Representation of Perfect Reconstruction Octave Decomposition Filter Banks with Set of Decimators {*2,4,4*} via Tree Structure

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Abstract—**In this letter, we prove that a filter bank with set of decimators {*2,4,4*} achieves perfect reconstruction if and only if it can be represented via a tree structure and each branch of the tree structure achieves perfect reconstruction.**

Index Terms—**filter bank, perfect reconstruction, tree structure.**

I. INTRODUCTION

IT is well known that the tree structure filter bank shown in figure 1b can be represented via an octave decomposition filter bank with set of decimators {*2,4,4*} shown in figure 1a, and perfect reconstruction can be achieved if and only if each branch of the tree structure achieves perfect reconstruction [1]-[4]. However, is the converse true? That is, given any perfect reconstruction octave decomposition filter bank with set of decimators {*2,4,4*} shown in figure 1a, can it be represented via a tree structure shown in figure 1b? In general, a perfect reconstruction non-uniform filter bank cannot be represented by a tree structure [11], [12]. This letter works on the problem of representing a perfect reconstruction octave decomposition filter bank with set of decimators {*2,4,4*} by a tree structure.

There are some advantages of representing an octave decomposition filter bank with set of decimators {*2,4,4*} via a tree structure, such as reducing the filter length in the filters [5], and improving the computation complexity and implementation speed [5]. In section II, we show how a perfect reconstruction octave decomposition filter bank with set of decimators {*2,4,4*} can be converted to a tree structure filter bank. Some illustrative examples are demonstrated in section III. Finally, a conclusion is given in section IV.

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II. REPRESENTATION OF OCTAVE DECOMPOSITION FILTER BANK WITH SET OF DECIMATORS {*2,4,4*} VIA A TREE STRUCTURE

Theorem 1: An octave decomposition filter bank with set of decimators {*2,4,4*} achieves perfect reconstruction if and only if it can be represented via a tree structure and each branch of the tree structure achieves perfect reconstruction.

Proof: Since the if part is well known [1]-[4], we only prove the only if part.

An octave decomposition filter bank with set of decimators {*2,4,4*} shown in figure 1a achieves perfect reconstruction if and only if $\exists c \in C$ and $\exists m \in Z$ such that:

$$
\begin{bmatrix}
\frac{1}{2} \cdot H_0(z) & \frac{1}{4} \cdot H_1(z) & \frac{1}{4} \cdot H_2(z) \\
0 & \frac{1}{4} \cdot H_1(z \cdot W) & \frac{1}{4} \cdot H_2(z \cdot W) \\
\frac{1}{2} \cdot H_0(z \cdot W^2) & \frac{1}{4} \cdot H_1(z \cdot W^2) & \frac{1}{4} \cdot H_2(z \cdot W^2) \\
0 & \frac{1}{4} \cdot H_1(z \cdot W^3) & \frac{1}{4} \cdot H_2(z \cdot W^3)
\end{bmatrix}
\cdot
\begin{bmatrix}
G_0(z) \\
G_1(z) \\
G_2(z)
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
c \cdot z^m \\
0 \\
0 \\
0
\end{bmatrix},
\tag{1}
$$

where

$$
W = \exp\left(-\frac{j \cdot 2 \cdot \pi}{4}\right) = -j. \tag{2}
$$

This directly implies that:

$$
\det\left(\begin{bmatrix}
H_1(z \cdot W) & H_2(z \cdot W) \\
H_1(z \cdot W^3) & H_2(z \cdot W^3)
\end{bmatrix}\right) = 0. \tag{3}
$$

Since (3) is true for all $z \in C$, by using the substitution $z = z \cdot W^{-1}$, we have:

$$
\det\left(\begin{bmatrix}
H_1(z) & H_2(z) \\
H_1(z \cdot W^2) & H_2(z \cdot W^2)
\end{bmatrix}\right)
$$

$$
= H_1(z) \cdot H_2(z \cdot W^2) - H_2(z) \cdot H_1(z \cdot W^2) \tag{4}
$$

$$
= 0.
$$

$$\Rightarrow \frac{H_1(z)}{H_2(z)} = \frac{H_1(z \cdot W^2)}{H_2(z \cdot W^2)} = R(z). \quad (5)$$

Let

$$H_i(z) = \sum_{l=0}^{3} z^{-l} \cdot E_{i,l}(z^4), \quad (6)$$

for $i = 1,2$,

then:

$$\frac{E_{1,0}(z^4) + z^{-2} \cdot E_{1,2}(z^4) + z^{-1} \cdot (E_{1,1}(z^4) + z^{-2} \cdot E_{1,3}(z^4))}{E_{2,0}(z^4) + z^{-2} \cdot E_{2,2}(z^4) + z^{-1} \cdot (E_{2,1}(z^4) + z^{-2} \cdot E_{2,3}(z^4))}$$

$$= \frac{E_{1,0}(z^4) + z^{-2} \cdot E_{1,2}(z^4) - z^{-1} \cdot (E_{1,1}(z^4) + z^{-2} \cdot E_{1,3}(z^4))}{E_{2,0}(z^4) + z^{-2} \cdot E_{2,2}(z^4) - z^{-1} \cdot (E_{2,1}(z^4) + z^{-2} \cdot E_{2,3}(z^4))} \quad (7)$$

$$= R(z).$$

$$\Rightarrow \frac{E_{1,0}(z^4) + z^{-2} \cdot E_{1,2}(z^4) + z^{-1} \cdot (E_{1,1}(z^4) + z^{-2} \cdot E_{1,3}(z^4))}{E_{2,0}(z^4) + z^{-2} \cdot E_{2,2}(z^4) + z^{-1} \cdot (E_{2,1}(z^4) + z^{-2} \cdot E_{2,3}(z^4))}$$

$$= \frac{E_{1,0}(z^4) + z^{-2} \cdot E_{1,2}(z^4)}{E_{2,0}(z^4) + z^{-2} \cdot E_{2,2}(z^4)} \quad (8)$$

$$= \frac{E_{1,1}(z^4) + z^{-2} \cdot E_{1,3}(z^4)}{E_{2,1}(z^4) + z^{-2} \cdot E_{2,3}(z^4)}$$

$$R(z).$$

Hence:

$$\frac{H_1(z)}{H_2(z)} = \frac{E_{1,0}(z^4) + z^{-2} \cdot E_{1,2}(z^4)}{E_{2,0}(z^4) + z^{-2} \cdot E_{2,2}(z^4)} = R(z). \quad (9)$$

As a result, there exist $F_1'(z)$, $F_0(z)$, and $F_1(z)$ such that $H_1(z) = F_1'(z) \cdot F_0(z^2)$ and $H_2(z) = F_1'(z) \cdot F_1(z^2)$, respectively. Hence, the octave decomposition filter bank with set of decimators $\{2,4,4\}$ shown in figure 1a can be represented via a tree structure shown in figure 1b.

Because the input-output behavior of these two representations remains unchanged, so if the octave decomposition filter bank with set of decimators $\{2,4,4\}$ achieves perfect reconstruction, the tree structure filter bank does also. Hence, it completes the proof. ∎

For the design of the synthesis filters in an octave decomposition filter bank with set of decimators $\{2,4,4\}$, if that filter bank system achieves perfect reconstruction, it is proved in Theorem 1 that it can be represented by a tree-structure filter bank and that tree-structure filter bank achieves perfect reconstruction. The synthesis filters in a tree-structure filter bank for the perfect reconstruction can be designed easily [7], and those filters in the tree-structure filter bank can be transformed to the filters in an octave decomposition filter bank with set of decimators $\{2,4,4\}$. Hence, the synthesis filters in an octave decomposition filter bank with set of decimators $\{2,4,4\}$ can be designed.

III. ILLUSTRATIVE EXAMPLES

A. Non-tree Structure Filter Bank

Consider an example of $H_0(z) = 1 + z^{-1}$, $H_1(z) = 1 - z^{-1}$, and $H_2(z) = z^{-2}$, respectively. Since $H_1(z)/H_2(z) = z^2 \cdot (1 - z^{-1})$, there do not exist $F_1'(z)$, $F_0(z)$, and $F_1(z)$ such that $H_1(z) = F_1'(z) \cdot F_0(z^2)$ and $H_2(z) = F_1'(z) \cdot F_1(z^2)$, respectively. Hence, this octave decomposition filter bank with set of decimators $\{2,4,4\}$ cannot be represented via a tree structure. By Theorem 1, this octave decomposition filter bank with set of decimators $\{2,4,4\}$ does not achieve perfect reconstruction.

It is worth to note that by converting the octave decomposition filter bank with set of decimators $\{2,4,4\}$ to a uniform filter bank shown in figure 2 [6]-[10], perfect reconstruction can be achieved. However, $G_{-1}'(z) \neq z^{-2} \cdot G_0'(z)$. This implies that the corresponding synthesis filter $G_0(z)$ shown in figure 1a is time varying.

B. Tree Structure Filter Bank

Consider another example with $H_0(z) = 1 + z^{-1}$, $H_1(z) = 1 - z^{-1} + z^{-2} - z^{-3}$, and $H_2(z) = 1 - z^{-1} - z^{-2} + z^{-3}$, respectively. Since $H_1(z)/H_2(z) = (1 + z^{-2})/(1 - z^{-2})$, there exist $F_1'(z)$, $F_0(z)$, and $F_1(z)$ such that $H_1(z) = F_1'(z) \cdot F_0(z^2)$ and $H_2(z) = F_1'(z) \cdot F_1(z^2)$, respectively. Hence, this octave decomposition filter bank with set of decimators $\{2,4,4\}$ can be represented via a tree structure. It can be checked easily that each branch in the tree structure can achieve perfect reconstruction. Hence, this octave decomposition filter bank with set of decimators $\{2,4,4\}$ can achieve perfect reconstruction.

IV. CONCLUSION

In this letter, we show that an octave decomposition filter bank with set of decimators $\{2,4,4\}$ achieves perfect reconstruction if and only if it can be represented via a tree structure and each branch of the tree structure achieves perfect reconstruction. The advantage of representing an octave decomposition filter bank with set of decimators $\{2,4,4\}$ via a tree structure is to reduce the computation complexity and provide a fast implementation for an octave decomposition filter bank with set of decimators $\{2,4,4\}$ [5].

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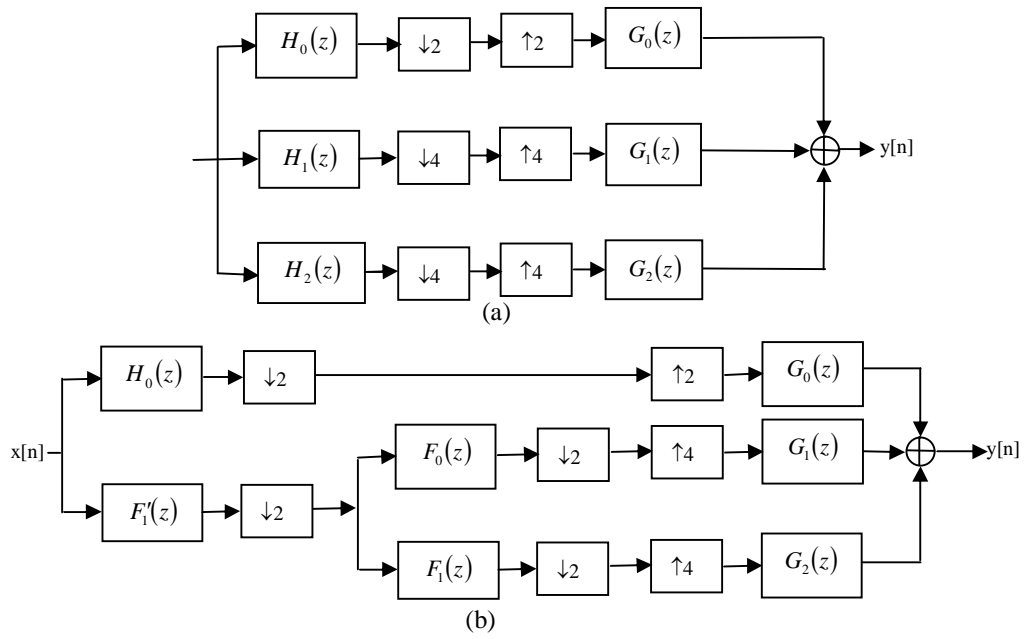
Fig. 1. (a) Octave decomposition filter bank with set of decimators {2,4,4}.
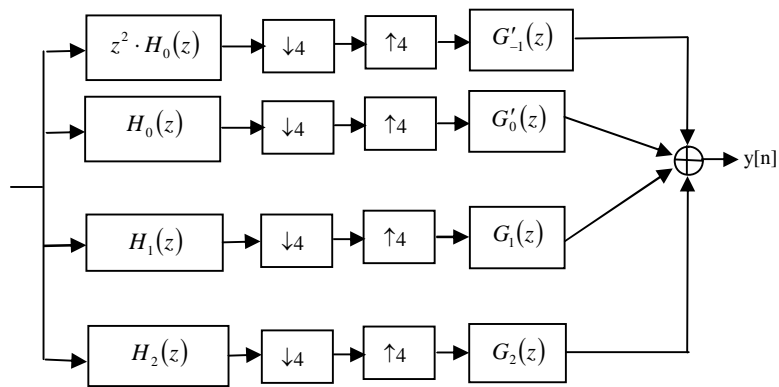(b) tree structure filter bank.

Fig. 2. Representation of octave decomposition filter bank with set of decimators
{2,4,4} via a uniform filter bank.