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Cyclic Meir-Keeler Contraction and Its Fractals

R. Pasupathi[a], A. K. B. Chand[a] and M. A. Navascués[b]

[a]Department of Mathematics, Indian Institute of Technology Madras, Chennai - 600036, India,
[b] Departmento de Matemática Aplicada, Escuela de Ingeniería y Arquitectura, Universidad de Zaragoza, C/ María de Luna 3 Zaragoza 50018, Spain.

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ABSTRACT
In present times, there has been a substantial endeavor to generalize the classical notion of iterated function system (IFS). We introduce a new type of non-linear contraction namely cyclic Meir-Keeler contraction, which is more generic than the famous Banach contraction. We show the perseverance and uniqueness of the fixed point for the cyclic Meir-Keeler contraction. Using this result, we propose the cyclic Meir-Keeler IFS in the literature for construction of fractals. Furthermore, we extend the theory of countable IFS and generalized IFS by using these cyclic Meir-Keeler contraction maps.

KEYWORDS
Fractal, Iterated function system, Cyclic Meir-Keeler contraction, Fixed point, Fractal interpolation function.

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1. Introduction

The notion of fractals was coined by Mandelbrot [12] in 1975 to bring most of objects or sets having self-affine (self-similar) characteristics in nature and sciences to a single platform. An object is said to be self-affine (self-similar) if it can be written as finite union of transformed copies of itself with different (fixed) contractions. Iterated function system (IFS) was conscripted by Hutchinson in his fundamental work [8] as a finite set of contraction maps defined on a compact set $\mathbb{K}$ of an Euclidean space $\mathbb{R}^n$, where $n \in \mathbb{N}$. The IFS is denoted by $\mathcal{I} = \{\mathbb{K}; f_1, f_2, \dots f_M\}$. By Gluing Lemma, one can find that there exists a compact set K satisfying the self-referential equation $K = \bigcup_{k=1}^{M} g_k(K)$, and K is referred as an invariant set or an attractor with respect to $\mathcal{I}$. Hutchinson's idea gives that the invariant compact set K is fully determined by $\mathcal{I}$, and also K is the limit of a sequence of prefractal sets that can be built by the members of $\mathcal{I}$. Following this work, Barnsley [2] popularized the theory of IFS by modelling and classifying a broad class of fractals through different types of IFSs. Since classical fractals like Cantor sets, Sierpiński triangles, dragon curves are traditionally seen as

Email: pasupathi4074@gmail.com, chand@iitm.ac.in, manavas@unizar.es

being produced by a process of successive microscopic refinement taken to the limit, it makes sense to approximate them through invariant sets associated with suitable IFSs. The existence of invariant sets of IFS proceeds from the Banach fixed point theorem in a complete metric space. IFS portrays a decisive role in the development and applications of fractal interpolation functions in approximation theory and geometric modelling, see for instance [4–6,9,13,17–19,25,26].

Many researchers have been worked on various extensions of this IFS structure to more generic spaces, generic contractions and with infinite number of maps or more broadly multifunction systems, etc. Infinite IFS was introduced by Wicks [27]. Miculescu and Mihail [15] investigated the shift space amalgamated to invariant sets of infinite IFSs associated with a complete metric spaces. Further, they introduced generalized iterated function system (GIFS), which abides of a finite number of Lipschitz contractions $g_1, \ldots, g_M : \mathbb{X}^l \to \mathbb{X}$, where $(\mathbb{X}, \tau)$ is a complete metric space and $M, l \in \mathbb{N}$, and study various properties of its attractor in [16]. Strobin and Swaczyna [24] sharpen the conclusions of Miculescu and Mihail by taking into account the GIFS consisting of φ-contractions. Leśniak studied a multivalued approach of infinite iterated function systems in [11]. Secelean [20,21] introduced the existence of a compact attractor for countable IFSs, and he studied the IFS composed of a countable family of F-contractions in [22]. Further, Secelean [23] investigated IFSs placed with generalized contractions on the product space $\mathbb{X}^J$ into $\mathbb{X}$, where $\mathbb{X}$ is a metric space and J is an arbitrary set of natural numbers.

The fixed point theory plays a very important role for the existence of invariant sets in different types of IFSs. In various mathematical problems, we need the existence of a solution, which can be put in an equivalent form as the existence of a fixed point for a suitable transformation. It shows that the existence of a fixed point is vital in different areas of mathematics and other sciences. Over the last 50 years many authors discussed variety of fixed point results and their applications (see for instance: [3], [7], [10]). Meir and Keeler [14] constructed a generalization of the contraction map, and demonstrated the perseverance of fixed point for their map, namely a Meir-Keeler contraction. Also, Dumitru studied the generalized IFS with Meir-Keeler type mappings. Kirk et. al [10] proposed cyclic contraction maps in several metric spaces, and proved the presence of proximity points and fixed points for these maps. It is found that cyclic contraction maps are not used in construction of IFSs and fractal functions in the literature. Therefore, we have proposed the Mier-Keeler cyclic contraction map, and proved the existence of its fixed point. Using this novel Mier-Keeler cyclic contraction, we have shown the existence fixed point for an IFS in the corresponding hyperspace. We have constructed a fractal function using the Mier-Keeler cyclic IFS. Further, we have extended our results to an IFS consisting of countable number of maps and to an IFS consists with generalized contractions from X^J into X.

In this paper, we construct a generalization of Meir-Keeler contraction by employing the concepts of cyclic contraction[10] and Meir-Keeler[14] contraction, namely cyclic Meir-Keeler contraction, and discuss the presence of the fixed point for this map and and its uniqueness in Section 3. This generalization is the strict generalization that is, every Meir-Keeler contraction is a cyclic Meir-Keeler contraction, but there exist some cyclic Meir-Keeler contractions which may not be a Meir-Keeler contraction (see Example 3.3). We introduce novel IFSs namely cyclic Meir-Keeler IFS and countable cyclic Meir-Keeler IFS, and establish the perseverance of the attractors of these IFSs and their uniqueness in Section 4. We validate the presence of the fixed point or fractal of the generalized cyclic Meir-Keeler contraction in Section 5 to construct a generalized cyclic Meir-Keeler IFS.

2. Preliminary Facts

In the current section, we revise the basic theory of IFS, countable IFS, Meir-Keeler contraction and fractal interpolation function that are required for our work. The details can be found in [2,10,14,20,21].
A map g on a metric space $(\mathbb{X}, \tau)$ is said to be contraction in nature if g ascertains

$$\tau(g(p), g(q)) \leq s\tau(p, q) \text{ for any } p, q \in \mathbb{X},\ s \in [0, 1),$$

where the contractivity factor of g is s. In 1922, Banach[1] established the most important and widely used fixed point result acknowledged as 'the Banach contraction criterion':

Theorem 2.1. *Suppose g is a contraction map on a complete metric space $(\mathbb{X}, \tau)$. Then g has a unique fixed point $\tilde{p}$ (say). This fixed point $\tilde{p}$ can be obtained from the convergence of a sequence $(g^m(p))_{m=1}^{\infty}$, i.e., $\lim_{m\to\infty} g^m(p) = \tilde{p}$, where $p \in \mathbb{X}$ is arbitrary.*

Theorem 2.1 confirms the perseverance of fixed point of several self-maps of complete metric spaces along with its uniqueness, and this provides an effective method to compute fixed points of these contractions. A generalization of the Banach contraction principle is given by Meir and Keeler[14] by using a cyclic map, which is described in the following:

Definition 2.2. *[10] Assume that $\{C_j\}_{j=1}^{r}$ is a finite collection of subsets of a metric space $(\mathbb{X}, \tau)$ with $C_{r+1} := C_1$. If a map $g : \bigcup_{j=1}^{r} C_j \to \bigcup_{j=1}^{r} C_j$ satisfies $g(C_j) \subseteq C_{j+1}$ for all $j \in \mathbb{N}_r$, then g is termed a cyclic map, where $\mathbb{N}_q$ denote the first q natural numbers .*

Definition 2.3. *[14] g is called a Meir-Keeler contraction map on a metric space $(\mathbb{X}, \tau)$ if for all $\zeta > 0$, there exists $\eta > 0$ such that for any $p, q \in \mathbb{X}$,*

$$\zeta \leq \tau(p, q) < \zeta + \eta \;\;\Rightarrow\;\; \tau(g(p), g(q)) < \zeta.$$

Definition 2.4. *g is called a contractive map on a metric space $(\mathbb{X}, \tau)$ if it satisfies*

$$\text{for all } \; p, q \in \mathbb{X} \text{ with } p \neq q \;\;\Rightarrow\;\; \tau(g(p), g(q)) < \tau(p, q).$$

Remark 2.5. *(i) Meir-Keeler contraction map is a more general formulation of Banach contraction map, i.e., every contraction map is a Meir-Keeler contraction map (for every $\zeta > 0$, choose $\eta = \frac{1-k}{k}\zeta$, where k is the contractivity factor of the Banach contraction map). But the converse need not be true. A Meir-Keeler contraction map is given in example 2.6, which is not a contraction map.*
(ii) Each Meir-Keeler contraction map is also contractive: choose $\zeta = \tau(p, q)$ for $p \neq q$ in the definition of Meir-Keeler contraction. It is easy to observe that each contractive map is continuous. Consequently, we confirm that

$$\textit{contraction} \Rightarrow \textit{Meir-Keeler contraction contractive} \Rightarrow \textit{continuous.}$$

(iii) In Section 3, we introduce the cyclic Meir-Keeler contraction which need not be continuous and hence need not be both contraction and contractive. Note that this

cyclic map is a generalization of Meir-Keeler contraction map.

Example 2.6. *Let $\mathbb{X} = [0,1] \cup \{4, 5, 8, 9, \ldots, 4m, 4m+1, \ldots\}$ be endowed with the Euclidean distance. Define a self-map g on $\mathbb{X}$ as*

$$g(p) = \begin{cases} p/4 & \text{for } p \in [0,1], \\ 0 & \text{for } p = 4m, \\ 1 - \frac{1}{m+3} & \text{for } p = 4m+1. \end{cases}$$

The function g is a Meir-Keeler contraction map (for the case $\zeta < 1$, choose $\eta = min\{\zeta, 1-\zeta\}$ and for $\zeta \geq 1$, choose $\eta = \zeta$). But g is not a contraction map.
Take $p_m = 4m$ and $q_m = 4m+1$. Therefore, $|p_m - q_m| = 1$ and

$$|g(p_m) - g(q_m)| = 1 - \frac{1}{m+3} = \frac{m+2}{m+3},$$

$$\Rightarrow \sup\{\frac{|g(p_m) - g(q_m)|}{|p_m - q_m|}\} = 1.$$

Theorem 2.7. *[14] Assume that g is a Meir-Keeler contraction map on a metric space $(\mathbb{X}, \tau)$, which is also complete. Then g has a unique fixed point $\bar{p} \in \mathbb{X}$, where $\bar{p}$ can be computed from a sequence as described in Theorem 2.1.*

We will introduce the cyclic Meir-Keeler contraction and prove the perseverance of its fixed point in Section 3.
Let $(\mathbb{X}, \tau)$ be a metric space and $\mathbb{H}(\mathbb{X})$ be the collection of all non-empty compact subsets of $\mathbb{X}$. For $p \in \mathbb{X}$ and $P, Q \in \mathbb{H}(\mathbb{X})$, define $\tau(p, Q) = \inf\{\tau(p,q) \ : \ q \in Q\}$ and $\mathbb{D}(P, Q) = \sup\{\tau(p, Q) \ : \ p \in P\}$. Clearly, both $\mathbb{D}(P,Q)$ and $\mathbb{D}(Q,P)$ exist and are non-negative. We define the Hausdorff metric between P and Q as

$$h(P,Q) = \max\{\mathbb{D}(P,Q), \mathbb{D}(Q,P)\}. \tag{1}$$

It is known that the Hausdorff metric space $(\mathbb{H}(\mathbb{X}), h)$ is complete (compact) whenever the metric space $(\mathbb{X}, \tau)$ is complete (compact) respectively.
A finite collection of contraction maps $(g_k)_{k=1}^{M}$ on a complete metric space $(\mathbb{X}, \tau)$ is called an *iterated function systems (IFS)*. This system induces a set valued map $\mathbb{G}$ known as Hutchinson map on $(\mathbb{H}(\mathbb{X}), h)$ as

$$\mathbb{G}(C) = \bigcup_{k=1}^{M} g_k(C),$$

where $g_k(C) = \{g_k(p) \ : \ p \in C\}$. Here $\mathbb{G}$ is also a contraction map on $(\mathbb{H}(\mathbb{X}), h)$ with the contraction factor $\max\{s_1, \ldots, s_M\}$, where s_k's are contractivity factors of g_k respectively. By the Banach contraction criterion, $\mathbb{G}$ has a unique fixed point K in $\mathbb{H}(\mathbb{X})$. i.e,

$$K = \mathbb{G}(K) = \bigcup_{k=1}^{M} g_k(K).$$

Moreover, $K = \lim_{m\to\infty} \mathbb{G}^m(C)$ for any $C \in \mathbb{H}(\mathbb{X})$, where $\mathbb{G}^m = \mathbb{G}\circ\mathbb{G}\circ\cdots$ m-times. This K is called the invariant set as $\mathbb{G}(K) = K$. It is also an attractor since it is the limit of any sequence $\{\mathbb{G}^m(C)\}_{m=1}^{\infty}$, $C \in \mathbb{H}(\mathbb{X})$.

A countable iterated function system (CIFS) is defined as a sequence of contraction maps $(g_k)_{k\geq 1}$ on a compact metric space $(\mathbb{X}, \tau)$ such that $\sup_k s_k < 1$, where s_k is the contractivity factors g_k for each k. The corresponding Hutchinson map $\mathbb{G}$ on $\mathbb{H}(\mathbb{X})$ is defined by

$$\mathbb{G}(C) = \overline{\bigcup_{k=1}^{\infty} g_k(C)}, \quad \forall C \in \mathbb{H}(\mathbb{X}).$$

Here the bar represents the topological closure of the countable union of all these sets. Since $\mathbb{G}(C)$ is closed in the compact metric space $\mathbb{X}$, $\mathbb{G}$ is a well-defined map, and it maps compact sets into compact sets. Since $\sup_k s_k < 1$, $\mathbb{G}$ is a contraction map on the complete metric space $(\mathbb{H}(\mathbb{X}), h)$. Consequently, $\mathbb{G}$ has a unique attractor K in $\mathbb{H}(\mathbb{X})$, and for any $C \in \mathbb{H}(\mathbb{X}), \mathbb{G}^m(C)$ converges to K, that is

$$K = \mathbb{G}(K) = \overline{\bigcup_{k=1}^{\infty} g_k(K)} \text{ and } \lim_{m\to\infty} \mathbb{G}^m(C) = K, \forall\, C \in \mathbb{H}(\mathbb{X}).$$

In the following, we provide an IFS whose fixed point or fractal is the graph of a function called as *fractal interpolation function (FIF).*
Consider an interpolating data set $\{(p_j, q_j) \in \mathbb{R}^2 : j \in \{0\} \cup \mathbb{N}_M\}$, where $-\infty < p_0 < p_1 < \cdots < p_M < \infty$. Let $J = [p_0, p_M]$ and $J_j = [p_{j-1}, p_j]$ for $j \in \mathbb{N}_M$. For $j \in \mathbb{N}_M$, the functions $L_j : J \to J_j$ are contraction homeomorphisms so that for all $p, q \in J$ and $0 \leq r_j < 1$,

$$|L_j(p) - L_j(q)| \leq r_j|p - q|; \quad L_j(p_0) = p_{j-1}, \quad L_j(p_M) = p_j.$$

Further, consider M continuous functions $F_j : J \times \mathbb{R} \to \mathbb{R}$ abiding by the following set of rules:

$$|F_j(p, q) - F_j(p, q')| \leq \alpha_j|q - q'|; \quad \forall p \in J, \quad q, q' \in \mathbb{R}, \; \alpha_j \in [0, 1),$$

$$F_j(p_0, q_0) = q_{j-1}, \quad F_j(p_M, q_M) = q_j.$$

Define $t_j : J \times \mathbb{R} \to J_j \times \mathbb{R}$ by

$$t_j(p, q) = (L_j(p), F_j(p, q)) \;\forall\; j \in \mathbb{N}_M.$$

The desired IFS for construction of a FIF is $\{J \times \mathbb{R},\; t_j : j \in \mathbb{N}_M\}$. Now, define the associated Hutchinson map $T : \mathbb{H}(\mathbb{X}) \to \mathbb{H}(\mathbb{X})$ as $T(C) = \bigcup_{j=1}^{M} t_j(C)$, Barnsley [2] proved the following fundamental result:

Theorem 2.8. *(i) The Hutchinson map T is a contraction map and it has a unique invariant set $G \in \mathbb{H}(\mathbb{X})$. Also, G is the graph of a continuous function $g : J \to \mathbb{R}$ verifying $g(p_j) = q_j$ for all $j \in \mathbb{N} \cup \{0\}$.*

(ii) Let $C^*(J) = \{g : J \to \mathbb{R} \mid g \text{ is continuous}, g(p_0) = q_0, g(p_M) = q_M\}$ *be endowed with the metric* ρ *induced from the uniform norm. Define a Read-Bajraktarevíc operator* T *on the complete metric space* $(C^*(J), \rho)$ *as* $(Tg)(p) = F_j(L_j^{-1}(p), g \circ L_j^{-1}(p)) \quad \forall p \in J_j, \quad j \in \mathbb{N}_M$*. Since the contraction factor of* T *is* $\alpha = \max\{|\alpha_j| : j \in \mathbb{N}_M\} < 1$*, the unique fixed point of* T *is* g *as described in (i), and it is described iteratively from the following equation:*

$$g(L_j(p)) = F_j(p, g(p)), p \in J, j \in \mathbb{N}_M.$$

Definition 2.9. *The above implicit function* g *in Theorem 2.8 is known as a fractal interpolation function which varies depending on the choice scale vector* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)$ *and the choice of* $F_j(p, q), j \in \mathbb{N}_M$*.*

3. Cyclic Meir-Keeler contractions

Definition 3.1. *Suppose that* $\{C_j\}_{j=1}^r$ *is a collection of non-null subsets of a metric space* $(\mathbb{X}, \tau)$ *with* $C_{r+1} = C_1$*. A map* g *on* $\bigcup_{j=1}^{r} C_j$ *to be christened as a cyclic Meir-Keeler contraction if it verifies the following two conditions:*

(1) $g(C_j) \subseteq C_{j+1}$ *for all* $j \in \mathbb{N}_r$*,*
(2) $\forall\ \zeta > 0,\ \exists\ \eta > 0$ *such that* $\zeta \leq \tau(p, q) < \zeta + \eta$ *implies* $\tau(g(p), g(q)) < \zeta \quad \forall p \in C_j,\ q \in C_{j+1}$ *and for all* $j \in \mathbb{N}_r$*.*

Example 3.2. *Let* $C_1 = [0, 1], \quad C_2 = [0, 3]$*. For* $m \geq 2$*, define* $g : C_1 \cup C_2 \to C_1 \cup C_2$ *by*

$$g(p) = \begin{cases} p/m & \text{for } p \in [0, 2], \\ 1/m & \text{for } p \in (2, 3]. \end{cases}$$

We confirm that g *is a cyclic Meir-Keeler contraction on* $C_1 \cup C_2$*. Observe that*

$$g(C_1) = [0, \frac{1}{m}] \subseteq C_2, \quad g(C_2) = [0, \frac{2}{m}] \subseteq C_1.$$

For $\zeta > 0$*, choose* $\eta = (m-1)\zeta$*. Let* $p \in C_1, \quad q \in C_2$ *and* $\zeta \leq |p - q| < m\zeta$*.*
(i) If $p \in [0, 1],\ q \in [0, 2]$*;*

$$|g(p) - g(q)| = \frac{1}{m}|p - q| < \frac{1}{m}(m\zeta) = \zeta.$$

(ii)If $p \in [0, 1],\ q \in (2, 3]$*;*

$$|g(p) - g(q)| = \frac{1}{m}|p - 1| \leq \frac{1}{m}|p - q| < \zeta.$$

Therefore, the above map g *is a cyclic Meir-Keeler contraction. Here* g *is not a Meir-Keeler contraction due to the fact that* g *is not a continuous function.*

Example 3.3. *Suppose $\mathbb{X} = \mathbb{R}$ embedded with the Euclidean distance. Take $C_1 = [0,1], C_2 = [\frac{1}{2}, \frac{3}{2}]$. Define a self-map g on $\mathbb{X}$ by*

$$g(p) = \begin{cases} 11/10 - p & \text{whenever } p \in [0, \frac{1}{10}], \\ 1 & \text{whenever } p \in [\frac{1}{10}, \frac{1}{2}], \\ 21/20 - p/10 & \text{whenever } p \in [\frac{1}{2}, \frac{3}{2}]. \end{cases}$$

Let $\zeta > 0$ and choose $\eta = \zeta$. Let $p \in C_1$ and $q \in C_2$ be arbitrary satisfying $\zeta \leq |p-q| < 2\zeta$.

Case 1. *Consider $p \in [0, \frac{1}{10}]$. Here $|g(p) - g(q)| = |(\frac{11}{10} - p) - (\frac{21}{20} - \frac{q}{10})| = |\frac{q}{10} - p + \frac{1}{20}|$. Since $0 \leq \frac{q}{10} - p + \frac{1}{20} \leq 1/5$ and $|p-q| \geq 2/5$, we obtain*

$$|g(p) - g(q)| \leq 1/5 \leq \frac{1}{2}|p-q| < \frac{1}{2}(2\zeta) = \zeta.$$

Case 2. *For $p \in [\frac{1}{10}, \frac{1}{2}]$, $|g(p) - g(q)| = |\frac{1}{20} - \frac{q}{10}| = \frac{1}{10}|\frac{1}{2} - q| \leq \frac{1}{10}|p-q| < \frac{1}{10}(2\zeta) < \zeta$.*

Case 3. *For $p \in [\frac{1}{2}, 1]$, $|g(p) - g(q)| = |(\frac{21}{20} - \frac{p}{10}) - (\frac{21}{20} - \frac{q}{10})| = \frac{1}{10}|p-q| < \frac{1}{10}(2\zeta) < \zeta$.*

Moreover, $g([0,1]) \subseteq [\frac{1}{2}, \frac{3}{2}]$ and $g([\frac{1}{2}, \frac{3}{2}]) \subseteq [0,1]$. Therefore, g proposed in this example is a cyclic Meir-Keeler contraction, and hence it is continuous also. Finally, consider $p, q \in [0, \frac{1}{10}]$. In this case,

$$|g(p) - g(q)| = \left|\frac{11}{10} - p - (\frac{11}{10} - q)\right| = |p-q|.$$

This implies g is not a Meir-Keeler contraction.

Theorem 3.4. *Let $\{C_j\}_{j=1}^{r}$ be a finite number of non-null closed subsets of a complete metric space $(\mathbb{X}, \tau)$. If $g : \bigcup_{j=1}^{r} C_j \to \bigcup_{j=1}^{r} C_j$ is a cyclic Meir-Keeler contraction map, then g has fixed point $\bar{p}$ (say) that is unique, and it is computed as $\lim_{m\to\infty} g^m(p) = \bar{p}$, $p \in \bigcup_{j=1}^{r} C_j$.*

Proof. Let $p_0 \in \bigcup_{j=1}^{r} C_j$, then there exists at least one $j \in \mathbb{N}_r$ such that $p_0 \in C_j$. Let $p_m = g^m(p_0)$, $m \in \mathbb{N}$. By cyclic condition of g, $p_1 = g(p_0) \in C_{j+1}$ and

$$\tau(p_1, p_2) = \tau(g(p_0), g(p_1)) < \tau(p_0, p_1).$$

Inductively, we can write

$$\tau(p_m, p_{m+1})) < \tau(p_{m-1}, p_m) \text{ for all } m \in \mathbb{N}. \tag{2}$$

Let $\tau_m = \tau(p_m, p_{m+1}), m \in \mathbb{N}$. By (2), $(\tau_m)_{m=1}^{\infty}$ is a strictly decreasing sequence and bounded below by 0, then $\lim \tau_m \downarrow \zeta$ (say). We need to prove $\zeta = 0$. If not, let $\zeta > 0$.

By definition of g, there exists $\eta > 0$ such that

$$\zeta \leq \tau(p,q) < \zeta + \eta \quad \Rightarrow \quad \tau(g(p), g(q)) < \zeta \ \ \forall p \in C_j, \ q \in C_{j+1} \text{ and } \forall j \in \mathbb{N}_r. \qquad (3)$$

The above one is not true for $p = p_m, q = q_{m+1}$. Therefore, we get contradiction, and hence $\zeta = 0$.

Now, we try to prove that $(p_m)_{m=1}^{\infty}$ is Cauchy. Suppose that there exists no $\zeta > 0$ such that $\limsup \tau(p_m, p_n) > 2\zeta$. Again by definition of g, there exists a $\eta > 0$ satisfying (3) for this ζ. It will be also true for η replaced by $\eta' = \min(\eta, \zeta)$.

Since $\lim_{m\to\infty} \tau_m = 0$, there exists $M > 0$ such that $d_M < \eta'/2r+1$. Select $m, n > M$ such that $\tau(p_m, p_n) > 2\zeta$. For $m \leq j \leq n$,

$$|\tau(p_m, p_j) - \tau(p_m, p_{j+1})| \leq \tau_j < \eta'/2r+1.$$

This implies since $\tau(p_m, p_{m+1}) < \zeta$ and $\tau(p_m, p_n) > \zeta + \eta'$, that there exists j_1, $m \leq j_1 \leq n$ such that

$$\zeta + \frac{2r\eta'}{2r+1} < \tau(p_m, p_{j_1}) < \zeta + \eta'.$$

From this, there exists j_2, $m \leq j_2 \leq n$ such that

$$\zeta + \frac{(2r-1)\eta'}{2r+1} < \tau(p_m, p_{j_2}) < \zeta + \frac{2\eta'}{2r+1}.$$

Continuing this process, there exists j_r, $m \leq j_r \leq n$ such that

$$\zeta + \frac{(r+1)\eta'}{2r+1} \ < \ \tau(p_m, p_{j_r}) \ < \ \zeta + \frac{(r+2)\eta'}{2r+1}$$

$$\Rightarrow \zeta + \frac{r\eta'}{2r+1} \ < \ \tau(p_m, p_{j_r-1}) \ < \ \zeta + \frac{(r+3)\eta'}{2r+1}.$$

Successively, we get

$$\zeta + \frac{2\eta'}{2r+1} < \tau(p_m, p_{j_r-(r-1)}) < \zeta + \eta'.$$

We now conclude

$$\zeta + \frac{2\eta'}{2r+1} \ < \ \tau(p_m, p_{j_r-k}) \ < \ \zeta + \eta' \ \text{ for all } k \in \mathbb{N}_{r-1} \cup \{0\}. \qquad (4)$$

Note that $p_m \in C_j$ for some $j \in \mathbb{N}_r$, there exists $k \in \{j_r, j_r - 1, \ldots, j_r - (r-1)\}$ such that $p_k \in C_{j+1}$. By hypothesis, $\tau(p_{m+1}, p_{k+1}) < \zeta$. Therefore,

$$\begin{aligned}\tau(p_m, p_k) &\leq \tau(p_m, p_{m+1}) + \tau(p_{m+1}, p_{k+1}) + \tau(p_{k+1}, p_k)\\ &\leq \frac{\eta'}{2r+1} \ + \ \zeta \ + \ \frac{\eta'}{2r+1} = \zeta + \frac{2\eta'}{2r+1}.\end{aligned}$$

which contradicts (4). Therefore $(p_m)_{m=1}^{\infty}$ is a Cauchy sequence in a complete metric space $(\mathbb{X}, \tau)$, and the sequence $(p_m)_{m=1}^{\infty}$ converges to a point $\bar{p}$ in $\mathbb{X}$. Then, we find infinitely many elements of the sequence $(p_m)_{m=1}^{\infty}$ remain in each C_j. Hence $\bar{p} \in \bigcap_{j=1}^{r} C_j$. If $p_m = \bar{p}$ for some m, then $\bar{p}$ is a fixed point of g, otherwise

$$\tau(p_{m+1}, g(\bar{p})) = \tau(g(p_m), g(\bar{p})) < \ \tau(p_m, \bar{p}) \text{ for } m \in \mathbb{N}.$$

We conclude $(p_{m+1})_{m=1}^{\infty}$ converges to $g(\bar{p})$ and hence, $\bar{p}$ is a fixed point of g .
Suppose there exist two different fixed points p and q, then $p, q \in \bigcap_{j=1}^{r} C_j$. Then, we have arrived at contradiction: $\tau(p, q) = \tau(g(p), g(q)) < \tau(p, q)$. □

4. Cyclic Meir-Keeler Iterated Function Systems

Lemma 4.1. *Let $P, Q \in \mathbb{H}(\mathbb{X})$. Then for given $p \in P$, we can find a $q \in Q$ such that $\tau(p, q) \leq h(P, Q)$.*

Proof. Let $p \in P$ be arbitrary. Since Q is compact, there exists $q \in Q$ for which $\tau(p, q) = \inf_{q' \in Q} \tau(p, q') \leq \mathbb{D}(P, Q) \leq h(P, Q)$. □

Lemma 4.2. *Suppose P is closed in a metric space $(\mathbb{X}, \tau)$ which is complete. Then, $\mathbb{H}(P)$ is closed in $(\mathbb{H}(\mathbb{X}), h)$.*

Proof. Since P is a closed in a complete metric space, P is complete. Therefore, $(\mathbb{H}(P), h)$ is also complete, and the result follows. □

Theorem 4.3. *Let $C_j \subset \mathbb{X}$ and $C_j \neq \emptyset$ for $j \in \mathbb{N}_r$ with $C_{r+1} = C_1$, where $(\mathbb{X}, \tau)$ is a metric space. Let $g_k : \bigcup_{j=1}^{r} C_j \to \bigcup_{j=1}^{r} C_j$, $k \in \mathbb{N}_M$ be continuous cyclic Meir-Keeler contraction maps. Then Hutchinson map $\mathbb{G} : \bigcup_{j=1}^{r} \mathbb{H}(C_j) \to \bigcup_{j=1}^{r} \mathbb{H}(C_j)$ defined by $\mathbb{G}(C) := \bigcup_{k=1}^{M} g_k(C)$ for every $C \in \bigcup_{j=1}^{r} \mathbb{H}(C_j)$, (where $g_k(C) = \{g_k(p) : p \in C\}$) is a cyclic Meir-Keeler contraction map with respect to the induced metric h defined in (1).*

Proof. Let $P \in \mathbb{H}(C_j)$ for some $j \in \mathbb{N}_r$. Since each g_k is cyclic, we have $\mathbb{G}(P) \subseteq C_{j+1}$. Since each g_k is continuous, then $\mathbb{G}(P)$ is a compact set. Thus $\mathbb{G}(P) \in \mathbb{H}(C_{j+1})$, and we conclude that $\mathbb{G}(\mathbb{H}(C_j)) \subseteq \mathbb{H}(C_{j+1})$ for each $j \in \mathbb{N}_r$. For given $\zeta > 0$, we can find a $\eta_k > 0$, $k \in \mathbb{N}_M$ for whoch the following is true:

$$\zeta \leq \tau(p, q) < \zeta + \eta_k \Rightarrow \tau(g_k(p), g_k(q)) < \zeta, \forall p \in C_j, q \in C_{j+1}, j \in \mathbb{N}_r.$$

Let $P \in \mathbb{H}(C_j), Q \in \mathbb{H}(C_{j+1})$ such that $\zeta \leq h(P, Q) < \zeta + \eta$, where $\eta = \min\{\eta_k : k \in \mathbb{N}_M\}$. Our claim is $h(\mathbb{G}(P), \mathbb{G}(Q)) < \zeta$.
Let $v \in \mathbb{G}(P)$ be arbitrary. Then there exists $l \in \mathbb{N}_M$ and $p \in P \subset C_j$ such that $v = g_l(p)$. By Lemma 4.1, there exists $q \in Q \subset C_{j+1}$ satisfying $\tau(p, q) \leq h(P, Q) < \zeta + \eta$. If $\tau(p, q) \geq \zeta$, then $\zeta \leq \tau(p, q) < \zeta + \eta$ and $p \in C_j, q \in C_{j+1}$ implies $\tau(g_l(p), g_l(q)) < \zeta$. Otherwise $\tau(p, q) < \zeta$, then $\tau(g_l(p), g_l(q)) < \tau(p, q) < \zeta$.
Therefore $\tau(v, \mathbb{G}(Q)) < \zeta$. Since $v \in \mathbb{G}(P)$ is arbitrary and $\mathbb{G}(P)$ is compact, we obtain

that $\mathbb{D}(\mathbb{G}(P),\mathbb{G}(Q)) < \zeta$. The proof of $\mathbb{D}(\mathbb{G}(Q),\mathbb{G}(P)) < \zeta$ follows from similar lines. Consequently, we obtain $h(\mathbb{G}(P),\mathbb{G}(Q)) < \zeta$ that verifies $\mathbb{G}$ is also a cyclic Meir-Keeler contraction. □

Corollary 4.4. *Let $C_j \subset \mathbb{X}$ and $C_j \neq \emptyset$ be closed for $j \in \mathbb{N}_r$ with $C_{r+1} = C_1$, where $(\mathbb{X},\tau)$ is a complete metric space. Let $g_k : \overset{r}{\underset{j=1}{\cup}} C_j \to \overset{r}{\underset{j=1}{\cup}} C_j$, $k \in \mathbb{N}_M$, be continuous cyclic Meir-Keeler contraction maps. Define $\mathbb{G} : \overset{r}{\underset{j=1}{\cup}} \mathbb{H}(C_j) \to \overset{r}{\underset{j=1}{\cup}} \mathbb{H}(C_j)$ by $\mathbb{G}(C) := \overset{M}{\underset{k=1}{\cup}} g_k(C)$ for every $C \in \overset{r}{\underset{j=1}{\cup}} \mathbb{H}(C_j)$. Then, the unique fixed point K of $\mathbb{G}$ satisfies the self-affine equation*

$$K = \mathbb{G}(K) = \overset{M}{\underset{k=1}{\cup}} g_k(K).$$

Moreover, the attractor can be obtained as $K = \lim_{m\to\infty} \mathbb{G}^m(C)$ for any $C \in \overset{r}{\underset{j=1}{\cup}} \mathbb{H}(C_j)$.

Proof. We know that since the original space $(\mathbb{X},\tau)$ is a complete, then so is the hyperspace $(\mathbb{H}(\mathbb{X}), h)$. According to Lemma 4.2, we find that all non-empty subsets $\mathbb{H}(C_j)$, $j \in \mathbb{N}_r$ are closed in $\mathbb{H}(\mathbb{X})$. Employing Theorem 3.4, we conclude that the proposed set-valued map $\mathbb{G}$ is a cyclic Meir-Keeler contraction on $\overset{r}{\underset{j=1}{\cup}} \mathbb{H}(C_j)$. The results pertaining to the unique fixed point K follow from Theorem 3.1. □

Definition 4.5. *Consider a finite collection of nonempty closed subsets $C_1, C_2 \dots C_r$, $r \in \mathbb{N}$ of a complete metric space $(\mathbb{X},\tau)$, where $C_{r+1} = C_1$. Let g_k, $k \in \mathbb{N}_M$ be a finite number of continuous cyclic Meir-Keeler contraction mappings on $\overset{r}{\underset{j=1}{\cup}} C_j$. Then, we call these collection $\{(\mathbb{X}, C_1, C_2, \dots C_r); g_k : k = 1, 2, \dots M\}$ as a cyclic Meir-Keeler IFS, and we denote it by I_{CMK}.*
Define the associated Hutchinson operator $\mathbb{G}$ on $\overset{r}{\underset{j=1}{\cup}} \mathbb{H}(C_j)$ for the above IFS I_{CMK} by $\mathbb{G}(C) = \overset{M}{\underset{k=1}{\cup}} g_k(C)$ for each $C \in \overset{r}{\underset{j=1}{\cup}} \mathbb{H}(C_j)$. According to Corollary 4.4, the fixed point K of $\mathbb{G}$ is given by $K = \lim_{m\to\infty} \mathbb{G}^m(C)$ for any $C \in \overset{r}{\underset{j=1}{\cup}} \mathbb{H}(C_j)$ and K is called the fractal of this cyclic Meir-Keeler IFS I_{CMK}.

Example 4.6. *Let $\mathbb{X} = \mathbb{R}^2$ and $C_1 = [-1,1] \times \mathbb{R}$, $C_2 = [-0.5,1] \times \mathbb{R}$. Let g_2 be the linear spline to the data $\{(0,1), (\frac{1}{5},6), (\frac{2}{5},2), (\frac{3}{5},7), (\frac{4}{5},1), (1,0)\}$. Let g_1 be the line joining $(0,1)$ and $(1,0)$. Let $p_j = \frac{j}{5}$ for $j \in \mathbb{N}_5$. Let $L_j : [p_0, p_5] \to [p_{j-1}, p_j]$ as $L_j(p) = a_j p + b_j$ for $j \in \mathbb{N}_5$. Define $\mathbb{G}_j(p,q) = \alpha_j q + g_2(L_j(p)) - g_1(p)$ for $j \in \mathbb{N}_5$. Define $g_j(p,q) = (L_j(p), F_j(p,q))$ for $j \in \mathbb{N}_5$. If the scaling factors are chosen as $\alpha_j = 0.3$ for $j \in \mathbb{N}_5$, then according to Corollary 4.4, the fixed point K of the cyclic Meir-Keeler IFS $I_{CMK} \equiv \{(\mathbb{X}, C_1, C_2); g_j, j \in \mathbb{N}_5\}$ is the required fractal function, and its graph is plotted in Figure 1 by using Section 2.*

In the following, we consider a countable collection of maps $\{g_m\}_{m=1}^{\infty}$ and $\mathbb{X}$ is compact.

Theorem 4.7. *Let $C_j \subset \mathbb{X}$ and $C_j \neq \emptyset$ be closed for $j \in \mathbb{N}_r$ with $C_{r+1} = C_1$, where $(\mathbb{X},\tau)$ is a complete metric space. Let $(g_k)_{k=1}^{\infty}$ be a sequence of cyclic Meir-Keeler*

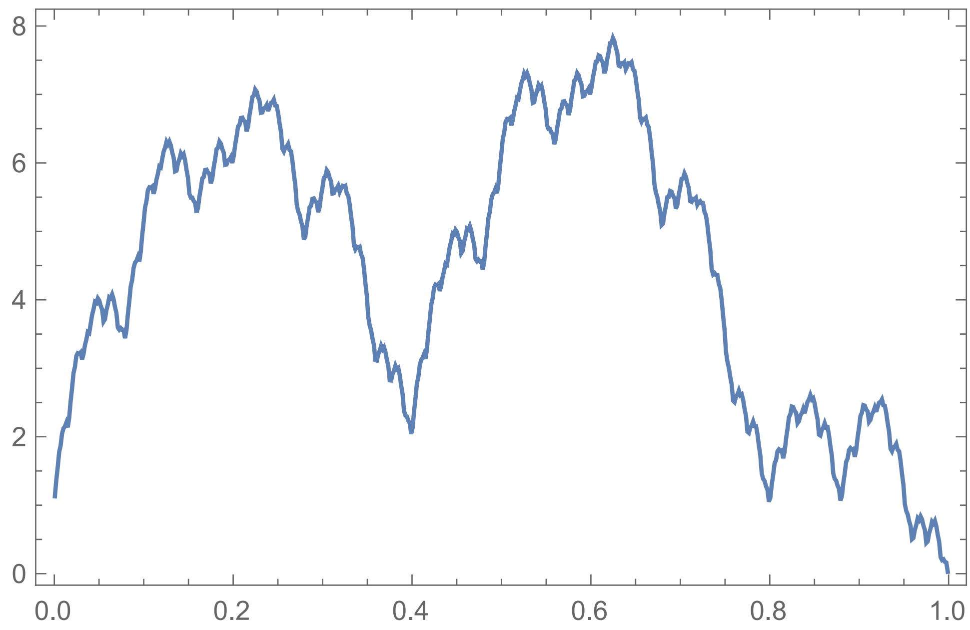

Figure 1. Attractor of cyclic IFS which is graph of a fractal function

contraction mappings on $\bigcup_{j=1}^{r} C_j$ *obeying the following condition:*
For any $\zeta > 0$*, there exists* $\eta > 0$ *for which*

$$\forall p \in C_j, q \in C_{j+1} \text{ and } \zeta \leq \tau(p,q) < \zeta + \eta \implies \sup_k \tau(g_k(p), g_k(q)) < \zeta. \tag{5}$$

Then, the set-valued map $\mathbb{G}$ *on* $\bigcup_{j=1}^{r} \mathbb{H}(C_j)$ *defined by* $\mathbb{G}(C) := \overline{\bigcup_{k\geqslant 1} g_k(C)}$ *for every* $C \in \bigcup_{j=1}^{r} \mathbb{H}(C_j)$ *is a cyclic Meir-Keeler contraction, where the bar is used to write the topological closure of the union of sets.*

Proof. Suppose $C \in \mathbb{H}(C_j)$ for some $j \in \mathbb{N}_r$. Since all $g_k, k \in \mathbb{N}$ are cyclic Meir-Keeler contraction maps, we have $\bigcup_{k\geqslant 1} g_k(C) \subseteq C_{j+1}$. Now C_{j+1} is closed, and $\mathbb{G}(C) = \overline{\bigcup_{k\geqslant 1} g_k(C)} \subseteq C_{j+1}$. Since $\mathbb{X}$ is compact, then so is C_{j+1}. Consequently, we obtain $\mathbb{G}(C) \in \mathbb{H}(C_{j+1})$. Therefore we have $\mathbb{G}$ is a cyclic map on $\{\mathbb{H}(C_j)\}_{j=1}^{r}$.
Suppose that for given $\zeta > 0$, there exists $\eta > 0$ such that it satisfies (5). Consider two sets P and Q such that $P \in \mathbb{H}(C_j), Q \in \mathbb{H}(C_{j+1})$ for some $j \in \mathbb{N}_r$ for which $\zeta \leq h(P,Q) < \zeta + \eta$. Using compactness of P and Q, we can find $p \in P$ and $q \in Q$ such that $\tau(p,q) \leq h(P,Q) < \zeta + \eta$.
Whenever $\zeta \leq \tau(p,q)$, we get $\zeta \leq \tau(p,q) < \zeta + \eta$. In this case for $p \in C_j, q \in C_{j+1}$, we have $\sup_k \tau(g_k(p), g_k(q)) < \zeta$. Otherwise if $\tau(p,q) < \zeta$, then $\tau(g_k(p), g_k(q)) < \tau(p,q) < \zeta$ for all $k \in \mathbb{N}$. Thus,

$$\sup_k \tau(g_k(p), g_k(q)) < \zeta \text{ for } p \in C_j, q \in C_{j+1}.$$

Therefore for any $p \in P$ we have $\inf_{q\in Q}\sup_{k} \tau(g_k(p), g_k(q)) < \zeta$. Since $P \in \mathbb{H}(C_j)$, we conclude that $\sup_{p\in P}\inf_{q\in Q}\sup_{k} \tau(g_k(p), g_k(q)) < \zeta$. Using this result, it is straight forward to see

$$\sup_{k} \mathbb{D}(g_k(P), g_k(Q)) = \sup_{k}\sup_{p\in P}\inf_{q\in Q} \tau(g_k(p), g_k(q)) \leq \sup_{p\in P}\inf_{q\in Q}\sup_{k} \tau(g_k(p), g_k(q)) < \zeta.$$

Similarly, $\sup_{k} \mathbb{D}(g_k(Q), g_k(P)) < \zeta$, and we obtain $\sup_k h(g_k(P), g_k(Q)) < \zeta$. Finally, using the standard property of Hausdorff metric, we get

$$h(\mathbb{G}(P), \mathbb{G}(Q)) = h(\overline{\underset{k\geqslant 1}{\cup} g_k(P)}, \overline{\underset{k\geqslant 1}{\cup} g_k(Q)}) \leq \sup_{k} h(g_k(P), g_k(Q)) < \zeta.$$

Thus, $\mathbb{G}$ verifies all the conditions of a cyclic Meir-Keeler contraction map on $\{\mathbb{H}(C_j)\}_{j=1}^{r}$. □

Countable Cyclic Meir-Keeler Iterated Function Systems: Suppose $C_j \subset \mathbb{X}$ and $C_j \neq \emptyset$ is closed for $j \in \mathbb{N}_r$ with $C_{r+1} = C_1$, where $(\mathbb{X}, \tau)$ is a compact metric space. Suppose a sequence of cyclic maps $(g_k)_{k\geqslant 1}$ on $\{C_j\}_{j=1}^{r}$ obeys the following:

For given $\zeta > 0$, there exists $\eta > 0$ such that for all $p \in C_j, q \in C_{j+1}$,

$$\zeta \leq \tau(p, q) < \zeta + \eta \text{ implies } \sup_{k} \tau(g_k(p), g_k(q)) < \zeta.$$

Then, $\{(\mathbb{X}, C_1, C_2, \dots C_r); g_k : k \in \mathbb{N}\}$ is termed as a *countable cyclic Meir-Keeler IFS*. In this case, the Hutchinson operator $\mathbb{G}$ is defined on $\overset{r}{\underset{j=1}{\cup}} \mathbb{H}(C_j)$ as

$$\mathbb{G}(C) = \overline{\bigcup_{k\geqslant 1} g_k(C)}.$$

Utilizing Theorem 4.7, we observe that $\mathbb{G}$ is a cyclic Meir-Keeler contraction on the complete metric space $\overset{r}{\underset{j=1}{\cup}} \mathbb{H}$ as it is closed in $(\mathbb{H}(\mathbb{X}), h)$. According to Theorem 3.4, we can find a unique non-empty set $K \in \overset{r}{\underset{j=1}{\cap}} \mathbb{H}(C_j)$ satisfying the self-referential relation

$$K = \mathbb{G}(K) = \overline{\bigcup_{k\geqslant 1} g_k(K)} = \lim_{m\to\infty} \mathbb{G}^m(C) \text{ for any non-empty } C \in \overset{r}{\underset{j=1}{\cup}} \mathbb{H}(C_j).$$

Here K is called the fractal or attractor of the countable cyclic Meir-Keeler IFS $\{(\mathbb{X}, C_1, C_2, \dots C_r);$
$g_k : k \in \mathbb{N}\}$.
Note that the functions in the countable cyclic Meir-Keeler IFS need not be continuous. The following is an example of a collection of non-continuous maps which forms a countable cyclic Meir-Keeler IFS.

Example 4.8. *We construct a sequence of functions using Example 3.2:*

Let $C_1 = [0,1], \quad C_2 = [0,3]$. *For all* $m \in \mathbb{N}$, *define* $g_m : C_1 \cup C_2 \to C_1 \cup C_2$ *by*

$$g_m(p) = \begin{cases} \frac{p}{m+1} & \text{if } p \in [0,2], \\ \frac{1}{m+1} & \text{if } p \in (2,3]. \end{cases}$$

Observe that

$$g_m(C_1) = [0, \frac{1}{m+1}] \subseteq C_2, \quad g_m(C_2) = [0, \frac{2}{m+1}] \subseteq C_1.$$

For $\zeta > 0$, *choose* $\eta = \zeta$. *Let* $p \in C_1, \quad q \in C_2$ *and* $\zeta \leq |p - q| < 2\zeta$.
(i) If $p \in [0,1],\ q \in [0,2]$, *then*

$$\sup_m |g_m(p) - g_m(q)| = \sup_m |\frac{p}{m+1} - \frac{q}{m+1}| = \sup_m \frac{1}{m+1}|p-q| < \frac{1}{2}(2\zeta) = \zeta.$$

(ii)If $p \in [0,1],\ q \in (2,3]$, *then*

$$\sup_m |g_m(p) - g_m(q))| = \sup_m |\frac{p}{m+1} - \frac{1}{m+1}| = \sup_m \frac{1}{m+1}|p-1| \leq \frac{1}{2}|p-q| < \zeta.$$

Therefore, the collection $\{g_m\}_{m=1}^{\infty}$ *is a countable cyclic Meir-Keeler IFS with each* g_m*'s being non-continuous.*

5. Generalized Cyclic Meir-Keeler IFS.

Definition 5.1. *Suppose that* $C_j \subset \mathbb{X}$ *and* $C_j \neq \emptyset$ *are closed for* $j \in \mathbb{N}_r$ *with* $C_{r+1} = C_1$, *where* $(\mathbb{X}, \tau)$ *is a metric space. We consider the metric on* $\mathbb{X}^l$ *for* $l \in \mathbb{N}$,

$$\bar{\tau}((p_1, \ldots, p_l), (q_1, \ldots, q_l)) = \max\{\tau(p_1, q_1), \ldots, \tau(p_l, q_l)\}.$$

The induced product map $g : (\overset{r}{\underset{j=1}{\cup}} C_j)^l \to \overset{r}{\underset{j=1}{\cup}} C_j$ *is called a generalized cyclic Meir-Keeler contraction on* $\{C_j\}_{j=1}^r$ *if* g *satisfies:*

1. *(1)* $g(C_j \times \ldots \times C_j) \subseteq C_{j+1}$ *for* $1 \leq j \leq r$,
2. *(2) for all* $\zeta > 0$, $\exists \eta > 0$ *such that for each* $i \in \mathbb{N}_l$ *and* $j_i \in \mathbb{N}_r, p_i \in P_{j_i}, q_i \in P_{j_i+1}$,
$\zeta \leq \bar{\tau}((p_1, \ldots, p_l), (q_1, \ldots, q_l)) < \zeta + \eta \implies \tau(g(p_1, ..., p_l), g(q_1, ..., q_l)) < \zeta$.

If this product map g *obeys only the first axiom* (1), *then* g *is termed as a generalized cyclic map.*

Note that, for $l = 1$, the generalized cyclic Meir-Keeler contraction coincide with the classical cyclic Meir-Keeler contraction. Thus, each cyclic Meir-Keeler contraction map is a generalized cyclic Meir-Keeler contraction map.

Example 5.2. *For* $0 \leq k < 1$, *let* $C_1 = [0, \frac{1}{2}]$ *and* $C_2 = [0,1]$.

Define $g : C_1 \cup C_2 \times C_1 \cup C_2 \to C_1 \cup C_2$ *by*

$$g(p,q) = \begin{cases} kp & \text{if } (p,q) \in [0, \frac{1}{2}] \times [0,1], \\ \frac{k}{2} & \text{if } (p,q) \in (\frac{1}{2}, \frac{3}{4}] \times [0,1], \\ \frac{k}{4} & \text{if } (p,q) \in (\frac{3}{4}, 1] \times [0,1]. \end{cases}$$

Observe that $g(C_1 \times C_1) \subseteq C_2$ *and* $g(C_2 \times C_2) \subseteq C_1$.
For $\zeta > 0$, *choose* $\eta = \frac{1-k}{k}\zeta$.
We want to prove for $j \in \{1,2\}$, $p_j \in C_j$, $q_j \in C_{j+1}$,

$$\zeta \leq \max\{|p_1 - q_1|,\ |p_2 - q_2|\} < \frac{\zeta}{k} \ \text{ implies } \ |f(p_1,p_2) - f(q_1,q_2)| < \zeta.$$

We discuss only one case: $(p_1,p_2) \in C_1 \times C_1$, $(q_1,q_2) \in C_2 \times C_2$. *The other cases follow similarly.*
Sub-case (i) : If $(q_1,q_2) \in [0, \frac{3}{4}] \times [0,1]$, *then*

$$|g(p_1,p_2) - g(q_1,q_2)| \leq k|p_1 - q_1| \leq k \max\{|p_1 - q_1|,\ |p_2 - q_2|\} < \zeta.$$

Sub-case (ii) : If $(q_1,q_2) \in (\frac{3}{4}, 1] \times [0,1]$, *then*

$$|g(p_1,p_2) - g(q_1,q_2)| \leq k|p_1 - \frac{1}{4}| \leq k|p_1 - q_1| \leq k \max\{|p_1 - q_1|,\ |p_2 - q_2|\} < \zeta.$$

Thus, g is a generalized cyclic Meir-Keeler contraction map.

Definition 5.3. *Let* $g : \mathbb{X}^l \to \mathbb{X}$ *a function for some* $l \in \mathbb{N}$. *Then we declare that* $p \in \mathbb{X}$ *is a fixed point for* g *if* $g(p,\ldots,p) = p$.

Theorem 5.4. *Let* $C_j \subset \mathbb{X}$ *and* $C_j \neq \emptyset$ *be closed for* $j \in \mathbb{N}_r$ *with* $C_{r+1} = C_1$, *where* $(\mathbb{X}, \tau)$ *is a complete metric space. If* $g : (\overset{r}{\underset{j=1}{\cup}} C_j)^l \to \overset{r}{\underset{j=1}{\cup}} C_j$ *is a generalized cyclic Meir-Keeler contraction map, then* g *possesses a unique fixed point* p^* *(say). Additionally, if for every* $p_0 \in \overset{r}{\underset{i=1}{\cup}} C_j$, *consider* $p_m := g(p_{m-1},\ldots,p_{m-1})$, $m \in \mathbb{N}$, *then the sequence* $(p_m)_{m=1}^{\infty}$ *converges to this fixed point* p^*.

Proof. Define $T : \overset{r}{\underset{j=1}{\cup}} C_j \to \overset{r}{\underset{j=1}{\cup}} C_j$ by $T(p) = g(p,...,p)$ for all $p \in \overset{r}{\underset{j=1}{\cup}} C_j$.
Now our claim is T is cyclic Meir-Keeler contraction on $\overset{r}{\underset{j=1}{\cup}} C_j$. Observe that $T(C_j) \subset g(C_j \times ... \times C_j)$ for each $j \in \mathbb{N}_r$. Since T is a generalized cyclic map, $g(C_j \times ... \times C_j) \subset C_{j+1}$, and this implies $T(C_j) \subset C_{j+1}$ for all $j \in \mathbb{N}_r$.
Since T is cyclic Meir-Keeler contraction, for given $\zeta > 0$, we can find a $\eta > 0$ so that for each $i \in \mathbb{N}_l$ and $j_i \in \mathbb{N}_r, p_i \in C_{j_i}, q_i \in C_{j_i+1}$,
$\zeta \leq \bar{\tau}((p_1,\ldots,p_l),(q_1,\ldots,q_l)) < \zeta + \eta \implies \tau(g(p_1,...,p_l), g(q_1,...,q_l)) < \zeta$.
Let $p \in C_j, q \in C_{j+1}$ for arbitrary $j \in \mathbb{N}_r$. Assume that $\zeta \leq \tau(p,q) < \zeta + \eta$. Then we get
$\zeta \leq \bar{\tau}((p,...,p),(q,...,q)) < \zeta + \eta$ which implies $\tau(g(p,...,p), g(q,...,q)) < \zeta$, and this condition is equivalent to $\tau(T(p),T(q)) < \zeta$. Thus, we have proved that T is a cyclic Meir-Keeler contraction on $\overset{r}{\underset{j=1}{\cup}} C_j$.

According to Theorem 3.4, T possesses a unique fixed point p^* for which $T(p^*) = p^*$ and for all $p \in \bigcup_{j=1}^{r} C_j, T^m(p) \to p^*$ as $m \to \infty$. Consequently, $g(p^*, ..., p^*) = p^*$ confirms that g has a fixed point p^*. For any $p \in \bigcup_{j=1}^{r} C_j$, define $p_1 := g(p, ..., p)$ and $p_m := g(p_{m-1}, ..., p_{m-1})$ for $m \in \mathbb{N}$. Then this sequence $(p_m)_{m=1}^{\infty}$ converges to this unique fixed point p^*.
For the uniqueness of this fixed point, if there exist two fixed points $p^*, q^* \in \bigcup_{j=1}^{r} C_j$ such that $g(p^*, ..., p^*) = p^*$ and $g(q^*, ..., q^*) = q^*$, then we can write $T(p^*) = p^*$ and $T(q^*) = q^*$. But the fixed point of T is unique, and hence $p^* = q^*$. □

Theorem 5.5. *Let $C_j \subset \mathbb{X}$ and $C_j \neq \emptyset$ for $j \in \mathbb{N}_r$ with $C_{r+1} = C_1$, where $(\mathbb{X}, \tau)$ is a metric space. If $g_k : (\bigcup_{j=1}^{r} C_j)^l \to \bigcup_{j=1}^{r} C_j$ are continuous generalized cyclic Meir-Keeler contraction maps, then the Hutchinson map $\mathbb{G} : (\bigcup_{j=1}^{r} \mathbb{H}(C_j))^l \to \bigcup_{j=1}^{r} \mathbb{H}(C_j)$ is again a generalized cyclic Meir-Keeler contraction map with respect to the induced Hausdorff metric h, where $\mathbb{G}$ is defined as $\mathbb{G}(Q_1 \times ... \times Q_l) := \bigcup_{k=1}^{M} g_k(Q_1 \times ... \times Q_l)$ for every $Q_1 \times ... \times Q_l \in (\bigcup_{j=1}^{r} \mathbb{H}(C_j))^l$ and $g_k(Q_1 \times ... \times Q_l) = \{g_k(p_1, ..., p_l) : p_j \in Q_j, \forall j \in \mathbb{N}_l\}$.*

Proof. Let $Q_i \in \mathbb{H}(C_j)$ for some $j \in \mathbb{N}_r$ and for each $i \in \mathbb{N}_l$. By generalized cyclic condition of $g_k^{\prime}s$,

$$\mathbb{G}(Q_1 \times ... \times Q_l) \subseteq \mathbb{G}(C_j \times ... \times C_j) \subseteq C_{j+1}$$

and by continuity of $g_k^{\prime}s$, $\mathbb{G}(Q_1 \times ... \times Q_l) \in \mathbb{H}(C_{j+1})$. This implies $\mathbb{G}(\mathbb{H}(C_j) \times ... \times \mathbb{H}(C_j)) \subseteq \mathbb{H}(C_{j+1})$ for each $j \in \mathbb{N}_r$.
We know that for given $\zeta > 0$, there exists $\eta_k > 0$ for each $k \in \mathbb{N}_M$ so that for each $i \in \mathbb{N}_l$ and $j_i \in \mathbb{N}_r, p_i \in C_{j_i}, q_i \in C_{j_i+1}, \zeta \leq \bar{\tau}((p_1, \ldots, p_l), (q_1, \ldots, q_l)) < \zeta + \eta_k \implies \tau(g_k(p_1, ..., p_l), g_k(q_1, ..., q_l)) < \zeta$.
Let $P_i \in \mathbb{H}(C_{j_i}), Q_i \in \mathbb{H}(C_{j_i+1})$ such that for each $i \in \mathbb{N}_l$ and corresponding $j_i \in \mathbb{N}_r$, assume that the following relation is true:

$$\zeta \leq \max_{1 \leq i \leq l} \{h(P_i, Q_i)\} < \zeta + \eta, \text{ where } \eta := \max\{\eta_k : k \in \mathbb{N}_M\}.$$

Let $z \in \mathbb{G}(P_1 \times ... \times P_l)$, then there exist $k \in \mathbb{N}_M$ and $p_i \in P_i$, for each $i \in \mathbb{N}_l$, such that $g_k(p_1, ..., p_l) = z$. By Lemma 4.1 there exists $q_i \in Q_i$ such that for each $i \in \mathbb{N}_l, \tau(p_i, q_i) \leq h(P_i, Q_i) < \zeta + \eta$.
If there exists $i \in \mathbb{N}_l$ such that $\zeta \leq \tau(p_i, q_i)$, then $\zeta \leq \max_{1 \leq i \leq l} \{\tau(p_i, q_i)\} < \zeta + \eta$, and in this case for $p_i \in C_{j_i}, q_i \in C_{j_i+1}$ for each $i \in \mathbb{N}_l$, we obtain

$$\forall k \in \mathbb{N}_M, \;\; \tau(g_k(p_1, ..., p_l), g_k(q_1, ..., q_l)) < \zeta.$$

Otherwise, for all $i \in \mathbb{N}_l, \tau(p_i, q_i) < \zeta$ and for the case all $i \in \mathbb{N}_l, p_i = q_i$ is trivial. Therefore assume that there exists $i \in \mathbb{N}_l$ such that $p_i \neq q_i$. Let $\eta = \bar{\tau}((p_1, ..., p_l), (q_1, ..., q_l))$. We know $p_i \in C_{j_i}, q_i \in C_{j_i+1}$ for each $i \in \mathbb{N}_l$ implies

$$\tau(g_k(p_1, ..., p_l), g_k(q_1, ..., q_l)) < \eta = \max_{1 \leq i \leq l} \{\tau(p_i, q_i)\} < \zeta.$$

From this we conclude that $\tau(g_k(p_1,...,p_l),\mathbb{G}(Q_1\times...\times Q_l))<\zeta$. Since $\mathbb{G}(\mathbb{X}_1\times...\times\mathbb{X}_m)$ is compact, we have $\mathbb{D}(\mathbb{G}(P_1\times...\times P_l),\mathbb{G}(Q_1\times...\times Q_l))<\zeta$. Similarly, one can prove that $\mathbb{D}(\mathbb{G}(Q_1\times...\times Q_l),\mathbb{G}(P_1\times...\times P_l))<\zeta$. Consequently, we conclude that $h(\mathbb{G}(P_1\times...\times P_l),\mathbb{G}(Q_1\times...\times Q_l))<\zeta$ to confirm G is a generalized cyclic Meir-Keeler contraction map. □

Corollary 5.6. *Let $C_j\subset\mathbb{X}$ and $C_j\neq\emptyset$ be non-empty for $j\in\mathbb{N}_r$ with $C_{r+1}=C_1$, where $(\mathbb{X},\tau)$ is a complete metric space. Let $g_k:(\bigcup_{j=1}^{r}C_j)^l\to\bigcup_{j=1}^{r}C_j$, $k\in\mathbb{N}_M$ be continuous generalized cyclic Meir-Keeler contraction maps. Define $\mathbb{G}:(\bigcup_{j=1}^{r}\mathbb{H}(C_j))^l\to\bigcup_{j=1}^{r}\mathbb{H}(C_j)$ by $\mathbb{G}(C_1\times\ldots\times C_l):=\bigcup_{k=1}^{M}g_k(C_1\times\ldots\times C_l)$, for every $C_1\times\ldots\times C_l\in(\bigcup_{j=1}^{r}\mathbb{H}(C_j))^l$. Then $\mathbb{G}$ has a unique fractal K (say). That is*

$$K=\mathbb{G}(K\times\ldots\times K)=\bigcup_{k=1}^{M}g_k(K\times\ldots\times K).$$

Moreover, for every $P_0\in\bigcup_{j=1}^{r}\mathbb{H}(C_j)$, consider $P_m:=\mathbb{G}(P_{m-1},\ldots,P_{m-1})$ for each $m\in\mathbb{N}$, then the sequence $(P_m)_{m=1}^{\infty}$ converges to K.

Definition 5.7. *Suppose that $C_j\subset\mathbb{X}$ and $C_j\neq\emptyset$ are closed for $j\in\mathbb{N}_r$ with $C_{r+1}=C_1$, where $(\mathbb{X},\tau)$ is a complete metric space. For $l\in\mathbb{N},M\in\mathbb{N}$, let $g_k:(\bigcup_{j=1}^{r}C_j)^l\to\bigcup_{j=1}^{r}C_j$, $k\in\mathbb{N}_M$ be the generalized cyclic Meir-Keeler contraction on $\{C_j\}_{j=1}^{r}$. Then, $I_{GCMK}=\{\mathbb{X}^l,(C_1,C_2,\ldots,C_r),;g_k,k\in\mathbb{N}_M\}$ is called a generalized cyclic Meir-Keeler IFS. According to Corollary 5.6, the fractal of I_{GCMK} is unique.*

6. Conclusion

In this paper, we introduced a new type of non-linear contraction namely cyclic Meir-Keeler contraction, which is a more general class of Banach contraction and we studied the existence and uniqueness of fixed point for the cyclic Meir-Keeler contraction mappings. For the application of fractal, we developed new IFSs consisting of cyclic Meir-Keeler contraction called cyclic Meir-Keeler IFS and countable cyclic Meir-Keeler IFS, which are strict generalization of classical Hutchinson-Barnsley theory of IFS and countable IFS respectively. The existence and uniqueness of attractor for these IFSs were proved. Finally, we proposed a generalized cyclic Meir-Keeler contraction to construct generalized cyclic Meir-Keeler IFS.

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