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Abstract
We study the dynamics of the exploitation of a natural resource distributed among and flowing between several nodes connected via a weighted, directed network. The network represents both the locations and the interactions of the resource nodes. A regulator decides to designate some of the nodes as natural reserves where no exploitation is allowed. The remaining nodes are assigned (one-to-one) to players, who will exploit the resource at the node. We show how the equilibrium exploitation and the resource stocks depend on the productivity of the resource sites, on the structure of the connections between the sites, and on the number and the preferences of the agents. The best locations to host nature reserves are identified according to the model's parameters, and we find that they correspond to the most central (in the sense of eigenvector centrality) nodes of a suitably redefined network that considers the nodes' productivity.

Keywords: Harvesting, spatial models, differential games, nature reserve

JEL Codes: Q28, C72, Q23, C61, R12, Q20, R11, C73

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ON COMPETITION FOR SPATIALLY DISTRIBUTED RESOURCES IN NETWORKS: AN EXTENDED VERSION

GIORGIO FABBRI*, SILVIA FAGGIAN†, AND GIUSEPPE FRENI‡

ABSTRACT. We study the dynamics of the exploitation of a natural resource distributed among and flowing between several nodes connected via a weighted, directed network. The network represents both the locations and the interactions of the resource nodes. A regulator decides to designate some of the nodes as natural reserves where no exploitation is allowed. The remaining nodes are assigned (one-to-one) to players, who will exploit the resource at the node. We show how the equilibrium exploitation and the resource stocks depend on the productivity of the resource sites, on the structure of the connections between the sites, and on the number and the preferences of the agents. The best locations to host nature reserves are identified according to the model's parameters, and we find that they correspond to the most central (in the sense of eigenvector centrality) nodes of a suitably redefined network that considers the nodes' productivity.

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1. Introduction

In the exploitation of common property and open access resources, externalities engender distortions that the social planner or agents may wish to strategically regulate or control. The question of how to estimate and correct such effects has produced a huge body of literature, in which resource stocks are usually assumed homogeneous in space. A relevant exception is given by metapopulation models (see e.g., Sanchirico and Wilen, 2005) that explicitly address the possibility that natural resource stocks can be *spatially distributed*, with various productive sites connected by non-homogeneous migration flows. Migratory fish provide the most obvious example of a moving distributed stock, but the same spatio-temporal structure is common to other resources, such as water and oil, which are often flowing between locations. Moreover, the same dynamics are shared by other non-natural stocks, such as "knowledge" or pollution, which may be generated in specific locations and afterward diffused to others.

In settings where a network of flows connects the resource extraction sites, do different productivities of the various sites and different intensities of migration flows map into a specific hierarchy of the sites? Does this hierarchy affect how the access of competing agents should be regulated and, in particular, where natural reserves should be placed?[1]

While these ranking questions have received scant attention in the metapopulation literature (however see Costello and Polaski, 2008), they are central in the network literature (for surveys, see Jackson and Zenou, 2015; Zenou, 2016), within which a prominent approach consists in studying the Nash equilibrium of *static* games where players are connected via a network of externalities, and in identifying the key players in this equilibrium by using network statistics (Ballester et al., 2006).

In this paper, we take a network perspective on common spatially distributed resources and develop a simple dynamic model where n nodes ($n \geq 2$) of a weighted directed network represent the n sites where the resource resides and evolves in time, while the weights on the edges give the interregional migrations rates of the resource. The n regions are heterogeneous not only because they are differently connected, but also because the growth rates of the resource are possibly different in different regions. The regulator's task is to assign extraction rights to $f < n$ agents to maximize a welfare function, which, for the most part, we take to be the sum of the agents' utilities. We assume that the regulator is constrained to assign at most one agent to a region. Following the assignment stage, the agents compete for the exploitation of the resource as in the classic Levhari and Mirman (1980) dynamic game, with four main differences: 1. time is continuous and the exploitation of the resource occurs continuously; 2. the stock of the resource is not homogeneous but distributed among the n regions; 3. the site productivities are assumed to be independent of the stocks; and, finally, 4. each agent can only access the resource through the single node to which they are assigned. Further, we assume the instantaneous utility functions of the agents are isoelastic, consistently with the bulk of the literature on the Levhari-Mirman game in continuous time. Our main aim is then to study how the structure of the network affects the regulator's decision.

[1]The same questions also apply in more general contexts, including mobile resources with environmental or amenity values whose reproduction process is affected by economic activities. For example, given that urban development is likely to worsen conditions at breeding sites of migratory or non-migratory birds that are capable of moving, ranking the sites can help inform zoning regulations and urban planning.

The assumption that sites' growth functions are linear may seem strong, yet it is valid in many cases including the following: whenever the resource is exhaustible; when the resource is renewable but the stock is collapsing (and, hence, concave growth functions with finite steepness at zero at the different sites can be harmlessly approximated by their tangent at the origin); when there exists no fixed factors causing decreasing returns, for example if the resource is "knowledge" (corresponding to a non-exhaustible pool of ideas).

Our first contribution is to describe the properties of a Nash equilibrium of the game; in particular, we show that:

(a) When agents are sufficiently "patient", in the generalized growth theory sense that their rate of discount is close to a critical discount rate (see e.g., McFadden, 1973, for a discussion of critical discount rates in optimal growth theory), and the network is fully connected, there exists a unique Markov perfect equilibrium (MPE) in linear strategies for the post-assignment dynamic game (see Theorem 1 and Theorem 2).

(b) The same linear MPE exists and is unique among linear strategies for the class of strongly connected networks if the state space is appropriately restricted[2] (see Section 3.3 and in particular Proposition 3 and Theorem 3).

(c) At this equilibrium, all agents, independently of the assignment, evaluate the different site stocks by means of a constant common vector of relative prices that proves to be the eigenvector centrality of a network that combines the migrations flows and the sites' net rates of growth. These two forces interact in determining the centrality of the sites (Section 4.4). Moreover, at the MPE, our patient agents restrain themselves the most when assigned to the most central sites but moving an agent from a peripheral to a more central node does not enhance resource conservation. In fact, the diminished extraction intensity of the agent only offsets the otherwise negative effects that a constant extraction rate from more productive sites would have caused on the resource's rate of growth.

Our second contribution is that we provide some comparative statics that show how the equilibrium outcome is affected by the choice of sites for natural reserves, the number of issued permits, and the network structure. We begin by showing that when the social planner compares equilibria for different choices of nodes where natural reserves are set, they find that the welfare of each agent who has obtained

[2]Many of our results continue to hold or have natural counterparts when source or sink nodes are added to the network, making it reducible. However, we think that each reducible network is reducible in its own way, and different kinds of reducibility give rise to different phenomena that cannot be captured within a single model.

a permit decreases in the centrality measure of his/her assigned node. Thus, an utilitarian planner always sets the reserves at the most central regions of the network (Section 4.1), and permits are always issued in order of centrality, starting from the most peripheral node.

Once consistent outside options are specified for the agents without permits, the last result serves as the basis for studying how the number of licenses impacts the planner welfare function. If a tradeoff exists between the utilities of inframarginal and marginal concessionaires, then the planner welfare function may have an internal maximum that can be interpreted as the optimal number of permits. However, paralleling analogous results obtained in other growth models with externalities (e.g., Tornell and Lane, 1999), we find that this tradeoff actually exists only if agents' intertemporal elasticity of substitution is below a given threshold. Agents with an elasticity of intertemporal substitution between 1 and that threshold increase their equilibrium extraction rate when a new license is granted, while agents with elasticity of intertemporal substitution above the threshold behave in the opposite way (Proposition 7). Here, the decline of the extraction rate is so strong that a higher number of agents contributes to mitigate the "tragedy of commons" effect. Under these circumstances, the regulator tends to maximize the number of issued permits.

We also analyze how the outcome changes when the parameters representing the network are varied (Section 4.2). In our model, the effects of varying the site productivities and the network density are mediated by the largest eigenvalue of the process that governs the stocks evolution in absence of exploitation. This eigenvalue coincides with the von Neumann rate of growth of the system (i.e., the maximum rate of growth of the resource) and it plays the same role as the productivity parameter in the standard aggregate linear growth model (Rebelo, 1991). The well-known fact that the above eigenvalue is an increasing function of the elements of the matrix representing the process can be used to single out the effect of increasing the site productivities, but not to study the effect of changing the weights of the network. Indeed, a change in a migration flow engenders a simultaneous change, which is equal but opposite in sign, in the net growth rate of the node from which the resource flows. Nevertheless, for symmetric networks with different (gross) productivities, we prove that the largest eigenvalue is a decreasing function of the elements of the adjacency matrix. The intuition behind this negative effect of increased mobility is that it prevents the accumulation of the stock in more productive sites. After having established these basic facts, we then show that, as it occurs for the number of permits, the effect on the equilibrium rate of growth of a change in the maximum rate of growth depends on the value of agents' elasticity of intertemporal substitution, so that an increase in the dominant eigenvalue can result in a decrease of the rate of growth.

Our paper is naturally related to the metapopulation literature (Hanski, 1999; Sanchirico and Wilen, 2005; Smith et al., 2009; Costello and Polaski, 2008, and others). A few papers in this stream have explored aspects of the problem of dynamic strategic interaction with distributed and moving resources, especially to evaluate whether management of the resources through a system of Territorial Use Rights (Territorial Use Right for Fishing, or "TURF", in the case of fisheries) can effectively mitigate the "tragedy of the commons" (see e.g., Kaffine and Costello, 2011, Costello et al., 2015, Herrera et al., 2016, Costello and Kaffine, 2018, Costello et al., 2019, de Frutos and Martin-Herran, 2019, Fabbri et al., 2020). For example, Kaffine and Costello (2011) have shown, using a discrete time model, that Territorial Use Rights coupled with profits sharing can effectively mitigate the overexploitation of moving resources. Costello et al. (2015) have extended the same model to show how partial enclosure of the commons can improve the welfare of the common property regime. Costello and Kaffine (2018) compared the relative efficiency of centralized versus decentralized management of a moving resource when users have heterogeneous preferences for conservation and the regulator has incomplete information about these preferences. Conversely, in a two-region model in continuous time, Fabbri et al. (2020) have suggested that modulating the access to the different sites through the assignment of Territorial Use Rights can be effective in increasing the rate of growth of moving collapsing resources, in a context of high harvesting effort. Apart for the choice of the time structure (continuous vs discrete), the model we study is similar to the N-patch discrete time model of Kaffine and Costello (2011). The models, however, differ in the specifications of both the production functions (linear vs. strictly concave) and the utility functions (isoelastic vs. linear). In particular, the linear specification of the instantaneous utility function significantly simplifies the dynamics in the Kaffine and Costello model, implying that the equilibrium path jumps immediately to the stationary state without any transitional dynamics.

Closely related to this article is the now extensive literature on differential games in resources economics surveyed in Clemhout and Wan (1994), Dockner et al. (2000) and Long (2011). In almost all the games considered in that literature, the state variable is scalar (an exception is Clemhout and Wan, 1985, which allows multispecies predator–prey interactions). Plourde and Yeung (1989) provide a continuous time version of the Levhari and Mirman (1980) dynamic game. A discussion of the MPE for the case of an exhaustible resource exploited by n agents can be found in Dockner et al. (2000) section 12.1.2. Clemhout and Wan (1985) contains various models of renewable resources, and also covers the one-dimensional case in which the reproduction function is linear. Linear versions of the renewable resource game have been used in development economics by Tornell and Velasco (1992), Tornell and Lane (1999) and others to show that the interaction of multiple powerful groups can hinder the rate of growth of poor countries. We discuss the connection with Tornell and Lane (1999), when we present our comparative statics results [3].

Our work is also broadly related to the network literature that connects the Nash equilibrium of static games to network statistics (e.g., centrality measures) (Ballester et al., 2006, Bramuillé et al., 2014, Allouch, 2015). We also connect the policy extraction function of the agents in the Markovian equilibrium of our dynamic game to a centrality measure of a network. The eigenvector centrality of our network is also related to the solution of a single-player game and hence to the Pareto efficient outcomes of the model and to the efficiency prices of optimal extraction plans. In the network literature, Elliott and Golub (2019) have recently studied a similar problem in a static framework that uses the eigenvector centrality of the nodes of the marginal benefits network to characterize the Pareto-efficient and Lindahl outcomes in the provision of a public good by a group of agents whose actions heterogeneously benefit each other.

Other works specifically study the role of networks in the management of natural resources. Currarini et al. (2016) survey various contributions in which network economics has been utilized in analyzing issues ranging from the pattern and speed of diffusion of new green technology to the structure and dynamics of international agreements, from the formation of links in building an environmental coalition to the role of infrastructural networks in the access to natural resources. Among those studies, that by İlkılıç (2011) is closest to the question we explore here. İlkılıç (2011) studies a static game in which a given number of users exploit multiple sources of a common pool [4], and each user faces marginal costs that are increasing in the total extraction from the site, due to the presence of source-specific congestion externalities. The main conclusion is that, in the unique Nash equilibrium of the game, the rate of extraction at each source is proportional to a centrality measure of the links

[3]There is also some connection with multisector growth models with externalities (e.g., Benhabib et al., 2000). In this literature, however, the focus is on competitive equilibria. Moreover, the production functions are specified differently (Cobb-Douglas vs. perfect substitutes).

[4]For the situation of a common pool with a network of players, we also mention the paper by Marco and Goetz (2017), in which a model with limited rationality and evolution of social norms is proposed.

of the source. Our model provides the basis for developing dynamic versions of the İlkılıç (2011) model. More recently, Kyriakopoulou and Xepapadeas (2018) studied the interaction between a global congestion externality and local positive externalities, reflecting collaboration links in the exploitation of a single resource by a given number of agents. They show that the equilibrium rate of extraction of agents is, in this case, proportional to their centrality in the local interactions network.

Finally, we note that while we framed our model as a common resource extraction game, it might also be interpreted as a discrete public goods contribution game where a group of agents are investing, for example in knowledge, to reach a target when there are externalities (imagine, for example, multiple connected laboratories trying to achieve a scientific breakthrough). In that case, the control variables must be interpreted as costly efforts that influence the state of the project. Homogeneous-stock versions of the model have been studied in, among others, Kessing (2007), who show that efforts are strategic complements in time, in Georgiadis (2015), who analyze optimal contracts for a generalized model in which the evolution of the project is stochastic, and in Cvitanić and Georgiadis (2016), who propose a budget-balanced mechanism that induces each agent to choose the first-best effort level. Distributed-stock versions of the model, which can be developed with the techniques used in this paper, may have applications in the decisions regarding the formation of teams, and allowing agents heterogeneous in terms of centrality in a network, they might be particularly relevant in deciding the optimal team composition.

The remainder of the paper is organized as follows. In Section 2 the model is described and preliminaries are discussed. Sections 3 contains the main results of the paper and the description of the Nash equilibrium. In Section 4 the role of the network structure is discussed with the aid of a variety of examples. Section 5 provides sketches of possible applications of the developed techniques to other problems while Section 6 contains the final remarks. The proofs of all analytic results are collected in Appendix A.

2. The model

We consider a common property resource that is diffused over a (possibly geographical) area and partitioned in subareas. The resource is *mobile* in space, meaning that it can move from one region to another in certain proportions, assumed as given. It can be renewable or non-renewable. Typical examples of such resources are livestock, (e.g., fish or birds), but also water or oil reservoirs, which often spread across

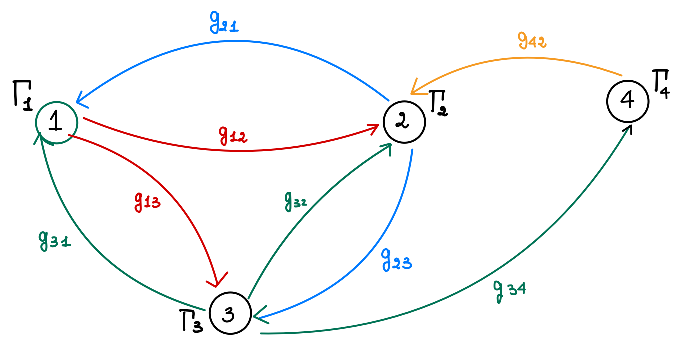


FIGURE 1. An example of strongly connected network.

regional or national borders. As a model example, we consider fish mobile, for instance, around different regional or national waters, in seas or oceans. We assume that these subareas are sufficiently distinct from one another and that each is sufficiently homogeneous that they can be represented as nodes of a network, and that the presence of an edge between two regions means that they are connected, with the weight on the edges representing the intensity of such connections. We additionally assume that any region can be reached from any other, directly or through an indirect path. The network then acts as a space diversification of the resource. Note also that the assumption of a strongly connected network is naturally satisfied when the resource is fish, in seas or oceans, as territorial waters (the nodes) are all connected.

The evolution system. Mathematically speaking, we consider a network $\mathcal{G}$, with n nodes – as many as the number of regions – that we assume to be directed and weighted. We denote the set of nodes by $N := \{1, .., n\}$, and with $g_{ij} \geq 0$, the weight upon the edge connecting a source node i and a target node j, with g_{ij} representing the intensity of the outflow from i to j, so that when $g_{ij} = 0$ and $g_{ji} = 0$, there are no direct paths between the two nodes. We assume:

(H1) $\mathcal{G}$ strongly connected, with $g_{ii} = 0$ for all $i \in N$;

that is, there exists in $\mathcal{G}$ a path connecting any two nodes with corresponding strictly positive coefficients g_{ij} and $\mathcal{G}$ has no loops.

We denote by G the (weighted) $n \times n$ adjacency matrix with elements g_{ij}, and $i, j \in N$, by e_i the i-th vector of the canonical basis on $\mathbb{R}^n$, and by $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^n$. We also denote by $\mathbb{R}_+$ the set of nonnegative real values.

For all $i \in N$, the quantity $X_i(t)$ stands for the mass at node i at time t, and $X(t)$ for the vector with components $X_1(t), .., X_n(t)$. The evolution in time of mass $X_i(t)$ on region i depends on several factors:

(a) the natural growth $\Gamma_i X_i(t)$ of the resource at time t at node i, embodied by the (constant) natural growth rate Γ_i; for renewable resources $\Gamma_i > 0$, while for non-renewable resources $\Gamma_i \leq 0$;

(b) the outflow of the resource from region i to a linked region j at time t, given by $g_{ij} X_i(t)$, so that the net inflow at location i is given by

$$\left(\sum_{j=1}^{n} g_{ji} X_j(t)\right) - \left(\sum_{j=1}^{n} g_{ij} X_i(t)\right) \quad = \quad \langle G e_i, X(t) \rangle - \left(\sum_{j=1}^{n} g_{ij}\right) X_i(t)$$

(c) the rate of extraction $c_i(t)$ at time t from region i.

As a whole, we then have for all i

$$\dot{X}_i(t) = \left(\Gamma_i - \sum_{j=1}^{n} g_{ij}\right) X_i(t) + \langle G e_i, X(t) \rangle - c_i(t).$$

If $A = (a_{ij})$ is the diagonal matrix of the net reproduction factors, namely

$$\begin{cases} a_{ij} = 0 & \text{if } i \neq j \\ a_{ii} \equiv a_i = \Gamma_i - \left(\sum_{j=1}^{j=n} g_{ij}\right), & \end{cases}$$

$c(t)$ is the vector with components $c_1(t), .., c_n(t)$, and x_0 is the vector of all initial stocks of the resource at the different nodes; then, the evolution of the system in vector form is given by:

$$\begin{cases} \dot{X}(t) = (A + G^{\top}) X(t) - c(t), \quad t > 0 \\ X(0) = x_0 \in \mathbb{R}^n_+. \end{cases} \tag{1}$$

In addition, we require the following positivity constraints:

$$c_i(t) \geq 0, \quad t \geq 0, i \in N \tag{2}$$

as well as

$$X_i(t) \geq 0, \quad t \geq 0, i \in N. \tag{3}$$

To exemplify what connection weights in G signify, we consider the particular case in which the resource moves toward less crowded areas, proportionally to the difference $X_i(t) - X_j(t)$ (Fick's first law). When such difference is positive, fish move from node i to node j, and when it is negative, from j to i. Then the weights g_{ij}

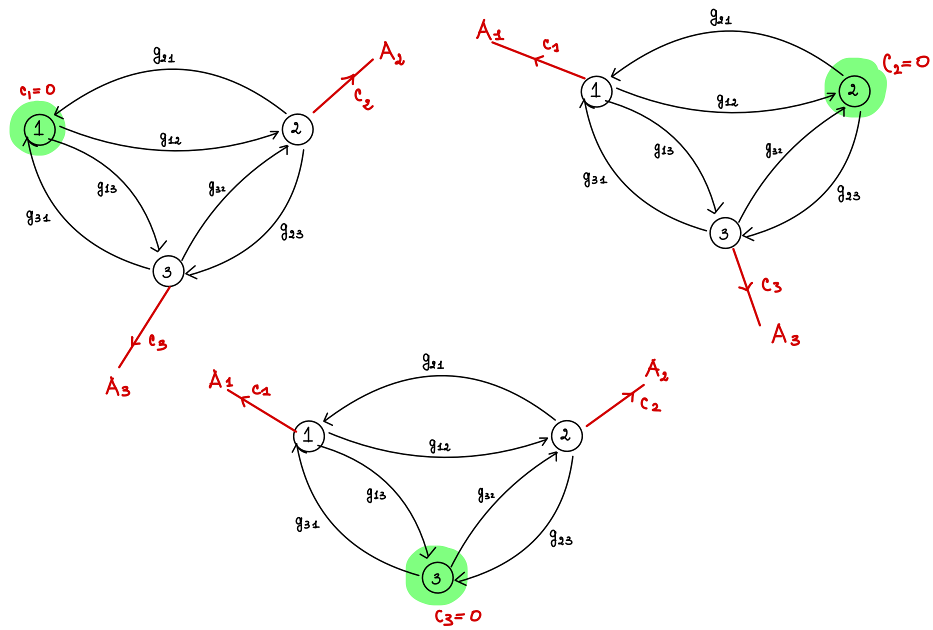


FIGURE 2. Two agents are differently allocated in a network with three nodes. The (only) reserve-node is highlighted in green, while red arrows represent the extractions of the two agents.

express the proportion mentioned above, and $g_{ij} = g_{ji}$, with the net inflow at node i given by

$$-\sum_{j=1}^{n} g_{ij}(X_i - X_j) = \sum_{j=1}^{n} g_{ij} X_j - \sum_{j=1}^{n} g_{ij} X_i.$$

Consequently, $G = G^\top$, then $A + G^\top = A + G$ and the problem simplifies. We will refer to this subcase as the *symmetric case*.

Harvesting Rules and Payoffs. We assume that the regulator wishes to devote some of the n regions to the reproduction of the resource (natural reserves), and to assign each of the remaining regions to an agent, for exclusive exploitation and enhancing a Territorial Use Right policy. More precisely, harvesting is prohibited in a subset M of N, while every node i with $i \in F := N \setminus M$ is exclusively assigned to agent i. We denote by f the number of elements of F, so that elements in M are $n - f$.

An example where $n = 3$ and $f = 2$ is represented in Figure 2, with all possible configurations for the one reserve.

Finally, we assume that agents strategically interact in a differential game where agent i maximizes the payoff

$$J_i(c_i) = \int_0^{+\infty} e^{-\rho t} u(c_i(t))dt, \quad i \in F \tag{4}$$

where

$$u(c) = \ln(c) \quad or \quad u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \qquad \sigma > 0, \ \sigma \neq 1$$

(the case of a logarithmic u stands for the case $\sigma = 1$), where $\rho \in \mathbb{R}$ is the discount rate.

2.1. **Primitives of the Network.** For reasons that will be made clear soon, eigenvalues and eigenvectors of the matrix $A + G$ will have an important role in our discussion. First, we observe that $A + G$ has an eigenvalue that is simple, real, strictly greater than the real parts of the other eigenvalues, and with a *positive* associated normalized eigenvector, as a consequence of the Perron–Frobenius Theorem and the fact that G (and then $A + G$) is irreducible. The transpose $A + G^\top$ enjoys similar properties. We then order the eigenvalues of $A + G$ as follows:

$$\lambda > \mathrm{Re}(\lambda_2) \geq \mathrm{Re}(\lambda_3) \geq ... \geq \mathrm{Re}(\lambda_n). \tag{5}$$

and call $\eta \in \mathbb{R}^n_+$ (respectively, $\zeta \in \mathbb{R}^n_+$), the right (resp., left) normalized eigenvectors of $A + G$, both associated to λ, and having both strictly positive coordinates. The rest of the section is then devoted to the interpretation of λ, η, and ζ and associated useful quantities.

2.1.1. *Trajectories in the long run.* In the case of null extraction ($c \equiv 0$) it is well known that the trajectories of the system converge to the direction of the eigenvector ζ associated to the dominant eigenvalue λ. Moreover, the trajectories starting on the ray through ζ remain steadily on the ray at all times. Then ζ represents the direction of convergence in the long run of the trajectories of the system with null extraction.

2.1.2. *Weighted Total Mass.* The total mass (or aggregate stock) of the resource is given by $\sum_{i=1}^n X_i(t)$. Nonetheless, we rather consider the *weighted* total mass given by

$$\langle X(t), \eta \rangle := \sum_{i=1}^n X_i(t)\eta_i.$$

namely, the scalar product of $X(t)$ and η. Since all η_i are strictly positive, indeed the two quantities have time-equivalent magnitude, as suggested by

$$\frac{1}{M}\langle X(t),\eta\rangle \leq \sum_{i=1}^{n} X_i(t) \leq \frac{1}{m}\langle X(t),\eta\rangle \leq \frac{M}{m}\sum_{i=1}^{n} X_i(t). \tag{6}$$

where $m = \min_i \eta_i$ and $M = \max_i \eta_i$. In particular, the total mass and the weighted total mass grow at the same rate.

2.1.3. *Growth Rate of the System.* In the absence of extraction, that is, for all $c_i = 0$, λ represents the total mass growth rate. Indeed, (1) implies $\langle \dot{X}(t),\eta\rangle = \lambda\langle X(t),\eta\rangle$, and then

$$\langle X(t),\eta\rangle = e^{\lambda t}\langle x_0,\eta\rangle.$$

Moreover, net reproduction rates a_i satisfy

$$a_i < \lambda, \quad \forall i \in F. \tag{7}$$

as the expansion in rows of the equality $(A+G)\eta = \lambda\eta$ gives

$$(\lambda - a_i)\eta_i = \sum_{j=1, j\neq i}^{n} g_{ij}\eta_j > 0, \tag{8}$$

with at least one of the g_{ij} strictly positive, as the network is strongly connected.

2.1.4. *Detrended Trajectory.* It is sometimes useful to consider the detrended trajectory of the system in the absence of extraction $Y(t) = e^{-\lambda t}X(t)$ so that $\dot{Y}(t) = (A + G^\top - \lambda I)Y(t)$ with null principal eigenvalue. Hence, $\langle \dot{Y}(t),\eta\rangle = \langle Y(t),(A+G-\lambda I)\eta\rangle = 0$ and

$$\langle Y(t),\eta\rangle \equiv \langle x_0,\eta\rangle, \quad \forall t \geq 0, \tag{9}$$

establishing that the state $X(t)$ has constant projection in time along the direction of η, magnified only by the growth factor $e^{\lambda t}$.

2.1.5. *Meaning of the eigenvector η.* We give three interpretations of the components η_i of eigenvector η. The first two strictly follow from the fact that they are the Von Neumann prices. However, the last one characterizes them in terms of eigenvector centralities of nodes in a specific migration matrix:

(1) The component η_i of η measures the *long-term productivity of the system at node i*. One way to establish this is to consider the detrended trajectory $Y^i(t)$ starting with a unitary mass concentrated in the i-th node, namely $x_0 = e_i$. Then, by (9)

$$\langle Y^i(t),\eta\rangle = \eta_i,$$

implying that the total mass, in the long run, is maximized when such unitary mass is allocated in the node where η_i is maximal.

(2) The minimal η_i signals also the best node to extract a mass so as to have minimal impact on the overall growth rate. More precisely, if every player extracts the resource proportionally to the (weighted) total mass, that is, $c_i = I_i(t)\langle X(t), \eta\rangle$, with $I(t) = (I_i(t))_i$ denoting the intensities of extraction at time t, then the evolution of the system becomes

$$\dot{X}(t) = (A + G^\top)X(t) - \langle X(t), \eta\rangle I(t)$$

which implies $\langle \dot{X}(t), \eta\rangle = g\langle X(t), \eta\rangle$ with

$$g = \lambda - \sum_{i=1}^{n} I_i(t)\eta_i$$

the new growth rate of the total mass. In particular, the rate g is a decreasing function of I_i with

$$\frac{\partial g}{\partial I_i} = -\eta_i, \tag{10}$$

implying that one has the lowest decrease in the growth rate when the resource is extracted at the node where η_i is minimal.

(3) The η_i's represents what network theory terms the *eigencentrality* of node i, not of the original $\mathcal{G}$ but of a related network $\mathcal{G}'$ whose adjacency matrix is $A + G$. Note that since $(A + G)\eta = \lambda\eta$, and the matrix $A - \lambda I$ is diagonal with all positive diagonal coefficients $\lambda - a_i$, one can rewrite

$$(\lambda I - A)^{-1}G\eta = \eta. \tag{11}$$

Then η is the dominant eigenvector (of eigenvalue 1) also of the *migration* network with adjacency matrix $(\lambda I - A)^{-1}G$, that is, where the coefficients of the original adjacency matrix G are magnified by reproduction rates: the i-th row of G is multiplied by $1/(\lambda - a_i)$, and flows are magnified by such factor; the greater a_i, the stronger the effect.

3. Existence of Markovian Equilibria

In our investigation of Nash equilibria of the game, we restrict the search to *Markovian* equilibria, that is, equilibria in which the strategies (the extraction rates) c_i of the agents are described as reaction maps, in real time, to the observed level of the stock $X(t)$ at time t, that is

$$c(t) = \psi(X(t)), \quad \text{with } c_i(t) = \psi_i(X(t)), \ \forall i \in N$$

(clearly, $\psi_i \equiv 0$, $i \in N \setminus F$) and the system evolves according to the *closed-loop equation* (briefly, CLE)

$$\begin{cases} \dot{X}(t) = (A + G^\top)X(t) - \psi(X(t)), & t > 0 \\ X(0) = x_0, \end{cases} \tag{12}$$

provided that such equation has a (unique) solution. We define such reaction maps as

$$\psi = (\psi_1, \psi_2, \ldots, \psi_n), \quad \psi_i : S \to [0, +\infty)$$

where S is a subset of $\mathbb{R}^n_+$ (possibly coinciding with $\mathbb{R}^n_+$, or with a cone contained in $\mathbb{R}^n_+$), depending on the data of the problem. We denote the set of admissible strategy profiles by

$$\mathbb{A} = \mathbb{A}_1 \times \mathbb{A}_2 \times \cdots \times \mathbb{A}_n,$$

where $\mathbb{A}_i$ is the collection of all (re)actions ψ_i of player i (or a null reaction in nodes with reserves). We denote by $X(t; \psi; x_0)$ or by $X^{\psi, x_0}(t)$ the solution of (12). We also adhere to the custom of denoting by ψ_{-i} all components of ψ different from the i-th, so that $\psi = (\psi_i, \psi_{-i})$.

The idea underlying the definition of a *consistent couple* below is the following. If the initial stock x lies in the subset S of the positive orthant and players select their strategies in $\mathbb{A}$, then the trajectory always remains in S. Now, let's assume that one or more agents playing a Markovian strategy deviate from their initial choice for a lapse of time. In any case, the stock is stirred within the set S, so if the agents' original choice was optimal for each of their maximization problems, it remains a feasible and optimal reaction from their new position in S.

Definition 1 **(Consistent couple)** *Assume that for any $x_0 \in S$, for any $\psi \in \mathbb{A}$, there exists a unique solution X^{ψ, x_0} to* (12)*, with $X^{\psi, x_0}(t) \in S$ for all $t \geq 0$. Then, the couple $(S, \mathbb{A})$ is said to be* consistent.

The existence of a consistent couple is not for free, and we will need to specify how it must be chosen for different networks (see section 3.3).

The next step is the definition of a Nash equilibrium, for which we will refer to the following.

Definition 2 **(Markovian Perfect Equilibrium)** *Assume the couple $(S, \mathbb{A})$ is consistent. We say that a strategy profile $\psi \in \mathbb{A}$, is a* Markovian Perfect Equilibrium (MPE) *if, for all $x_0 \in S$ and for all $i \in F$, the control $c_i(t) = \psi_i(X_i^{\psi, x_0}(t))$ is optimal for the problem of Player i, given by: the state equation* (1) *in which $c_j(t) = \psi_j(X(t))$*

for every $j \neq i$; the constraints (2)*; the functional $J_i(c_i)$ given by* (4)*, to be maximized over the set of admissible controls $\mathbb{A}_i$.*

Note that if the problem is set at a consistent couple $(S, \mathbb{A})$, a Markovian Nash equilibrium ψ in $\mathbb{A}$ is subgame perfect by definition: if a player deviates (purposefully or mistakenly) from ψ, they cannot leave the set S, and the strategy profile ψ remains feasible and Nash from the state reached in S. Hence, adding to the previous remarks, identifying a suitable consistent couple $(S, \mathbb{A})$ is crucial to ensure subgame perfection of the equilibria that we compute.

The above notwithstanding, we will proceed by initially assuming that the problem can be set at a consistent couple $(S, \mathbb{A})$, and that the equilibrium lies in $\mathbb{A}$ (Theorem 1 in section 3.1), and by later identifying a consistent couple in different sets of data, so that the assumptions of Theorem 1 are satisfied (section 3.3).

3.1. **Dynamic Programming.** In the forthcoming Theorem 1 and the subsequent remarks, we establish the existence of an MPE, computing an explicit formula for the equilibrium, the welfare of players, and other relevant quantities. To this end, we solve the problem of player i by means of Dynamic Programming, as we outline below:

(1) We define V_i, the value function – or *welfare* – of player i, as the highest overall utility of player i among those achieved with a choices of his/her strategy c_i, namely

$$V_i(x) = \sup_{c_i \in \mathbb{A}_i} J_i(c_i; x)$$

where x is the initial stock of the resource and the notation $J_i(c_i; x)$ points out that dependence in J_i;

(2) Given the strategies c_j, with $j \neq i$ as known, we associate with the problem of player i a Hamilton-Jacobi-Bellman (HJB) equation, namely

$$\rho v(x) = \max_{c_i \geq 0} \left\{ u(c_i) - c_i \frac{\partial v}{\partial x_i}(x) \right\} + \langle x, (A+G)\nabla v(x) \rangle - \sum_{j \in F - \{i\}} \left(\frac{\partial v}{\partial x_j}(x) \right) c_j \quad (13)$$

to which V_i is a (candidate) solution;

(3) We establish the relationship between the maximizing control c_i^* and the value function V_i, $u'(c_i^*) = \frac{\partial V_i}{\partial x_i}(x)$; thus, at every moment, the marginal utility from extraction is equal to the marginal cost of having a smaller amount of the resource in the future at node i. Note that, in both cases of power function and logarithmic utility, u is invertible on the positive real axis, so that the

previous relationship can be rewritten as

$$c_i^* = \psi_i(x) \equiv (u')^{-1} \left(\frac{\partial V_i}{\partial x_i}(x) \right) \tag{14}$$

which becomes a closed-loop formula for c_i^*, once V_i is known;

(4) Finally, in Theorem 1 we exhibit the welfares V_i's and a strategy profile $c^* = (c_i^*, c_{-i}^*)$, with linear dependence on the observed stock $X(t)$ that fulfills the above properties for every $i \in F$, thus an MPE. Specifically, we provide an analytic formula for c^* as a function of $X(t)$.

Before detailing these steps, we introduce some useful notation. We define ξ as the vector with components $\xi_i = \eta_i^{-1}$ if $i \in F$, and $\xi_i = 0$ otherwise, and $\xi\, \eta^\top$ the $n \times n$ matrix obtained by multiplying the column vector ξ by the row vector $\eta^\top$, in symbols

$$\xi = \sum_{i \in F} \eta_i^{-1} e_i, \qquad \xi\, \eta^\top = (\xi_i \eta_j)_{ij}. \tag{15}$$

Finally we set

$$\theta := \frac{\rho + (\sigma - 1)\lambda}{1 + (\sigma - 1) f}. \tag{16}$$

Theorem 1 *Assume $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, with $\sigma > 0, \sigma \neq 1$, $\theta > 0$. Assume also that $(S, \mathbb{A})$ is a consistent couple and that $\psi^* \colon S \to \mathbb{R}_+^n$ given by*

$$\psi_i^*(x) = \frac{\theta}{\eta_i} \langle x, \eta \rangle, \text{ for all } i \in F, \qquad \psi_i^*(x) = 0, \text{ for all } i \notin F. \tag{17}$$

is a strategy profile in $\mathbb{A}$. Then:

(i) ψ^* is an MPE of the game in the sense of Definition 2;

(ii) the welfare of agent i along such equilibrium is

$$V_i(x) = \frac{\theta^{-\sigma} \eta_i^{\sigma - 1}}{1 - \sigma} \langle x, \eta \rangle^{1-\sigma}; \tag{18}$$

(iii) If $X^*(t) = X^{\psi^*, x_0}(t)$ is the trajectory at the equilibrium then

$$\langle X^*(t), \eta \rangle = e^{gt} \langle x_0, \eta \rangle \tag{19}$$

with

$$g = \lambda - \theta f = \frac{\lambda - f\rho}{1 + (\sigma - 1) f}, \tag{20}$$

The reader will find the proof of Theorem 1 in Appendix A.

Several remarks are due here:

(1) The same result, with due changes, applies to the case of logarithmic utility $u(c) = \ln(c)$. Although we do not discuss this in detail, it can be proven that the associated MPE is obtained from (17) setting $\theta = \rho$, corresponding to the choice $\sigma = 1$ in (16). Consequently, the welfare of agent i reads as

$$V_i(x) = \frac{1}{\rho}\left[\ln\left(\frac{\rho}{\eta_i}\langle x, \eta\rangle\right) + \lambda - f\rho\right].$$

(2) For all choices of u, the extraction ψ_i^* and the welfare V_i of player i are greater at nodes i, with a smaller η_i. This behavior is consistent with (10), as the overall growth rate g is damaged more at nodes with a greater η_i, resulting in a future lesser utility. Moreover, agents appear to self-regulate, extracting less when at a more central node. However, because in equation (20), the rate of growth is independent from the agent's assignment, this propensity does not give the regulator a tool to promote the conservation of the resource: the diminished intensity of extraction that the agents optimally apply at more central nodes just offsets the negative effects of exploiting more productive sites.

(3) Note that (18) implies $\frac{\partial V_i(x)}{\partial x_j} = \theta^{-\sigma}\eta_i^{\sigma-1}\langle x, \eta\rangle^{-\sigma}\eta_j$ so that for any couple of indices j, k in N one has

$$\frac{\frac{\partial V_i(x)}{\partial x_j}}{\frac{\partial V_i(x)}{\partial x_k}} = \frac{\eta_j}{\eta_k}, \tag{21}$$

where the left-hand side represents the relative shadow prices of the resources at nodes j and k, as evaluated by player i. Nonetheless, the right-hand side does not depend on i, implying that every player gives the same relative evaluation of stocks, independently of the node where they stand.

(4) Equation (19) says that the weighted total mass $\langle X^*(t), \eta\rangle$ of the resource at equilibrium grows with rate g, which equals the natural growth rate λ diminished by a quantity proportional to both θ and the number of players f. As a consequence of (6), g is also the growth rate in the long run of the aggregate stock $\sum_i X_i(t)$.

3.2. **Stability.** We are now interested in the long-term behavior of the stock, particularly in establishing if the stock tends to stabilize over time around certain values at the different nodes. Technically speaking, we address convergence of transitional dynamics towards a potential steady state. To do so, it is useful to describe the equilibrium trajectory through the eigenvectors/eigenvalues of the matrix of the CLE

(12), in which ψ^* is implemented, that is, $A+G^\top-\theta\,\xi\eta^\top$. Indeed, by means of (15), the MPE (17) can be written in vector form as

$$\psi^*(x)=\theta\langle x,\eta\rangle\xi=\theta\,\xi\eta^\top x,$$

and (12) becomes

$$\begin{cases} \dot X(t)=(A+G^\top-\theta\,\xi\eta^\top)X(t), & t>0 \\ X(0)=x_0. \end{cases} \tag{22}$$

(for both power and logarithmic utilities, with $\theta=\rho$ for the logarithmic utility). Now we set

$$\theta_1=\frac{\lambda-\mathrm{Re}(\lambda_2)}{f}, \tag{23}$$

(noting that $0<\theta<\theta_1$ is equivalent to $g>\mathrm{Re}(\lambda_2)$), and $E=\ \xi\eta^\top$. The properties of E are listed in Lemma 2 in the Appendix, and in particular, they imply the following result.

Lemma 1 *Assume* $0<\theta<\theta_1$*.*

(i) $A+G-\theta E^\top$ *has normalized eigenvector* η *associated with the eigenvalue* $g=\lambda-\theta f$*; as a consequence, there exists a normalized eigenvector* $\hat\zeta$ *of* $A+G^\top-\theta E$ *associated to the same eigenvalue* g*.*

(ii) Consider the base $\{\zeta,v_2,\dots,v_n\}$ *of generalized eigenvectors of* $A+G^\top$*, associated to the eigenvalues* $\{\lambda,\lambda_2,\dots,\lambda_n\}$*. Then,* $A+G^\top-\theta E$ *has eigenvalues* $\{g,\lambda_2,\dots,\lambda_n\}$ *associated respectively with eigenvectors* $\{\hat\zeta,v_2,\dots,v_n\}$*.*

The above lemma implies that the extraction process modifies only the direction of the principal eigenvector of the matrix $A+G^\top$, which changes from ζ to $\hat\zeta$, and the associated eigenvalue, which decreases from λ to g. The remaining eigenvalues and eigenvectors remain the same.

Since both g and $\hat\zeta$ depend continuously on θ, and $\zeta>0$, then there exists $\theta_2>0$ such that $\hat\zeta\equiv\hat\zeta(\theta)$ is definitively positive for all $\theta<\theta_2$. We then set

$$\theta_2=\sup\{\theta:\hat\zeta(s)>0,\forall s\in[0,\theta]\}.$$

Then, in the next proposition, we discuss the decomposition of the equilibrium trajectory along the eigenvectors directions, and its definitive convergence within the positive orthant (see also Lemma A.1 in Appendix A).

Proposition 1 *Assume* $0<\theta<\theta_1$*, let* X^* *be the equilibrium trajectory described in Theorem 1. Then, there exists* $\hat\alpha\geq0$ *such that the detrended trajectory*

$X^*(t)e^{-gt}$ satisfies

$$\lim_{t\to+\infty} X^*(t)e^{-gt} = \hat{\alpha}\,\hat{\zeta}. \tag{24}$$

If in addition $\theta < \theta_2$, the trajectory enters definitively the positive orthant.

The proof can be found in Appendix A, and has the following consequences and interpretations:

(1) *Stock in the long run.* When g is (remains) the eigenvalue with greatest real part among $\{g, \lambda_2, \dots, \lambda_n\}$, the equilibrium trajectory X^* converges toward the direction of the associated eigenvector $\hat{\zeta}$, meaning that in the long run, the stock X^* tends to be distributed in the various nodes proportionally to the components of $\hat{\zeta}$.

(2) *Players are patient.* Note that convergence within the positive orthant is guaranteed by a sufficiently small θ, expressed by the condition $0 < \theta < \min\{\theta_1, \theta_2\}$. This has a straightforward interpretation regarding logarithmic utility ($\theta = \rho$), since requiring that θ is sufficiently small is equivalent to assuming that the agents are sufficiently patient. If instead $\sigma \neq 1$ (i.e., the elasticity of intertemporal substitution is different from 1) and the number of agents is given, $\theta \approx 0$ means $\rho \approx \hat{\rho} \equiv (1-\sigma)\lambda$. In optimal growth theory, the value $\hat{\rho}$ gives the critical discount rate (i.e., the minimum discount rate for which an optimal solution exists, in the case of one player). The case of an exogenous growth rate is dealt with in Brock and Gale (1969), while the case of a linear technology, the one relevant here, is treated extensively in McFadden (1973)[5]. Since with a linear technology there is a trade-off between the growth rate and the intensity of consumption, we can think that agents for whom $\rho \approx \hat{\rho}$ are patient in the generalized sense that they prefer a high growth rate over immediate consumption. The peculiarity in our multiagent setting is that the sign of the difference $\rho - \hat{\rho}$ is not necessarily positive but depends on the sign of the denominator in the formula defining θ.

3.3. **Subgame Perfection.** According to Definitions 1 and 2, the result of Theorem 1 needs to be completed with the identification of a consistent couple $(S, \mathbb{A})$ for the problem, namely, a set of initial data S and a set of strategy profiles $\mathbb{A}$ such that, when starting from S, the stock vector $X(t)$ can never be stirred out of S when players choose strategies/reactions in $\mathbb{A}$.

[5]In both these studies, time is discrete. However, analogous results hold in continuous time.

The existence and choice of the consistent couple depends on the primitives of the problem, such as the intensity of the connections between nodes, represented by the matrix G, and the magnitude of θ described by (16). We then separate the discussion into the case of a fully connected and the (merely) strongly connected networks.

3.3.1. *Fully Connected Networks.* We first discuss the case in which the network is fully connected: all nodes are connected by positive edges, in both directions, and hypothesis (H1) is replaced by the stronger

(H2) $\mathcal{G}$ is fully connected, i.e. $g_{ij} > 0$, $\forall i, j \in N$ with $i \neq j$, and $g_{ii} = 0, \forall i \in N$.

Under this assumption, as large an S as possible can be chosen, that is

$$S = \mathbb{R}^n_+,$$

and coupled with set of strategies $\mathbb{A}_i$ for player i, $i \in F$, given by

$$\mathbb{A}_i := \left\{ \psi_i \colon \mathbb{R}^n_+ \to [0, +\infty) \ : \ \begin{array}{r} (i)\, \psi_i \text{ is Lipschitz-continuous} \\ (ii)\, \psi_i(x) \leq \left\langle (A + G^\top)x, e_i \right\rangle \\ \text{for all } x \in \mathbb{R}^n_+ \text{ such that } x_i = 0. \end{array} \right\} \tag{25}$$

When $i \notin F$ we assume $\mathbb{A}_i$ contains only the null strategy. Note that the request (i) that ψ_i is Lipschitz-continuous[6] implies that the CLE (12) has a unique (classical) solution, $X(t)$; whereas the condition (ii) ensures that, when the mass at node i is null, the extraction ψ_i can be at most as much as the inflow at node i from the other nodes, so that the mass at node i remains non-negative at all times.

Clearly $(S, \mathbb{A})$ defined above is a consistent couple as the (unique) solution of the CLE (12) starting at a point of the positive orthant x_0 remains there contained at all subsequent times, for all choices of a strategy profile in $\mathbb{A}$.

However, is the strategy profile described in (17) admissible? The answer is affirmative for certain values of the data, as we specify below.

Proposition 2 *The strategy profile ψ^* described in* (17) *lies in $\mathbb{A}$ (and consequently, ψ^* is an MPE) if and only if*

$$0 < \theta \leq g_{ij} \frac{\eta_i}{\eta_j}, \quad \forall i \in F,\, j \in N,\, i \neq j. \tag{26}$$

Specifically, if (26) *is violated, there exist initial data $x_0 \in \mathbb{R}^n_+$ such that the trajectory $X^*(\cdot)$ starting at x_0 leaves the positive orthant $\mathbb{R}^n_+$ at some times.*

[6]Namely, there is constant $L > 0$ such that $|\psi_i(x) - \psi_i(y)| \leq L|x - y|$, for all $x, y \in \mathbb{R}^n_+$. A Lipschitz-continuous function is continuous and differentiable almost anywhere (with respect to Lebesgue measure) and with a bounded derivative, at the points where it exists.

The proof is in Appendix A.

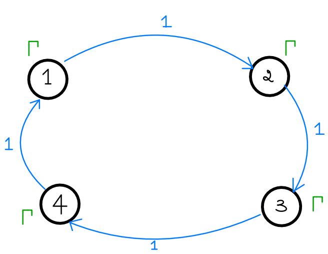
Again, the condition (26) says that θ needs to be small for ψ^* to be admissible, meaning we should assume that agents are sufficiently patient in extracting the resource. The critical value for θ is

$$\hat{\theta} = \min \left\{ g_{ij} \frac{\eta_i}{\eta_j} \ : \ i \in F, \, j \in N, \, i \neq j \right\}$$

with $\hat{\theta} > 0$, due to the full connection of the network.

3.3.2. *Strongly Connected Networks.* For a strongly connected network where some g_{ij}, $i \neq j$ are null, Proposition 2 implies that there exist some initial positions x_0 from which the trajectory X^* solving (22) is not feasible, i.e. $X^*(t)$ leaves the positive orthant, at least for some times t. An example of this fact follows.

Example. Consider a network with $n \geq 4$ nodes, and in which node i is only connected to node $i+1$ (and node n only to node 1) with edge weight 1, and all natural growth rates $\Gamma_i = \Gamma$. (Note that with such symmetry nodes are fundamentally indistinguishable.)



We have $\lambda = \Gamma$, $\eta = \zeta = \frac{1}{\sqrt{n}} \sum_{j=1}^n e_j$, and the candidate MPE, out of the reserves, is $\psi_i^*(x) = \theta \sum_{j=1}^n x_j$. Assume X^* is the candidate equilibrium trajectory and $1 \in F$, so that

$$\dot{X}_1^*(t) = (\Gamma - 1) X_1^* + X_n^* - \theta \sum_{j=1}^n X_j^*(t),$$

and consider the initial condition $x_0 = e_3$. Then $(X_1^*)'(0) = -\theta X_3(0) = -\theta$ and the trajectory leaves immediately the positive orthant for every $\theta > 0$.

Nonetheless, exhibiting a consistent couple is possible, at least in presence of some *effort constraints*, which introduce in the model the feature that extraction is more difficult when the resource is less abundant. More precisely, we consider the standard Schaefer catch functions $\beta_i E_i x_i$, where the stock x_i is bilinearly combined with player i 's effort E_i, and the catchability parameter β_i, $\beta_i \geq 0$ for all $i \in F$. Assuming player i 's total capacity for effort is finite and normalized to 1, we derive the constraints

$$\psi_i(x) \leq \beta_i x_i$$

We define $\mathbb{A}^\beta = \mathbb{A}_1^{\beta_1} \times \cdots \times \mathbb{A}_n^{\beta_n}$, where the strategy set of player i is

$$\mathbb{A}_i^{\beta_i} := \left\{ \psi_i \colon S \to [0, +\infty) \ : \ \begin{array}{l} (i) \, \psi_i \text{ is Lipschitz-continuous} \\ (ii) \, \psi_i(x) \leq \beta_i x_i, \ \text{for all } x \in S. \end{array} \right\}$$

Strategies in $\mathbb{A}_j^{\beta_j}$ at reserve nodes $j \notin F$ are chosen null.

We now establish the existence of a cone S in $\mathbb{R}^n_+$ such that $(S, \mathbb{A}^\beta)$ is a consistent couple. For a positive defined matrix $P \in \mathbb{R}^n \times \mathbb{R}^n$ and a real constant d, we define the ellipsoid depending on P, d and the positive eigenvector ζ

$$E(P, d) := \left\{ x \in \mathbb{R}^n_+ \ : \ x^\top P x \le d \right\},$$

and the positive cone through the origin containing it

$$S^* = \{ry : r > 0, y \in \zeta + E(P, d)\}. \tag{27}$$

Note that for d small enough, $E(P, d)$ is an arbitrarily small neighborhood of the origin, regardless of the choice of P. Consequently, for a small enough d, both $\zeta + E(P, d)$ and S^* are entirely contained in $(0, +\infty)^n$.

PROPOSITION 3 *Let ψ^* be the strategy profile described in* (17)*, and* $X^*(t) := X(t; x_0, \psi^*)$ *the associated trajectory. Let additionally S^* be the cone defined in* (27) *and contained in* $(0, +\infty)^n$*, and let* $0 < \theta < \min\{\theta_1, \theta_2\}$*. Then, for small enough catchability parameters β_i (with $\beta_i > 0$) and extraction intensity θ (θ possibly smaller than* $\min\{\theta_1, \theta_2\}$*), the couple* $(S^*, \mathbb{A}^\beta)$ *is consistent, the strategy profile ψ^* is admissible, and hence ψ^* is an MPE in* $(S^*, \mathbb{A}^\beta)$*.*

See Appendix A for the proof.

3.4. **Uniqueness of the equilibrium.** Finding all the MPE for our problem would require simultaneously solving f interdependent partial differential equations of type (13), one for every player. Obviously, when the state variable is scalar, the system reduces to a system of ordinary differential equations, for which standard uniqueness results can sometimes be used (e.g., Cvitanić and Georgiadis, 2016), but with a state variable dimension of at least 2 we are not aware of any uniqueness result for systems of PDE of such general form. Nonetheless, assuming that linear strategies are salient because of their simplicity and that, therefore, players are more likely to coordinate on this kind of equilibria rather than on (eventually existing) alternative equilibria with more complex structure, we analyze uniqueness among linear strategies.

Regarding the strategy profile ψ^* given by (17), we are able to prove that:

(i) when the network is fully connected (i.e., in the unconstrained problem), ψ^* is the unique linear MPE for the problem on $(\mathbb{R}^n_+, \mathbb{A})$;

(ii) for a general strongly connected network (i.e., for the problem with effort constraints), ψ^* is the unique linear MPE for the problem on $(S^*, \mathbb{A}^\beta)$, if $\beta > 0$ and $\theta > 0$ are small enough.

Theorem 2 *Assume (H2), i.e.* $\mathcal{G}$ *fully connected, and* $\theta \in (0, \hat{\theta})$ *and that* $f < n$ *(i.e., there exists at least a reserve). Then the hypotheses of Theorem 1 are verified at the consistent couple* $(\mathbb{R}^n_+, \mathbb{A})$ *and the strategy profile* ψ^* *there described is the unique linear MPE of the game, namely, the unique form* $\psi_j(x) = \langle w^j, x \rangle$*, with* $w^j \in \mathbb{R}^n_+$*.*

The long and nontrivial proof of this statement is contained in the Appendix.

The presence of reserves is quite primal in the topic of this paper so the assumption $f < n$ in Theorem 2 can be considered not particularly strong. However, note that this assumption is indispensable: one can construct examples of systems with $f = n$ (and that verify the other assumptions of the theorem) in which more than one linear equilibrium can be constructed. However, these are specific examples that likely only take place on a set of measure zero on the space of admissible parameters.

A counterpart of the result proved in Theorem 2 can additionally be stated for the second consistent couple characterized in Proposition 3, namely, the cone and the effort-constrained controls. Here, we do not need to restrict to fully connected networks but need to work with sufficiently small efforts β_i. The result is as follows.

Theorem 3 *Suppose that the hypotheses of Proposition 3 are verified. If* $\theta \in (0, \theta_1)$ *and* $\beta > 0$ *are small enough the MPE given in Theorem 1 and Proposition 3 is the unique linear MPE of the game on* $(S^*, \mathbb{A}^\beta)$*.*

A sketch of the proof, very similar and even simpler than the one of Theorem 2, can be found in Appendix A.

4. Comparative statics

4.1. **Optimal Location of the Reserves.**

Here, we assume that the number f of extraction permits are given and that the intervention of the regulator is limited to deciding where natural reserves are placed among the available n regions. This decision is made at the beginning of the game and never changed afterwards.

We assume the regulator aggregates preferences in a Benthamian way, that is, they act to maximize the sum of the utilities of all players, knowing that agents will choose their strategies according to Theorem 1. We assume also that the equilibrium strategies are admissible for the given set of data, for any choice of reserves placement. The regulator then compares the outcome of different equilibria associated to different placements of the $n - f$ reserves and chooses the configuration maximizing the sum of utilities of players.

In symbols, if $\mathcal{F} = \{F \subset N : |F| = f\}$ describes all possible subsets of N having f elements, he/she maximizes with respect to $F \in \mathcal{F}$

$$W(x, F) = \sum_{i \in F} V_i(x). \tag{28}$$

where value functions V_i are those described in Theorem 1, and x is the initial distribution of the resource through the nodes. Thus, the following result is a corollary of Theorem 1.

PROPOSITION 4 *In the hypotheses of Theorem 1, assume the strategies profile ψ^* is admissible at x_0 for any choice of $F \in \mathcal{F}$. Then the social welfare W defined in (28) is maximized if the natural reserves are built at a subset F of nodes i where η_i are highest. If $F_{MAX} \in \mathcal{F}$ is one of such choices, then*

$$W(x_0) = \frac{\theta^{-\sigma}}{1-\sigma} \langle x, \eta \rangle^{1-\sigma} \sum_{i \in F_{MAX}} \eta_i^{\sigma-1}. \tag{29}$$

The proof is straightforward. This result has a clear explanation. In Section 2.1 we showed that η_i measures the long term productivity at node i, and in (21) that every player gives the same evaluation of stocks independently of the node where they stand. Moreover, as observed in Section 3.2, “patient” agents (i.e., agents whose optimal rate of extraction θ is small) prefer to consume dividends rather than the stock. Given that resource flows extracted in the different regions are perfect substitutes, it clearly follows that the regulator must optimally preserve those units of stocks that prospectively have a higher productivity.

4.2. **Comparative Growth Rates.** We now analyze how the long-term growth rate of the resource stocks, namely g given by (20), changes with respect to the parameters of the system.

We start by varying reproduction rates Γ_i, embodying local productivity advancements, and the entries of the adjacency matrix G, describing flows among nodes. As equation (20) shows, the largest eigenvalue of the matrix $A + G$ (i.e., the maximum rate of growth of the system) is the medium through which these “technological” parameters impact the equilibrium rate of growth. Hence, the two relevant questions to ask are: how does the largest eigenvalue change with changes in the “technological” parameters? How does a change in the largest eigenvalue affect the rate of growth?

To answer the latter question so as to convey economic intuitions, it is useful to rewrite equation (20) explicitly considering the dependence of the equilibrium extraction rate θ on the largest eigenvalue λ:

$$g = \lambda - \theta(\lambda) f. \tag{30}$$

Then, recalling that θ is given by (16), we derive

$$\frac{d\theta}{d\lambda} = -\frac{1-\sigma}{1-(1-\sigma)f}, \tag{31}$$

which is certainly positive if $(1-\sigma) < 0$, while it has the opposite sign of $1-(1-\sigma)f$, that is the denominator in (16), if $(1-\sigma) > 0$[7].

Thus, there are two different cases in which the agents exploit the resource more intensively when λ is greater: when $(1-\sigma) < 0$ and when $(1-\sigma) > 0$ and $1-(1-\sigma)f < 0$. However, these two cases differ, because in the former $\frac{d\theta}{d\lambda} = \frac{1}{f-(1-\sigma)^{-1}} < \frac{1}{f}$, while in the latter $\frac{d\theta}{d\lambda} > \frac{1}{f}$. Therefore, under the latter parameter configuration there is a disproportionate increase of the agents equilibrium extraction rates implying $\frac{dg}{d\lambda} = 1 - f\frac{d\theta}{d\lambda} < 0$. In the context of a development model with common and multiple private stocks, Tornell and Lane (1999) dubbed this disproportionate agents reaction *the voracity effect.*

On this basis, recalling that Theorem 1 ensures the existence of an MPE when $\theta > 0$, so that numerator and denominator in (16) bear necessarily the same sign, the following two regimes can be identified:

(a) A *standard regime*, in which $\rho - (1-\sigma)\lambda > 0$ and $1 + (\sigma - 1)f > 0$. These same conditions can be reformulated as follows:

$$\rho - (1-\sigma)\lambda > 0, \text{ and } (\sigma \geq 1, f \geq 1) \vee \left(0 < \sigma < 1,\ 1 \leq f < \frac{1}{1-\sigma}\right), \tag{32}$$

where with $\sigma = 1$ we intend the case of logarithmic utility.

(b) A *voracious regime* with negative numerator and denominator, i.e. $\rho - (1-\sigma)\lambda < 0$ and $1 + (\sigma - 1)f < 0$, which can be reformulated as

$$\rho - (1-\sigma)\lambda < 0,\ 0 < \sigma < 1,\ f > \frac{1}{1-\sigma}. \tag{33}$$

We note that a positive sign of $\rho - (1-\sigma)\lambda$, only holding in the standard regime, is a necessary and sufficient condition for a finite value function regarding a single player ($f = 1$), and regarding a social planner maximizing the sum of utilities of players. Moreover, as λ represents the asymptotic growth rate of the resource under null extraction, the result is consistent with the parallel condition $\rho - (1-\sigma)A > 0$ in the standard single-player/social-planner AK-models (for extraction or growth).

[7]We recall that in the case of $\sigma = 1$ the rate of extraction coincides with the rate of discount and is, therefore, independent from λ.

The role of the condition $\rho - (1-\sigma)\lambda > 0$ remains the same in a game with a general number f of players. Conversely, in the game with f players, when $\rho - (1-\sigma)\lambda > 0$ and $\sigma < 1$ hold, the condition $1 + (\sigma - 1)f > 0$ implies a restriction on the number of players, as necessarily $f < \frac{1}{1-\sigma}$. This condition descends from the fact that each agent, in solving their control problem, perceives a maximum rate of growth of the resource that is given by $\lambda - (f-1)\theta$. Hence their value function is finite if and only if $\rho + (\sigma - 1)[\lambda - (f-1)\theta] > 0$. Given the value of θ, this is equivalent to $1 + (\sigma - 1)f > 0$. Similar conditions for the aggregate cases with $A = 0$ are given in Dockner et al. (2000).

In the voracious regime (33), the outcome for the game and the social planner problem diverge: over an infinite horizon, the social planner problem is not well defined, as some strategies engender an infinite value function; conversely, a Nash equilibrium for the game does exist, as players at the equilibrium tend to overexploit the resource, hence reducing their future earnings, and keeping their value functions finite.

The two regimes recur, giving rise to significantly different system behaviors[8]. We have already specifically shown that, while in the standard regime a greater value of λ implies a greater rate of growth, in the voracious regime, whenever the value of λ is greater, the agents always coordinate on a Nash equilibrium with a lower growth rate. Formally, we have the following proposition.

Proposition 5 *In the assumptions of Theorem 1, consider the growth rate g of the system described in (20), as a function of the dominant eigenvalue λ. Then:*

(i) *g is strictly increasing in λ, in the standard regime* (32)*;*
(ii) *g is strictly decreasing in λ, in the voracious regime* (33)*.*

With these preliminaries in place and given the well-known result that the largest eigenvalue of an irreducible Metzler matrix is an increasing function of its elements, the comparative statics regarding the reproduction rate Γ_i is now a straightforward corollary of the above proposition.

Corollary 1 *In the assumptions of Theorem 1, consider the growth rate g of the system described in (20), as a function of the reproduction rate Γ_i. Then:*

(i) *g is strictly increasing in Γ_i, in the standard regime* (32)*;*

[8]On whether one regime is more realistic than the other, the empirical evidence is inconclusive. Some studies report an elasticity of intertemporal substitution $\frac{1}{\sigma}$ close to zero (see for instance Hall, 1988 and Best et al., 2020), while others report values greater than 1 (e.g. Gruber, 2013).

(ii) *g is strictly decreasing in Γ_i, in the voracious regime* (33).

Next, we analyze how the growth rate g changes with changes in the entries of the adjacency matrix G. Note that we cannot use the fact that λ is an increasing function of the elements of the matrix $A+G$. Indeed, a change in a migration flow engenders a simultaneous change that is equal but opposite in sign, in the net growth rate of the node from which the resource flows. Nevertheless, for symmetric networks, we can establish the following result.

PROPOSITION 6 *Assume G is symmetric, i.e. $g_{ij} = g_{ji}$. Consider the dominant eigenvalue λ of the matrix $A+G$ as a function of g_{ij} $(i, j \in N,\ i \neq j)$. Then λ is a nonincreasing function of g_{ij}. Moreover, λ is strictly decreasing if $\Gamma_i \neq \Gamma_j$.*

The intuition behind the negative effect of increased mobility on the maximum growth rate λ is that it prevents the accumulation of the stock in more productive sites. Having proved this, we can again invoke Proposition 5 to conclude that, as for the local reproduction rates, the manner in which this effect translates on the equilibrium rate of growth depends on which of the two above regimes prevails. In particular, we have the following result.

COROLLARY 2 *Assume G is symmetric, i.e. $g_{ij} = g_{ji}$. In the assumptions of Theorem 1, consider the growth rate g of the system described in (20), as a function of g_{ij} $(i, j \in N,\ i \neq j)$. Then:*

(*i*) *g is a nonincreasing function of g_{ij}, in the standard regime* (32)*;*
(*ii*) *g is a nondecreasing function of g_{ij}, in the voracious regime* (33).

Finally, we analyze how the equilibrium growth rate of the system, g, is affected by the number f of issued extraction permits. The conclusions once again diverge in the standard and voracious regimes.

PROPOSITION 7 *In the assumptions of Theorem 1, consider the growth rate g of the system described in (20), as a function of the number of players f. Then:*

(i) *g is strictly decreasing in f, in the standard regime* (32)*;*
(ii) *g is strictly increasing in f, in the voracious regime* (33).

The picture emerging from Proposition 7 is pretty transparent. In the standard regime a "tragedy of commons" mechanism prevails: the higher the number of agents the quicker they tend to appropriate the resource to avoid being preceded by the others. Conversely, in the voracious regime a higher number of players interferes with the voracious baseline attitude so that the greater the competition the more resource survival is guaranteed.

4.3. **Endogenizing the number of permits.** Presently, the number of agents involved in the game, i.e. the number f of issued extraction permits, has been given exogenously. In this section, we briefly discuss the case in which the social regulator can choose such number f^* to maximize the overall welfare W described in (29).

We assume there are $f_M < n$ potential concessionaires, hence, the regulator chooses a number f in the set $\{0, 1, \ldots, f_M\}$. Note that the outside option for the players must be specified differently for the two cases $\sigma \geq 1$ and $\sigma < 1$. In the former, indeed, the value function $V_i(x)$ of an agent with extraction rights is unbounded below (it goes to $-\infty$, if $x \to 0$), and $-\infty$ is the only consistent outside option. This implies that with $\sigma \geq 1$, the only possibility for the regulator to keep the value of the welfare function above $-\infty$ is to choose $f = f_M$. In the latter case, $V_i(x)$ is bounded below by zero, and so the outside option can be consistently set at zero – as under these circumstances no agent with a permit has ever an incentive to quit the game, while an outsider always wants to join the game. Henceforth, therefore, we limit our analysis to the case of $\sigma < 1$.

Let us order the sites according to their eigenvector centrality, with site 1 the most peripheral one. Proposition 4 implies that issuing the permits in increasing order of centrality is a necessary condition for welfare maximization. Therefore, when $f \leq f_M$ permits are issued, the welfare function for $f > 0$ can be written as

$$W(f) = \frac{[\theta(f)]^{-\sigma}}{1-\sigma} \langle x, \eta \rangle^{1-\sigma} \sum_{i=1}^{f} (\eta_i)^{\sigma-1}, \tag{34}$$

where $\theta(f) = \frac{\rho-(1-\sigma)\lambda}{1-(1-\sigma)f} > 0$, and we can set $W(0) = 0$.

Recalling Proposition 7, comparative statics analysis with respect to the number of permits can be performed under the two (mutually exclusive) sets of assumption, standard (32) and voracious (33).

To study the maximum of the function in equation (34), we first note that

$$\frac{\partial \theta}{\partial f} = \frac{1-\sigma}{(1-(1-\sigma)f)^2} (\rho - (1-\sigma)\lambda), \tag{35}$$

so that $\theta(f+1) > \theta(f)$ holds in the standard case, while $\theta(f+1) < \theta(f)$ if voracity prevails. Then, we split the effect of issuing an additional permit into the sum of two terms

$$W(f+1) - W(f) = \frac{[\theta(f+1)]^{-\sigma} - [\theta(f)]^{-\sigma}}{1-\sigma} \langle x, \eta \rangle^{1-\sigma} \sum_{i=1}^{f} (\eta_i)^{\sigma-1} + \\ + \frac{[\theta(f+1)]^{-\sigma}}{1-\sigma} \langle x, \eta \rangle^{1-\sigma} (\eta_{f+1})^{\sigma-1}, \quad (36)$$

where the first term measures the effects on the value functions of the inframarginal concessionaires and the second the effect on the value function of the marginal concessionaire. As in the voracious regime an additional agent creates *positive* externalities on inframarginal concessionaires, we have the immediate implication that in this case the optimal number f^* of permits is f_M. In the standard regime, instead, the externalities on inframarginal concessionaires are negative and the optimal number f^* of permits can be smaller than f_M. We show in Proposition 8 that, in the latter case, $W(f+1) - W(f)$ is positive for $f = 0$ and can change sign at most once for $f > 0$. Therefore, if $f^* < f_M$, a standard interpretation in terms of public goods applies: the surplus of the planner (who in our case places equal weights on all players) is maximized at the maximum f for which the marginal benefit is still greater than the *sum* of marginal costs the group of agents incurs.

PROPOSITION 8 *In the assumptions of Theorem 1, consider the social welfare $W(f)$ described in (34), as a function of the number of players f. Then:*

(i) *Assume the standard regime* (32), *with* $0 < \sigma < 1$, $\rho - (1-\sigma)\lambda > 0$, $f \in [1, \frac{1}{1-\sigma})$. *Then there exists* $f^* \in \{1, .., f_M\}$ *such that* W *is, on* $\{1, .., f_M\}$, *increasing for* $f < f^*$, *and decreasing for* $f > f^*$, *with maximum at* f^*.

(ii) *In the voracious regime* (33), *the maximum of* $W(f)$ *is reached at* $f^* = f_M$.

4.4. **Hierarchy of Nodes.** In section 4.1 we widely discussed how all agents, at our equilibrium (17), evaluate the different sites' stocks by means of a vector of relative prices η, independently of the assignment. Nodes are ordered in a hierarchy determined by the magnitude of the components of η: the smaller η_i, the greater the welfare of player i. In section 2.1.5 we also noted that the η_i represent the eigencentrality of node i (not of the original network $\mathcal{G}$ but) of a strongly connected network $\mathcal{G}'$ associated to the matrix $A + G$, thus combining the migration flows and the net rates of growth at the sites. In this section we analyze how these two forces interact in determining the eigencentrality η of the sites.

4.4.1. *A Two-region Example.* The joint role of flow coefficients g_{ij} and of net reproduction rates a_i (or, equivalently, of reproduction rates Γ_i) in determining eigencentrality preferences is already apparent in an example with two regions and a single player. Assume that

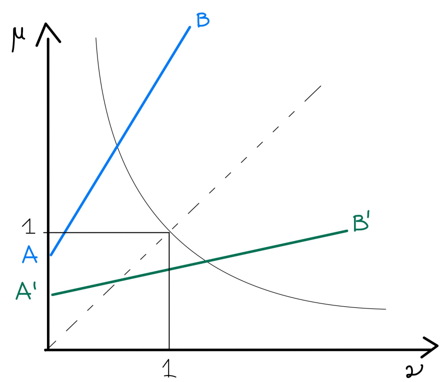
$$A + G = \begin{bmatrix} a_1 & g_{12} \\ g_{21} & a_2 \end{bmatrix}$$

and, without loss of generality, $a_1 \geq a_2$ and $\eta = (\mu, 1)^\top$. As a consequence of Proposition 4, the reserve is best set at node 1 when $\mu > 1$, at node 2 when $\mu < 1$ and indifferently at node 1 or 2 when $\mu = 1$. Note also that, given that there is a single agent, conditional on the assignment, the outcome is always Pareto efficient. To better understand the role of the parameters we expand

$$\begin{bmatrix} a_1 & g_{12} \\ g_{21} & a_2 \end{bmatrix} \begin{bmatrix} \mu \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} \mu \\ 1 \end{bmatrix},$$

to obtain $a_1 + \frac{g_{12}}{\mu} = g_{21}\mu + a_2 \iff \mu = \frac{g_{12}}{g_{21}}\frac{1}{\mu} + \frac{a_1 - a_2}{g_{21}}$, and represent the solution of the last equation as the intersection in the plane (ν, μ) of the line $\mu = \frac{g_{12}}{g_{21}}\nu + \frac{a_1-a_2}{g_{21}}$ and the hyperbola $\mu\nu = 1$, as depicted in Figure 3. When the component μ of the intersection sits above the bisectrix of the first orthant, the reserve is best set at node 1 (otherwise at node 2). This fact is influenced both by the slope g_{12}/g_{21} depending on the flows intensity between regions, as well as by the intercept $\frac{a_1-a_2}{g_{21}}$ on the μ-axis, depending on the net productivities.

FIGURE 3. AB e $A'B'$ represent two instances of equation $\mu = \frac{g_{12}}{g_{21}}\nu + \frac{a_1-a_2}{g_{21}}$, with OA (or OA') equal to $\frac{a_1-a_2}{g_{21}}$. In the case of AB, the reserve is best placed at node 1, while in the case of $A'B'$ at node 2.



4.4.2. *An example with a unique breeding ground.* Suppose the stock is distributed over three nodes and that node 3 is the only breeding ground for the resource, with positive growth rate $\Gamma_3 > 0$, while $\Gamma_1 = \Gamma_2 = 0$. To simplify the analysis, we suppose the intensity of the migration between nodes is given by a weighted undirected network, where $0 < a < 1$ is the weight of the connection between nodes 1 and 2, and $1-a$ the weight on the links between the breeding place (node 3) and nodes 1 and 2. Moreover, we assume $\Gamma_3 = (1-a)$. Since

$$A + G = \begin{pmatrix} -1 & a & 1-a \\ a & -1 & 1-a \\ 1-a & 1-a & -(1-a) \end{pmatrix},$$

it can be easily verified that the dominant eigenvalue and the corresponding eigenvector are given by

$$\lambda = (\sqrt{2}-1)(1-a), \quad \eta = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)^{\top}.$$

Assuming the two acting agents have a logarithmic utility with discount $\rho > 0$, the (candidate) optimal strategy of an agent assigned to nodes 1 or 2 is $c_1 = c_2 = \rho(x_1 + x_2 + \sqrt{2}x_3)$, while that of an agent assigned to region 3 is $c_3 = \rho(\frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2 + x_3)$. In view of Proposition 4, if the regulator maximizes the sum of the agents' utilities and the profile

$$(\psi_1(x), \psi_2(x), \psi_3(x)) = \Big(\rho(x_1 + x_2 + \sqrt{2}x_3), \rho(x_1 + x_2 + \sqrt{2}x_3),\ 0\Big)$$

is feasible, then the reserve should definitely be placed at node 3. Said differently, protecting the only breeding ground is welfare-maximizing for the regulator.

To complete the discussion, we analyze feasibility of the above strategy profile, in the absence of effort constraints. Proposition 2 implies the above strategy is feasible if and only if

$$\rho \leq \frac{1}{\sqrt{2}}\min\{a, 1-a\}.$$

Therefore, the equilibrium exists if the agents are sufficiently patient. Furthermore, by Theorem 2, such equilibrium is the unique linear MPE.

4.4.3. *Networks with equal net reproduction rates.* We now analyze the hierarchy of nodes when they are all are equally productive. We start with an example with three regions and two agents, where nodes bear the same net productivity, and are connected with different weights, as described in

$$A+G=\begin{pmatrix} -2b & a & a \\ b & -2b & b \\ b & b & -2b \end{pmatrix},$$

where $a>b>0$. The dominant eigenvalue and the corresponding eigenvector are

$$\lambda=(\hat{\eta}_1-1)b, \quad \eta=(\hat{\eta}_1,\ 1,\ 1)^\top,$$

where $\hat{\eta}_1>1$ is the positive solution of the equation $\eta_1^2+\eta_1-2a/b=0$. We also assume logarithmic utilities, with discount rate ρ. According to Proposition 4, the best choice for the regulator is to place the reserve at node 1; moreover,

$$(\psi_1(x),\psi_2(x),\psi_3(x))=(0,\ \rho(\hat{\eta}_1x_1+x_2+x_3),\ \rho(\hat{\eta}_1x_1+x_2+x_3))$$

is the unique linear MPE, when agents are patient enough to satisfy the condition $\rho\le\frac{1}{\hat{\eta}_1}b$ consistently with Proposition 2. In this case, protecting the site with the largest migration routes is the welfare-maximizing policy.

The above result can be generalized as follows to networks with identical net productivities, i.e., with

$$a_i=\Gamma_i-\sum_{j=1}^n g_{ij}\equiv a \quad \text{for all } i\in N.$$

We denote the Perron–Frobenius eigenvalue for G by $\lambda_\circ$ and the associated normalized eigenvector by $\eta_\circ$. In this context, $A+G=aI+G$, and the eigenvectors of G and $aI+G$ are the same, implying $\eta=\eta_\circ$ (with η, $\eta_\circ$ associated, respectively, to eigenvalues λ, $\lambda_\circ=\lambda-a$). Hence, when nodes are equally productive, all sites are ranked according to the eigenvector centrality $\eta_\circ$ of the migration network $\mathcal{G}$, with η_i higher when node i is better connected to the other nodes. Differently said, when nodes are undifferentiated with respect to productivity, the migration network rules the hierarchy.

4.4.4. *Fully Connected Networks with Equal Flows.* We now investigate what impacts the hierarchy of the nodes when we assume that the structure of the network is neutral. To this extent, we assume a complete symmetry of the network, i.e., that different nodes are all connected to one another with the same intensity of connection $\alpha>0$. That translates into $g_{ij}=\alpha$ for all $i\neq j$, and $g_{ii}=0$. Combining the i-th and the ℓ-th row of equation (8), one obtains

$$\eta_\ell=\frac{a_i-\lambda-\alpha}{a_\ell-\lambda-\alpha}\eta_i,$$

so that from $a_\ell - \lambda - \alpha < 0$ (see (7)) one derives

$$\eta_\ell \geq \eta_i \quad \Longleftrightarrow \quad a_\ell \geq a_i \quad \Longleftrightarrow \quad \Gamma_\ell \geq \Gamma_i.$$

Thus, when the network structure is neutral, nodes are ordered in decreasing order of (natural or net) productivity.

4.4.5. *More on hierarchy of general networks.* In the analysis of the previous subcases, we established a monotonic relationship between Γ_i, η_i° and η_i. But is this a general rule? For example, assume a node i has, with respect to another node j, a greater centrality in the network $\mathcal{G}$, namely $\eta_i^\circ \geq \eta_i^\circ$, and a greater reproduction rate and $\Gamma_i \geq \Gamma_j$. Is the reserve then better placed at node i than at node j? Namely, do we have $\eta_i \geq \eta_j$? The answer is in the negative, as explained by means of the following example. Consider the network described by $\Gamma_1 = 1, \quad \Gamma_2 = 1 + b, \quad \Gamma_3 = 0,$

$$G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}, \quad A + G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & b & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

with $b > 0$. By explicit calculation one has $\lambda_\circ = \sqrt[3]{2}$ and $\eta^\circ = \mu^\circ/|\mu^\circ|$ with $\mu^\circ = (2^{-2/3}, 2^{-1/3}, 1)^\top$. Note that $\eta_2^\circ > \eta_1^\circ$ and $\Gamma_2 > \Gamma_1$, that is, node 2 precedes node 1 both in productivity (natural and net) and centrality. Nonetheless, $\eta_1 > \eta_2$ for some choices of positive b, as we show next. To this extent, if $\eta = \mu/|\mu|$, with $\mu = (1, \mu_2, \mu_3)^\top$, then μ satisfies

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & b & 1 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \mu_2 \\ \mu_3 \end{pmatrix},$$

whose expansion implies

$$\mu_1 = 1, \quad \mu_2 = \lambda, \quad \mu_3 = \lambda(\lambda - b), \quad b = \lambda - \frac{2}{\lambda(\lambda + 2)}.$$

Note that the last equation implies in particular that b is an increasing function of λ and vice versa. A direct calculation shows that for $b = 0$ one has $\lambda(0) \simeq 0.8$, so that by continuity $\lambda(0) < \lambda(b) < 1$ for small positive b. Hence $\eta_1 > \eta_2$ and a reserve is better set at node 1 rather than at node 2. Thus, the relationship between the hierarchy dictated by the eigencentrality η and the productivity/network structure is complex and generally nonmonotonic.

Finally, we interpret of eigencentrality η_i as a measure of productivity and connectiveness not only of the i-th node, but also of the nodes more directly connected to it. In the previous example we set $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = 2 + a$. In this case

$$\mu_1 = 1, \quad \mu_2 = \lambda, \quad \mu_3 = \lambda^2, \quad a = \lambda - \frac{2}{\lambda^2},$$

with λ an increasing function of a, moreover for $a = -1$ one has $\mu_1 = \mu_2 = \mu_3 = 1$, and $\lambda = 1$, so that $\lambda > 1$ if and only if $a > -1$. Therefore

$$\mu_1 < \mu_2 < \mu_3 \text{ for } a > -1, \quad \text{and} \quad \mu_1 > \mu_2 > \mu_3 \text{ for } a < -1.$$

Hence, an increasing reproduction rate Γ_3 not only increases η_3, making (definitively) node 3 the most central, but also influences the centrality η_2 of node 2, which is more directly connected to it than node 1.

5. Extensions of the Model and Other Applications

We have derived our results in the context of games of extraction of a (possibly renewable) resource, such as fish. However, they can be more broadly applied to games with distributed state variables, such as growth models with local capital goods and externalities, pollution games with spatially distributed stocks and, with some adaptation, contribution games on networks.

5.1. **Growth Models with Externalities.** First, we consider a growth model in which production is distributed among the n nodes of a network. Assume that all local production functions are linear and of type $y_i = \Gamma_i k_i$, that capitals decay at rates δ_i, and that production generates non-negative externalities on stocks of the other nodes, linearly depending on production levels; namely, $h_{ji}\Gamma_j k_j$ is the externality generated by node-j on node-i, where h_{ij} are given nonnegative coefficients. The state equation at node-i is then given by

$$\dot{k}_i(t) = (\Gamma_i - \delta_i)k_i(t) + \sum_{j \neq i} h_{ji}\Gamma_j k_j(t) - c_i(t),$$

and in matricial form the system dynamics is given by

$$\dot{k}(t) = [(I + H^\top)\Gamma - D]k(t) - c(t)$$

where Γ is the diagonal matrix of productivities Γ_i, $H = (h_{ij})$, and D is the (also diagonal) matrix of decay rates δ_i. This system is of type (1) when we set

$$A = \Gamma - D, \; G = \Gamma H \text{ and } f = n.$$

If we assume that player i maximizes (4), then under the hypothesis of Theorem 1 a linear MPE of the game is given by

$$\psi^*(k) = \frac{\rho - (1-\sigma)\lambda}{1-(1-\sigma)n} \frac{\langle k, \eta \rangle}{\eta_i}, \text{ for all } i,$$

where λ is still the dominant eigenvalue of the matrix $[(I + H^\top)\Gamma - D]$ and η the associated positive eigenvector. Comparative statics is much the same as in the original model, with the exception that increasing links h_{ij} always imply an increasing eigenvalue λ. However, as noted in Section 3.4, the uniqueness of the linear equilibrium should be further investigated, as with $f = n$ the above MPE is probably only generically the unique linear MPE equilibrium of the game[9].

In the above model, all externalities are non-negative. However, extensions of the Perron-Frobenius theory to matrices with some negative entries (for example, eventually positive or eventually exponentially positive matrices, see e.g. Noutnos and Tsatsomeros, 2008) can be used to extend the analysis to cases in which positive and negative externalities coexist.

5.2. **Pollution Games.** The model can be also applied to study spatial diffusion of pollution. A most direct interpretation takes the state variables as local measures of environmental quality and the weighted sum $\langle X(t), \eta \rangle$ as the corresponding aggregate measure of environmental quality. The model then becomes a dynamic pollution game in which "clearness" moves across different locations and is depleted by the agents' local economic activities. Here, the dominant eigenvalue represents the overall rate of natural regeneration. Therefore, it is natural to assume $\lambda \geq 0$. However, if $g = \lambda - \theta f \leq 0$, the environmental quality is either stationary or declining, so that carrying capacities (i.e., saturation points) need not be explicitly modeled. Hence, (18) implies that the best overall outcome is reached when stocking of the resource (the reserves) takes place at nodes with the greatest eigencentralities η_i or, equivalently, pollution is null at those nodes.

In a further generalization of the model, the direct costs of pollution could be considered by assuming stock-dependent utility functions.

5.3. **Contribution Games.** Finally, an adaptation of our technique can be utilized to generalize the dynamic contribution game in Cvitanić and Georgiadis (2016) (see also Georgiadis (2015)). There, the authors assume that f agents exert costly efforts

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[9] Assuming to exemplify logarithmic utility functions, in case $\Gamma - D = rI$, where $r \in R$, $\psi^{**}(k) = \rho k_i + \sum_{j \neq i} h_{ji} \Gamma_j k_j(t)$, for all i, is a second linear MPE of the game. This can be easily verified with the procedure used in Theorem 1.

to reach a target associated to a prize V. The target is an exogenously given level $\bar{q}$ of the state of a project, represented by a one-dimensional state variable $X(t)$

In a generalized version with a distributed stock, one can consider an $n-$ dimensional vector $X(t)$ of states, built up at a subset of the n nodes of a network by the agents efforts, and a scalar function $q(X(t)) : \mathbb{R}^n_+ \to \mathbb{R}_+$ giving the state of the project. The stocks are mobile through connected nodes (e.g., one can think of knowledge that spreads through a network), giving rise to an evolution system of type (1) [10]. The possibility of free riding is so allowed.

A simple case occurs when $\dot{q}(X(t)) \geq 0$, meaning progress is irreversible, and there is no progress without efforts, so that $\dot{q}(X(t)) = 0$ if $c_i(t) \equiv 0$. This translates into requiring that $q(X(t)) = \langle X(t), \eta \rangle$ and that the matrix $A + G$ has dominant eigenvalue $\lambda = 0$, as $\langle X'(t), \eta \rangle = \lambda \langle X(t), \eta \rangle$ when $c_i(t) \equiv 0$. Thus, setting at $\bar{q} > 0$ the completion of the project and assuming the cost function at node i is given by the convex function $b_i(c_i)$, the problem of player i is to maximize

$$J^i(c_i; x) = -\int_0^{T_{c_i}} e^{-\rho t} b_i(c_i(t))dt + e^{-\rho T_{c_i}} \alpha_i V$$

where T_{c_i} is the strategy-dependent time when the goal is reached (possibly infinite).

Setting $q(t) \equiv q(X(t)) = \langle X(t), \eta \rangle$ and proceeding like in Theorem 1 in Cvitanić and Georgiadis (2016), one can establish that a solution for the system of HBJ equations that characterizes the MPE of a scalar model whose state equation is

$$\dot{q} = \sum_{i=1}^{f} \eta_i c_i,$$

exists and, provided $\bar{q}$ is so small that at least an agent has incentive to complete the project even without any contributions from the other agents, is unique in a large class of functions. Theorem 1 can then be utilized to verify that if $v_i(q)$ are the value functions at the MPE of the aggregate model, then $V_i(x) = v_i(\langle x, \eta \rangle)$ are the value functions in an MPE when the stocks are distributed.

To illustrate with a simple example, in which it is possible to characterize the solution analytically, assume all cost functions are quadratic and uniform across the nodes: $b_i(c_i(t)) = \frac{\gamma}{2} c_i(t)^2$, for all i, with γ a positive constant, and that the regulator wants to pick up a single agent with the objective to minimize the project's completion time. The system of PDEs characterizing the MPE is then given by the f equations,

[10] Of course, the signs of the control variables c_i in the state equation have to be reversed

one for each agent i,

$$\rho V_i(x) = \langle x, (A+G)DV_i(x)\rangle + \left\langle \sum_{j\neq i} c_j e_j, DV_i(x) \right\rangle + \langle c_i e_i, DV_i(x)\rangle - \frac{\gamma}{2}c_i^2,$$

where, in order that all Hamiltonian functions are maximized, we must set

$$\langle e_i, DV_i(x)\rangle = \gamma c_i$$

and, for all $j \neq i$,

$$\langle e_j, DV_j(x)\rangle = \gamma c_j.$$

This system must be satisfied on the intersection of the first orthant with the set $\langle x, \eta\rangle \leq \bar{q}$. Moreover, $V_i(x) = \alpha_i V$ for all i on the boundary $\langle x, \eta\rangle = \bar{q}$.

The corresponding system of HJB equations for the aggregate model,

$$\rho v_i(q) = \frac{1}{2\gamma}\eta_i^2 \left(\frac{dv_i(q)}{dq}\right)^2 + \frac{1}{\gamma}\frac{dv_i(q)}{dq}\sum_{j\neq i}\eta_j^2 \frac{dv_j(q)}{dq}, \tag{37}$$

is defined on interval $0 \leq q \leq \bar{q}$ and the solution must satisfy the f boundary conditions $v_i(\bar{q}) = \alpha_i V$.

Given that if $\alpha_i = 1$, and $\alpha_j = 0$ for $j \neq i$, only agent i can have an incentive to exert an effort, then, $v_j(q) = 0$ for $j \neq i$, and thus system (37) reduces to the scalar ODE

$$\gamma\rho v_i(q) = \frac{1}{2}\eta_i^2 \left(\frac{dv_i(q)}{dq}\right)^2, \tag{38}$$

for which the solution is a quadratic function provided $V \geq \frac{\gamma\rho}{2\eta_i^2}\bar{q}^2$ (i.e., agent i has an incentive to complete the project if and only if the ratio $\frac{\bar{q}^2}{V}$ is sufficiently small). In particular, by equating the coefficients, one obtains the formula for the value function

$$v_i(q) = \frac{\gamma\rho}{2\eta_i^2}q^2 + m_i q + \frac{\eta_i^2}{2\gamma\rho}m_i^2, \tag{39}$$

where $m_i > 0$ is determined by using the boundary condition

$$V = \frac{\gamma\rho}{2\eta_i^2}\bar{q}^2 + m_i\bar{q} + \frac{\eta_i^2}{2\gamma\rho}m_i^2. \tag{40}$$

Thus, in the model with a distributed stock, the value function for the agent i is obtained immediately:

$$V_i(x) = \frac{\gamma\rho}{2\eta_i^2}\langle x, \eta\rangle^2 + m_i\langle x, \eta\rangle + \frac{\eta_i^2}{2\gamma\rho}m_i^2. \tag{41}$$

Note now that the boundary condition (40) implies that $\eta_i^2 m_i$ is increasing in the eigencentrality of the node and that from the Hamiltonian condition it follows that

$$\langle e_i, DV_i(x)\rangle = \frac{\gamma\rho}{\eta_i^2}\eta_i \langle x, \eta\rangle + m_i\eta_i = \gamma c_i.$$

By substituting this into the state equation, the CLE, when the regulator picks up agent i, satisfies

$$\dot{X}(t) = (A + G^T)X(t) + [\frac{\rho}{\eta_i}\langle X(t), \eta\rangle + \frac{m_i}{\gamma}\eta_i)]e_i,$$

and therefore,

$$\left\langle \dot{X}(t), \eta \right\rangle = \rho \langle X(t), \eta\rangle + \frac{m_i}{\gamma}\eta_i^2.$$

The most central agent is thus the one completing the project in the shortest time and, therefore, the one the regulator should select.

6. Concluding remarks

The main aim of this paper was to explore, using a simple framework with heterogeneous regions and a given number of agents, how the structure of the migration network affects competition for spatially distributed moving resources. We found that if the regulator's objective is to maximize the unweighted sum of the utilities of the agents, and they are constrained to assign no more than one agent to each region, then the reserves should be localized in the most central regions. Here, the relevant centrality measure is given by the eigenvector centrality of a derived network obtained by magnifying the links of each node in the original migration network by a factor that is increasing in the productivity of the node itself.

Although in our analysis both the agents and the regulator in our analysis care only about consumption of the resource, our model provides a basis for more general analyses in which preferences for conservation are considered, introducing, for example, the resource stocks in the utility functions of the agents and/or in the regulator welfare function. A theme of this analysis will be how the role of the regulator is enhanced under the new hypotheses.

In a different vein, the role of the regulator could additionally be examined in more general contexts in which a "bad" extreme equilibrium coexists with the interior equilibrium. For example, an extreme equilibrium can be expected to exist in variants of our model if the extracted resource can be stored (e.g., Kremer and Morcom, 2000). In this case, a spatially structured policy could be a useful tool to eliminate the incentives that might potentially lead the agents to coordinate on the "bad" outcome.

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Appendix A. Proofs

Proof of Theorem 1. We initially take the perspective of player i, active at node i. For all other players, we assume that they play a Markovian strategy, described by

$$c_j(t) = d_j \langle X(t), \eta \rangle, \text{ with } j \in F - \{i\},$$

where d_j are non-negative real numbers. With this choice, the HJB equation (13) can be rewritten as

$$\rho v(x) = \frac{\sigma}{1-\sigma} \left(\frac{\partial v}{\partial x_i} \right)^{1-\frac{1}{\sigma}} + \langle x, (A+G)\nabla v(x) \rangle - \langle x, \eta \rangle \sum_{j \in F - \{i\}} \left(\frac{\partial v}{\partial x_j} \right) d_j$$

with maximum attained at

$$c_i = \left(\frac{\partial v}{\partial x_j} \right)^{-\frac{1}{\sigma}}. \tag{42}$$

Step 1: we search for a solution of HJB equation of type

$$v(x) = \frac{b_i}{1-\sigma} \langle x, \eta \rangle^{1-\sigma}, \quad \text{with} \quad \nabla v(x) = b_i \langle x, \eta \rangle^{-\sigma} \eta \tag{43}$$

where b_i is a suitable positive real number. Substituting v and its partial derivatives into the HJB equation, we derive that v is a solution if and only if

$$b_i = \frac{1}{\eta_i} \left(\frac{\sigma \eta_i}{\rho - \lambda (1-\sigma) + (1-\sigma) \sum_{j \in F - \{i\}} \eta_j d_j} \right)^{\sigma}.$$

Note that the quantity above will be proven well defined (i.e., the argument of power σ is nonnegative) once (44) is established.

Step 2: Markovian equilibrium. From (42) follows

$$c_i(t) = (b_i \eta_i)^{-\frac{1}{\sigma}} \langle X(t), \eta \rangle,$$

moreover, at the equilibrium one has $d_i = (b_i \eta_i)^{-\frac{1}{\sigma}}$, implying

$$d_i = \frac{1}{\eta_i} \frac{\rho - \lambda (1-\sigma)}{1 - (1-\sigma) f} = \frac{\theta}{\eta_i}, \text{ and } b_i = \eta_i^{\sigma-1} \theta^{-\sigma} \tag{44}$$

from which formulas (17) and (18) derive.

Step 3: Closed loop equation. Note that along the equilibrium trajectories, $c(t) = \theta \langle X(t), \eta \rangle \xi = \theta \, \xi \eta^\top X(t)$, so that the evolution system can be rephrased as in (22). Statement (*iii*) follows from

$$\langle \dot{X}(t), \eta \rangle = \langle X(t), (A+G)\eta \rangle - \langle X(t), \eta \rangle \langle \xi, \eta \rangle = \langle X(t), \eta \rangle (\lambda - \theta f)$$

where $\lambda - \theta f = g = (\lambda - f\rho)(1 + (\sigma - 1) f)^{-1}$.

Step 4: Best response. We verify now that the feedback strategy (17) is the best response for Player i, when the other players choose ψ_j, with $j \neq i$, as in (17). Then the problem of Player i is maximizing (4) under the dynamics

$$\begin{cases} \dot{X}(t) = (A + G^\top - \theta \, \xi^i \eta^\top) X(t) - c_i(t) e_i, & t > 0 \\ X(0) = x_0. \end{cases} \tag{45}$$

where the vector ξ^i coincides with ξ except for the i-th component, which is set equal to 0, namely $\xi^i_\ell = \xi_\ell$ for all $\ell \neq i$, and $\xi^i_i = 0$.

Set $c_i^*(t) = \psi(X^*(t))$ and let $c_i(t)$ be any other admissible control, with $X^*(t)$ and $X(t)$, respectively, the associated trajectories. Now we consider the quantity $(c_i^*(t) - c_i(t)) \frac{\partial v}{\partial x_i}(X^*(t))$ and use the fact that $c_i^*(t)$ realizes the maximum in (42) with $d_j = \theta/\eta_j$, and $p = \nabla v(X^*(t))$ to derive

$$\frac{1}{1-\sigma} \left(c_i^*(t)^{1-\sigma} - c_i(t)^{1-\sigma} \right) \geq (c_i^*(t) - c_i(t)) \frac{\partial v}{\partial x_i}(X^*(t)) \tag{46}$$

Next, observe that adding and subtracting $\big\langle (A + G^\top - \theta \quad \xi^i \eta^\top)(X^*(t) - X(t)), \nabla v(X^*(t))\big\rangle$ and making use of (45), the right-hand side in (46) equals

$$\begin{aligned} &\big\langle (A + G^\top - \theta\, \xi^i \eta^\top)(X^*(t) - X(t)), \nabla v(X^*(t))\big\rangle - \big\langle (\dot X^*(t) - \dot X(t)), \nabla v(X^*(t))\big\rangle \\ &= \big\langle X^*(t) - X(t), (A + G - \theta \eta (\xi^i)^\top) \nabla v(X^*(t))\big\rangle - \big\langle (\dot X^*(t) - \dot X(t)), \nabla v(X^*(t))\big\rangle. \end{aligned} \tag{47}$$

Recalling (43) and (19), we have

$$\nabla v(X^*(t)) = b_i \langle X^*(t), \eta\rangle^{-\sigma} \eta = b_i e^{-\sigma g t} \langle x_0, \eta\rangle^{-\sigma} \eta.$$

Using this expression and the fact that $(A + G - \theta\eta(\xi^i)^\top)\eta = (\lambda - \theta(f-1))\eta$, the expression in (47) can be written as

$$= b_i \langle x_0, \eta\rangle^{-\sigma} e^{-\sigma g t}\Big[\big\langle X^*(t) - X(t), [\lambda - \theta(f-1)]\eta\big\rangle - \big\langle (\dot X^*(t) - \dot X(t)), \eta\big\rangle\Big]$$

Thus, utilizing these estimates and integrating (46) on $[0, T]$ for $T > 0$, we obtain

$$\begin{aligned} &\int_0^T \frac{e^{-\rho t}}{1-\sigma}\left(c_i^*(t)^{1-\sigma} - c_i(t)^{1-\sigma}\right) dt \geq \\ &b_i \langle x_0, \eta\rangle^{-\sigma}\Big[\int_0^T e^{-(\sigma g + \rho)t}\big\langle X^*(t) - X(t), (\lambda - \theta(f-1))\eta\big\rangle dt - \int_0^T e^{-(\sigma g+\rho)t}\big\langle (\dot X^*(t) - \dot X(t)), \eta\big\rangle dt\Big] \end{aligned} \tag{48}$$

and, integrating by parts the last term, the right-hand side equals

$$\begin{aligned} &= b_i \langle x_0, \eta\rangle^{-\sigma}\Big[\int_0^T e^{-(\sigma g + \rho)t}\big\langle X^*(t) - X(t), \big(\lambda - \theta(f-1)\big)\eta\big\rangle dt + \\ &\quad - e^{-(\rho + \sigma g)T}\left\langle (X^*(T) - X(T)), \eta\right\rangle - \int_0^T e^{-(\sigma g+\rho)t}\big\langle (X^*(t) - X(t)), (\sigma g + \rho)\eta\big\rangle dt\Big] \\ &= b_i \langle x_0, \eta\rangle^{-\sigma} e^{-(\rho+\sigma g)T}\left\langle (X(T) - X^*(T)), \eta\right\rangle \geq -b_i \langle x_0, \eta\rangle^{-\sigma} e^{-(\rho + \sigma g)T}\left\langle X^*(T), \eta\right\rangle \end{aligned} \tag{49}$$

where the last equality is a consequence of $\sigma g + \rho = \lambda - \theta(f-1)$, and the last inequality a consequence of $\langle X(T), \eta\rangle \geq 0$, as $X(T)$ is admissible and hence non-negative. Now $e^{-(\rho+\sigma g)T}\langle X^*(T), \eta\rangle = e^{-(\rho+\sigma g)T} e^{gT}\langle x_0, \eta\rangle$ decreases to 0, as T tends to $+\infty$, as

$$g(1-\sigma) - \rho = -\theta < 0. \tag{50}$$

Thus, taking the limit as T tends to $+\infty$ of the inequalities (48)(49), implies

$$\int_0^{+\infty} e^{-\rho t}\frac{c_i^*(t)^{1-\sigma}}{1-\sigma} dt \geq \int_0^{+\infty} e^{-\rho t}\frac{c_i(t)^{1-\sigma}}{1-\sigma} dt,$$

that is, the optimality of $c_i^*(t)$. (Note that limits exist, as integrals are monotonic in T).

□

Lemma 2 *Let $E = \xi\eta^\top$. The matrix E has an eigenvalue f with multiplicity 1, associated to the (right) eigenvector ξ and left eigenvector η; moreover it has eigenvalue 0 with multiplicity $n-1$. All eigenvectors of E (respectively, $E^\top$) associated to the zero eigenvalue are orthogonal to ξ (respectively, η).*

The above statements can be proven by induction, and we omit the easy (but rather tedious) proof.

Proof of Lemma 1. The proof of (i) is trivial. To prove (ii), we first show that all v_i with $i = 2, \dots, n$ are orthogonal to η. The property is well known but we detail it here for the reader's convenience. Consider the eigenvalue λ_2. Assume the associated Jordan block has dimension m, and let $v_2^1, \dots, v_2^m$ the associated generalized eigenvectors. It is then known that, if $V_2^j = \{v \in \mathbb{R}^n \mid (A + G^\top - \lambda_2 I)^j v = 0\}$, with $j \in \{1, 2, \dots, m\}$, a suitable choice of the v_2^j's is $v_2^1 \in V_2^m - V_2^{m-1}$, and $v_2^j = (A + G^\top - \lambda_2 I)^{j-1} v_2^1$, for all j. According to our initial notation, we can set $v_2 := v_2^1, v_3 := v_2^2, \dots, v_{m+1} := v_2^m$.

Now we show that every v_2^j, associated to the eigenvalue λ_2 is orthogonal to η. Indeed $(A + G^\top - \lambda_2 I)^m v_2^1 = 0$ implies, for every $j = 1, \dots, m$

$$0 = \eta^\top (A + G^\top - \lambda_2)^j (A + G^\top - \lambda_2)^{m-j} v_2^1 = (\lambda - \lambda_2)^j \eta^\top \; v_2^{m-j+1}$$

and since $\lambda \neq \lambda_2$ then $\eta^\top \; v_2^{m-j+1} = 0$, implying η is orthogonal to v_2^j for all j. We can then proceed similarly for all the other eigenvalues of $A + G^\top$ different from λ_2, and thus obtain a complete base of generalized eigenvectors $\{\zeta, v_2, \dots, v_n\}$.

Now v_i, $i = 2, \dots, n$, is an eigenvector of $A + G^\top - \theta E$ of eigenvalue λ_i if and only if

$$(A + G^\top - \theta E) v_i = \lambda_i v_i \iff E v_i = 0 \iff \eta^\top v_i = 0,$$

which is a consequence of Lemma 2. □

The proof of Lemma A.1 and Proposition 1 are well known facts (see, e.g., chapter 1 in Colonius and Kliemann (2014)) and we provide them here for the reader's convenience.

Lemma A.1 In the assumptions of Proposition 1, there exist continuous coefficients m_i, linear in x_0, with $\lim_{t\to\infty} m_i(t) e^{-\varepsilon t} = 0$ for all $\varepsilon > 0$, and such that

$$X^*(t) = m_1(t) e^{gt} \hat{\zeta} + \sum_{i=2}^{n} e^{\mathrm{Re}(\lambda_i) t} m_i(t) v_i. \tag{51}$$

Proof. If J is the real Jordan form of the matrix $A + G^\top - \theta E$, then there exists a real invertible matrix P such that $P^{-1}(A + G^\top - \theta E)P = J$. Consequently, there exist real coefficients β_i such that

$$X^*(t) = e^{t(A+G^\top - \theta E)}x = Pe^{tJ}\left(\sum_{i=1}^n \langle x_0, v_i\rangle P^{-1}v_i\right) = P\sum_{i=1}^n \beta_i e^{Jt}P^{-1}v_i. \tag{52}$$

It follows then from the general theory (see for instance Section 1.3 of Colonius and Kliemann (2014)) that $e^{Jt}P^{-1}v_i = e^{\mathrm{Re}(\lambda_i)t}M_i(t)P^{-1}v_i$ where $M_i(t)$ is a block matrix (that is non-zero only on the Jordan block related to λ_i) whose coefficients are products of sinus and cosinus functions of t and of polynomials of t with maximum degree the dimensions of the generalized eigenspace. As $Pe^{Jt}P^{-1}v_i$ is again an element of the generalized eigenspace associated to λ_i, it can be written as a linear combination of the eigenvectors related to the same generalized eigenspace, with the coefficient having the same described behavior as t, and then the claim follows. □

Proof of Proposition 1. The proof is entirely based on Lemma A.1 and follows from (51), once we observe that $M_i(t)$ for a simple eigenvalue is just a real coefficient. □

Proof of Proposition 2. If we specify condition (ii) of (25) for $x = e_j$ and for $j \neq i$ we get (26); hence, it is necessary. However, if we suppose that (26) is verified, given $x = \sum_{j\neq i} x_j e_j$ for some $x_j \geq 0$, we have

$$\psi_i(x) = \sum_{j\neq i} x_j \frac{\theta}{\eta_i}\eta_j \leq \sum_{j\neq i} x_j \frac{\frac{g_{ij}\eta_j}{\eta_i}}{\eta_i}\eta_j = \langle x, Ge_i\rangle = \left\langle G^\top x, e_i\right\rangle + \left\langle (A + G^\top)x, e_i\right\rangle$$

wherein for the inequality we utilized (26) and in the last equality we utilized that $x_i = 0$ and A is diagonal so that $\langle Ax, e_i\rangle = 0$. Therefore, (26) is also sufficient.

To prove the last claim, observe that the condition (26) is equivalent to requiring that the matrix of system (22) (having nondiagonal terms $g_{ij} - \theta\eta_i\xi_j$), is indeed a Metzler matrix, that is, a matrix with non-negative off-diagonal coefficients. This is equivalent to establishing that the system is positive, that is, it has solutions contained in the positive orthant $\mathbb{R}^n_+$ for all initial conditions $x \in \mathbb{R}^n_+$ (see, for example, Farina and Rinaldi, 2000, Chapter 2): as soon as a such a condition is violated, there exist trajectories of the system starting at some $x_0 \in \mathbb{R}^n_+$ which comes out of the positive orthant $\mathbb{R}^n_+$. □

Proof of Proposition 3. For simplicity, we prove the assertion for the case of all $\beta_i \equiv \beta > 0$ (for the general case the adjustment is minimal). When $\beta = 0$, the agents can only choose to fish null amounts at every node and, arguing as in Proposition 1, whatever the initial condition $x_0 \in \mathbb{R}^n_+$, the system converges to the vector ζ. By making use of the decomposition of Lemma A.1, and of the fact that $\theta < \min\{\theta_1, \theta_2\}$, the projection of the detrended trajectory $X^*(t)e^{-\lambda t}$ on the $(n-1)$-dimensional space $\zeta^\perp$, has eigenvalues with negative part and then (see Bitsoris, 1991), the set $E(P, d)$ is (positively) invariant for the projected system and the vector field of the velocities on the boundary of $E(P, d)$ is strictly inward.

By continuity there exists $\bar{\beta} > 0$ such that, for any $\beta \in [0, \bar{\beta}]$, the projection of any vector field of the velocities satisfying

$$[A + G^T - \beta]X(t) \leq \dot{X}(t) \leq [A + G^T]X(t), \tag{53}$$

on the boundary of S^* is inward and then, for any choice of the strategies in $\mathbb{A}^\beta$, the system remains in S^*. This prove that $(S^*, \mathbb{A}^\beta)$ is a consistent couple so if the equilibrium described in Theorem 1 is admissible, then it is also subgame perfect.

Since $S^* \subseteq (0, +\infty)^n$ (and it does not touch the boundary of $\mathbb{R}^n_+$, except at the origin) there exist constants r_i such that, for any $x \in S^*$,

$$r_i \left(\sum_{j \neq i} x_j \eta_j \right) \leq x_i$$

and then, for θ and β_i's such that $\theta < \beta \min_i (\eta_i / r_i)$, we have

$$\psi(x) = \frac{\theta}{\eta_i} \langle x, \eta \rangle \leq \frac{\theta}{\eta_i} \frac{x_i}{r_i} \leq \beta x_i$$

and the set of strategy and the equilibrium described in Theorem 1 is admissible, and hence a Markov perfect equilibrium.

□

Proof of Theorem 2. Assume that $\hat{\psi}_j(x) = \langle w^j, x \rangle$, with $w^j \in \mathbb{R}^n_+$, $j \in N$ is a linear MPE in $\mathbb{A}$. We want to show that necessarily $\hat{\psi} = \psi^*$. Note that when starting at $x = e_j$, the extraction rate $\langle w^j, e_i \rangle = w^j_i$, which implies $w^j_i \geq 0$, for all $i \in N$. Then we define the square non-negative matrices

$$W := \sum_{j \in N} e_j (w^j)^\top, \qquad W_{-i} := \sum_{j \in N, j \neq i} e_j (w^j)^\top$$

so that the stock evolves with law $\dot{X} = (A + G^\top - W)X$. Given that $\hat{\psi}$ is admissible (it lies in $\mathbb{A}$ by hypothesis) at every initial stock $x_0 \in \mathbb{R}^n_+$, then $X^{\psi, x_0}(t) \geq 0$ for all $t \geq 0$

(see for example Farina and Rinaldi, 2000, Chapter 2[11]), implying that $A + G^\top - W$ is a Metzler matrix. Since the w^j's are positive, $A + G^\top - W_{-i} = A + G^\top - W + w^i e_i$ is *a fortiori* a Metzler matrix.

We now take the viewpoint of player i: assume that the other players stick to the choice $\hat{\psi}_{-i}$, and solve the problem for player i to maximize (4) for $c_i \in \mathbb{A}_i$, when subject to

$$\dot{X}(t) = (A + G^\top - W_{-i})X(t) - c_i(t)e_i, \quad X(0) = x_0$$

and under the constraint $X_i(t) \geq 0$ for all $t \geq 0$.[12]

The case $A + G - W^\top$ is irreducible. We treat the case for $u(c) = (1-\sigma)^{-1}c^{1-\sigma}$, the proof for logarithmic utility is very similar. As a first step we assume $A+G-W^\top$ is irreducible. Then, *a fortiori*, $A + G - W_{-i}^\top$ is irreducible. The Perron–Frobenius theorem implies that $A + G - W^\top$ ($A + G - W_{-i}^\top$) has a simple, real eigenvalue $\hat{\lambda}$ ($\hat{\lambda}^i$) strictly greater than all other eigenvalues' real parts, and associated to the unique strictly positive eigenvector $\hat{\eta}$ ($\hat{\eta}^i$).

The problem of agent i is associated to an HJB equation of type (13) where c_j is replaced by $\psi_j(x)$ for every $j \neq i$. Now set $\hat{\eta}^i = (\hat{\eta}^i_1, \hat{\eta}^i_2, \dots, \hat{\eta}^i_n)$, and

$$b = \left(\frac{\sigma}{\rho - \hat{\lambda}^i(1-\sigma)}\right)^\sigma (\hat{\eta}^i_i)^{\sigma-1}, \text{ and } \theta^i = \frac{\rho - \hat{\lambda}^i(1-\sigma)}{\sigma}. \tag{54}$$

Similar to Theorem 1, one can verify that a solution of this HJB equation is given by $v(x) = b(1-\sigma)^{-1}\langle x, \hat{\eta}^i\rangle^{1-\sigma}$, and moreover that (14) implies that the only candidate optimal extraction policy is

$$c_i^* = \frac{\theta^i}{\hat{\eta}^i_i}\left\langle \hat{\eta}^i, x\right\rangle,$$

Optimality can be proved by means of a standard verification argument, as in Theorem 1. Since c_i^* is the only optimizer, then $\hat{\psi}_i(x)$ coincides with c_i^*, implying

$$A + G - W^\top = (A + G - W_{-i}^\top) - \theta^i E_i.$$

[11] Observe that in Farina and Rinaldi (2000), the authors often work under the hypothesis of non-negativity of the matrices. Clearly, all the results we utilize straightforwardly generalize to the case of Metzler matrices. This is true also for other mentioned texts.

[12] Observe that, since $A + G - W_{-i}^\top$ is a Metzler matrix, the constraint $X_i \geq 0$ (together with the non-negativity of the initial datum) is enough to ensure that all the components of X remain non-negative.

where $E_i = \frac{1}{\hat{\eta}_i^i}\, \hat{\eta}^i e_i^\top$. Note: $E_i\, \hat{\eta}^i = \hat{\eta}^i$ so that $\hat{\eta}^i$ is a strictly positive eigenvector of both the right-hand and left-hand sides of the above identity. Since by Perron-Frobenius's theorem the positive eigenvector is unique, necessarily

$$\hat{\eta} \equiv \hat{\eta}^i, \quad \text{and} \quad \hat{\lambda} = \hat{\lambda}^i - \theta^i = \frac{\hat{\lambda}^i - \rho}{\sigma}, \tag{55}$$

and that can equally be proven for all $i \in F$. Hence, $\theta^i \equiv \rho - \hat{\lambda}(1-\sigma)$ for all i, and

$$W^\top = \sum_{i\in F} \theta^i E_i \Rightarrow W^\top \hat{\eta} = (\rho - (1-\sigma)\hat{\lambda}) f\, \hat{\eta}.$$

so that, by difference, $\hat{\eta}$ is also a positive eigenvector of $A+G$, i.e. $(A+G)\hat{\eta} = (\rho - (1-\sigma)\hat{\lambda}) f \hat{\eta} + \hat{\lambda}\hat{\eta}$. But then, necessarily, $\hat{\eta} \equiv \eta$, and $(\rho - (1-\sigma)\hat{\lambda}) f + \hat{\lambda} \equiv \lambda$. Therefore, $\lambda = \theta^i f + \frac{\rho - \theta^i}{1-\sigma}$, and $\theta^i = \frac{\rho + (\sigma-1)\lambda}{1+(\sigma-1)f} = \theta$. Thus, we have proven that $\psi^* \equiv \hat{\psi}$.

The case $A+G-W^\top$ reducible. We pass now to the analysis of the case $A+G-W^\top$ reducible. To simplify the notation we set $M = A + G^\top - W$ and, for any agent i, $M_{-i} = A + G^\top - W + e_i {w^i}^\top$. Barring a permutation (i.e., changing the names of the locations), we can assume that $M^\top = A + G - W^\top$ is in its Frobenius form (see (1.7.1) page 38 of Bapat and Raghavan (1997)), that in this case reads as

$$M^\top = \begin{bmatrix} M_1^\top & M_{21}^\top & \dots & M_{K1}^\top \\ 0 & M_2^\top & \dots & M_{K2}^\top \\ \vdots & 0 & \ddots & \\ 0 & 0 & 0 & M_K^\top \end{bmatrix} \tag{56}$$

with irreducible submatrices $M_k^\top$ on the diagonal (some $M_{Ki}^\top$ can contain zeros). Since G has strictly positive and no-extraction arises in reserves, all the matrix elements from a reserve to any other location need to be strictly positive so reserves are among the locations associated to $M_K^\top$, although the same block may partly also refer to fishing locations.

As before, we denote by $\hat{\lambda}$ the dominant eigenvalue of $M^\top$ (even if not reducible it is a Metzler matrix), and by $\hat{\eta}$ one of the associated eigenvectors; by $\hat{\lambda}^i$ the dominant eigenvalue of $M_{-i}^\top$, and by $\hat{\eta}^i$ one of the associated eigenvectors. Moreover, we denote by λ_k the dominant eigenvalue of M_k for any $k = 1, \dots, K$.

After these introductory elements we divide the rest of the proof in two steps.

Step 1: if there exists $i \in F$ for which $\hat{\eta}^i > 0$, then the only linear equilibrium is the one described in Theorem 1.

Indeed, arguing as in the case of an irreducible M, we obtain again $w^i = (\theta^i/\hat{\eta}_i^i)\hat{\eta}^i$, where θ_i is given by (54), and moreover that $\hat{\eta}^i$ is an eigenvector of the matrix $M^\top$.

Since $\hat{\eta}^i > 0$ by assumption it needs to be the unique dominant eigenvector of $M^\top$ (Theorem 11, page 36 of Farina and Rinaldi (2000)) and in particular to coincide with $\hat{\eta}$.

We look now at the behavior of other agents. We call agent-jk any agent j fishing (at the node j) in the subset of nodes k. Its extraction vector at the given equilibrium is denoted by $w^j = (w_{jk}^{(1)}, w_{jk}^{(2)}, \dots, w_{jk}^{(K)})^\top$, where $w_{jk}^{(h)}$ is a vector with as many coordinates as the dimension of block M_h. We first look at a possible agent in one of the nodes related to K.

First observe that (since KK is the down-right block and since we already proved that the unique dominant eigenvector of $M^\top$ is strictly positive), one needs to have $\lambda_K = \hat{\lambda}$. Moreover, $\hat{\lambda}$ is the maximum of all λ_k, thus $\lambda_K \geq \lambda_k$ for all k. Now the control problem for agent-jK is associated to the matrix

$$M_{-j}^\top = \begin{bmatrix} M_1^\top & M_{21}^\top & \dots & M_{K1}^\top \\ 0 & M_2^\top & \dots & M_{K2}^\top \\ \vdots & 0 & \ddots & \\ 0 & 0 & 0 & M_K^\top \end{bmatrix} + \begin{bmatrix} 0 & \dots & 0 & w_{jK}^{(1)} \\ 0 & \dots & 0 & w_{jK}^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & w_{jK}^{(K)} \end{bmatrix}. \tag{57}$$

The Perron's eigenvalue of $M_K^\top + w_{jK}^{(K)}$ is higher than λ_K and then of all λ_k. Then the dominant eigenvalue of $M_{-j}^\top$ is strictly positive, unique and it is associated to the eigenvalue of $M_K^\top + w_{jK}^{(K)}$. Arguing again as above, one computes the optimal closed-loop control of agent-jK and verifies that $w^j = (\theta^j / \hat{\eta}_j^j)\hat{\eta}$ (where $\hat{\eta}$ is the Frobenius eigenvector of $M^\top$).

We show now that the same holds for any other agent-jk, $k \neq K$. We first show that there exists a unique eigenvector associated to $\hat{\lambda}^j$ and it is strictly positive.

We consider the matrix

$$\begin{bmatrix} a_{jk} & m_{Kjk}^\top \\ g_{jkK} & M_K^\top \end{bmatrix}$$

which is the proper principal submatrix of $M_{-j}^\top$ obtained by removing all rows and corresponding columns that are not in the K block and are not jk. Its dominant eigenvalue is strictly greater than λ_K but, at the same time (see Theorem 1.7.4 of Bapat and Raghavan (1997)) it is smaller than $\hat{\lambda}^j$ so $\hat{\lambda}_j > \hat{\lambda} = \lambda_K$. This implies that all matrices of type $[\hat{\lambda}^j I - M_k^\top]$ have strictly positive inverse (see Theorem 1.7.2 page 35 of Bapat and Raghavan, 1997).

This fact is sufficient to show that all the components of any eigenvector $\hat{\eta}^j$ associated to $\hat{\lambda}^j$ (which is ex-post unique, see Theorem 11, page 36 of Farina and Rinaldi (2000)) are strictly positive.

For the components related to the block K we have:

$$\hat{\eta}_K^j = \hat{\eta}^j(j)[\hat{\lambda}_j I - M_{KK}^\top]^{-1} g_{jkK}, \tag{58}$$

where $\hat{\eta}_K^j$ is the part of the eigenvector $\hat{\eta}^j$ corresponding to the areas in K, $\hat{\eta}_K^j(j)$ is the j-th component of the same eigenvector, and g_{jkK} is the vector of inflows from the K part. Therefore, prices in this subset are either all positive or all zero. Proceeding recursively, the same alternative occurs for all prices in the other blocks of the matrix. So eventually $\hat{\eta}_j > 0$, and this is true for all agents.

Following the same argument as in the irreducible case, we first compute the optimal closed loop controls of all agents and see that, for all j, w^j is indeed the Frobenius's eigenvector of $M^\top$ and then verify that the w^j are indeed, except for multiplication factors, the unique strictly positive eigenvectors of $A + G$. Uniqueness of the multiplication factors is found in the proof of Theorem 1, so that uniqueness of the linear equilibrium is proven.

Step 2: There exists an agent i for which $\hat{\eta}^i > 0$.

In the case in which agents concentrate in the subset K of nodes, the matrix is irreducible. Then we can assume w.l.o.g. that agents are distributed also in regions other than those in the subset K. If $\lambda_K > \lambda_k$, then, arguing as in Step 1, all $\hat{\eta}^i$'s are strictly positive and coinciding with $\hat{\eta}$. In the opposite case, there exists k^*, with $k^* \neq K$ such that $\lambda_{k^*} \geq \mathrm{Re}\lambda_k$, for all $k = 1, \ldots, K$.

Now consider an agent-ik^*, that is, operating in region i and belonging to subset k^*. Arguing as in Step 1, we see that the matrix $M_{-i} = M_{k^*}^\top + w_{ik^*}^{(k^*)} e_i^\top$ has eigenvalue $\hat{\lambda}^i$ with $\hat{\lambda}^i > \lambda_{k^*}$. Thus, all matrices of type $[\hat{\lambda}^i I - M_k^\top]$ with $k \neq k^*$ have a positive inverse.

In particular (58) holds:

$$\hat{\eta}_K^i = \hat{\eta} i(i)[\hat{\lambda}_i I - M_K^\top]^{-1} g_{ik^*K}. \tag{59}$$

Substituting backwards into the eigenvector equation for the $K - k^* - 1$ block, we find a similar equation. Proceeding iteratively until we reach k^*, we see that all prices in the areas "downstream" of k^* are either all positive, if the component of the eigenvector at ik^* is positive, or all zero, if the same component is zero. However, this component cannot be zero, since otherwise one would have to find a non-strictly positive eigenvector of the irreducible matrix

$$M_{k^*k^*}^\top + w_{ik^*}^{(k^*)} e_{ik^*}^\top.$$

Since the components of the eigenvector at ik^* are positive, all prices "upstream" of k^* are also positive. We therefore have an agent with positive prices. □

Proof of Theorem 3 (Sketch). The existence part of the statement is of course proved in Proposition 3. We sketch here the proof of the uniqueness which follows the same arguments we utilized in the proof of Theorem 2.

We assume a linear MPE, that is, a set of strategies of the form $\hat{\psi}_j(x) = \langle w^j, x \rangle$, with $w^j \in \mathbb{R}^n_+$, $j \in N$.

First, arguing as in the proof of Proposition 3 (remaining, thanks to the same proposition, the trajectories in the cone which is contained in $(0, +\infty)^n$) we know that, along the admissible trajectory, we can always estimate each component of x as the product of any other component and some suitable positive constant. In particular, there exist constants $s^j_m > 0$ (which depend only on the cone structure so can be chosen independently of β), such that

$$x_j \le s^j_m x_m$$

Since we work here under the constraint $\hat{\psi}_j(x) = \langle w^j, x \rangle \le \beta x_j$ (and, as proven in the proof of Theorem 2, all the w^j_m are non-negative) we have in particular, for all $m \in N$, $w^j_m x_m \le \beta x_j$ so that we obtain

$$w^j_m \le \beta \frac{x_j}{x_m} \le \beta s^j_m.$$

If one chooses $\beta > 0$ small enough, by continuity, since the matrix $A + G$ has a simple dominant eigenvalue associated to a strictly positive eigenvector, the same properties hold for the matrix $A + G - W^\top$ and for all the matrices $A + G - W_i^\top$. Then, after choosing a θ small enough to ensure (as in the proof of Proposition 3) that the candidate equilibrium is admissible, we can argue exactly as in the proof of Theorem 2 in the case where $A + G - W^\top$ is irreducible. □

Proof of Proposition 4. We choose F a certain subset of N made of f elements and we denote by ψ_F the related equilibrium introduced in Theorem 1 (in the text of Theorem 1 it was called simply ψ but here we underline the role of the choice of F). The corresponding value for the social welfare W is

$$W(x_0) = \theta^{-\sigma} \langle x_0, \eta \rangle^{1-\sigma} \sum_{i \in F} \frac{\eta_i^{\sigma-1}}{1-\sigma}.$$

In this expression, the only part that depends on the choice of F is the sum. Each term of the sum is a (strictly) decreasing function of the corresponding η_i so the highest value of the sum and of the social welfare is reached when we choose to authorize fishing in the nodes i where η_i is the lowest and then to establish the natural reserves in the nodes where the η_i are the highest. □

Proof of Proposition 5*.* The proof of the proposition is given in the text that precedes it. □

Proof of Proposition 6*.* We first check the effect of an ϵ increase of $\mathbf{g}_{ij}$, with $i \neq j$, on the value of λ. To this extent, fix $\epsilon > 0$ and define $M_{ij} := (e_i\, e_j^\top + e_j\, e_i^\top) - (e_i\, e_i^\top + e_j\, e_j^\top)$, and note that the system matrix changes from $A + G$ to $A + G + \epsilon M_{ij}$. Note that this last matrix can be written as the sum of two Metzler matrices

$$A + G + \epsilon M_{ij} = [A - \epsilon(e_i\, e_i^\top + e_j\, e_j^\top)] + [G + \epsilon(e_i\, e_j^\top + e_j\, e_i^\top)]$$

so that it is itself a Metzler matrix. Moreover, M_{ij} is a negative-semidefinite matrix so that $\langle x, M_{ij}x\rangle \leq 0$ for all $x \in \mathbb{R}^n$. We denote by η_ϵ its Perron-Frobenius eigenvector of norm 1, and by λ_ϵ the associated Perron-Frobenius eigenvalue. Since the network matrix is symmetric, we can utilize the variational characterization of eigenvalues (see for instance Corollary III.1.2 of Bhatia, 2013) so that

$$\begin{aligned}
\max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\langle x, (A+G+\epsilon M_{ij})x\rangle}{|x|^2} = \lambda_\epsilon &= \frac{\langle \eta_\epsilon, (A+G+\epsilon M_{ij})\eta_\epsilon\rangle}{|\eta_\epsilon|^2} \\
&= \frac{\langle \eta_\epsilon, (A+G)\eta_\epsilon\rangle}{|\eta_\epsilon|^2} + \epsilon \frac{\langle \eta_\epsilon, M_{ij}\eta_\epsilon\rangle}{|\eta_\epsilon|^2} \\
\leq \max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\langle x, (A+G)x\rangle}{|x|^2} \;+\; \epsilon \frac{\langle \eta_\epsilon, M_{ij}\eta_\epsilon\rangle}{|\eta_\epsilon|^2} &\leq \max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\langle x, (A+G)x\rangle}{|x|^2} = \lambda \qquad (60)
\end{aligned}$$

This means that $\frac{d\lambda}{d\mathbf{g}_{i,j}} \leq 0$. □

Proof of Proposition 7*.* Calculating the derivative of the expression (20) w.r.t. f, we obtain

$$\frac{dg}{df} = -\frac{\rho - (1-\sigma)\lambda}{(1-(1-\sigma)f)^2} = -\frac{\theta}{(1-(1-\sigma)f)}.$$

We are interested in the case $\theta > 0$, and the claim follows immediately. □

Proof of Proposition 8*.* We build a differentiable function $\hat{W}(s)$ coinciding with $W(s)$ on $\mathbb{N}$. Set $\phi : (0, +\infty) \to [0, +\infty)$, $\phi(t) = \sum_{j=1}^n (\eta_j)^{\sigma-1}\chi_{]j-1,j]}(t)$ where χ_I is the indicator function of the interval I, that is $\chi_I(t) = 1$ when $t \in I$ and 0 elsewhere. Note that

$$\phi(j) = \eta_j^{\sigma-1} \quad \text{and} \quad \int_0^j \phi(t)dt = \sum_{i=1}^j (\eta_i)^{\sigma-1}.$$

Then, the differentiable function

$$\hat{W}(s) := \frac{\theta^{-\sigma}(s)}{1-\sigma}\langle x,\eta\rangle^{1-\sigma}\int_0^s \phi(t)dt,$$

coincides with $W(s)$ for all $s \in \{1,\dots,f_M\}$. Now

$$\frac{\partial \hat{W}(s)}{\partial s} = \frac{\langle x,\eta\rangle^{1-\sigma}[\theta(s)]^{-\sigma}}{1-\sigma}\left(-\sigma\frac{\theta'(s)}{\theta(s)}\int_0^s \phi(t)dt + \phi(s)\right)$$

and if we set for $s > 0$

$$L(s) = \frac{\phi(s)}{\int_0^s \phi(t)dt}, \quad R(s) = \sigma\frac{\theta'(s)}{\theta(s)} = \frac{\sigma(1-\sigma)}{1-(1-\sigma)s}$$

then $\partial\hat{W}(s)/\partial s \geq 0$ iff $L(s) \geq R(s)$.

In the *standard regime* (i), $L(1) = 1$ and, for $\sigma < 1$, $L(s)$ is a piecewise continuous and decreasing function of s, as $\forall s \geq 0, h \geq 0$

$$\phi(s) \geq \phi(s+h), \quad \text{and} \quad \int_0^s \phi(t)dt \leq \int_0^{s+h} \phi(t)dt.$$

. Instead, $R(1) = 1-\sigma$ and $R(s)$ is increasing to $+\infty$, as $s \to \frac{1}{1-\sigma}$. Since $L(1) > R(1)$, there exists $\hat{s} > 1$ such that

$$\frac{\partial \hat{W}(s)}{\partial s} \geq 0, \;\; s \in (1,\hat{s}], \quad \text{e} \quad \frac{\partial \hat{W}(s)}{\partial s} \leq 0, \;\; s \in [\hat{s}, \frac{1}{1-\sigma}[$$

Hence the maximum value of $W(s)$ is reached at $f^* = f_M$ if $\lfloor \hat{s} \rfloor + 1 \geq f_M$ (being $\lfloor \hat{s} \rfloor$ the integer part of $\hat{s}$) or, if $\hat{s}\rfloor + 1 < f_M$ in one of the two points $\lfloor \hat{s} \rfloor$ or $\lfloor \hat{s} \rfloor + 1$.

In the *voracious regime* (ii), one has $L(s) \geq 0$ for all s, while $R(s) < 0$ for all s, then $W'(s) > 0$ for all s and $f^* = f_M$. □

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