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Theoretical Derivation of Synthesized Rational Formula and Runoff Analysis Using High-Resolution Rainfall Data

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THEORETICAL DERIVATION OF SYNTHESIZED RATIONAL FORMULA AND RUNOFF ANALYSIS USING HIGH-RESOLUTION RAINFALL DATA

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ABSTRACT

In order to clarify the physical meanings of conceptual rainfall-runoff models and obtain those models as simple-to-use form, the rational formula, which is a traditional and widely used conceptual model for water resources management and flood control is theoretically derived as explicit analytical formulas in this paper. The authors derived the general basic equation for the runoff in a hill slope and declare that the rational formula is derived when the flow resistance takes the steady velocity such as saturated Darcy's law or under steady state condition. Synthesized Rational Formula can be expressed by superposing the analytical formula of rational formula because the obtained analytical formula of the rational formula is linear. Since rainfall is described as a summation of unit step functions as same as the general rainfall data, derived analytical formulas can accept any temporal resolution rainfall data without any modification. As an example of the calculations with synthesized rational formula, numerical simulations are conducted in paper. The results show that the temporal resolution of rainfall data cause the significant difference in the peak discharge from the one calculated by the rational formula which use the average rainfall intensity in duration of rainfall.

1. INTRODUCTION

In urban areas, because of the peak of rainfall runoff appears very rapidly, a localized torrential downpour causes great damage[e.g.1)] to the city. Kimura, et al.[2)] estimated the effects of configuration factor of runoff/flood mechanism and simulation on the simulation results for real-time urban flood prediction. They indicated that rainfall data is short as much as possible will be needed besides mesh data with necessary resolution for ensuring the analysis accuracy. On this point, the X-band polarimetric (multi paramater) RAdar Information Network (X-RAIN)[3,4)] has been developed and started operation in Japan from 2010. Since so, high spatiotemporal resolution rainfall data can be obtained. It is prospective that we can actualize to

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precipitate runoff discharge with a much better precision than that before using distributed physical models with the rainfall data.

On the other hand, in practice, conceptual models are used in almost medium and small size rivers and sewage system development plan for runoff analysis in Japan. As a representative conceptual model, rational formula was mostly used for medium and small size rivers plan building. As a method for calculating the hydrograph, superimpose method is used [5)]. The rational formula is theoretically derived from the basic equation of runoff in a simple hill slope and the synthesized rational formula named by the authors, which is the hydrograph method using analytical formula of the rational formula, is proposed in this paper. Because of rainfall data form is superimposed from single rectangular, analytical solution of every rectangular data can be obtained and easy to apply to simulate with high temporal resolution rainfall data. The proposed synthesized rational formula is used in this study for urban area runoff analysis and the effects of temporal resolution of rainfall data on runoff analysis result is evaluated.

2. ANALYTICAL SOLUTION OF RATIONAL FORMULA AND PHYSICAL DERIVATION OF SYNTHESIZED RATIONAL FORMULA

We will discuss about the runoff process in a simplified hill slope as shown in figure 1. The analytical solution of rational formula is theoretically delivered from the governing equation of runoff in the slope.

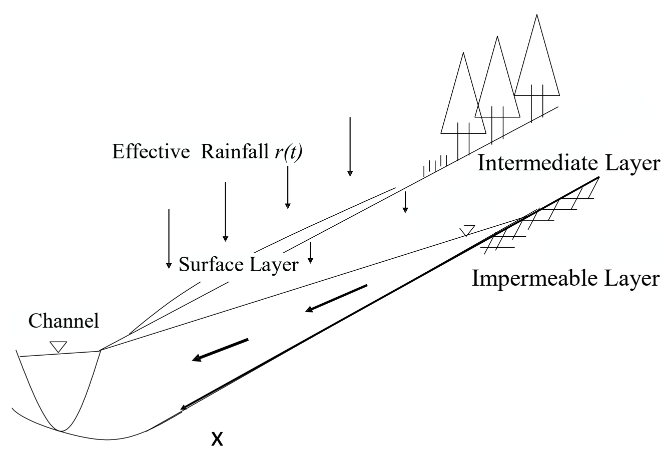

Figure 1 Image of Simplified Runoff Process Hill Slope and Channel.

2.1 Rainfall Runoff at a Simplified Hill Slope

Simplified hill slope where its inclination is uniformly i and it rains uniformly from the top to the end of the slope although the rainfall intensity will change in a time scale ($r(t)$).

a).Continuity Equation and Generalized Law of Motion

The runoff process in the simplified slope is expressed based on Kinematic Wave method. To express various runoff modes in a slope flow such as Darcy’s law, Manning’s average velocity and saturated and unsaturated seepage flow, the authors express the generalized law of motion as eq. 1, which determines velocity along a hill as exponent of water ponding depth. This exponent expresses all runoff modes generally from low water flow to high water flow. Where h is a water ponding depth in the surface soil layer [m], v is a flow velocity [m/h], q is a discharge per unit length along bottom of the slope [m^2], r is an effective rainfall intensity [m/h] and α and m are parameters express characteristics of a basin. Eq. 1 is the equation of continuity.

$$v = \alpha h^m, \quad q = vh = \alpha h^{m+1} \tag{1}$$

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = r(t) \tag{2}$$

b). Governing Equation of Rainfall Runoff at a Simplified Hill Slope

By rearranging eq. 1 and eq. 2, the eq. 3 obtained, which is the basic equation for runoff in the slope equation.

$$\frac{\partial q}{\partial t} + a q^{\frac{m}{m+1}} \frac{\partial q}{\partial x} = a q^{\frac{m}{m+1}} r(t) \tag{3}$$

Where,

$$a = (m+1)\alpha^{\frac{1}{m+1}} \tag{4}$$

2.2 Analytic Solution of Rational Formula

a). Rational Formula Indicates Flow and Law of Resistance

Kure and Yamada[6)] showed that the existing conceptual runoff model such as tank model and storage function model are derived from eq. 3 by taking each appropriate assumption for models. As the assumption, we take equation 5 as a law of motion and derive the rational formula as explicit formula.

$$v = \alpha h^{0} = \alpha = Const. \tag{5}$$

Eq. 5 means the law of flow resistance m=0, that is, cross-sectional average velocity v in a slope flow is constant without regard to water depth. This status occurs in a saturated Darcy's law or steady condition. When the law of flow resistance takes equation 5, the basic equation (eq. 3) is expressed as the partial differential equation with generation term such as eq. 7 by substituting eq. 6.

$$a = (0+1)\alpha^{\frac{0}{0+1}} = \alpha = v \tag{6}$$

This eq. 7 expresses the runoff phenomena in a hill slope with the various flows generally in the condition of steady velocity. The analytical formula of the rational formula is derived by solving this eq. 7. The right side of eq. 7 is the mathematical generation term.

$$\frac{\partial q}{\partial t} + v\frac{\partial q}{\partial x} = vr(t) \tag{7}$$

b). Analytical Formula of Rational Formula with Rectangular Rainfall Data

Initial condition and boundary condition are given as follows. The initial unit discharge per width is uniformly zero. And the unit discharge per width in the top of the hill (where x = 0) is also zero. As shown in figure 2, the rainfall data to be given is expressed as a summation of unit-step functions eq. 8 for solving the eq. 7. It is the single rectangular rainfall that has arbitrary duration of rain. Generally the rainfall data are the data discretized per unit observation time and expressed as combination of rectangular in a hyetograph. The shape of rainfall data to be given (eq. 8) is the completely same as the unit part of a rainfall data that we generally obtain and use. Therefore the combination of the rainfall can count as general rainfall data that has arbitrary resolution of time.

$$r(t) = r_n(t) = r_n^{ave}\{U(t - t_n) - U(t - (t_n + t_r)\} \tag{8}$$

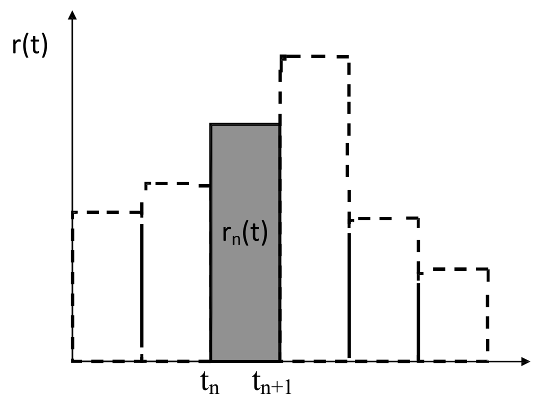

Figure 2 Image of Given Rectangular Rainfall Dara

We can solve this problem by applying the Laplace transform with the I.C. and the B.C. and obtain the formula as eq. 9.

$$q\text{-}_n(t) = r_n^{ave} v \left[\begin{Bmatrix} (t - t_n)H[t - t_n] - \\ \left(t - (t_n + t_r)\right)H[t - (t_n + t_r)] \end{Bmatrix} - \begin{Bmatrix} \left(t - t_n - \frac{x}{v}\right) H \left[t - t_n - \frac{x}{v}\right] - \\ \left(t - (t_n + t_r) - \frac{x}{v}\right) H \left[t - (t_n + t_r) - \frac{x}{v}\right] \end{Bmatrix} \right] \tag{9}$$

Where,

$$H(x - a) = \begin{Bmatrix} 1 \ (x > a) \\ 0 \ (x < a) \end{Bmatrix} \tag{10}$$

L is length of slope [m], Lr is length of channel [m], t_n is starting time of a rain, t_r is duration of a rain, r^{ave} is average rainfall intensity in duration of a rain and H is Heaviside step function.

$$Q = q \cdot L_r = L_r \cdot v(t - t_n) \cdot r_{ave} = L_r L \cdot r_{ave} \tag{11}$$

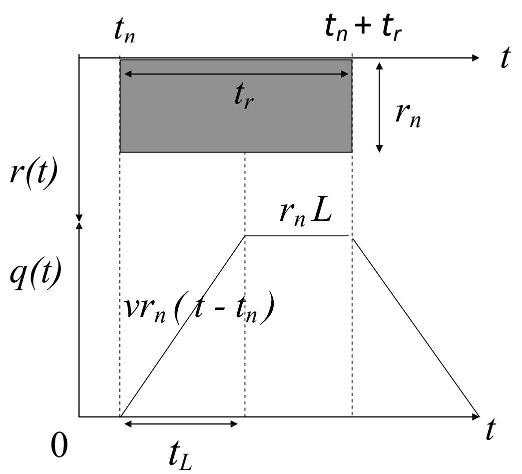

Figure 3 Image of Hydrograph of Case: $t_r > t_L$

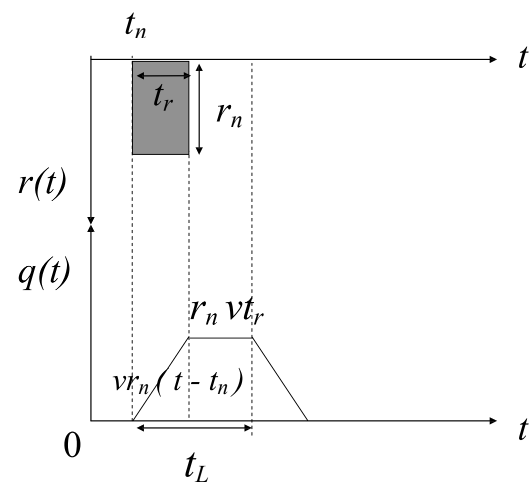

Figure 4 Image of Hydrograph of Case: $t_r < t_L$

As showing in figure 3 and figure 4, this formula (eq. 9) is classified into 2 cases depending on whether the duration of rain (t_r) is smaller or bigger than the arrival time of flood ($t_L = L/v$). Here the arrival time is defined as the length of slope (L) divided by the cross-sectional average velocity (v). This formula shows the same peak discharge as the one calculated by the rational formula when t_r is larger than t_L. Also it expresses another case that the peak discharge is smaller than the one of the rational formula when t_r is smaller than t_L. Since both cases express same phenomena and the shapes of their hydrograph are the same, we can say that eq. 9 is the more general analytical formula of the rational formula.

2.3 Explicit Solution of Synthesized Rational Formula with Solution of Rational Formula

Arbitrary rainfall data is expressed by superposing each time parts of the data, which are express as eq. 12 with considering time. Since eq. 9, which is the analytical formula for the rational formula against to the unit part of the arbitrary rainfall, is a linear solution, the hydrograph toward this general arbitrary rainfall data is expressed by superposing eq. 9 as showing in eq. 13. The authors call this method synthesized rational formula. Synthesized rational formula, eq. 13, is the method that can calculate the hydrograph with the analytical formula of the rational formula.

$$r(t) = r_1(t) + r_2(t) + \cdots + r_{rtime}(t) \tag{12}$$

$$q(t) = q_1(t) + q_2(t) + \ldots.. + q_{rtime}(t) = \sum_{n=1}^{rtime} q_n(t)$$

The notable point here is since rainfall is given as same form as real rainfall data (eq. 8 and eq. 12), if time resolution of data is defined, it can be directly applied to this formulas and the hydrograph is easily calculated. And it is interesting that eq. 13 which is an analytical formula of the rational formula is also expressed as a summation of unite-step functions as well as rainfall.

3. EFFECTS OF TEMPORAL RESOLUTION OF RAINFALL DATA ON RUNOFF ANALYSIS

As we know, urban area runoff has a sensitive reaction to rainfall event. The obtained synthesized rational formula can easily handle any temporal resolution rainfall data because it is the analytical formula for the rainfall runoff that its hyetograph is rectangular, which is the same as real rainfall data. To research on the effects of temporal resolution of rainfall data in runoff calculations, the authors conducted the numerical simulation with various temporal resolution rainfall data, including random generated data and X-RAIN composite data (temporal resolution is 1min.) as examples of numerical simulation with synthesized rational formula.

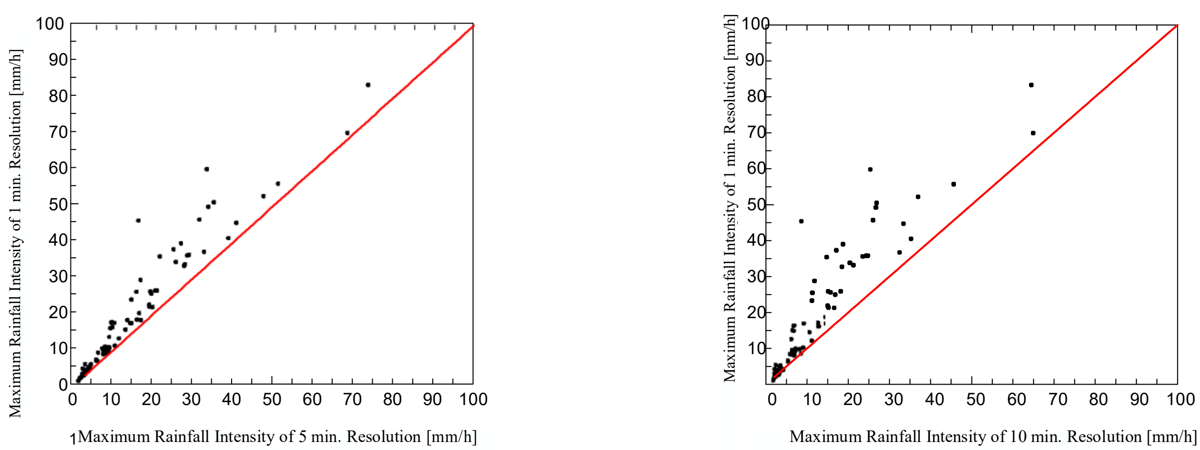

Figure 5 Relation of Maximum Rainfall Intensity with 1min., 5min., 10min. Resolution

3.1 Rainfall Date with Different Temporal Resolution

Figure 5 shows the temporal character of rainfall data. The data is X-RAIN observed 237 rainfall events all over Japan (27 areas) from Aug. 2011 to Sep. 2011. The figure shows 1min maximum bserved composite value; 5min maximum moving averaged value and 10min. maximum moving averaged value. The maximum value of 1min. resolution is 1~3 times and 1~5 times to that of 5min. and 10min. resolution respectively. The 1min. resolution maximum value is much larger than that of 10min and it can be said that rainfall intensity fluctuates every minute or more frequently.

3.2 Effects of Temporal Resolution of Rainfall Data on runoff analysis

In this section, the authors estimate the effects of different temporal resolution rainfall data on runoff analysis results applying the derived theoretical solution of synthesized rational formula to the same rainfall event. The conditions approved by existing research[7)] for calculations are used in this study. They are, the length of the slope is 30[m] as average scale; assume urban area runoff that the time of peak appears is 5minutes to 180 minutes (synthesized rational formula can be applied) and varied the velocity to fit the peak appear time. Average 1min. resolution rainfall data to different resolution data as rainfall input set.

a). Simulation Using Composite X-RAIN Data

Choose X-RAIN observed rainfall events composite data in the 6 areas of AMeDAS stations and average them to 5min., 10min., 30min. and 60min. resolution rainfall data for calculation. Figure 6 shows an example of results. Compare to the result of hydrograph with 1min. resolution data, the result of that with 60min. resolution has a conspicuous lower discharge. There is about 30% difference of peak discharge between the two results. Even using 10min. or 30min. resolution rainfall data, there also have about 10-20% difference of peak discharge to the result using 1min. resolution rainfall data.

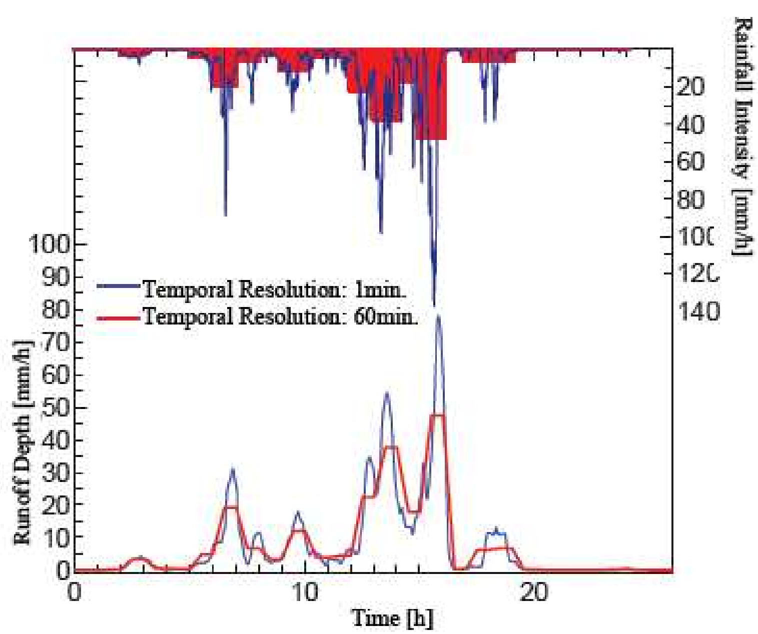

Figure 6 An Example of Calculated hydrograph with different Resolution Rainfall Data

b). Simulation Using Randomly Generated Rainfall Data

Authors set 5 different resolution of time for rainfall data. (1min., 5 min., 10min., 20min. and 60min.). 1min. resolution rainfall data are generated randomly based on uniform distribution. Taking an average of 1minutes rainfall data to generate other temporal resolution rainfall data. Authors also set 3 rainfall data cases that differ in maximum and minimum values of uniform distribution for generating random numbers, these are {0, 100}, {20, 80}, {40, 60}. The population mean of uniform distributions for all 3 cases are set as 50 [mm/h], which is the target value of the designing sewage facility in Tokyo Metropolitan. That is there are 3 rainfall intensity range cases contain 5 different time scale rainfall data. Moreover authors conducted each calculation in 7 different velocities: 5, 10, 20, 30, 40, 50, and 60 [m/h] assuming the rainfall runoff phenomena especially in urban area. The length of the hill slope is set as 30 [m]. Multiple calculations were conducted in every case to remove the bias due to the random numbers. The results of cases are showed in figure 7. The fluctuation of the rainfall is lessening in a hydrograph. However there should be some difference between the hydrograph calculated using the average rainfall and the one using the rainfall with high temporal resolution. Figure 8 shows the ratio of the peak discharge calculated by the rational formula using the average rainfall to the one calculated by synthesized rational formula using various time resolution rainfall data. The horizontal axes of graphs are the average rainfall divided by the flow velocity. Those graphs show that the larger the velocity is and the higher the time resolution of rainfall is, the larger peak discharge is calculated with synthesized rational formula than the one by the rational formula. It means if there is some fluctuation in rainfall and the time resolution of rainfall data is not enough there is a possibility that we cannot calculate the peak discharge correctly unless using the not average rainfall data but high time resolution rainfall data especially in an area where water flows quickly such as urbanized areas. Due to the existing fieldwork research[7)], the peak of hydrograph appears in 20 minutes in a completely urbanized area. Therefore, this kind of area, high temporal resolution (e.g. 1min.) will be needed for predicting peak discharge with accuracy.

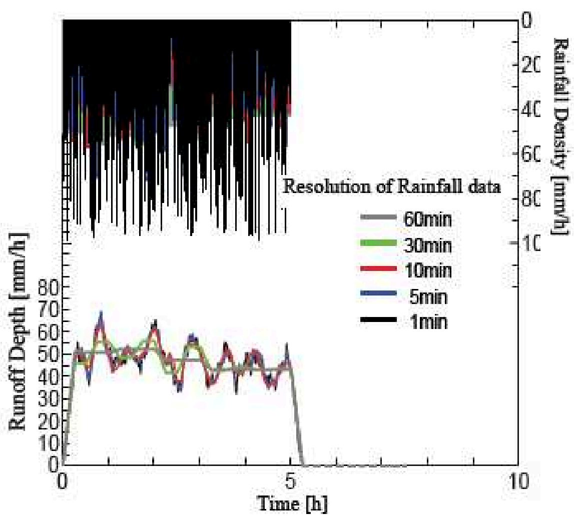

Figure 7 Comparison of Hydrograph

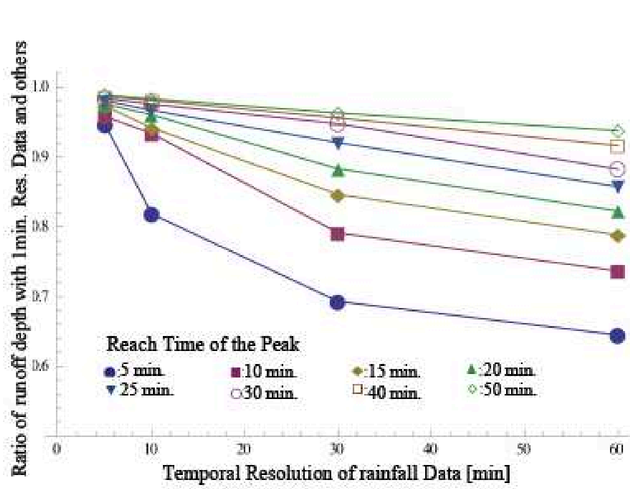

Figure 8 Comparison of Peak Runoff Depth

4. COMPARISON OF LINER-NONLINEAR SIMULATION RESULTS

The linear model, synthesized rational formula can obtain nonlinearity by using the calculated runoff depth to vary average velocity of flow. In this section, the linear and nonlinear calculation results will be compared, especially, the increase part of hydrograph.
Figure 9 is: i) Calculation by using linear model, synthesized rational formula; ii) Obtain nonlinearity by using the calculated runoff depth to vary average velocity of flow; iii) Calculation by using nonlinear model, Yamada Model. From the figure, the nonlinearity of increase part of hydrograph is expressed. The discharge calculated by nonlinear model is smaller than that by linear model at the increasing part of hydrograph. During the increasing of hydrograph, how to understand the relation between the safety and the nonlinearity is a significant issue for flood control and water resources management.

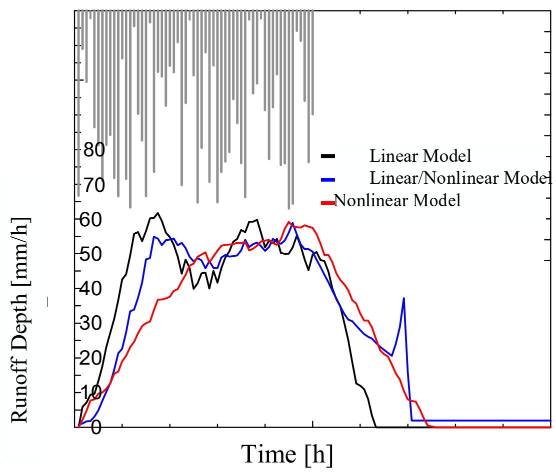

Figure 9 Hydrograph Calculated by Linear and Nonlinear Models

5. CONCLUSIONS

(1) The rational formula can be derived from the equation for rainfall runoff phenomena in a hill slope under conditions as follow: the flow resistance takes saturated Darcy's low or steady condition.
(2) The explicit analytical formula of the rational formula was derived for rainfall runoff in a hill slope especially when a rainfall shape is rectangular as same as the general rainfall data. Synthesized rational formula can be expressed as summation of the analytical formulas of the rational formula since they are linear solution. Synthesized rational formula is a simple-to-apply model, which can use any temporal resolution rainfall data directly to calculate the hydrograph.
(3) Through X-RAIN data analysis, we found the maximum value of 1min. resolution is 1~3 times, 1~5 times, 1~9 times to that of 5min. , 10min. and 1hour, respectively.
(4) Calculated hygrograph using derived synthesized rational formula with different temporal resolution virtual rainfall data and X-RAIN composite rainfall data in this study. The peak discharge decrease the lower temporal resolution rainfall data used.

(5) The shorter reach time of peak discharge, the more effects of temporal resolution of rainfall data on runoff analysis. It can be said that high temporal resolution rainfall data for runoff analysis with accuracy in urbanized basin area is needed.

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