)}(x) &= 1, \\ L_1^{(\alpha)}(x) &= \alpha + 1 - x, \\ L_2^{(\alpha)}(x) &= \frac{1}{2}\Big((\alpha+1)(\alpha+2) - 2(\alpha+2)x + x^2\Big). \end{aligned}$$

Generalized Laguerre polynomials satisfy several recurrence equalities, see [2] [p. 241], between them, for example,

$$xL_n^{(\alpha+1)}(x) = (n+\alpha+1)L_n^{(\alpha)}(x) - (n+1)L_{n+1}^{(\alpha)}(x). \tag{2}$$

Rodrigues' formula was initially introduced for Legendre polynomials by Olinde Rodrigues in 1816. The name "Rodrigues formula" was given by Heine in 1878, after Hermite pointed out in 1865 that Rodrigues was the first to discover it instead of Ivory and Jacobi. The term is also used to describe similar formulas for other orthogonal polynomials, mainly Laguerre and Hermite polynomials and many other sequences of orthogonal functions. These are also called the Rodrigues formula (or Rodrigues' type formula) for that case, especially when the resulting sequence is polynomial.

By means of fractional calculi, several generalizations of Rodrigues formula have appeared in the literature in the last years. We present some of them in the next lines.

Several important special functions can be expressed as derivatives of complex order of elementary function, see, for example, [3]. The derivative of a complex order $\nu$ of a complex function $f$ of a complex variable $z$ is defined by the generalized Cauchy integral,

$$f^{(\nu)}(z) = \left(\frac{d}{dz}\right)^{\nu} f(z) = \frac{\Gamma(\nu+1)}{2\pi i}\int_{\gamma} f(w)(w-z)^{-\nu-1}dw,$$

under some assumptions about $\nu$, $f$ and the path $\gamma$ [3,4] [p. 113]. In the case of Laguerre functions, we have

$$L_{\nu}^{(\mu)}(z) = \frac{z^{-\mu}}{\Gamma(1+\nu)}e^{z}\left(\frac{d}{dz}\right)^{\nu}(z^{\nu+\mu}e^{-z}),$$

which coincides with the Rodrigues' Formula (1) for $\nu = n$.

In [5], certain Laguerre polynomials of arbitrary orders are defined. The fractional Caputo derivative $D^{\alpha}$ of order $\alpha \in (n-1, n]$ of a function $f$ is given there by

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt, \qquad x \geq 0,$$

see [5] [Definition 1.3]. The author defines the Laguerre polynomials $Ł_{\alpha}^{\beta}$ of order $\alpha > 0$ by

$$Ł_{\alpha}^{\beta}(x) = \frac{x^{-\beta}e^{x}}{\Gamma(1+\alpha)}D^{\alpha}(e^{-x}x^{\alpha+\beta}), \qquad x > 0,\ \beta > -1,$$

and proves

$$\lim_{\alpha\to n^+} Ł_{\alpha}^{\beta}(x) = \lim_{\alpha\to n^-} Ł_{\alpha}^{\beta}(x) = L_n^{(\beta)}(x).$$

A wide generalization of Rodrigues' formula is treated in [6]. The author considers the Riemann–Liouville integral to include a large numbers of special functions, in particular Laguerre polynomials and functions [6] [Section 1].

In [7], authors use a generalization of the Rodrigues' formula to define a new special function. They study some of its properties, some recurrence relations, orthogonality property, and the continuation to the Rodrigues' formula of the Laguerre polynomials as a limit case. In addition, the confluent hypergeometric representation is given.

In this paper, we consider the Weyl and Riemann–Liouville fractional calculi, $W_+^{\alpha}$ and $D_+^{\alpha}$ with $\alpha \in \mathbf{R}$ in the half real line in the second section. In Section 3, Theorem 1, we show the following fractional Rodrigues' formulae:

$$\begin{aligned} M(\alpha,\nu+1,z) &= \tfrac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)}z^{-\nu}e^{z}D_+^{-\alpha}(t^{\nu-\alpha}e^{-t})(z), \\ U(\alpha,\nu+1,z) &= z^{-\nu}e^{z}W_+^{-\alpha}(t^{\nu-\alpha}e^{-t})(z), \end{aligned} \tag{3}$$

where $M(\alpha, \nu+1, z)$ and $U(\alpha, \nu+1, z)$ are the confluent hypergeometric functions,

$$\begin{aligned} M(\alpha,\nu,z) &:= \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} \frac{\Gamma(\nu)}{\Gamma(\nu+j)} \frac{z^j}{j!}; \\ U(\alpha,\nu,z) &:= \frac{\Gamma(1-\nu)}{\Gamma(\alpha+1-\nu)} M(\alpha,\nu,z) + \frac{\Gamma(\nu-1)}{\Gamma(\alpha)} z^{1-\nu} M(\alpha+1-\nu, 2-\nu, z). \end{aligned}$$

This theorem extends [8] [Theorem 3] and completes the picture given in formulae [6] [(8)–(12)], where the author only considers the Riemann–Liouville fractional calculus.

In the particular case $-\alpha = n \in \mathbf{N}$, the confluent hypergeometric functions are essentially the Laguerre polynomials and we get a second fractional Rodrigues' formula,

$$L_n^{(\alpha-n)}(x) = \frac{(-1)^n}{n!} e^x W_+^{\alpha}(t^n e^{-t})(x), \quad x \geq 0,$$

in Theorem 2. As a consequence, we get a new integral addition formula for Laguerre polynomials in Corollary 1. We also obtain a integral representation of $W_+^{\alpha}(t^n e^{-\lambda t})$, i.e.,

$$W_+^{\alpha}(r^n e^{-\lambda r})(t) = \lambda^{\alpha} e^{-\lambda t} t^n + \lambda^{\alpha} \sum_{k=1}^{n} (-1)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k)} \binom{n}{k} \int_t^{\infty} (r-t)^{k-1} r^{n-k} e^{-\lambda r} dr,$$

and apply it to get a new integral representation of $L_n^{(\alpha-n)}(t)$, i.e.,

$$L_n^{(\alpha-n)}(t) = \frac{(-1)^n}{n!} e^t \sum_{k=0}^{n} (-1)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \binom{n}{k} W_+^{-k}\left(r^{n-k} e^{-r}\right)(t), \qquad t > 0.$$

in Theorem 3. All of these results show the deep and interesting connection between fractional calculi, in particular Weyl fractional derivation, and Laguerre polynomials.

In the last section, we introduce new fractional Lebesgue space $\mathcal{T}_p^{(\alpha)}(t^{\mu} + t^{\alpha})$ which are contained in $L^p(\mathbf{R}^+)$ with $0 \leq \mu \leq \alpha$ and $p \geq 1$. Note that we understand that $\mathcal{T}_p^{(0)}(t^0 + t^0) = L^p(\mathbf{R}^+)$. As in the classical case, we show that the space $\mathcal{T}_p^{(\alpha)}(t^{\mu} + t^{\alpha})$ for $p > 1$ is module for the algebra $\mathcal{T}_1^{(\alpha)}(t^{\mu} + t^{\alpha})$ (Theorem 4). This family of function spaces contains as a particular case some spaces which have appeared previously in the literature [9–13]. Finally, we present some special functions which belong to these fractional Lebesgue spaces $\mathcal{T}_p^{(\alpha)}(t^{\mu} + t^{\alpha})$ in Remark 2.

## 2. Weyl and Riemann–Liouville Fractional Calculi

We denote by $\mathcal{D}_+$ the set of test functions of compact support in $[0, \infty)$, $\mathcal{D}_+ \equiv C_c^{\infty}([0, \infty))$ and by $\mathcal{S}_+$ the Schwartz class on $[0, \infty)$, i.e., functions which are infinitely differentiable, which verifies

$$\sup_{t \geq 0} \left| t^m \frac{d^n}{dt^n} f(t) \right| < \infty,$$

for any $m, n \in \mathbf{N} \cup \{0\}$.

**Definition 1.** *Given $f \in \mathcal{S}_+$, the Weyl fractional integral of $f$ of order $\alpha > 0$ is defined by*

$$W_+^{-\alpha} f(u) := \frac{1}{\Gamma(\alpha)} \int_u^{\infty} (t-u)^{\alpha-1} f(t) dt, \qquad u \geq 0,$$

*with $\alpha > 0$. This operator $W_+^{-\alpha} : \mathcal{S}_+ \to \mathcal{S}_+$ is one to one, and its inverse, $W_+^{\alpha}$, is the Weyl fractional derivative of order $\alpha$, and*

$$W_+^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^{\infty} (s-t)^{n-\alpha-1} f(s) ds, \qquad t \geq 0,$$

*holds with $n = [\alpha] + 1$; see, for example, [14,15].*

It is easy to check that, if $\alpha = n \in \mathbf{N}$, then $W_+^\alpha f = (-1)^\alpha f^{(\alpha)} = (-1)^n \dfrac{d^n}{dt^n}$ and $W_+^{\alpha+\beta} f = W_+^\alpha(W_+^\beta f)$ with $\alpha, \beta \in \mathbf{R}$, $W_+^0 = Id$ and $f \in \mathcal{S}_+$ and

$$W_+^\alpha(f_r) = r^\alpha (W_+^\alpha f)_r, \tag{4}$$

where $f_r(s) := f(rs)$ for $r > 0$; see more details in [14,15].

**Example 1.** *Let $\lambda \in \mathbf{C}^+$ and $e_\lambda(s) := e^{-\lambda s}$ with $s \geq 0$. It is clear that $e_\lambda \in \mathcal{S}_+$ and*

$$W_+^{-\alpha}(e_\lambda)(s) = \lambda^{-\alpha} e^{-\lambda s}, \qquad s \geq 0.$$

*Then, $W_+^\alpha e_\lambda(s) = \lambda^\alpha e^{-\lambda s}$ for $\alpha \in \mathbf{R}$ and $s \geq 0$. In Theorem 3, we give an integral expression of $W_+^\alpha(r^n e^{-\lambda r})$.*

**Proposition 1.** *Take $\alpha \in \mathbf{R}$ and $f \in \mathcal{S}_+$ then :*

$$W_+^\alpha(sf(s))(t) = tW_+^\alpha f(t) - \alpha W_+^{\alpha-1} f(t) \qquad t > 0.$$

**Proof.** If $\alpha < 0$, it is shown in [14] [p. 246]; if $\alpha > 0$, we have

$$W_+^{-\alpha}(sW_+^\alpha f(s) - \alpha W_+^{\alpha-1} f(s))(t) = tf(t) + \alpha W_+^{-1} f(t) - \alpha W_+^{-1} f(t) = tf(t)$$

with $t > 0$. $\square$

The usual convolution product $*$ on $\mathbf{R}^+$ is defined by

$$(f * g)(t) := \int_0^t f(t-s)g(s)ds, \qquad t \in \mathbf{R}^+,$$

for functions $f, g$ which are "good enough", for example, absolutely integrable functions. For functions $f, g \in \mathcal{S}_+$, the following integral equality for the convolution product holds

$$\begin{aligned} W_+^\alpha(f * g)(s) &= \int_0^s W_+^\alpha g(r) \int_{s-r}^s \tfrac{(t+r-s)^{\alpha-1}}{\Gamma(\alpha)} W_+^\alpha f(t)dt\, dr \\ &- \int_s^\infty W_+^\alpha g(r) \int_s^\infty \tfrac{(t+r-s)^{\alpha-1}}{\Gamma(\alpha)} W_+^\alpha f(t)dt\, dr \end{aligned} \tag{5}$$

for $s \in \mathbf{R}^+$ ([10,12] [Proposition 1.2]).

**Definition 2.** *Given $f \in \mathcal{S}_+$, Riemann–Liouville fractional integral of order $\alpha > 0$ is defined by*

$$D_+^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \qquad t \geq 0.$$

*If $\alpha = n \in \mathbf{N}$, the Riemann–Liouville derivative Riemann–Liouville of order $n$, $D_+^n$, is the usual derivative and, if $\alpha \in \mathbf{R}^+ \backslash \mathbf{N}$, Riemann–Liouville fractional derivative of order $\alpha$ is defined by*

$$D_+^\alpha f(t) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)ds, \qquad t \geq 0,$$

*with $n = [\alpha] + 1$, see, for example, [14–16].*

If $f \in \mathcal{S}_+$, then $D_+^{-\alpha} D_+^{-\beta} f = D_+^{-(\alpha+\beta)} f$ for $\alpha, \beta \geq 0$, but, in general, it is false that $D_+^{\alpha} D_+^{\beta} f = D_+^{\alpha+\beta} f$ with $\alpha, \beta \in \mathbf{R}$. For example,

$$D_+^{-1} D_+^{1}(e^{-s})(t) = -\int_0^t e^{-s} ds = e^{-t} - 1 \neq e^{-t}.$$

If $\alpha > 0$ and $f, g \in \mathcal{S}_+$, we apply the Fubini theorem to get that

$$\int_0^\infty W_+^{-\alpha} f(t) g(t) dt = \int_0^\infty f(t) D_+^{-\alpha} g(t) dt$$

and, if we apply $W_+^{-\alpha}(W_+^{\alpha} g) = g$, we obtain the following "integration by parts" formula:

$$\int_0^\infty f(t) g(t) dt = \int_0^\infty D_+^{-\alpha} f(t) W_+^{\alpha} g(t) dt. \tag{6}$$

We will use both equalities in the following sections. Note that Formula (6) shows the dual behaviour between both fractional calculi.

## 3. Fractional Rodrigues' Formulae for Confluent Hypergeometric Functions

Two linearly independent solutions of Kummer's differential equation

$$z w''(z) + (\nu - z) w'(z) - \alpha w(z) = 0, \qquad \alpha, \nu \in \mathbf{R}$$

are given by confluent hypergeometric functions $M(-\alpha, 1+\nu, z)$ (also written by ${}_1F_1(-\alpha, 1+\nu, z)$) and $U(-\alpha, 1+\nu, z)$, see definitions in the Introduction, Formula (3). In the particular case of $-\alpha = n \in \mathbf{N}$, we have that

$$L_n^{(\nu)}(z) = \frac{\Gamma(\nu+n+1)}{\Gamma(1+\nu) n!} M(-n, 1+\nu, z) = \frac{(-1)^n}{n!} U(-n, \nu+1, z), \qquad z \in \mathbf{C}.$$

There is a big amount of equalities which confluent hypergeometric functions verify. We consider the following ones:

$$\begin{aligned} \tfrac{d^n}{dz^n}\left(e^{-z} z^{c-1} M(a,c,z)\right) &= (-1)^n (1-c)_n e^{-z} z^{c-n-1} M(a-n, c-n, z), \\ \tfrac{d^n}{dz^n}\left(e^{-z} z^{c-a+n-1} U(a,c,z)\right) &= (-1)^n e^{-z} z^{c-a-1} U(a-n, c, z). \end{aligned} \tag{7}$$

The following integral representations hold:

$$\begin{aligned} M(\alpha,\nu,z) &= \tfrac{\Gamma(\nu) z^{1-\nu}}{\Gamma(\alpha)\Gamma(\nu-\alpha)} \int_0^z e^t t^{\alpha-1} (z-t)^{\nu-\alpha-1} dt, \qquad \Re\nu > \Re\alpha > 0; \\ U(\alpha,\nu,z) &= \tfrac{z^{1-\nu}}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1} (z+t)^{\nu-\alpha-1} dt, \qquad \Re\nu > 0, \Re z > 0, \end{aligned} \tag{8}$$

and also the recurrence relation

$$U(\alpha,\nu,z) - U(\alpha,\nu-1,z) - \alpha U(\alpha+1,\nu,z) = 0, \tag{9}$$

see all these formulae and much more in [2] [Chapter VI].

The part (i) of the next theorem includes [8] [Theorem 3] for $\alpha < 0$ and $\nu > -1$ and for $\alpha \in \mathbf{R}$ and $\nu > \alpha - 1$ are presented in [6] [Section 1]. We include the proof of both parts to avoid the lack of completeness.

**Theorem 1.** *Given $\alpha \in \mathbf{R}$ and $\Re z > 0$, the following equalities hold:*

*(i)* $M(\alpha, \nu+1, z) = \dfrac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)} z^{-\nu} e^z D_+^{-\alpha}(t^{\nu-\alpha} e^{-t})(z)$, *for* $\nu > \alpha - 1$.

*(ii)* $U(\alpha, \nu+1, z) = z^{-\nu} e^z W_+^{-\alpha}(t^{\nu-\alpha} e^{-t})(z)$, *for* $\nu \in \mathbf{R}$.

**Proof.** (i) Let $\nu > \alpha - 1$. For $\alpha = 0$, $M(0, \nu+1, z) = 1$ and the equality holds. Take $\alpha > 0$ and

$$\begin{aligned} M(\alpha, \nu+1, z) &= \frac{\Gamma(\nu+1)z^{-\nu}}{\Gamma(\alpha)\Gamma(\nu+1-\alpha)} \int_0^z e^t t^{\alpha-1}(z-t)^{\nu-\alpha} dt \\ &= \frac{\Gamma(\nu+1)z^{-\nu}}{\Gamma(\alpha)\Gamma(\nu+1-\alpha)} e^z \int_0^z e^{-t}(z-t)^{\alpha-1} t^{\nu-\alpha} dt \\ &= \frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)} z^{-\nu} e^z D_+^{-\alpha}(t^{\nu-\alpha}e^{-t})(z), \end{aligned}$$

for $z > 0$ and by holomorphy $\Re z > 0$. Finally, take $\alpha < 0$. We write $n = [-\alpha] + 1$

$$D_+^{-\alpha}(t^{\nu-\alpha}e^{-t})(z) = \frac{d^n}{dz^n}(D_+^{-n-\alpha}(t^{\nu-\alpha}e^{-t}))(z),$$

and we apply the above equality for $\alpha > 0$ to get

$$D_+^{-n-\alpha}(t^{\nu-\alpha}e^{-t})(z) = e^{-z} z^{\nu+n} \frac{\Gamma(\nu+1-\alpha)}{\Gamma(\nu+n+1)} M(n+\alpha, \nu+n+1, z),$$

for $z > 0$. We apply the first formula in (7) to get

$$\begin{aligned} D_+^{-\alpha}(t^{\nu-\alpha}e^{-t})(z) &= \frac{\Gamma(\nu+1-\alpha)}{\Gamma(\nu+n+1)} \frac{d^n}{dz^n}\left(e^{-z}z^{\nu+n} M(n+\alpha, \nu+n+1, z)\right)(z) \\ &= \frac{\Gamma(\nu+1-\alpha)}{\Gamma(\nu+n+1)} e^{-z} z^{\nu} (-1)^n (-\nu-n)_n M(\alpha, \nu+1, z) \\ &= \frac{\Gamma(\nu+1-\alpha)}{\Gamma(\nu+1)} e^{-z} z^{\nu} M(\alpha, \nu+1, z), \end{aligned}$$

and we get the equality for $z > 0$ and then $\Re z > 0$.
(ii) Let $\nu \in \mathbf{R}$. For $\alpha = 0$, $U(0, \nu+1, z) = 1$ and the equality holds. Take $\alpha > 0$. We apply the second formula in (8) to get

$$\begin{aligned} U(\alpha, \nu+1, z) &= \frac{z^{-\nu}}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1}(z+t)^{\nu-\alpha} dt = \frac{z^{-\nu}e^z}{\Gamma(\alpha)} \int_z^\infty e^{-t}(t-z)^{\alpha-1} t^{\nu-\alpha} dt \\ &= z^{-\nu} e^z W_+^{-\alpha}(t^{\nu-\alpha}e^{-t})(z), \end{aligned}$$

for $z > 0$ and by holomorphy $\Re z > 0$. Finally, take $\alpha < 0$ and $n = [-\alpha] + 1$. Firstly, we consider $n = 1$, i.e., $-1 < \alpha < 0$. By the second formula in (7) for $n = 1$ and the above equality for $\alpha > 0$, we get that

$$\begin{aligned} U(\alpha, \nu+1, z) &= (-1)e^z z^{-(\nu-\alpha-1)} \frac{d}{dz}\left(e^{-z} z^{\nu-\alpha} U(1+\alpha, \nu+1, z)\right) \\ &= (-1)e^z z^{-(\nu-\alpha-1)} \frac{d}{dz}\left(z^{-\alpha} W_+^{-(1+\alpha)}(t^{\nu-\alpha-1}e^{-t})(z)\right) \\ &= \alpha e^z z^{-\nu} W_+^{-(1+\alpha)}(t^{\nu-(1+\alpha)}e^{-z}) + e^z z^{-(\nu-1)} W_+^{-\alpha)}(t^{\nu-1-\alpha)}e^{-z}) \\ &= \alpha U(1+\alpha, \nu+1, z) + e^z z^{-(\nu-1)} W_+^{-\alpha}(t^{\nu-1-\alpha}e^{-z}), \end{aligned}$$

for $z > 0$. By the recurrence relation (9), we have that

$$U(\alpha, \nu+1, z) = \alpha U(1+\alpha, \nu+1, z) + U(\alpha+1, \nu, z),$$

and we conclude that $U(\alpha, \nu+1, z) = e^z z^{-(\nu-1)} W_+^{-\alpha}(t^{\nu-1-\alpha}e^{-z})$, for $z > 0$ and $\Re z > 0$. Now, we suppose that the equality holds for $n+1 > -\alpha > n$, and we claim that it also is true for $n+1 > -\alpha > n$. Again, by the second formula in (7) for $n=1$ and our hypothesis, we get that

$$\begin{aligned} U(\alpha,\nu+1,z) &= (-1)e^z z^{-(\nu-\alpha-1)}\frac{d}{dz}\left(e^{-z}z^{\nu-\alpha}U(1+\alpha,\nu+1,z)\right) \\ &= \alpha e^z z^{-\nu} W_+^{-(1+\alpha)}(t^{\nu-(1+\alpha)}e^{-z}) + e^z z^{-(\nu-1)} W_+^{-\alpha}(t^{\nu-1-\alpha}e^{-z}) \\ &= \alpha U(1+\alpha,\nu+1,z) + e^z z^{-(\nu-1)} W_+^{-\alpha}(t^{\nu-1-\alpha}e^{-z}), \end{aligned}$$

for $z>0$. Again, by the recurrence relation (9), we conclude that

$$U(\alpha,\nu+1,z) = e^z z^{-(\nu-1)} W_+^{-\alpha}(t^{\nu-1-\alpha}e^{-z}),$$

for $z>0$ and $\Re z > 0$, and the proof is finished. □

## 4. Fractional Rodrigues' Formulae for Laguerre Polynomials

Now, first we check $L_n^{(\alpha-n)}(x) = \frac{(-1)^n}{n!}e^x W_+^\alpha(t^n e^{-t})(x)$ with $\alpha \in \mathbf{R}$ and $x \geq 0$. In the case $\alpha = 0$, the equality holds directly (see [2] [p. 240]) since

$$L_n^{(-n)}(x) = \frac{(-x)^n}{n!}L_0^{(n)}(x) = \frac{(-x)^n}{n!},$$

and then

$$\frac{(-1)^n}{n!}e^x W_+^0(t^n e^{-t})(x) = \frac{(-x)^n}{n!} = L_n^{(-n)}(x).$$

Analogously, we have $L_0^{(\alpha)}(x) = 1$.

**Theorem 2.** *Let $\alpha \in \mathbf{R}$, $n \in \mathbf{N}$. Then,*

$$L_n^{(\alpha-n)}(x) = \frac{(-1)^n}{n!}e^x W_+^\alpha(t^n e^{-t})(x), \quad x \geq 0.$$

**Proof.** We give a proof by induction. Take $\alpha > 0$; for $n=1$, we apply Proposition 1 to get

$$W_+^\alpha(te^{-t})(x) = xW_+^\alpha(e^{-t})(x) - \alpha W_+^{\alpha-1}(e^{-t})(x) = e^{-x}(x-\alpha) = -L_1^{(\alpha-1)}(x).$$

Taking the case $n+1$ and again by Proposition 1, we obtain

$$W_+^\alpha(t^{n+1}e^{-t})(x) = xW_+^\alpha(t^n e^{-t})(x) - \alpha W_+^{\alpha-1}(t^n e^{-t})(x), \qquad x > 0$$

and, using induction hypothesis,

$$W_+^\alpha(t^{n+1}e^{-t})(x) = e^{-x}n!(-1)^n\left(xL_n^{(\alpha-n)}(x) - \alpha L_{n+1}^{(\alpha-1-n)}(x)\right), \qquad x > 0.$$

We apply the recurrence Formula (2) and

$$\begin{aligned} W_+^\alpha(t^{n+1}e^{-t})(x) &= e^{-x}n!(-1)^n(-1)(n+1)L_{n+1}^{(\alpha-n-1)}(x) \\ &= e^{-x}(-1)^{n+1}(n+1)!L_{n+1}^{(\alpha-(n+1))}(x) \end{aligned}$$

and we get the equality. In the case $\alpha < 0$, we work in a similar way. □

**Remark 1.** *In the case $\alpha = m \in \mathbf{N}$, we obtain an equivalent formula to*

$$L_m^{(\beta)}(x) = \frac{x^{-\beta}e^x}{m!}\frac{d^m}{dx^m}(x^{m+\beta}e^{-x}).$$

*To check this, we use $L_n^{(-k)}(x) = (-x)^k\dfrac{(n-k)!}{n!}L_{n-k}^{(k)}(x)$, $1 \leq k \leq n$, ([2] p. 240).*

*If $\lambda > 0$, we also get*

$$\begin{aligned} W_+^{\alpha}(t^n e^{-\lambda t})(x) &= \lambda^{\alpha-n} e^{-\lambda x} (-1)^n n! L_n^{(\alpha-n)}(\lambda x) \\ &= \lambda^{\alpha} e^{-\lambda x} \left( \sum_{m=0}^{n} \lambda^{m-n} (-1)^{n-m} \binom{\alpha}{n-m} \frac{n!}{m!} x^m \right). \end{aligned}$$

*The addition formula for Laguerre polynomials states that*

$$L_n^{(\alpha+\beta+1)}(t+r) = \sum_{j=0}^{n} L_j^{(\alpha)}(t) L_{n-j}^{(\beta)}(r), \qquad t, r \in \mathbf{R}.$$

*([2] [p. 249]). The following corollary shows a new integral addition formula for Laguerre polynomials.*

**Corollary 1.** *Let $\alpha \in \mathbf{R}$ and $n, m \in \mathbf{N}$. Then,*

$$\begin{aligned} e^{-s} L_{n+m+1}^{(\alpha-(n+m+1))}(s) &= \int_s^{\infty} e^{-r} L_n^{(\alpha-n)}(r) \int_s^{\infty} \frac{(t+r-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-t} L_m^{(\alpha-m)}(t) dt\, dr \\ &- \int_0^{s} e^{-r} L_n^{(\alpha-n)}(r) \int_{s-r}^{s} \frac{(t+r-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-t} L_m^{(\alpha-m)}(t) dt\, dr \end{aligned}$$

*for $s \in \mathbf{R}^+$.*

**Proof.** We write $p_n(s) = \dfrac{s^n}{n!} e^{-s}$ for $s \in \mathbf{R}^+$. Note that $p_n * p_m = p_{n+m+1}$ for $n, m \in \mathbf{N}$. By Theorem 2, $W^{\alpha}(p_n)(s) = (-1)^n e^{-s} L_n^{(\alpha-n)}(s)$ for $\alpha, s \in \mathbf{R}$ and $n \in \mathbf{N}$. Finally, we apply Formula (5) to conclude the equality. □

Now, we want to give another representation, an integral representation of $W_+^{\alpha}(t^n e^{-\lambda t})$ and $L_n^{(\alpha-n)}$. To do this, we check the following Lemma about the Pochhammer symbol $(\alpha)_j$, where

$$(\alpha)_j := \alpha(\alpha+1)\dots(\alpha+j-1) = \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)}, \qquad j \in \mathbf{N}, \quad \alpha \in \mathbf{C}.$$

**Lemma 1.** *Let $l \in \mathbf{N}$ and $\alpha \in \mathbf{C}$. Then,*

$$\sum_{k=1}^{l} (-1)^k \frac{(\alpha)_k}{(k-1)!} \binom{l}{k} = \frac{(-\alpha)_l}{(l-1)!}. \tag{10}$$

**Proof.** Since

$$\frac{(-\alpha)_l}{(l-1)!} = (-1)^l \frac{\alpha(\alpha-1)\dots(\alpha-l+1)}{(l-1)!},$$

it is enough to prove that $P_l$ is a polynomial in $\alpha$ of degree $l$, the leading coefficient $(-1)^l/l!$ and roots $\{0, 1, \dots, l-1\}$ where

$$P_l(\alpha) := \sum_{k=1}^{l} (-1)^k \frac{(\alpha)_k}{(k-1)!} \binom{l}{k} = \sum_{k=1}^{l} (-1)^k \frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{(k-1)!} \binom{l}{k}.$$

Let $j \in \{1, \dots, l-1\}$ and we consider the polynomial:

$$x^{j-1}(1-x)^l = (-1)^l \sum_{k=0}^{l} \binom{l}{k} (-1)^k x^{k+j-1}.$$

We derive $j$ times and evaluate in $x = 1$ to obtain

$$\sum_{k=0}^{l} \binom{l}{k} (-1)^k (k+j-1)\dots k = 0,$$

and then

$$\begin{aligned} P_l(j) &= \sum_{k=1}^{l} (-1)^k \frac{j(j+1)\dots(j+k-1)}{(k-1)!} \binom{l}{k} \\ &= \frac{1}{(j-1)!} \sum_{k=0}^{l} (-1)^k k(k+1)\dots(j+k-1) \binom{l}{k} = 0, \end{aligned}$$

and we conclude the proof. □

**Theorem 3.** *If $n \in \mathbf{N}$ and $\lambda, \alpha > 0$, we have that*

$$W_+^{\alpha}(r^n e^{-\lambda r})(t) = \lambda^{\alpha} \sum_{k=0}^{n} (-1)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \binom{n}{k} W_+^{-k}\Big(r^{n-k} e^{-\lambda r}\Big)(t), \qquad t > 0,$$

*and*

$$L_n^{(\alpha-n)}(t) = \frac{(-1)^n}{n!} e^t \sum_{k=0}^{n} (-1)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \binom{n}{k} W_+^{-k}\Big(r^{n-k} e^{-r}\Big)(t), \qquad t > 0.$$

**Proof.** By Lemma 1, we have that

$$\frac{n!}{j!l} \sum_{k=1}^{l} (-1)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k)} \binom{l}{k} = (-1)^l \frac{n!}{j!} \binom{\alpha}{l},$$

for $l \geq 1$ and $0 \leq j \leq n$. By the second remark in Theorem 2 and taking in the last expression $l = n - j$, we have that

$$\begin{aligned} \lambda^{-\alpha} e^{\lambda t} W_+^{\alpha}(r^n e^{-\lambda r})(t) &= t^n + \sum_{j=0}^{n-1} \lambda^{j-n} (-1)^{n-j} \binom{\alpha}{n-j} \frac{n!}{j!} t^j \\ &= t^n + \sum_{j=0}^{n-1} \lambda^{j-n} t^j \frac{n!}{j!(n-j)} \sum_{k=1}^{n-j} (-1)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k)} \binom{n-j}{k} \\ &= t^n + \sum_{k=1}^{n} (-1)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k)} \binom{n}{k} \sum_{j=0}^{n-k} \lambda^{j-n} t^j \binom{n-k}{j} (n-1-j)! \end{aligned}$$

due to $\frac{n!}{j!}\binom{n-j}{k} = (n-j)!\binom{n}{k}\binom{n-k}{j}$.

In the other hand, we apply Newton's formula, to get the equality:

$$\begin{aligned} \int_t^{\infty} (r-t)^{k-1} r^{n-k} e^{-\lambda r} dr &= \sum_{j=0}^{n-k} \binom{n-k}{j} t^j \int_t^{\infty} (r-t)^{n-j-1} e^{-\lambda r} dr \\ &= e^{-\lambda t} \sum_{j=0}^{n-k} \binom{n-k}{j} t^j \lambda^{j-n} (n-1-j)! \end{aligned}$$

with $t \geq 0$. From here, we have that

$$W_+^{\alpha}(r^n e^{-\lambda r})(t) = \lambda^{\alpha} e^{-\lambda t} t^n + \lambda^{\alpha} \sum_{k=1}^{n} (-1)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k)} \binom{n}{k} \int_t^{\infty} (r-t)^{k-1} r^{n-k} e^{-\lambda r} dr,$$

and we obtain the equality. As a consequence, we apply Theorem 2 and the first equality to obtain the second one. □

As a corollary of this theorem, we show an equality considered in [14] [p. 315]. However, first, we need to comment some aspects about Laplace transform. Given $f \in \mathcal{S}_+$, the Laplace transform of $f$, $\mathcal{L}(f)$, is given by

$$\mathcal{L}(f)(z) := \int_0^\infty f(t)e^{-zt}dt, \qquad \Re z \geq 0,$$

see, for example [17]. If $f$ is a function in two variables $f = f(t,s)$, $\mathcal{L}(f;t)$ and $\mathcal{L}(f;s)$ are Laplace transforms, if there exist, in each parameter.

If we apply equality (6) in the integral representation of the Laplace transform, we get that, for $\alpha > 0$

$$\mathcal{L}(f)(z) = \int_0^\infty W_+^\alpha f(t) S_\alpha(t,z)dt, \qquad \Re z \geq 0, \tag{11}$$

with $f \in \mathcal{S}_+$ and where

$$S_\alpha(t,z) := D_+^{-\alpha}(e_z)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-zs}ds \qquad \Re z,\ t \geq 0.$$

In fact, function $S_\alpha(t,z)$ may be defined for any $z \in \mathbf{C}$ and

$$S_{\alpha+\beta}(t,z) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} S_\alpha(s,z)ds, \qquad z \in \mathbf{C}, t > 0. \tag{12}$$

**Corollary 2.** *Let $a, \alpha > 0$, $\lambda > a$ and $n \in \mathbf{N} \cup \{0\}$. Then, we have that*

$$\frac{1}{(\lambda - a)^{n+1}} = \frac{\lambda^\alpha}{n!}\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \mathcal{L}(S_{\alpha+k}(r,-a)r^{n-k})(\lambda).$$

**Proof.** By the equality (6) and Theorem 3, we get that

$$\begin{aligned}\frac{1}{(\lambda - a)^{n+1}} &= \int_0^\infty \frac{t^n}{n!}e^{at}e^{-\lambda t}dt = \frac{1}{n!}\int_0^\infty W_+^\alpha(r^n e^{-\lambda r})(t) D_+^{-\alpha}(e^{ar})(t)dt \\ &= \frac{\lambda^\alpha}{n!}\sum_{k=0}^n (-1)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k)}\binom{n}{k}\int_0^\infty S_\alpha(t,-a)\int_t^\infty (r-t)^{k-1} r^{n-k} e^{-\lambda r} dr dt.\end{aligned}$$

We apply the Fubini theorem to get

$$\begin{aligned}\frac{1}{(\lambda - a)^{n+1}} &= \frac{\lambda^\alpha}{n!}\sum_{k=0}^n (-1)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k)}\binom{n}{k}\int_0^\infty r^{n-k}e^{-\lambda r}\int_0^r (r-t)^{k-1} S_\alpha(t,-a)dtdr \\ &= \frac{\lambda^\alpha}{n!}\sum_{k=0}^n (-1)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\binom{n}{k}\int_0^\infty r^{n-k}e^{-\lambda r} S_{\alpha+k}(r,-a)dr,\end{aligned}$$

and we conclude the equality. □

## 5. Fractional Lebesgue Spaces

The well-known Hardy inequality states that

$$\int_0^\infty |W_+^{-\alpha} f(t)|^p dt \leq \left(\frac{\Gamma(\frac{1}{p})}{\Gamma(\alpha + \frac{1}{p})}\right)^p \int_0^\infty t^{\alpha p}|f(t)|^p dt, \tag{13}$$

for $p \geq 1$ and $\alpha \geq 0$, see, for example, [18] [pp. 244–245].

For $0 \le \mu \le \alpha$, we introduce a family of subspaces $\mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha)$ that are contained in $L^p(\mathbf{R}^+)$.

**Definition 3.** *For $\alpha > 0$, let the Banach space $\mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha)$ be defined as the completion of the Schwartz class $\mathcal{S}_+$ in the norm*

$$||f||_{(\mu,\alpha),p} := \frac{1}{\Gamma(\alpha+1)} \left( \int_0^\infty |W_+^\alpha f(t)|^p (t^\mu + t^\alpha)^p dt \right)^{\frac{1}{p}}.$$

We understand that $\mathcal{T}_p^{(0)}(t^0 + t^0) = L^p(\mathbf{R}^+)$ and $|| \quad ||_{(0,0),p} = || \quad ||_p$. The case $p = 1$, $\mu = 0$ and $\alpha \in \mathbf{N}$ were introduced in [9], and, for $p = 1$, $0 \le \mu \le \alpha$ and $\alpha > 0$ in [10] [Section 1]. Finally, the case $p > 1$, $\mu = \alpha$ and $\alpha > 0$ were considered in [11] [Definition 2.1] and [13] [Section 1.2]. In [19], other families of function spaces are studied connected with a kernel function $k$ on $[0, \infty)$.

In the next proposition, we present some results for spaces $\mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha)$ which extends [11] [Proposition 2.2].

**Proposition 2.** *Take $p \ge 1$ and $\beta > \alpha \ge \mu > 0$.*

*(i)* *The operator $D_{\mu,\alpha} : \mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha) \to L^p(\mathbf{R}^+)$ defined by*

$$D_{\mu,\alpha} f(t) := \frac{1}{\Gamma(\alpha+1)} (t^\mu + t^\alpha) W_+^\alpha f(t), \quad t \in \mathbf{R}^+,\ f \in \mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha),$$

*is an isometry whose inverse operator $(D_{\mu,\alpha})^{-1} : L^p(\mathbf{R}^+) \to \mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha)$ is given by*

$$(D_{\mu,\alpha})^{-1} f(t) = W_+^{-\alpha}((t^\mu + t^\alpha)^{-1} f)(t), \quad t \in \mathbf{R}^+\ f \in \mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha).$$

*(ii)* $\mathcal{T}_p^{(\beta)}(t^{\mu+\beta-\alpha} + t^\beta) \hookrightarrow \mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha) \hookrightarrow L^p(\mathbf{R}^+)$.

*(iii)* *If $p > 1$ and $p'$ satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, then the dual of $\mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha)$ is $\mathcal{T}_{p'}^{(\alpha)}(t^\mu + t^\alpha)$, where the duality is given by*

$$\langle f, g \rangle_{\mu,\alpha} = \frac{1}{\Gamma(\alpha+1)^2} \int_0^\infty W_+^\alpha f(t) W_+^\alpha g(t) (t^\mu + t^\alpha)^2 dt = \langle D_{\mu,\alpha} f, D_{\mu,\alpha} g \rangle_0$$

*for $f \in \mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha)$, $g \in \mathcal{T}_{p'}^{(\alpha)}(t^\mu + t^\alpha)$.*

**Proof.** (i) By definition, we have

$$\|D_{\mu,\alpha} f\|_p = \frac{1}{\Gamma(\alpha+1)} \left( \int_0^\infty ((t^\mu + t^\alpha)|W_+^\alpha f(t)|)^p dt \right)^{1/p} = \|f\|_{(\mu,\alpha),p}.$$

(ii) Let $f \in \mathcal{S}_+$ and $0 < \mu \le \alpha < \beta$. For $p = 1$, see [10] [Section 1, p. 16]. Let $1 < p < \infty$, then

$$\begin{aligned} \|f\|_{(\mu,\alpha),p}^p &= \frac{1}{(\Gamma(\alpha+1))^p} \int_0^\infty (t^\mu + t^\alpha)^p |(W^{-(\beta-\alpha)}(W^\beta f))|^p(t) dt \\ &\le \left( \frac{\Gamma(\frac{1}{p})}{\Gamma(\alpha+1)\Gamma(\beta-\alpha+\frac{1}{p})} \right)^p \int_0^\infty t^{(\beta-\alpha)p} (t^\mu + t^\alpha)^p |W^\beta f|^p(t) dt \\ &\le \left( \frac{\Gamma(\frac{1}{p})\Gamma(\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha+\frac{1}{p})} \right)^p \|f\|_{(\mu+\beta-\alpha,\beta),p}^p \end{aligned}$$

where we have used Hardy's inequality (13). The part (iii) is a straightforward consequence of (i) and the duality of $L^p(\mathbf{R}^+)$. $\square$

Note that, in fact,

$$\|f\|_{(\mu,\alpha),p} = \|D_{\mu,\alpha} f\|_p, \qquad \langle f, g\rangle_{\nu,\alpha} = \langle D_{\mu,\alpha} f, D_{\mu,\alpha} g\rangle_0, \tag{14}$$

for $f \in \mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha)$, $g \in \mathcal{T}_{p'}^{(\alpha)}(t^\mu + t^\alpha)$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

**Example 2.** *In this example, we consider some functions which belong (or not) to $\mathcal{T}_p^{(\alpha)}(t^\alpha + t^\mu)$ for $p \geq 1$ and $0 \leq \mu \leq \alpha$.*

*(i)* *Note that $t^\beta \notin \mathcal{T}_p^{(\alpha)}(t^\alpha + t^\mu)$ for $\beta \in \mathbf{C}$ due to $t^\beta$ does not belong to $L^p(\mathbf{R}^+)$.*

*(ii)* *For $0 < \gamma < \delta$ and $a > 0$, it is well known that $W_+^{-\gamma}(a+t)^{-\delta} = \frac{\Gamma(\delta-\gamma)}{\Gamma(\delta)}(t+a)^{\gamma-\delta}$; see, for example, [20] [p. 201]. With this formula, it is easy to check that*

$$W_+^\alpha (a+t)^{-\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}(t+a)^{-(\alpha+\beta)}.$$

*Write $f(t) := (a+t)^{-\beta}$ and then*

$$||f||^p_{(\mu,\alpha),p} = C_{\alpha,p}\int_0^\infty \frac{(t^\alpha + t^\mu)^p}{|(t+a)^{(\alpha+\beta)p}|}dt \leq C_{\alpha,p}\int_0^\infty \frac{1}{(t+a)^{p\beta}}dt < \infty,$$

*and we conclude functions $(a+t)^{-\beta} \in \mathcal{T}_p^{(\alpha)}(t^\alpha + t^\mu)$ for $\beta > 1/p$ and $a > 0$.*

*(iii)* *We define functions $j_c(t) := \dfrac{(1-t)^{c-1}}{\Gamma(c)}\chi_{(0,1)}(t)$ for $t \geq 0$. It is easy to check that $W_+^{-\alpha}(j_c) = j_{\alpha+c}$ for $\alpha > 0$. Then, $j_c \in \mathcal{T}_p^{(\alpha)}(t^\alpha + t^\mu)$ if and only if $c > \alpha + 1 - \frac{1}{p}$.*

Note that, for $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha + t^\mu)$ for $p, \alpha \geq 1$, then $f \in C(\mathbf{R}^+)$, $\lim_{t\to\infty} f(t) = 0$ and

$$\sup_{t>0} t^p |f(t)| \leq C_{\alpha,p}\|f\|_{(\mu,\alpha),p}, \qquad f \in \mathcal{T}_p^{(\alpha)}(t^\alpha + t^\mu),$$

where $C_{\alpha,p}$ is independent of $f$, compare with [11] [Proposition 2.4].

The following theorem extends the case $\beta = \alpha$, which was proved in [13] [Proposition 4.1.9] and $p = 1$ in [10] [Proposition 1.4].

**Theorem 4.** *Take $p \geq 1$ and $\alpha \geq \mu > 0$. Then,*

$$\mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha) * \mathcal{T}_1^{(\alpha)}(t^\mu + t^\alpha) \hookrightarrow \mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha),$$

*i.e., $\|f * g\|_{(\mu,\alpha),p} \leq C_{\alpha,p}\|f\|_{(\mu,\alpha),p}\|g\|_{(\mu,\alpha),1}$ for $f \in \mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha)$, and $g \in \mathcal{T}_1^{(\alpha)}(t^\mu + t^\alpha)$, where $C_{\alpha,p}$ is a constant independent of $f$ and $g$.*

**Proof.** Take $p > 1$ and $f, g \in \mathcal{S}_+$. Then, we apply Formula (5) to get

$$\begin{aligned} |W^\alpha(f*g)(s)|^p \quad &\leq \quad C_{\alpha,p}\left(\int_0^s |W^\alpha g(r)| \int_{s-r}^s (t+r-s)^{\alpha-1}|W^\alpha f(t)|\,dt\,dr\right)^p \\ &\quad + C_{\alpha,p}\left(\int_s^\infty |W^\alpha g(r)| \int_s^\infty (t+r-s)^{\alpha-1}|W^\alpha f(t)|\,dt\,dr\right)^p \end{aligned}$$

for $s \geq 0$. From now, we write all constants by $C_{\alpha,p}$, which may be different in each line.

We start with the second summand. By the Minkowski inequality ([21] [p. 200]), we get that

$$\begin{aligned}
&\left(\int_0^\infty (s^\alpha+s^\mu)^p \left(\int_s^\infty |W^\alpha g(r)| \int_s^\infty (t+r-s)^{\alpha-1}\, |W^\alpha f(t)|\, dt\, dr\right)^p ds\right)^{\frac{1}{p}} \\
\le\ & \int_0^\infty |W^\alpha g(r)| \left(\int_0^r (s^\alpha+s^\mu)^p \left(\int_s^\infty (t+r-s)^{\alpha-1}\, |W^\alpha f(t)|\, dt\right)^p ds\right)^{\frac{1}{p}} dr.
\end{aligned}$$

In the inner integral, we apply the Hölder integral with $\dfrac{1}{p}+\dfrac{1}{q}=1$ to obtain:

$$\begin{aligned}
&\int_s^\infty (t+r-s)^{\alpha-1}\, |W^\alpha f(t)|\, dt \\
\le\ & \left(\int_s^\infty \frac{(t+r-s)^{(\alpha-1)q}}{(t^\alpha+t^\mu)^q}\, dt\right)^{\frac{1}{q}} \left(\int_s^\infty (t^\alpha+t^\mu)^p\, |W^\alpha f(t)|^p\, dt\right)^{\frac{1}{p}} \\
\le\ & \|f\|_{(\mu,\alpha),p} \left(\int_s^\infty \frac{(t+r-s)^{(\alpha-1)q}}{t^{\alpha q}}\, dt\right)^{\frac{1}{q}}.
\end{aligned}$$

We change the variable $t-s=u$ to have that

$$\int_s^\infty \frac{(t+r-s)^{(\alpha-1)q}}{t^{\alpha q}}\, dt = \int_0^\infty \left(\frac{u+r}{u+s}\right)^{\alpha q} \left(\frac{1}{u+r}\right)^q du \le \int_0^\infty \frac{r^{\alpha q}}{s^{\alpha q}}\, \frac{1}{(u+r)^q}\, du = C_q \frac{r^{\alpha q}}{s^{\alpha q}\, r^{\frac{q}{p}}},$$

where we have applied that $(u+r)/(u+s)^{-1} \le r/s$ for $u>0$ and $0\le s\le r$. Finally, as $0\le\mu\le\alpha$, we get that

$$\left(\int_0^r (s^\alpha+s^\mu)^p \left(\int_s^\infty (t+r-s)^{\alpha-1}\, |W^\alpha f(t)|\, dt\right)^p ds\right)^{\frac{1}{p}} \le C_{\alpha,p} \|f\|_{(\mu,\alpha),p}\, r^\alpha.$$

Now, we consider the first integral in the bound of $|W^\alpha(f*g)(s)|^p$. Again, by the Minkowski inequality, we obtain that

$$\begin{aligned}
&\left(\int_0^\infty (s^\alpha+s^\mu)^p \left(\int_0^s |W^\alpha g(r)| \int_{s-r}^s (t+r-s)^{\alpha-1}\, |W^\alpha f(t)|\, dt\, dr\right)^p ds\right)^{\frac{1}{p}} \\
\le\ & \int_0^\infty |W^\alpha g(r)| \left(\int_r^\infty (s^\alpha+s^\mu)^p \left(\int_{s-r}^s (t+r-s)^{\alpha-1}\, |W^\alpha f(t)|\, dt\right)^p ds\right)^{\frac{1}{p}} dr.
\end{aligned}$$

We split the interval $(r,\infty)$ in two parts $(r,2r)\cup[2r,\infty)$. In the first summand, as $s\le 2r$, then $(s^\alpha+s^\mu)^p \le 2^{\alpha p}\,(r^\alpha+r^\mu)^p$, and

$$\begin{aligned}
&\left(\int_r^{2r} (s^\alpha+s^\mu)^p \left(\int_{s-r}^s (t+r-s)^{\alpha-1}\, |W^\alpha f(t)|\, dt\right)^p ds\right)^{\frac{1}{p}} \\
\le\ & 2^\alpha (r^\alpha+r^\mu) \left(\int_r^{2r} \left(\int_{s-r}^s (t+r-s)^{\alpha-1}\, |W^\alpha f(t)|\, dt\right)^p ds\right)^{\frac{1}{p}}.
\end{aligned}$$

We change the variable $s-r=x$, and by the Hardy inequality (13), we get that

$$\le 2^\alpha (r^\alpha+r^\mu) \left(\int_0^\infty \left(\int_x^\infty (t-x)^{\alpha-1}\, |W^\alpha f(t)|\, dt\right)^p dx\right)^{\frac{1}{p}} \le C_{\alpha,p} (r^\alpha+r^\mu) \|f\|_{(\mu,\alpha),p}.$$

In the second summand, we change the variable $x = s - r \geq r$, and then

$$
\begin{aligned}
&\left(\int_r^\infty ((x+r)^\alpha + (x+r)^\beta)^p \left(\int_x^{r+x} (t-x)^{\alpha-1} |W^\alpha f(t)| \, dt\right)^p dx\right)^{\frac{1}{p}} \\
\leq \quad & 2^\alpha \left(\int_r^\infty (x^\alpha + x^\beta)^p \left(\int_x^{r+x} (t-x)^{\alpha-1} |W^\alpha f(t)| \, dt\right)^p dx\right)^{\frac{1}{p}}.
\end{aligned}
$$

We change the variable $t - x = u$, in the inner integral, and we apply the Minkowski inequality to get

$$
\begin{aligned}
& 2^\alpha \left(\int_r^\infty (x^\alpha + x^\beta)^p \left(\int_0^r u^{\alpha-1} |W^\alpha f(u+x)| \, du\right)^p dx\right)^{\frac{1}{p}} \\
\leq \quad & 2^\alpha \int_0^r u^{\alpha-1} \left(\int_r^\infty (x^\alpha + x^\beta)^p |W^\alpha f(u+x)|^p \, dx\right)^{\frac{1}{p}} du, \\
\leq \quad & 2^\alpha \int_0^r u^{\alpha-1} \left(\int_r^\infty ((x+u)^\alpha + (x+u)^\beta)^p |W^\alpha f(u+x)|^p \, dx\right)^{\frac{1}{p}} du, \\
\leq \quad & 2^\alpha \int_0^r u^{\alpha-1} \left(\int_{u+r}^\infty (s^\alpha + s^\beta)^p |W^\alpha f(s)|^p \, ds\right)^{\frac{1}{p}} du \leq C_{\alpha,p} r^\alpha \|f\|_{(\mu,\alpha),p}.
\end{aligned}
$$

Finally, we join every summand to get that

$$\|f * g\|_{(\mu,\alpha),p} \leq \|f\|_{(\mu,\alpha),p} \|g\|_{(\mu,\alpha),1},$$

and the proof is finished. □

**Remark 2.** *(a) We consider the confluent hypergeometric functions $U(\alpha,\nu,z)$ treated in Section 3, and we defined*

$$u_{\alpha,\nu}(z) := z^\nu e^{-z} U(\alpha,\nu,z), \qquad z > 0.$$

*By Theorem 1 (ii), functions $u_{\alpha,\nu} \in \mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha)$ for $\nu > \alpha - \mu - \frac{1}{p}$ with $\alpha \geq \mu \geq 0$ and $p \geq 1$.*
*(b) For $n \geq 0$, we write functions $q_n(t) := \dfrac{t^n}{n!} e^{-t}$. Note that $q_0(t) = e^{-t}$ and $q_n = (q_{n-1} * q_0)$ for $n \geq 1$. It is straightforward to check that $q_0 \in \mathcal{T}_1^{(\alpha)}(t^\mu + t^\alpha)$,*

$$||q_0||_{(\mu,\alpha),1} = 1 + \frac{\Gamma(\mu+1)}{\Gamma(\alpha+1)},$$

*$q_0 \in \mathcal{T}_p^{(\alpha)}(t^\mu + t^\alpha)$ and*

$$||q_0||_{(\mu,\alpha),p} \leq \frac{2}{\Gamma(\alpha+1)p^\mu} \left(\frac{2\Gamma(\alpha p+1)}{p}\right)^{\frac{1}{p}},$$

*for $p > 1$. By Theorems 2 and 4, we conclude that*

$$\left(\int_0^\infty (t^\alpha + t^\mu)^p e^{-tp} |L_n^{(\alpha-n)}(t)|^p dt\right)^{\frac{1}{p}} \leq C_{\alpha,p}^n \frac{2}{p^\mu} \left(\frac{2\Gamma(\alpha p+1)}{p}\right)^{\frac{1}{p}} \left(1 + \frac{\Gamma(\mu+1)}{\Gamma(\alpha+1)}\right)^n,$$

*for $p \geq 1$, $n \in \mathbf{N}$ and $\alpha \geq \mu \geq 0$.*

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