Sasikumar, K.

An Analytical Solution for Optimal Resource Allocation for Fuzzy Demands with Piecewise Linear Membership Functions

AN ANALYTICAL SOLUTION FOR OPTIMAL RESOURCE ALLOCATION FOR FUZZY DEMANDS WITH PIECEWISE LINEAR MEMBERSHIP FUNCTIONS

Sasikumar K.[1]

Abstract: An analytical solution is derived for the optimal resource allocation problem in which the users specify the requirements for the resource in an imprecise manner. The demands of the users are expressed by means of fuzzy sets with piecewise linear membership functions. The maximum requirement, the maximum fully acceptable shortage in the maximum requirement, and the minimum acceptable allocation for any user define the nature of the membership function of the fuzzy set of demand for that user. This membership function is viewed as a representation of the variation of satisfaction level of the user against any possible allocation. The objective of the optimal allocation problem may then be interpreted as the maximization of the minimum satisfaction level among all the users in the system subject to the constraints on the total resource available for allocation and the maximum requirements of the users. The resulting optimization problem is a nonlinear one and analytical expressions are derived for the optimal solution by considering the different sub-problems based on the relative magnitudes of the total resource available and the various requirement levels of each user.

Keywords: Resource allocation, Optimization, Fuzzy demand, Satisfaction level, Analytical optimal solution

INTRODUCTION

Resource allocation problems are very frequently encountered in many fields of science, engineering, management and economics. When the resource is not sufficient to meet the demands of all users it has to be optimally allocated to the users. An optimal allocation of the resource aims at maximizing (or minimizing) a measure of performance which depends on the various activity levels of the users. A variety of optimal allocation problems have been addressed in literature (e.g. Ibaraki and Katoh, 1988). In the present study, an allocation problem is formulated with users expressing their demands by means of fuzzy sets and analytical solution is derived for that optimal allocation problem.

ALLOCATION PROBLEM FORMULATION

The fuzzy demand of a user i is represented as a fuzzy set X_i with monotonically non-decreasing piecewise linear membership function. The membership function is shown in Fig. 1. This membership function can be mathematically expressed as follows:

1 Assistant Professor, Department of Civil Engineering, National Institute of Technology Calicut, P. O. NIT Campus, Calicut, Kerala, India, PIN – 673601, E-mail: sasikumar@nitc.ac.in

$$\mu_{X_i}(x_i) = \begin{cases} 0 & 0 \le x_i \le x_i^L \\ \dfrac{x_i - x_i^L}{x_i^U - x_i^L} & x_i^L \le x_i \le x_i^U \\ 1 & x_i^U \le x_i \le R_i \end{cases} \qquad (1)$$

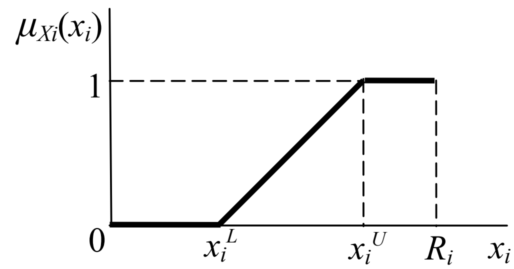

Fig. 1 Membership Function for the Fuzzy Demand

Here, R_i is the maximum requirement of resource for user *i*, $R_i - x_i^U$ maximum shortage of resource fully acceptable to user *i*, x_i^L is the minimum acceptable requirement for user *i*. The fuzzy set X_i is the set of allocations acceptable to the user. Any allocation x_i in the range 0 to R_i has a degree of membership $\mu_{X_i}(x_i)$ in the fuzzy set of *acceptable allocations* X_i. Here the degree of membership $\mu_{X_i}(x_i)$ may be interpreted as a measure of satisfaction the user *i* derives from an allocation x_i. Since $R_i - x_i^U$ is the maximum shortage of resource fully acceptable to user *i,* any allocation in the range x_i^U to R_i is assigned a degree of membership 1 indicating full satisfaction level. On the other hand, since x_i^L is the minimum acceptable requirement for user *i*, all allocations in the range 0 to x_i^L are assigned a degree of membership 0 indicating zero satisfaction level. Further, a linear variation of the degree of membership is assumed in the present study for any allocation in the range x_i^L to x_i^U. That is, the satisfaction level of user *i* increases linearly from 0 to 1 as the allocation x_i increases from x_i^L to x_i^U. In other words, any shortage $R_i - x_i$ ($x_i^L < x_i < x_i^U$) is also acceptable to user *i* with a lesser degree of satisfaction than that of the fully acceptable shortage $R_i - x_i^U$ .Viewing the degree of membership as satisfaction level shall help us to get better insight into the allocation process. The user does not specify the requirement in a precise manner but specifies it through a range over which a preference structure is assigned by means of the membership function. The allocation problem can now be stated as follows:

A quantity $Q \in R^+$ of a resource is to be allocated to a set of *n* users whose demands are expressed by means of fuzzy sets X_i with monotonically non-decreasing piecewise linear membership functions as given by Eq.(1) such that the minimum satisfaction level among all the users is maximum.

The fuzzy decision *D* regarding the allocation may be considered as a confluence of the fuzzy sets of demand X_i (Bellman and Zadeh, 1970) and mathematically *D* is expressed as the intersection of the fuzzy sets X_i as follows:

$$D = \bigcap_{i=1}^{n} X_i \qquad (2)$$

The membership function $\mu_D(\mathbf{X})$ for the fuzzy decision D may be expressed using the definition of membership function for the intersection of fuzzy sets as follows:

$$\mu_D(\mathbf{X}) = \min_i \left\{ \mu_{X_i}(x_i) \right\} \qquad (3)$$

where $\mathbf{X} = \{x_1, x_2, \cdots, x_n\}$ denotes the vector of admissible allocations satisfying the constraints on the total resource availability and the resource requirement of each user. An optimal resource allocation vector $\mathbf{X}^*$ is chosen as that particular allocation $\mathbf{X}$ which maximizes $\mu_D(\mathbf{X})$ and may be mathematically expressed as follows:

$$\mu_D(\mathbf{X}^*) = \max_{\mathbf{X}} \left\{ \mu_D(\mathbf{X}) \right\} \qquad (4)$$

The optimal fuzzy decision problem was formulated as an optimization problem by Zimmermann (1978, 1985) and many researchers have used it in diversified fields of resource allocations (e.g., Feng, 1983; Kindler, 1992; Rao, 1993; Sakawa 1995; Sasikumar and Mujumdar, 1998). An equivalent formulation considering the membership function as a representation for variation of satisfaction level is presented next.

Let λ denote the minimum satisfaction level among all users corresponding to an allocation vector **X** (i.e., $\lambda = \mu_D(\mathbf{X})$ from Eq.(3) **).** Since all satisfaction levels shall be greater than or equal to the minimum satisfaction level λ, it is possible to express this using the following inequality:

$$\mu_{X_i}(x_i) \geq \lambda \qquad \forall i \qquad (5)$$

The resource allocation problem may now be stated as an optimization problem as follows:

$$\text{Maximize} \quad \lambda$$

$$\text{subject to}$$

$$\mu_{X_i}(x_i) \geq \lambda \qquad \forall i \qquad (6)$$

$$\sum_{i=1}^{n} x_i \leq Q \qquad (7)$$

$$0 \leq x_i \leq R_i \qquad \forall i \qquad (8)$$

$$0 \leq \lambda \leq 1 \qquad (9)$$

The Constraint (6) defines λ to be the minimum satisfaction level, Constraint (7) implies that the total allocation shall be less than or equal to the resource available, Constraint (8) denotes the lower and upper bounds of allocation for each user, and Constraint (9) represents the lower and upper bounds of the minimum satisfaction level λ. The expression for $\mu_{X_i}(x_i)$ as given in Eq. (1) may be re-written as a single expression using the absolute values of variables (Piskunov, 1974) as follows:

$$\mu_{X_i}(x_i) = \frac{1}{2}\left[1+\frac{1}{2l_i}\left(x_i - x_i^L + \left|x_i - x_i^L\right|\right) - \left|1-\frac{1}{2l_i}\left(x_i - x_i^L + \left|x_i - x_i^L\right|\right)\right|\right] \quad (10)$$

where $l_i = x_i^U - x_i^L$. The optimization problem may now be restated as follows:

Maximize λ

subject to

$$\frac{1}{2}\left[1+\frac{1}{2l_i}\left(x_i - x_i^L + \left|x_i - x_i^L\right|\right) - \left|1-\frac{1}{2l_i}\left(x_i - x_i^L + \left|x_i - x_i^L\right|\right)\right|\right] \geq \lambda \quad \forall i \quad (11)$$

$$\sum_{i=1}^{n} x_i \leq Q \quad (12)$$

$$0 \leq x_i \leq R_i \quad \forall i \quad (13)$$

$$0 \leq \lambda \leq 1 \quad (14)$$

The Constraint (11) is a nonlinear constraint and hence the optimization problem is a nonlinear optimization problem that may demand complicated solution procedures. But it is interesting to note that the above optimization problem may be solved in a much easier way and even analytical expressions may be derived for the optimal solution for this problem. The remaining Sections focus on the derivation of the analytical optimal solution for the resource allocation problem.

SOLUTION OF THE ALLOCATION PROBLEM

As a first step towards deriving the analytical solution, we may subdivide the problem into different cases depending on the relative magnitude of *Q* and the total resource requirement of all users. The different cases are as follows:

Case (i) : $\sum x_i^L < Q < \sum x_i^U$. Case (ii) : $Q = \sum x_i^L$. Case (iii) : $0 \leq Q < \sum x_i^L$
Case (iv) : $Q = \sum x_i^U$. Case (v) : $\sum x_i^U < Q < \sum R_i$. Case (vi) : $Q = \sum R_i$
Case (vii) : $Q > \sum R_i$

Except the first case all the remaining cases may be solved trivially. The Case (i) is considered first.

Case (i) $\sum x_i^L < Q < \sum x_i^U$

In this case only the linear portion of the membership function between $x_i = x_i^L$ and $x_i = x_i^U$ is of interest. Hence the optimization problem may be expressed as follows:

Maximize λ

Subject to

$$\frac{x_i - x_i^L}{x_i^U - x_i^L} \geq \lambda \quad \forall\, i \tag{15}$$

$$\sum_{i=1}^{n} x_i \leq Q \tag{16}$$

$$x_i^L \leq x_i \leq x_i^U \quad \forall\, i \tag{17}$$

$$0 \leq \lambda \leq 1 \tag{18}$$

Any linear programming routine may be used to solve this problem. However, it is also possible to obtain analytical optimal solution for this problem by observing the following facts. Let us consider an allocation scenario in which all the users are at the same satisfaction level, say, λ^* and the resource is fully allocated ($\because \sum x_i^L < Q < \sum x_i^U$). It may be noted that λ^* also corresponds to the minimum satisfaction level. In order to bring any user k to a higher satisfaction level than λ^*, the resource has to be taken from at least one of the remaining users, say user j, and allocate it to user k, as the total resource Q is already allocated. Even though this additional allocation increases the satisfaction level of user k, the satisfaction level of user j now reduces to λ_1 from λ^*, and the new minimum satisfaction level now becomes λ_1 which is smaller than the previous minimum satisfaction level λ^*. If Δx_j is the resource taken from user j to allot to user k, it can be inferred from the linear portion the membership function of user j that the decrease ($\lambda^* - \lambda_1$) in satisfaction of user j is $\Delta x_j \big/ (x_j^U - x_j^L)$. Hence it may be concluded that when the allocation scenario is such that all the users are at the same satisfaction level, the minimum satisfaction level attains its maximum value compared to that of any other possible allocation scenarios. This is an important observation in deriving the analytical expressions for the optimum allocation. The analytical expressions for the optimum allocation may be derived as follows.

Consider two users i and i+1. Let the superscript ' * ' denote the optimum solution. Since the satisfaction levels of all users shall be the same at the optimum (that is, maximized minimum (max-min)) satisfaction level we may write the following equality.

$$\frac{x_i^* - x_i^L}{l_i} = \frac{x_{i+1}^* - x_{i+1}^L}{l_{i+1}} \qquad i = 1 \text{ to } (n-1) \tag{19}$$

Denoting $a_i = 1/l_i$, and $b_i = -x_i^L / l_i$, Eq. (19) may be re-written as

$$a_i\, x_i^* + b_i = a_{i+1}\, x_{i+1}^* + b_{i+1} \qquad i = 1 \text{ to } (n-1) \tag{20}$$

$$a_i\, x_i^* - a_{i+1}\, x_{i+1}^* = b_{i+1} - b_i \qquad i = 1 \text{ to } (n-1) \tag{21}$$

Eq. (21) when applied to all the successive pairs of users starting from users 1 and 2 results in

(n–1) independent linear equations with n unknowns. Since $\sum x_i^L < Q < \sum x_i^U$ all the resource Q shall be allocated and the following equality holds.

$$\sum_{i=1}^{n} x_i^* = Q \qquad (22)$$

Eqs. (21) and (22) provide n independent linear equations with n unknowns and may be expressed in matrix form as follows:

$$\begin{bmatrix} a_1 & -a_2 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & a_2 & -a_3 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & a_3 & -a_4 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & a_{n-1} & a_n \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ \vdots \\ x_{n-1}^* \\ x_n^* \end{Bmatrix} = \begin{Bmatrix} b_2 - b_1 \\ b_3 - b_2 \\ b_4 - b_3 \\ \vdots \\ b_n - b_{n-1} \\ Q \end{Bmatrix} \qquad (23)$$

Eq. (23) may be solved for the vector of unknown resource allocations. But a simple transformation of the variables may be used to obtain the solutions directly. New variables y_i^* are defined as follows:

$$y_i^* = x_i^* - x_i^L \qquad (24)$$

Eq. (19) may now be rewritten as

$$\frac{y_i^*}{l_i} = \frac{y_{i+1}^*}{l_{i+1}} \qquad (25)$$

Introducing the variable $c_{i,i+1} = l_i / l_{i+1}$, Eq. (25) may be expressed as

$$y_i^* = c_{i,i+1}\, y_{i+1}^* \qquad (26)$$

Expanding Eq. (26) for i = 1 to (n – 1) we get the following set of equations:

$$\left.\begin{aligned} y_1^* &= c_{12}\, y_2^* \\ y_2^* &= c_{23}\, y_3^* \\ &\cdots\cdots\cdots \\ y_{n-1}^* &= c_{n-1,n}\, y_n^* \end{aligned}\right\} \qquad (27)$$

Using Eqs. (27), all the variables from y_2^* through y_n^* may be expressed in terms of y_1^* as follows:

$$\left.\begin{aligned} y_2^* &= y_1^* / c_{12} \\ y_3^* &= y_2^* / c_{23} = y_1^* / (c_{12}\, c_{23}) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n^* &= y_{n-1}^* / c_{n-1,n} = y_1^* / (c_{12}\, c_{23}\, c_{34} \cdots\cdots c_{n-1,n}) \end{aligned}\right\} \qquad (28)$$

Eq. (22) may be rewritten in terms of y_i^* using Eq. (24) as follows:

$$\sum_{i=1}^{n} x_i^* \quad = \quad \sum_{i=1}^{n} (y_i^* + x_i^L) \quad = \quad Q \qquad (29)$$

$$\therefore \quad \sum_{i=1}^{n} y_i^* \quad = \quad Q - \sum_{i=1}^{n} x_i^L \qquad (30)$$

Substituting for y_2^* through y_n^* in terms of y_1^* using Eqs. (28) in Eq. (30) and solving for y_1^* gives an analytical solution for y_1^* as follows.

$$y_1^* \quad = \quad \frac{Q - \sum_{i=1}^{n} x_i^L}{\left(1 + \frac{1}{c_{12}} + \frac{1}{c_{12}\ c_{23}} + \frac{1}{c_{12}\ c_{23}\ c_{34}} + \cdots\cdots + \frac{1}{c_{12}\ c_{23}\ c_{34} \cdots\cdots\ c_{n-1,n}} \right)} \qquad (31)$$

$$\text{i.e.,} \qquad y_1^* \quad = \quad \frac{Q - \sum_{i=1}^{n} x_i^L}{\left(1 + \sum_{i=2}^{n} \left(\prod_{k=2}^{i} \frac{1}{c_{k-1,k}} \right) \right)} \qquad (32)$$

The remaining unknowns y_2^* through y_n^* may be obtained using Eq. (28). Now the unknown optimal resource allocations may be estimated from Eq. (24) as follows:

$$x_i^* \quad = \quad y_i^* + x_i^L \qquad \forall\, i \qquad (33)$$

It may be noted that at the optimum point all users shall be at the same satisfaction level and its value (that is, the max-min satisfaction level) λ^* may be estimated for any *i* using the following expression:

$$\lambda^* \quad = \quad \frac{y_i^*}{l_i} \qquad (34)$$

Thus, the Eqs. (28), (32), (33), and (34) together provide the analytical expression for optimal solution for the optimization problem posed under Case (i). When the membership functions of the fuzzy demands are identical for all the users (in other words, all the users have identical preference structure for their demands), it may be shown using the analytical solution (Eqs. (32) and (33)) that each user gets the same allocation *Q*/*n* as follows:

$$y_1^* \quad = \quad \frac{Q - \sum_{i=1}^{n} x_i^L}{\left(1 + \sum_{i=2}^{n} \left(\prod_{k=2}^{i} \frac{1}{c_{k-1,k}} \right) \right)} \quad = \quad \frac{Q - n\, x_1^L}{\left(1 + (n-1) \right)} \quad = \quad \frac{Q}{n} - x_1^L \quad = \quad \frac{Q}{n} - (x_1^* - y_1^*)$$

$$\therefore \quad x_1^* \quad = \quad \frac{Q}{n}$$

From Eqs. (28) the following relation may be obtained:

$$x_i^* \quad = \quad \frac{Q}{n} \qquad \forall\, i$$

It is also intuitively obvious in this case that the allocations shall be the same and equal to *Q*/*n* because there is no justification for any bias in allocation as long as identical preference structure of fuzzy demand exists among the users.

Illustrative Example for Case (i) : Table 1 gives the data for resource allocation problem with 25 users and a total resource Q = 1000 units. It may be noted that (Σx_i^L=375) < (Q=1000) < (Σx_i^U=1425). Also shown in the Table are the values of y^*, optimal resource allocations x_i^*, satisfaction levels μ_i at optimum point, and the max-min satisfaction level λ^*. It is easy to verify that any other allocation shall result in a minimum satisfaction level which is less than the max-min satisfaction level λ^* = 0.5952.

Table 1 Data and Optimal Solutions for the Resource Allocation Problem

User					Optimal Solutions			
i	x_i^L	x_i^U	l_i	$c_{i,i+1}$	y_i^*	$x_i^* = y_i^* + x_i^L$	μ_i	λ^*
1	5	10	5	-	2.976	7.976	0.5952	0.5952
2	5	15	10	1/2	5.952	10.952	0.5952	
3	5	20	15	2/3	8.928	13.928	0.5952	
4	5	25	20	3/4	11.905	16.905	0.5952	
5	5	30	25	4/5	14.881	19.881	0.5952	
6	10	20	10	5/2	5.952	15.952	0.5952	
7	10	30	20	1/2	11.905	21.905	0.5952	
8	10	40	30	2/3	17.857	27.857	0.5952	
9	10	50	40	3/4	23.810	33.810	0.5952	
10	10	60	50	4/5	29.762	39.762	0.5952	
11	15	25	10	5/1	5.952	20.952	0.5952	
12	15	35	20	1/2	11.905	26.905	0.5952	
13	15	45	30	2/3	17.857	32.857	0.5952	
14	15	55	40	3/4	23.810	38.810	0.5952	
15	15	65	50	4/5	29.762	44.762	0.5952	
16	20	40	20	5/2	11.905	31.905	0.5952	
17	20	60	40	1/2	23.810	43.810	0.5952	
18	20	80	60	2/3	35.714	55.714	0.5952	
19	20	100	80	3/4	47.619	67.619	0.5952	
20	20	120	100	4/5	59.524	79.524	0.5952	
21	25	50	25	4/1	14.881	39.881	0.5952	
22	25	75	50	1/2	29.762	54.762	0.5952	
23	25	100	75	2/3	44.643	69.643	0.5952	
24	25	125	100	3/4	59.524	84.524	0.5952	
25	25	150	125	4/5	74.405	99.405	0.5952	
Sum	**375**	**1425**				**1000**		

Case (ii) : $Q = \sum x_i^L$: Under this case, multiple solutions exist for the optimal allocations with λ^* = 0. One possible optimal solution as given by the method for Case (i) is $y_i^* = 0,\ x_i^* = x_i^L \ \ \forall i$ and λ^* = 0. The general optimal solution may be expressed in parametric form as $x_i^* = \alpha_i Q$ where $\sum \alpha_i = 1$, and $0 \le \alpha_i \le 1$.

Case (iii) : $0 \le Q < \sum x_i^L$: In this case, the resource Q is not sufficient to meet the total minimum acceptable demand $\sum x_i^L$ of all users. There shall be at least one user whose satisfaction level is zero implying λ^* = 0. Multiple solutions may be expressed by means of parametric form as in Case (ii), that is, as $x_i^* = \alpha_i Q$ where $\sum \alpha_i = 1$, and $0 \le \alpha_i \le 1$. One possible optimal allocation may be given as follows:

$$x_i^* = \frac{x_i^L}{\sum x_i^L} Q = \alpha_i Q \qquad \forall i \qquad (34)$$

Here the allocation to each user i is proportional to the minimum acceptable level x_i^L set by the user i. It may be noted that $\sum x_i^* = Q$.

Case (iv) : $Q = \sum x_i^U$: In this case there exists a unique optimal solution with λ^* = 1. The optimal solution as obtained from the Case (i) is $y_i^* = l_i = x_i^U - x_i^L,\ \ x_i^* = x_i^U \ \ \forall i$. Here the solution is unique because any allocation more than x_i^U to any user i shall result in lesser allocation than x_j^U to at least one other user j resulting in a minimum satisfaction level less than the present optimal level λ^* = 1.

Case (v) : $\sum x_i^U < Q < \sum R_i$: In this case the optimum satisfaction level λ^* is 1, and the optimum allocations may be expressed in parametric form as follows with restrictions on α_i as in Case (ii):

$$x_i^* = x_i^U + \alpha_i (Q - \sum x_i^U) \qquad \forall i \qquad (35)$$

Case (vi) : $Q = \sum R_i$: Here the resource is just sufficient to meet the requirements of all users and the optimum satisfaction level λ^* is 1, and the optimum allocation is $x_i^* = R_i \ \forall i$.

Case (vii) : $Q > \sum R_i$: In the case, the resource available is more than the total requirement. The optimal solution remains the same as in Case (vi) and the unused resource is $Q - \sum R_i$.

Cases (i) through (vii) exhaust all possibilities of the general resource allocation problem posed by the Constraints (11) through (14) with the objective of maximizing the minimum satisfaction level among the users.

CONCLUSIONS

Analytical solutions may be obtained for the optimum resource allocation problem in which the users specify their requirements in an imprecise manner by means of fuzzy sets. The user specifies the resource requirement in terms of the maximum requirement R_i, the maximum shortage fully acceptable $R_i - x_i^U$, and the minimum acceptable requirement x_i^L. The membership function for the fuzzy demand for the resource may be interpreted as the variation of satisfaction level the user attains against any particular allocation. Linear variation of satisfaction is assumed for any allocation in the range of x_i^L to x_i^U. The objective in arriving at an optimal allocation policy is to maximize the minimum satisfaction level among all the users. Even though the resulting optimization problem is a nonlinear one, it is possible to derive analytical expressions for the optimal solutions by suitably considering the various cases of seven sub-problems. Hence, the resource allocation problem with fuzzy demands as specified in the study may be directly solved using the analytical expressions for the optimal solutions.

REFERENCES

Bellman, R. E., and L. A. Zadeh. 1970. "Decision-making in a fuzzy environment." *Manage. Sci*., 17(4), B-141 – B-164.

Feng, Y. 1983. "A method using fuzzy mathematics to solve the vector maximum problem." *Fuzzy Sets and Systems*, 9, 129 – 136.

Ibaraki, T., and N. Katoh. 1988. *Resource Allocation Problems*. The MIT Press.

Kindler, J. 1992. "Rationalizing water requirements with aid of fuzzy allocation model." *J. Water Resour. Plng. and Mgmt*., ASCE, 118(3), 308 – 323.

Piskunov, N. 1974. *Differential and Integral Calculus, Vol. I*, Mir Publishers, Moscow.

Rao, S. S. 1993. "Optimization using fuzzy set theory," in *Structural Optimization: Status and Promise*, ed. by M. P. Kamath. Progress in Astronautics and Aeronautics, A. R. Seebass (Editor in Chief), AIAA, Vol. 150, pp. 637 – 661.

Sakawa, M. 1995. "Fuzzy programming," in *Fuzzy Systems for Management*, ed. by K. Asai. IOS Press, Inc. Bruke, pp. 57 – 78.

Sasikumar, K., and P. P. Mujumdar. 1998. "Fuzzy Optimization Model for Water Quality Management of a River System." *J. of Water Resour. Plng. and Mgmt.*, 124(2), ASCE, pp. 79 – 88.

Zimmermann, H. J. 1978. "Fuzzy programming and linear programming with several objective functions." *Fuzzy Sets and Systems*, 1, pp. 45 – 55.

Zimmermann, H. J. 1985. "Application of fuzzy set theory to mathematical programming." *Information Sciences*, 36, pp. 29 – 58.