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**An, Hyunuk; Ichikawa, Yutaka; Tachikawa, Yasuto; Shiiba, Michiharu**  
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**Modified Picard Iteration Method for Solving Multi-**  
**Dimensional Saturated-Unsaturated Flow Equation**

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# A NUMERICAL METHOD COMBINING THE IADI METHOD WITH THE MODIFIED PICARD ITERATION METHOD FOR SOLVING MULTI-DIMENSIONAL SATURATED-UNSATURATED FLOW EQUATION

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## ABSTRACT

Numerical models for simulation of saturated-unsaturated flow are increasingly being used in water resources evaluation and environmental management. Because of a superior mass balance, the modified Picard iteration method is widely used for solving saturated-unsaturated flow equation. However, the computational cost becomes considerably large when calculating multi-dimensional flow properties in wide region with finer grid resolution as it is necessary to make a large matrix calculation to get solution. To reduce this computational cost, in this study, we proposed a numerical method combining the iterative alternating direction implicit (IADI) method with the modified Picard iteration method for solving multi-dimensional saturated-unsaturated flow. There is no requirement to make a large matrix calculation in the process of the proposed method. Two and three dimensional slope simulations under rainfall conditions are conducted by the proposed method and the modified Picard iteration method. It is found that the proposed method is faster than the conventional modified Picard method while the difference of the simulation results is not significant. Further it is observed that the new method is more efficient for simulating large regions with finer resolution.

*Keywords:* Richards Equation, Modified Picard Method, IADI Method, Saturated-Unsaturated Flow

## 1. INTRODUCTION

Numerical simulation of saturated-unsaturated subsurface flow is being widely used in many branches of science and engineering including environment engineering, agricultural engineering, and hydrology. The saturated-unsaturated subsurface flow is governed by the Richards equation (Richards, 1931) which is written as follows

$$\frac{\partial \theta}{\partial t} = \nabla \cdot K(\psi) \nabla \psi + \frac{\partial K}{\partial z} \quad (1)$$

Where  $\psi$  is the pressure head,  $\theta$  is the volumetric moisture content,  $K(\psi)$  is hydraulic conductivity,  $t$  is time,  $z$  is denotes the vertical dimension, assumed positive upward. Also it is assumed that appropriate constitutive relationships between  $\theta$  and  $\psi$  and between  $K$  and

$\psi$  are available. Eq. (1) is called as the mixed form Richards equation including both  $\theta$  and  $\psi$ . The mixed form Richards equation has advantages over the  $\psi$ -based and  $\theta$ -based Richards equation, because the mixed-form is more mass conservative than the  $\psi$ -based and  $\theta$ -based forms (Celia et al. 1990; Clement et al. 1994). Celia et al. (1990) shows that the modified Picard iteration method for the mixed form Richards equation is fully mass conserving in the unsaturated zone.

However, the computational cost becomes considerably large when calculating multi-dimensional flow properties in wide region with finer grid resolution as it is necessary to make a large matrix calculation to get solution. In this study, we reduce these computational costs by combining the iterative alternating direction implicit (IADI) method (Rubin. 1968) with the modified Picard iteration method. Because the calculation in the process of the IADI method is conducted implicitly for one direction (dimension) even though the simulation is multi-dimensional, it can be solved efficiently. To validate the proposed numerical method, two and three dimensional slope simulations are conducted by the proposed method and the modified Picard iteration method and the result of the two methods are compared.

## 2. NUMERICAL METHOD

### Modified iteration Picard method

The two dimensional Richards equation is written as follows

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left[ K \frac{\partial \psi}{\partial x} \right] + \frac{\partial}{\partial z} \left[ K \left( \frac{\partial \psi}{\partial z} + 1 \right) \right] \quad (2)$$

where  $x$  denotes the horizontal dimension,  $z$  denotes the vertical dimension. The modified Picard iteration method developed by Celia et al. (1990) is based on a fully implicit (backward Euler) time approximation as follows

$$\frac{\theta^{n+1,m+1} - \theta^n}{\Delta t} = \frac{\partial}{\partial x} \left[ K^{n+1,m} \frac{\partial \psi^{n+1,m+1}}{\partial x} \right] + \frac{\partial}{\partial z} \left[ K^{n+1,m} \left( \frac{\partial \psi^{n+1,m+1}}{\partial z} + 1 \right) \right] \quad (3)$$

where the superscripts  $n$  and  $m$  denote time level and iteration level, and the source/sink term has been ignored for simplicity. The moisture content at the new time step and a new iteration level ( $\theta^{n+1,m+1}$ ) is replaced with the truncated Taylor series expansion with respect to  $\psi$ , about the expansion point  $\psi^{n+1,m}$  as follows

$$\theta^{n+1,m+1} = \theta^{n+1,m} + \left. \frac{d\theta}{d\psi} \right|^{n+1,m} (\psi^{n+1,m+1} - \psi^{n+1,m}) + 0(\delta^2) \quad (4)$$

If the higher-order terms are neglected in Eq. (4) and substituting this equation into Eq. (3) gives

$$C^{n+1,m} \frac{\psi^{n+1,m+1} - \psi^{n+1,m}}{\Delta t} + \frac{\theta^{n+1,m} - \theta^n}{\Delta t} = \frac{\partial}{\partial x} \left[ K^{n+1,m} \frac{\partial \psi^{n+1,m+1}}{\partial x} \right] + \frac{\partial}{\partial z} \left[ K^{n+1,m} \left( \frac{\partial \psi^{n+1,m+1}}{\partial z} + 1 \right) \right] \quad (5)$$

where  $C = d\theta/d\psi$  is the specific moisture capacity function. Eq. (5) can be rewritten in the

terms of the increment in iteration  $\delta^m = \psi^{n+1,m+1} - \psi^{n+1,m}$  as follows

$$\begin{aligned} & \frac{C^{n+1,m}}{\Delta t} \delta^m + \frac{\partial}{\partial x} \left[ K^{n+1,m} \frac{\partial \delta^m}{\partial x} \right] + \frac{\partial}{\partial z} \left[ K^{n+1,m} \frac{\partial \delta^m}{\partial z} \right] \\ & = -\frac{\theta^{n+1,m} - \theta^n}{\Delta t} + \frac{\partial}{\partial x} \left[ K^{n+1,m} \frac{\partial \psi^{n+1,m}}{\partial x} \right] + \frac{\partial}{\partial z} \left[ K^{n+1,m} \left( \frac{\partial \psi^{n+1,m}}{\partial z} + 1 \right) \right] \end{aligned} \quad (6)$$

Eq. (6) defines the modified Picard iteration method of Celia et al. (1990).  $\delta^m$  is the unknown dependent variable and this simultaneous equation is able to be solved by making a matrix calculation. In case of multi-dimensional simulation for large region with fine grid resolution, a large matrix calculation is required in the process of calculating simultaneous equation.

The iterative process of Eq. (6) continues until the difference between the calculated values of the moisture content of two successive iteration levels becomes less than the tolerance  $\delta_\theta$  (Huang et al. 1996) as follows

$$|\theta^{n+1,m+1} - \theta^{n+1,m}| \leq \delta_\theta \quad (7)$$

In this study,  $\delta_\theta = 0.00001$  is used. Huang et al. (1996) showed that this convergence criterion based on moisture content is more efficient in a computation than based on pressure head especially when the soil hydraulic characteristics are highly nonlinear.

### IADI Method and Its Coupling with the Modified Picard Method

Iterative alternating direction implicit (IADI) method of Rubin (1968) is an algorithm for  $\psi$ -based form Richards equation as follows

$$C(\psi) \frac{\partial \psi}{\partial t} = \frac{\partial}{\partial x} \left[ K \frac{\partial \psi}{\partial x} \right] + \frac{\partial}{\partial z} \left[ K \left( \frac{\partial \psi}{\partial z} + 1 \right) \right] \quad (8)$$

The linearized expressions of Eq. (8) based on the IADI method is written as follows

$$\begin{aligned} & C_{i,j}^{n+1,2m} \frac{\psi_{i,j}^{n+1,2m+1} - \psi_{i,j}^n}{\Delta t} + I_m \overline{K}_n (\psi_{i,j}^{n+1,2m+1} - \psi_{i,j}^{n+1,2m}) \\ & = \frac{\partial}{\partial x} \left[ K^n \frac{\partial \psi^{n+1,2m+1}}{\partial x} \right]_{i,j} + \frac{\partial}{\partial z} \left[ K^n \left( \frac{\partial \psi^{n+1,2m}}{\partial z} + 1 \right) \right]_{i,j} \end{aligned} \quad (9)$$

$$\begin{aligned} & C_{i,j}^{n+1,2m} \frac{\psi_{i,j}^{n+1,2m+2} - \psi_{i,j}^n}{\Delta t} + I_m \overline{K}_n (\psi_{i,j}^{n+1,2m+2} - \psi_{i,j}^{n+1,2m+1}) \\ & = \frac{\partial}{\partial x} \left[ K^n \frac{\partial \psi^{n+1,2m+1}}{\partial x} \right]_{i,j} + \frac{\partial}{\partial z} \left[ K^n \left( \frac{\partial \psi^{n+1,2m+2}}{\partial z} + 1 \right) \right]_{i,j} \end{aligned} \quad (10)$$

$$\overline{K}_{i,j}^n = K_{i-1/2,j}^n + K_{i+1/2,j}^n + K_{i,j-1/2}^n + K_{i,j+1/2}^n \quad (11)$$

where subscript  $i$  and  $j$  means the spatial coordinates of the point in  $x$  and  $z$  axis respectively, and  $I_m$  is the iteration parameter, computed for each iteration. In this study,  $I_m = 0.55^m$  is used according to Weeks (2004). Because Eqs (9) and (10) are one-directional implicit forms, there is no requirement to make a large matrix calculation for solving the equations even though in a multi-dimension.

The Richards equation can be efficiently solved by using IADI method. But as this scheme is an algorithm for the  $\psi$ -based form Richards equation, it is often faces to a problem of mass conservation error. To take advantage of two numerical methods, IADI method and the modified Picard iteration method, we propose a numerical method combining both two methods. The proposed method applied to Eq. (2) is written as follows

$$\begin{aligned} & C_{i,j}^{n+1,2m} \frac{\psi_{i,j}^{n+1,2m+1} - \psi_{i,j}^{n+1,2m}}{\Delta t} + \frac{\theta_{i,j}^{n+1,2m} - \theta_{i,j}^n}{\Delta t} + I_m \overline{K_n} (\psi_{i,j}^{n+1,2m+1} - \psi_{i,j}^{n+1,2m}) \\ &= \frac{\partial}{\partial x} \left[ K^{n+1,2m} \frac{\partial \psi^{n+1,2m+1}}{\partial x} \right]_{i,j} + \frac{\partial}{\partial z} \left[ K^{n+1,2m} \left( \frac{\partial \psi^{n+1,2m}}{\partial z} + 1 \right) \right]_{i,j} \end{aligned} \quad (12)$$

$$\begin{aligned} & C_{i,j}^{n+1,2m+1} \frac{\psi_{i,j}^{n+1,2m+2} - \psi_{i,j}^{n+1,2m+1}}{\Delta t} + \frac{\theta_{i,j}^{n+1,2m+1} - \theta_{i,j}^n}{\Delta t} + I_m \overline{K_n} (\psi_{i,j}^{n+1,2m+2} - \psi_{i,j}^{n+1,2m+1}) \\ &= \frac{\partial}{\partial x} \left[ K^{n+1,2m} \frac{\partial \psi^{n+1,2m+1}}{\partial x} \right]_{i,j} + \frac{\partial}{\partial z} \left[ K^{n+1,2m} \left( \frac{\partial \psi^{n+1,2m+2}}{\partial z} + 1 \right) \right]_{i,j} \end{aligned} \quad (13)$$

These equations are rewritten as follows

$$\begin{aligned} & C_{i,j}^{n+1,2m} \frac{\delta^{2m}}{\Delta t} + \frac{\partial}{\partial x} \left[ K^{n+1,2m} \frac{\partial \delta^{2m}}{\partial x} \right]_{i,j} + I_m \overline{K_n} \delta^{2m} \\ &= -\frac{\theta_{i,j}^{n+1,2m} - \theta_{i,j}^n}{\Delta t} + \frac{\partial}{\partial x} \left[ K^{n+1,2m} \frac{\partial \psi^{n+1,2m}}{\partial x} \right]_{i,j} + \frac{\partial}{\partial z} \left[ K^{n+1,2m} \left( \frac{\partial \psi^{n+1,2m}}{\partial z} + 1 \right) \right]_{i,j} \end{aligned} \quad (14)$$

$$\begin{aligned} & C_{i,j}^{n+1,2m+1} \frac{\delta^{2m+1}}{\Delta t} + \frac{\partial}{\partial z} \left[ K^{n+1,2m} \frac{\partial \delta^{2m+1}}{\partial z} \right]_{i,j} + I_m \overline{K_n} \delta^{2m+1} \\ &= -\frac{\theta_{i,j}^{n+1,2m+1} - \theta_{i,j}^n}{\Delta t} + \frac{\partial}{\partial x} \left[ K^{n+1,2m} \frac{\partial \psi^{n+1,2m+1}}{\partial x} \right]_{i,j} + \frac{\partial}{\partial z} \left[ K^{n+1,2m} \left( \frac{\partial \psi^{n+1,2m+1}}{\partial z} + 1 \right) \right]_{i,j} \end{aligned} \quad (15)$$

Eqs (14) and (15) are one-directional implicit forms and based on mixed-form Richards equation.  $\delta^{2m}$  and  $\delta^{2m+1}$  are the unknown dependent variables and the convergence criterion is as follows

$$|\theta^{n+1,2m+2} - \theta^{n+1,2m}| \leq \delta_\theta \quad (16)$$

The numerical method using Eqs (14) and (15) was confirmed by conducting two dimensional slope simulation.

## Modifying IADI method

Extending Eqs (14) and (15) for a three-dimensional field, it is written as follows

$$\begin{aligned}
 & C_{i,j,k}^{n+1,3m} \frac{\delta^{3m}}{\Delta t} + \frac{\partial}{\partial x} \left[ K^{n+1,3m} \frac{\partial \delta^{3m}}{\partial x} \right]_{i,j,k} + I_m \overline{K}_n \delta^{3m} \\
 &= -\frac{\theta_{i,j,k}^{n+1,3m} - \theta_{i,j,k}^n}{\Delta t} + \frac{\partial}{\partial x} \left[ K^{n+1,3m} \frac{\partial \psi^{n+1,3m}}{\partial x} \right]_{i,j,k} + \frac{\partial}{\partial y} \left[ K^{n+1,3m} \frac{\partial \psi^{n+1,3m}}{\partial y} \right]_{i,j,k} + \frac{\partial}{\partial z} \left[ K^{n+1,3m} \left( \frac{\partial \psi^{n+1,3m}}{\partial z} + 1 \right) \right]_{i,j,k}
 \end{aligned} \tag{17}$$

$$\begin{aligned}
 & C_{i,j,k}^{n+1,3m+1} \frac{\delta^{3m+1}}{\Delta t} + \frac{\partial}{\partial y} \left[ K^{n+1,3m} \frac{\partial \delta^{3m+1}}{\partial y} \right]_{i,j,k} + I_m \overline{K}_n \delta^{3m+1} \\
 &= -\frac{\theta_{i,j,k}^{n+1,3m+1} - \theta_{i,j,k}^n}{\Delta t} + \frac{\partial}{\partial x} \left[ K^{n+1,3m+1} \frac{\partial \psi^{n+1,3m+1}}{\partial x} \right]_{i,j,k} + \frac{\partial}{\partial y} \left[ K^{n+1,3m+1} \frac{\partial \psi^{n+1,3m+1}}{\partial y} \right]_{i,j,k} + \frac{\partial}{\partial z} \left[ K^{n+1,3m+1} \left( \frac{\partial \psi^{n+1,3m+1}}{\partial z} + 1 \right) \right]_{i,j,k}
 \end{aligned} \tag{18}$$

$$\begin{aligned}
 & C_{i,j,k}^{n+1,3m+2} \frac{\delta^{3m+2}}{\Delta t} + \frac{\partial}{\partial z} \left[ K^{n+1,3m+2} \left( \frac{\partial \delta^{3m+2}}{\partial z} + 1 \right) \right]_{i,j,k} + I_m \overline{K}_n \delta^{3m+2} \\
 &= -\frac{\theta_{i,j,k}^{n+1,3m+2} - \theta_{i,j,k}^n}{\Delta t} + \frac{\partial}{\partial x} \left[ K^{n+1,3m+2} \frac{\partial \psi^{n+1,3m+2}}{\partial x} \right]_{i,j,k} + \frac{\partial}{\partial y} \left[ K^{n+1,3m+2} \frac{\partial \psi^{n+1,3m+2}}{\partial y} \right]_{i,j,k} + \frac{\partial}{\partial z} \left[ K^{n+1,3m+2} \left( \frac{\partial \psi^{n+1,3m+2}}{\partial z} + 1 \right) \right]_{i,j,k}
 \end{aligned} \tag{19}$$

where subscript  $i, j$ , and  $k$  are spatial coordinates of the point in  $x, y$ , and  $z$  axis respectively. We conducted three dimensional slope simulation using Eqs (17)~(19). However, the calculation using these equations was unstable and the iterative calculation did not converge. The reason of this instability has not been made clear, but we supposed that the terms calculated implicitly are not enough to solve the equation in a three dimension simulation. The two and three dimensional equations, Eqs (14)~(15), (17)~(19), are one-directional implicit forms. The half of the direction are implicitly calculated in the two-dimensional equations, Eqs (14) and (15). On the other hand, the one of three of the directions are implicitly calculated in the three-dimensional equations, Eqs (17)~(19). Therefore, it is supposed to be needed to calculate implicitly over the half of the direction terms for stabilizing the iterative process.

To stabilize the iterative calculation in a three-dimensional simulation, we modified the IADI method part of the proposed method. It is assumed that Eq. (17) is calculated according to the order shown in Fig. 1. Then, when the red region of Fig. 1 is being calculated, the values of the next iteration step in the blue region were already solved. Hence, we can use these values for calculating the red region. According to this idea, we modified Eq. (17) as

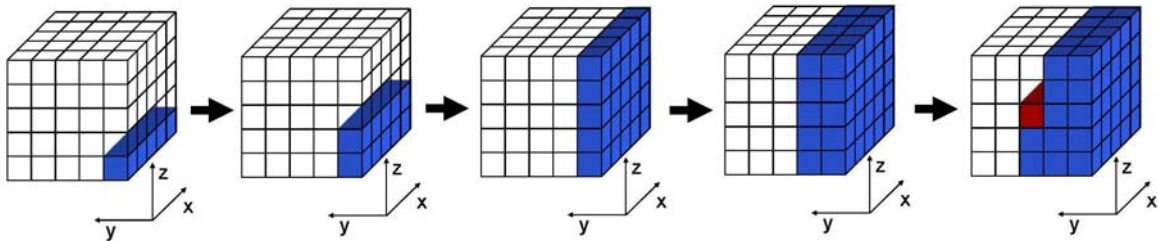


Figure 1 the order of the calculation for equation (17) in modified IADI method

$$\begin{aligned}
& C_{i,j,k}^{n+1,3m} \frac{\delta^{3m}}{\Delta t} + \frac{\theta_{i,j,k}^{n+1,3m} - \theta_{i,j,k}^n}{\Delta t} \\
&= \frac{1}{\Delta x^2} \left[ K_{i-1/2,j,k}^{n+1,3m} \psi_{i-1,j,k}^{n+1,3m+1} + K_{i-1/2,j,k}^{n+1,3m} \psi_{i,j,k}^{n+1,3m+1} + K_{i+1/2,j,k}^{n+1,3m} \psi_{i,j,k}^{n+1,3m+1} + K_{i+1/2,j,k}^{n+1,3m} \psi_{i+1,j,k}^{n+1,3m+1} \right] \\
&+ \frac{1}{\Delta y^2} \left[ K_{i,j-1/2,k}^{n+1,3m} \psi_{i,j-1,k}^{n+1,3m+1} + K_{i,j-1/2,k}^{n+1,3m} \psi_{i,j,k}^{n+1,3m+1} + K_{i,j+1/2,k}^{n+1,3m} \psi_{i,j,k}^{n+1,3m+1} + K_{i,j+1/2,k}^{n+1,3m} \psi_{i,j-1,k}^{n+1,3m+1} \right] \\
&+ \frac{1}{\Delta z^2} \left[ K_{i,j,k-1/2}^{n+1,3m} \psi_{i,j,k-1}^{n+1,3m+1} + K_{i,j,k-1/2}^{n+1,3m} \psi_{i,j,k}^{n+1,3m+1} + K_{i,j,k+1/2}^{n+1,3m} \psi_{i,j,k}^{n+1,3m+1} + K_{i,j,k+1/2}^{n+1,3m} \psi_{i,j,k+1}^{n+1,3m+1} \right] \\
&+ \frac{1}{\Delta z} \left[ K_{i,j,k+1/2}^{n+1,3m} - K_{i,j,k-1/2}^{n+1,3m} \right]
\end{aligned} \tag{20}$$

This equations is rewritten as follows

$$\begin{aligned}
& C_{i,j,k}^{n+1,3m} \frac{\delta^{3m}}{\Delta t} + \frac{1}{\Delta x^2} \left[ K_{i-1/2,j,k}^{n+1,3m} \delta_{i-1,j,k}^{3m} + K_{i-1/2,j,k}^{n+1,3m} \delta_{i,j,k}^{3m} + K_{i+1/2,j,k}^{n+1,3m} \delta_{i,j,k}^{3m} + K_{i+1/2,j,k}^{n+1,3m} \delta_{i+1,j,k}^{3m} \right] \\
&+ \frac{K_{i,j-1/2,k}^{n+1,3m} \delta_{i,j,k}^{3m}}{\Delta y^2} + \frac{K_{i,j,k-1/2}^{n+1,3m} \delta_{i,j,k}^{3m}}{\Delta z^2} \\
&= - \frac{\theta_{i,j,k}^{n+1,3m} - \theta_{i,j,k}^n}{\Delta t} \\
&+ \frac{1}{\Delta x^2} \left[ K_{i-1/2,j,k}^{n+1,3m} \psi_{i-1,j,k}^{n+1,3m} + K_{i-1/2,j,k}^{n+1,3m} \psi_{i,j,k}^{n+1,3m} + K_{i+1/2,j,k}^{n+1,3m} \psi_{i,j,k}^{n+1,3m} + K_{i+1/2,j,k}^{n+1,3m} \psi_{i+1,j,k}^{n+1,3m} \right] \\
&+ \frac{1}{\Delta y^2} \left[ K_{i,j-1/2,k}^{n+1,3m} \psi_{i,j-1,k}^{n+1,3m+1} + K_{i,j-1/2,k}^{n+1,3m} \psi_{i,j,k}^{n+1,3m+1} + K_{i,j+1/2,k}^{n+1,3m} \psi_{i,j,k}^{n+1,3m+1} + K_{i,j+1/2,k}^{n+1,3m} \psi_{i,j-1,k}^{n+1,3m+1} \right] \\
&+ \frac{1}{\Delta z^2} \left[ K_{i,j,k-1/2}^{n+1,3m} \psi_{i,j,k-1}^{n+1,3m+1} + K_{i,j,k-1/2}^{n+1,3m} \psi_{i,j,k}^{n+1,3m+1} + K_{i,j,k+1/2}^{n+1,3m} \psi_{i,j,k}^{n+1,3m+1} + K_{i,j,k+1/2}^{n+1,3m} \psi_{i,j,k+1}^{n+1,3m+1} \right] \\
&+ \frac{1}{\Delta z} \left[ K_{i,j,k+1/2}^{n+1,3m} - K_{i,j,k-1/2}^{n+1,3m} \right]
\end{aligned} \tag{21}$$

Eq. (20) is implicit for not only  $x$  direction but also the half of  $y$  and  $z$  directions. The terms of  $\psi_{i,j-1,k}^{n+1,3m+1}$  and  $\psi_{i,j,k-1}^{n+1,3m+1}$  in the right side of Eq. (21) are already calculated values and  $\delta^{3m}$  is the only unknown dependent variable in Eq. (21). Therefore, the equation implicit for  $x$  direction and the half of  $y$  and  $z$  directions is solved in the process of this method but this equation can be solved by calculating the equation implicit for only one direction. The equation of  $y$  and  $z$  directions, Eqs (18) and (19), are also modified in the same way as the equation of  $x$  direction.

This numerical method combining the modified IADI method with modified Picard iteration method is a newly developed method by this study.

### 3. NUMERICAL EXPERIMENTS

#### Two-dimensional simulations

To validate the proposed computation method, the two-dimensional slope simulations are conducted by using the modified Picard iteration method and the proposed method. The

simulation condition is shown in Fig. 2. Soil slope, 10m long, 1m thickness and 20 degree inclined, is concerned. A constant water level is maintained at the lower end and the bottom side and the upper end are no flux boundaries. The slope surface is under rainfall condition and Fig. 2(b) shows the rainfall intensity. Table 1 shows the four grid size conditions used in the simulation. The time step is 10 sec and the simulation length is 10 hours. The soil water retention and hydraulic conductivity were described as follows

$$\frac{\theta - \theta_r}{\theta_s - \theta_r} = \left( \frac{\psi'}{\psi_0} + 1 \right) \exp\left( -\frac{\psi'}{\psi_0} \right), \quad \psi' = \begin{cases} \psi & (\psi < 0) \\ 0 & (\psi \geq 0) \end{cases} \quad (22)$$

$$K = K_s \left( \frac{\theta - \theta_r}{\theta_s - \theta_r} \right)^\beta \quad (23)$$

where  $\theta_s$  and  $\theta_r$  are the saturated and residual water contents respectively;  $K_s$  is the saturated hydraulic conductivity;  $\psi_0$  is air entry pore-water pressure head;  $\beta$  is the parameter determining the shape of unsaturated hydraulic conductivity curve. In this study,  $\theta_s=0.5$ ,  $\theta_r=0.1$ ,  $\psi_0=-0.25\text{m}$ ,  $K_s=0.00001\text{m/s}$ ,  $\beta=3.0$  are used.

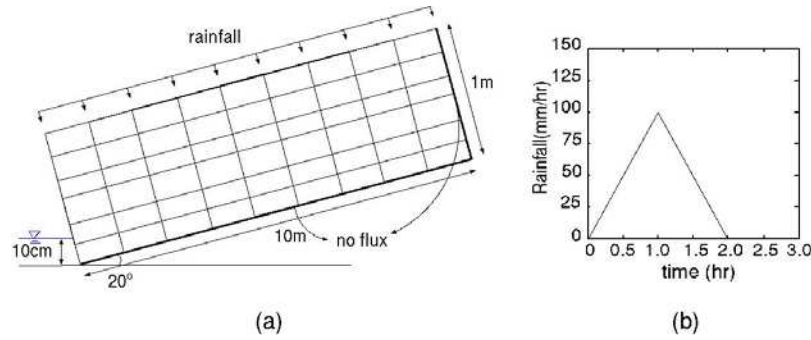


Figure 2 Simulation condition: (a) slope area, (b) rainfall intensity

Table 1 Grid size conditions for two-dimensional simulation

	$\Delta X$	$\Delta Z$	Number of mesh
Case1	0.1 m	0.05 m	25000
Case2	0.05 m	0.05 m	50000
Case3	0.05 m	0.25 m	100000
Case4	0.25 m	0.25 m	200000

Figure 3 shows the error which is defined as follows

$$Error = \frac{|\theta_{MP} - \theta_{Co}|}{\theta_{MP}} \quad (24)$$

where  $\theta_{MP}$  is the moisture content calculated by the modified Picard iteration method,  $\theta_{Co}$  is the moisture content calculated by the proposed method. Figure 3(a) is the average error and Figure 3(b) is the maximum error at all grids points. The peak value of the average errors



are around  $5.0 \times 10^{-5}$  and the peak value of the maximum errors are around  $1.4 \times 10^{-3}$ . Both values are small enough to be neglected.

Figure 4(a) shows computation times of the modified Picard iteration method and of the proposed method for each case. Figure 4(b) shows the ratio of the computation time of the proposed method to that of the modified Picard iteration method. In this simulation, the proposed method is faster from 1.5 to 5 times than the modified Picard iteration method.

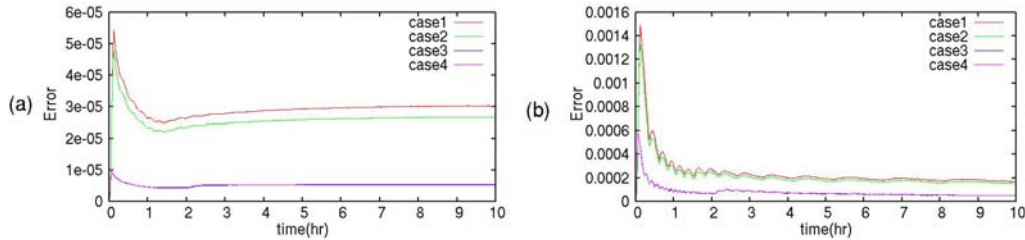


Figure 3 Error of moisture content: (a) average value, (b) maximum value in all grids

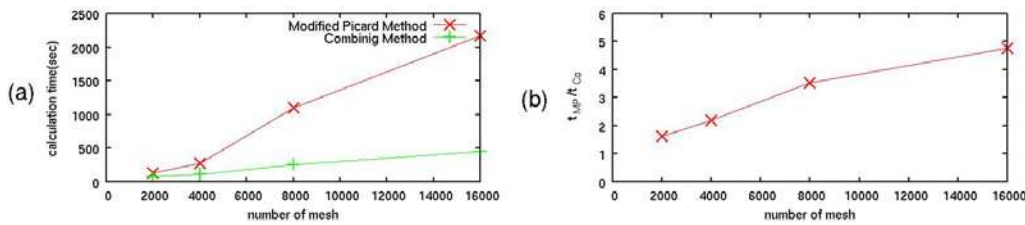


Figure 4 Computation times in two-dimensional simulations

### Three-dimensional simulations

The three-dimensional simulation condition is shown in Fig. 5. Soil slope, 10m long, 5m wide, 1m thickness and 20 degree inclined, is concerned. The half of the surface is under rainfall condition so that the subsurface flow becomes three-dimensional flow. The Rainfall intensity is shown in Fig. 2(b). A constant water level is maintained at the lower end and the both lateral sides, the bottom side, and the upper end are no flux boundaries. Table 3 shows the four grid size conditions used in the simulation. The time step is 10 sec and the simulation length is 10 hours. The soil water retention and hydraulic conductivity were same in the two-dimensional simulations.

Table 3 Grid size conditions for three-dimensional simulation

	$\Delta x$	$\Delta y$	$\Delta z$	Number of mesh
Case1	0.2 m	0.2 m	0.05 m	25000
Case2	0.2 m	0.1 m	0.05 m	50000
Case3	0.1 m	0.1 m	0.05 m	100000
Case4	0.1 m	0.1 m	0.025 m	200000

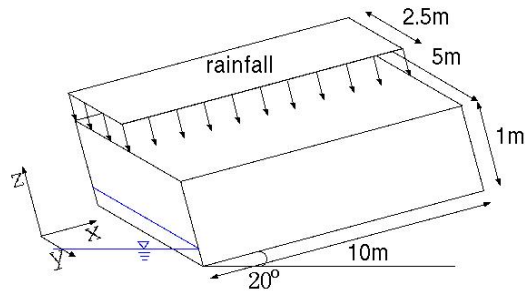


Figure 5 Simulation condition

Figure 6 shows the error which is defined as Eq. (24). The peak value of the average errors are around  $2.5 \times 10^{-4}$  and the peak value of the maximum errors are around  $6.5 \times 10^{-3}$ . Both values are small enough to be neglected.

Figure 7(a) shows computation times of the modified Picard iteration method and of the proposed method for each case. Figure 7(b) shows the ratio of the computation time of the combining method to that of the modified Picard iteration method. In this simulation, the proposed method is faster from 6 to 8 times than the modified Picard iteration method.

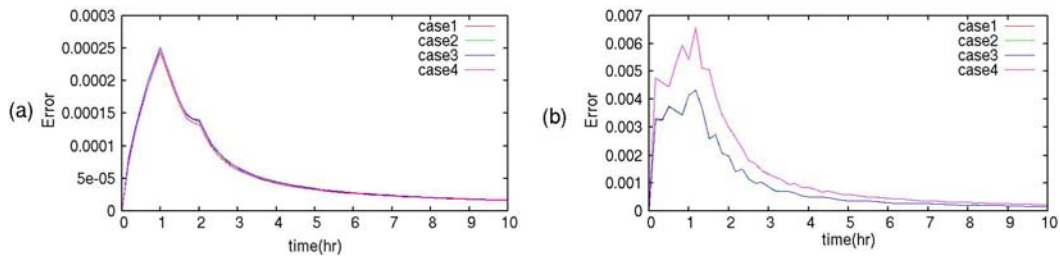


Figure 6 Error of moisture content: (a) average value, (b) maximum value in all grids

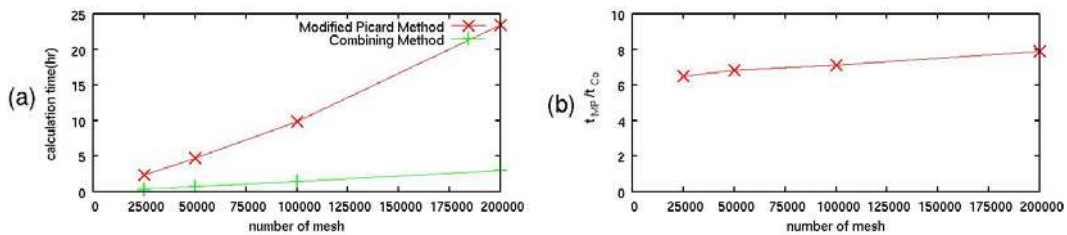


Figure 7 Computation times in three-dimensional simulations

#### 4. CONCLUSION

This study proposed a new computation method combining the iterative alternating direction implicit (IADI) method with the modified Picard iteration method for solving multi-dimensional saturated-unsaturated flow. There is no requirement to make a large matrix calculation in the process of the proposed method.

To validate the proposed method, the two and three dimensional slope simulations were conducted. The proposed method was stable in the two-dimensional simulations. But in the three-dimensional simulation, it was unstable and iterative calculation did not converge. To stabilize the iterative calculation in the three-dimensional simulations, we modified the IADI method part of the proposed method. This method combining the modified IADI

method with the modified Picard iteration method is applicable in the two and three dimensional simulations.

The two and three dimensional slope simulations under rainfall conditions were conducted by the proposed method and the modified Picard iteration method. It is found that the proposed method is faster than the conventional modified Picard method while the difference of the simulation results is not significant. Further it is observed that the new method is more efficient for simulating with finer resolution.

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