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SHAPE ANALYTICITY AND SINGULAR PERTURBATIONS FOR LAYER POTENTIAL OPERATORS

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Abstract. We study the effect of regular and singular domain perturbations on layer potential operators for the Laplace equation. First, we consider layer potentials supported on a diffeomorphic image $\phi(\partial\Omega)$ of a reference set $\partial\Omega$ and we present some real analyticity results for the dependence upon the map ϕ. Then we introduce a perforated domain $\Omega(\epsilon)$ with a small hole of size ϵ and we compute power series expansions that describe the layer potentials on $\partial\Omega(\epsilon)$ when the parameter ϵ approximates the degenerate value $\epsilon = 0$.

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1. Introduction

Potential theory is a valuable tool to analyze boundary value problems for elliptic differential equations and systems, both to deduce theoretical results and to perform numerical computations. Indeed, layer potentials can be used to convert boundary value problems into systems of integral equations that are often easier to study than the original problems. In recent times, potential theoretic techniques have been successfully employed to analyze boundary value problems on perturbed domains. In view of this application, it is important to understand what happens to the layer potentials when we perturb the support of integration. In this paper we look at this problem in the terms of the following question: what is the regularity of the maps that take the perturbation parameters to the corresponding layer potential operators?

To try to give an answer, we will consider the layer potentials related to the Laplace equation and we will study two different kind of perturbations, one that we call "regular," because we don't have loss of regularity in the perturbed sets, and one that we call "singular," because we do have some kind of loss of regularity in the perturbed sets. More specifically, as an example of a regular perturbation we will have layer potentials supported on a set $\phi(\partial\Omega)$ that is a diffeormorphic image of the boundary $\partial\Omega$ of a reference set Ω. In this case the perturbation parameter is the map ϕ and our goal is to understand the regularity of the map that takes ϕ, which we think as an element of a suitable Banach space of functions, to the corresponding layer potential operators, which we think as elements of suitable operator spaces. Instead, to make an example of a singular perturbation, we will analyze layer potentials supported on a set $\partial\Omega(\epsilon)$ with $\Omega(\epsilon)$ obtained removing from a fixed domain Ω an interior portion of size $\epsilon > 0$. This perturbation is "singular" because for $\epsilon = 0$ the set $\Omega(\epsilon)$ loses regularity on account of

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a removed point in its iterior. Also in this second case our goal is to understand the regularity of the map that takes the perturbation parameter –in this case ϵ– to the layer potential operators. In particular, we will focus on the situation where ϵ varies in a neighborhood of zero.

The interest for the regularity of this kind of maps can be motivated by the applications that they have in the framework of inverse scattering problems (which can be treated with variational or potential theoretic methods, or with the alternative approach of Kress and Päivärinta [34], see also Haddar and Kress [28] and Le Louër [48]). For example, in the works [54–56] of Potthast, we may find Fréchet differentiability results for certain layer potentials related to the Helmholtz equation. Charalambopoulos [5] obtained similar results, but for the layer potentials related to the elastic scattering problem. In the sense introduced above, the perturbations considered by Potthast and Charalambopoulos are of regular type: they consider a reference set of class C^2 that is perturbed into a new set that remains of class C^2. The regularity of the sets allows them to keep the analysis in the context of Schauder spaces. In [9, 10], instead, Costabel and Le Louër opt for the framework of Sobolev spaces to study the case of electromagnetic boundary integral operators. The family of layer potentials considered by Costabel and Le Louër is actually quite general and includes the usual boundary integral operators occurring in time-harmonic potential theory. More recently, Ivanyshyn Yaman and Le Louër [31] used the Piola transform to simplify the approach of [9, 10, 56].

Now, all the papers listed in the previous paragraph deal with differentiability properties and, indeed, regularity results that go beyond the differentiability seem to be much rarer in literature. There are some examples though. For instance, Ammari, Fitzpatrick, Kang, Ruiz, Yu, and Zhang consider a one-parametric regular perturbation and compute in [1] a series expansion of the single layer potential as a function of the perturbation parameter. Another example is given by the recent work on the "shape holomorphy" by Henríquez and Schwab [30], where the authors consider the layer potential operators supported on a C^2 Jordan curve in $\mathbb{R}^2$. In Henríquez and Schwab's paper a suitable parametrization of the Jordan curve plays the role of the (regular) perturbation parameter, which they think as an element in a complex Banach space, and, among other results, they show that the Calderón projector of the two-dimensional Laplacian is an holomorphic map of such parametrization. The idea of "shape holomorphy" was previously introduced in the papers by Jerez-Hanckes, Schwab, and Zech [32], dedicated to the electromagnetic wave scattering problem, and by Cohen, Schwab, and Zech [6], about the stationary Navier-Stokes equations.

Also the present paper's goal is to discuss regularity properties beyond the differentiability. More specifically, our aim is to prove real analyticity results. So, for example, in the first part of the paper, where we consider layer potentials supported on a diffeomorphic image $\phi(\partial\Omega)$ of a reference set $\partial\Omega$, we show that the map that takes ϕ to the corresponding layer potential operators is real analytic. The results of this first part are a direct consequence of the work of Lanza de Cristoforis and Rossi in [44, 45] and they can be compared with the holomorphy results proven by Henríquez and Schwab in [30]. Indeed, real analytic maps can be extended to holomorphic maps between reasonable complexifications of the underlying Banach spaces (see, e.g., the monograph of Hàyes and Johanis [29] and the references therein, see also the paragraph after Corollary 3.3). Although the restriction to the two-dimensional case might not be essential in Henríquez and Schwab paper, we also remark that here we consider all dimensions $n \geq 2$.

Probably the first analyticity results of this type were obtained for the Cauchy integral in the paper of Coifman and Meyer [7] and later by Lanza de Cristoforis and Preciso in [43]. The above mentioned papers [44, 45] were dedicated to the layer potentials for the Laplace and Helmholtz equations and served as a starting point for Lanza de Cristoforis and collaborators to extend this research topic in many different directions. For example, in [12] the authors considered a family of fundamental solutions of second order constant coefficient differential operators and proved that the corresponding layer potentials depend real analytically jointly on the parametrization of the support, the density, and the coefficients of the operators. We also mention [11], where a similar result was obtained for higher order operators, and [41], for the case of periodic layer potentials. Moreover, analyticity properties of the layer potentials have been exploited by the authors to analyze the shape dependence of physical quantities arising in fluid mechanics, material sciences, and scattering theory (see [17, 18, 46, 47]).

So, we might say that, as long as it concerns the problem proposed in this paper, regular perturbations are the subject of several works. Singular perturbations, instead, are widely studied in relation to boundary value problems and inverse problems (see, for example, Ammari and Kang [2], Ammari, Kang, and Lee [3], Maz'ya, Movchan, and Nieves [49], Maz'ya, Nazarov, and Plamenevskii [50, 51], and the references therein), but seem to be less studied in relation with the regularity of the layer potential operators maps. An exception is the work carried out by Lanza de Cristoforis and his collaborators with the development of the so called "functional analytic approach" (see the seminal papers [35–37], see also [16] and the references therein). To illustrate an application of the functional analytic approach we consider a domain $\Omega(\epsilon)$ with a hole of size ϵ. We first show that we can write the layer potential operators in terms of real analytic maps of ϵ, which are defined in an open neighborhood of $\epsilon = 0$, and of continuous elementary functions of ϵ, which might be not smooth, or even singular for $\epsilon = 0$. Then we focus on the analytic maps and we show how we can compute explicitly the coefficients of the corresponding power series expansions. The technique to compute such coefficients is inspired by the work in [21], where the computation was carried out in the case of a Dirichlet problem in a domain with a small hole (we incidentally note that a recent paper [24] by Feppon and Ammari presents a result comparable with that of [21]).

We observe that the presence of singular functions in the formula for the layer potentials may prevent these from being analytic functions of ϵ around $\epsilon = 0$. These specific singular functions are, however, completely known and in many cases restrictions to the positive values of ϵ have analytic continuations also for $\epsilon \leq 0$ (this is not surprising, one may think, for example, to the function $\epsilon \mapsto |\epsilon|$). Some consequences of these continuation properties are studied in the papers [19] and [20].

All results of the paper are presented in the framework of Schauder spaces, but we could very well have opted for Sobolev spaces instead. One reason to choose the Schauder environment is that it appears to be convenient when we apply our results to problems with nonlinear boundary conditions (as in Lanza de Cristoforis [38, 40]).

In conclusion of this introduction we like to stress that the aim of the paper is that of providing a well-organized toolbox of instruments for the analysis of perturbed boundary value problems. The reader may find applications of these instruments in papers by Lanza de Cristoforis (see, e.g., [36, 37, 39]), by the authors, Rogosin, and Pukhtaievych (see, e.g., [21, 22, 58]) and in the book [16]. From a very general point of view the main idea in the applications is that, having a boundary value problem transformed into an equivalent system of integral equations and knowing how the layer potentials depend on the perturbation parameter, we can recover some information on how the solution of the original boundary value problem depends on the perturbation parameter. In the conclusion section of the paper we will further comment on this point.

The paper is organized as follows. In Section 2 we introduce some notation, mainly related to layer potentials. In Section 3 we recall the results of Lanza de Cristoforis and Rossi [44, 45] on regular domain perturbations and we deduce some other analyticity results. We also include a paragraph where we discuss the relation between real analyticity and holomorphy. In Section 4, we consider singular domain perturbations and, after having deduced representations in terms of known elementary functions and real analytic maps, we show an explicit and constructive way to compute all the coefficients of the corresponding power series expansions. Finally, in Section 5 we give an example of how the results of Section 4 may be applied to a perturbed boundary value problem and we discuss some future developments.

2. Layer potentials for the Laplace equation

In this section, we introduce the layer potentials (and associated operators) for the Laplace equation. In order to do so, we fix

$$n \in \mathbb{N} \setminus \{0, 1\}$$

and we take

$$\alpha \in]0,1[\text{ and a bounded open connected subset } \tilde{\Omega} \text{ of } \mathbb{R}^n \text{ of class } C^{1,\alpha}.$$

For the definition of sets and functions of the Schauder class $C^{j,\alpha}$ ($j \in \mathbb{N}$) we refer, e.g., to Gilbarg and Trudinger [27].

Let G_n be the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$G_n(x) := \begin{cases} -\frac{1}{s_2}\log|x| & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n = 2\,, \\ \frac{1}{(n-2)s_n}|x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n \geq 3\,, \end{cases}$$

where s_n denotes the $(n-1)$-dimensional measure of the unit sphere in $\mathbb{R}^n$. The function G_n is well-known to be a fundamental solution of $-\Delta := -\sum_{j=1}^n \partial_{x_j}^2$. For the sake of simplicity, we will sometimes use the notation $\partial_j := \partial_{x_j}$.

We now introduce the single layer potential. If $\mu \in C^0(\partial\tilde{\Omega})$, we set

$$\mathcal{S}_{\tilde{\Omega}}[\mu](x) := \int_{\partial\tilde{\Omega}} G_n(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n\,, \tag{1}$$

where $d\sigma$ denotes the area element of a $(n-1)$-dimensional manifold imbedded in $\mathbb{R}^n$. As is well-known, if $\mu \in C^0(\partial\tilde{\Omega})$, then $\mathcal{S}_{\tilde{\Omega}}[\mu]$ is continuous in $\mathbb{R}^n$. Moreover, if $\mu \in C^{0,\alpha}(\partial\tilde{\Omega})$, then the function $\mathcal{S}^{\mathrm{int}}_{\tilde{\Omega}}[\mu] := \mathcal{S}_{\tilde{\Omega}}[\mu]_{|\overline{\tilde{\Omega}}}$ belongs to $C^{1,\alpha}(\overline{\tilde{\Omega}})$, and the function $\mathcal{S}^{\mathrm{ext}}_{\tilde{\Omega}}[\mu] := \mathcal{S}_{\tilde{\Omega}}[\mu]_{|\mathbb{R}^n\setminus\tilde{\Omega}}$ belongs to $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \tilde{\Omega})$. As usual, $\overline{A}$ denotes the closure of a set A.

Similarly, we introduce the double layer potential. If $\psi \in C^0(\partial\tilde{\Omega})$, we set

$$\mathcal{D}_{\tilde{\Omega}}[\psi](x) := -\int_{\partial\tilde{\Omega}} \nu_{\tilde{\Omega}}(y)\cdot\nabla G_n(x-y)\psi(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n\,, \tag{2}$$

where $\nu_{\tilde{\Omega}}$ denotes the outer unit normal to $\partial\tilde{\Omega}$ and the symbol "$\cdot$" denotes the scalar product in $\mathbb{R}^n$. In the above definition of the double layer potential $\mathcal{D}_{\tilde{\Omega}}[\psi]$, the symbol $\nabla G_n(x-y)$ must be understood as $(\nabla G_n)(x-y)$, and thus

$$\nabla_y(G_n(x-y)) = -(\nabla G_n)(x-y) = -\nabla G_n(x-y)\,.$$

Incidentally, we note that the function $G_n(x-y)$ has for $x = y$ a singularity of order $n-2$ when $n \geq 3$ and a logarithmic singularity when $n = 2$. Its gradient $\nabla G_n(x-y)$ has a singularity of order $n-1$ for all $n \geq 2$, but if $\tilde{\Omega}$ is of class $C^{1,\alpha}$ and we take x and y in $\partial\tilde{\Omega}$, then we can see that the singularity of $\nu_{\tilde{\Omega}}(y)\cdot\nabla G_n(x-y)$ is of order $n-1-\alpha$. As a consequence, the functions in the integrals of (1) and (2) are integrable in the classical sense also for $x \in \partial\tilde{\Omega}$ and we don't need to understand the integrals as principle values. For more details we refer the reader to [16, §4.3, §4.4].

As is well known, if $\psi \in C^{1,\alpha}(\partial\tilde{\Omega})$ the restriction $\mathcal{D}_{\tilde{\Omega}}[\psi]_{|\tilde{\Omega}}$ extends to a function $\mathcal{D}^{\mathrm{int}}_{\tilde{\Omega}}[\psi]$ in $C^{1,\alpha}(\overline{\tilde{\Omega}})$ and the restriction $\mathcal{D}_{\tilde{\Omega}}[\psi]_{|\mathbb{R}^n\setminus\overline{\tilde{\Omega}}}$ extends to a function $\mathcal{D}^{\mathrm{ext}}_{\tilde{\Omega}}[\psi]$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\tilde{\Omega})$. We observe that the symbols $\mathcal{D}^{\mathrm{int}}_{\tilde{\Omega}}[\psi]$ and $\mathcal{D}^{\mathrm{ext}}_{\tilde{\Omega}}[\psi]$ denote the extensions of the restrictions of the double layer potential to the closure of the interior and of the exterior of $\tilde{\Omega}$, respectively.

Next, we introduce two operators associated with the boundary trace of the double layer potential and of the normal derivative of the single layer potential. Let

$$\mathcal{K}_{\tilde{\Omega}}[\psi](x) := -\int_{\partial\tilde{\Omega}} \nu_{\tilde{\Omega}}(y)\cdot\nabla G_n(x-y)\psi(y)\,d\sigma_y \qquad \forall x \in \partial\tilde{\Omega}\,, \tag{3}$$

for all $\psi \in C^{1,\alpha}(\partial\tilde{\Omega})$, and

$$\mathcal{K}'_{\tilde{\Omega}}[\mu](x) := \int_{\partial\tilde{\Omega}} \nu_{\tilde{\Omega}}(x)\cdot\nabla G_n(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\tilde{\Omega}\,, \tag{4}$$

for all $\mu \in C^{0,\alpha}(\partial\tilde{\Omega})$. As it is well-known from classical potential theory, $\mathcal{K}_{\tilde{\Omega}}$ is a compact operator from $C^{1,\alpha}(\partial\tilde{\Omega})$ to itself and $\mathcal{K}'_{\tilde{\Omega}}$ is a compact operator from $C^{0,\alpha}(\partial\tilde{\Omega})$ to itself (see Schauder [59, 60]). Also, the operators $\mathcal{K}_{\tilde{\Omega}}$ and $\mathcal{K}'_{\tilde{\Omega}}$ are adjoint one to the other with respect to the duality on $C^{1,\alpha}(\partial\tilde{\Omega}) \times C^{0,\alpha}(\partial\tilde{\Omega})$ induced by the inner product of the Lebesgue space $L^2(\partial\tilde{\Omega})$ (cf., e.g., Kress [33, Chap. 4]). Moreover, the following jump formulas, describing the boundary behavior of the layer potentials with the corresponding boundary operators, hold.

$$
\begin{aligned}
\mathcal{D}^{\mathrm{int}}_{\tilde{\Omega}}[\psi]_{|\partial\tilde{\Omega}} &= -\frac{1}{2}\psi + \mathcal{K}_{\tilde{\Omega}}[\psi] && \forall \psi \in C^{1,\alpha}(\partial\tilde{\Omega}),\\
\mathcal{D}^{\mathrm{ext}}_{\tilde{\Omega}}[\psi]_{|\partial\tilde{\Omega}} &= \frac{1}{2}\psi + \mathcal{K}_{\tilde{\Omega}}[\psi] && \forall \psi \in C^{1,\alpha}(\partial\tilde{\Omega}),\\
\nu_{\tilde{\Omega}} \cdot \nabla\mathcal{S}^{\mathrm{int}}_{\tilde{\Omega}}[\mu]_{|\partial\tilde{\Omega}} &= \frac{1}{2}\mu + \mathcal{K}'_{\tilde{\Omega}}[\mu] && \forall \mu \in C^{0,\alpha}(\partial\tilde{\Omega}),\\
\nu_{\tilde{\Omega}} \cdot \nabla\mathcal{S}^{\mathrm{ext}}_{\tilde{\Omega}}[\mu]_{|\partial\tilde{\Omega}} &= -\frac{1}{2}\mu + \mathcal{K}'_{\tilde{\Omega}}[\mu] && \forall \mu \in C^{0,\alpha}(\partial\tilde{\Omega})
\end{aligned}
$$

(see, e.g., Folland [26, Chap. 3]).

Finally, we also set

$$
\mathcal{V}_{\tilde{\Omega}}[\mu](x) := \mathcal{S}_{\tilde{\Omega}}[\mu](x) \qquad \forall x \in \partial\tilde{\Omega}, \tag{5}
$$

for all $\mu \in C^{0,\alpha}(\partial\tilde{\Omega})$, and

$$
\mathcal{W}_{\tilde{\Omega}}[\psi](x) := -\nu_{\tilde{\Omega}}(x) \cdot \nabla\mathcal{D}^{\mathrm{ext}}_{\tilde{\Omega}}[\psi](x) = -\nu_{\tilde{\Omega}}(x) \cdot \nabla\mathcal{D}^{\mathrm{int}}_{\tilde{\Omega}}[\psi](x) \qquad \forall x \in \partial\tilde{\Omega}, \tag{6}
$$

for all $\psi \in C^{1,\alpha}(\partial\tilde{\Omega})$ (see, e.g., [16, Thm. 4.31 (iii)]). Clearly, $\mathcal{V}_{\tilde{\Omega}}[\mu] \in C^{1,\alpha}(\partial\tilde{\Omega})$ for all $\mu \in C^{0,\alpha}(\partial\tilde{\Omega})$ and $\mathcal{W}_{\tilde{\Omega}}[\psi] \in C^{0,\alpha}(\partial\tilde{\Omega})$ for all $\psi \in C^{1,\alpha}(\partial\tilde{\Omega})$.

3. Regular perturbations and shape analyticity

In this section we consider layer potentials supported on the diffeomorphic image of a reference set. We show some results of Lanza de Cristoforis and Rossi [44, 45] on the real analyticity of the maps that take the parametrization to the corresponding layer potentials. From these results we deduce some analyticity results for the corresponding operators. For the definition and properties of analytic operators, we refer to Prodi and Ambrosetti [57, p. 89] and to Deimling [23, p. 150]. Here we just recall that if $\mathcal{X}$, $\mathcal{Y}$ are real Banach spaces, and if F is an operator from an open subset $\mathcal{W}$ of $\mathcal{X}$ to $\mathcal{Y}$, then F is real analytic in $\mathcal{W}$ if for every $x_0 \in \mathcal{W}$ there exist $r > 0$ and continuous symmetric m-linear operators A_m from $\mathcal{X}^m$ to $\mathcal{Y}$ such that $\sum_{m\geq 1} \|A_m\| r^m < \infty$ and $F(x_0 + h) = F(x_0) + \sum_{m\geq 1} A_m(h, \ldots, h)$ for $\|h\|_{\mathcal{X}} \leq r$.

We now introduce the geometry of the problem. We fix

$$
\begin{gathered}
\alpha \in \,]0,1[\text{ and a bounded open connected subset } \Omega \text{ of } \mathbb{R}^n \text{ of class } C^{1,\alpha}\\
\text{such that } \mathbb{R}^n \setminus \overline{\Omega} \text{ is connected.}
\end{gathered} \tag{7}
$$

To consider shape perturbations of layer potential operators, we take the set Ω of (7) as a reference set. Then we introduce a specific class $\mathcal{A}^{1,\alpha}_{\partial\Omega}$ of $C^{1,\alpha}$-diffeomorphisms from $\partial\Omega$ to $\mathbb{R}^n$: $\mathcal{A}^{1,\alpha}_{\partial\Omega}$ is the set of functions of class $C^{1,\alpha}(\partial\Omega, \mathbb{R}^n)$ that are injective and have injective differential at all points of $\partial\Omega$. By Lanza de Cristoforis and Rossi [45, Lem. 2.2, p. 197] and [44, Lem. 2.5, p. 143], we can see that $\mathcal{A}^{1,\alpha}_{\partial\Omega}$ is open in $C^{1,\alpha}(\partial\Omega, \mathbb{R}^n)$. Moreover, for all $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega}$ the Jordan-Leray separation theorem ensures that $\mathbb{R}^n \setminus \phi(\partial\Omega)$ has exactly two open connected components (see, e.g., Deimling [23, Thm. 5.2, p. 26] and [16, §A.4]). We denote by $\mathbb{I}[\phi]$ the bounded connected component of $\mathbb{R}^n \setminus \phi(\partial\Omega)$ and by $\mathbb{E}[\phi]$ the unbounded one. Then, we have $\mathbb{E}[\phi] = \mathbb{R}^n \setminus \overline{\mathbb{I}[\phi]}$ and $\overline{\mathbb{E}[\phi]} = \mathbb{R}^n \setminus \mathbb{I}[\phi]$ (see Figure 1).

We will think at the diffeomorphism ϕ as a point in the Banach space $C^{1,\alpha}(\partial\Omega, \mathbb{R}^n)$ and we want to see that the maps that take $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega} \subseteq C^{1,\alpha}(\partial\Omega, \mathbb{R}^n)$ to the operators

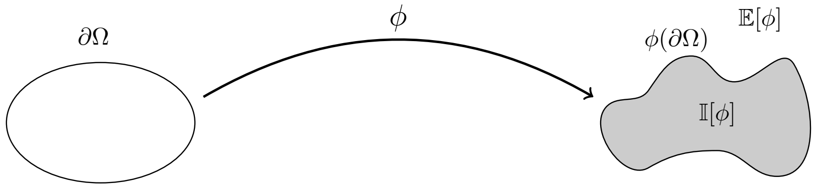

FIGURE 1. *The diffeomorphism* $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega}$ *and the* ϕ*-dependent sets* $\phi(\partial\Omega)$, $\mathbb{I}[\phi]$ *and* $\mathbb{E}[\phi]$.

$\mathcal{V}_{\mathbb{I}[\phi]}$, $\mathcal{K}_{\mathbb{I}[\phi]}$, $\mathcal{K}'_{\mathbb{I}[\phi]}$, and $\mathcal{W}_{\mathbb{I}[\phi]}$ are, in a sense, real analytic. We observe, however, that these operators are elements of spaces that depend on ϕ. For example, $\mathcal{V}_{\mathbb{I}[\phi]}$ belongs to

$$\mathcal{L}(C^{0,\alpha}(\phi(\partial\Omega)), C^{1,\alpha}(\phi(\partial\Omega)))\,.$$

So, to have real analytic maps between fixed Banach spaces we "pull-back" the operators to the reference set $\partial\Omega$. For example, for a diffeomorphism $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega}$, we denote by $\mathcal{V}_\phi$ the operator that takes a density function $\mu \in C^{0,\alpha}(\partial\Omega)$ to $\mathcal{V}_{\mathbb{I}[\phi]}[\mu \circ \phi^{(-1)}] \circ \phi$. Namely, we set

$$\mathcal{V}_\phi[\mu] := \mathcal{V}_{\mathbb{I}[\phi]}[\mu \circ \phi^{(-1)}] \circ \phi \qquad \forall \mu \in C^{0,\alpha}(\partial\Omega)\,.$$

Then we see that $\mathcal{V}_\phi$ is an element of the space

$$\mathcal{L}(C^{0,\alpha}(\partial\Omega), C^{1,\alpha}(\partial\Omega))\,,$$

which does not depend on ϕ, and it makes sense to ask if the map $\phi \mapsto \mathcal{V}_\phi$ is real analytic. Similarly, we denote by $\mathcal{K}_\phi$ the element of $\mathcal{L}(C^{1,\alpha}(\partial\Omega), C^{1,\alpha}(\partial\Omega))$ such that

$$\mathcal{K}_\phi[\psi] := \mathcal{K}_{\mathbb{I}[\phi]}[\psi \circ \phi^{(-1)}] \circ \phi \qquad \forall \psi \in C^{1,\alpha}(\partial\Omega)\,,$$

we denote by $\mathcal{K}'_\phi$ the element of $\mathcal{L}(C^{0,\alpha}(\partial\Omega), C^{0,\alpha}(\partial\Omega))$ defined by

$$\mathcal{K}'_\phi[\mu] := \mathcal{K}'_{\mathbb{I}[\phi]}[\mu \circ \phi^{(-1)}] \circ \phi \qquad \forall \mu \in C^{0,\alpha}(\partial\Omega)\,,$$

and by $\mathcal{W}_\phi$ the element of $\mathcal{L}(C^{1,\alpha}(\partial\Omega), C^{0,\alpha}(\partial\Omega))$ defined by

$$\mathcal{W}_\phi[\psi] := \mathcal{W}_{\mathbb{I}[\phi]}[\psi \circ \phi^{(-1)}] \circ \phi \qquad \forall \psi \in C^{1,\alpha}(\partial\Omega)\,.$$

In the following Lemma 3.1 we present some results from Lanza de Cristoforis and Rossi [44, 45].

Lemma 3.1. *Let* α, Ω *be as in* (7). *Then the following statements hold.*

(i) *The map from* $\mathcal{A}^{1,\alpha}_{\partial\Omega} \times C^{0,\alpha}(\partial\Omega)$ *to* $C^{1,\alpha}(\partial\Omega)$ *that takes a pair* (ϕ, μ) *to the function* $\mathcal{V}_{\mathbb{I}[\phi]}[\mu \circ \phi^{(-1)}] \circ \phi$ *is real analytic.*

(ii) *The map from* $\mathcal{A}^{1,\alpha}_{\partial\Omega} \times C^{1,\alpha}(\partial\Omega)$ *to* $C^{1,\alpha}(\partial\Omega)$ *that takes a pair* (ϕ, ψ) *to the function* $\mathcal{K}_{\mathbb{I}[\phi]}[\psi \circ \phi^{(-1)}] \circ \phi$ *is real analytic.*

(iii) *The map from* $\mathcal{A}^{1,\alpha}_{\partial\Omega} \times C^{0,\alpha}(\partial\Omega)$ *to* $C^{0,\alpha}(\partial\Omega)$ *that takes a pair* (ϕ, μ) *to the function* $\mathcal{K}'_{\mathbb{I}[\phi]}[\mu \circ \phi^{(-1)}] \circ \phi$ *is real analytic.*

(iv) *The map from* $\mathcal{A}^{1,\alpha}_{\partial\Omega} \times C^{1,\alpha}(\partial\Omega)$ *to* $C^{0,\alpha}(\partial\Omega)$ *that takes a pair* (ϕ, ψ) *to the function* $\mathcal{W}_{\mathbb{I}[\phi]}[\psi \circ \phi^{(-1)}] \circ \phi$ *is real analytic.*

By Lemma 3.1 we deduce the validity of the following theorem, where we show that the operators $\mathcal{V}_\phi$, $\mathcal{K}_\phi$, $\mathcal{K}'_\phi$, and $\mathcal{W}_\phi$ depend real analytically on ϕ.

Theorem 3.2. *Let* α, Ω *be as in* (7). *Then the following statements hold.*

(i) *The map from* $\mathcal{A}^{1,\alpha}_{\partial\Omega}$ *to* $\mathcal{L}(C^{0,\alpha}(\partial\Omega), C^{1,\alpha}(\partial\Omega))$ *that takes* ϕ *to* $\mathcal{V}_\phi$ *is real analytic.*

(ii) *The map from* $\mathcal{A}^{1,\alpha}_{\partial\Omega}$ *to* $\mathcal{L}(C^{1,\alpha}(\partial\Omega), C^{1,\alpha}(\partial\Omega))$ *that takes* ϕ *to* $\mathcal{K}_\phi$ *is real analytic.*

(iii) *The map from* $\mathcal{A}^{1,\alpha}_{\partial\Omega}$ *to* $\mathcal{L}(C^{0,\alpha}(\partial\Omega), C^{0,\alpha}(\partial\Omega))$ *that takes* ϕ *to* $\mathcal{K}'_\phi$ *is real analytic.*

(iv) *The map from* $\mathcal{A}^{1,\alpha}_{\partial\Omega}$ *to* $\mathcal{L}(C^{1,\alpha}(\partial\Omega), C^{0,\alpha}(\partial\Omega))$ *that takes* ϕ *to* $\mathcal{W}_\phi$ *is real analytic.*

Proof. We prove only statement (i). The proof of statements (ii)-(iv) can be effected similarly and is accordingly left to the reader. By Lemma 3.1 the map

$$\mathcal{A}^{1,\alpha}_{\partial\Omega} \times C^{0,\alpha}(\partial\Omega) \ni (\phi,\mu) \mapsto \mathcal{V}^\sharp(\phi,\mu) := \mathcal{V}_{\mathbb{I}[\phi]}[\mu \circ \phi^{(-1)}] \circ \phi \in C^{1,\alpha}(\partial\Omega)$$

is real analytic. Since $\mathcal{V}^\sharp$ is linear and continuous with respect to the variable μ, we have

$$\mathcal{V}_{\phi^\sharp} = d_\mu \mathcal{V}^\sharp(\phi^\sharp,\mu^\sharp) \qquad \forall (\phi^\sharp,\mu^\sharp) \in \mathcal{A}^{1,\alpha}_{\partial\Omega} \times C^{0,\alpha}(\partial\Omega)\,.$$

Since the right-hand side equals a partial Fréchet differential of a map which is real analytic by Lemma 3.1 (i), the right-hand side is analytic on $(\phi^\sharp,\mu^\sharp)$. Hence $(\phi^\sharp,\mu^\sharp) \mapsto \mathcal{V}_{\phi^\sharp}$ is real analytic on $\mathcal{A}^{1,\alpha}_{\partial\Omega} \times C^{0,\alpha}(\partial\Omega)$ and, since it does not depend on $\mu^\sharp$, we conclude that it is real analytic on $\mathcal{A}^{1,\alpha}_{\partial\Omega}$. □

Theorem 3.2 is a direct consequence of Lemma 3.1 in Lanza de Cristoforis and Rossi [44], but the new formulation has some advantages. For example, we can now recover from a different perspective the result of Henríquez and Schwab [30] on the shape holomorphy of the Calderón projector, and actually we can extend it from the 2-dimensional case to any dimension $n \geq 2$. So, as in Henríquez and Schwab [30], we now introduce the element $\mathcal{C}_\phi$ of $\mathcal{L}(C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega), C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega))$ defined by

$$\mathcal{C}_\phi := \begin{pmatrix} \frac{1}{2}I - \mathcal{K}_\phi & \mathcal{V}_\phi \\ \mathcal{W}_\phi & \frac{1}{2}I + \mathcal{K}'_\phi \end{pmatrix}$$

for all $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega}$. In other words, if $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega}$ and $(\psi,\mu) \in C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)$, then

$$\mathcal{C}_\phi[\psi,\mu] = \left(\frac{1}{2}\psi - \mathcal{K}_\phi[\psi] + \mathcal{V}_\phi[\mu], \mathcal{W}_\phi[\psi] + \frac{1}{2}\mu + \mathcal{K}'_\phi[\mu]\right).$$

The operator $\mathcal{C}_\phi$ is called Calderón projector. By Theorem 3.2, we immediately deduce the validity of the following corollary, where we show that the map that takes ϕ to $\mathcal{C}_\phi$ is real analytic (for the case of arbitrary dimension $n \geq 2$).

Corollary 3.3. *Let α, Ω be as in* (7)*. Then the map acting from the space $\mathcal{A}^{1,\alpha}_{\partial\Omega}$ to $\mathcal{L}(C^{1,\alpha}(\partial\Omega)\times C^{0,\alpha}(\partial\Omega), C^{1,\alpha}(\partial\Omega)\times C^{0,\alpha}(\partial\Omega))$ that takes ϕ to the bounded linear operator $\mathcal{C}_\phi$ is real analytic.*

We can now see that the map $\phi \mapsto \mathcal{C}_\phi$ has a holomorphic extension. Indeed, it is well known that for a real vector space X we can consider the complexified vector space $\widetilde{X} = X + iX$ with the operations

$$(x+iy) + (u+iv) = (x+u) + i(y+v)$$

and

$$(a+ib)(x+iy) = (ax - by) + i(bx + ay)$$

for all $x,y,u,v \in X$ and $a,b \in \mathbb{R}$. If in addition X is a normed space, with norm denoted by $\|\cdot\|_X$, then we might want to equip $\widetilde{X}$ with a norm as well. How to define a norm on $\widetilde{X}$ is not, however, a trivial task. We can see, for example, that the function

$$n(x+iy) = \sqrt{\|x\|^2_X + \|y\|^2_X}$$

is a norm on $\widetilde{X}$ only when the norm of X comes from an inner product (we can verify that if $n(\cdot)$ is positive homogeneous, then $\|\cdot\|_X$ has the parallelogram property). In [61] Taylor proposed to consider the function

$$\|x+iy\|_{\widetilde{X}} = \sup_{\Phi \in \overline{\mathbb{B}}_{X^*}} \sqrt{\Phi(x)^2 + \Phi(y)^2}, \tag{8}$$

where $\overline{\mathbb{B}}_{X^*}$ is the closed unit ball in $X^* := \mathcal{L}(X,\mathbb{R})$ (see also Michal and Wyman [52]). We can verify that $\|\cdot\|_{\widetilde{X}}$ is a norm on $\tilde{X}$ and that $\tilde{X}$ with the norm $\|\cdot\|_{\widetilde{X}}$ is complete as soon as X is complete. We can also see that the norm in (8) can be written as

$$\|x+iy\|_{\widetilde{X}} = \sup_{t\in[0,2\pi]} \|(\cos t)x + (\sin t)y\|_X$$

(cf. Muños, Sarantopoulos, and Tonge [53, Eq. (1)]). In addition, every *reasonable* norm $\|\cdot\|'_{\widetilde{X}}$ on $\widetilde{X}$ that satisfies the conditions

$$\|x\|'_{\widetilde{X}} = \|x\|_X \quad \forall x \in X$$

and

$$\|x+iy\|'_{\widetilde{X}} = \|x-iy\|'_{\widetilde{X}} \quad \forall x, y \in X$$

is equivalent to $\|\cdot\|_{\widetilde{X}}$ (cf. Muños, Sarantopoulos, and Tonge [53, Prop. 3]).

For what concerns this paper, we deduce that $C^{1,\alpha}(\partial\Omega, \mathbb{C})$ (the space of $C^{1,\alpha}$ complex valued functions on $\partial\Omega$) coincides algebraically with the complexified space $\widetilde{C^{1,\alpha}(\partial\Omega)}$ and the standard norm on $C^{1,\alpha}(\partial\Omega, \mathbb{C})$, which is *reasonable* in the sense introduced above, is equivalent to the norm defined by (8). Similarly, we have

$$\widetilde{C^{1,\alpha}(\partial\Omega, \mathbb{R}^n)} = C^{1,\alpha}(\partial\Omega, \mathbb{C}^n)$$

algebraically and with equivalent norms, and the complexification of the real Banach space $\mathcal{L}(C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega), C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega))$ coincides algebraically with

$$\mathcal{L}(C^{1,\alpha}(\partial\Omega, \mathbb{C}) \times C^{0,\alpha}(\partial\Omega, \mathbb{C}), C^{1,\alpha}(\partial\Omega, \mathbb{C}) \times C^{0,\alpha}(\partial\Omega, \mathbb{C}))$$

and has an equivalent norm.

Then, from Corollary 3.3 and from Hájek and Johanis [29, Thm. 171, p. 75] (see also Bochnak [4, Thm. 5]) we readily deduce the following result of the shape holomorphy of the Calderón projector.

Corollary 3.4. *Let α, Ω be as in (7). There exist an open subset $\widetilde{\mathcal{A}}^{1,\alpha}_{\partial\Omega}$ of $C^{1,\alpha}(\partial\Omega, \mathbb{C}^n)$ such that $\mathcal{A}^{1,\alpha}_{\partial\Omega} = \widetilde{\mathcal{A}}^{1,\alpha}_{\partial\Omega} \cap C^{1,\alpha}(\partial\Omega, \mathbb{R}^n)$ (that is, $\mathcal{A}^{1,\alpha}_{\partial\Omega}$ is the subset of the real valued functions of $\widetilde{\mathcal{A}}^{1,\alpha}_{\partial\Omega}$) and a holomorphic map $\widetilde{\mathcal{C}}$ from $\widetilde{\mathcal{A}}^{1,\alpha}_{\partial\Omega}$ to*

$$\mathcal{L}(C^{1,\alpha}(\partial\Omega, \mathbb{C}) \times C^{0,\alpha}(\partial\Omega, \mathbb{C}), C^{1,\alpha}(\partial\Omega, \mathbb{C}) \times C^{0,\alpha}(\partial\Omega, \mathbb{C}))$$

such that $\widetilde{\mathcal{C}}[\phi] = \mathcal{C}_\phi$ for all $\phi \in \mathcal{A}^{1,\alpha}_{\partial\Omega}$.

4. Singular perturbations

In this section we consider the effect of a singular perturbation produced by a small perforation in the domain that is bounded by the support of integration.

We fix

$$\begin{aligned}
&\alpha \in\,]0,1[\text{ and two bounded open connected subsets } \Omega^o,\ \Omega^i \text{ of } \mathbb{R}^n \text{ of class } C^{1,\alpha},\\
&\text{such that their exteriors } \mathbb{R}^n \setminus \overline{\Omega^o} \text{ and } \mathbb{R}^n \setminus \overline{\Omega^i} \text{ are connected,}\\
&\text{and the origin 0 of } \mathbb{R}^n \text{ belongs both to } \Omega^o \text{ and to } \Omega^i.
\end{aligned} \tag{9}$$

Here the superscript "o" stands for "outer domain" and the superscript "i" stands for "inner domain." We take

$$\epsilon_0 := \sup\{\theta \in\,]0,+\infty[\, : \epsilon\overline{\Omega^i} \subseteq \Omega^o,\ \forall \epsilon \in\,]-\theta,\theta[\}, \tag{10}$$

and we define the perforated domain $\Omega(\epsilon)$ by setting

$$\Omega(\epsilon) := \Omega^o \setminus \epsilon\overline{\Omega^i}$$

for all $\epsilon \in\,]-\epsilon_0, \epsilon_0[$. Clearly, when ϵ tends to zero, the set $\Omega(\epsilon)$ degenerates to the punctured domain $\Omega^o \setminus \{0\}$ (see Figure 2).

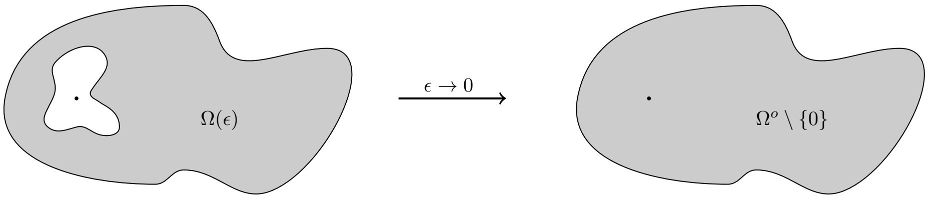

FIGURE 2. *The perforated set $\Omega(\epsilon)$ and the limiting punctured set $\Omega^o \setminus \{0\}$.*

We observe that the regularity of the set Ω was playing a crucial role when dealing with the regular perturbations of Section 3, but not here. Here, we could very well relax the conditions on Ω^i and Ω^o and take two Lipschitz domains instead. If that was our choice, we should consider integral operators in the framework of Sobolev spaces instead of Schauder spaces (as in [8]). For the sake of simplicity in the presentation, we prefer to keep working with domains of class $C^{1,\alpha}$ and with Schauder spaces.

4.1. **The operator $\mathcal{V}_{\Omega(\epsilon)}$**

Our aim is to study the maps that take $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus \{0\}$ to the operators $\mathcal{V}_{\Omega(\epsilon)}$, $\mathcal{K}_{\Omega(\epsilon)}$, $\mathcal{K}'_{\Omega(\epsilon)}$, and $\mathcal{W}_{\Omega(\epsilon)}$. We see, however, that these operators are defined on spaces that depend on the parameter ϵ. For example, for every fixed $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus \{0\}$ the operator $\mathcal{V}_{\Omega(\epsilon)}$ is an element of $\mathcal{L}(C^{0,\alpha}(\partial\Omega(\epsilon)), C^{1,\alpha}(\partial\Omega(\epsilon)))$ (we remind that $\mathcal{V}_{\Omega(\epsilon)}$ is the restriction of the single layer to the boundary of $\Omega(\epsilon)$, see definition (5)). Then, to describe the dependence of $\mathcal{V}_{\Omega(\epsilon)}$ upon ϵ we "pull-back" the operator to the boundary of the fixed domains $\partial\Omega^o$ and $\partial\Omega^i$. That is, we define

$$\mathcal{V}^o_\epsilon[\theta^o, \theta^i](x) := \mathcal{V}_{\Omega(\epsilon)}[\mu_\epsilon](x) \qquad \forall x \in \partial\Omega^o\,,$$
$$\mathcal{V}^i_\epsilon[\theta^o, \theta^i](t) := \mathcal{V}_{\Omega(\epsilon)}[\mu_\epsilon](\epsilon t) \qquad \forall t \in \partial\Omega^i\,,$$

with

$$\mu_\epsilon(x) := \begin{cases} \theta^o(x) & \text{if } x \in \partial\Omega^o\,, \\ \theta^i(x/\epsilon) & \text{if } x \in \partial(\epsilon\Omega^i)\,, \end{cases} \tag{11}$$

for all $(\theta^o, \theta^i) \in C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)$. So, in a sense, we identify functions of $C^{0,\alpha}(\partial\Omega(\epsilon))$ and $C^{1,\alpha}(\partial\Omega(\epsilon))$ with elements in the product spaces $C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)$ and $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$, respectively. Then we set

$$\mathcal{V}_\epsilon := (\mathcal{V}^o_\epsilon, \mathcal{V}^i_\epsilon)$$

and we observe that, for every $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus \{0\}$, the operator $\mathcal{V}_\epsilon$ is an element of a space that does not depend on ϵ, namely

$$\mathcal{V}_\epsilon \in \mathcal{L}(C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i), C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i))\,.$$

In the following Theorem 4.1 we describe $\mathcal{V}_\epsilon$ as a matrix operator with entries written in terms of analytic maps and elementary functions of ϵ. As we shall see, the proof of Theorem 4.1 exploits certain real analyticity results for integral operators with real analytic kernels (cf. [42]) and computations based on the Taylor series expansion of the kernel. In what follows we will often use the equality

$$\partial^k_\epsilon(F(\epsilon x)) = \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| = k}} \frac{k!}{\beta!} x^\beta (D^\beta F)(\epsilon x), \tag{12}$$

which holds for all $k \in \mathbb{N}$, $\epsilon \in \mathbb{R}$, $x \in \mathbb{R}^n$, and for all functions F analytic in a neighborhood of ϵx. Here, if $\beta \in \mathbb{N}^n$, then $(D^\beta F)(y)$ denotes the partial derivative of multi-index β of the function F evaluated at $y \in \mathbb{R}^n$.

Theorem 4.1. *Let α, Ω^o, Ω^i be as in (9). Let ϵ_0 be as in (10). There exist real analytic maps*

$$\begin{aligned}]-\epsilon_0, \epsilon_0[\to & \ \mathcal{L}(C^{0,\alpha}(\partial\Omega^i), C^{1,\alpha}(\partial\Omega^o)) \\ \epsilon \mapsto & \ \mathcal{V}^{o,i}_\epsilon \end{aligned}$$

and

$$\begin{aligned}]-\epsilon_0, \epsilon_0[\to & \ \mathcal{L}(C^{0,\alpha}(\partial\Omega^o), C^{1,\alpha}(\partial\Omega^i)) \\ \epsilon \mapsto & \ \mathcal{V}^{i,o}_\epsilon \end{aligned}$$

such that

$$\mathcal{V}_\epsilon = \begin{pmatrix} \mathcal{V}_{\Omega^o} & |\epsilon|^{n-1}\mathcal{V}^{o,i}_\epsilon \\ \mathcal{V}^{i,o}_\epsilon & |\epsilon|\,\mathcal{V}_{\Omega^i} - \delta_{2,n}\frac{|\epsilon|\log|\epsilon|}{2\pi}\mathrm{Int}_{\partial\Omega^i} \end{pmatrix} \tag{13}$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus \{0\}$, where

$$\mathrm{Int}_{\partial\Omega^i}[\theta^i] := \int_{\partial\Omega^i} \theta^i \, d\sigma \quad \forall \theta^i \in C^{0,\alpha}(\partial\Omega^i)\,.$$

Moreover, the following statements hold.

(i) *The coefficients $\mathcal{V}^{o,i}_{(k)}$ of the power series expansion $\mathcal{V}^{o,i}_\epsilon = \sum_{k=0}^\infty \epsilon^k \mathcal{V}^{o,i}_{(k)}$ with ϵ in a neighborhood of 0 are given by*

$$\mathcal{V}^{o,i}_{(k)}[\theta^i](x) := (-1)^k \sum_{\substack{\beta\in\mathbb{N}^n \\ |\beta|=k}} \frac{1}{\beta!}(D^\beta G_n)(x) \int_{\partial\Omega^i} s^\beta \theta^i(s)\, d\sigma_s$$

for all $k \in \mathbb{N}$, $x \in \partial\Omega^o$, and $\theta^i \in C^{0,\alpha}(\partial\Omega^i)$.

(ii) *The coefficients $\mathcal{V}^{i,o}_{(k)}$ of the power series expansion $\mathcal{V}^{i,o}_\epsilon = \sum_{k=0}^\infty \epsilon^k \mathcal{V}^{i,o}_{(k)}$ with ϵ in a neighborhood of 0 are given by*

$$\mathcal{V}^{i,o}_{(k)}[\theta^o](t) := (-1)^k \sum_{\substack{\beta\in\mathbb{N}^n \\ |\beta|=k}} \frac{1}{\beta!} t^\beta \int_{\partial\Omega^o} (D^\beta G_n)(y)\theta^o(y)\, d\sigma_y$$

for all $k \in \mathbb{N}$, $t \in \partial\Omega^i$, and $\theta^o \in C^{0,\alpha}(\partial\Omega^o)$.

Proof. Let $(\theta^o, \theta^i) \in C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)$, $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus \{0\}$. By a computation based on the theorem of change of variable in integrals we have

$$\begin{aligned} \mathcal{V}^o_\epsilon[\theta^o, \theta^i](x) &= \int_{\partial\Omega^o} G_n(x-y)\theta^o(y)\, d\sigma_y + |\epsilon|^{n-1} \int_{\partial\Omega^i} G_n(x-\epsilon s)\theta^i(s)\, d\sigma_s \\ &= \mathcal{V}_{\Omega^o}[\theta^o](x) + |\epsilon|^{n-1} \int_{\partial\Omega^i} G_n(x-\epsilon s)\theta^i(s)\, d\sigma_s \end{aligned}$$

for all $x \in \partial\Omega^o$. Similarly, we can compute that

$$\begin{aligned} &\mathcal{V}^i_\epsilon[\theta^o, \theta^i](t) \\ &\quad = \int_{\partial\Omega^o} G_n(\epsilon t - y)\theta^o(y)\, d\sigma_y + |\epsilon| \int_{\partial\Omega^i} G_n(t-s)\theta^i(s)\, d\sigma_s - \delta_{2,n}\frac{|\epsilon|\log|\epsilon|}{2\pi}\int_{\partial\Omega^i} \theta^i(s)\, d\sigma_s \\ &\quad = \int_{\partial\Omega^o} G_n(\epsilon t - y)\theta^o(y)\, d\sigma_y + |\epsilon|\; \mathcal{V}_{\Omega^i}[\theta^i](t) - \delta_{2,n}\frac{|\epsilon|\log|\epsilon|}{2\pi}\int_{\partial\Omega^i} \theta^i(s)\, d\sigma_s \end{aligned}$$

for all $t \in \partial\Omega^i$, where we have also used the equality

$$G_n(\epsilon\xi) = |\epsilon|^{2-n} G_n(\xi) - \delta_{2,n}\frac{1}{2\pi}\log|\epsilon| \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}\,, \forall \epsilon \neq 0\,.$$

Then equality (13) holds with

$$\mathcal{V}^{o,i}_\epsilon[\theta^i](x) := \int_{\partial\Omega^i} G_n(x-\epsilon s)\theta^i(s)\, d\sigma_s \qquad \forall \theta^i \in C^{0,\alpha}(\partial\Omega^i)\,,\ \forall x \in \partial\Omega^o$$

and

$$\mathcal{V}^{i,o}_\epsilon[\theta^o](t) := \int_{\partial\Omega^o} G_n(\epsilon t - y)\theta^o(y)\, d\sigma_y \qquad \forall \theta^o \in C^{0,\alpha}(\partial\Omega^o)\,,\ \forall t \in \partial\Omega^i\,.$$

By the regularity results for the integral operators with real analytic kernel of [42] and by the same argument we have used in the proof of Theorem 3.2, we can see that the maps $\epsilon \mapsto \mathcal{V}^{o,i}_\epsilon$ and $\epsilon \mapsto \mathcal{V}^{i,o}_\epsilon$ are real analytic from $]-\epsilon_0, \epsilon_0[$ to $\mathcal{L}(C^{0,\alpha}(\partial\Omega^i), C^{1,\alpha}(\partial\Omega^o))$ and from $]-\epsilon_0, \epsilon_0[$ to $\mathcal{L}(C^{0,\alpha}(\partial\Omega^o), C^{1,\alpha}(\partial\Omega^i))$, respectively.

Then we can locally express $\epsilon \mapsto \mathcal{V}^{o,i}_\epsilon$ with its Taylor series. In particular, we have

$$\mathcal{V}^{o,i}_\epsilon = \sum_{k=0}^{\infty} \epsilon^k \frac{1}{k!} (\partial^k_\epsilon \mathcal{V}^{o,i}_\epsilon)_{|\epsilon=0}$$

for ϵ in a neighborhood of 0 and we can prove statement (i) computing the derivatives $(\partial^k_\epsilon \mathcal{V}^{o,i}_\epsilon)_{|\epsilon=0}$. With the help of equation (12) we can see that

$$\partial^k_\epsilon \left(\int_{\partial\Omega^i} G_n(x - \epsilon s)\theta^i(s)\, d\sigma_s \right) = (-1)^k \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=k}} \frac{k!}{\beta!} \int_{\partial\Omega^i} s^\beta (D^\beta G_n)(x - \epsilon s)\theta^i(s)\, d\sigma_s\,.$$

Accordingly

$$\partial^k_\epsilon \left(\int_{\partial\Omega^i} G_n(x - \epsilon s)\theta^i(s)\, d\sigma_s \right)_{|\epsilon=0} = (-1)^k \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=k}} \frac{k!}{\beta!} (D^\beta G_n)(x) \int_{\partial\Omega^i} s^\beta \theta^i(s)\, d\sigma_s\,,$$

and statement (i) follows.

Similarly, to verify statement (ii) we have to compute the derivatives $(\partial^k_\epsilon \mathcal{V}^{i,o}_\epsilon)_{|\epsilon=0}$. Again, with the help of (12) we see that

$$\partial^k_\epsilon \left(\int_{\partial\Omega^o} G_n(\epsilon t - y)\theta^o(y)\, d\sigma_y \right) = \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=k}} \frac{k!}{\beta!} \int_{\partial\Omega^o} t^\beta (D^\beta G_n)(\epsilon t - y)\theta^o(y)\, d\sigma_y\,,$$

accordingly

$$\begin{aligned}\partial^k_\epsilon \left(\int_{\partial\Omega^o} G_n(\epsilon t - y)\theta^o(y)\, d\sigma_y \right)_{|\epsilon=0} &= \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=k}} \frac{k!}{\beta!} t^\beta \int_{\partial\Omega^o} (D^\beta G_n)(-y)\theta^o(y)\, d\sigma_y \\ &= (-1)^k \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=k}} \frac{k!}{\beta!} t^\beta \int_{\partial\Omega^o} (D^\beta G_n)(y)\theta^o(y)\, d\sigma_y\,,\end{aligned}$$

and statement (ii) follows. □

4.2. The operator $\mathcal{K}_{\Omega(\epsilon)}$

We proceed with the boundary operator $\mathcal{K}_{\Omega(\epsilon)}$, which is the restriction of the double layer potential to the boundary of $\Omega(\epsilon)$ (see definition (3)). In a way that resembles what we did above for the single layer potential, we set

$$\begin{aligned}\mathcal{K}^o_\epsilon[\theta^o, \theta^i](x) &:= \mathcal{K}_{\Omega(\epsilon)}[\psi_\epsilon](x) && \forall x \in \partial\Omega^o\,,\\ \mathcal{K}^i_\epsilon[\theta^o, \theta^i](t) &:= \mathcal{K}_{\Omega(\epsilon)}[\psi_\epsilon](\epsilon t) && \forall t \in \partial\Omega^i\,,\end{aligned}$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus \{0\}$ and $(\theta^o, \theta^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$, where

$$\psi_\epsilon(x) := \begin{cases} \theta^o(x) & \text{if } x \in \partial\Omega^o\,,\\ \theta^i(x/\epsilon) & \text{if } x \in \partial(\epsilon\Omega^i)\,. \end{cases} \tag{14}$$

Then we denote by $\mathcal{K}_\epsilon$ the element of $\mathcal{L}(C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i), C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i))$ defined by

$$\mathcal{K}_\epsilon := (\mathcal{K}^o_\epsilon, \mathcal{K}^i_\epsilon) \quad \forall \epsilon \in \,]-\epsilon_0, \epsilon_0[\, \setminus \{0\}\,.$$

We have the following.

Theorem 4.2. *Let α, Ω^o, Ω^i be as in* (9). *Let ϵ_0 be as in* (10). *There exist real analytic maps*

$$\begin{aligned}]-\epsilon_0,\epsilon_0[\,\to\, & \mathcal{L}(C^{1,\alpha}(\partial\Omega^i), C^{1,\alpha}(\partial\Omega^o)) \\ \epsilon \mapsto\, & \mathcal{K}^{o,i}_\epsilon \end{aligned}$$

and

$$\begin{aligned}]-\epsilon_0,\epsilon_0[\,\to\, & \mathcal{L}(C^{1,\alpha}(\partial\Omega^o), C^{1,\alpha}(\partial\Omega^i)) \\ \epsilon \mapsto\, & \mathcal{K}^{i,o}_\epsilon \end{aligned}$$

such that

$$\mathcal{K}_\epsilon = \begin{pmatrix} \mathcal{K}_{\Omega^o} & \epsilon|\epsilon|^{n-2}\mathcal{K}^{o,i}_\epsilon \\ \mathcal{K}^{i,o}_\epsilon & -\mathcal{K}_{\Omega^i} \end{pmatrix} \tag{15}$$

for all $\epsilon \in \,]-\epsilon_0,\epsilon_0[\,\setminus\{0\}$. Moreover, the following statements hold.

(i) *The coefficients $\mathcal{K}^{o,i}_{(k)}$ of the power series expansion $\mathcal{K}^{o,i}_\epsilon = \sum_{k=0}^\infty \epsilon^k \mathcal{K}^{o,i}_{(k)}$ with ϵ in a neighborhood of* 0 *are given by*

$$\mathcal{K}^{o,i}_{(k)}[\theta^i](x) := (-1)^k \sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=k}} \frac{1}{\beta!}(\nabla D^\beta G_n)(x)\cdot\int_{\partial\Omega^i} \nu_{\Omega^i}(s)s^\beta\theta^i(s)\,d\sigma_s$$

for all $k\in\mathbb{N}$, $x\in\partial\Omega^o$, and $\theta^i\in C^{1,\alpha}(\partial\Omega^i)$.

(ii) *The coefficients $\mathcal{K}^{i,o}_{(k)}$ of the power series expansion $\mathcal{K}^{i,o}_\epsilon = \sum_{k=0}^\infty \epsilon^k \mathcal{K}^{i,o}_{(k)}$ with ϵ in a neighborhood of* 0 *are given by*

$$\mathcal{K}^{i,o}_{(k)}[\theta^o](t) := (-1)^k \sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=k}} \frac{1}{\beta!}t^\beta\int_{\partial\Omega^o} \nu_{\Omega^o}(y)\cdot(\nabla D^\beta G_n)(y)\theta^o(y)\,d\sigma_y$$

for all $k\in\mathbb{N}$, $t\in\partial\Omega^i$, and $\theta^o\in C^{1,\alpha}(\partial\Omega^o)$.

Proof. Let $(\theta^o,\theta^i)\in C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)$, $\epsilon\in\,]-\epsilon_0,\epsilon_0[\,\setminus\{0\}$. By the theorem of change of variables in integrals and equality

$$\nu_{\epsilon\Omega^i}(\epsilon s) = \mathrm{sgn}(\epsilon)\nu_{\Omega^i}(s) \qquad \forall s\in\partial\Omega^i\,,$$

we can see that

$$\begin{aligned} \mathcal{K}^o_\epsilon[\theta^o,\theta^i](x) &= \mathcal{K}_{\Omega^o}[\theta^o](x) + |\epsilon|^{n-1}\mathrm{sgn}(\epsilon)\int_{\partial\Omega^i}\nu_{\Omega^i}(s)\cdot\nabla G_n(x-\epsilon s)\theta^i(s)\,d\sigma_s \\ &= \mathcal{K}_{\Omega^o}[\theta^o](x) + \epsilon|\epsilon|^{n-2}\int_{\partial\Omega^i}\nu_{\Omega^i}(s)\cdot\nabla G_n(x-\epsilon s)\theta^i(s)\,d\sigma_s \end{aligned}$$

for all $x\in\partial\Omega^o$. Moreover, by equality

$$\nabla G_n(\epsilon\eta) = \mathrm{sgn}(\epsilon)|\epsilon|^{1-n}\nabla G_n(\eta) \qquad \forall\epsilon\in\mathbb{R}\setminus\{0\},\,\forall\eta\in\mathbb{R}^n\setminus\{0\},$$

we can compute that

$$\mathcal{K}^i_\epsilon[\theta^o,\theta^i](t) = -\int_{\partial\Omega^o}\nu_{\Omega^o}(y)\cdot\nabla G_n(\epsilon t - y)\theta^o(y)\,d\sigma_y - \mathcal{K}_{\Omega^i}[\theta^i](t) \qquad \forall t\in\partial\Omega^i\,.$$

Then equality (15) holds with

$$\mathcal{K}^{o,i}_\epsilon[\theta^i](x) := \int_{\partial\Omega^i}\nu_{\Omega^i}(s)\cdot\nabla G_n(x-\epsilon s)\theta^i(s)\,d\sigma_s \qquad \forall\theta^i\in C^{1,\alpha}(\partial\Omega^i)\,,\ \forall x\in\partial\Omega^o$$

and

$$\mathcal{K}_\epsilon^{i,o}[\theta^o](t) := -\int_{\partial\Omega^o} \nu_{\Omega^o}(y)\cdot\nabla G_n(\epsilon t - y)\theta^o(y)\,d\sigma_y \qquad \forall\theta^o\in C^{1,\alpha}(\partial\Omega^o)\,,\ \forall t\in\partial\Omega^i\,.$$

By the regularity results for the integral operators with real analytic kernel of [42] and by the same argument we have used in the proof of Theorem 3.2, we can see that the maps $\epsilon\mapsto\mathcal{K}_\epsilon^{o,i}$ and $\epsilon\mapsto\mathcal{K}_\epsilon^{i,o}$ are real analytic from $]-\epsilon_0,\epsilon_0[$ to $\mathcal{L}(C^{1,\alpha}(\partial\Omega^i),C^{1,\alpha}(\partial\Omega^o))$ and from $]-\epsilon_0,\epsilon_0[$ to $\mathcal{L}(C^{1,\alpha}(\partial\Omega^o),C^{1,\alpha}(\partial\Omega^i))$, respectively.

Then we can locally express $\epsilon\mapsto\mathcal{K}_\epsilon^{o,i}$ with its Taylor series. In particular, we have

$$\mathcal{K}_\epsilon^{o,i} = \sum_{k=0}^{\infty}\epsilon^k\frac{1}{k!}(\partial_\epsilon^k\mathcal{K}_\epsilon^{o,i})_{|\epsilon=0}$$

for ϵ in a neighborhood of 0 and we can prove statement (i) computing the derivatives $(\partial_\epsilon^k\mathcal{K}_\epsilon^{o,i})_{|\epsilon=0}$. With the help of equation (12) we can see that

$$\begin{aligned}\partial_\epsilon^k&\left(\int_{\partial\Omega^i}\nu_{\Omega^i}(s)\cdot\nabla G_n(x-\epsilon s)\theta^i(s)\,d\sigma_s\right)\\&=(-1)^k\sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=k}}\frac{k!}{\beta!}\int_{\partial\Omega^i}\nu_{\Omega^i}(s)\cdot(\nabla D^\beta G_n)(x-\epsilon s)s^\beta\theta^i(s)\,d\sigma_s\,,\end{aligned}$$

and accordingly

$$\begin{aligned}\partial_\epsilon^k&\left(\int_{\partial\Omega^i}\nu_{\Omega^i}(s)\cdot\nabla G_n(x-\epsilon s)\theta^i(s)\,d\sigma_s\right)_{|\epsilon=0}\\&=(-1)^k\sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=k}}\frac{k!}{\beta!}(\nabla D^\beta G_n)(x)\cdot\int_{\partial\Omega^i}\nu_{\Omega^i}(s)s^\beta\theta^i(s)\,d\sigma_s\end{aligned}$$

and statement (i) follows.

Similarly, to verify statement (ii) we have to compute the derivatives $(\partial_\epsilon^k\mathcal{K}_\epsilon^{i,o})_{|\epsilon=0}$. Again, with the help of (12) we see that

$$\begin{aligned}\partial_\epsilon^k&\left(\int_{\partial\Omega^o}\nu_{\Omega^o}(y)\cdot\nabla G_n(\epsilon t-y)\theta^o(y)\,d\sigma_y\right)\\&=\sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=k}}\frac{k!}{\beta!}\int_{\partial\Omega^o}\nu_{\Omega^o}(y)\cdot(\nabla D^\beta G_n)(\epsilon t-y)t^\beta\theta^o(y)\,d\sigma_y\,,\end{aligned}$$

and accordingly

$$\begin{aligned}\partial_\epsilon^k&\left(-\int_{\partial\Omega^o}\nu_{\Omega^o}(y)\cdot\nabla G_n(\epsilon t-y)\theta^o(y)\,d\sigma_y\right)_{|\epsilon=0}\\&=-\sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=k}}\frac{k!}{\beta!}\int_{\partial\Omega^o}\nu_{\Omega^o}(y)\cdot(\nabla D^\beta G_n)(-y)t^\beta\theta^o(y)\,d\sigma_y\\&=(-1)^k\sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=k}}\frac{k!}{\beta!}t^\beta\int_{\partial\Omega^o}\nu_{\Omega^o}(y)\cdot(\nabla D^\beta G_n)(y)\theta^o(y)\,d\sigma_y\end{aligned}$$

and statement (ii) follows. □

4.3. The operator $\mathcal{K}'_{\Omega(\epsilon)}$

We now turn to $\mathcal{K}'_{\Omega(\epsilon)}$, the boundary operator related with the normal derivative of the single layer potential (see definition (4)). We set

$$\mathcal{K}_\epsilon'^{o}[\theta^o,\theta^i](x) := \mathcal{K}'_{\Omega(\epsilon)}[\mu_\epsilon](x) \qquad \forall x\in\partial\Omega^o\,,$$

$$\mathcal{K}'^{i}_{\epsilon}[\theta^o,\theta^i](t) := \mathcal{K}'_{\Omega(\epsilon)}[\mu_\epsilon](\epsilon t) \qquad \forall t \in \partial\Omega^i\,,$$

for all $\epsilon \in]-\epsilon_0,\epsilon_0[\setminus\{0\}$ and $(\theta^o,\theta^i) \in C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i)$, with μ_ϵ as in (11). Then we define

$$\mathcal{K}'_\epsilon := (\mathcal{K}'^{o}_{\epsilon},\mathcal{K}'^{i}_{\epsilon}) \quad \forall \epsilon \in]-\epsilon_0,\epsilon_0[\setminus\{0\}$$

and we note that $\mathcal{K}'_\epsilon$ is an element of $\mathcal{L}(C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i), C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i))$. We can prove the following.

Theorem 4.3. *Let α, Ω^o, Ω^i be as in* (9). *Let ϵ_0 be as in* (10). *There exist real analytic maps*

$$\begin{aligned}]-\epsilon_0,\epsilon_0[&\to \mathcal{L}(C^{0,\alpha}(\partial\Omega^i), C^{0,\alpha}(\partial\Omega^o)) \\ \epsilon &\mapsto \mathcal{K}'^{o,i}_{\epsilon} \end{aligned}$$

and

$$\begin{aligned}]-\epsilon_0,\epsilon_0[&\to \mathcal{L}(C^{0,\alpha}(\partial\Omega^o), C^{0,\alpha}(\partial\Omega^i)) \\ \epsilon &\mapsto \mathcal{K}'^{i,o}_{\epsilon} \end{aligned}$$

such that

$$\mathcal{K}'_\epsilon = \begin{pmatrix} \mathcal{K}'_{\Omega^o} & |\epsilon|^{n-1}\mathcal{K}'^{o,i}_{\epsilon} \\ \mathrm{sgn}(\epsilon)\mathcal{K}'^{i,o}_{\epsilon} & -\mathcal{K}'_{\Omega^i} \end{pmatrix} \tag{16}$$

for all $\epsilon \in]-\epsilon_0,\epsilon_0[\setminus\{0\}$. Moreover, the following statements hold.

(i) *The coefficients $\mathcal{K}'^{o,i}_{(k)}$ of the power series expansion $\mathcal{K}'^{o,i}_{\epsilon} = \sum_{k=0}^{\infty}\epsilon^k\mathcal{K}'^{o,i}_{(k)}$ with ϵ in a neighborhood of* 0 *are given by*

$$\mathcal{K}'^{o,i}_{(k)}[\theta^i](x) := (-1)^k \sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=k}} \frac{1}{\beta!}\nu_{\Omega^o}(x)\cdot(\nabla D^\beta G_n)(x)\int_{\partial\Omega^i} s^\beta\theta^i(s)\,d\sigma_s$$

for all $k\in\mathbb{N}$, $x\in\partial\Omega^o$, and $\theta^i\in C^{0,\alpha}(\partial\Omega^i)$.

(ii) *The coefficients $\mathcal{K}'^{i,o}_{(k)}$ of the power series expansion $\mathcal{K}'^{i,o}_{\epsilon} = \sum_{k=0}^{\infty}\epsilon^k\mathcal{K}'^{i,o}_{(k)}$ with ϵ in a neighborhood of* 0 *are given by*

$$\mathcal{K}'^{i,o}_{(k)}[\theta^o](t) := (-1)^k \sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=k}} \frac{1}{\beta!}t^\beta\nu_{\Omega^i}(t)\cdot\int_{\partial\Omega^o}(\nabla D^\beta G_n)(y)\theta^o(y)\,d\sigma_y$$

for all $k\in\mathbb{N}$, $t\in\partial\Omega^i$, and $\theta^o\in C^{0,\alpha}(\partial\Omega^o)$.

Proof. Let $(\theta^o,\theta^i)\in C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i)$, $\epsilon\in]-\epsilon_0,\epsilon_0[\setminus\{0\}$. By a straightforward computation based on the theorem of change of variable in integrals we can see that

$$\mathcal{K}'^{o}_{\epsilon}[\theta^o,\theta^i](x) = \mathcal{K}'_{\Omega^o}[\theta^o](x) + |\epsilon|^{n-1}\int_{\partial\Omega^i}\nu_{\Omega^o}(x)\cdot\nabla G_n(x-\epsilon s)\theta^i(s)\,d\sigma_s \quad \forall x\in\partial\Omega^o\,,$$

$$\mathcal{K}'^{i}_{\epsilon}[\theta^o,\theta^i](t) = -\mathrm{sgn}(\epsilon)\int_{\partial\Omega^o}\nu_{\Omega^i}(t)\cdot\nabla G_n(\epsilon t-y)\theta^o(y)\,d\sigma_y - \mathcal{K}'_{\Omega^i}[\theta^i](t) \quad \forall t\in\partial\Omega^i\,.$$

Then (16) holds with

$$\mathcal{K}'^{o,i}_{\epsilon}[\theta^i](x) := \int_{\partial\Omega^i}\nu_{\Omega^o}(x)\cdot\nabla G_n(x-\epsilon s)\theta^i(s)\,d\sigma_s \qquad \forall x\in\partial\Omega^o\,,\ \forall\theta^i\in C^{0,\alpha}(\partial\Omega^i)\,,$$

$$\mathcal{K}'^{i,o}_{\epsilon}[\theta^o](t) := -\int_{\partial\Omega^o}\nu_{\Omega^i}(t)\cdot\nabla G_n(\epsilon t-y)\theta^o(y)\,d\sigma_y \quad \forall t\in\partial\Omega^i\,,\ \forall\theta^o\in C^{0,\alpha}(\partial\Omega^o)\,.$$

By the regularity results for the integral operators with real analytic kernel of [42] (see also the argument in the proof of Theorem 3.2) we can see that the maps $\epsilon\mapsto\mathcal{K}'^{o,i}_{\epsilon}$ and $\epsilon\mapsto\mathcal{K}'^{i,o}_{\epsilon}$ are real analytic from $]-\epsilon_0,\epsilon_0[$ to $\mathcal{L}(C^{0,\alpha}(\partial\Omega^i),C^{0,\alpha}(\partial\Omega^o))$ and from $]-\epsilon_0,\epsilon_0[$ to $\mathcal{L}(C^{0,\alpha}(\partial\Omega^o),C^{0,\alpha}(\partial\Omega^i))$, respectively.

To verify statement (i) we compute

$$\partial_\epsilon^k \left(\int_{\partial\Omega^i} \nu_{\Omega^o}(x)\cdot\nabla G_n(x-\epsilon s)\theta^i(s)\,d\sigma_s \right) = (-1)^k \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=k}} \frac{k!}{\beta!} \int_{\partial\Omega^i} \nu_{\Omega^o}(x)\cdot(\nabla D^\beta G_n)(x-\epsilon s)s^\beta\theta^i(s)\,d\sigma_s\,,$$

and accordingly

$$\partial_\epsilon^k \left(\int_{\partial\Omega^i} \nu_{\Omega^o}(x)\cdot\nabla G_n(x-\epsilon s)\theta^i(s)\,d\sigma_s \right)_{|\epsilon=0} = (-1)^k \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=k}} \frac{k!}{\beta!} \nu_{\Omega^o}(x)\cdot(\nabla D^\beta G_n)(x)\int_{\partial\Omega^i} s^\beta\theta^i(s)\,d\sigma_s\,.$$

To verify statement (ii), we note that we have

$$\partial_\epsilon^k \left(\int_{\partial\Omega^o} \nu_{\Omega^i}(t)\cdot\nabla G_n(\epsilon t-y)\theta^o(y)\,d\sigma_y \right) = \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=k}} \frac{k!}{\beta!} \int_{\partial\Omega^o} \nu_{\Omega^i}(t)\cdot(\nabla D^\beta G_n)(\epsilon t-y)t^\beta\theta^o(y)\,d\sigma_y$$

and accordingly

$$\partial_\epsilon^k \left(-\int_{\partial\Omega^o} \nu_{\Omega^i}(t)\cdot\nabla G_n(\epsilon t-y)\theta^o(y)\,d\sigma_y \right)_{|\epsilon=0} = -\sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=k}} \frac{k!}{\beta!} \int_{\partial\Omega^o} \nu_{\Omega^i}(t)\cdot(\nabla D^\beta G_n)(-y)t^\beta\theta^o(y)\,d\sigma_y$$
$$= (-1)^k \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=k}} \frac{k!}{\beta!} t^\beta \nu_{\Omega^i}(t)\cdot\int_{\partial\Omega^o} (\nabla D^\beta G_n)(y)\theta^o(y)\,d\sigma_y\,.$$

□

4.4. The operator $\mathcal{W}_{\Omega(\epsilon)}$

The last operator to consider is $\mathcal{W}_{\Omega(\epsilon)}$ (see definition (6)). As usual, we define

$$\mathcal{W}^o_\epsilon[\theta^o,\theta^i](x) := \mathcal{W}_{\Omega(\epsilon)}[\psi_\epsilon](x) \qquad \forall x\in\partial\Omega^o\,,$$
$$\mathcal{W}^i_\epsilon[\theta^o,\theta^i](t) := \mathcal{W}_{\Omega(\epsilon)}[\psi_\epsilon](\epsilon t) \qquad \forall t\in\partial\Omega^i\,,$$

for all $\epsilon\in\,]-\epsilon_0,\epsilon_0[\,\setminus\{0\}$ and $(\theta^o,\theta^i)\in C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)$, with ψ_ϵ as in (14). Then we take

$$\mathcal{W}_\epsilon := (\mathcal{W}^o_\epsilon, \mathcal{W}^i_\epsilon) \quad \forall\epsilon\in\,]-\epsilon_0,\epsilon_0[\,\setminus\{0\}$$

and we wish to describe the map $\epsilon\mapsto\mathcal{W}_\epsilon$ from $]-\epsilon_0,\epsilon_0[\,\setminus\{0\}$ to the space of operators $\mathcal{L}(C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i), C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i))$.

Theorem 4.4. *Let α, Ω^o, Ω^i be as in* (9). *Let ϵ_0 be as in* (10). *There exist real analytic maps*

$$\begin{aligned}]-\epsilon_0,\epsilon_0[&\to \mathcal{L}(C^{1,\alpha}(\partial\Omega^i), C^{0,\alpha}(\partial\Omega^o)) \\ \epsilon &\mapsto \mathcal{W}^{o,i}_\epsilon \end{aligned}$$

and

$$\begin{aligned}]-\epsilon_0,\epsilon_0[&\to \mathcal{L}(C^{1,\alpha}(\partial\Omega^o), C^{0,\alpha}(\partial\Omega^i)) \\ \epsilon &\mapsto \mathcal{W}^{i,o}_\epsilon \end{aligned}$$

such that

$$\mathcal{W}_\epsilon = \begin{pmatrix} \mathcal{W}_{\Omega^o} & \epsilon|\epsilon|^{n-2}\mathcal{W}^{o,i}_\epsilon \\ \mathrm{sgn}(\epsilon)\mathcal{W}^{i,o}_\epsilon & |\epsilon|^{-1}\mathcal{W}_{\Omega^i} \end{pmatrix} \tag{17}$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus \{0\}$. *Moreover, the following statements hold.*

(i) *The coefficients* $\mathcal{W}^{o,i}_{(k)}$ *of the power series expansion* $\mathcal{W}^{o,i}_\epsilon = \sum_{k=0}^\infty \epsilon^k \mathcal{W}^{o,i}_{(k)}$ *with* ϵ *in a neighborhood of* 0 *are given by*

$$\mathcal{W}^{o,i}_{(k)}[\theta^i](x) := (-1)^{k+1} \sum_{\substack{\beta\in\mathbb{N}^n \\ |\beta|=k}} \sum_{l=1}^n \frac{1}{\beta!}(\nu_{\Omega^o}(x))_l (\nabla D^\beta \partial_l G_n)(x) \cdot \int_{\partial\Omega^i} \nu_{\Omega^i}(s) s^\beta \theta^i(s)\, d\sigma_s$$

for all $k \in \mathbb{N}$, $x \in \partial\Omega^o$, *and* $\theta^i \in C^{1,\alpha}(\partial\Omega^i)$.

(ii) *The coefficients* $\mathcal{W}^{i,o}_{(k)}$ *of the power series expansion* $\mathcal{W}^{i,o}_\epsilon = \sum_{k=0}^\infty \epsilon^k \mathcal{W}^{i,o}_{(k)}$ *with* ϵ *in a neighborhood of* 0 *are given by*

$$\mathcal{W}^{i,o}_{(k)}[\theta^o](t) := (-1)^{k+1} \sum_{\substack{\beta\in\mathbb{N}^n \\ |\beta|=k}} \sum_{l=1}^n \frac{1}{\beta!} t^\beta (\nu_{\Omega^i}(t))_l \int_{\partial\Omega^o} \nu_{\Omega^o}(y) \cdot (\nabla D^\beta \partial_l G_n)(y)\theta^o(y)\, d\sigma_y$$

for all $k \in \mathbb{N}$, $t \in \partial\Omega^i$, *and* $\theta^o \in C^{1,\alpha}(\partial\Omega^o)$.

Proof. Let $(\theta^o, \theta^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$, $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus \{0\}$. By the theorem of change of variable in integrals we can compute that

$$\begin{aligned} \mathcal{W}^o_\epsilon[\theta^o, \theta^i](x) &= \mathcal{W}_{\Omega^o}[\theta^o](x) - |\epsilon|^{n-1}\mathrm{sgn}(\epsilon)\nu_{\Omega^o}(x) \cdot \nabla_x \int_{\partial\Omega^i} \nu_{\Omega^i}(s) \cdot \nabla G_n(x - \epsilon s)\theta^i(s)\, d\sigma_s \\ &= \mathcal{W}_{\Omega^o}[\theta^o](x) - \epsilon|\epsilon|^{n-2} \sum_{l=1}^n (\nu_{\Omega^o}(x))_l \int_{\partial\Omega^i} \nu_{\Omega^i}(s) \cdot \nabla\partial_l G_n(x - \epsilon s)\theta^i(s)\, d\sigma_s \end{aligned}$$

for all $x \in \partial\Omega^o$, and

$$\mathcal{W}^i_\epsilon[\theta^o, \theta^i](t) = -\mathrm{sgn}(\epsilon) \sum_{l=1}^n (\nu_{\Omega^i}(t))_l \int_{\partial\Omega^o} \nu_{\Omega^o}(y) \cdot \nabla\partial_l G_n(\epsilon t - y)\theta^o(y)\, d\sigma_y + |\epsilon|^{-1}\mathcal{W}_{\Omega^i}[\theta^i](t)$$

for all $t \in \partial\Omega^i$. Then (17) holds with

$$\mathcal{W}^{o,i}_\epsilon[\theta^i](x) := -\sum_{l=1}^n (\nu_{\Omega^o}(x))_l \int_{\partial\Omega^i} \nu_{\Omega^i}(s) \cdot \nabla\partial_l G_n(x - \epsilon s)\theta^i(s)\, d\sigma_s$$

for all $x \in \partial\Omega^o$ and $\theta^i \in C^{1,\alpha}(\partial\Omega^i)$, and

$$\mathcal{W}^{i,o}_\epsilon[\theta^o](t) := -\sum_{l=1}^n (\nu_{\Omega^i}(t))_l \int_{\partial\Omega^o} \nu_{\Omega^o}(y) \cdot \nabla\partial_l G_n(\epsilon t - y)\theta^o(y)\, d\sigma_y$$

for all $t \in \partial\Omega^i$, $\theta^o \in C^{1,\alpha}(\partial\Omega^o)$.

By the regularity results for the integral operators with real analytic kernel of [42] (see also the argument in the proof of Theorem 3.2) we can verify that the maps $\epsilon \mapsto \mathcal{W}^{o,i}_\epsilon$ and $\epsilon \mapsto \mathcal{W}^{i,o}_\epsilon$ are real analytic from $]-\epsilon_0, \epsilon_0[$ to $\mathcal{L}(C^{1,\alpha}(\partial\Omega^i), C^{0,\alpha}(\partial\Omega^o))$ and from $]-\epsilon_0, \epsilon_0[$ to $\mathcal{L}(C^{1,\alpha}(\partial\Omega^o), C^{0,\alpha}(\partial\Omega^i))$, respectively.

To verify statement (i) we compute

$$\begin{aligned} &\partial^k_\epsilon \Big(-\sum_{l=1}^n (\nu_{\Omega^o}(x))_l \int_{\partial\Omega^i} \nu_{\Omega^i}(s) \cdot \nabla\partial_l G_n(x - \epsilon s)\theta^i(s)\, d\sigma_s \Big) \\ &\qquad = (-1)^{k+1} \sum_{\substack{\beta\in\mathbb{N}^n \\ |\beta|=k}} \sum_{l=1}^n \frac{k!}{\beta!}(\nu_{\Omega^o}(x))_l \int_{\partial\Omega^i} \nu_{\Omega^i}(s) \cdot (\nabla D^\beta \partial_l G_n)(x - \epsilon s) s^\beta \theta^i(s)\, d\sigma_s\,, \end{aligned}$$

and accordingly

$$\partial_\epsilon^k\bigg(-\nu_{\Omega^o}(x)\cdot\nabla_x\int_{\partial\Omega^i}\nu_{\Omega^i}(s)\cdot\nabla G_n(x-\epsilon s)\theta^i(s)\,d\sigma_s\bigg)_{|\epsilon=0}$$
$$=(-1)^{k+1}\sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=k}}\sum_{l=1}^n\frac{k!}{\beta!}(\nu_{\Omega^o}(x))_l(\nabla D^\beta\partial_l G_n)(x)\cdot\int_{\partial\Omega^i}\nu_{\Omega^i}(s)s^\beta\theta^i(s)\,d\sigma_s\,.$$

To verify statement (ii) we compute

$$\partial_\epsilon^k\bigg(-\sum_{l=1}^n(\nu_{\Omega^i}(t))_l\int_{\partial\Omega^o}\nu_{\Omega^o}(y)\cdot\nabla\partial_l G_n(\epsilon t-y)\theta^o(y)\,d\sigma_y\bigg)$$
$$=-\sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=k}}\sum_{l=1}^n\frac{k!}{\beta!}(\nu_{\Omega^i}(t))_l\int_{\partial\Omega^o}\nu_{\Omega^o}(y)\cdot(\nabla D^\beta\partial_l G_n)(\epsilon t-y)t^\beta\theta^o(y)\,d\sigma_y$$

and accordingly

$$\partial_\epsilon^k\bigg(-\sum_{l=1}^n(\nu_{\Omega^i}(t))_l\int_{\partial\Omega^o}\nu_{\Omega^o}(y)\cdot\nabla\partial_l G_n(\epsilon t-y)\theta^o(y)\,d\sigma_y\bigg)_{|\epsilon=0}$$
$$=-\sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=k}}\sum_{l=1}^n\frac{k!}{\beta!}(\nu_{\Omega^i}(t))_l\int_{\partial\Omega^o}\nu_{\Omega^o}(y)\cdot(\nabla D^\beta\partial_l G_n)(-y)t^\beta\theta^o(y)\,d\sigma_y$$
$$=(-1)^{k+1}\sum_{\substack{\beta\in\mathbb{N}^n\\|\beta|=k}}\sum_{l=1}^n\frac{k!}{\beta!}t^\beta(\nu_{\Omega^i}(t))_l\int_{\partial\Omega^o}\nu_{\Omega^o}(y)\cdot(\nabla D^\beta\partial_l G_n)(y)\theta^o(y)\,d\sigma_y\,.$$

□

In conclusion of this section, we note that, putting together the results obtained for the operators $\mathcal{V}_\epsilon$, $\mathcal{K}_\epsilon$, $\mathcal{K}'_\epsilon$ and $\mathcal{W}_\epsilon$, we may also describe the map that takes ϵ to the (pull-back of) the corresponding Calderón projector.

5. Some final remarks

We give a simple example of how the results of Section 4 may be applied. Here we can only show the most important outlines of the arguments involved, for the detailed analysis of similar problems we refer the reader to [21] and [16]. Suppose we want to describe the dependence on ϵ of the solution $u_\epsilon\in C^{1,\alpha}(\overline{\Omega(\epsilon)})$ of the mixed problem

$$\begin{cases}\Delta u_\epsilon(x)=0 & \forall x\in\Omega(\epsilon)\,,\\ u_\epsilon(x)=f^o(x) & \forall x\in\partial\Omega^o\,,\\ -\nu_{\epsilon\Omega^i}(x)\cdot\nabla u_\epsilon(x)=f^i(x/\epsilon) & \forall x\in\epsilon\partial\Omega^i\,,\end{cases}\tag{18}$$

for a fixed datum $(f^o,f^i)\in C^{1,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i)$. In particular, we are interested in ϵ that approaches 0 and $\epsilon\Omega^i$ that shrinks to a point. We may proceed as follows: First we observe that problem (18) has at most one solution for all $\epsilon\in]0,\epsilon_0[$ (this can be proven by a standard energy argument). Then we look for a solution written as a combination of a single layer potential and a constant function. That is, we take

$$u_\epsilon(x)=\mathcal{S}_{\Omega(\epsilon)}[\mu_\epsilon](x)+\xi_\epsilon\qquad\forall x\in\overline{\Omega(\epsilon)}\tag{19}$$

for some function $\mu_\epsilon\in C^{0,\alpha}(\partial\Omega(\epsilon))$ and some real number ξ_ϵ. If we rescale the restriction of μ_ϵ on $\epsilon\partial\Omega^i$ and write

$$\mu_\epsilon(x):=\begin{cases}\theta^o_\epsilon(x) & \text{if } x\in\partial\Omega^o\,,\\ \theta^i_\epsilon(x/\epsilon) & \text{if } x\in\epsilon\partial\Omega^i\,,\end{cases}$$

we see that the function in (19) is a solution of (18) whenever the triple $(\theta^o_\epsilon, \theta^i_\epsilon, \xi_\epsilon)$ is a solution of the system of integral equations

$$\mathcal{M}_\epsilon \begin{pmatrix} \theta^o_\epsilon \\ \theta^i_\epsilon \\ \xi_\epsilon \end{pmatrix} = \begin{pmatrix} f^o \\ f^i \end{pmatrix},$$

with

$$\mathcal{M}_\epsilon := \begin{pmatrix} \mathcal{V}_{\Omega^o} & 0 & \mathbb{I} \\ \mathcal{K}'^{i,o}_{(0)} & \frac{1}{2}\mathbb{I} - \mathcal{K}'_{\Omega^i} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \epsilon^{n-1}\sum_{k=0}^\infty \epsilon^k \mathcal{V}^{o,i}_{(k)} & 0 \\ \sum_{k=1}^\infty \epsilon^k \mathcal{K}'^{i,o}_{(k)} & 0 & 0 \end{pmatrix}$$

(cf. Theorems 4.1 and 4.3). The right space to let the matrix operator $\mathcal{M}_\epsilon$ act on is

$$\mathcal{X} := C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times \mathbb{R}$$

with

$$C^{0,\alpha}(\partial\Omega^o)_0 = \left\{\theta \in C^{0,\alpha}(\partial\Omega^o) \colon \int_{\partial\Omega^o} \theta \, d\sigma = 0\right\},$$

because we can see that

$$\begin{pmatrix} \mathcal{V}_{\Omega^o} & 0 & \mathbb{I} \\ \mathcal{K}'^{i,o}_{(0)} & \frac{1}{2}\mathbb{I} - \mathcal{K}'_{\Omega^i} & 0 \end{pmatrix}$$

is invertible from $\mathcal{X}$ to

$$\mathcal{Y} := C^{1,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)$$

while

$$\begin{pmatrix} 0 & \epsilon^{n-1}\sum_{k=0}^\infty \epsilon^k \mathcal{V}^{o,i}_{(k)} & 0 \\ \sum_{k=1}^\infty \epsilon^k \mathcal{K}'^{i,o}_{(k)} & 0 & 0 \end{pmatrix}$$

is "small" in the operator norm of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ as $\epsilon \to 0$. We deduce that $\mathcal{M}_\epsilon$ is invertible for ϵ small and so the solution u_ϵ exists and it can be written as in (19) for ϵ sufficiently small. Moreover, using the Neumann series theorem and working out some algebra, we may compute the power series expansion of the inverse operator $\mathcal{M}_\epsilon^{-1}$ and then derive the power series expansion of the triple $(\theta^o_\epsilon, \theta^i_\epsilon, \xi_\epsilon)$ as a function of the perturbation parameter ϵ. This being done, we resort to the representation formula (19) and combining the expansion obtained for $(\theta^o_\epsilon, \theta^i_\epsilon, \xi_\epsilon)$ with that of the single layer potential of Theorem 4.1 we obtain an expansion for the solution u_ϵ of problem (18).

As mentioned above, the details of the computation go beyond the aim of this paper, but similar ideas were used for example in [21] to obtain all the terms of the series expansion for the solution of a Dirichlet problem in a perforated 2-dimensional domain. Now, the results of Section 4 can be applied in any dimension $n \geq 2$ and to different boundary conditions. We may, for example, recover the result of Feppon and Ammari [24], which are based on layer potentials with Dirichlet Green function as a kernel, and make them available also when the Green function is not explicitly given. We observe, indeed, that formulas similar to those of Section 4 are available in specific dimensions and geometric settings in Ammari, Kang, and Lee [3, Lem. 3.3], Feppon and Ammari [25, Prop. 2.3], [24, Prop. 2.5], and in a number of previous papers by the authors (see in the introduction of this paper).

We believe, however, that the systematic presentation provided here is a useful toolbox in view of future developments. In particular, the simple ideas illustrated above for the case of a mixed boundary value problem may be extended to problems with nonlinear boundary conditions. We might, for example, replace the last condition of (18) with a condition in the form

$$-\nu_{\epsilon\Omega^i}(x) \cdot \nabla u_\epsilon(x) = F^i(\epsilon, x/\epsilon, u_\epsilon(x)),$$

where F^i is a map that, in some sense, "preserves" the real analyticity. For similar problems it is known that the solution can be written in terms of analytic maps and (possibly singular) elementary functions of ϵ (see, e.g., Lanza de Cristoforis [38, 40], see also [13–15] for an elastic counterpart). Using the results of this paper and the known explicit formulas for the series expansions of the composition operators (see Valent [62, Chap. 2, §5]), we may now try to compute explicit series expansions also in the case of nonlinear conditions. This will be the subject of future investigation.

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