Conference Paper, Published Version

Kim, Tae Beom; Choi, Sung-Uk

Algorithm for 2D Finite Element Modeling of Bed Elevation Change in a Natural River

ALGORITHM FOR 2D FINITE ELEMENT MODELING OF BED ELEVATION CHANGE IN A NATURAL RIVER

Tae Beom Kim[1] and Sung-Uk Choi[2]

[1] Post-doctoral Researcher, School of Civil & Environmental Engineering, Yonsei University
134 Shinchon-dong, Seodaemun-gu, Seoul, 120-749, Korea, e-mail: geo108@naver.com
[2] Professor, School of Civil & Environmental Engineering, Yonsei University
134 Shinchon-dong, Seodaemun-gu, Seoul, 120-749, Korea, e-mail: schoi@yonsei.ac.kr

ABSTRACT

The purpose of this study is to develop a 2D finite element model which is capable of predicting the time-dependent bed elevation change of the stream. The shallow water equations and the Exner's equation are solved with 2D Characteristic Dissipative-Galerkin method and Bubnov-Galerkin method, respectively. The developed model is a decoupled model in a sense that the bed elevation does not change simultaneously with the flow during each computational time step. Solving the Exner's equation provides the time-dependent bed elevation change based on the equilibrium sediment load. The impacts of the secondary flows in a curved channel and the gravity force due to the geographic change in the direction of sediment transport are taken into account for the accurate spatial variation of equilibrium sediment loads. In order to incorporate these effects into the numerical model, the spatial variation of mean velocity and the topography are necessary. However, estimating the variation of spatially-continuous variables at one node of FEM grid is very difficult. A new FEM algorithm for the Exner's equation, proposed herein, estimates the equilibrium sediment load not at the node but within the element. For validation, the developed model is applied to 140° bended laboratory channel data at Delft Hydraulics Laboratory and aggradation data of Soni et al.'s (1980) experiment. The simulated results agree well with the measured data. Presently, this model is restricted to the case with uniform sediment, neglecting armoring or grain sorting effects.

Keywords: finite element method (FEM), Exner's Equation, Morphological Change

1. INTRODUCTION

A natural alluvial river tends to keep a stable equilibrium state between flow and river bed. When one of two regimes is altered, the other reacts and finally achieves a new stable state. The response like this may be occurred when the flow characteristics are changed through the curved reach from the straight one or when the sediment higher than that the flow is capable of transporting is loaded from the upstream reach. The former generates the erosion near the outer bank and the deposition near the inner bank. The latter generates the aggradation of river bed, and then induces the change of flow characteristics. The prediction of such a river bed change is a very important and challenging task for practical engineers and scientists.

In predicting the morphological change of a curved channel, a 1D model is not appropriate since the transverse bed profile can not be explained. A 3D model may be the best choice. Nevertheless 3D model is not readily applicable to many engineering problems even nowadays due to tremendous computational expense, complexity of numerical computations, and necessity of sufficiently detailed observation data for calibration and verification. These necessitate the introduction of 2D model.

Since the past, 2D models of the bed deformation have used the finite difference method as numerical tool. After a few days, the finite volume method began to be used since it is excellent in the aspect of mass conservation. Recently, 2D FDM or FVM models using curvilinear or body fitted coordinate system are developed (Kassem and Chaudhry, 2002; Duc et al., 2004; Wu 2004) since Cartesian coordinate systems may not accurately represent the irregular channel shape and can induce inaccurate simulation results. It is well known that the finite element method provides more flexibility in handling spatial domain than FDM or FVM. However, the finite element model of bed morphological change is not sufficiently and commonly used and developed compared with the finite difference or finite volume models.

In this study, a numerical model capable of predicting the time variation of the bed elevation change is developed. The shallow water equations and the Exner's equation are solved by the finite element method. The shallow water equations are solved with 2D Characteristic Dissipative-Galerkin scheme proposed by Ghanem (1995) which is one of the Streamline-Upwind / Petrov-Galerkin (SU/PG) scheme. The Exner's equation is solved with classical Bubnov-Galerkin scheme. A new FEM algorithm for the Exner's equation is introduced in this study, in which estimates the equilibrium sediment load not at a node but within an element. For validation, the developed model is applied to 140° bended laboratory channel data at Delft Hydraulics Laboratory (Struiksma, 1983) and straight channel data for bed aggradation due to sediment overloading (Soni et al., 1980).

The numerical model developed in the present study is based upon the decoupled modeling approach assuming that the interaction between flow and bed is ignorable during the computational time step. Also, the model is restricted to beds of uniform sediment without armoring or grain sorting effects.

2. GOVERNING EQUATIONS

Flow equations

For the flow analysis, the following 2D shallow water equations with the effective stress terms are adopted:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial y} + \frac{\partial \mathbf{D_x}}{\partial x} + \frac{\partial \mathbf{D_y}}{\partial y} + \mathbf{F} = \mathbf{0} \tag{1}$$

where

$$\mathbf{U}^T = \begin{pmatrix} h & p & q \end{pmatrix} \tag{2}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ gh - \dfrac{p^2}{h^2} & 2\dfrac{p}{h} & 0 \\ -\dfrac{pq}{h^2} & \dfrac{q}{h} & \dfrac{p}{h} \end{bmatrix} \tag{3}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{pq}{h^2} & \frac{q}{h} & \frac{p}{h} \\ gh - \frac{q^2}{h^2} & 0 & 2\frac{q}{h} \end{bmatrix} \tag{4}$$

$$\mathbf{D_x} = \begin{bmatrix} 0 \\ -2\nu_t \frac{\partial p}{\partial x} \\ -\nu_t \left(\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} \right) \end{bmatrix} \tag{5}$$

$$\mathbf{D_y} = \begin{bmatrix} 0 \\ -\nu_t \left(\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} \right) \\ -2\nu_t \frac{\partial q}{\partial y} \end{bmatrix} \tag{6}$$

$$\mathbf{F} = \begin{bmatrix} 0 \\ gh \frac{\partial z_b}{\partial x} + \frac{gn^2}{h^{7/3}} p\sqrt{p^2+q^2} \\ gh \frac{\partial z_b}{\partial y} + \frac{gn^2}{h^{7/3}} q\sqrt{p^2+q^2} \end{bmatrix} \tag{7}$$

where h is flow depth, p and q are discharge per unit width in x- and y-directions, respectively, g is gravitational acceleration, z_b is bed elevation measured from a certain datum, n is Manning's roughness coefficient, and ν_t is turbulent viscosity. Herein, the following parabolic eddy viscosity model is used:

$$\nu_t = \frac{\kappa}{6} U_* h \tag{8}$$

where U_* is shear velocity, and κ is von Kármán constant ($\approx$0.4).

Bed elevation change equations

In order to estimate the bed elevation change, the following Exner's equation is solved:

$$\left(1 - p'\right)\frac{\partial z_b}{\partial t} + \frac{\partial q_{tx}}{\partial x} + \frac{\partial q_{ty}}{\partial y} = 0 \tag{9}$$

where p' is porosity, and q_{tx} and q_{ty} are the x- and y-components of total sediment load per unit width which are expressed as following, respectively.

$$q_{tx} = q_t \cos\Phi \,; \quad q_{ty} = q_t \cos\Phi \tag{10}$$

where Φ is the counterclockwise angle of sediment transport direction from the positive *x*-axis , and q_t is total sediment load per unit width. In the present study, Engelund and Hansen's (1972) formula and Soni et al.'s (1980) formula, which are expressed as Eq. 11 and Eq. 12, respectively, are used for the total sediment load.

$$q_t = 0.05\gamma_s V^2 \left[\frac{d}{g\left(\gamma_s/\gamma - 1\right)}\right]^{1/2} \left[\frac{\tau_0}{\left(\gamma_s - \gamma\right)d_{50}}\right]^{3/2} \tag{11}$$

where V is depth-averaged flow velocity, γ is specific weight of fluid, γ_s is specific weight of sediment, d_{50} is the median grain diameter, and τ_0 is bed shear stress.

$$q_t = aV^b \tag{12}$$

where *a* and *b* have values as 0.00145 and 5.0, respectively for the sediment size used in Soni et al.'s (1980) experiment.

In laterally sloping bed, sediment transport direction in Eq. 10 is not identical with the direction of bed shear stress due to the gravity force acting on the particles. This effect can be expressed as following (Koch and Flokstra, 1980):

$$\tan\Phi = \frac{\sin\alpha - \dfrac{1}{f_s\theta_*}\dfrac{\partial z_b}{\partial y}}{\cos\alpha - \dfrac{1}{f_s\theta_*}\dfrac{\partial z_b}{\partial x}} \tag{13}$$

where α is the direction of bed shear stress, f_s is shape factor ranging from 1 to 2, and θ_* is the dimensionless Shield's parameter expressed as following:

$$\theta_* = \frac{n^2V^2}{h^{1/3}\left(\dfrac{\gamma_s}{\gamma} - 1\right)d_{50}} \tag{14}$$

In a curved channel, the difference of centrifugal forces between the upper and the lower layer of flow induces the secondary flow, also known as helical or spiral flow. The secondary flow causes the direction of bed shear stress to deviate from the direction of the mean flow velocity. Therefore, it is necessary to reflect the effect of secondary flow on the sediment transport direction in a curved channel. This effect can be introduced in Eq. 13 as following:

$$\alpha = \tan^{-1}\left(\frac{v}{u}\right) + \tan^{-1}\left(\frac{Fh}{R_c}\right) \tag{15}$$

where *u* and *v* are the *x*- and *y*-components of the depth-averaged velocity, respectively, R_c is the local radius of curvature of the streamline, and *F* is the parameter defined as following (Jansen, 1979):

$$F = \frac{2}{\kappa^2}\left(1 - \frac{n\sqrt{g}}{\kappa h^{1/6}}\right) \tag{16}$$

In Eq. 15, the second term of the right side is the deviation of the bed shear stress from the streamlines due to the secondary flow in a curved channel. Since in some cases the inertia of the secondary flow has to be accounted for, it is necessary to reflect the inertia effect using a inertial adaptation equation (Struiksma et al., 1985) as following:

$$\beta \frac{C}{\sqrt{g}} h \frac{\partial I}{\partial s} + I = \frac{hV}{R} \tag{17}$$

$$I = \frac{hV}{R_c} \tag{18}$$

where C is Chézy coeffient, I is a measure of the intensity of the secondary flow, s is the streamwise coordinate, β is a given coefficient normally between 0.4 and 2.0, for which De Vriend (1981) proposed about 1.3 and Struiksma et al. (1985) used 0.6, and R is the local radius of curvature of the streamline calculated as following:

$$\frac{1}{R} = \frac{1}{V^3}\left[\left(u^2 \frac{\partial v}{\partial x} + uv \frac{\partial v}{\partial y}\right) - \left(uv \frac{\partial u}{\partial x} + v^2 \frac{\partial u}{\partial y}\right)\right] \tag{19}$$

3. NUMERICAL METHODS

Flow equations

To solve the sallow water equations numerically, the Streamline-Upwind/Petrov-Galerkin (SU/PG) scheme is used. The SU/PG scheme employs the weighting function as following:

$$N_i^* = N_i + \omega \Delta x \frac{\partial N_i}{\partial x} \mathbf{W_x} + \omega \Delta y \frac{\partial N_i}{\partial y} \mathbf{W_y} \tag{20}$$

where N_i is basis or shape function for the i-th node, N_i^* is weighting function for the i-th node, ω is weighting coefficient, and $\mathbf{W_x}$ and $\mathbf{W_y}$ are weighting matrices in the x- and y-directions, respectively. In this study, following weighting matrices suggested by Ghanem (1995) are used:

$$\mathbf{W_x} = \frac{\mathbf{A}}{\sqrt{\mathbf{A}^2 + \mathbf{B}^2}}, \quad \mathbf{W_y} = \frac{\mathbf{B}}{\sqrt{\mathbf{A}^2 + \mathbf{B}^2}} \tag{21}$$

The inverse of the square root of the matrix can be determined by Cayley-Hamilton theorem (Hoger and Carlson, 1984). In estimation of Δx and Δy, following expressions suggested by Katopodes (1984), which can be applied to the distorted elements, are used:

$$\Delta x = 2\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial x}{\partial \eta}\right)^2}\ ,\ \ \Delta y = 2\sqrt{\left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2} \tag{22}$$

where ξ and η are isoparametric coordinates. Then, the weighted residual equation of the shallow water equations takes the form of

$$\begin{aligned}&\int_\Omega \left(N_i + \omega \Delta x \frac{\partial N_i}{\partial x}\mathbf{W_x} + \omega \Delta y \frac{\partial N_i}{\partial y}\mathbf{W_y} \right) \\ &\times \left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial y} + \frac{\partial \mathbf{D_x}}{\partial x} + \frac{\partial \mathbf{D_y}}{\partial y} + \mathbf{F} \right) d\Omega = 0\end{aligned} \tag{23}$$

The resulting nonlinear equations are linearized by using the Newton-Raphson method, and the global matrix is solved by the frontal solution algorithm for unsymmetric matrices proposed by Hood (1976).

Bed elevation change equations

The weighted residual equation of the Exner's equations for bed elevation change is as following:

$$\int_\Omega N^* \left[\frac{\partial z_b}{\partial t} + \frac{1}{1-p'}\left(\frac{\partial q_{tx}}{\partial x} + \frac{\partial q_{ty}}{\partial y} \right) \right] d\Omega = 0 \tag{24}$$

The time derivative is replaced by finite difference form and the finite element approximation to the solution or variables is applied. Then, the following equation in matrix form can be obtained:

$$\mathbf{A\Delta z_b} = \frac{\Delta t}{1-p'}\left(\mathbf{Bq_{tx}} + \mathbf{Cq_{ty}}\right) \tag{25}$$

$$A_{ij} = \int_{\Omega^e} \left(N_i^* N_j\right) d\Omega^e \tag{26}$$

$$B_{ij} = \int_{\Omega^e} \left(N_i^* \frac{\partial N_j}{\partial x} \right) d\Omega^e \tag{27}$$

$$C_{ij} = \int_{\Omega^e} \left(N_i^* \frac{\partial N_j}{\partial y} \right) d\Omega^e \tag{28}$$

where $\mathbf{\Delta z_b}$, $\mathbf{q_{tx}}$, and $\mathbf{q_{ty}}$ are the vector of variation of bed elevation, and the vector of x- and y-components of total sediment load per unit width, respectively. In order to use Eq. 25, q_{tx} and q_{ty} defined on every each node are necessary. q_{tx} and q_{ty} can be estimated in Eq. 10 and the sediment transport direction, Φ, can be estimated through Eq. 13~19. Here, partial derivatives of spatial variables such as bed elevation and flow velocity components are necessary. Gradient of spatially continuous variables at any one node can not be defined in the discretized finite element mesh. Therefore, new algorithm to estimate the equilibrium sediment load not at a node but within an element is proposed.

The following equation in matrix form can be obtained by using the Green's Theorem in Eq. 24 and the finite element approximation to the solution:

$$\mathbf{A}\Delta\mathbf{z_b} = \frac{\Delta t}{1-p'}(\mathbf{D}-\mathbf{F}) \tag{29}$$

$$D_{ij} = \int_{\Omega^e}\left(\frac{\partial N_i^*}{\partial x}q_{tx} + \frac{\partial N_i^*}{\partial y}q_{ty}\right)d\Omega^e \tag{30}$$

$$F_i = \int_{\Gamma^e}\left[N_i^*\left(n_x q_{tx} + n_y q_{ty}\right)\right]d\Gamma^e \tag{31}$$

where Γ^e means the boundary of an element. As shown in Eq. 30 and 31, q_{tx} and q_{ty} can be defined within an element or along element boundaries. Therefore, spatially continuous variables can also be estimated within an element by using finite element approximation. In Eq. 24, time derivative can be replace by time-difference approximation of the general form

$$\frac{z_b^{n+1} - z_b^n}{\Delta t} = \theta\left(\frac{\partial z_b}{\partial t}\right)^{n+1} + (1-\theta)\left(\frac{\partial z_b}{\partial t}\right)^n \tag{32}$$

where n is the know time level, n+1 is the unknown time level, and θ is the implicitness parameter.

4. MODEL TESTS

140° Curved channel

For validation, the developed model is applied 140° curved channel data from Delft Hydraulics Laboratory (DHL) (Struiksma, 1983). The main parameters of DHL experiment reported by Struiksma et al. (1985) are: water discharge = 0.061 m^3/s, flume width = 1.5 m, water depth = 0.1 m, flow velocity 0.41 m/s, slope = 0.203 %, Chézy coefficient = 28.8, median grain diameter = 0.45 mm, bend radius = 12 m, bend length = 29.32 m. The sediment is considered uniform sand (σ_g = 1.19). Since the meander length is more than 10 times the width, it does not fit the features of a freely meandering stream (Struiksma et al., 1985). The duration of the flow in the flume which was initially flat in the lateral direction was long enough to establish equilibrium bed topography. Bed-level fluctuations were smoothed out by averaging a large number of independent soundings.

Since the initial condition implies a subcritical flow with Froude number of 0.414, the discharge and the flow depth were imposed at the upstream and downstream boundaries, respectively. For numerical stability, the straight part was extended to 10 and 15 times of the channel width at the upstream and downstream parts, respectively. The bed elevation change at upstream and downstream boundaries was ignored. Porosity of the sediment was assumed to be 0.4. Figure 1 shows the variation of the bed profile along the longitudinal lines. In the figure, both profiles show sediment deposition and erosion near the inner and outer banks, respectively. Good agreement is also found in the location and the amounts of maximum deposition and erosion between simulated and measured data as well as the bed profile tendency. However, after their peaks, the fluctuation of the computed bed profile is slightly overestimated compared with the measured profile.

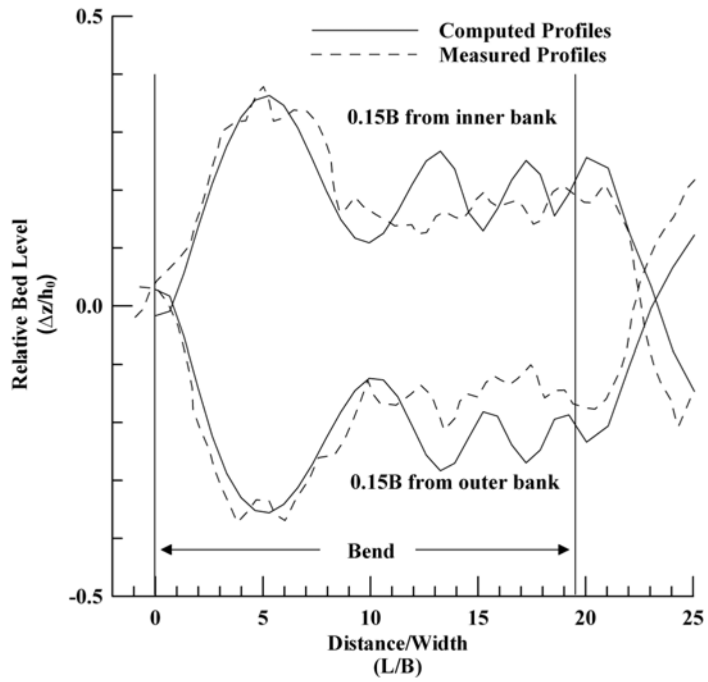
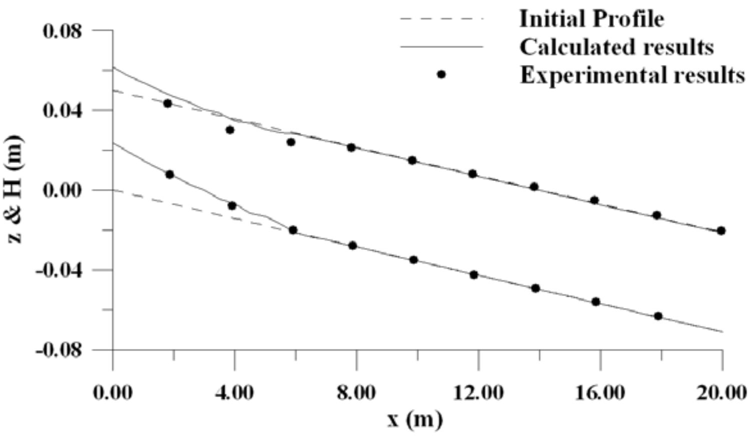


Figure 1 Comparison between the computed and measured longitudinal bed profiles for DHL experiment (Struiksma, 1983).

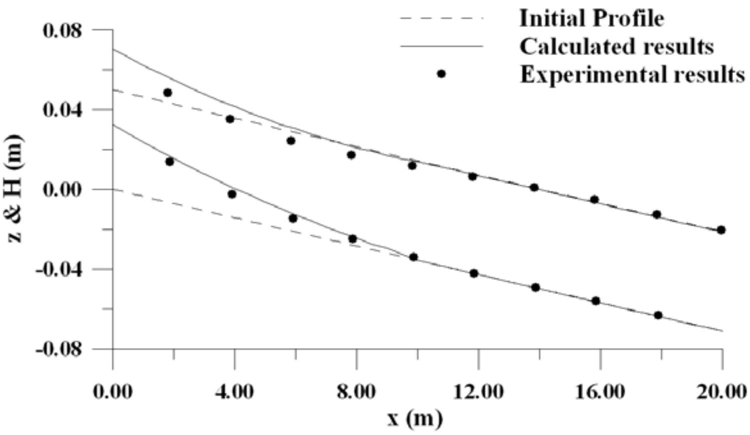
Bed aggradation

To verify the applicability of boundary condition, the developed model was applied to Soni et al.'s (1980) experiment in which they carried out bed aggradation test due to overloaded sediment in a straight flume in order to provide information on bed and water surface profiles. The experiment were conducted in a 0.2 m wide, 0.5 m deep, and 30 m long recirculatory tilting flume located in the Hydraulics Laboratory of the University of Roorkee, Roorkee, India. The bed and injected material was uniform sand with a median diameter of 0.32 mm. After the establishment of uniform flow conditions for a given discharge and slope, the sediment supply rate was increased to a predetermined value by continuously feeding excess sediment at the upstream end of the flume at a constant rate. The bed and water surface profiles were recorded using a point gage at every 2.5 m length at intervals generally varying from 10 ~ 20 minutes. The measured profiles were averaged because of the presence of ripples and dunes on the bed.

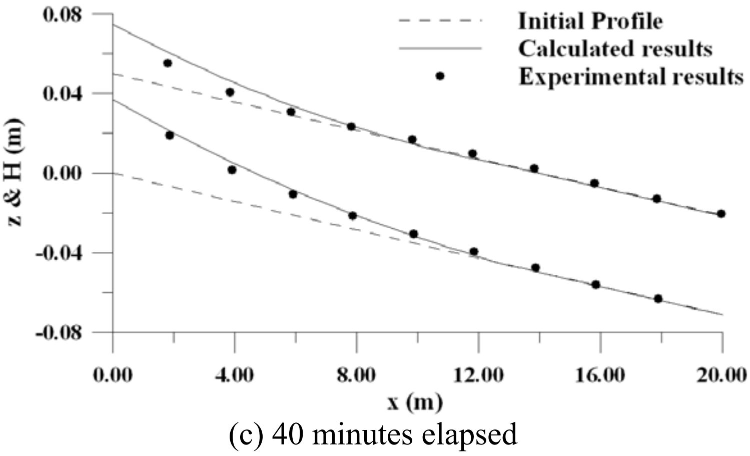
For the numerical simulation, the experiment with 4 times more than equilibrium sediment load rate was selected. Initial bed slope, water discharge and water depth were 0.00356, 0.02 m^2/s, and 0.05 m, respectively. Figure 2 shows the results of the computed bed and water surface elevation profiles at various times. Due to the overloaded sediment influx in the upstream boundary, bed elevation increases and the range of bed elevation change is extended toward downstream in time. Simulated results agree well with experimental data by Soni et al. (1980). Although water surface profiles are slightly different from the experimental data, it is satisfactory.



(a) 15 minutes elapsed



(b) 30 minutes elapsed



(c) 40 minutes elapsed

Figure 2 Bed and water surface profiles for Soni et al.'s (1980) experiment

5. CONCLUSION

In the present paper, a 2D finite element model was proposed for numerical simulation of bed elevation change. In order to alleviate the numerical instability encountered in advection-dominated flows, the Streamline Upwind/Petrov- Galerkin method was used for the numerical analysis of the shallow water equations. For evaluating the bed elevation change, the Exner's equation was solved by using a newly proposed FEM algorithm in which estimates the equilibrium sediment load not at a node but within an element. Both effects by secondary flows in a curved channel and gravity force acting on a sediment grain in a laterally sloping bed were reflected in the model.

The developed model was applied to two experimental cases: 140° curved channel at Delft Hydraulics Laboratory and Soni et al.'s (1980) bed aggradation experiment. It was found that the model predicts the bed elevation change in a curved channel properly. The resulting flow field induces sediment erosion and deposition near the outer and inner banks, respectively. Also, through the application of Soni et al.'s (1980) experiment, the model got the confidential results for the sediment boundary condition.

ACKNOWLEDGMENTS

This research was supported by a grant (code No. 2-3-3) from Sustainable Water Resources Research Center of 21st Century Frontier Research Program.

REFERENCES

De Vriend, H.J. (1981), *Steady flow in shallow channel bends*, Communications on Hydraulics, Department Civil Engineering, Delft University of Technology, The Netherlands.

Duc, B.M., Wenka, T. and Rodi, W. (2004), Numerical modeling of bed deformation in laboratory channels, *Journal of Hydraulic Engineering*, *ASCE*, 130(9), pp.894-904.

Engelund, F. and Hansen, E. (1972), *A Monograph on Sediment Transport in Alluvial Streams*, Teknisk Forlag, Copenhagen, 62 pp.

Ghanem, A.H.M. (1995), *Two-dimensional finite element modeling of flow in aquatic habitats*, Ph.D. Thesis, University of Alberta, Alberta.

Hoger, A. and Carlson, D.E. (1984), Determination of the stretch and rotation in the polar decomposition of the deformation gradient, *Quarterly of Applied Mathematics*, 42(1), pp.113-117.

Hood, P. (1976), Frontal solution program for unsymmetric matrices, *International Journal of Numerical Methods in Engineering*, 10, pp. 379-399.

Jansen, P.P. (1979), *Principles of River Engineering*, Pitman, London, 509 pp.

Kassem, A.A. and Chaudhry, M.H. (2002), Numerical modelling of bed evolution in channel bend, *Journal of Hydraulic Engineering*, *ASCE*, 128(5), pp.507-514.

Katopodes, N.D. (1984), Two-dimensional surges and shocks in open channels, *Journal of Hydraulic Engineering, ASCE*, 110(6), pp.794-812.

Koch, F.G. and Flokstra, C. (1980), Bed level computations for curved alluvial channels, *Proceedings of XIXth Congress of IAHR*, New Delhi, India, pp.357-364.

Soni, J.P., Garde, R.J. and Ranga Raju, K.G. (1980), Aggradation in streams due to overloading, *Journal of the Hydraulics Division, ASCE*, 106(HY1), pp.117-132.

Struiksma, N. (1983), Results of movable bed experiments in the DHL curved flume, Report on Experimental Investigation, *TOW Report R657-XVIII/M1771*, Delft Hydraulics Laboratory, The Netherlands.

Struiksma, N., Olesen, K.W., Flokstra, C. and De Vriend, H.J. (1985), Bed deformation in curved alluvial channels, *Journal of Hydraulic Research*, 23(1), pp.57-79.

Wu, W. (2004), Depth-averaged two-dimensional numerical modeling of unsteady flow and nonuniform sediment transport in open channels, *Journal of Hydraulic Engineering*, *ASCE*, 130(10), pp.1013-1024.