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A REYNOLDS AVERAGED THEORY OF TURBULENT SHEAR FLOWS OVER STABLE SINUSOIDAL BEDS AND FORMATION OF SAND WAVES

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ABSTRACT

A new theory of turbulent shear flow over a wavy bed is developed using the Reynolds averaged Navier-Stokes (RANS) and the time-averaged continuity equations to address (1) the characteristics of free surface profiles over stable sinusoidal sand beds and (2) the instability criterion of erodible beds leading to the formation of sand waves. In the first case, there exists a spatial lag between the free surface and the bed profiles; and if the flow depth is reduced, accumulation of heaved wave in the free surface is developed. In the second case, the curves of the Froude number F_m versus nondimensional wave number β decide a stability zone. For $F_m < 0.8$, the bed remains stable with the formation of dunes; while for $F_m \geq 0.8$, the bed remains unstable with the formation of standing waves and antidunes.

Keywords: alluvial dynamics, channel flow, river dynamics, sediment transport, stability

1. INTRODUCTION

The flow characteristics over bed-forms and their formation have been well explored. Anderson (1953) analyzed the fully developed sand waves, using potential flow over a sinusoidal bed and sediment transport as bed-load. Kennedy (1963) analyzed the 2D stability of dunes and antidunes applying potential flow solution. To produce an unstable wave, he defined a spatial lag between the local sediment transport rate and the local bed velocity. Later, Reynolds (1965) modified Kennedy's theory by 3D stability analysis, which was supplemented by Engelund and Fredsøe (1971), assuming a sediment transport model of suspension. Considering real fluid flow, Engelund and Hansen (1966) developed a stability theory of flow over a sinusoidal sand-bed. They assumed a departure in pressure distribution resulting from the vertical acceleration of fluid induced by the sinusoidal bed. Hayashi (1970) provided an improved justification for the spatial lag following a 2D stability analysis based on potential fluid flow. Engelund (1970) proposed a model of the sediment transport based on the vorticity transport equation of 2D real fluid flow and a diffusion equation of suspended sediment load. Smith (1970) applied the same approach but limited to the flow of low Froude numbers. Later, Fredsøe (1974) extended Engelund's (1970) work by introducing the effect of the local bed slope on the bed-load transport rate. Richards (1980) gave a stability theory that envisages the occurrence of two modes of instability of formation of ripples and dunes.

Based on the RANS equations, a theory of turbulent shear flow over a wavy sand-bed is developed addressing (i) the characteristics of turbulent shear flow over a stable sinusoidal bed and (ii) the instability of a plane erodible bed leading to the formation of sand waves.

2. THE REYNOLDS AVERAGED NAVIER-STOKES (RANS) EQUATIONS

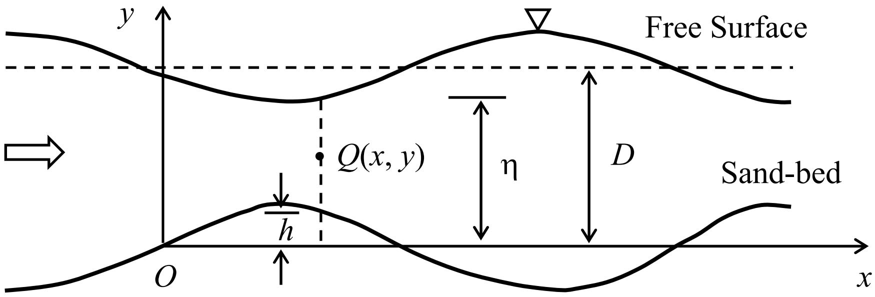

Figure 1 Definition sketch of flow over an undulating sand-bed

Figure 1 shows the definition sketch. The x-axis, having an origin O at convenient point, is along the mean bed-level and the y-axis is vertically upwards. For erodible sand-bed, the bed elevation is a function of x and time t, say $h(x, t)$, where h is the height of sand wave. The elevation of wavy free surface profile is also a function of x and t, say $\eta(x, t)$. However, the mean flow depth D is constant. Due to the gradual variation of the bed undulation, the maximum amplitude $|h|$ is small compared to the horizontal scale of the bed-forms and $|\partial h/\partial x|$ << 1. Likewise, $|\eta|$ must be small and $|\partial\eta/\partial x|$ << 1. The instantaneous velocity components (u, v) at a point $Q(x, y)$ is split into time-averaged part ($\overline{u}$, $\overline{v}$) and fluctuation part (u', v') as

$$u(x,y,t)=\overline{u}(x,y,t)+u'(x,y,t)\,,\ v(x,y,t)=\overline{v}(x,y,t)+v'(x,y,t) \tag{1}$$

The continuity equations for ($\overline{u}$, $\overline{v}$) and (u', v') are

$$\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{v}}{\partial y}=0\,,\ \ \frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0 \tag{2}$$

The exact RANS equations of 2D turbulent flow are

$$\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}=-\frac{\partial\overline{P}}{\partial x}+\frac{\partial\tau}{\partial y}+\upsilon\frac{\partial^2\overline{u}}{\partial y^2}-\frac{\partial(\overline{u'^2})}{\partial x} \tag{3a}$$

$$\frac{\partial\overline{v}}{\partial t}+\overline{u}\frac{\partial\overline{v}}{\partial x}+\overline{v}\frac{\partial\overline{v}}{\partial y}=-\frac{\partial\overline{P}}{\partial y}+\frac{\partial\tau}{\partial x}+\upsilon\frac{\partial^2\overline{v}}{\partial x^2}-\frac{\partial(\overline{v'^2})}{\partial y}-g \tag{3b}$$

where $\overline{P}\,(x, y, t)$ is the time-averaged hydrostatic pressure relative to mass density of fluid ρ, $\tau(x, y, t)$ is $-\overline{u'v'}$, that is the Reynolds shear stress relative to ρ, υ is the kinematic viscosity of fluid and g is the gravitational acceleration. Eqs. 2 - 3b form an undetermined system, since there are six dependent parameters (namely $\overline{u}$, $\overline{v}$, u', v', $\overline{P}$ and τ) against four equations.

3. TURBULENCE ASSUMPTIONS

The gradients of the Reynolds stresses along x are nearly zero. Thus, one obtains

$$\partial\tau/\partial x\approx 0\,,\ \ \partial(\overline{u'^2})/\partial x\approx 0\,,\ \ \partial(\overline{v'^2})/\partial x\approx 0 \tag{4}$$

The second assumption is on the law of variation of $\overline{u}$ with y in turbulent flow. A single averaged distribution of $\overline{u}$ is assumed following the $1/p$-th power law, where p is usually taken as 7 for the turbulent flow over a rigid boundary. Thus, one can write

$$\bar{u} = U_0(x,t)\left(\frac{y-h}{\eta-h}\right)^{1/p} \quad (5)$$

where U_0 is the maximum velocity at $y = \eta$. A theoretical approximation by the power law may be sought in the turbulent stresses that dominate the viscous stresses in Eqs. 3a and 3b. Thus, one makes the assumption

$$\left|\upsilon\,\partial^2\bar{u}/\partial y^2\right| << \left|\partial\tau/\partial y\right|, \quad \left|\upsilon\,\partial^2\bar{v}/\partial x^2\right| << \left|\partial\tau/\partial x\right| \approx 0 \quad (6)$$

for a slowly varying variable, $\zeta = (y-h)^{1/p}$, where $p > 1$. Since $y - h = \zeta^p$ and $dy = p\zeta^{p-1}d\zeta$, the first condition in Eq. 6 becomes

$$\left|\partial^2\bar{u}/\partial\zeta^2\right| << (p\zeta^{p-1}/\upsilon)\left|\partial\tau/\partial\zeta\right| \quad (7)$$

so that

$$\left|\partial^2\bar{u}/\partial\zeta^2\right| \le \varepsilon \ll (p\delta^{p-1}/\upsilon)\min\{\left|\partial\tau/\partial\zeta\right|\} \text{ with } \zeta \ge \delta \quad (8)$$

where ε and $\delta \ge 0$ are small constants. The left hand side of the above inequality implies that $-\varepsilon \le \partial^2\bar{u}/\partial\zeta^2 \le \varepsilon$, whose appropriate solution is

$$\bar{u} = A(x,t) + B(x,t)\zeta + \theta\varepsilon\zeta^2 \text{ with } \zeta \ge \delta \text{ and } -0.5 \le \theta \le 0.5 \quad (9)$$

where θ is an uncertainty function. As $\bar{u} \to 0$ for $\zeta \to 0$, $A(x, t) = 0$. Setting $B(x, t) = U_0(x, t)/(\eta-h)^{1/p}$ and dropping the uncertain small term containing $\theta\varepsilon$, one gets $\bar{u}$ in the form of Eq. 5, that remains valid for $\zeta \to 0$ or $y \to h$. In Eq. 5, the term U_0 represents the maximum velocity at a flow section, which can be related to the depth-averaged velocity $U(x, t)$ as

$$U(x,t) = \frac{1}{\eta-h}\int_h^\eta \bar{u}\,dy = \frac{p}{1+p}U_0(x,t) \quad (10)$$

Thus, using Eq. 10, Eq. 5 can be written as a function of U as

$$\bar{u} = \frac{1+p}{p}U(x,t)\left(\frac{y-h}{\eta-h}\right)^{1/p} \quad (11)$$

The continuity equation, Eq. 2, then yields

$$\bar{v} = -(\eta-h)\frac{\partial U}{\partial x}\left(\frac{y-h}{\eta-h}\right)^{(1+p)/p} \quad (12)$$

4. FLOW ASSUMPTIONS

The free surface profile possesses a curvature with insignificant streamwise gradient. It implies that as $|\partial h/\partial x| \approx 0$, $|\partial\eta/\partial x| \approx 0$. By Eq. 2, the advective vertical acceleration is

$$\bar{u}\frac{\partial\bar{v}}{\partial x} + \bar{v}\frac{\partial\bar{v}}{\partial y} = \bar{u}\frac{\partial\bar{v}}{\partial x} - \bar{v}\frac{\partial\bar{u}}{\partial x} = \bar{u}^2\frac{\partial(\tan\psi)}{\partial x} \approx \bar{u}^2\kappa \quad (13)$$

where $\tan\psi$ is the slope of a streamline through the point $Q(x, y)$ $(= \bar{v}/\bar{u})$ and κ is the curvature of the streamline through the point Q, such that $\kappa(h) \approx \partial^2 h/\partial x^2$ and $\kappa(\eta) \approx \partial^2\eta/\partial x^2$.

Following the Boussinesq theory, one can assume a linear variation of κ between the curvatures κ(*h*) and κ(η) at levels *h* and η (that is $h \le y \le \eta$), respectively, so that

$$\kappa = \kappa(h) + [\kappa(\eta) - \kappa(h)]\frac{y-h}{\eta-h} \tag{14}$$

With this value of κ in Eq. 13 and $\overline{u}$ given by Eq. 5, Eq. 3b is integrated with respect to *y* and the resulting equation is

$$\overline{P} = \overline{P}_0 + g(\eta - y) - U^2(\eta - h)\left(\frac{1+p}{p}\right)^2 \left\{ \frac{p}{2+p}\kappa(h)\left[\left(\frac{y-h}{\eta-h}\right)^{(2+p)/p} - 1\right] + \frac{p}{2(p+1)}[\kappa(\eta)-\kappa(h)]\left[\left(\frac{y-h}{\eta-h}\right)^{(2+p)/p} - 1\right]\right\} - \overline{v'^2} \tag{15}$$

where $\overline{P}_0$ is the value of $\overline{P}$ at $y = \eta$. The above equation yields $\partial\overline{P}/\partial x$, noting that the contribution of $\overline{v'^2}$ is negligible due to negligible variations of turbulence stresses, as given in Eq. 4. The gravity, curvature of flow and 1/*p*-th power law of variation of streamwise velocity with height contribute to the expression for $\partial\overline{P}/\partial x$. The expression for $\partial\overline{P}/\partial x$ is used in the momentum equation, Eq. 3a.

5. DEPTH-AVERAGED EQUATIONS

Taking the depth-averaged continuity equation, Eq. 2, and using Eq. 10, one can write

$$\frac{D\eta}{Dt} - \frac{Dh}{Dt} = \overline{v}\Big|_h^\eta = -\int_h^\eta \frac{\partial\overline{u}}{\partial x}dy = -\frac{\partial}{\partial x}[(\eta-h)U] + \overline{u}(x,\eta,t)\frac{\partial\eta}{\partial x} - \overline{u}(x,h,t)\frac{\partial h}{\partial x} \tag{16}$$

where $D(\cdot)/Dt = \partial(\cdot)/\partial t + \overline{u}\,\partial(\cdot)/\partial x$. Eq. 16 thus reduces to

$$\frac{\partial}{\partial t}(\eta-h) + \frac{\partial}{\partial x}[(\eta-h)U] = 0 \tag{17}$$

For depth averaging Eq. 3a, from Eq. 15 one gets

$$\int_h^\eta \frac{\partial\overline{P}}{\partial x}dy = g(\eta-h)\frac{\partial\eta}{\partial x} + \gamma\frac{\partial}{\partial x}\left[U^2(\eta-h)^2\kappa(\eta) + \frac{p}{2(p+1)}\kappa(h)\right] \tag{18}$$

where $\gamma = (p+1)^2/[p(3p+2)]$. Similarly, for the advective acceleration by partially integrating the third term of the left hand side of Eq. 3a using Eqs. 2 and 11, one gets

$$\int_h^\eta \left(\frac{\partial\overline{u}}{\partial t} + \overline{u}\frac{\partial\overline{u}}{\partial x} + \overline{v}\frac{\partial\overline{u}}{\partial y}\right)dy = \frac{\partial}{\partial t}[(\eta-h)U] + \frac{\partial}{\partial x}\int_h^\eta \overline{u}^2 dy = \frac{\partial}{\partial t}[(\eta-h)U] + \sigma\frac{\partial}{\partial x}[(\eta-h)U] \tag{19}$$

where $\sigma = (p+1)^2/[p(p+2)]$. Using Eq. 4, the integration of Eq. 3a with respect to *y*, yields

$$\frac{\partial}{\partial t}[(\eta-h)U] + \sigma\frac{\partial}{\partial x}[(\eta-h)U] + \gamma\frac{\partial}{\partial x}\left\{(\eta-h)^2U^2\left[\kappa(\eta) + \frac{p}{2(p+1)}\kappa(h)\right]\right\} + g(\eta-h)\frac{\partial\eta}{\partial x} + gn^2\frac{U^2}{(\eta-h)^3} = 0 \tag{20}$$

where n is the Manning roughness coefficient. In the above derivation, the Reynolds stress $\tau(y)$ is assumed to vanish at $y = h$ and η. The bed shear stress τ_0 is represented in Eq. 20 by applying the Manning equation locally as $\rho u_\tau^2 = \tau_0 = \rho g n^2 U^2 (\eta - h)^{1/3}$; where u_τ is the shear velocity at distance x. Differentiating of Eq. 20, one obtains an alternative form of Eq. 20 as

$$\frac{\partial U}{\partial t} + (2\sigma - 1) U \frac{\partial U}{\partial x} + (\sigma - 1)\frac{U^2}{\eta - h} \cdot \frac{\partial}{\partial x}(\eta - h) + \gamma(\eta - h) U^2 \left[\frac{\partial \kappa(\eta)}{\partial x} + \frac{p}{2(p+1)} \frac{\partial \kappa(h)}{\partial x} \right]$$
$$+ 2\gamma U \left[\kappa(\eta) + \frac{p}{2(p+1)} \kappa(h) \right] \frac{\partial}{\partial x}[(\eta - h)U] + g\frac{\partial \eta}{\partial x} + g n^2 \frac{U^2}{(\eta - h)^{4/3}} = 0 \quad (21)$$

Eq. 20 or 21 can be viewed a generalization of the Saint Venant equation, considering $1/p$-th power law of velocity and the curvature of streamlines. For application of Eq. 21, it is assumed that $p \approx 7$, yielding $\sigma \approx 1$ and $\gamma \approx 2/5$. In these approximations, the coefficient of the second term of the left hand side in Eq. 21 is unity, while the third term becomes negligible.

6. FREE SURFACE PROFILES OVER STABLE UNDULATING SAND-BEDS

For steady flow over an undulating sand-bed, the flow depth h and the depth-averaged velocity U are invariant of t; and the continuity equation, Eq. 17, yields

$$(\eta - h) U = q \quad (22)$$

where q is the discharge per unit width. Eliminating U from Eq. 21 with the aid of Eq. 22, yields the differential equation of the wavy free surface profile as

$$\frac{d^3\eta}{dx^3} - \frac{5}{2q^2} \cdot \frac{q^2 - g(\eta - h)^3}{(\eta - h)^2} \cdot \frac{d\eta}{dx} + \frac{7}{16} \cdot \frac{d^3 h}{dx^3} + \frac{5}{2(\eta - h)^2} \cdot \frac{dh}{dx} + \frac{5}{2} \cdot \frac{g n^2}{(\eta - h)^{7/3}} = 0 \quad (23)$$

If the bed has a sinusoidal form as $h = a \sin(kx)$, where a is the amplitude and k is the wave number, Eq. 23 in nondimensional form is given by

$$\frac{d^3\hat{\eta}}{d\hat{x}^3} - \frac{5}{2\beta^2} \cdot \frac{F^2 - [1 + \alpha(\hat{\eta} - \sin\hat{x})]}{F^2 [1 + \alpha(\hat{\eta} - \sin\hat{x})]^2} \cdot \frac{d\hat{\eta}}{d\hat{x}} - \frac{7}{16}\cos\hat{x} + \frac{5}{2\beta^2} \cdot \frac{\cos\hat{x}}{[1 + \alpha(\hat{\eta} - \sin\hat{x})]^2}$$
$$+ \frac{5}{2} \cdot \frac{\varphi}{\alpha\beta^3} \cdot \frac{1}{[1 + \alpha(\hat{\eta} - \sin\hat{x})]^{7/3}} = 0 \quad (24)$$

where α is the nondimensional amplitude of bed-form ($= a/D$), β is the wave number with respect to mean flow depth ($= kD$), F is the Froude number [$= q/(gD^3)^{0.5}$], φ is the bed characteristic parameter ($= n^2 g/D^{1/3}$), $\hat{x}$ is the nondimensional horizontal distance ($= kx$) and $\hat{\eta}$ is the nondimensional vertical distance [$= (\eta - D)/(\alpha D)$].

A typical numerical experiment was conducted for the values of $\alpha = 0.1$, $\beta = 13$, $F = 0.2$ and $\varphi = 4\times10^{-3}$. Eq. 24 was then solved by the Runge-Kutta method. A satisfactory solution was obtained for the initial values of $\hat{\eta} = 0.8$, $d\hat{\eta}/d\hat{x} = -0.71$ and $d^2\hat{\eta}/d\hat{x}^2 = -0.002$ at the origin $\hat{x} = 0$. The wavy free surface profile computed is shown in Figure 2(a). It is evident that the spatial lag is $\hat{x} = 3$. In another numerical experiment, the value of β was reduced to 9.5 keeping the other parameters unchanged. It means that the mean flow depth D is reduced. The values that yield closest to periodicity were found to be $\hat{\eta} = 0.8$, $d\hat{\eta}/d\hat{x} = -1.41$ and $d^2\hat{\eta}/d\hat{x}^2 = -0.003$. The profile of the free surface is plotted in Figure 2(b), where the peaks of the waves definitely show periodic groups of waves in heaving motion.

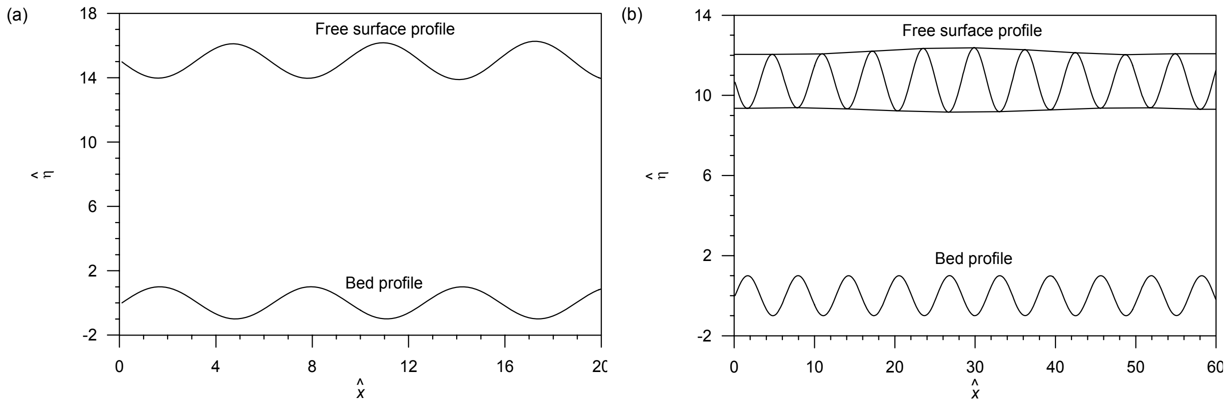

Figure 2 (a) Nondimensional free surface profile for α = 0.1, β = 13, F = 0.2 and φ = 4×10^{-3}; and (b) nondimensional free surface profile for α = 0.1, β = 9.5, F = 0.2 and φ = 4×10^{-3}

7. FORMATION OF SAND WAVES

When the flow Froude number exceeds a certain lower limit, erosion of sand-bed begins. The total-load transport q_T per unit time and width is thus assumed to be given by

$$q_T = q_B + \int_h^\eta \bar{u} c dy \tag{25}$$

where q_B is the bed-load transport rate and c is the concentration of sand suspension. The total-load satisfies the Exner's sediment continuity equation. It is

$$\frac{\partial q_T}{\partial x} = -(1-\rho_0)\frac{\partial h}{\partial t} - \frac{\partial}{\partial t}\int_h^\eta c(x,y,t)dy = -(1-\rho_0)\frac{\partial h}{\partial t} - \frac{\partial}{\partial t}[(\eta - h)C] \tag{26}$$

where ρ_0 is the porosity of bed sand and $C(x,t)$ is the depth-averaged concentration given by

$$C(x,t) = \frac{1}{\eta - h}\int_h^\eta c(x,y,t)dy \tag{27}$$

Due to bed slope, modified bed-load q_B equation of Meyer-Peter and Müller is

$$q_B = 8\sqrt{(s-1)gd^3}\left[\frac{\tau_0}{(s-1)\rho g d} - \mu\frac{\partial h}{\partial x} - 0.047\right]^{3/2} \tag{28}$$

where s is the relative density of sand, d is the median sediment and μ is the particle frictional coefficient (~ 0.1). The bed shear stress is obtained from the Manning equation as $\tau_0 = \rho g n^2 U^2/(\eta - h)^{1/3}$. Sediment concentration c has an advection-diffusion equation of the type

$$\frac{Dc}{Dt} = w_s\frac{\partial c}{\partial y} + \left(\varepsilon_x\frac{\partial^2 c}{\partial x^2} + \varepsilon_y\frac{\partial^2 c}{\partial y^2}\right) \tag{29}$$

where w_s is the terminal fall velocity of sand, ε_x is the turbulent diffusivity in x-direction and ε_y is the turbulent diffusivity in y-direction. The diffusivities ε_x and ε_y are dependent on flow conditions. Thackston and Krenkel (1967) estimated ε_x as

$$\varepsilon_x = 7.25u_\tau D(U/u_\tau)^{1/4} = 7.25g^{3/8}n^{3/4}UD^{7/8} \tag{30}$$

On the other hand, Lane and Kalinske (1941) estimated ε_y as

$$\varepsilon_y = u_\tau D/15 = 0.066 g^{1/2} n U D^{5/6} \tag{31}$$

For the present analysis from Eq. 26, the quantity of interest is the depth-averaged concentration C. Thus, using Eq. 2 into Eq. 29 and integrating between limits h to η, yields

$$\int_h^\eta \frac{Dc}{Dt} dy = \frac{\partial}{\partial t}[(\eta - h)C] + \frac{\partial}{\partial x}\int_h^\eta \bar{u} c dy \tag{32}$$

In the flow domain, the time-averaged velocity $\bar{u}$ increases with height y, while c diminishes. Hence, in Eq. 32, it can be assumed that $\bar{u}\, c \approx UC$, replacing the velocity and the concentration by the averaged values. The integral of the right hand side in Eq. 29 equals

$$\left(w_s c + \varepsilon_y \frac{\partial c}{\partial y} \right)_h^\eta + \varepsilon_x \int_h^\eta \frac{\partial^2 c}{\partial x^2} dy \approx \varepsilon_x \frac{\partial^2}{\partial x^2}[(\eta - h)C] \tag{33}$$

In the above, the first term vanishes as there is no net vertical sediment flux across the extreme levels at $y = h$ and $y = \eta$. Eq. 29 thus leads to

$$\frac{\partial}{\partial t}[(\eta - h)C] + \frac{\partial}{\partial x}[(\eta - h)UC] = \varepsilon_x \frac{\partial^2}{\partial x^2}[(\eta - h)C] \tag{34}$$

Thus, using Eqs. 25, 28 and 34 into the Exner equation, that is Eq. 26, finally yields

$$(1-\rho_0)\frac{\partial h}{\partial t} + \varepsilon_x \frac{\partial^2}{\partial x^2}[(\eta - h)C] + 12[(s-1)gd^3]^{0.5}\left[\frac{n^2U^2}{(\eta - h)^{1/3}(s-1)d} - \mu\frac{\partial h}{\partial x} - 0.047\right]^{0.5}$$
$$\times\left\{\frac{n^2}{(\eta - h)^{1/3}(s-1)d}\left[2U\frac{\partial U}{\partial x} - \frac{1}{3}\cdot\frac{U^2}{\eta - h}\left(\frac{\partial \eta}{\partial x} - \frac{\partial h}{\partial x}\right)\right] - \mu\frac{\partial^2 h}{\partial x^2}\right\} = 0 \tag{35}$$

Eqs. 17, 21 (with $\sigma = 1$ and $\gamma = 2/5$), 34 and 35 constitute the equation of perturbed flow due to erosion of bed. In Eq. 21, $\kappa(h) = \partial^2 h/\partial x^2$ and $\kappa(\eta) = \partial^2\eta/\partial x^2$ are taken. To investigate sand wave propagation, the above set of equations to the first order is linearized as

$$\frac{\partial \eta}{\partial t} - \frac{\partial h}{\partial t} + D\frac{\partial u}{\partial x} + U_m\left(\frac{\partial \eta}{\partial x} - \frac{\partial h}{\partial x}\right) = 0 \tag{36a}$$

$$\frac{\partial U}{\partial t} + U_m\frac{\partial U}{\partial x} + \frac{2}{5}DU_m^2\left(\frac{\partial^3 \eta}{\partial x^3} + \frac{7}{16}\cdot\frac{\partial^3 h}{\partial x^3}\right) + g\frac{\partial \eta}{\partial x} + \frac{gn^2U_m^2}{D^3} = 0 \tag{36b}$$

$$\frac{\partial C}{\partial t} + U_m\frac{\partial C}{\partial x} + C_0\frac{\partial U}{\partial x} = \varepsilon_x\frac{\partial^2 C}{\partial x^2} \tag{36c}$$

$$(1-\rho_0)\frac{\partial h}{\partial t} + \varepsilon_x D\frac{\partial^2 C}{\partial x^2} + G\left\{\frac{n^2U_m}{(s-1)dD^{1/3}}\left[2\frac{\partial U}{\partial x} - \frac{1}{3}\cdot\frac{U_m}{D}\left(\frac{\partial \eta}{\partial x} - \frac{\partial h}{\partial x}\right)\right] - \mu\frac{\partial^2 h}{\partial x^2}\right\} = 0 \tag{36d}$$

where G is $12[n^2gd^2U_m^2D^{-1/3} - 0.047(s-1)gd^3]^{0.5}$ and C_0 is the initial average concentration that may occur due to mean flow velocity U_m. If an exponential distribution of C_0 is assumed with ε_y given by Eq. 31, the average concentration C_0 is

$$C_0 = 4.853\times10^{-4}[(g^2n^4U_m^4)/(w_s^4D^{2/3})] \tag{37}$$

For propagating wave type solution of the linear system of differential equations [Eqs. 36a - 36d with Eq. 37], the solution must be of the form

$$(\eta, h, U, C) = (\overline{E}, \overline{H}, \overline{U}, \overline{C})\exp(-\lambda t)\exp(ikx) \tag{38}$$

where it is imperative that $Re(\lambda) > 0$ for bounded waves to propagate. Here, $Re(\lambda)$ denotes the real part of λ. In this case, the moving wavy bed-form is $h = \overline{H}\exp[-Re(\lambda)t]\exp\{i[kx - Im(\lambda)t]\}$ and the flow variables remain bounded for $t > 0$, where $Im(\lambda)$ denotes the imaginary part of λ. However, for $t \to \infty$, the return to zero value of h is physically inhibited by weakening erosion process, resulting in wavy bed-forms for all times. By substitution of Eq. 36b, noting that the constant term (last term) in Eq. 36b has no role in such a stable solution analysis; the following linear algebraic equations are therefore obtained:

$$(-\lambda + ikU_m)(\overline{E} - \overline{H}) + ikD\overline{U} = 0 \tag{39a}$$

$$8i\left(5kg - 2k^3DU_m^2\right)\overline{E} - 7ik^3DU_m^2\overline{H} + 40(-\lambda + ikU_m)\overline{U} = 0 \tag{39b}$$

$$ikC_0\overline{U} + (-\lambda + ikU_m + \varepsilon_x k^2)\overline{C} = 0 \tag{39c}$$

$$-(1-\rho_0)\lambda\overline{H} - \varepsilon_x Dk^2\overline{C} + G\left\{\frac{ikn^2U_m}{(s-1)dD^{4/3}}\left[2D\overline{U} - \frac{1}{3}U_m(\overline{E} - \overline{H})\right] + \mu k^2\overline{H}\right\} = 0 \tag{39d}$$

Eliminating $\overline{E}$, $\overline{H}$, $\overline{U}$ and $\overline{C}$ from Eqs. 39a – 39d, one gets the quartic equation for λ:

$$(\lambda - ikU_m - \varepsilon_x k^2)\left\{\left[(1-\rho_0)\frac{\lambda}{G} - \mu k^2\right]\left[(\lambda - ikU_m)^2 + k^2D\left(-\frac{2}{5}Dk^2U_m^2 + g\right)\right] + \frac{n^2k^2U_m}{(s-1)dD^{1/3}}\right.$$
$$\left.\times\left(2\lambda - \frac{7}{3}ikU_m\right)\left(-\frac{23}{40}Dk^2U_m^2 + g\right)\right\} - \varepsilon_x\frac{k^4C_0D}{G}(\lambda - ikU_m)\left(-\frac{23}{40}Dk^2U_m^2 + g\right) = 0 \tag{40}$$

For formation of sand waves, the real parts of all the four roots must be positive. In terms of nondimensional quantities as $X = \lambda(D/g)^{0.5}$, $\beta = kD$, $F_m = U_m/(gD)^{0.5}$, $\varphi_0 = D\varphi/[(s-1)d]$, $\varphi_A = 0.083/[(s-1)(F_m^2\varphi_0 - 0.047)]^{0.5}$, $\varepsilon = \varepsilon_x/(gD^3)^{0.5} = 7.25\varphi^{3/8}F_m$ and $C_0 = 4.853\times10^{-4}(F_m^2/\varphi^2)(u_\tau/w_s)^4$, Eq. 40 can then be written as a quartic equation of X. It is

$$(-X + i\beta F_m + \varepsilon\beta^2)\left\{\left[(1-\rho_0)\frac{X}{\varphi_A} - \mu\beta^2\right]\left[(X - i\beta F_m)^2 + \beta^2\left(-\frac{2}{5}\beta^2F_m^2 + 1\right)\right]\right.$$
$$\left.+\varphi_0\beta^2F_m\left(2X - \frac{7}{3}i\beta F_m\right)\left(-\frac{23}{40}\beta^2F_m^2 + 1\right)\right\} + \varepsilon\frac{C_0\beta^2}{\varphi_A}(X - i\beta F_m)\left(-\frac{23}{40}\beta^2F_m^2 + 1\right) = 0 \tag{41}$$

Eq. 41 has four complex roots. Reasonable values of the different parameters are selected as $\varphi = 2.5\times10^{-3}$, $\varphi_0 = 600\varphi$ and $u_\tau/w_s = 0.6$ for the computation of the four roots of X for different values of wave number β and Froude number F_m. It transpires that all four roots have positive real parts when the points (β, F_m) in the β-F_m plot, shown in Figure 3, lie in a curved band forming a zone in which bed-forms remain unstable without becoming unbounded in time. This zone, where significant sediment transport takes place as bed-load and suspended-load, contains the experimental data of antidunes and standing waves, having higher values of F_m (> 0.8). The zone shrinks to an asymptotic critical line at $F_m = 0.177$, when β becomes large. Below this theoretical value no root of Eq. 41 exists and bed erosion is inhibited due to significant reduction of flow velocity. If C_0 is set equal to zero, the transport process is due to bed-load only. In this case, the lower boundary of the unstable zone degenerates into the asymptotic line $F_m = 0.177$, that is the *lower limit for dune formation*. For validation, experimental data used are due to Tison (1949), Tsubaki et al. (1953), Brooks (1954), Barton and Lin (1955), Plate (1957), Laursen (1958), Simons et al. (1961), Kennedy (1961a), Kennedy (1961b) and Guy et al. (1966). Importantly, a group of data of dunes, for $\beta < 3$, is apparent to lie in the inner stable zone, while other group of data with $\beta > 3$ appears to lie in the outer stable zone following the critical limit line and becoming independent of F_m, although small numbers of dune data remain within the unstable zone. However, in Figure 3, it is evident that in the dune zone, the stability limits change considerably by the influence of gravity. This is in accordance with the fact that the bed-load transport is the principal sediment transport mechanism in the dunal regime, whereas the formation of antidune is significantly associated with the suspended-load of sediment transport. However, this is in conformity with the present curves in general being well supported by the experimental data.

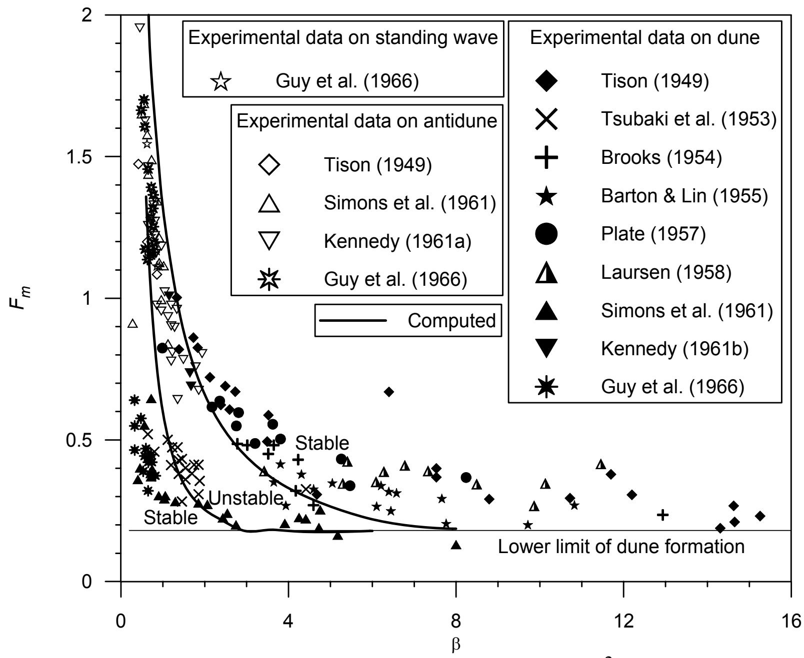

Figure 3 Diagram of stability of sand waves for $\varphi = 2.5\times10^{-3}$, $\varphi_0 = 600\varphi$ and $u_\tau/w_s = 0.6$.

8. CONCLUSIONS

A novel theory of turbulent shear flow over a wavy sand-bed has been evolved to deal with two important problems: (1) The characteristics of free surface profiles over stable sinusoidal sand-beds and (2) the instability principle of plane sand-beds leading to the formation of sand waves. Two basic equations obtained (Eqs. 17 and 20 or 21) can be regarded as the generalization of the Saint Venant equations of motion. In case of shear flow over a stable sinusoidal sand-bed, the free surface profile lags the bed profile, and when the flow depth decreases an accumulation of heaved waves in the free surface is formed. In case of instability of a horizontal plane sand-bed, at higher Froude numbers F_m (> 0.8), the bed-forms remain unstable as standing waves and antidunes, while at lower F_m (with no suspended-load), the instability zone is extended to the lower limit of $F_m = 0.177$.

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