

Mikhail Guzev, Chengzhi Qi*, Jiping Bai and Kairui Li

Equilibrium equations and boundary conditions of strain gradient theory in arbitrary curvilinear coordinates

Abstract: Equilibrium equations and boundary conditions of the strain gradient theory in arbitrary curvilinear coordinates have been obtained. Their special form for an axisymmetric plane strain problem is also given.

Keywords: arbitrary curvilinear coordinates; boundary conditions; equilibrium equations; strain gradient model.

DOI 10.1515/jmbm-2014-0018

1 Introduction

In the early 1960s, Toupin [1, 2] and Mindlin [3, 4] proposed a strain gradient theory, which suggests that energy depends not only on deformations, but also on gradient of deformations. Since then, various extensions or versions of the theory have been developed and applied to different problems [5–8]. In the past 20 years there has been a growing interest in various versions of this theory in the field of mechanics due to its successful application to a number of problems pertaining to current technologies including nanotechnology. Classical continuum theories are not capable of addressing problems for which the size effect due to the underlying microstructure is important to include in a phenomenological description. Moreover localization of strain due to material softening cannot be captured. Both these issues have been

considered successfully within internal length or strain gradient theories, as has been demonstrated by Aifantis et al. for plasticity [9–12] and elasticity [13–16], as well as by other authors including Fleck and Hutchinson [17–19], Chambon et al. [20], Lurie and co-workers [21, 22], and Zhao et al. [23, 24]. Possible application of this theory to the macrodescription of materials was given by Qi et al. [25], and the relation of kinematical parameters with internal geometrical material characteristics was demonstrated by Guzev [26, 27].

The strain gradient theory developed by Toupin [1] and Mindlin [3] was formulated in Cartesian rectangular coordinates. For problems which require curvilinear coordinates, the corresponding equilibrium equations and boundary conditions cannot be obtained automatically from the theory. Therefore, an appropriate form of equilibrium equations and boundary conditions of strain gradient theories in curvilinear coordinates would be desirable to obtain. An effort in this direction has been made by Zhao and Pedrosa [28], who adopted the approach proposed by Eringen [29] suggesting that the transition from rectangular coordinates to any of the curvilinear coordinates follows two rules: (a) the partial differentiation symbol (∂) must be replaced by the covariant differentiation symbol (∇); and (b) the repeated indices must be on diagonal positions. However, a systematic derivation following this approach was not explicitly provided in Eringen's book [29].

This task was undertaken during our study where equilibrium equations and boundary conditions of the strain gradient theory in arbitrary curvilinear coordinates are provided. The axisymmetric plane strain problem for long cylinder was also considered in detail as a special case. The structure of the present article is as follows. Section 2 summarizes the principle of virtual work used for derivation of equilibrium equations and boundary conditions of strain gradient theory. Section 3 provides the form of equilibrium equations in curvilinear coordinates, while Section 4 lists the corresponding curvilinear coordinate form of associated boundary conditions. Finally, in Section 5, the formalism is specialized for a plane strain problem of a pressurized long cylinder.

*Corresponding author: Chengzhi Qi, Beijing High Institution Research Center for Engineering Structures and New Materials, School of Civil and Transportation Engineering, Beijing University of Civil Engineering and Architecture, Beijing 100044, China, e-mail: qichengzhi@bucea.edu.cn

Mikhail Guzev: Institute for Applied Mathematics, Far Eastern Branch, Russian Academy of Sciences, Vladivostok 690041, Russia

Jiping Bai: Faculty of Computing, Engineering and Science, University of South Wales, Pontypridd CF37 1DL, UK

Kairui Li: Beijing High Institution Research Center for Engineering Structures and New Materials, Beijing University of Civil Engineering and Architecture, Beijing 100044, China

2 Principle of virtual work

The governing equations of the gradient theory can be derived using the principle of virtual work. It states that, for any given material volume V , the virtual work done on V by material forces and sources exterior to V (i.e., the external work) is equal to the virtual work generated within V (i.e., the internal work). External work is assumed to be done by the macroscopic body force \mathbf{F}_b , and the macroscopic surface traction \mathbf{F}_s :

$$W_{\text{ext}} = \int_V \mathbf{F}_b \cdot \delta \mathbf{u} dV + \int_{\Gamma} \mathbf{F}_s \cdot \delta \mathbf{u} d\Gamma \quad (1)$$

where δ denotes variation; $\delta \mathbf{u}$ denotes virtual displacement; and Γ denotes boundary of V .

For the description of reversible deformations, the elastic strain tensor $\boldsymbol{\varepsilon}$ and its gradient $\nabla \boldsymbol{\varepsilon}$ may be taken as internal variables.

The internal work W_{int} is assumed to arise from the action of elastic stress tensor $\boldsymbol{\sigma}$, as well as the generalized force \mathbf{T} conjugate to $\nabla \boldsymbol{\varepsilon}$:

$$\begin{aligned} W_{\text{int}} &= \int_V \boldsymbol{\sigma} \cdot \delta \boldsymbol{\varepsilon} dV + \int_V \mathbf{T} \cdot \delta \nabla \boldsymbol{\varepsilon} dV = \int_V \sigma_{ij} \cdot \delta \varepsilon_{ij} dV + \int_V T_{ijk} \cdot \delta \varepsilon_{ij,k} dV \\ &= \int_V \sigma_{ij} \cdot \delta \varepsilon_{ij} dV + \int_{\Gamma} T_{ijk} \delta \varepsilon_{ij} n_k dV - \int_V T_{ijk,k} \cdot \delta \varepsilon_{ij} dV \\ &= - \int_V \sigma_{ij,j} \cdot \delta u_i dV + \int_{\Gamma} \sigma_{ij} n_j \delta u_i d\Gamma + \int_{\Gamma} T_{ijk} \delta \varepsilon_{ij} n_k dV \\ &\quad + \int_V T_{ijk,kj} \cdot \delta u_i dV - \int_{\Gamma} T_{ijk,k} \delta u_i n_k dV, \end{aligned} \quad (2)$$

where the symbol $(,)$ as usual denotes partial differentiation. The internal work can now be written in the form

$$\begin{aligned} W_{\text{int}} &= - \int_V (\sigma_{ij,j} - T_{ijk,kj}) \cdot \delta u_i dV + \int_{\Gamma} [(\sigma_{ij} - T_{ijk,k}) n_j] \delta u_i d\Gamma \\ &\quad + \int_{\Gamma} T_{ijk} \delta \varepsilon_{ij} n_k d\Gamma. \end{aligned} \quad (3)$$

The principle of virtual work expressed as

$$W_{\text{ext}} = W_{\text{int}}, \quad (4)$$

demands

$$\begin{aligned} \int_V (\sigma_{ij,j} - T_{ijk,kj} + F_{bi}) \delta u_i dV + \int_{\Gamma} (F_{si} - \sigma_{ij} - T_{ijk,k} n_j) \delta u_i d\Gamma \\ + \int_{\Gamma} T_{ijk} \delta \varepsilon_{ij} n_k d\Gamma = 0. \end{aligned} \quad (5)$$

The equilibrium equations may then be obtained from the above variational statement as

$$\sigma_{ij,j} - T_{ijk,kj} + F_{bi} = 0, \quad (6)$$

and the boundary conditions can also be obtained by utilizing the second and third integrals on the left-hand side of Eq. (5).

In forming boundary conditions it is necessary to note that the external boundary Γ may be divided into two parts: the surface boundary Γ_{σ} for static forces and the other one Γ_u for displacements. Then the boundary conditions on the static boundary Γ_{σ} can be obtained by utilizing the second and third integrals on the left-hand side of Eq. (5). On the kinematic surface Γ_u , the normal and tangential derivatives are not independent [12]. In our work, we have not dealt with kinematic condition but have paid more attention to the systematic derivation of equilibrium equations and the static boundary conditions in arbitrary curvilinear coordinates. Though this case is simpler than the situation given in Ref. [12], it allows us to demonstrate in detail the mathematical tools of the continuum.

3 Equilibrium equations in arbitrary curvilinear coordinates

The equilibrium equation given by Eq. (6) may be written in an equivalent form as

$$\frac{\partial}{\partial x_j} \left[\sigma_{ij} - \frac{\partial}{\partial x_k} T_{ijk} \right] + f_i = 0 \quad (7)$$

where $f_i = F_{bi}$ denotes the components of the body force.

Overall stress is formed as a result of the superposition of the usual second-order stress field σ_{ij} and an additional gradient stress field $\partial T_{ijk} / \partial x_k$. It is written in the rectangular coordinates x_i . Let us now introduce the curvilinear coordinates z^k linked with x_i by means of the transformation [29]:

$$z^j = z^j(x_1, x_2, x_3), \quad z^i = z^i(\bar{x}). \quad (8)$$

By assuming that this kind of relation between x_i and z^k is one-to-one, we can uniquely invert Eq. (8) to obtain

$$x_i = x_i(z^1, z^2, z^3), \quad x_i = x_i(\bar{z}). \quad (9)$$

In Eqs. (8) and (9) coordinates are specified by superscripts and subscripts which, in geometrical terminology, means accounting for contravariant and covariant vectors, respectively. There is no distinction between superscripts and subscripts for the rectangular coordinates: $x_i = x^i$. This supposition is not contradictory because in curvilinear coordinates, transition from superscripts to subscripts is carried out with the corresponding metric tensor, which for Cartesian coordinates is the identity $g_{ij}(\bar{x}) = \delta_{ij}$ matrix. Then, the following operations of

raising and lowering indices: $\frac{\partial}{\partial x_i} = g^{ik}(\bar{x}) \frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^i}$ and $\sigma^{ij}(\bar{x}) = g^{ik}(\bar{x}) g^{mj}(\bar{x}) \sigma_{km}(\bar{x}) = \sigma_{ij}(\bar{x}) = g^{ik}(\bar{x}) \sigma_k^j(\bar{x}) = \sigma_i^j(\bar{x})$ are valid where the components $g^{ik}(\bar{x}) \equiv (g^{-1})_{ik}(\bar{x}) = \delta^{ik} = \delta_i^k = \delta_{ik}$ are equal to the identity matrix when the indices repeat, and equal to zero otherwise (i.e., the usual symbol of Kronecker delta).

In transition from rectangular coordinate systems to curvilinear systems the components of the first-, second-, and third-rank tensors $f_k(x^i)$, $\sigma_{ij}(x^i)$, $T_{ijk}(x^i)$ are transformed according to the following rules:

$$\begin{aligned} f_\alpha(\bar{z}) &= \frac{\partial x^i}{\partial z^\alpha} f_i(\bar{x}), \quad \sigma_{\alpha\beta}(\bar{z}) = \frac{\partial x^i}{\partial z^\alpha} \frac{\partial x^k}{\partial z^\beta} \sigma_{ik}(\bar{x}), \\ T_{\alpha\beta\mu}(\bar{z}) &= \frac{\partial x^i}{\partial z^\alpha} \frac{\partial x^k}{\partial z^\beta} \frac{\partial x^n}{\partial z^\mu} T_{ikn}(\bar{x}), \quad f_k(\bar{x}) = \frac{\partial z^\alpha}{\partial x^k} f_\alpha(\bar{z}), \\ \sigma_{ik}(\bar{x}) &= \frac{\partial z^\alpha}{\partial x^k} \frac{\partial z^\beta}{\partial x^i} \sigma_{\alpha\beta}(\bar{z}), \quad T_{ijk}(\bar{x}) = \frac{\partial z^\alpha}{\partial x^i} \frac{\partial z^\beta}{\partial x^j} \frac{\partial z^\mu}{\partial x^k} T_{\alpha\beta\mu}(\bar{z}), \end{aligned} \quad (10)$$

The metric tensor $g_{ij}(\bar{z})$ in curvilinear coordinates is given as:

$$g_{\alpha\beta}(\bar{z}) = \frac{\partial x^k}{\partial z^\alpha} \frac{\partial x^l}{\partial z^\beta}. \quad (11)$$

This permits us to compute the square of the length element $(ds)^2 = g_{\alpha\beta}(\bar{z}) dz^\alpha dz^\beta$ and to determine the transition from covariant to contravariant components of tensors; in particular,

$$\begin{aligned} \sigma_\beta^\alpha(\bar{z}) &= g^{\alpha\gamma}(\bar{z}) \sigma_{\gamma\beta}(\bar{z}), \quad T_\beta^{\alpha\gamma}(\bar{z}) = g^{\alpha\gamma}(\bar{z}) g^{\nu\mu}(\bar{z}) T_{\beta\gamma\mu}(\bar{z}), \\ g^{\alpha\beta}(\bar{z}) &= \frac{\partial z^\alpha}{\partial x^k} \frac{\partial z^\beta}{\partial x^k}. \end{aligned} \quad (12)$$

The quantity $g^{\alpha\beta}(\bar{z})$ denotes the inverse of the metric tensor, i.e., $g^{\alpha\beta}(\bar{z}) g_{\beta\gamma}(\bar{z}) = \delta_\gamma^\alpha$.

The equilibrium equations in terms of objects [Eq. (12)] can be expressed as follows.

The derivative $\partial/\partial x_i$ is related to $\partial/\partial z^\alpha$ as per the following expression:

$$\frac{\partial}{\partial x_i} = \frac{\partial z^\alpha}{\partial x_i} \frac{\partial}{\partial z^\alpha}. \quad (13)$$

By using Eqs. (10) and (13), we have

$$\begin{aligned} \frac{\partial}{\partial x_j} \sigma_{ij}(\bar{x}) &= \frac{\partial}{\partial x^j} \sigma_{ij}(\bar{x}) = \frac{\partial z^\alpha}{\partial x^j} \frac{\partial}{\partial z^\alpha} \left[\frac{\partial z^\beta}{\partial x^i} \frac{\partial z^\gamma}{\partial x^j} \sigma_{\beta\gamma}(\bar{z}) \right] \\ &= \frac{\partial z^\alpha}{\partial x^j} \frac{\partial}{\partial z^\alpha} \left[\frac{\partial z^\beta}{\partial x^i} \right] \frac{\partial z^\gamma}{\partial x^j} \sigma_{\beta\gamma}(\bar{z}) + \frac{\partial z^\alpha}{\partial x^j} \frac{\partial z^\beta}{\partial x^i} \frac{\partial}{\partial z^\alpha} \left[\frac{\partial z^\gamma}{\partial x^j} \right] \\ &\quad \sigma_{\beta\gamma}(\bar{z}) + \frac{\partial z^\alpha}{\partial x^j} \frac{\partial z^\beta}{\partial x^i} \frac{\partial z^\gamma}{\partial x^j} \frac{\partial \sigma_{\beta\gamma}(\bar{z})}{\partial z^\alpha}. \end{aligned} \quad (14)$$

Noting that in the first and last terms on the right-hand side of Eq. (14) the convolution by index j gives the components of the inverse of metric tensor of Eq. (12), we have

$$\begin{aligned} \frac{\partial z^\alpha}{\partial x^j} \frac{\partial}{\partial z^\alpha} \left[\frac{\partial z^\beta}{\partial x^i} \right] \frac{\partial z^\gamma}{\partial x^j} \sigma_{\beta\gamma}(\bar{z}) + \frac{\partial z^\alpha}{\partial x^j} \frac{\partial z^\beta}{\partial x^i} \frac{\partial z^\gamma}{\partial x^j} \frac{\partial \sigma_{\beta\gamma}(\bar{z})}{\partial z^\alpha} \\ = g^{\alpha\gamma}(\bar{z}) \frac{\partial}{\partial z^\alpha} \left[\frac{\partial z^\beta}{\partial x^i} \right] \sigma_{\beta\gamma}(\bar{z}) + g^{\alpha\beta}(\bar{z}) \frac{\partial z^\gamma}{\partial x^j} \frac{\partial \sigma_{\beta\gamma}(\bar{z})}{\partial z^\alpha}. \end{aligned} \quad (15)$$

The derivative $\frac{\partial}{\partial z^\alpha} [\dots]$ is obtained from the equality $\frac{\partial x^i}{\partial z^\gamma} \frac{\partial z^\beta}{\partial x^i} = \delta_\gamma^\beta$ by means of differentiation

$$\frac{\partial^2 x^i}{\partial z^\alpha \partial z^\gamma} \frac{\partial z^\beta}{\partial x^i} + \frac{\partial x^i}{\partial z^\gamma} \frac{\partial}{\partial z^\alpha} \left[\frac{\partial z^\beta}{\partial x^i} \right] = 0.$$

From that we obtain

$$\frac{\partial}{\partial z^\alpha} \left[\frac{\partial z^\beta}{\partial x^i} \right] = \frac{\partial^2 x^n}{\partial z^\alpha \partial z^\gamma} \frac{\partial z^\beta}{\partial x^n} \frac{\partial z^\gamma}{\partial x^i}.$$

The last equation may be written through the Christoffel symbols $\Gamma_{\alpha\gamma}^\beta = \Gamma_{\alpha\gamma}^\beta(\bar{z})$ of the second kind:

$$\Gamma_{\alpha\gamma}^\beta = \frac{\partial^2 x^n}{\partial z^\alpha \partial z^\gamma} \frac{\partial z^\beta}{\partial x^n}, \quad \Gamma_{\alpha\gamma}^\beta = \Gamma_{\gamma\alpha}^\beta, \quad \frac{\partial}{\partial z^\alpha} \left[\frac{\partial z^\beta}{\partial x^i} \right] = -\Gamma_{\alpha\gamma}^\beta \frac{\partial z^\gamma}{\partial x^i}. \quad (16)$$

Substituting Eqs. (15) and (16) into Eq. (14), we obtain

$$\begin{aligned} \frac{\partial}{\partial x_j} \sigma_{ij}(\bar{x}) &= \frac{\partial z^\beta}{\partial x^i} g^{\alpha\gamma}(\bar{z}) \left[\frac{\partial \sigma_{\beta\gamma}(\bar{z})}{\partial z^\alpha} - \Gamma_{\beta\alpha}^\nu \sigma_{\nu\gamma}(\bar{z}) - \Gamma_{\gamma\alpha}^\nu \sigma_{\beta\nu}(\bar{z}) \right] \\ &= \frac{\partial z^\beta}{\partial x^i} g^{\alpha\gamma}(\bar{z}) \nabla_\alpha \sigma_{\beta\gamma}(\bar{z}). \end{aligned} \quad (17)$$

The expression in the square brackets defines the covariant partial derivative [22]:

$$\nabla_\alpha \sigma_{\beta\gamma} \equiv \frac{\partial \sigma_{\beta\gamma}(\bar{z})}{\partial z^\alpha} - \Gamma_{\alpha\beta}^\nu \sigma_{\nu\gamma}(\bar{z}) - \Gamma_{\alpha\gamma}^\nu \sigma_{\beta\nu}(\bar{z}).$$

It is clear that

$$\begin{aligned} g^{\alpha\gamma}(\bar{z}) \nabla_\alpha \sigma_{\beta\gamma}(\bar{z}) &= \frac{\partial g^{\alpha\gamma}(\bar{z}) \sigma_{\beta\gamma}(\bar{z})}{\partial z^\alpha} - \frac{\partial g^{\alpha\gamma}(\bar{z})}{\partial z^\alpha} \sigma_{\beta\gamma}(\bar{z}) \\ &\quad - \Gamma_{\alpha\beta}^\nu g^{\alpha\gamma}(\bar{z}) \sigma_{\nu\gamma}(\bar{z}) - \Gamma_{\alpha\gamma}^\nu g^{\alpha\gamma}(\bar{z}) \sigma_{\beta\nu}(\bar{z}). \end{aligned} \quad (18)$$

From Eq. (12) it follows that

$$\frac{\partial g^{\alpha\gamma}(\bar{z})}{\partial z^\beta} = \frac{\partial}{\partial z^\beta} \left[\frac{\partial z^\alpha}{\partial x^k} \right] \frac{\partial z^\gamma}{\partial x^k} + \frac{\partial z^\alpha}{\partial x^k} \frac{\partial}{\partial z^\beta} \left[\frac{\partial z^\gamma}{\partial x^k} \right].$$

Substitution of Eq. (16) into the above equation yields

$$\frac{\partial g^{\alpha\gamma}(\bar{z})}{\partial z^\beta} = -\Gamma_{\nu\beta}^\alpha g^{\nu\gamma}(\bar{z}) - \Gamma_{\nu\beta}^\gamma g^{\alpha\nu}(\bar{z}). \quad (19)$$

Combination of Eqs. (18) and (19) gives

$$g^{\alpha\gamma}(\bar{z}) \nabla_\alpha \sigma_{\beta\gamma}(\bar{z}) = \frac{\partial g^{\alpha\gamma}(\bar{z}) \sigma_{\beta\gamma}(\bar{z})}{\partial z^\alpha} + \Gamma_{\alpha\nu}^\alpha g^{\nu\gamma}(\bar{z}) \sigma_{\beta\gamma}(\bar{z}) - \Gamma_{\alpha\beta}^\nu g^{\alpha\gamma}(\bar{z}) \sigma_{\nu\gamma}(\bar{z}). \quad (20)$$

By using the operation for raising index in Eq. (18), Eq. (20) can then be expressed in the form

$$g^{\alpha\gamma}(\bar{z}) \nabla_\alpha \sigma_{\beta\gamma}(\bar{z}) = \frac{\partial \sigma_{\beta\gamma}^\alpha(\bar{z})}{\partial z^\alpha} + \Gamma_{\nu\alpha}^\alpha \sigma_{\beta\gamma}^\nu(\bar{z}) - \Gamma_{\alpha\beta}^\nu \sigma_{\nu\gamma}^\alpha(\bar{z}). \quad (21)$$

Using the covariant derivative for the stress tensor

$$\nabla_\alpha \sigma_{\beta\gamma}^\nu(\bar{z}) \equiv \frac{\partial \sigma_{\beta\gamma}^\nu(\bar{z})}{\partial z^\alpha} + \Gamma_{\nu\alpha}^\nu \sigma_{\beta\gamma}^\nu(\bar{z}) - \Gamma_{\beta\alpha}^\nu \sigma_{\nu\gamma}^\nu(\bar{z}), \quad (22)$$

it turns out that Eq. (21) is equivalent to the following equation

$$g^{\alpha\gamma}(\bar{z}) \nabla_\alpha \sigma_{\beta\gamma}(\bar{z}) = \nabla_\alpha \sigma_{\beta\gamma}^\alpha(\bar{z}). \quad (23)$$

Now let us consider the term $\partial^2 T_{ijk} / \partial x_j \partial x_k$ in Eq. (7). The tensor $T_{ijk} = T_{ijk}(\bar{x})$ is a third-order tensor, and its components are transformed according to the rule of Eq. (12). Passing on to curvilinear coordinates, we may write the derivative of $T_{ijk}(\bar{x})$ in the following form:

$$\begin{aligned} \frac{\partial}{\partial x_k} T_{ijk}(\bar{x}) &= \frac{\partial z^\alpha}{\partial x^k} \frac{\partial}{\partial z^\alpha} \left[\frac{\partial z^\beta}{\partial x^i} \frac{\partial z^\gamma}{\partial x^j} \frac{\partial z^\mu}{\partial x^k} T_{\beta\gamma\mu}(\bar{z}) \right] \\ &= \frac{\partial z^\alpha}{\partial x^k} \frac{\partial}{\partial z^\alpha} \left[\frac{\partial z^\beta}{\partial x^i} \right] \frac{\partial z^\gamma}{\partial x^j} \frac{\partial z^\mu}{\partial x^k} T_{\beta\gamma\mu}(\bar{z}) \\ &+ \frac{\partial z^\alpha}{\partial x^k} \frac{\partial z^\beta}{\partial x^i} \frac{\partial}{\partial z^\alpha} \left[\frac{\partial z^\gamma}{\partial x^j} \right] \frac{\partial z^\mu}{\partial x^k} T_{\beta\gamma\mu}(\bar{z}) \\ &+ \frac{\partial z^\alpha}{\partial x^k} \frac{\partial z^\beta}{\partial x^i} \frac{\partial z^\gamma}{\partial x^j} \frac{\partial}{\partial z^\alpha} \left[\frac{\partial z^\mu}{\partial x^k} \right] T_{\beta\gamma\mu}(\bar{z}) \\ &+ \frac{\partial z^\alpha}{\partial x^k} \frac{\partial z^\beta}{\partial x^i} \frac{\partial z^\gamma}{\partial x^j} \frac{\partial}{\partial z^\alpha} [T_{\beta\gamma\mu}(\bar{z})]. \end{aligned} \quad (24)$$

By using the expressions for the connection coefficients (the Christoffel symbols) given by Eq. (16) and the metric tensor given by Eq. (12) we obtain

$$\begin{aligned} \frac{\partial T_{ijk}(\bar{x})}{\partial x_k} &= g^{\alpha\mu}(\bar{z}) \frac{\partial z^\beta}{\partial x^i} \frac{\partial z^\gamma}{\partial x^j} \left[\frac{\partial T_{\beta\gamma\mu}(\bar{z})}{\partial z^\alpha} - \Gamma_{\beta\alpha}^\nu T_{\nu\gamma\mu}(\bar{z}) \right. \\ &\quad \left. - \Gamma_{\gamma\alpha}^\nu T_{\beta\nu\mu}(\bar{z}) - \Gamma_{\mu\alpha}^\nu T_{\beta\gamma\nu}(\bar{z}) \right]. \end{aligned} \quad (25)$$

The expression in the square brackets defines the covariant partial derivative of the tensor $T_{\beta\gamma\mu}(\bar{z})$:

$$\nabla_\alpha T_{\beta\gamma\mu}(\bar{z}) \equiv \frac{\partial T_{\beta\gamma\mu}(\bar{z})}{\partial z^\alpha} - \Gamma_{\beta\alpha}^\nu T_{\nu\gamma\mu}(\bar{z}) - \Gamma_{\gamma\alpha}^\nu T_{\beta\nu\mu}(\bar{z}) - \Gamma_{\mu\alpha}^\nu T_{\beta\gamma\nu}(\bar{z}).$$

Then Eq. (25) may be written as

$$\frac{\partial}{\partial x_k} T_{ijk}(\bar{x}) = g^{\alpha\mu}(\bar{z}) \frac{\partial z^\beta}{\partial x^i} \frac{\partial z^\gamma}{\partial x^j} \nabla_\alpha T_{\beta\gamma\mu}(\bar{z}). \quad (26)$$

Further differentiation of Eq. (26) results in

$$\begin{aligned} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} T_{ijk}(\bar{x}) &= \frac{\partial z^\gamma}{\partial x^j} \frac{\partial}{\partial z^\gamma} \left[g^{\alpha\mu}(\bar{z}) \frac{\partial z^\beta}{\partial x^i} \frac{\partial z^\gamma}{\partial x^j} \nabla_\alpha T_{\beta\gamma\mu}(\bar{z}) \right] \\ &= \frac{\partial z^\gamma}{\partial x^j} \frac{\partial}{\partial z^\gamma} [g^{\alpha\mu}(\bar{z})] \frac{\partial z^\beta}{\partial x^i} \frac{\partial z^\gamma}{\partial x^j} \nabla_\alpha T_{\beta\gamma\mu}(\bar{z}) \\ &+ \frac{\partial z^\gamma}{\partial x^j} g^{\alpha\mu}(\bar{z}) \frac{\partial}{\partial z^\gamma} \left[\frac{\partial z^\beta}{\partial x^i} \right] \frac{\partial z^\gamma}{\partial x^j} \nabla_\alpha T_{\beta\gamma\mu}(\bar{z}) \\ &+ \frac{\partial z^\gamma}{\partial x^j} g^{\alpha\mu}(\bar{z}) \frac{\partial z^\beta}{\partial x^i} \frac{\partial}{\partial z^\gamma} \left[\frac{\partial z^\gamma}{\partial x^j} \right] \nabla_\alpha T_{\beta\gamma\mu}(\bar{z}) \\ &+ \frac{\partial z^\gamma}{\partial x^j} g^{\alpha\mu}(\bar{z}) \frac{\partial z^\beta}{\partial x^i} \frac{\partial z^\gamma}{\partial x^j} \frac{\partial}{\partial z^\gamma} [\nabla_\alpha T_{\beta\gamma\mu}(\bar{z})]. \end{aligned} \quad (27)$$

The derivative $\partial g^{\alpha\mu} / \partial z^\gamma$ is obtained from Eq. (19); then by using Eq. (16), Eq. (27) has the form

$$\begin{aligned} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} T_{ijk}(\bar{x}) &= g^{\alpha\mu}(\bar{z}) g^{\nu\gamma}(\bar{z}) \frac{\partial z^\beta}{\partial x^i} \left[\frac{\partial \nabla_\alpha T_{\beta\gamma\mu}(\bar{z})}{\partial z^\nu} - \Gamma_{\alpha\nu}^\kappa \nabla_\kappa T_{\beta\gamma\mu}(\bar{z}) \right. \\ &\quad \left. - \Gamma_{\beta\nu}^\kappa \nabla_\alpha T_{\kappa\gamma\mu}(\bar{z}) - \Gamma_{\gamma\nu}^\kappa \nabla_\alpha T_{\beta\kappa\mu}(\bar{z}) - \Gamma_{\mu\nu}^\kappa \nabla_\alpha T_{\beta\gamma\kappa}(\bar{z}) \right]. \end{aligned} \quad (28)$$

The expression in the square brackets given in Eq. (28) defines the covariant derivative of tensor $\nabla_\alpha T_{\beta\gamma\mu}(\bar{z})$.

$$\begin{aligned} \nabla_\nu \nabla_\alpha T_{\beta\gamma\mu}(\bar{z}) &= \frac{\partial \nabla_\alpha T_{\beta\gamma\mu}(\bar{z})}{\partial z^\nu} - \Gamma_{\alpha\nu}^\kappa \nabla_\kappa T_{\beta\gamma\mu}(\bar{z}) - \Gamma_{\beta\nu}^\kappa \nabla_\alpha T_{\kappa\gamma\mu}(\bar{z}) \\ &\quad - \Gamma_{\gamma\nu}^\kappa \nabla_\alpha T_{\beta\kappa\mu}(\bar{z}) - \Gamma_{\mu\nu}^\kappa \nabla_\alpha T_{\beta\gamma\kappa}(\bar{z}). \end{aligned}$$

Summarizing the results of the above operations, we obtain

$$\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} T_{ijk}(\bar{x}) = g^{\alpha\mu}(\bar{z}) g^{\nu\gamma}(\bar{z}) \frac{\partial z^\beta}{\partial x^i} \nabla_\nu \nabla_\alpha T_{\beta\gamma\mu}(\bar{z}). \quad (29)$$

The property of Eq. (19) allows us to write Eq. (29) in the following form:

$$\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} T_{ijk}(\bar{x}) = \frac{\partial z^\beta}{\partial x^i} \nabla_\nu \nabla_\alpha [g^{\alpha\mu}(\bar{z}) g^{\nu\gamma}(\bar{z}) T_{\beta\gamma\mu}(\bar{z})]. \quad (30)$$

By using the operation of raising index in Eq. (12), we express Eq. (30) in the form

$$\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} T_{ijk}(\bar{x}) = \frac{\partial z^\beta}{\partial x^i} \nabla_\nu \nabla_\alpha T_{\beta}^{\nu\alpha}(\bar{z}). \quad (31)$$

Combining Eqs. (17), (23), and (31) with Eq. (7) gives the following result:

$$\frac{\partial}{\partial x_j} \sigma_{ij}(\bar{x}) + \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} T_{ijk}(\bar{x}) + f_i = \frac{\partial z^\beta}{\partial x^i} [\nabla_\alpha \sigma_\beta^\alpha(\bar{z}) + \nabla_\alpha \nabla_\nu T_{\beta}^{\nu\alpha}(\bar{z})] + f_i = 0.$$

Then, passing to components $f_k(\bar{z})$ according to Eq. (10), we obtain the equilibrium equations in arbitrary curvilinear coordinates:

$$\frac{\partial}{\partial x_j} \sigma_{ij}(\bar{x}) + \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} T_{ijk}(\bar{x}) + f_i = \nabla_\alpha \sigma_\beta^\alpha(\bar{z}) + \nabla_\alpha \nabla_\nu T_{\beta}^{\nu\alpha}(\bar{z}) + f_\beta(\bar{z}) = 0. \quad (32)$$

This establishes the derivation of the equation of equilibrium in arbitrary curvilinear coordinates. The derivation in this case is new and distinct from the work of Zhao and Pedrosa [28] in that it is not based on heuristic prescription and the derivation is performed in arbitrary curvilinear coordinate system.

4 Boundary conditions in arbitrary curvilinear coordinates

In order to determine the form of boundary conditions correctly, we firstly decompose the derivative $\partial\Phi/\partial x_i$ of an arbitrary function Φ into tangential and normal components. To this end, we express the surface in the parametric form $x_j = x_j(u, v)$, $j=1, 2, 3$, where $z^1=u$ and $z^2=v$ are curvilinear coordinates. Secondly, we introduce metric tensors $G_{\alpha\beta} = \frac{\partial x_j}{\partial z^\alpha} \frac{\partial x_j}{\partial z^\beta}$, ($\alpha, \beta=1, 2$), and prove that $\partial\Phi/\partial x_i$ may be decomposed into tangential and normal components in the following form

$$\frac{\partial\Phi}{\partial x_j} = n_j \frac{\partial\Phi}{\partial n} + G^{\gamma\beta} \frac{\partial x_j}{\partial z^\gamma} \frac{\partial\Phi}{\partial z^\beta}, \quad (33)$$

where $G_{\alpha\beta}$ and $G^{\beta\gamma}$ are mutually inverse metric tensors. $G_{\alpha\beta} G^{\beta\gamma} = \delta_\alpha^\gamma$ and n_j are the components of the normal to surface unit vector.

In order to establish Eq. (33), we decompose $\partial\Phi/\partial x_i$ into components along the normal direction \bar{n} and tangential directions $\bar{\tau}_1$ and $\bar{\tau}_2$.

$$\nabla\Phi = \bar{n}f^0 + \bar{\tau}_1 f^1 + \bar{\tau}_2 f^2, \quad (34)$$

where

$$f^0 = \nabla\Phi \cdot \bar{n} = \frac{\partial\Phi}{\partial n}. \quad (35)$$

In order to determine f^1 and f^2 , we express $\bar{\tau}_\alpha$ as

$$\bar{\tau}_\alpha = \left(\frac{\partial x_1}{\partial z^\alpha}, \frac{\partial x_2}{\partial z^\alpha}, \frac{\partial x_3}{\partial z^\alpha} \right). \quad (36)$$

The dot product of Eq. (34) with $\bar{\tau}_\alpha$ gives

$$\begin{aligned} f^1 \bar{\tau}_1 \cdot \bar{\tau}_1 + f^2 \bar{\tau}_1 \cdot \bar{\tau}_2 &= \nabla\Phi \cdot \bar{\tau}_1, \\ f^1 \bar{\tau}_2 \cdot \bar{\tau}_1 + f^2 \bar{\tau}_2 \cdot \bar{\tau}_2 &= \nabla\Phi \cdot \bar{\tau}_2. \end{aligned} \quad (37)$$

Because of $G_{\alpha\beta} = \frac{\partial x_j}{\partial z^\alpha} \frac{\partial x_j}{\partial z^\beta} = \bar{\tau}_\alpha \cdot \bar{\tau}_\beta$, Eq. (37) may be expressed as

$$G_{\alpha\beta} f^\beta = \nabla\Phi \cdot \bar{\tau}_\alpha. \quad (38)$$

The solution of Eq. (38) is

$$f^\gamma = G^{\gamma\alpha} \nabla\Phi \cdot \bar{\tau}_\alpha. \quad (39)$$

By substituting Eqs. (35) and (39) into Eq. (34) we obtain

$$\nabla\Phi = \bar{n} \frac{\partial\Phi}{\partial n} + G^{\gamma\alpha} \bar{\tau}_\gamma \nabla\Phi \cdot \bar{\tau}_\alpha. \quad (40)$$

Expressing Eq. (40) in component form gives Eq. (33)

$$(\nabla\Phi)_j = \frac{\partial\Phi}{\partial x_j} = n_j \frac{\partial\Phi}{\partial n} + G^{\gamma\alpha} \frac{\partial x_j}{\partial z^\gamma} \frac{\partial\Phi}{\partial z^\alpha} = n_j \frac{\partial\Phi}{\partial n} + G^{\gamma\alpha} \frac{\partial x_j}{\partial z^\gamma} \frac{\partial\Phi}{\partial z^\alpha}.$$

By using $d\Gamma = \sqrt{G} dz_1 dz_2$ and $G = \det\|G_{\alpha\beta}\|$, we transform the third integral on the left-hand side of Eq. (5)

$$\begin{aligned} \int_\Gamma T_{ijk} \delta \varepsilon_{ij} n_k d\Gamma &= \int_\Gamma (T_{ijk} n_k) \frac{\partial \delta u_i}{\partial x_j} d\Gamma \\ &= \int_S (T_{ijk} n_k) \left[n_j \frac{\partial \delta u_i}{\partial n} + G^{\gamma\alpha} \frac{\partial x_j}{\partial z^\gamma} \frac{\partial \delta u_i}{\partial z^\alpha} \right] \sqrt{G} dz_1 dz_2 \\ &= \int_\Gamma (T_{ijk} n_k) n_j \frac{\partial \delta u_i}{\partial n} d\Gamma + \int_S (T_{ijk} n_k) G^{\gamma\alpha} \frac{\partial x_j}{\partial z^\gamma} \frac{\partial \delta u_i}{\partial z^\alpha} \sqrt{G} dz_1 dz_2 \end{aligned} \quad (41)$$

where S is the area of variation of z^1 and z^2 . Because of the arbitrariness of $\partial \delta u_i / \partial n$, from the first term in the last expression of Eq. (41) we obtain the first boundary condition

$$T_{ijk} n_j n_k = 0. \tag{42}$$

Next, we continue transforming the second term in the last expression of Eq. (41) as

$$\begin{aligned} & \int_S (T_{ijk} n_k) G^{\gamma\alpha} \frac{\partial x_j}{\partial z^\gamma} \frac{\partial \delta u_i}{\partial z^\alpha} \sqrt{G} dz_1 dz_2 \\ &= \int_S \frac{\partial}{\partial z^\alpha} \left[(T_{ijk} n_k) G^{\gamma\alpha} \frac{\partial x_j}{\partial z^\gamma} \delta u_i \sqrt{G} \right] dz_1 dz_2 \\ & - \int_S \frac{\partial}{\partial z^\alpha} \left[(T_{ijk} n_k) G^{\gamma\alpha} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} \right] \delta u_i dz_1 dz_2 \\ &= \int_\Gamma \text{div} \bar{F} d\Gamma - \int_S \frac{\partial}{\partial z^\alpha} \left[(T_{ijk} n_k) G^{\gamma\alpha} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} \right] \delta u_i dz_1 dz_2, \tag{43} \end{aligned}$$

where $\text{div} \bar{F} = \frac{1}{\sqrt{G}} \frac{\partial}{\partial z^\alpha} (F^\alpha \sqrt{G})$, $F^\alpha = (T_{ijk} n_k) G^{\gamma\alpha} \frac{\partial x_j}{\partial z^\gamma} \delta u_i$.

According to Stokes' theorem [4], the divergence integral over the above closed surface is zero, hence Eq. (43) may be written as

$$\begin{aligned} & \int_S (T_{ijk} n_k) G^{\gamma\alpha} \frac{\partial x_j}{\partial z^\gamma} \frac{\partial \delta u_i}{\partial z^\alpha} \sqrt{G} dz_1 dz_2 = \\ & - \int_\Gamma \frac{1}{\sqrt{G}} \frac{\partial}{\partial z^\alpha} \left[(T_{ijk} n_k) G^{\gamma\alpha} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} \right] \delta u_i d\Gamma. \tag{44} \end{aligned}$$

Hence,

$$W_{\text{int}} = - \int_V (\sigma_{ij,j} - T_{ijk,kj}) \delta u_i dV + \int_\Gamma [(\sigma_{ij} - T_{ijk,k}) n_j - \kappa_i] \delta u_i d\Gamma \tag{45}$$

where $\kappa_i = \frac{1}{\sqrt{G}} \frac{\partial}{\partial z^\beta} \left[(T_{ijk} n_k) G^{\beta\gamma} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} \right]$.

The quantity κ_i may be further expressed through T_{ijk} and surface characteristic parameters as follows:

$$\begin{aligned} & \frac{\partial}{\partial z^\beta} \left[T_{ijk} n_k G^{\gamma\beta} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} \right] = \frac{\partial T_{ijk}}{\partial z^\beta} n_k G^{\gamma\beta} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} \\ & + T_{ijk} \frac{\partial n_k}{\partial z^\beta} G^{\gamma\beta} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} + T_{ijk} n_k \frac{\partial G^{\gamma\beta}}{\partial z^\beta} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} \\ & + T_{ijk} n_k G^{\gamma\beta} \frac{\partial^2 x_j}{\partial z^\beta \partial z^\gamma} \sqrt{G} + T_{ijk} n_k G^{\gamma\beta} \frac{\partial x_j}{\partial z^\gamma} \frac{\partial \sqrt{G}}{\partial z^\beta}. \tag{46} \end{aligned}$$

The last several terms may be expressed in terms of surface invariants.

According to differential geometry, we have the following equation

$$\frac{\partial n_k}{\partial z^\beta} = -b_{\beta\gamma} G^{\gamma\alpha} \frac{\partial x_k}{\partial z^\alpha} \tag{47}$$

where $b_{\beta\gamma}$ denotes fundamental quantities of the second kind.

Using Eq. (19) in which we replace $g \rightarrow G$, we have

$$\frac{\partial G^{\gamma\beta}}{\partial z^\beta} = -G^{\nu\alpha} \Gamma_{\alpha\beta}^{\nu} \Gamma_{\beta\gamma}^{\nu} G^{\gamma\beta}.$$

Therefore, the third and fourth terms on the right-hand side of Eq. (46) is equal to

$$\begin{aligned} & T_{ijk} n_k \left[\frac{\partial G^{\gamma\beta}}{\partial z^\beta} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} + G^{\gamma\beta} \frac{\partial^2 x_j}{\partial z^\beta \partial z^\gamma} \sqrt{G} \right] \\ &= T_{ijk} n_k \left[-G^{\nu\alpha} \Gamma_{\alpha\beta}^{\nu} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} - \Gamma_{\beta\nu}^{\nu} G^{\gamma\beta} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} + G^{\gamma\beta} \Gamma_{\beta\gamma}^{\sigma} \frac{\partial x_j}{\partial z^\sigma} \sqrt{G} \right] \\ &= -T_{ijk} n_k G^{\nu\alpha} \Gamma_{\alpha\beta}^{\nu} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} = -T_{ijk} n_k G^{\nu\alpha} \frac{\partial x_j}{\partial z^\gamma} \frac{\partial \sqrt{G}}{\partial z^\alpha} \tag{48} \end{aligned}$$

In view of the relation $\Gamma_{\alpha\beta}^{\beta} = \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial z^\alpha}$, the last terms on the right-hand side of Eqs. (46) and (48) are mutually cancelled. Substituting Eq. (47) into Eq. (46) yields

$$\begin{aligned} & \frac{\partial}{\partial z^\beta} \left[T_{ijk} n_k G^{\gamma\beta} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} \right] \\ &= \frac{\partial T_{ijk}}{\partial z^\beta} n_k G^{\gamma\beta} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} - T_{ijk} b_{\beta\nu} G^{\nu\alpha} \frac{\partial x_k}{\partial z^\alpha} G^{\gamma\beta} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} \\ &= \left[\frac{\partial T_{ijk}}{\partial z^\beta} n_k - T_{ijk} b_{\beta\nu} G^{\nu\alpha} \frac{\partial x_k}{\partial z^\alpha} \right] G^{\gamma\beta} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G}. \tag{49} \end{aligned}$$

Hence,

$$\begin{aligned} \kappa_i &= \frac{1}{\sqrt{G}} \frac{\partial}{\partial z^\beta} \left[(T_{ijk} n_k) G^{\gamma\beta} \frac{\partial x_j}{\partial z^\gamma} \sqrt{G} \right] \\ &= \left[\frac{\partial T_{ijk}}{\partial z^\beta} n_k - T_{ijk} b_{\beta\nu} G^{\nu\alpha} \frac{\partial x_k}{\partial z^\alpha} \right] G^{\gamma\beta} \frac{\partial x_j}{\partial z^\gamma}. \tag{50} \end{aligned}$$

The second boundary condition can be expressed as follows from Eq. (45)

$$(\sigma_{ij} - T_{ijk,k}) n_j - \kappa_i = 0 \tag{51}$$

with κ_i given by Eq. (50).

5 Equations for axisymmetric plane strain problems

Consider a long cylinder with initial inner radius a and external radius b , respectively, as shown in Figure 1. It is assumed that the cylinder is subjected to axisymmetric loading, so that the stress-strain state is axisymmetric. It is also assumed that the length of the cylinder is much larger than the cylinder thickness, so that the generalized

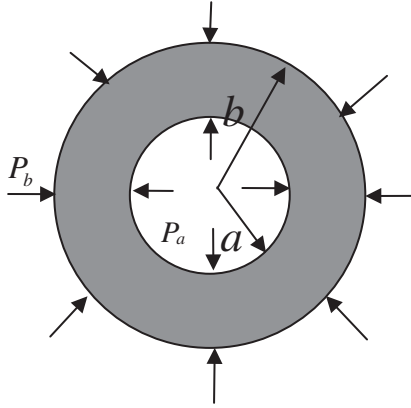


Figure 1 Model of the cylinder.

plane strain conditions apply and all the components of strain and strain gradient associated with the axis coordinate are zero.

For the description of the stress-strain state, a cylindrical coordinate system is adopted. In this case the variables involved are

$$z^1=r, \quad z^2=\theta, \quad z^3=z; \quad x_1=x=r \cos \theta, \quad x_2=y=r \sin \theta, \quad x_3=z \quad (52)$$

and the element of arc in the cylindrical system is given by

$$ds^2=dr^2+r^2d\theta^2+dz^2,$$

such that

$$g_{11}=g^{11}=1, \quad g_{22}=\frac{1}{g^{22}}=r^2, \quad g_{33}=g^{33}=1, \quad g_{kl}=0 \quad (k \neq l). \quad (53)$$

The corresponding Christoffel symbols of the second kind have the following values in cylindrical coordinates

$$\Gamma_{12}^2=\Gamma_{21}^2=1/r, \quad \Gamma_{22}^1=-r, \quad \text{all others being zero.} \quad (54)$$

In strain gradient theory, it is assumed that in addition to the conventional Eulerian strain ε_{ij} and Cauchy stresses σ_{ij} , strain gradients η_{ijk} and their work-conjugate higher-order stresses τ_{ijk} are also present in the material body [1, 4]. The strains and strain gradients are defined, respectively, by

$$\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2, \quad \eta_{ijk}=u_{k,ij} \quad (55)$$

where u_i denotes displacement (ε_{ij} and η_{ijk} are symmetric with respect to the indices i and j).

Within the framework of linear elasticity, the following generalized Hooke's law type of relation between σ_{ij} and ε_{ij} and between τ_{ijk} and η_{ijk} are assumed [2, 4, 8]

$$\begin{aligned} \sigma_{ij} &= \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \\ \tau_{ijk} &= l^2 [a_1(\eta_{ipp} \delta_{jk} + \eta_{jpp} \delta_{ik}) + a_2(\eta_{ppi} \delta_{jk} + 2\eta_{kpp} \delta_{ij} + \eta_{ppj} \delta_{ik}) \\ &\quad + a_3 \eta_{ppk} \delta_{ij} + a_4 \eta_{ijk} + a_5(\eta_{kji} + \eta_{kij})] \end{aligned} \quad (56)$$

where λ and μ are the usual Lamé constants, l is the internal length scale due to the introduction of strain gradients, and a_i ($i=1, \dots, 5$) are higher-order elastic constants associated with gradient terms in a material.

For the present axisymmetric problem of pressurized cylinder, the displacement u_i does not depend on θ and z coordinates, and is a function only of the r coordinate. Under plane strain conditions, only six components of the conventional strains ε_{ij} and strain gradients η_{ijk} [17] are nonzero: $\varepsilon_r, \varepsilon_\theta, \eta_{rrr}, \eta_{\theta\theta r}, \eta_{r\theta\theta}$ and $\eta_{\theta r\theta}$, which can be expressed in terms of radial displacement u_r in the following form:

$$\begin{aligned} \varepsilon_r &= u_{r,r}, \quad \varepsilon_\theta = \frac{u_r}{r}, \\ \eta_{rrr} &= u_{r,rr}, \quad \eta_{\theta\theta r} = \frac{1}{r^2}(ru_{r,r} - u), \quad \eta_{r\theta\theta} = \eta_{\theta r\theta} = \frac{1}{r} \left(u_{r,r} - \frac{u_r}{2r} \right). \end{aligned} \quad (57)$$

By neglecting body forces, Eq. (32) gives the following equilibrium equation for the radial direction:

$$\begin{aligned} \frac{\partial}{\partial r} \left[\sigma_r - \frac{\partial \tau_{rr}}{\partial r} - \frac{1}{r} (\tau_{rr} - \tau_{\theta\theta r} - \tau_{r\theta\theta}) \right] \\ + \frac{1}{r} \left[\sigma_r - \sigma_\theta - \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{\theta r\theta}}{\partial r} - \frac{1}{r} (\tau_{rr} - 2\tau_{\theta\theta r} - 2\tau_{r\theta\theta} - \tau_{\theta r\theta}) \right] = 0. \end{aligned} \quad (58)$$

The boundary conditions of Eqs. (42) and (51) on the inner $r=a$ and outer $r=b$ radii of the cylinder take the following form, respectively.

$$\begin{cases} T_a = -\sigma_{rr} + \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} (-\tau_{rr} - \tau_{\theta\theta r}) = p_a \\ R_{r=a} = \tau_{rr} = 0 \end{cases} \quad (59)$$

$$\begin{cases} T_b = \sigma_{rr} - \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} (-\tau_{rr} + 2\tau_{r\theta\theta} + \tau_{\theta r\theta}) = -p_b \\ R_b = \tau_{rr} = 0 \end{cases} \quad (60)$$

Here, we will follow Zhao et al. [24] and consider the values of a_i as following: $a_1=-c$, $a_2=\frac{7}{4}c$, $a_3=-c$, $a_4=3c$, and $a_5=-c$ where c denotes a single gradient-dependent elastic parameter. Substituting Eq. (57) into Eq. (56) yields

$$\begin{cases} \sigma_r = \frac{E}{(1-2\nu)(1+\nu)} \left[(1-\nu) \frac{\partial u_r}{\partial r} + \nu \frac{u_r}{r} \right] \\ \sigma_\theta = \frac{E}{(1-2\nu)(1+\nu)} \left[\nu \frac{\partial u_r}{\partial r} + (1-\nu) \frac{u_r}{r} \right] \end{cases} \quad (61)$$

and

$$\begin{cases} \tau_{rr} = cl^2 \left(5 \frac{\partial^2 u_r}{\partial r^2} + 4 \frac{\partial u_r}{r \partial r} - \frac{13 u_r}{4 r^2} \right) \\ \tau_{r\theta\theta} = \tau_{\theta r\theta} = cl^2 \left(\frac{3}{4} \frac{\partial^2 u_r}{\partial r^2} + \frac{11}{4} \frac{\partial u_r}{r \partial r} - \frac{7 u_r}{4 r^2} \right) \\ \tau_{rzz} = \tau_{zrz} = cl^2 \left(\frac{3}{4} \frac{\partial^2 u_r}{\partial r^2} + \frac{3}{4} \frac{\partial u_r}{r \partial r} - \frac{1 u_r}{2 r^2} \right) \\ \tau_{\theta\theta r} = cl^2 \left(\frac{3}{2} \frac{\partial^2 u_r}{\partial r^2} + \frac{7}{2} \frac{\partial u_r}{r \partial r} - \frac{11 u_r}{4 r^2} \right) \\ \tau_{zzr} = cl^2 \left(\frac{3}{2} \frac{\partial^2 u_r}{\partial r^2} + \frac{3}{2} \frac{\partial u_r}{r \partial r} - \frac{5 u_r}{4 r^2} \right) \end{cases} \quad (62)$$

It is clear from Eqs. (58–62) that not only the equilibrium equations, but also the boundary conditions are very complex. It follows that it is difficult to obtain analytical solutions of the problem.

6 Conclusions

In this article, the equilibrium equations and boundary conditions of the strain gradient theory in arbitrary curvilinear coordinates are obtained. An axisymmetric problem is also considered as a special case. The obtained equilibrium equations and boundary conditions of the strain gradient theory may serve as a basis for solving related analytical and numerical problems.

Acknowledgments: The work presented in this article is carried out under the support of grant RFFI no 11-01-00357-a, NSFC grant No. 51174012, the Project of Construction of Innovative Teams and Teacher Career Development for Universities and Colleges Under Beijing Municipality (IDHT20130512), and the “973” Key State Research Program (grant no 2015CB0578005).

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