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Comparison Between Analytical Water Wave Theories and Measurements Made in Mexican Ports

COMPARISON BETWEEN ANALYTICAL WATER WAVE THEORIES AND MEASUREMENTS MADE IN MEXICAN PORTS

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ABSTRACT

This paper presents four computer programs to calculate the physical characteristics of the progressive wave using linear, Stokes second order, Stokes third order, and cnoidal of Keulegan and Patterson theories: profile of water surface, celerity, wave length, horizontal and vertical orbital velocity, horizontal and vertical orbital acceleration, horizontal and vertical displacements, and under water surface pressure; in some theories energy, potency, group velocity and mass transport. An equation was developed to relate directly the full elliptical integrals of first and second class with their modulus, necessary for the cnoidal computer program. Results after application these four computer programs validate the application region proposal by Littman and Struik and the experimental works of Le Méhauté, Divoky and Lyn in the Tetra Tech wave tank, for waves with periods of 1 to 12 seconds and $0.244 \text{ m/s}^2 \geq d/T^2 \geq 0.015 \text{ m/s}^2$. Importance of this four computer programs establish oneself not necessary use graphical solutions, for example Wiegel's curves for the determination of the physical characteristics progressive wave and were compared results between different theoretical wave models and direct measurements made in some Mexican ports.

Keywords: water wave theories, Mexican ports, full elliptical integrals

1. INTRODUCTION

Since the mathematical point of view, there is no a general solution to describe the real behaviour of waves; even in the simplest cases, certain approximations must be realised. The mathematical approximations to study the movements of waves are as varied as the same physical aspects.

Actually, the math treatments which pretend to analyse the waves, use the resources of their physical and math behaviour with linear and not linear problems; being the main difficulty the fact that one of the frontiers, called free surface, is one of the questions. The waves movements are as varied and complex that any attempt for a unique classification, may lead to errors.

Solution methods

Depending upon the problem itself and the rank of values of the parameters H/L, H/d and L/d, three math approximations can be used for the solution.

Linear approach (linearization). The simplest case about wave's theories is of course the linear theory, in which the terms of convective inertia are completely ignored. These theories are valid only when H/L, H/d and L/d are small, i.e. for short fullness and short length waves within deep waters. The linear equations are held to math solutions that wave's linear theories use for an extreme variety of waves movements.

Taylor series. Some solutions as a series of potencies in function of a small parameter compared with others can also be found. This small quantity is H/L for deep waters or H/d for low waters. In the first case (development within terms of H/L or Stokes waves), the first term of the series is a solution regarding to the linear equations. In the second case, the first term of the series is already a solution to non-linear equations, being these the conidial waves. The calculus of the series consecutive terms is pretty laborious; hence these methods are used in a very short number of cases.

Numeric methods. It may happen that there is no as solution a continuous and regular profile of condition, in such a case a numerical solution where the differentials are substituted by finite differences is frequently used. This happens for large values of H/d and L/d, where the non-linear terms such as [ρu (∂u/∂x)] are relatively large comparing with local inertia [ρ(∂u/∂t)]._Of course, in this case a numerical method can be used to solve a linear equations system; e.g. the relaxation method is used to study the movement of small waves in a pool. Also an analytical solution of a non-linear equations system may be found in some particular cases. Nevertheless, it must keep in mind that these three methods of solution and the rank of application given in each case show rather a tendency than a general rule.

Besides the three methods described which focus onto a complete deterministic solution regarding wave's problem, the description of the sea status generally involves the use of random functions. The math operations which are used in this case (like the harmonic analysis) generally imply that waves follows linear laws, necessary requirements in order to accept that the superimpose principle is valid. Therefore, such approximations lose their validity to describe this sea status in reduced waters (large values for H/d and L/d) and in the waves breakpoint.

2. DEVELOPMENT

Water wave theories aspects

The problem over the wave surface in a eulerian system of co-ordinates implies three questions ($\eta, p, \vec{v}$). It starts with the supposition that a periodic and progressive wave which travels over the horizontal sea bed is characterised by a profile of stationary condition,

$$\eta, \vec{v}, p = f(x - Ct) \tag{1}$$

Where C is a constant equal to the wave speed (in fact, it has noticed that under certain conditions, the wave profile in shallow waters happens to be unsteady, asymmetrical and it degenerates in a succession of smaller waves). The problem thus focuses in solving a system to satisfy the equations of continuity, movement amount and certain frontier conditions. Nevertheless, these are not enough to solve the non-linear problem two more considerations are quite necessary. This leads to the discussion of the problem about the rotation and the mass transport, since they are related.

In the first place, it is observed that a stationary flow of arbitrary speed distribution can always be superimposed to a given waves movement, so inducing to any distribution of arbitrary mass transport speed (Dubreuil-Jacotin, 1934). The considerations over the calculus of wave's movement are inherent to the conditions which prevail upon the hypothesis used in the mass transport calculus.

The wave movement can be obtained considering there is no mass transport at all. These are the closed orbits theories, like the exact solution of Gerstner (1809) in deep waters and the potency series solution of Boussinesq, Kravtchenko and Dauber (1957) for shallow waters. As a result of this hypothesis, the movement is rotational and the vortex function presents in opposite direction to the particle rotation; i.e. in opposite direction to the one that should be expected physically under the influence of a shear effort because of the wind blowing into the waves direction.

The wave movement can also be accepted as non-rotational, in such case a mass transport distribution is presented as result of the non-linear function (Wehausen and Laitone, 1960). These are the waves Stokesians theories, which include those from Stokes (1847), Levi-Civita (1925), Struik (1926) and Nekrassov (1951). Even though there is a mass transport distribution, which is function of the vertical co-ordinate, the integration constant is frequently determined assuming that the average mass transportation is void because of the continuity condition, i.e. a stationery flow is superimposed so the mass transport average speed equals zero.

Longuet-Higgins (1953) demonstrated the value of the viscous forces in the bottom in order to explain the fact that the mass transport on the sea bed is always into the wave's direction. In addition to the first hypothesis about the rotation or mass transport, another extra condition is required. E.g. for monochromatic progressive waves, a status solution is intended to establish, as to the potential function could be $\varphi = f(x - Ct)$, where C is a constant equal to the wave speed, in such a case the solution is unique. Even though the stationary status solutions are alike, C is undetermined and to obtain it, another condition is required. The average horizontal speed during a wave period in a given situation may be set as zero, but the mass transport is then the minimum.

Generally another condition is preferred; it consists in assuming that the quantity of average movement along the wave length is zero by the addition of some stationary movement; in such a case another expression for C is obtained, which consists in a different mass transportation. This way it is clear that the calculus of wave's theories is held to certain arbitrariness factors in consequence to various hypotheses used to lead to different values for C.

Description of programs

LINEAR THEORY. The most elemental theory which describes the elements of the short fullness waves was developed by Airy in 1845.Its basic meaning consists in its easy application and in the great wave region that comprises. Mathematically, the linear theory, also known as Airy's theory, may be considered as a first approach of some complete theoretical description regarding wave's behaviour.

Program description. The Turbo-Pascal, version 6.0 programming language was used (likewise the rest of the programs). This program requires the following data to be introduced: period, depth, height and phase angle of waves, as the depth "z" of the particle to be analysed. In the first part of the program, the Newton-Raphson method is used to determine the wave length. In its second part, the thirteen physical characteristics of waves, providing of equations with hyperbolic functions, are calculated: profile of water surface, celerity, wave length, group velocity, horizontal and vertical orbital velocity, horizontal and vertical orbital acceleration, horizontal and vertical displacements, under water surface pressure, energy and potency.

STOKES SECOND ORDER THEORY. Some simplifications made into the linear theory, reduce the precision which pretends to represent waves with math models; other theories have been developed, their objective is to reduce this difference between the reality and its math representation. They can receive the name as superior order theories. A group of these sorts of theories, assigned to finite fullness waves, receive the generic name of Stokes Theories, which go from the second to the fifth order of nearness. When finite fullness waves are presented, i.e. if the wave's fullness is big comparing with its length, the non-linear terms about the basic equations must be considered. Stokes, in 1880, studied this kind of waves, which has got various levels of nearness, depending upon the extension of the math terms introduced in the progress of the expressions. The equations so obtained by this theory present a lot of terms, coefficients, etc. and these are the reasons why different researchers have not always achieved the same result. The demonstration of the series convergence used during the obtaining of the equations, was realised by Levi-Civita in 1925, for infinite depth waters and by Struik, in 1926, for finite depth waters, with several corrections made by Wolf in 1944. The math model used by Stokes and by many other researchers afterwards consisted in extending the speed potential close to the level of quiet waters, getting a non-linear surface condition for the potential in the survey of such level, which is integrated by infinite series containing partial derivatives of the potential. The deduction of the expressions starts from the three equations of Navier-Stokes and Bernoulli equation (already indicated in the Linear Theory) adding the speed factor, which is now considered in this theory. Combining the former equations with the given condition, the following relation can be obtained:

$$g\frac{\partial \phi}{\partial z}+\frac{\partial^2 \phi}{\partial t^2}+2\hat{\nabla}\phi \bullet \hat{\nabla}\frac{\partial \phi}{\partial t}+\frac{1}{2}\hat{\nabla}\left(\hat{\nabla}\phi \bullet \hat{\nabla}\phi\right)=0 \qquad (2)$$

With the purpose to obtain a potential ϕ to satisfy this expression, some developments in Taylor Series are used, until the potential ϕ of superior order equals the original potential plus a series of terms of correction. Under this deductive scheme the equations that permit to establish the physical characteristics of the finite fullness waves for progressive waves have been obtained with the Stokes second order theory.

Program description. The Turbo-Pascal, version 6.0 programming language was used (likewise the rest of the programs). This program requires the following data to be introduced: period, depth, height and phase angle of waves, as the depth "z" of the particle to be analysed. The eleven physical characteristics of waves, providing of equations with hyperbolic functions, are calculated: profile of water surface, celerity, wave length, horizontal and vertical orbital velocity, horizontal and vertical orbital acceleration, horizontal and vertical displacements, under water surface pressure, and mass transport.

STOKES THIRD ORDER THEORY. Stokes extended his works, considering terms of third order in the used series; in 1959, Skelbreia reused these works, giving them a distinct presentation.

Program description. The Turbo-Pascal, version 6.0 programming language was used. This program requires the following data to be introduced: period, depth, height and phase angle of waves, as the depth "z" of the particle to be analysed. The nine physical characteristics of waves, providing of equations with hyperbolic functions, are calculated: profile of water surface, celerity, wave length, horizontal and vertical orbital velocity, horizontal and vertical orbital acceleration, and horizontal and vertical displacements.

CNOIDAL THEORY. Since 1877, Boussinesq organised the first previews about wave's studies in shallow waters and with finite fullness. However, Korteweg and Devries, in 1895, were who first developed the cnoidals wave's theory. The term "cnoidal" is applied due to the free surface of waves is expressed through the elliptic jacobians functions. This theory proposes that the shape of the free surface of waves and the other characteristics are given by the curves of the three elliptic jacobians functions. There is no just one cnoidal theory; the bibliography presents several alike which are not identical. Like stokesians theories, since all the cnoidals representations are unfinished series, the order of nearness is important because certain factors equal zero in the low order theory. There are two kinds of cnoidal theories; the oldest is intuitive in nature while the newest theories are integers and more severe, all being non-rotational ones. The intuitive elementary theory is by Korteweg and de Vries (1895); the first and second terms of the series are deducted but one scheme for the extension to higher order terms is not presented, being unique they found terms. The severe theories are by Keulegan and Patterson (1940) and others; all of them are based upon a disturbance expansion developed by Friedrichs in 1948. The work of Keller confirms the results of Korteweg and de Vries, while Laitone and Chappelear studies give a higher order term. Unfortunately, even when the rigour predominates, the actual theories are different after the third term. The theories by Keulegan and Patterson are not mathematically consistent, since several terms of second order are unvalued while the third order ones are included, regardless it can be the most physically attractive. Using the following equations system, proposed by Keulegan and Patterson,

$$m = k^2 = \frac{H/d}{\left(2L + 1 - \frac{Z_t}{d}\right)} \tag{3}$$

$$\left(2L + 1 - \frac{Z_t}{d}\right)E(k) = \left(2L + 2 - \frac{H}{d}\right)K(k) \tag{4}$$

$$\frac{Z_t}{d} = \frac{H}{d[kK(k)]^2}\{K(k)[K(k) - E(k)]\} + 1 - \frac{H}{d} \tag{5}$$

was obtained the following equation, who related directly the full elliptical integrals of first K (k) and second E (k) class with their modulus k,

$$k^2 = m = \frac{E(k)\frac{H}{d}}{\left\{\left\{\frac{H}{d}\left[\frac{K(k)[K(k)-E(k)]}{[kK(k)]^2} + \frac{1}{m} - 1\right]\right\} + 2 - \frac{H}{d}\right\}K(k)} \tag{6}$$

Program description. The Turbo-Pascal, version 6.0 programming language was used (it was necessary to deduct and use the above equation). This program requires the following data to be introduced: depth, height and phase angle of waves, as the depth “z” of the particle to be analysed. In the first part of the program, the Newton-Rap son method is used to determine the wave length. In its second part, the nine physical characteristics of waves, providing of equations with elliptic jacobians functions, are calculated: profile of water surface, celerity, wave length, horizontal and vertical orbital velocity, horizontal and vertical orbital acceleration, under water surface pressure and period.

3. CONCLUSIONS

Validation of programs

After processing a big number of run-downs with the four programs already commented, it is possible to resume some outstanding aspects for each exposed theories. Is important indicates not include the 43 used equations relatives to 4 wave theories because require a big space.

LINEAR THEORY. Through the processes in ‘x’, ‘z’, it was observed that some closed circular trajectories were generated in deep waters and closed elliptic trajectories in middle and low waters, which proves the theoretical settings. The pressure under the free surface of the water is incremented in non-linear way. As the particles speeds being analysed as their speed-up, were going down from the free surface of the water to the bottom of the sea.

STOKES SECOND ORDER THEORY. The free surface presents an uneven shape, being detected by drawing these values, where in general, sharper “peaks” and flatten valleys, along with a central horizontal part are observed. Analysing the trajectory of the particles resulted very obvious that the starting and ending points do not coincide, i.e. a moving site to the front was presented, thus indicating the presence of open trajectories in this theory. Comparing the speed of the particles presented by both theories, it was noticed a difference among the final values from 10 to 30%, in agreement to the waves zone where they were calculated. From the comparative studies of the given values by both theories concerning the speed-up of the particles, a big difference was found, from 30 to 100%, depending upon the component and the analysed wave’s zone.

STOKES THIRD ORDER THEORY. Wiegel (1950), Morrison (1951), Suquet and Wallet (1953) and Savage (1954) have measured experimentally the waves speed for certain ranks and have compared with the Stokes third order theory and Linear theory. In general, the obtained figures show that the long fullness waves travel faster than the short fullness ones. The work of Suquet and Wallet do not include data for d/L <0.2, though the work of Morrison does.

CNOIDAL THEORY. This theory is appropriated in progressive periodical waves in waters which depth is under 1/10 of their length. Littman proposes the region of validity for the cnoidal theory shown in next figure 1 in function of the parameters H/d and C^2/gd, where a zone of application for Stokes (Struik) theories is shown and the limit for the lonely wave theory. Whit the computer programs development in this work, this region was validated for both, cnoidal and stokes theories.

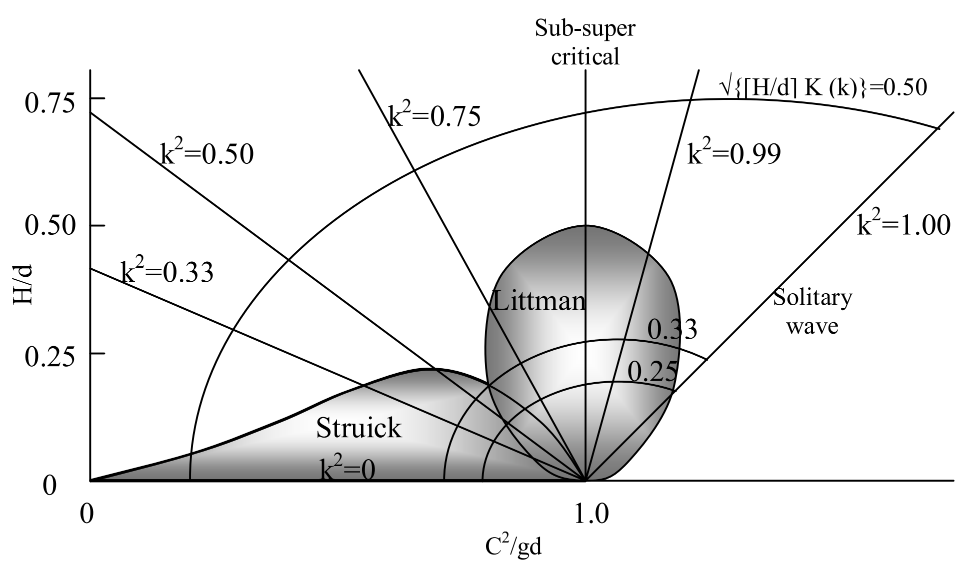

Figure 1 Approximates regions of validity for existents profiles for cnoidal waves (Littman) and Stokes waves (Struik).

Le Méhauté, Divoky and Lin developed measures upon waves speed in the waves tank

Tetra Tech, with waves ranks $0.244\ m/s^2 \geq d/T^2 \geq 0.015\ m/s^2$ of 32 meters of length, square across section of 1.219 meters and waves generator with a speed control in a rank from 1 to 12 seconds of period. These results were compared with the calculus of the programs proposed in this work for the free surface profile, speeds and accelerations of the particles and it was observed that although none of these theories adjust precisely, the most accurate was the Keulegan and Patterson cnoidal theory.

Comparison with measurements on Mexican ports

Measurements of the basic characteristics of the water wave throughout 12 months were made in the following Mexican ports: Acapulco, Lázaro Cardenas, Manzanillo, Salina Cruz and Zihuatanejo, obtaining itself that the theories of the water waves that approached the measurements of fields more are the following ones: Stokes third order theory with a 48%, cnoidal theory with a 29%, Stokes second order theory with a 15% and the linear theory with a 8%.

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