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Magnonic Goos–Hänchen Effect Induced by 1D Solitons

Victor Laliena and Javier Campo**

The spin wave spectral problem is solved in terms of the spectrum of a diagonalizable operator for a class of magnetic states that includes several types of domain walls and the chiral solitons of monoaxial helimagnets. Focusing on these latter solitons, it is shown that the spin waves reflected and transmitted by them suffer a lateral displacement analogous to the Goos-Hänchen effect of optics. The displacement is a fraction of the wavelength, but can be greatly enhanced by using an array of well separated solitons. Contrarily to the Goos–Hänchen effect recently studied in some magnetic systems, which takes place at the interfaces between different magnetic systems, the effect predicted here takes place at the soliton position, which is interesting for applications since solitons can be created at different places and moved across the material by suitable means. Moreover, the effect predicted here is not particular to monoaxial helimagnets, but it is generic of 1D solitons, although it is accidentally absent in the domain walls of ferromagnets with uniaxial anisotropy. Even though in this work the dipolar interaction is ignored for simplicity, we argue that the Goos–Hänchen shift is also present when it is taken into account.

1. Introduction

Magnonics has been a subject of much interest in recent years since it is a promising field that could transform the design of devices for information technology.[1] Replacing electric currents by spin waves as information carriers in electronic devices would imply a large reduction of heat production and energy consumption, due to the absence of Joule heating. Conceptual designs of devices based on spin waves have already been proposed.[2,3] One of the main challenges with spin waves involves their control and manipulation. This control can be achieved in part by using the magnetic modulations of nanometric scale that are (meta)stable in some materials: domain walls, skyrmions, or the chiral solitons of monoaxial helimagnets. These solitonic states appear easily in chiral magnets, characterized by the presence of an important Dzyaloshinskii–Moriya interaction (DMI). Domain walls and their magnonics, with and without DMI, are being extensively studied.[4–12] Comparatively, monoaxial helimagnets, in which the DMI acts only along one axis (the DMI axis), have received much less attention, although many experimental and theoretical results concerning their equilibrium[13–31] and dynamical[32–44] properties have been obtained.

Generically, the theoretical study of spin waves in noncollinear states faces some mathematical difficulties related to the nature of the spin wave equation. The problem is not merely technical, but rather raises the question of whether a spectral representation for the spin waves exists in general, that is, whether a general solution of the linearized Landau–Lifshitz–Gilbert (LLG) equation can be expressed as a combination of well defined spin wave modes.

In this work we develop a method that provides the solution of the spectral spin wave problem in terms of the spectrum of a diagonalizable linear operator, for special cases including the domain walls of many systems and the isolated soliton and the chiral soliton lattice (CSL) of monoaxial helimagnets. By applying this method to the spin wave scattering by a soliton in a monoaxial helimagnet, we predict the existence of a lateral displacement of the scattered waves analogous to the Goos–Hänchen effect of optics. We argue that this lateral shift is a generic feature of the scattering by 1D solitons. Before presenting the method, in Section 2 we analyze the general problem of magnonics, proving that the spectral representation of spin waves should exist in general. In Section 3 we present the method, which applies to the interesting class of problems described above. In Section 4 the method is applied to the characterization of the spin wave spectrum of an isolated soliton in a monoaxial helimagnet. Section 5 is devoted to the detailed study of the scattering of spin waves by these chiral solitons, and to the analysis of the Goos–Hänchen effect. Finally, in Section 6 we summarize the conclusions. Additional developments and results are gathered in Supporting Information.[45]

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2. General Features

Consider a generic magnetic system described by a magnetization vector field $\boldsymbol{M} = M_s\boldsymbol{n}$, with constant modulus, M_s, and direction given by the unit vector $\boldsymbol{n}$. Its energy is given by an energy functional $\mathcal{E}[\boldsymbol{n}]$. The stationary states are those at which the

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variational derivative of $\mathcal{E}[\boldsymbol{n}]$ vanishes. The (meta)stable states are the local minima of $\mathcal{E}[\boldsymbol{n}]$, a subset of the stationary states. Let $\boldsymbol{n}_0$ be one stationary point of the energy. Small deviations from $\boldsymbol{n}_0$ can be written in terms of two real fields, ξ_1 and ξ_2, as

$$\boldsymbol{n} = (1 - \xi_1^2 - \xi_2^2)^{1/2}\boldsymbol{n}_0 + \xi_1 \boldsymbol{e}_1 + \xi_2 \boldsymbol{e}_2 \tag{1}$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{n}_0\}$ is an orthonormal triad. These two fields can be grouped into a two-component field, ξ, represented by the column vector $\xi = [\xi_1, \xi_2]^{\mathrm{T}}$. We use the notation

$$(f, g) = \int \mathrm{d}^3 r\, f^*(\boldsymbol{r}) g(\boldsymbol{r}) \tag{2}$$

for the scalar product of two functions and

$$\langle \xi, \eta \rangle = (\xi_1, \eta_1) + (\xi_2, \eta_2) \tag{3}$$

for the scalar product of the two-component fields ξ and η.

Let us expand $\mathcal{E}[\boldsymbol{n}]$ in powers of ξ_i to quadratic order

$$\mathcal{E} = \mathcal{E}_0 + A\langle \xi, K\xi \rangle + O(\xi^3) \tag{4}$$

The linear term in ξ vanishes because $\boldsymbol{n}_0$ is a stationary state. The constant A has dimensions of energy per unit length and K is a Hermitian operator of the form

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{12}^{\dagger} & K_{22} \end{pmatrix} \tag{5}$$

where $K_{\alpha\beta}$ are linear second order integro-differential operators, with K_{11} and K_{22} Hermitian. Second order means that the action of the operator involves second order derivatives of ξ, but not higher order derivatives. The integral part of the operator is due to the nonlocal dipolar interaction.

To avoid the complications induced by the boundary conditions in finite bodies, we consider an infinite system, which approximates a large enough system. Any disturbance, ξ, of the (meta)stable state, $\boldsymbol{n}_0$, has a finite energy, and therefore $\langle \xi, K\xi \rangle$ has to be finite. This condition is satisfied if the components of ξ are square integrable functions. Hence, K acts on the space of two-component fields whose components are square integrable functions defined in the whole $\mathbb{R}^3$.[46] Thus, the boundary conditions are that $\xi(\boldsymbol{r})$ has to vanish rapidly enough as $|\boldsymbol{r}| \to \infty$.

Since $\boldsymbol{n}_0$ is a (meta)stable state, K has to be positive (semi) definite, because any small variation described by ξ has to increase the energy. Had K not be positive semidefinite, there would exist a variation ξ such that $\langle \xi, K\xi \rangle < 0$, and therefore this variation would decrease the energy. The positivity of K requires that K_{11} and K_{22} be positive (semi)definite, as is easily seen by considering variations ξ with $\xi_2 = 0$ and $\xi_1 = 0$, respectively. In addition, the positivity of K imposes certain constraints on K_{12} which are analyzed in the Supporting Information.[45]

The oscillations of the magnetization about the (meta)stable state obey the LLG equation

$$\partial_t \boldsymbol{n} = \gamma \boldsymbol{B}_{\mathit{eff}} \times \boldsymbol{n} + \alpha \boldsymbol{n} \times \partial_t \boldsymbol{n} \tag{6}$$

where $\boldsymbol{B}_{\mathit{eff}} = -(1/M_s)\delta\mathcal{E}/\delta\boldsymbol{n}$ is the effective field, γ the electron gyromagnetic constant, and α the Gilbert damping parameter, which we ignore in this work, setting $\alpha = 0$. Let us pick up some characteristic parameter of the system with units of inverse length, q_0, and introduce the constant $\omega_0 = \gamma 2Aq_0^2/M_s$, with dimensions of inverse time. Considering the small oscillations about $\boldsymbol{n}_0$ given by Equation (1), we expand the LLG equation in powers of ξ. The zeroth order term vanishes since $\boldsymbol{n}_0$ is a stationary point. To linear order we obtain

$$\partial_t \xi = \Omega\xi \tag{7}$$

where $\Omega = (\omega_0/q_0^2)JK$, with

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{8}$$

and I is the identity operator.

One standard way of solving Equation (7) is to find a complete set of eigenstates of Ω, which are solutions of the spectral equation

$$\Omega\xi = \nu\xi \tag{9}$$

where ν is the corresponding eigenvalue. Then $\exp(\nu t)\xi$ is a solution of Equation (7). The general solution of Equation (7) is a linear superposition of these solutions if they form a complete set. However, since Ω is not an anti-Hermitian (not even a normal) operator, it is not guaranteed that a complete set of eigenstates exists, and therefore the general solution of the spin wave equation may not be a linear superposition of eigenmodes.

We give here a formal solution to this problem in the case that K has a strictly positive spectrum (that is, its spectrum lies in the positive real axis and is separated from zero by a finite gap). In this case $K^{1/2}$ is a Hermitian invertible operator. Multiplying both sides of the spectral equation (9) by $K^{1/2}\Omega$, and bearing in mind that $\Omega = (\omega_0/q_0^2)JK$, we obtain

$$(\omega_0^2/q_0^4)Q\, K^{1/2}\, \xi = \nu^2 K^{1/2}\xi \tag{10}$$

where $Q = K^{1/2}JKJK^{1/2}$ is a Hermitian operator. It is easily checked that Q is negative definite,[47] so that $\nu^2 < 0$, and $\nu = \mathrm{i}\omega$, with ω real. Since Ω is a real operator, $-\mathrm{i}\omega$ is also an eigenvalue. This implies that each non zero eigenvalue of Q has an even degeneracy. To make the discussion simple, avoiding irrelevant complications with notation, let us assume that the degeneracy of the nonzero eigenvalues of Q is two.

We have seen in the previous paragraph that each eigenstate ξ of Ω gives an eigenstate, $K^{1/2}\xi$, of Q. The reciprocal is not true, due to the degeneracy of the spectrum of Q. Let η be any eigenstate of Q belonging to the 2D subspace associated to the eigenvalue $-\omega^2$. The state $K^{-1/2}\eta$ is not necessarily an eigenstate of Ω, but the 2D space spanned by the two states $K^{-1/2}\eta$ and $\Omega K^{-1/2}\eta$ is invariant under the action of Ω, since

$$\Omega\left(\Omega K^{-1/2}\eta\right) = K^{-1/2}Q\eta = -\omega^2 K^{-1/2}\eta \tag{11}$$

Then, Ω can be diagonalized within this 2D invariant subspace, and therefore the state $K^{-1/2}\eta$ gives two eigenstates of Ω, characterized by an index $\sigma = \pm 1$, and given by

$$\xi^{(\sigma)} = \mathrm{i}\sigma\,\omega K^{-1/2}\eta + \Omega K^{-1/2}\eta \tag{12}$$

The corresponding eigenvalues are $\mathrm{i}\sigma\omega$, where $\omega > 0$ is the positive square root of ω^2.

Now we notice that $\eta^{(\sigma)} = K^{1/2}\xi^{(\sigma)}$ are two eigenstates of Q that span the subspace of eigenstates associated to the eigenvalue $-\omega^2$. In this way we get a one to one correspondence between the eigenstates of Ω and Q. Since the eigenstates of Q are complete, so are the eigenstates of Ω. More details are given in Supporting Information.[45]

We have proved that the eigenstates of Ω have the form $\xi^{(i\sigma)} = K^{-1/2}\eta^{(i\sigma)}$, where $\eta^{(i\sigma)}$, with $\sigma = \pm 1$, are two degenerate eigenstates of Q which span the 2D subspace associated to the eigenvalue $-\omega_i^2$. The index i labels the eigenvalues of Q, and σ gives the degeneracy. The $\xi^{(i\sigma)}$ satisfy the orthogonality property

$$\left\langle \xi^{(i\sigma)}, K\xi^{(j\sigma')} \right\rangle = N_i^{\sigma\sigma'}\delta_{ij} \tag{13}$$

where the constant $N_i^{\sigma\sigma'}$ depends on how the eigenstates of Q are normalized.[45]

We conclude that for a (meta)stable state for which the spectrum of K is strictly positive, the spectrum of Ω lies on the imaginary axis and its eigenstates form a complete set. This implies that the general solution of the spin wave equation can be obtained as a linear superposition of these eigenmodes.

It is not clear that the operator Q is useful in practice to compute the spectrum of Ω, but it certainly serves to show that, at least in the case that the spectrum of K is strictly positive, Ω has a complete set of eigenstates in terms of which the general solution of the spin wave equation can be expressed.

If K has zero modes, as it happens in solitonic states like domain walls, 1D chiral solitons, and skyrmions, $K^{-1/2}$ is not defined, and the argument presented above is problematic. In the Section 3 we will show how to overcome the difficulty in the case $K_{12} = 0$.

The spectral problem for Ω is easy if the four operators $K_{\alpha\beta}$ commute, as in the ferromagnetic and helical states of monoaxial helimagnets,[48] and in the domain wall of anisotropic ferromagnets.[4] The reason is that in this case the problem is reduced to finding the spectrum of one Hermitian operator (e.g., K_{11}) and the diagonalization of a 2×2 matrix.[45]

The following relatively simple exactly solvable example may be illustrative of the general formalism described in this section.

An example: uniformly magnetized state in an anisotropic ferromagnet

Let us consider a ferromagnet with uniaxial anisotropy, of easy-axis type, along the z axis and under the action of an external magnetic field applied also along the z axis. Its energy is given by

$$\mathcal{E} = \int \mathrm{d}^3 r\left(A\sum_i \partial_i \boldsymbol{n}\cdot\partial_i \boldsymbol{n} - K_\mathrm{u}(\boldsymbol{z}\cdot\boldsymbol{n}) - \mu_0 M_\mathrm{s} H \boldsymbol{z}\cdot\boldsymbol{n} - \frac{\mu_0}{2}M_\mathrm{s}^2\boldsymbol{h}_m\cdot\boldsymbol{n}\right) \tag{14}$$

where A is the exchange stiffness constant, $K_u \geqslant 0$ is the anisotropy constant, M_s is the constant magnetization modulus, H the intensity of the applied field, μ_0 the vacuum permeability, and $\boldsymbol{h}_\mathrm{m}$ the magnetostatic field, in units of M_s, which is the solution of the equations

$$\nabla\times\boldsymbol{h}_\mathrm{m} = 0, \quad \nabla\cdot\boldsymbol{h}_\mathrm{m} = -\nabla\cdot\boldsymbol{n} \tag{15}$$

The effective field is given by

$$\boldsymbol{B}_{\mathit{eff}} = \frac{2A}{M_\mathrm{s}}\left(\nabla^2\boldsymbol{n} + q_a^2(\boldsymbol{z}\cdot\boldsymbol{n})\boldsymbol{z} + q_\mathrm{e}^2\boldsymbol{z} + q_\mathrm{m}^2\boldsymbol{h}_\mathrm{m}\right) \tag{16}$$

where we have introduced the quantities $q_\mathrm{a}^2 = K_u/A$, $q_\mathrm{e}^2 = \mu_0 M_\mathrm{s} H/(2A)$, and $q_m^2 = \mu_0 M_\mathrm{s}^2/(2A)$, which have the dimensions of inverse square length. Notice that q_m is the inverse of the exchange length.

Let us consider a spherical system with a very large radius, which eventually will be sent to infinity. The uniformly magnetized state, with magnetization pointing along the z direction, $\boldsymbol{n}_0 = \boldsymbol{z}$, is an equilibrium state, since it is well known that in this case the magnetostatic field is constant, $\boldsymbol{h}_\mathrm{m} = -(1/3)\boldsymbol{z}$. Let us study the variations (1) about this equilibrium state. Evidently we can choose $\boldsymbol{e}_1 = \boldsymbol{x}$ and $\boldsymbol{e}_2 = \boldsymbol{y}$. By expanding the energy in powers of ξ_α up to second order we obtain the K operator. It will determine the stability of the equilibrium state and the dynamics of the spin waves about $\boldsymbol{n}_0$. Details on the computations, which involve the solution of equations (15) in powers of ξ_α up to second order, are given in Supporting Information.[45] We obtain

$$\mathcal{E} = VA\left(-q_\mathrm{a}^2 - q_\mathrm{e}^2 + \frac{1}{3}q_\mathrm{m}^2\right) + A\int \mathrm{d}^3 r \sum_{\alpha,\beta=1}^{2}\xi_\alpha(\boldsymbol{r})\left(K_{\alpha\beta}\xi_\beta\right)(\boldsymbol{r}) + O(\xi^3) \tag{17}$$

where V is the volume of the system, the linear term in ξ vanishes since $\boldsymbol{n}_0$ is an equilibrium state, and

$$\left(K_{\alpha\beta}\xi_\beta\right)(\boldsymbol{r}) = \delta_{\alpha\beta}\left(-\nabla^2 + q^2\right)\xi_\beta(\boldsymbol{r}) + \frac{q_m^2}{4\pi}\int \mathrm{d}^3 r' \frac{x_\alpha - x'_\alpha}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\partial_{x'_\beta}\xi_\beta(\boldsymbol{r}') \tag{18}$$

where $q^2 = q_\mathrm{a}^2 + 2q_\mathrm{e}^2 - q_\mathrm{m}^2/3$ and we use the obvious notation $x_1 = x$ and $x_2 = y$.

The four operators $K_{\alpha\beta}$ commute and are diagonal in the Fourier basis. Denoting the Fourier transform of $\xi_\alpha(\boldsymbol{r})$ by $\tilde{\xi}_\alpha(\boldsymbol{k})$, where $\boldsymbol{k}$ is the wave vector, the action of the K operator is reduced to the multiplication of the two component field $\tilde{\xi}(\boldsymbol{k})$ by the 2×2 matrix $\tilde{K}$ whose matrix elements are

$$\tilde{K}_{\alpha\beta} = (k^2 + q^2)\delta_{\alpha\beta} + q^2\frac{k_\alpha k_\beta}{k^2} \tag{19}$$

where $k_1 = k_x$, $k_2 = k_y$, and $k^2 = |\boldsymbol{k}|^2$. The eigenvalues of $\tilde{K}$ are

$$\varepsilon_1 = k^2 + q^2, \quad \varepsilon_2 = k^2 + q^2 + q_\mathrm{m}^2(k_x^2 + k_y^2)/k^2 \tag{20}$$

The eigenvectors can be easily obtained, and from them $\tilde{K}^{1/2}$ is readily computed.

The minimum eigenvalue of $\tilde{K}$ is q^2, attained at $k^2 = 0$. Then K is positive definite, and thus the uniformly magnetized state is stable, if $q^2 > 0$, that is if $q_\mathrm{a}^2 + 2q_\mathrm{e}^2 > q_\mathrm{m}^2/3$. Thus, stability is obtained if the combined effect of anisotropy and applied field overcomes the tendency of the magnetostatic interaction to avoid the magnetic poles that appear on the surface of the uniformly magnetized material.

The linearization of the LLG equation around the uniform magnetization $\boldsymbol{n}_0 = \boldsymbol{z}$ gives two coupled linear equations for ξ_α that can be written in terms of the two component field ξ as Equation (7). After Fourier transform, the action of Ω is reduced to multiplication by the 2×2 matrix

$$\tilde{\Omega} = \frac{2A\gamma}{M_s} \begin{pmatrix} -q_m^2 k_x k_y / k^2 & -k^2 - q^2 - k_y^2/k^2 \\ k^2 + q^2 + k_y^2/k^2 & q_m^2 k_x k_y / k^2 \end{pmatrix} \tag{21}$$

Its two eigenvalues are $\nu_\pm = \pm i\sqrt{\omega_1\omega_2}$, with $\omega_1 = (2\gamma A/M_s)\varepsilon_1$ and $\omega_2 = (2\gamma A/M_s)\varepsilon_2$. Since ε_1 and ε_2 are positive if the uniform state is stable, the eigenvalues of Ω are purely imaginary, what means that the small perturbations of the magnetization associated to the eigenmodes oscillate around the equilibrium magnetization with constant amplitude. The frequencies $\omega = \sqrt{\omega_1\omega_2}$ give the well known dispersion relation of spin waves in a ferromagnet.[49]

The eigenvectors of $\tilde{\Omega}$ can be readily obtained without using the Q operator, which in the Fourier basis is given by the 2×2 matrix $\tilde{Q} = \tilde{K}^{1/2} J \tilde{K} J \tilde{K}^{1/2}$. This matrix is indeed proportional to the 2×2 identity matrix, with a proportionality coefficient given by $-\varepsilon_1\varepsilon_2$, as it has to be. Obviously, the general formalism can also be applied to get the eigenvectors of $\tilde{\Omega}$, as follows: any constant two component vector $\tilde{\eta}$ is an eigenvector of $\tilde{Q}$, so pick up any of them and construct the two eigenvectors of $\tilde{\Omega}$ as in Equation (12)

$$\tilde{\xi}^{(\sigma)} = i\sigma\sqrt{\omega_1\omega_2}\,\tilde{K}^{-1/2}\tilde{\eta} + \tilde{\Omega}\tilde{K}^{-1/2}\tilde{\eta} \tag{22}$$

This shows in a concrete case the close relationship between the spectrum of the Hermitian operator Q and the non-Hermitian operator Ω.

3. The Case $K_{12} = 0$

The main goal of this paper is to address problems in which the $K_{\alpha\beta}$ do not commute, focusing on the cases were $K_{12} = 0$, for which we develop a solution. Examples include 1D solitonic states as the isolated soliton and the CSL of monoaxial helimagnets, and the domain walls of some systems with DMI. In this last instance the problem has been addressed recently via perturbation theory, splitting Ω_2 as the sum of an operator that commutes with Ω_1 plus a perturbation.[10] This may be a reasonable approach, especially if the unperturbed operator can be treated analytically, provided it can be guaranteed that the perturbation does not originate new bound states. Here we develop the non perturbative approach.

Let us define $\Omega_1 = (\omega_0/q_0^2)K_{11}$ and $\Omega_2 = (\omega_0/q_0^2)K_{22}$. As shown in the previous section, the eigenvalues of Ω for a (meta) stable state are complex conjugate pairs of purely imaginary numbers, $i\omega$, with ω real. Only the zero modes, if there are any, can be unpaired. In components, the spectral equation for Ω gives

$$\Omega_2\xi_2 = -i\omega\,\xi_1, \quad \Omega_1\xi_1 = i\omega\,\xi_2 \tag{23}$$

Substituting the values of ξ_1 and ξ_2 given explicitly by one of these equations into the other, we obtain

$$\Omega_2\Omega_1\xi_1 = \omega^2\xi_1, \quad \Omega_1\Omega_2\xi_2 = \omega^2\xi_2 \tag{24}$$

These two equations are compatible since $\Omega_2\Omega_1$ and $\Omega_1\Omega_2$ have the same spectrum: if ξ_1 is an eigenfunction of $\Omega_2\Omega_1$ then $\Omega_1\xi_1$ is an eigenfunction of $\Omega_1\Omega_2$ with the same eigenvalue; the same is true interchanging 1 and 2. The case $\omega = 0$ is special: if ξ_1 is an eigenfunction of $\Omega_2\Omega_1$ with zero eigenvalue, we have an eigenstate of Ω just taking $\xi_2 = 0$. Again, the statement is valid interchanging 1 and 2.

The K operator of a (meta)stable state may be gapless or even have zero modes. When $K_{12} = 0$ the zero modes or the gapless modes are generically associated to one operator, say K_{11}, and K_{22} has a gap. For instance, the magnetization of 1D solitons lies on a plane, and the soliton is described by the dependence of an angle, say θ, on a coordinate, say x. Then, the equilibrium magnetization is a function of x through $\theta(x)$, $\boldsymbol{n}_0(x) = \boldsymbol{n}_0(\theta(x))$. Let us define $\boldsymbol{e}_1 = \partial_\theta \boldsymbol{n}_0$, with a normalization factor if necessary, and $\boldsymbol{e}_2 = \boldsymbol{n}_0 \times \boldsymbol{e}_1$. A spatial translation of the soliton is described by the function $\theta(x+a)$. If a is infinitesimal, the translation corresponds to a distortion of the magnetization given by $a\theta'\boldsymbol{e}_1$, where the prime means derivative with respect to x, since, performing the Taylor expansion in a, we have

$$\boldsymbol{n}_0\big(\theta(x+a)\big) = \boldsymbol{n}_0\big(\theta(x)\big) + a\theta'(x)\boldsymbol{e}_1 + O(a^2) \tag{25}$$

where we used $\boldsymbol{n}_0' = \theta'\boldsymbol{e}_1$. Therefore, the infinitesimal translation is equivalent to a distortion of the non translated soliton given by $\xi_1 = a\theta'$ and $\xi_2 = 0$. The variation of energy due to this distortion is given by $Aa^2(\theta', K_{11}\theta')$. This has to vanish since the energy does not change by a translation. And since K_{11} is a Hermitian positive semidefinite operator, this implies $K_{11}\theta' = 0$. Thus, the translational symmetry of the soliton implies that θ' is a zero mode of K_{11}. But it gives no condition on K_{22}, since the distortion associated to the translation does not involve ξ_2. Hence, K_{22} is generically strictly positive in the phase diagram region where the soliton is stable. The appearance of a zero mode in the spectrum of K_{22} signals the boundary of the soliton stability region.[32]

Hence Ω_2 is a Hermitian positive definite invertible operator, and so it is its square root. It is clear that the non-Hermitian operator $\Omega_2\Omega_1$ is related by a similarity transformation to the Hermitian positive semidefinite operator $\Lambda = \Omega_2^{1/2}\Omega_1\Omega_2^{1/2}$, since $\Lambda = S^{-1}\Omega_2\Omega_1 S$, with $S = \Omega_2^{-1/2}$. Hence $\Omega_2\Omega_1$ and Λ have the same spectrum, and if $\{f_i\}$ is a complete set of orthonormal eigenfunctions of Λ, then $\{\psi_i = \sqrt{N_i\omega_0}\,\Omega_2^{1/2} f_i\}$, where N_i is a normalization constant, is a complete set of (nonorthonormal) eigenfunctions of $\Omega_2\Omega_1$ satisfying[50]

$$(\psi_i, \omega_0\,\Omega_2^{-1}\,\psi_j) = N_i\,\delta_{ij} \tag{26}$$

Each eigenfunction, ψ_i, of $\Omega_2\Omega_1$ gives rise to two eigenstates of Ω. To see this, let ω_i^2 be the eigenvalue corresponding to ψ_i, and let us notice that the 2D space spanned by the states

$$\begin{pmatrix} 0 \\ \Omega_2^{-1}\psi_i \end{pmatrix}, \quad \Omega\begin{pmatrix} 0 \\ \Omega_2^{-1}\psi_i \end{pmatrix} = \begin{pmatrix} -\psi_i \\ 0 \end{pmatrix} \tag{27}$$

is invariant under the action of Ω, since

$$\Omega\begin{pmatrix} -\psi_i \\ 0 \end{pmatrix} = \omega_i^2\begin{pmatrix} 0 \\ \Omega_2^{-1}\psi_i \end{pmatrix} \tag{28}$$

where we used the relation $\Omega_1\psi_i = \Omega_2^{-1}\Omega_2\Omega_1\psi = \omega_i^2\Omega_2^{-1}\psi_i$. Therefore, Ω can be diagonalized within this 2D subspace. The restriction of Ω to the subspace, in the basis (27), is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ \omega_i^2 & 0 \end{pmatrix} \tag{29}$$

and therefore the eigenvalues are $\pm i\sigma\omega_i$, where ω_i is the positive square root of ω_i^2 and $\sigma = \pm 1$. The corresponding eigenstates are easily obtained and have the form

$$\xi^{(i\sigma)} = \begin{pmatrix} \psi_i \\ i\sigma\omega_i\Omega_2^{-1}\psi_i \end{pmatrix} \tag{30}$$

They satisfy the following normalization condition, obtained from Equation (26)

$$\left\langle \xi^{(i\sigma)}, G\xi^{(j\sigma')} \right\rangle = \left(1 + \sigma\sigma' \frac{\omega_i^2}{\omega_0^2}\right) N_i \delta_{ij} \tag{31}$$

where

$$G = \begin{pmatrix} \omega_0\Omega_2^{-1} & 0 \\ 0 & \omega_0^{-1}\Omega_2 \end{pmatrix} \tag{32}$$

The completeness of the set $\{\psi_i\}$ implies the completeness of the set $\{\xi^{(i\sigma)}\}$: for any given ξ we have $\xi = \sum_{i\sigma} c_{i\sigma}\xi^{(i\sigma)}$, where

$$c_{i\sigma} = \frac{1}{2N_i}\left[\left(\psi_i, \omega_0\Omega_2^{-1}\xi_1\right) - i\sigma \frac{\omega_0}{\omega_i}\left(\psi_i, \xi_2\right)\right] \tag{33}$$

Notice that, although the eigenstates (30) involve $\Omega_2^{-1}\psi_i$, actually it is not necessary to invert Ω_2, since for $\omega_i \neq 0$ the spectral equation $\Omega_2\Omega_1\psi_i = \omega_i^2\psi_i$ implies $\Omega_2^{-1}\psi_i = (1/\omega_i^2)\Omega_1\psi_i$, while for $\omega_i = 0$ the lower component of the eigenstate obviously vanishes.

In summary, we have obtained the eigenstates $\xi^{i\sigma}$ of Ω in terms of the eigenfunctions ψ_i of the diagonalizable operator $\Omega_2\Omega_1$, for the cases in which $K_{12} = 0$, which allows us to solve a number of important problems.

Incidentally, let us notice that Equations (30)–(33) can be taken as a starting point to quantization, by imposing canonical commutation relations on ξ_1 and ξ_2, which are derived from the algebra of angular momentum satisfied by the quantized components of $\boldsymbol{n}$.

4. Spin Wave Spectrum of Solitons in a Monoaxial Helimagnet

In the remaining part of the paper we apply the method of Section 3 to the analysis of the spin waves in presence of an isolated soliton in a monoaxial helimagnet. We use a Cartesian coordinate system given by the orthonormal triad $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$. The system is characterized by an energy functional $\mathcal{E}[\boldsymbol{n}] = 2A\int d^3r W$, with

$$W = \frac{1}{2}\sum_{i=x,y,z} \partial_i \boldsymbol{n}\cdot\partial_i \boldsymbol{n} - q_0 \boldsymbol{z}\cdot(\boldsymbol{n}\times\partial_z \boldsymbol{n}) - \frac{1}{2}q_0^2\kappa(\boldsymbol{z}\cdot\boldsymbol{n})^2 - q_0^2 h \boldsymbol{y}\cdot\boldsymbol{n} \tag{34}$$

The successive terms of the right-hand side represent a FM exchange interaction, a uniaxial DMI along the $\boldsymbol{z}$ axis, an easy-plane ($\kappa < 0$) uniaxial magnetic anisotropy (UMA) along the DMI axis, and a Zeeman interaction with an applied magnetic field perpendicular to the DMI axis, given by $h\boldsymbol{y}$. For simplicity, we ignore the magnetostatic energy. However, we argue below that the main qualitative result of the present work, namely, the presence of a spatial shift in the spin wave packets scattered by the soliton, holds also in the presence of the magnetostatic interaction. The constant q_0 is proportional to the ratio between the DMI and FM exchange interaction strengths, and plays the role of the q_0 parameter introduced in Section 2, and κ and h are dimensionless. The numerical results discussed in this work correspond to $\kappa = -5.0$ and $h = 1.0$, unless other values are explicitly quoted.

The Sine–Gordon soliton is a stationary point of the energy, given by $\boldsymbol{n}_0 = -\sin\varphi\,\boldsymbol{x} + \cos\varphi\,\boldsymbol{y}$, with $\varphi(z) = 4\arctan[\exp(z/\Delta)]$, where $\Delta = 1/(q_0\sqrt{h})$ is the soliton width. Notice that $\boldsymbol{n}_0$ lies on the plane perpendicular to the DMI axis. The solitons are metastable below a certain value of h that depends on the DMI and UMA strengths,[32] and they condense into a CSL for h below the critical field $h_c = \pi^2/16$.[31] Notice that the soliton obeys the Sine–Gordon equation since the uniaxial anisotropy acts along the DMI axis. If there were an additional anisotropy along an axis perpendicular to the DMI axis, as it happens if the material is under some mechanical stress, the soliton would obey the double Sine–Gordon equation.[15]

Let us consider the propagation of spin waves in the presence of one soliton on an otherwise ferromagnetic state. Taking $\boldsymbol{e}_1 = \boldsymbol{z}\times\boldsymbol{n}_0$ and $\boldsymbol{e}_2 = \boldsymbol{z}$, so that ξ_1 and ξ_2 describe the in-plane and out-of-plane oscillations, respectively, K_{12} vanishes and Ω_1 and Ω_2 are given by

$$\Omega_1 = \frac{\omega_0}{q_0^2}\left[-\nabla^2 + U_1 + q_0^2 h\right] \tag{35}$$

$$\Omega_2 = \frac{\omega_0}{q_0^2}\left[-\nabla^2 + U_2 + q_0^2(h - \kappa)\right] \tag{36}$$

where $U_1 = -(1/2)\varphi'^2$ and $U_2 = -(3/2)\varphi'^2 + 2q_0\varphi'$ are even functions of z which decay exponentially to zero when $|z| \to \infty$, since $\varphi'(z) = 2/[\Delta\cosh(z/\Delta)]$. These functions are independent of κ, but depend on h through Δ. They are displayed in **Figure 1** (left) for the case $\kappa = -5$ and $h = 1.5$.

The soliton stability requires that Ω_1 and Ω_2 be positive semidefinite. The region in the (κ, h) plane where this condition holds has been obtained by Laliena et al.[32] It turns out that, as discussed generically in the Section 3, Ω_1 has a zero mode associated to the translational invariance of the soliton, and, in the stability region, Ω_2 has a positive spectrum separated from zero by a gap, so that it is invertible.

Since U_1 and U_2 are independent of x and y, the operators Ω_1 and Ω_2 are partially diagonalized by a Fourier transform in x and y. Given that x and y enter the problem in a symmetric way, to simplify the notation we consider only the x dependence, writing the eigenfunctions of $\Omega_2\Omega_1$ as

$$\psi_{k_x}(x, z) = \exp(ik_x x)\phi_{k_x}(z) \tag{37}$$

The general case is obtained from the expressions reported in this paper by replacing k_x^2 by $k_x^2 + k_y^2$ and $k_x x$ by $k_x x + k_y y$ in

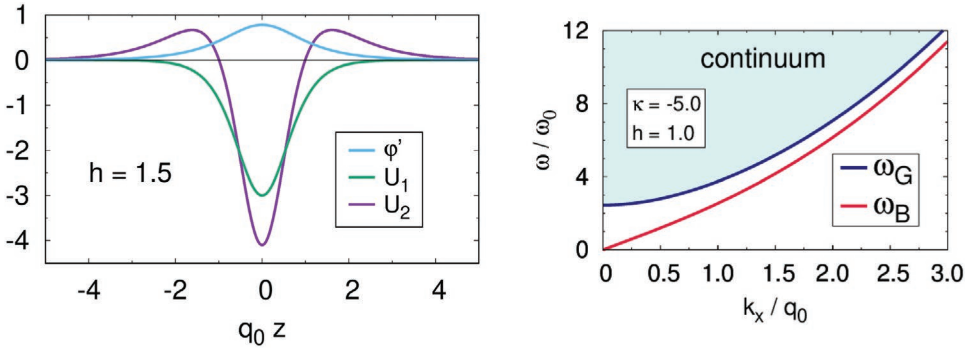

Figure 1. Left: soliton profile, φ', and the potentials U_1 and U_2 for $\kappa = -5.0$ and $h = 1.5$. Right: spin wave spectrum. The red line is the dispersion relation of the gapless branch, and the blue line signals the gap

the expressions below. After the partial Fourier transform, the spectral problem becomes

$$\Omega_{2k_x}\Omega_{1k_x}\phi_{k_x} = \omega^2\phi_{k_x} \tag{38}$$

where ω^2 is a function of k_x^2 and

$$\Omega_{1k_x} = \frac{\omega_0}{q_0^2}\left[-\partial_z^2 + U_1 + q_0^2 h + k_x^2\right] \tag{39}$$

$$\Omega_{2k_x} = \frac{\omega_0}{q_0^2}\left[-\partial_z^2 + U_2 + q_0^2(h-\kappa) + k_x^2\right] \tag{40}$$

The eigenfunctions, $\phi_{k_x i}$, labeled by i, satisfy a normalization condition analogous to Equation (26). Since $\Omega_{2k_x}\Omega_{1k_x}$ is a real operator symmetric under the parity transformation, $z \to -z$, the eigenfunctions can be chosen to be real and of definite parity (either even or odd functions).

The spectral problems were solved numerically as a function of k_x, on a box $-L \leqslant z \leqslant L$ with Dirichlet boundary conditions at $z = \pm L$. Some details about the numerical methods can be found in Supporting Information.[45] The spectrum for $\kappa = -5$ and $h = 1.0$ is displayed in Figure 1 (right). Insight about the spectrum is obtained by studying the asymptotic properties of the eigenfunctions as $z \to \pm\infty$, as follows.

For $z \to \pm\infty$ the "potentials" U_1 and U_2 vanish exponentially and the spectral equation (38) becomes asymptotically

$$\left[\partial_z^2 - k_x^2 - q_0^2(h-\kappa)\right]\left[\partial_z^2 - k_x^2 - q_0^2 h\right]\phi_{k_x} = \frac{q_0^4\omega^2}{\omega_0^2}\phi_{k_x} \tag{41}$$

The solutions are exponential functions that can in general be written as $\exp(ik_z z)$, for some k_z. Equation (41) imposes a relation between k_z^2 and ω^2, which can be written as $\omega^2 = \omega_2\omega_1$, where

$$\omega_1 = \omega_0\left(\frac{k_z^2 + k_x^2}{q_0^2} + h\right) \tag{42}$$

$$\omega_2 = \omega_0\left(\frac{k_z^2 + k_x^2}{q_0^2} + h - \kappa\right) \tag{43}$$

The relation can be inverted to obtain

$$\frac{k_z^2}{q_0^2} = -\left(h + \frac{k_x^2}{q_0^2} - \frac{\kappa}{2}\right) \pm \left(\frac{\omega^2}{\omega_0^2} + \frac{\kappa^2}{4}\right)^{1/2} \tag{44}$$

Remember that $\kappa < 0$, so that each term within brackets in the above expression is positive.

Bound states in the z direction require $k_z^2 < 0$ (imaginary k_z). There are two possibilities: either the minus sign is taken in Equation (44), in which case there is no restriction in ω, or the plus sign is taken and $\omega < \omega_G(k_x)$, where

$$\omega_G(k_x) = \omega_0\left[(k_x^2/q_0^2 + h)(k_x^2/q_0^2 + h - \kappa)\right]^{1/2} \tag{45}$$

In the latter case the bound states are below the gap, while in the former bound states above the gap are possible. The numerical results show that, for fixed k_x, there is a single bound state, of even parity, with frequency below the gap. At $k_x = 0$ it is the zero mode associated to the translation invariance of the soliton, and has $\omega = 0$ and $k_z = i/\Delta$. Thus, the branch of states bound to the soliton is gapless. The dispersion relation of this branch of the spectrum, $\omega_B(k_x)$, is the red line of Figure 1 (right).

Continuum states, unbounded in all directions, have k_z real, which requires the plus sign in Equation (44) and $\omega > \omega_G(k_x)$. Thus, continuum states have a gap given by Equation (45). Taking into account these two conditions, Equation (44) provides the dispersion relation for the continuum states

$$k_z = q_0\left[\left(\frac{\omega^2}{\omega_0^2} + \frac{\kappa^2}{4}\right)^{1/2} - \left(\frac{k_x^2}{q_0^2} + h - \frac{\kappa}{2}\right)\right]^{1/2} \tag{46}$$

The dispersion relation is reciprocal, despite the fact that the system has chiral interactions. Nonreciprocal spin wave propagation, usually associated to chirality, is absent in the isolated soliton and in the CSL, because it requires that Ω contains first order derivatives, which is not the case. One can see, by deriving the generic form of the K operator associated to Equation (34), that nonreciprocal propagation takes place in monoaxial helimagnets only in states whose magnetic moments have a nonvanishing projection onto the DMI axis.

The continuum states can be conveniently labeled by the wave number $\pm k_z$, with $k_z \geqslant 0$. The two degenerate values of the wave number, $\pm k_z$, are combined to make the eigenfunctions real and with definite parity. Thus, the continuum states are actually labeled by k_z and the parity, denoted by the symbols e (even) and o (odd). Hence, we write $\phi_{k_x k_z}^{(e)}(z)$ and $\phi_{k_x k_z}^{(o)}(z)$

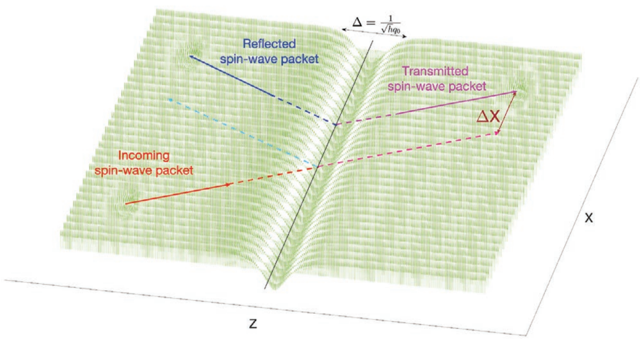

Figure 2. Schematic view of the scattering.

for the continuum eigenfunctions of $\Omega_{2k_x}\Omega_{1k_x}$. Their asymptotic behavior for $z \to \pm\infty$ is given by

$$\phi^{(e)}_{k_x k_z}(z) \approx \cos(k_z z \pm \delta_0) \tag{47}$$

$$\phi^{(o)}_{k_x k_z}(z) \approx \sin(k_z z \pm \delta_1) \tag{48}$$

where the phase shifts δ_0 and δ_1 depend, in general, on k_z and k_x. Continuum states start at $k_z = 0$, where $\omega = \omega_G(k_x)$, and fill the whole frequency region above the gap.

Notice that the phase shifts of the Ω eigenstates [Equation (30)] are those of $\phi^{(p)}_{k_x k_z}$, $p = e, o$, since in the asymptotic region $|z| \to \infty$ we have

$$\Omega_{2k_x} \approx (\omega_0/q_0^2)\left(-\partial_z^2 + q_0^2(h-\kappa) + k_x^2\right) \tag{49}$$

and then

$$\Omega^{-1}_{2k_x}\,\phi^{(p)}_{k_x k_z} \approx \frac{1}{k^2 + q_0^2(h-\kappa)}\,\phi^{(p)}_{k_x k_z} \tag{50}$$

Hence, apart of some normalization constant, the eigenstates of Ω are given by

$$\xi^{(p,\sigma)}_{k_x k_z}(x,z) \approx \begin{pmatrix} 1 \\ \dfrac{\mathrm{i}\sigma\omega}{k^2 + q_0^2(h-\kappa)} \end{pmatrix} \mathrm{e}^{\mathrm{i}k_x x}\phi^{(p)}_{k_x k_z}(z). \tag{51}$$

The scattering properties are therefore determined solely by the eigenfunctions of $\Omega_2\Omega_1$.

Summarizing, the spectrum, displayed in Figure 1 (right), contains a continuum of states unbounded in all directions, with frequencies above a gap given by $\omega_G(k_x)$. Below the gap there is a gapless branch of states, with frequency $\omega_B(k_x)$, consisting of waves bounded to the soliton position, that is, decaying exponentially as $z \to \pm\infty$, but unbounded in the other directions. For each fixed k_x, this is the only bound state of $\Omega_{2k_x}\Omega_{1k_x}$.

5. Spin Wave Scattering and the Goos–Hänchen Effect

Let us analyze in more detail the continuum states. They are used to describe the scattering of a spin wave packet by the soliton, which results in the emergence of one reflected and one transmitted wave packet (the scattered waves). **Figure 2** illustrates schematically the scattering process. Although $\Omega_{2k_x}\Omega_{1k_x}$ is not Hermitian, nor second order in derivatives, the concepts of scattering theory are valid since they rely only on the asymptotic properties of the wave equation. Details on this are provided in Supporting Information, where some well known relations of the usual 1D scattering theory are derived for the present case.[45]

The phase shifts are computed numerically from the asymptotic behavior given by Equations (47) and (48) and the boundary condition at $z = L$. Since the eigenfunctions $\phi^{(e)}_{k_x k_z}(z)$ and $\phi^{(o)}_{k_x k_z}(z)$ vanish at $z = L$ (Dirichlet boundary conditions), the asymptotic behavior implies $k_z L + \delta_0 = (n_0 + 1/2)\pi$ and $k_z L + \delta_1 = n_1\pi$, where n_0 and n_1 are the integers that make $0 \leqslant \delta_0,\ \delta_1 \leqslant \pi$. For each eigenfunction, the value of k_z is obtained from the corresponding eigenvalue, ω^2, through the dispersion relation (46). In this way, δ_0 and δ_1 are determined as a function of k_x and k_z. The phase shifts for $k_x = 0$ are shown as a function of k_z in **Figure 3** (left). In contrast with the domain wall of the uniaxial ferromagnet,[4] which is reflectionless, the reflection coefficient, $R = \sin^2(\delta_0 - \delta_1)$,[45] does not vanish since $\delta_0 \neq \delta_1$.

It is curious that, although it has been demonstrated only for some classes of Schrödinger operators, and $\Omega_{2k_x}\Omega_{1k_x}$ is not a Schrödinger operator, the phase shifts agree, for each fixed k_x, with the thesis of Levinson's theorem,[51] which states that $[\delta_0(0) - \delta_0(\infty)]/\pi + 1/2$ and $[\delta_1(0) - \delta_1(\infty)]/\pi$ are equal to the number of bound states of the respective parities. The agreement follows from $\delta_0 = \pi/2$ and $\delta_1 = 0$ for $k_z = 0$, from $\delta_0 = \delta_1 = 0$ for $k_z \to \infty$, and from the existence, for each fixed k_x, of just one bound state, with even eigenfunction. This is the state corresponding

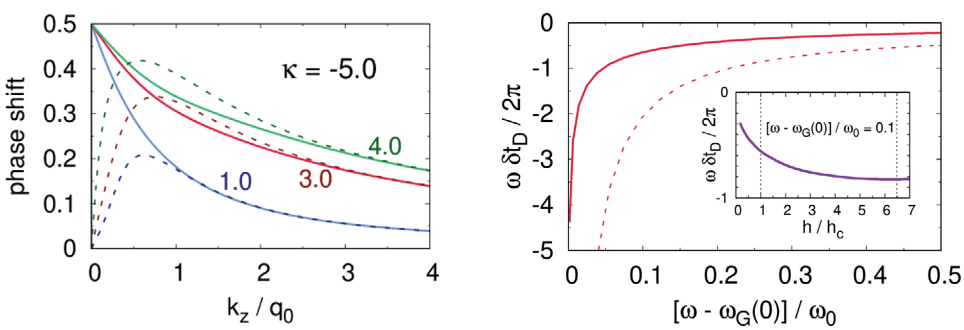

Figure 3. Left: phase shifts δ_0 (continuous lines) and δ_1 (broken lines) versus k_z for $k_x = 0$ for the displayed values of h. Right: time delay in units of the wave period $(2\pi/\omega)$ versus frequency $(\omega - \omega_G(0))$. The broken line is the Wigner causal bound. Inset: time delay versus h/h_c for $\omega - \omega_G(0) = 0.1\,\omega_0$.

to the gapless branch (red line of Figure 1, right), which is bound in the z direction. This means that, for given k_x, it is a bound state of $\Omega_{2k_x}\Omega_{1k_x}$.

The dependence of the phase shifts on the frequency introduces a time delay in the scattered (reflected and transmitted) waves given by $\delta t_D = d(\delta_0 + \delta_1)/d\omega$.[52] It is indeed an advance time, since we obtain $\delta t_D < 0$. This is usually the case when the scattering potential is repulsive, so that we may conclude that the soliton repels the spin waves. It was shown by Wigner that causality implies the bound $\delta t_D \geqslant -(2ak + 1)/kv$, where a is the range of the potential, $k^2 = k_x^2 + k_z^2$, and $v = d\omega/dk$ is the group velocity.[52] In our case we may reasonably estimate the bound taking $a = \Delta$. The product $\omega\delta t_D$ versus $\omega - \omega_G$ is shown in Figure 3 (right) for $k_x = 0$. The Wigner bound (broken line) is well satisfied. The delay time is appreciable for frequencies close to ω_G and, as the inset shows, decreases with the magnetic field strength.

The non trivial dependence of $\Omega_{2k_x}\Omega_{1k_x}$ on k_x induces a k_x dependence of the phase shifts, which causes a displacement of the scattered waves (reflected and transmitted) perpendicular to the DMI axis, z. That is, if the center of a wave packet of narrow cross section impinges the soliton at a point x, the scattered wave packets (both reflected and transmitted) leave the soliton centered at a point $x + \delta x_s$, where

$$\delta x_s = -\frac{\partial}{\partial k_x}(\delta_0 + \delta_1) \tag{52}$$

This relation is derived from a stationary phase analysis of the scattered wave.[53] See Supporting Information for a derivation in the present case.[45]

This very interesting effect is analogous to the well known Goos–Hänchen effect of optics,[54] in which a light beam reflected at the interface of two different media suffers a lateral displacement given by an expression similar to that of Equation (52). Recently, the Goos–Hänchen effect for spin waves has been theoretically studied at interfaces that separate different magnetic media.[55–61] Experimental evidence of the effect at the edge of a Permalloy film has been reported by Stigloher et al.[62] To our knowledge, the kind of Goos–Hänchen effect predicted here, induced by a magnetic modulation instead of an interface, has not been considered before.

Before continuing the analysis, it is worthwhile to mention that in the quantum mechanical scattering of a particle by a 1D potential the phase shifts are independent of the transverse wave vector, k_x, since k_x enters the Hamiltonian as a multiple of the identity operator. Therefore, the scattered matter waves suffer no lateral shift.

The Goos–Hänchen shift predicted here is caused by magnetic modulations, not by interfaces, and is due to the noncommutativity of Ω_{1k_x} and Ω_{2k_x}. If they commute, then there is a complete set of eigenfunctions common to $\Omega_{2k_x}\Omega_{1k_x}$, Ω_{1k_x}, and Ω_{2k_x}. But the eigenfunctions of Ω_{1k_x} (or Ω_{2k_x}) are independent of k_x, because k_x enters these operators through a multiple of the identity, as seen in Equations (39) and (40). Then the phase shifts are independent of k_x, and the lateral shift given by Equation (52) vanishes. Therefore, there is no Goos–Hänchen effect if Ω_{1k_x} and Ω_{2k_x} commute.

We do not find it easy to give a physical meaning to the commutativity or noncommutativity of Ω_1 and Ω_2. The generic situation is that they do not commute, hence let us try to understand what commutativity means. By analogy with Equations (39) and (40), we write in general $\Omega_i = -d^2/dz^2 + U_i + a_i$, where $i = 1, 2$, the functions U_i vanish as $|z| \to \infty$, and the a_i are real constants. First, let us point out that commutativity means that the potentials U_1 and U_2 differ by a constant. Since both U_1 and U_2 vanish as $|z| \to \infty$, the constant has to be zero. Hence, commutativity is equivalent to $U_1 = U_2$. Let us consider a localized fluctuation of the equilibrium magnetization along the $\boldsymbol{e}_1$ direction only, so that $\xi_1 = g$, where g is a function localized on a region of the material, and $\xi_2 = 0$. Let us call $\delta\mathcal{E}_1$ the energy of this fluctuation. Consider another fluctuation along $\boldsymbol{e}_2$, with the same amplitude, so that $\xi_1 = 0$ and $\xi_2 = g$, and let its energy be $\delta\mathcal{E}_2$. If Ω_1 and Ω_2 commute we have $U_1 = U_2$ and therefore

$$\delta\mathcal{E}_2 - \delta\mathcal{E}_1 = (a_2 - a_1)\int g^2 dz \tag{53}$$

Thus, the difference of the energies associated to fluctuations in the $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ directions with the same amplitude does not depend on the place where the fluctuations take place (for instance, whether the fluctuations take place in the middle of the soliton or far from it). This is so because the integral of g^2 is invariant under translations $g(z) \to g(z + c)$. Reciprocally, suppose that a system has this property: the difference in the energy of fluctuations of the same amplitude along $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ is independent of the place where the fluctuation takes place. This difference is

$$\delta\mathcal{E}_2 - \delta\mathcal{E}_1 = \int (U_2 - U_1 + a_2 - a_1) g^2 dz \tag{54}$$

If the above expression is invariant under translations of the function g, then the term in brackets within the integral has to be a constant. But this means that U_1 and U_2 differ by a constant, so that they are equal and Ω_1 and Ω_2 do commute. Hence, we have shown that Ω_1 and Ω_2 commute if and only if the difference of the energy associated to fluctuations of the magnetization along $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, of the same amplitude, does not depend on the place where the fluctuations are located. This is obviously the case in the uniform state, but for modulated states this property is nontrivial. It holds in the case of a domain wall in a uniaxial ferromagnet, but not in the chiral soliton of monoaxial helimagnets.

Since Ω_1 and Ω_2 do commute in the case of the domain wall of uniaxial ferromagnets with no DMI,[4] this soliton does not induce the Goos–Hänchen effect. The addition of an interfacial DMI, as in the model studied by Borys et al.,[10] removes the commutativity of Ω_{1k_x} and Ω_{2k_x}. Therefore, a Goos–Hänchen effect has to appear in this kind of domain wall, which has to be attributed to the DMI. Borys et al. did not address this question since they consider only the spin waves in 1D. To our knowledge, the Goos–Hänchen effect has not been analyzed yet in domain walls with DMI. It can be done following the ideas of this work.

The shift δx_s that we obtain in the case of the isolated soliton of a monoaxial helimagnet is a fraction of the wavelength in the x direction, $\lambda_x = 2\pi/k_x$. However, δx_s is additive as the wave is transmitted across an array of well separated solitons, and therefore the shift can be enhanced by a large factor, provided

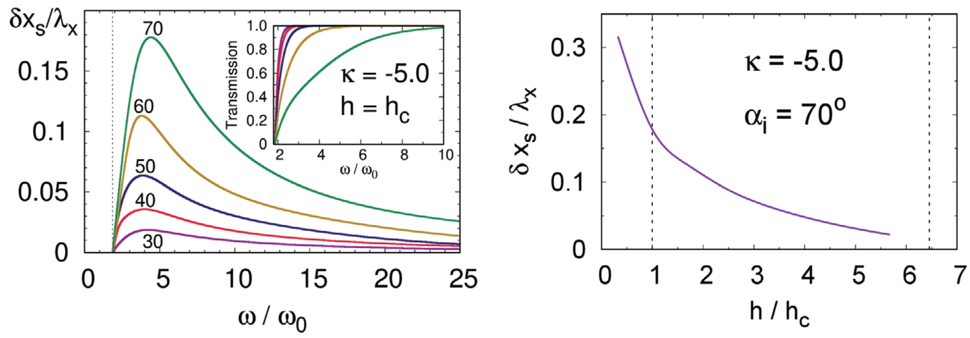

Figure 4. Left: Goos–Hänchen shift for several incidence angles, in degrees, for $h = h_c$. The vertical dashed line marks $\omega_G(0)$. Inset: transmission coefficient for the same angles. Right: maximum Goos–Hänchen displacement for incidence angle $\alpha_i = 70°$ versus h/h_c, for $\kappa = -5.0$. The vertical dashed lines mark the critical field ($h = h_c$) and the destabilizing field.

the transmission coefficient is high enough. **Figure 4** (left) displays $\delta x_s/\lambda_x$ as a function of frequency for several values of the incidence angle, for the critical field $h = h_c$, at which solitons can be easily created. The inset shows the transmission coefficient for the same angles. We see that there is a range of frequencies and incidence angles for which $\delta x_s/\lambda_x \approx 0.1$ and the transmission coefficient is very close to one. This means that δx_s can be enhanced to several tens of wavelengths.

The magnitude of the shift decreases with the applied field, which acts as a control parameter. The maximum shift for an incidence angle of 70° versus the applied field is displayed in Figure 4 (right). The shifts are higher below the critical field, where the ferromagnetic state that hosts the soliton is metastable. The vertical dashed lines signal the position of the critical field ($h/h_c = 1$ and of the field strength at which the soliton becomes unstable. The displacement vanishes at this destabilizing field.

The prediction of the Goos–Hänchen shift presented here has not taken into account the dipolar interaction. For a 1D texture modulated in the z direction in an infinite system, the dipolar interaction introduces a non trivial k_x dependence of the $K_{\alpha\beta}$ operators (including K_{12}, which in this case does not vanish). This means that the eigenfunctions of Ω, and thus the phase shifts, will depend on k_x, and therefore a Goos–Hänchen shift will be induced. There is no reason to suspect that the contribution of the dipolar interaction cancels the shift obtained by ignoring it (this seems extremely unlikely), but there remains the interesting question of whether it enhances or diminishes the magnitude of the shift. Furthermore, it is also clear that a Goos–Hänchen shift will be induced by the domain wall of anisotropic ferromagnets if the dipolar interaction is included. Again, it has to be determined if the addition of DMI enhances or reduces the magnitude of the shift.

It is worthwhile to stress again that the Goos–Hänchen displacement predicted here is not particular to monoaxial helimagnets, but it is expected to be induced by dipolar interactions in any 1D soliton, and, if dipolar interactions are ignored, in any soliton for which Ω_{1k_x} and Ω_{2k_x} do not commute, for instance in domain walls with DMI.[10] It is important also to remark that this kind of Goos–Hänchen effect does not take place at the interface between two different magnetic media, but at the soliton position. For potential applications, this has the advantage that solitons can be created at different locations and moved across the material by the application of magnetic fields or polarized currents.[32]

6. Conclusion

The scattering of spin waves by a 1D magnetic soliton has been analyzed in some important cases, which include the domain walls of ferromagnets with uniaxial anisotropy, which in addition may have some kind of DMI, and the solitons of monoaxial helimagnets. In general, as a result of the scattering by the soliton, the incoming wave packet gives rise to two scattered waves, one reflected and one transmitted. The exception is the scattering by the domain wall of uniaxial ferromagnets, which is reflectionless.

The phase shifts picked up by the scattered waves depend on the component of the wave vector perpendicular to the direction of the magnetic modulation (the transverse component). This dependence induces a lateral shift of the scattered waves (both reflected and transmitted) given by Equation (52). The shift is analogous to the lateral shift of the light waves reflected on the interface that separates two different media, known in optics as the Goos–Hänchen effect.

The dependence of the phase shift on the wave vector transverse component, which does not happen in usual quantum systems with 1D potentials, can be traced mathematically to the noncommutativity of the components of the wave operator, Ω_1 and Ω_2. These two operators do commute in the case of the domain wall of ferromagnets with uniaxial anisotropy, which means that the Goos–Hänchen effect does not take place in these systems. However, this commutativity can be seen as a coincidence typical of this system, and it is removed by the presence of other interactions, like the DMI. Therefore, the Goos–Hänchen effect predicted here can be considered a generic feature of spin wave scattering by 1D solitons.

In the analysis presented in this work the dipolar interaction is not taken into account. However, the contribution of the dipolar interaction gives to the spin wave operator Ω a nontrivial dependence on the wave vector transverse component, and this dependence will be transferred to the phase shift, thus inducing a Goos–Hänchen shift. It is difficult to believe that the shift induced by the dipolar interaction cancels the Goos–Hänchen shift obtained here. It would nevertheless be very interesting to determine whether the dipolar interaction enhances or reduces it. This discussion on the role of the dipolar interaction reinforces the idea that the Goos–Hänchen effect is induced by any 1D soliton, including those of uniaxial ferromagnets, if dipolar interactions are taken into account.

Finally, let us stress that the displacement is induced by a magnetic soliton, and not by the reflection at an interface, as happens in optics and also in magnonics. This has the advantage that the soliton can be created and moved across the material by suitable means. The fact that the displacement takes place also in the transmitted waves, and that it is additive, allows it to be enhanced by a large factor by using an array of well separated solitons. All this is very interesting from the point of view of applications.

Supporting Information

Supporting Information is available from the Wiley Online Library or from the author.

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Keywords

chiral helimagnets, magnetic solitons, magnonics

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[1] A. V. Chumak, V. I. Vasyuchka, A. A. Serga, B. Hillebrands, *Nat. Phys* **2015**, *11*, 453.
[2] A. V. Chumak, A. A. Serga, B. Hillebrands, *Nat. Commun.* **2014**, *5*, 4700.
[3] T. Schneider, A. A. Serga, B. Leven, B. Hillebrands, R. L. Stamps, M. P. Kostylev, *Appl. Phys. Lett.* **2008**, *92*, 022505.
[4] J. M. Winter, *Phys. Rev.* **1961**, *124*, 452.
[5] A. A. Thiele, *Phys. Rev. B* **1973**, *7*, 391.
[6] R. Hertel, W. Wulfhekel, J. Kirschner, *Phys. Rev. Lett.* **2004**, *93*, 257202.
[7] Y. Le Maho, J.-V. Kim, G. Tatara, *Phys. Rev. B* **2009**, *79*, 174404.
[8] F. Garcia-Sanchez, P. Borys, A. Vansteenkiste, J.-V. Kim, R. L. Stamps, *Phys. Rev. B* **2016**, *89*, 224408.
[9] J.-V. Kim, R. L. Stamps, R. E. Camley, *Phys. Rev. Lett* **2016**, *117*, 197204.
[10] P. Borys, F. Garcia-Sanchez, J.-V. Kim, R. L. Stamps, *Adv. Electron. Mater.* **2016**, *2*, 1500202.
[11] N. J. Whitehead, S. A. R. Horsley, T. G. Philbin, A. N. Kuchko, V. V. Kruglyak, *Phys. Rev. B* **2017**, *96*, 064415.
[12] B. W. Zingsem, M. Farle, R. L. Stamps, R. E. Camley, *Phys. Rev. B* **2019**, *99*, 214429.
[13] D. Song, L. Wang, W. Wang, F. Zheng, J. Tang, S. Wang, C. Zhu, J. Caron, A. Kovcs, Z. Liu, D. Mandrus, M. Tian, H. Du, R. Dunin-Borkowski, *Phys. Rev. B* **2020**, *102*.
[14] M. Ohkuma, M. Mito, Y. Kousaka, T. Tajiri, J. Akimitsu, J. Kishine, K. Inoue, *Appl. Phys. Lett.* **2020**, *117*.
[15] G. W. Paterson, A. A. Tereshchenko, S. Nakayama, Y. Kousaka, J. Kishine, S. McVitie, A. S. Ovchinnikov, I. Proskurin, Y. Togawa, *Phys. Rev. B* **2020**, *101*, 184424.
[16] Y. Togawa, J. Kishine, P. Nosov, T. Koyama, G. Paterson, S. McVitie, Y. Kousaka, J. Akimitsu, M. Ogata, A. Ovchinnikov, *Phys. Rev. Lett.* **2019**, *122*, 017204.
[17] V. Laliena, Y. Kato, G. Albalate, J. Campo, *Phys. Rev. B* **2018**, *98*, 144445.
[18] Y. Masaki, R. Aoki, Y. Togawa, Y. Kato, *Phys. Rev. B* **2018**, *98*, 100402(R).
[19] E. Clements, R. Das, M.-H. Phan, L. Li, V. Keppens, D. Mandrus, M. Osofsky, H. Srikanth, *Phys. Rev. B* **2018**, *97*, 214438.
[20] H. Han, L. Zhang, D. Sapkota, N. Hao, L. Ling, H. Du, L. Pi, C. Zhang, D. Mandrus, Y. Zhang, *Phys. Rev. B* **2017**, *96*, 094439.
[21] E. Clements, R. Das, L. Li, P. Lampen-Kelley, M.-H. Phan, V. Keppens, D. Mandrus, H. Srikanth, *Sci. Rep.* **2017**, *7*, 6545.
[22] J.-I. Yonemura, Y. Shimamoto, T. Kida, D. Yoshizawa, Y. Kousaka, S. Nishihara, F. Goncalves, J. Akimitsu, K. Inoue, M. Hagiwara, Y. Togawa, *Phys. Rev. B* **2017**, *96*, 184423.
[23] V. Laliena, J. Campo, Y. Kousaka, *Phys. Rev. B* **2017**, *95*, 224410.
[24] V. Laliena, J. Campo, Y. Kousaka, *Phys. Rev. B* **2016**, *94*, 094439.
[25] M. Shinozaki, S. Hoshino, Y. Masaki, J. Kishine, Y. Kato, *J. Phys. Soc. Jpn.* **2016**, *85*, 074710.
[26] K. Tsuruta, M. Mito, H. Deguchi, J. Kishine, Y. Kousaka, J. Akimitsu, K. Inoue, *Phys. Rev. B* **2016**, *93*, 104402.
[27] V. Laliena, J. Campo, J. Kishine, A. Ovchinnikov, Y. Togawa, Y. Kousaka, K. Inoue, *Phys. Rev. B* **2016**, *93*, 134424.
[28] J. Kishine, I. Bostrem, A. Ovchinnikov, V. Sinitsyn, *Phys. Rev. B* **2014**, *89*, 014419.
[29] N. Ghimire, M. McGuire, D. Parker, B. Sipos, S. Tang, J.-Q. Yan, B. Sales, D. Mandrus, *Phys. Rev. B* **2013**, *87*, 104403.
[30] Y. Togawa, T. Koyama, T. Takayanagi, S. Mori, Y. Kousaka, J. Akimitsu, S. Nishihara, K. Inoue, A. Ovchinnikov, J. Kishine, *Phys. Rev. Lett.* **2012**, *108*, 107202.
[31] I. Dzyaloshinskii, *Sov. Phys. J. Exp. Theor.Phys.* **1964**, *19*, 960.
[32] V. Laliena, S. Bustingorry, J. Campo, *Sci. Rep.* **2020**, *10*, 20430.
[33] F. Goncalves, Y. Shimamoto, T. Sogo, G. Paterson, Y. Kousaka, Y. Togawa, *Appl. Phys. Lett.* **2020**, *116*, 012403.
[34] J. Kishine, A. S. Ovchinnikov, *Phys. Rev. B* **2020**, *101*, 184425.
[35] D. Yoshizawa, Y. Sawada, Y. Kousaka, J.-I. Kishine, Y. Togawa, M. Mito, K. Inoue, J. Akimitsu, T. Nakano, Y. Nozue, M. Hagiwara, *Phys. Rev. B* **2019**, *100*, 104413.
[36] J. Kishine, V. Sinitsyn, I. Bostrem, I. Proskurin, F. Goncalves, Y. Togawa, A. Ovchinnikov, *Phys. Rev. B* **2019**, *100*, 024411.
[37] Y. Shimamoto, F. Goncalves, T. Sogo, Y. Kousaka, Y. Togawa, *Appl. Phys. Lett.* **2019**, *115*, 104415.
[38] A. Tereshchenko, A. Ovchinnikov, I. Proskurin, E. Sinitsyn, J.-I. Kishine, *Phys. Rev. B* **2018**, *97*, 184303.
[39] F. Goncalves, T. Sogo, Y. Shimamoto, I. Proskurin, V. Sinitsyn, Y. Kousaka, I. Bostrem, J. Kishine, A. Ovchinnikov, Y. Togawa, *Phys. Rev. B* **2018**, *98*, 144407.

[40] F. J. T. Goncalves, T. Sogo, Y. Shimamoto, Y. Kousaka, J. Akimitsu, S. Nishihara, K. Inoue, D. Yoshizawa, M. Hagiwara, M. Mito, R. L. Stamps, I. G. Bostrem, V. E. Sinitsyn, A. S. Ovchinnikov, J. Kishine, Y. Togawa, *Phys. Rev. B* **2017**, *95*, 104415.
[41] J. Kishine, I. Proskurin, I. G. Bostrem, A. S. Ovchinnikov, V. E. Sinitsyn, *Phys. Rev. B* **2016**, *93*, 054403.
[42] J. Kishine, I. Bostrem, A. Ovchinnikov, V. Sinitsyn, *Phys. Rev. B* **2012**, *86*, 214426.
[43] J.-I. Kishine, A. Ovchinnikov, I. Proskurin, *Phys. Rev. B* **2010**, *82*, 064407.
[44] I. Bostrem, J.-I. Kishine, A. Ovchinnikov, *Phys. Rev. B* **2008**, *78*, 064425.
[45] Supporting Information to this article.
[46] Strictly speaking, the operators K_{ij} act on the subspace of the square integrable functions defined in the whole $\mathbb{R}^3$ which are two times differentiable and whose derivatives up to the second order are square integrable functions. This subspace is dense in the space of square integrable functions, which means that any square integrable function can be attained as a limit of functions which belong to the subspace. This is well known in the theory of linear operators [?].
[47] For any $\xi \neq 0$ we have $\langle \xi, K^{1/2}JKJK^{1/2}\xi\rangle = \langle K^{1/2}\xi, JKJK^{1/2}\xi\rangle = -\langle JK^{1/2}\xi, KJK^{1/2}\xi\rangle = -\langle \eta, K\eta\rangle < 0$, since K is positive definite. We defined $\eta = JK^{1/2}\xi$ and we used the hermiticity of $K^{1/2}$ and the antihermiticity of J.
[48] V. Laliena, J. Campo, *Phys. Rev. B* **2017**, *96*, 134420.
[49] A. G. Gurevich, G. A. Melkov, *Magnetization Oscillations and Waves*, CRC Press, Boca Raton, FL **1996**.
[50] These results can be easily obtained by noticing that $\Omega_2\Omega_1$ is a Hermitian positive (semi)definite operator with respect to the scalar product $((f,g)) = (f, \omega_0 \Omega_2^{-1} g)$. Therefore, the eigenvalues of $\Omega_2\Omega_1$ are real and non-negative and its eigenfunctions are ortohogonal with respect to the ((,)) product, what amounts to Equation (26).
[51] M. S. de Bianchi, *J. Math. Phys.* **1994**, *35*, 2719.
[52] E. Wigner, *Phys. Rev.* **1955**, *98*, 145.
[53] K. Artmann, *Ann. Phys.* **1948**, *437*, 87.
[54] F. Goos, H. Hänchen, *Ann. Phys.* **1947**, *436*, 333.
[55] Y. S. Dadoenkova, N. N. Dadoenkova, I. L. Lyubchanskii, M. L. Sokolovskyy, J. W. Kos, J. Romero-Vivas, M. Krawczyk, *Appl. Phys. Lett.* **2012**, *101*, 042404.
[56] P. Gruszecki, J. Romero-Vivas, Y. S. Dadoenkova, N. N. Dadoenkova, I. L. Lyubchanskii, M. Krawczyk, *Appl. Phys. Lett.* **2014**, *105*, 242406.
[57] P. Gruszecki, Y. S. Dadoenkova, N. N. Dadoenkova, I. L. Lyubchanskii, J. Romero-Vivas, K. Y. Guslienko, M. Krawczyk, *Phys. Rev. B* **2015**, *92*, 054427.
[58] P. Gruszecki, M. Mailyan, O. Gorobets, M. Krawczyk, *Phys. Rev. B* **2017**, *95*, 014421.
[59] M. Mailyan, P. Gruszecki, O. Gorobets, M. Krawczyk, *IEEE Trans. Magn.* **2017**, *53*, 11.
[60] Z. Wang, Y. Cao, P. Yan, *Phys. Rev. B* **2019**, *100*, 064421.
[61] W. Zhen, D. Deng, *Opt. Commun.* **2020**, *474*, 126067.
[62] J. Stigloher, T. Taniguchi, H. S. Körner, M. Decker, T. Moriyama, T. Ono, C. H. Back, *Phys. Rev. Lett.* **2018**, *121*, 137201.