Electric field noise and feedback control for trapped ions

Master's Thesis
University of Turku
Theoretical Physics
2022
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Leppälä, Timo Electric field noise and feedback control for trapped ions

Master's Thesis, 30 pages
Theoretical Physics
July 2022

Noise from the environment often poses a problem when we want to study quantum systems, because it leads to decoherence, dissipation and heating, which are detrimental for quantum properties of the system such as entanglement. Entanglement is one of the main features of quantum systems that sets them apart from classical ones, and it is something that we want to utilize in quantum information processing. Therefore it is important to understand what the noise does to the system, and to find ways to reduce the effects of the noise. In particular, we investigate how electric field noise affects two trapped ions, and whether those effects could be mitigated by continuously measuring the noise, and applying active feedback. We derive a master equation for the two ions affected by noise, and another master equation that includes the feedback process. We apply these master equations to a geometric phase gate that can be used to create entangled states between two qubits. We calculate numerically the fidelity of the phase gate, which tells us how close the state after the gate operation is to the ideal case without noise. We study how to choose the delay between measuring the noise and feeding it back on the ions to maximize the fidelity.

Keywords: ion trap, electric field, noise, stochastic process, feedback, phase gate

TURUN YLIOPISTO
Fysiikan ja tähtitieteen laitos

Leppälä, Timo Sähkökenttien häiriöiden aiheuttama loukutettujen ionien lämpeneminen ja sen vähentäminen takaisinkytkennän avulla

Pro Gradu, 30 sivua
Teoreettinen Fysiikka
Heinäkuu 2022

Ympäristöstä peräisin olevat häiriöt aiheuttavat usein ongelmia kvanttisysteemeitä tutkittaessa, sillä ne johtavat dekoherenssiin ja lämpenemiseen, mitkä ovat haitallisia systeemin kvanttiominaisuuksille kuten lomittumiselle. Lomittuminen on yksi keskeisiä kvanttisysteemien ominaisuuksia, joka erottaa ne klassisista systeemeistä, ja sitä halutaan hyödyntää kvanttilaskennassa. Tämän vuoksi on tärkeää ymmärtää, miten häiriöt vaikuttavat systeemiin, ja miten niiden vaikutusta voi vähentää. Tässä työssä erityisesti tutkitaan, miten sähkökenttien häiriöt vaikuttavat kahteen samassa loukussa oleviin ioneihin. Lisäksi selvitetään, voiko näiden häiriöiden vaikutusta vähentää mittaamalla häiriöt, ja syöttämällä mittaustulos aktiivisesti takaisin systeemiin. Näille kahdelle tilanteelle johdetaan yhtälöt, jotka kuvaavat ionien tilan kehitystä. Näitä yhtälöitä sovelletaan geometriseen vaiheporttiin, jonka avulla voidaan luoda lomittuneita tiloja kahden kubitin välille. Vaiheportin suorituskykyä mitataan laskemalla numeerisesti sen fideliteetti, mikä kertoo miten lähellä vaiheportin suorituksen jälkeinen tila on ideaalista tapausta, jossa häiriöitä ei ole. Häiriöiden mittaamisen ja takaisinsyötön välisen viiveen vaikutus suorituskykyyn selvitetään, jotta se voidaan valita siten, että vaiheportin suorituskyky maksimoidaan.

Asiasanat: ioniloukku, sähkökenttä, stokastinen prosessi, takaisinkytkentä, vaiheportti

Contents

Introduction

Trapped ions offer a good way to study quantum dynamics, because they are well controllable and well isolated systems. To a good approximation, their motion can be modeled as a quantum harmonic oscillator, and two of the internal electronic levels of the ion, a ground state and an excited state, can be used as a two-level system, where the transitions of the electron between the two levels can be driven using a laser with a frequency that is resonant with the energy difference of the two levels. Both of these systems in the trapped ion can be manipulated and also coupled to each other by using lasers. These features also make trapped ions useful for quantum information processing, because the internal states of multiple ions in an ion trap can each represent one quantum bit. The motion of the ions is coupled by the Coulomb repulsion, and their coupled motion can be used perform operations between them. [1]

We want to take advantage of the quantum features of the system like superposition or entanglement, but they are fragile against noise coming from the environment, which turns superpositions and entangled states into classical states through decoherence, dissipation and heating [2]. This is why it is important to understand where the noise comes from and how it affects the system, and to find ways to mitigate those effects. Even at zero temperature and under vacuum conditions, trapped ions are still coupled to the electromagnetic field, which makes them susceptible to electric field noise. The electric fields influence the motion of the ions, causing heating. In quantum information processing, this heating from electric field noise is often one of the limiting factors of multi-qubit gate fidelities [3].

Multi-qubit gates are needed to create entangled states between qubits, and one type of gate that can achieve this with trapped ions is a geometric phase gate. Geometric gates take advantage of the geometric phase [4, 5], which is a phase that depends only on geometric properties of the evolution, making the gates more resistant to errors. The first gates to take advantage of the geometric phase were proposed in [6, 7]. The acquired phase depends on the qubit states of both ions, which means the qubits can be entangled if one of the qubits is initially in a superposition.

In this thesis we investigate how electric field noise affects two ions in an ion trap. We derive a master equation for the average state of the two ions, when the noise is modeled as a Gaussian stochastic process. Then we see if we could counter the effects of the noise by first measuring the noise, and then feeding it back on the ions with the opposite sign. We derive another master equation for the average state that includes this feedback process. Finally, we apply these two master equations to a two-qubit geometric phase gate to see what kind of impact the noise and feedback have on its performance.

We describe the coupled motion of the ions using normal modes, and we find that the electric field noise causes heating of the two normal modes. The heating of a normal mode can be reduced with the feedback only if the delay between the measurement and the feedback is chosen to be close to an integer multiple of the oscillation period of that normal mode, otherwise the heating rate is increased. The feedback can also be applied to the geometric phase gate, and again with the right choice of the delay, the fidelity of the gate can be improved because the heating is reduced.

1 Ion Traps

An ion trap is a device that uses electric and possibly magnetic fields to trap an ion at a specific location [8]. The most common types of ion traps are the Paul trap [9], which uses time-dependent electric fields, and the Penning trap [10], which uses static electric and magnetic fields to trap the ion. Earnshaw's theorem [11] states that an ion cannot be trapped with static electric fields alone. For example, trapping a positively charged ion would require a potential that increases in all three dimensions as the ion tries to move away from the trapping location. We could try to find such a potential, for example one that is quadratic in all three coordinates

$$\Phi(x, y, z) = \alpha x^2 + \beta y^2 + \gamma z^2. \tag{1}$$

To trap the ion, all the coefficients would have to be positive. The electric field of this potential is given by $\boldsymbol{E} = -\nabla\Phi$. But now, because of Gauss's law in free space $\nabla \cdot \boldsymbol{E} = 0$, this leads to the Laplace equation

$$\Delta\Phi(x, y, z) = -2(\alpha + \beta + \gamma) = 0, \tag{2}$$

and we can see that if two of the coefficients are positive, the last one has to be negative, making the trapping unstable in one of the coordinates. The Paul trap gets around this issue by using time varying electric fields instead. The potential is

$$\Phi(x, y, z, t) = \alpha x^2 + \beta y^2 + \gamma z^2 + \cos(\omega t)(\alpha' x^2 + \beta' y^2 + \gamma' z^2), \tag{3}$$

and to satisfy the Laplace equation, one possible choice of coefficients is

$$\begin{aligned} -(\alpha + \beta) &= \gamma, \\ \alpha' &= -\beta', \\ \gamma' &= 0, \end{aligned} \tag{4}$$

which gives a static electric field in the z direction, and an oscillating field in the x and y directions, resulting in a linear trap. From Newton's second law, we get for the z direction

$$\begin{aligned} m\ddot{z} &= F_z = eE_z = -e\frac{\partial\Phi}{\partial z} = -2e(\gamma + \gamma' \cos(\omega t))z, \\ \ddot{z} &= -\frac{2e\gamma}{m}z, \end{aligned} \tag{5}$$

where m is the mass of the ion, F_z is the z-component of the force from the electric field z-component E_z on the ion, and e is the charge of the ion. This is an equation of motion for a simple harmonic oscillator. The motion in the x and y directions is more complicated because of the oscillating potential. This causes the ion to have small oscillations at the frequency ω, called micromotion, and larger oscillations at a lower frequency, called secular motion [12]. The micromotion can usually be neglected, and the ion can be approximated as a simple harmonic oscillator even in the x and y directions.

We are mainly interested in the z direction, because multiple ions can be placed along the z-axis in a linear trap by making the potential weaker in the z direction compared to the x and y directions [13]. The ions will then experience the Coulomb potential from each other in addition to the trapping potential. For small oscillations, we can still approximate this potential as quadratic, and the ions as coupled harmonic oscillators.

2 Electric field noise in ion traps

We consider two ions of the same mass m, position operators q_j, in the same harmonic trap potential, that are affected by external forces $F_j(t)$. It was shown in [14] that you can write the Hamiltonian of this system as

$$\begin{aligned} H(t) &= H_+^0 + F_+(t)q_+ + H_-^0 + F_-(t)q_- + \widetilde{f}(t), \\ H_\pm^0 &= \frac{p_\pm^2}{2} + \frac{1}{2}\Omega_\pm^2 q_\pm^2, \\ F_\pm(t) &= \frac{1}{\sqrt{2m}}(F_1(t) \mp F_2(t)), \\ \widetilde{f}(t) &= \frac{q_0}{2}(F_2(t) - F_1(t)), \end{aligned} \tag{6}$$

by using the mass-weighted normal-mode coordinates

$$q_\pm = \sqrt{m}((q_1 - q_1^{(0)}) \mp (q_2 - q_2^{(0)})), \tag{7}$$

where $q_j^{(0)}$ is the equilibrium position of the ion j, and the conjugate momenta $p_\pm = -i\hbar\partial/\partial q_\pm$. $\Omega_\pm$ is the angular frequency of the normal mode $\pm$, and $q_0 = q_2^{(0)} - q_1^{(0)}$ is the distance between the two equilibrium positions. In the center-of-mass mode

($-$), the two ions move in the same direction, and in the stretch mode ($+$), they move in the opposite directions. These normal modes act as independent harmonic oscillators, and their frequencies are related by the equation $\Omega_+ = \sqrt{3}\Omega_-$.

In addition to the electric fields used to trap the ions, there may be other electric fields that affect the motion of the ions. We consider a fluctuating electric potential $\Phi(t, \boldsymbol{r})$ like in [15], which gives additional terms $e\Phi(t, \boldsymbol{r}_j + \boldsymbol{e}_q q_j)$ to the Hamiltonian, where $\boldsymbol{r}_j$ is the equilibrium position of the ion j, and $\boldsymbol{e}_q$ is a unit vector in the direction of the trap axis. This can be expanded as a Taylor series, where the first two terms are $e\Phi(t, \boldsymbol{r}_j + \boldsymbol{e}_q q_j) \approx e\Phi(t, \boldsymbol{r}_j) + e\boldsymbol{e}_q \cdot \nabla\Phi(t, \boldsymbol{r}_j) q_j$. Now $-\boldsymbol{e}_q \cdot \nabla\Phi(t, \boldsymbol{r}_j) = E_j$ can be identified as the electric field component in the direction $\boldsymbol{e}_q$ at the position of the ion j. eE_j is then the force on the ion from this electric field [16]. The first term in the Taylor expansion doesn't affect the motion of the ion, so the effect of the potential $\Phi(t, \boldsymbol{r})$ is a fluctuating force on both ions. We assume that this fluctuating force can be described by a stochastic process $\xi_j(t)$ that is Gaussian with zero mean [17]. There could also be a deterministic force $f_j(t)$ affecting the ion j, so the total force $F_j(t)$ on the ion is

$$F_j(t) = f_j(t) + \xi_j(t). \tag{8}$$

We will now investigate how this noise from the fluctuating potential affects the two normal modes. The evolution of the system is given by the von Neumann equation

$$\frac{\partial}{\partial t}\rho(t) = \frac{1}{i\hbar}[H(t), \rho(t)], \tag{9}$$

with the Hamiltonian (6). We will calculate the average state of the system $\overline{\rho}(t) = E[\rho(t)]$, where the average is taken over both $\xi_1(t)$ and $\xi_2(t)$. For a system with one ion subject to Gaussian noise, this has already been done in [18]. We apply the same approach to two ions, with two different Gaussian processes that may be correlated, and the dynamics of the system depend on those correlations. The solution of the equation (9) can be written as

$$\rho(t) = U_{0_+}U_{q_+}U_{p_+}U_{0_-}U_{q_-}U_{p_-}U_f\rho(0)U_f^\dagger U_{p_-}^\dagger U_{q_-}^\dagger U_{0_-}^\dagger U_{p_+}^\dagger U_{q_+}^\dagger U_{0_+}^\dagger, \tag{10}$$

where the unitary operators are

$$\begin{aligned} U_{0_\pm}(t) &= e^{\frac{1}{i\hbar}H_\pm^0 t}, \\ U_{q_\pm}(t) &= e^{\frac{1}{i\hbar}P_\pm(t)q_\pm}, \\ U_{p_\pm}(t) &= e^{\frac{1}{i\hbar}Q_\pm(t)p_\pm}, \\ U_f(t) &= e^{\frac{1}{i\hbar}\int_0^t f(s)\mathrm{d}s}, \end{aligned} \tag{11}$$

and the time-dependent functions in the exponents are

$$\begin{aligned} P_\pm(t) &= \int_0^t F_\pm(s)\cos(\Omega_\pm s)\,\mathrm{d}s\,, \\ Q_\pm(t) &= \frac{1}{\Omega_\pm}\int_0^t F_\pm(s)\sin(\Omega_\pm s)\,\mathrm{d}s\,, \\ f(t) &= \frac{q_0}{2}(f_2(t) - f_1(t)). \end{aligned} \tag{12}$$

The coupling between the forces and the position operators comes from the operators $U_{q_\pm}(t)$ and $U_{p_\pm}(t)$. The term $\widetilde{f}(t)$ in the Hamiltonian (6) comes from the equilibrium positions of the ions when switching to the normal-mode coordinates, and it does not depend on the position operators. Later the deterministic forces $f_j(t)$ can depend on the internal states of the ions, so this $\widetilde{f}(t)$ is not only a global phase term. However, we assume that the noise does not depend on the internal states. This means that the stochastic part of $\widetilde{f}(t)$ in the von Neumann equation (9) vanishes, because $\xi_j(t)$ commute with the state $\rho(t)$. Therefore we have not included the stochastic processes $\xi_j(t)$ in $U_f(t)$.

To get the average evolution of the system, we take the average of the von Neumann equation (9)

$$\begin{aligned} \frac{\partial}{\partial t}\overline{\rho} =& \frac{1}{i\hbar}\big[H_+^0 + H_-^0, \overline{\rho}\big] + \frac{1}{i\hbar}E\big[[F_+q_+ + F_-q_-, \rho]\big] + \frac{1}{i\hbar}[f, \overline{\rho}] \\ =& \frac{1}{i\hbar}\big[H_+^0 + H_-^0, \overline{\rho}\big] + \frac{1}{i\hbar}[f_+q_+ + f_-q_-, \overline{\rho}] + \frac{1}{i\hbar}[f, \overline{\rho}] \\ &+ \frac{1}{i\hbar\sqrt{2m}}\big[q_- + q_+, E[\xi_1\rho]\big] + \frac{1}{i\hbar\sqrt{2m}}\big[q_- - q_+, E[\xi_2\rho]\big], \end{aligned} \tag{13}$$

where $f_\pm(t) = \frac{1}{\sqrt{2m}}(f_1(t) \mp f_2(t))$ are the deterministic parts of the forces affecting the normal modes. To calculate the averages $E[\xi_j(t)\rho(t)]$, we can use the Furutsu-Novikov formula [19–21]

$$E[\xi_j(t_1)\rho(t_2)] = \sum_{k=1}^{2} \int_0^{t_2} \alpha_{jk}(t_1 - s) E\left[\frac{\delta\rho(t_2)}{\delta\xi_k(s)}\right] \mathrm{d}s\,, \tag{14}$$

where $\alpha_{jk}(t-s) = E[\xi_j(t)\xi_k(s)]$ is the noise correlation function. The functional derivatives of $\rho(t)$ with respect to the stochastic processes $\xi_j(s)$ can be calculated using the expression (10) for the state. This calculation can be found in Appendix A, and the resulting expressions are

$$\begin{aligned}
\frac{\delta\rho(t)}{\delta\xi_1(s)} =& \frac{1}{i\hbar\sqrt{2m}}\left[q_- \cos(\Omega_-(t-s)) - \frac{p_-}{\Omega_-}\sin(\Omega_-(t-s)), \rho(t)\right] \\
&+ \frac{1}{i\hbar\sqrt{2m}}\left[q_+ \cos(\Omega_+(t-s)) - \frac{p_+}{\Omega_+}\sin(\Omega_+(t-s)), \rho(t)\right] \\
&+ \frac{1}{i\hbar\sqrt{2m}}\left[\frac{\sin(\Omega_- s)}{\Omega_-}P_-(t) + \frac{\sin(\Omega_+ s)}{\Omega_+}P_+(t), \rho(t)\right], \\
\frac{\delta\rho(t)}{\delta\xi_2(s)} =& \frac{1}{i\hbar\sqrt{2m}}\left[q_- \cos(\Omega_-(t-s)) - \frac{p_-}{\Omega_-}\sin(\Omega_-(t-s)), \rho(t)\right] \\
&- \frac{1}{i\hbar\sqrt{2m}}\left[q_+ \cos(\Omega_+(t-s)) - \frac{p_+}{\Omega_+}\sin(\Omega_+(t-s)), \rho(t)\right] \\
&+ \frac{1}{i\hbar\sqrt{2m}}\left[\frac{\sin(\Omega_- s)}{\Omega_-}P_-(t) - \frac{\sin(\Omega_+ s)}{\Omega_+}P_+(t), \rho(t)\right],
\end{aligned} \tag{15}$$

These no longer depend on the stochastic processes, so when taking the average in the equation (14), we only need to take the average of the state $\rho(t)$. It may seem like the $P_\pm(t)$ terms still depends on the noise, but the stochastic processes ξ_j in them commute with the state $\rho(t)$, so the stochastic part of the terms vanish. After taking the average of these, the equation (14) gives us

$$\begin{aligned}
E[\xi_j(t)\rho(t)] =& \frac{1}{i\hbar\sqrt{2m}}\Bigg((\Theta^+_{j1}(t) - \Theta^+_{j2}(t))[q_+, \overline{\rho}] + (\Theta^-_{j1}(t) + \Theta^-_{j2}(t))[q_-, \overline{\rho}] \\
&- \frac{1}{\Omega_+}(\Sigma^+_{j1}(t) - \Sigma^+_{j2}(t))[p_+, \overline{\rho}] - \frac{1}{\Omega_-}(\Sigma^-_{j1}(t) + \Sigma^-_{j2}(t))[p_-, \overline{\rho}] \\
&+ \frac{1}{\Omega_+}(A^+_{j1}(t) - A^+_{j2}(t))[P_+, \overline{\rho}] + \frac{1}{\Omega_-}(A^-_{j1}(t) + A^-_{j2}(t))[P_-, \overline{\rho}]\Bigg),
\end{aligned} \tag{16}$$

where the time-dependent functions are

$$
\begin{aligned}
\Theta^{\pm}_{jk}(t) &= \int_0^t \alpha_{jk}(t-s)\cos(\Omega_{\pm}(t-s))\,\mathrm{d}s = \int_0^t \alpha_{jk}(s)\cos(\Omega_{\pm}s)\,\mathrm{d}s\,, \\
\Sigma^{\pm}_{jk}(t) &= \int_0^t \alpha_{jk}(t-s)\sin(\Omega_{\pm}(t-s))\,\mathrm{d}s = \int_0^t \alpha_{jk}(s)\sin(\Omega_{\pm}s)\,\mathrm{d}s\,, \\
A^{\pm}_{jk}(t) &= \int_0^t \alpha_{jk}(t-s)\sin(\Omega_{\pm}s)\,\mathrm{d}s\,.
\end{aligned}
\tag{17}
$$

Now using this expression for $E[\xi_j(t)\rho(t)]$ in the averaged von Neumann equation (13), we finally get a master equation for the average state $\overline{\rho}(t)$:

$$
\begin{aligned}
\frac{\partial}{\partial t}\overline{\rho} =& \frac{1}{i\hbar}\left[H^0_+ + H^0_-, \overline{\rho}\right] + \frac{1}{i\hbar}[f_+q_+ + f_-q_-, \overline{\rho}] + \frac{1}{i\hbar}[f, \overline{\rho}] \\
& - \frac{1}{2m\hbar^2}(\Theta^+_{11} - 2\Theta^+_{12} + \Theta^+_{22})[q_+, [q_+, \overline{\rho}]] - \frac{1}{2m\hbar^2}(\Theta^-_{11} - \Theta^-_{22})[q_+, [q_-, \overline{\rho}]] \\
& + \frac{1}{2m\hbar^2\Omega_+}(\Sigma^+_{11} - 2\Sigma^+_{12} + \Sigma^+_{22})[q_+, [p_+, \overline{\rho}]] + \frac{1}{2m\hbar^2\Omega_-}(\Sigma^-_{11} - \Sigma^-_{22})[q_+, [p_-, \overline{\rho}]] \\
& - \frac{1}{2m\hbar^2\Omega_+}(A^+_{11} - 2A^+_{12} + A^+_{22})[q_+, [P_+, \overline{\rho}]] - \frac{1}{2m\hbar^2\Omega_-}(A^-_{11} - A^-_{22})[q_+, [P_-, \overline{\rho}]] \\
& - \frac{1}{2m\hbar^2}(\Theta^+_{11} - \Theta^+_{22})[q_-, [q_+, \overline{\rho}]] - \frac{1}{2m\hbar^2}(\Theta^-_{11} + 2\Theta^-_{12} + \Theta^-_{22})[q_-, [q_-, \overline{\rho}]] \\
& + \frac{1}{2m\hbar^2\Omega_+}(\Sigma^+_{11} - \Sigma^+_{22})[q_-, [p_+, \overline{\rho}]] + \frac{1}{2m\hbar^2\Omega_-}(\Sigma^-_{11} + 2\Sigma^-_{12} + \Sigma^-_{22})[q_-, [p_-, \overline{\rho}]] \\
& - \frac{1}{2m\hbar^2\Omega_+}(A^+_{11} - A^+_{22})[q_-, [P_+, \overline{\rho}]] - \frac{1}{2m\hbar^2\Omega_-}(A^-_{11} + 2A^-_{12} + A^-_{22})[q_-, [P_-, \overline{\rho}]].
\end{aligned}
\tag{18}
$$

Based on numerical simulations, the terms $[q_\pm, [q_\pm, \overline{\rho}]]$ are the main source of heating. The effect of the terms with a position and a momentum operator is somewhat unclear, because for small noise strength they seem to have little effect, and for large noise strength the numerical simulation breaks. This breaking seems to happen because of the finite size of the Hilbert space that we have to use in simulations, when in reality it should be infinite.

Now we will take a look at what the effect of the noise is without deterministic forces. The master equation will simplify to

$$
\begin{aligned}
\frac{\partial}{\partial t}\overline{\rho} =& \frac{1}{i\hbar}\left[H_+^0 + H_-^0, \overline{\rho}\right] \\
& - \frac{1}{2m\hbar^2}(\Theta_{11}^+ - 2\Theta_{12}^+ + \Theta_{22}^+)[q_+,[q_+,\overline{\rho}]] - \frac{1}{2m\hbar^2}(\Theta_{11}^- - \Theta_{22}^-)[q_+,[q_-,\overline{\rho}]] \\
& + \frac{1}{2m\hbar^2\Omega_+}(\Sigma_{11}^+ - 2\Sigma_{12}^+ + \Sigma_{22}^+)[q_+,[p_+,\overline{\rho}]] + \frac{1}{2m\hbar^2\Omega_-}(\Sigma_{11}^- - \Sigma_{22}^-)[q_+,[p_-,\overline{\rho}]] \\
& - \frac{1}{2m\hbar^2}(\Theta_{11}^+ - \Theta_{22}^+)[q_-,[q_+,\overline{\rho}]] - \frac{1}{2m\hbar^2}(\Theta_{11}^- + 2\Theta_{12}^- + \Theta_{22}^-)[q_-,[q_-,\overline{\rho}]] \\
& + \frac{1}{2m\hbar^2\Omega_+}(\Sigma_{11}^+ - \Sigma_{22}^+)[q_-,[p_+,\overline{\rho}]] + \frac{1}{2m\hbar^2\Omega_-}(\Sigma_{11}^- + 2\Sigma_{12}^- + \Sigma_{22}^-)[q_-,[p_-,\overline{\rho}]].
\end{aligned}
\tag{19}
$$

In the white noise limit, where the noise is delta correlated in time, the correlation function $\alpha_{jk}(t-s) = C_{jk}\delta(t-s)$, where the matrix C describes the correlation in the noise between the positions of the two ions, and the strength of the noise. With these correlations, the functions (17) are

$$
\begin{aligned}
\Theta_{jk}^{\pm} &= \int_0^t C_{jk}\delta(s)\cos(\Omega_\pm s)\,\mathrm{d}s = \frac{1}{2}C_{jk}, \\
\Sigma_{jk}^{\pm} &= \int_0^t C_{jk}\delta(s)\sin(\Omega_\pm s)\,\mathrm{d}s = 0,
\end{aligned}
\tag{20}
$$

and the master equation becomes

$$
\begin{aligned}
\frac{\partial}{\partial t}\overline{\rho} =& \frac{1}{i\hbar}\left[H_+^0 + H_-^0, \overline{\rho}\right] \\
& - \frac{1}{4m\hbar^2}(C_{11} - 2C_{12} + C_{22})[q_+,[q_+,\overline{\rho}]] - \frac{1}{4m\hbar^2}(C_{11} - C_{22})[q_+,[q_-,\overline{\rho}]] \\
& - \frac{1}{4m\hbar^2}(C_{11} - C_{22})[q_-,[q_+,\overline{\rho}]] - \frac{1}{4m\hbar^2}(C_{11} + 2C_{12} + C_{22})[q_-,[q_-,\overline{\rho}]].
\end{aligned}
\tag{21}
$$

Figure 1 shows the heating of the normal modes as a result of the noise in a numerical simulation for different correlation coefficients c, when the matrix $C = D\left(\begin{smallmatrix}1 & c\\ c & 1\end{smallmatrix}\right)$, and the noise strength $D = 0.025\hbar m\Omega_-^2$. This form of the matrix assumes that the noise strength is the same for both ions, and that also makes the cross terms ($[q_\pm,[q_\mp,\overline{\rho}]]$) in the master equation vanish. The mass and frequency dependence in the noise strength D and the scaling of the axes make all the figures independent of the ion mass m and the frequency Ω_-. Both normal modes start in the ground

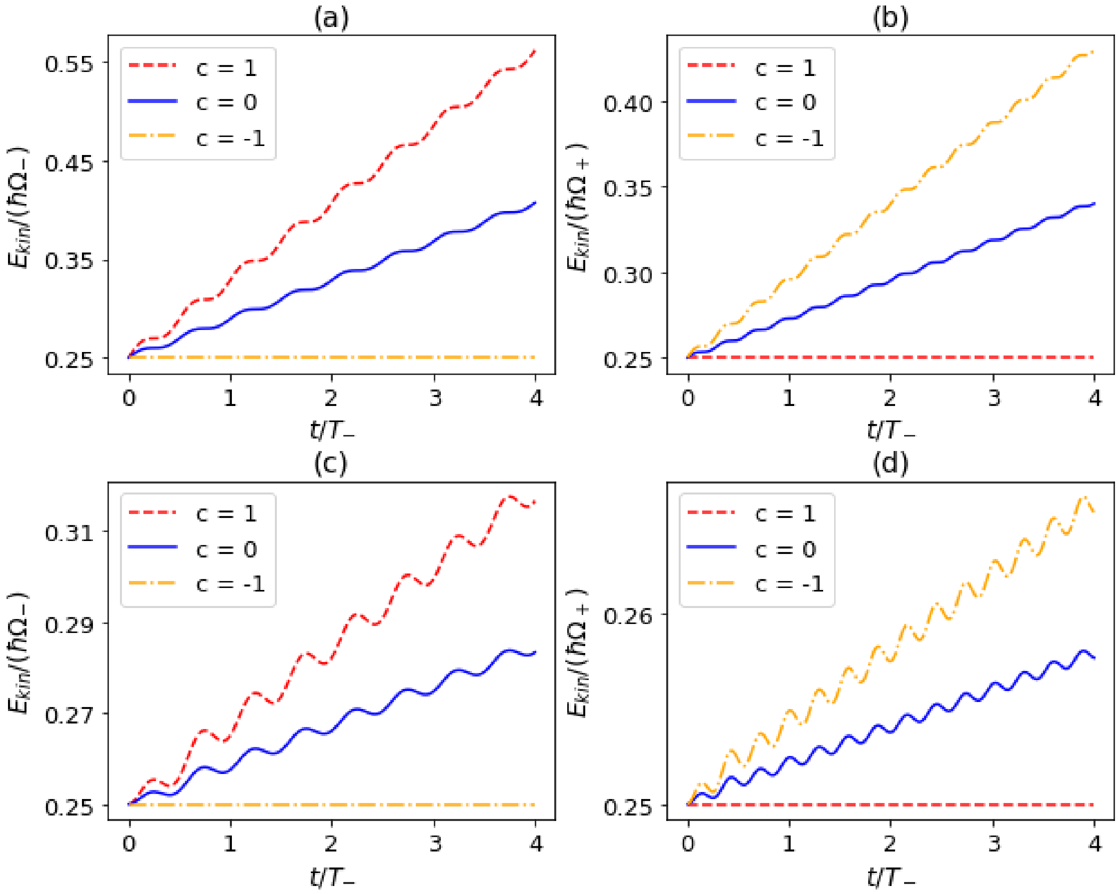

Figure 1. Kinetic energy of the normal modes over time for different values of the correlation coefficient c, which describes the correlation in the noise between the positions of the two ions. The left column is for the center-of-mass mode $(-)$, and the right column is for the stretch mode $(+)$. In (a) and (b), the noise is white noise. In (c) and (d), the noise is Ornstein-Uhlenbeck noise with the correlation time $0.3T_-$, where T_- is the period of the center-of-mass mode. The normal modes always start in the ground state.

state. Figures 1(a) and 1(b) show the increase in the kinetic energy of the center-of-mass and stretch modes, respectively, for the white noise. If the noise is perfectly correlated ($c = 1$), only the center-of-mass mode heats up, and the stretch mode stays in the ground state. For anticorrelated noise ($c = -1$), it is the other way around. If the noise is uncorrelated ($c = 0$), both normal modes experience heating, but at a slower rate compared to the two previous cases. The heating is also slower for the stretch mode because it has a higher frequency.

For a finite correlation time, we choose an Ornstein-Uhlenbeck process with the correlation function $\alpha_{jk}(t-s) = C_{jk}\Gamma e^{-\Gamma|t-s|}/2$, where Γ is the inverse of the correlation time. Figures 1(c) and 1(d) show the heating of the normal modes for this process. The matrix C is the same as before, and the correlation time $1/\Gamma = 0.3T_{-}$. The Ornstein-Uhlenbeck noise causes the normal modes to heat up slower than the white noise, even though the noise strength is the same. The longer the correlation time is, the slower the heating is.

3 Feedback

Now we will investigate if the heating caused by the noise in the last section could be reduced through quantum feedback control [22, 23]. This means somehow measuring the fluctuations in the electric field, and then feeding back the measurement result to the system in a way that cancels some or all of the effects of the noise. Feeding back the noise on the ions adds a new term to the forces $F_j(t)$:

$$F_j(t) = f_j(t) + \xi_j(t) + \int_0^t h(s)\xi_j(t-s)\,\mathrm{d}s\,, \tag{22}$$

where $h(t)$ is some response function that determines how the measured noise ξ_j from times between 0 and t is fed back at time t. We choose the response function $h(s) = -\delta(s-\tau)$, which results in feeding back the noise with the opposite sign after a delay τ. At times before the delay τ, there is no feedback, so the forces are now defined as

$$F_j(t) = \begin{cases} f_j(t) + \xi_j(t) & t < \tau \\ f_j(t) + \xi_j(t) - \xi_j(t-\tau) & t \geq \tau, \end{cases} \tag{23}$$

so the system evolves according to the master equation (18) at times $0 \leq t < \tau$. From now on we assume that $t \geq \tau$ to find out the effects of the feedback. The same solution (10) can be used again by using the new forces (23). We start again by taking the average of the von Neumann equation (9)

$$\begin{aligned} \frac{\partial}{\partial t}\overline{\rho} =& \frac{1}{i\hbar}\left[H_+^0 + H_-^0, \overline{\rho}\right] + \frac{1}{i\hbar}[f_+ q_+ + f_- q_-, \overline{\rho}] + \frac{1}{i\hbar}[f, \overline{\rho}] \\ &+ \frac{1}{i\hbar\sqrt{2m}}\big[q_- + q_+, E[(\xi_1(t) - \xi_1(t-\tau))\rho(t)]\big] \\ &+ \frac{1}{i\hbar\sqrt{2m}}\big[q_- - q_+, E[(\xi_2(t) - \xi_2(t-\tau))\rho(t)]\big], \end{aligned} \tag{24}$$

and then using the Furutsu-Novikov formula (14)

$$E[(\xi_j(t) - \xi_j(t-\tau))\rho(t)] = \sum_{k=1}^{2} \int_0^t (\alpha_{jk}(t-s) - \alpha_{jk}(t-\tau-s)) E\left[\frac{\delta\rho(t)}{\delta\xi_k(s)}\right] \mathrm{d}s\,. \tag{25}$$

We need to calculate the functional derivatives again, keeping in mind that the feedback parts of the forces are 0 before time τ, so the lower limit for the integrals in the functions $P_\pm(t)$ and $Q_\pm(t)$ in (12) is τ for the feedback part. This results in the expressions

$$
\begin{aligned}
\frac{\delta\rho(t)}{\delta\xi_1(s)} =& \frac{1}{i\hbar\sqrt{2m}}\left[q_- \cos(\Omega_-(t-s)) - \frac{p_-}{\Omega_-}\sin(\Omega_-(t-s)), \rho(t)\right] \\
&- \frac{1}{i\hbar\sqrt{2m}}\left[q_- \cos(\Omega_-(t-\tau-s)) - \frac{p_-}{\Omega_-}\sin(\Omega_-(t-\tau-s)), \rho(t)\right] \\
&+ \frac{1}{i\hbar\sqrt{2m}}\left[q_+ \cos(\Omega_+(t-s)) - \frac{p_+}{\Omega_+}\sin(\Omega_+(t-s)), \rho(t)\right] \\
&- \frac{1}{i\hbar\sqrt{2m}}\left[q_+ \cos(\Omega_+(t-\tau-s)) - \frac{p_+}{\Omega_+}\sin(\Omega_+(t-\tau-s)), \rho(t)\right] \\
&+ \frac{1}{i\hbar\sqrt{2m}}\left[\frac{\sin(\Omega_- s) - \sin(\Omega_-(s+\tau))}{\Omega_-}P_-(t), \rho(t)\right] \\
&+ \frac{1}{i\hbar\sqrt{2m}}\left[\frac{\sin(\Omega_+ s) - \sin(\Omega_+(s+\tau))}{\Omega_+}P_+(t), \rho(t)\right], \\
\frac{\delta\rho(t)}{\delta\xi_2(s)} =& \frac{1}{i\hbar\sqrt{2m}}\left[q_- \cos(\Omega_-(t-s)) - \frac{p_-}{\Omega_-}\sin(\Omega_-(t-s)), \rho(t)\right] \\
&- \frac{1}{i\hbar\sqrt{2m}}\left[q_- \cos(\Omega_-(t-\tau-s)) - \frac{p_-}{\Omega_-}\sin(\Omega_-(t-\tau-s)), \rho(t)\right] \\
&- \frac{1}{i\hbar\sqrt{2m}}\left[q_+ \cos(\Omega_+(t-s)) - \frac{p_+}{\Omega_+}\sin(\Omega_+(t-s)), \rho(t)\right] \\
&+ \frac{1}{i\hbar\sqrt{2m}}\left[q_+ \cos(\Omega_+(t-\tau-s)) - \frac{p_+}{\Omega_+}\sin(\Omega_+(t-\tau-s)), \rho(t)\right] \\
&+ \frac{1}{i\hbar\sqrt{2m}}\left[\frac{\sin(\Omega_- s) - \sin(\Omega_-(s+\tau))}{\Omega_-}P_-(t), \rho(t)\right] \\
&- \frac{1}{i\hbar\sqrt{2m}}\left[\frac{\sin(\Omega_+ s) - \sin(\Omega_+(s+\tau))}{\Omega_+}P_+(t), \rho(t)\right],
\end{aligned} \tag{26}
$$

that hold for $s \leq t-\tau$. For $s > t-\tau$, the functional derivatives are the same as without feedback, given by equation (15), because at time t the noise after time $t-\tau$ has not been fed back yet. For more details, see Appendix B. Keeping in mind this dependence on s, using the functional derivatives in the formula (25) results in

$$
\begin{aligned}
E[(\xi_j(t) - \xi_j(t-\tau))\rho(t)] =& \frac{1}{i\hbar\sqrt{2m}}\Bigg((\Theta^+_{j1} - \Theta^+_{j2})[q_+, \overline{\rho}] + (\Theta^-_{j1} + \Theta^-_{j2})[q_-, \overline{\rho}] \\
&- \frac{1}{\Omega_+}(\Sigma^+_{j1} - \Sigma^+_{j2})[p_+, \overline{\rho}] - \frac{1}{\Omega_-}(\Sigma^-_{j1} + \Sigma^-_{j2})[p_-, \overline{\rho}] \\
&+ \frac{1}{\Omega_+}(A^+_{j1} - A^+_{j2})[P_+, \overline{\rho}] + \frac{1}{\Omega_-}(A^-_{j1} + A^-_{j2})[P_-, \overline{\rho}]\Bigg),
\end{aligned} \tag{27}
$$

which has the same form as without feedback (16), but now the time-dependent functions are more complicated:

$$
\begin{aligned}
\Theta^{\pm}_{jk}(t,\tau) &= \int_0^t (\alpha_{jk}(s) - \alpha_{jk}(s-\tau))\cos(\Omega_\pm s)\,\mathrm{d}s \\
&\quad - \int_\tau^t (\alpha_{jk}(s) - \alpha_{jk}(s-\tau))\cos(\Omega_\pm (s-\tau))\,\mathrm{d}s\,, \\
\Sigma^{\pm}_{jk}(t,\tau) &= \int_0^t (\alpha_{jk}(s) - \alpha_{jk}(s-\tau))\sin(\Omega_\pm s)\,\mathrm{d}s \\
&\quad - \int_\tau^t (\alpha_{jk}(s) - \alpha_{jk}(s-\tau))\sin(\Omega_\pm (s-\tau))\,\mathrm{d}s\,, \\
A^{\pm}_{jk}(t,\tau) &= \int_0^t (\alpha_{jk}(t-s) - \alpha_{jk}(t-\tau-s))\sin(\Omega_\pm s)\,\mathrm{d}s \\
&\quad - \int_0^{t-\tau} (\alpha_{jk}(t-s) - \alpha_{jk}(t-\tau-s))\sin(\Omega_\pm (s+\tau))\,\mathrm{d}s\,.
\end{aligned}
\tag{28}
$$

Because the averages (27) have the same form as before (16), this also means that the master equation has the same form as before. The master equation with feedback is then (18), with the time-dependent functions (28).

We will again study the white noise limit without deterministic forces. With the correlation function $\alpha_{jk}(t-s) = C_{jk}\delta(t-s)$, the functions (28) become

$$
\begin{aligned}
\Theta^{\pm}_{jk}(\tau) &= C_{jk}\left(\frac{1}{2} - \cos(\Omega_\pm\tau) - \left(0 - \frac{1}{2}\right)\right) = C_{jk}(1-\cos(\Omega_\pm\tau)), \\
\Sigma^{\pm}_{jk}(\tau) &= -C_{jk}\sin(\Omega_\pm\tau),
\end{aligned}
\tag{29}
$$

and using them in (19), we get the master equation

$$
\begin{aligned}
\frac{\partial}{\partial t}\overline{\rho} =& \frac{1}{i\hbar}\left[H^0_+ + H^0_-, \overline{\rho}\right] \\
&- \frac{1-\cos(\Omega_+\tau)}{2m\hbar^2}\Big((C_{11} - 2C_{12} + C_{22})[q_+,[q_+,\overline{\rho}]] + (C_{11} - C_{22})[q_-,[q_+,\overline{\rho}]]\Big) \\
&- \frac{\sin(\Omega_+\tau)}{2m\hbar^2\Omega_+}\Big((C_{11} - 2C_{12} + C_{22})[q_+,[p_+,\overline{\rho}]] + (C_{11} - C_{22})[q_-,[p_+,\overline{\rho}]]\Big) \\
&- \frac{1-\cos(\Omega_-\tau)}{2m\hbar^2}\Big((C_{11} - C_{22})[q_+,[q_-,\overline{\rho}]] + (C_{11} + 2C_{12} + C_{22})[q_-,[q_-,\overline{\rho}]]\Big) \\
&- \frac{\sin(\Omega_-\tau)}{2m\hbar^2\Omega_-}\Big((C_{11} - C_{22})[q_+,[p_-,\overline{\rho}]] + (C_{11} + 2C_{12} + C_{22})[q_-,[p_-,\overline{\rho}]]\Big).
\end{aligned}
\tag{30}
$$

We could have already said that in the limit of instant feedback $\tau \to 0$, the noise

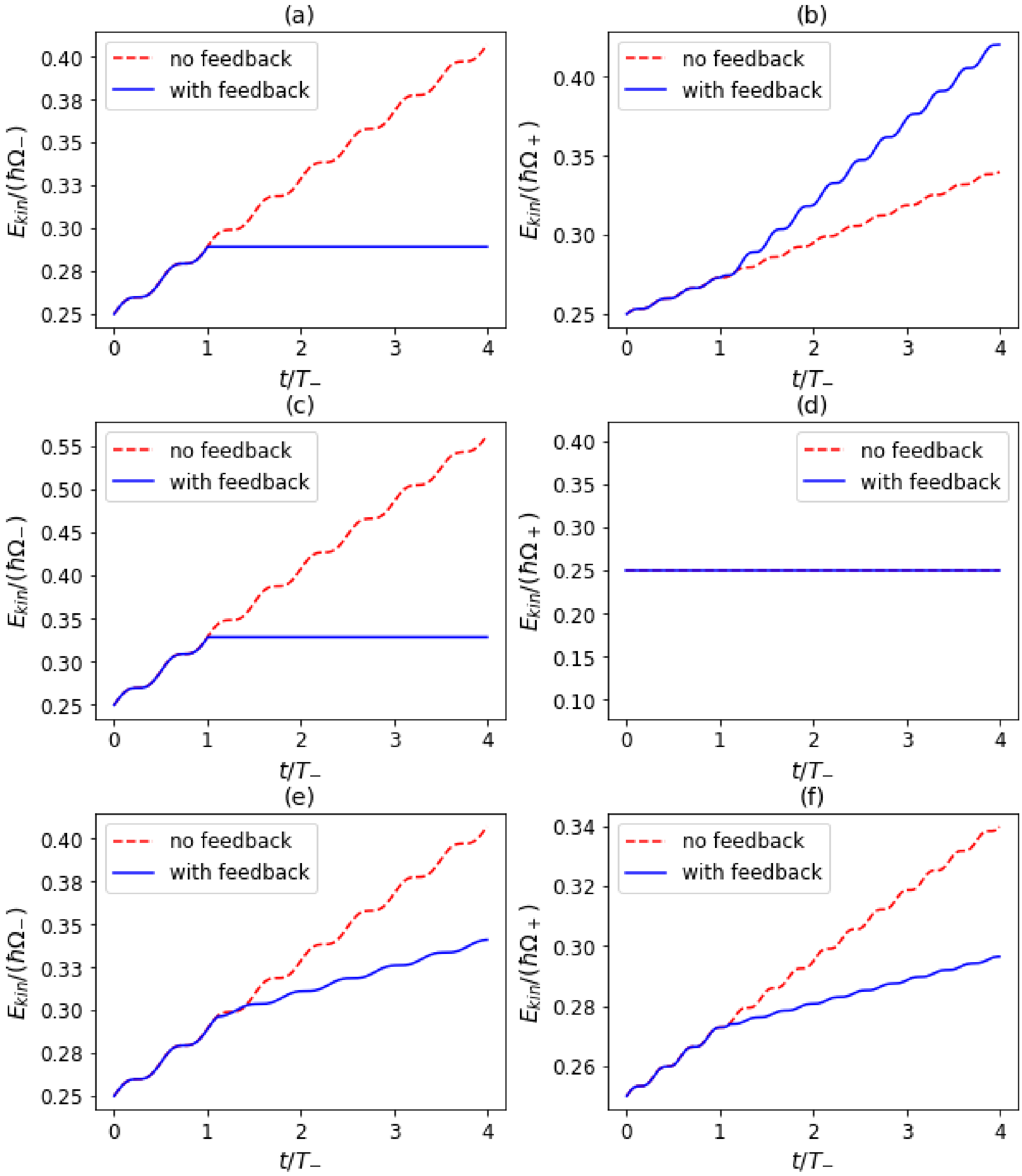

Figure 2. Kinetic energy of the normal modes over time when feedback is used, and the ions are affected by white noise. The left column is for the center-of-mass mode ($-$), and the right column for the stretch mode ($+$). For (a) and (b), the feedback delay τ is equal to the period of the center-of-mass mode T_-, and the noise is uncorrelated ($c = 0$). For (c) and (d), the feedback delay is the same, but the noise is perfectly correlated ($c = 1$). For (e) and (f), the noise is uncorrelated, and the feedback delay $\tau = 1.1T_-$. Both normal modes always start in the ground state.

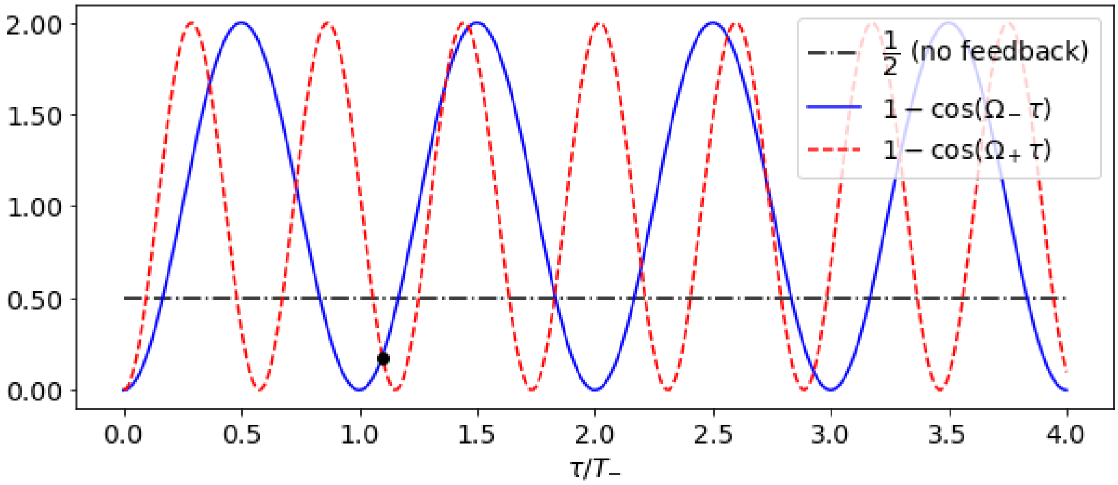

Figure 3. Comparison of the coefficients that appear in the master equation for the terms that are the main source of heating as a function of the feedback delay τ, for white noise. The blue and red curves are the coefficients for the center-of-mass and stretch modes, respectively. The black dot marks one of the lowest points where the coefficients for the feedback are lower for both normal modes than the coefficient without feedback.

is completely canceled by the feedback just by looking at the forces (23). However, processing the measurement signal and feeding it back is going to take some amount of time, so it would be good if we can find a delay $\tau > 0$ that still reduces heating. Now with the master equation (30), we can study how the choice of the delay τ affects the system in more detail by comparing it to the one without feedback (21). If we consider only one of the normal modes, the coefficients (29) vanish if the delay is equal to any integer multiple of the period $T_\pm = 2\pi/\Omega_\pm$, so the feedback would cancel the noise completely in that normal mode. This can be seen for the delay $\tau = T_-$ in figure 2(a), where the heating of the center-of-mass mode stops once the feedback starts. However, from 2(b) we can see that for this uncorrelated noise ($c = 0$), this choice of the delay makes the other normal mode heat up faster. But as we found out in the last section, if the noise is correlated or anticorrelated, the noise affects only one of the normal modes, and in that case we can also completely cancel the noise in the other normal mode. This is shown in figures 2(c) and 2(d) for the delay $\tau = T_-$ and correlation coefficient $c = 1$.

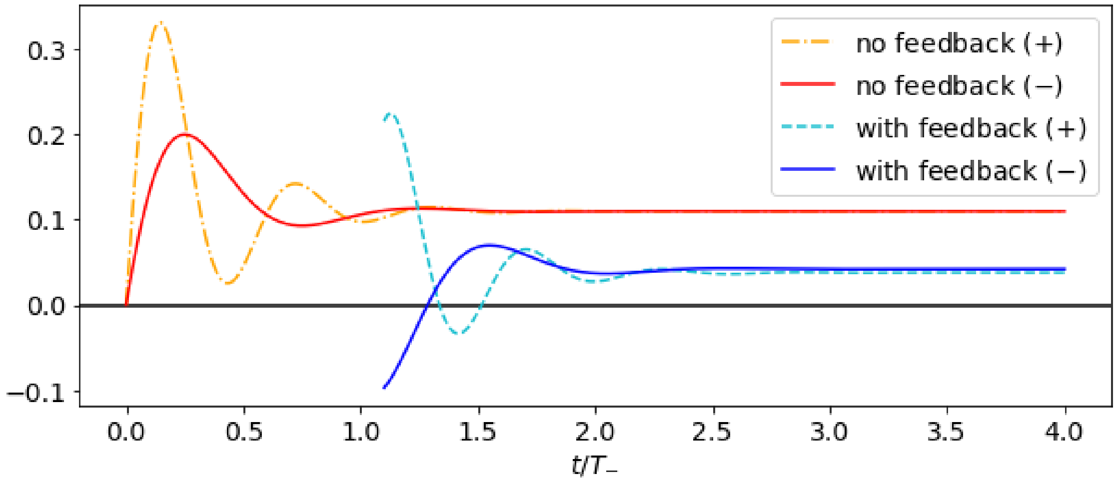

Figure 4. Comparison of the coefficients that appear in the master equation for the terms that are the main source of heating as a function of time for the center-of-mass $(-)$ and stretch $(+)$ modes, for Ornstein-Uhlenbeck noise with the correlation time $0.3T_{-}$. The feedback starts at $1.1T_{-}$.

We could also try to find a delay that is close to some integer multiple of both T_{-} and T_{+} to possibly cancel most of the noise in both normal modes, even when the noise is uncorrelated. To achieve the least amount of heating, it would be good to keep the delay as short as possible, because the system is fully affected by the noise until the time $t = \tau$. The terms in the master equation that are the main source of heating have the coefficients $\Theta^{\pm}_{jk}$, so we compare the functions $\Theta^{\pm}_{jk}(\tau)/C_{jk} = 1 - \cos(\Omega_{\pm})$ to the ones without feedback $\Theta^{\pm}_{jk}/C_{jk} = 1/2$. For white noise, these coefficients do not depend on time. Figure 3 shows these coefficients as a function of the feedback delay τ for both normal modes. Whenever the coefficients are smaller than the constant $1/2$, the heating is slower than without feedback. We can see that the coefficients for both normal modes get smaller than that at around $\tau \approx 1.10T_{-}$, which we have marked as a black dot in figure 3. At that point, the delay is close to an integer multiple of both periods at the same time, with $\tau \approx 1.10T_{-} = 1.10\sqrt{3}T_{+} \approx 1.91T_{+}$. The heating for this delay $\tau = 1.1T_{-}$ is shown in figures 2(e) and 2(f), and it does slow down the heating in both normal modes.

For the Ornstein-Uhlenbeck noise, the coefficients $\Theta^{\pm}_{jk}$ also depend on time in

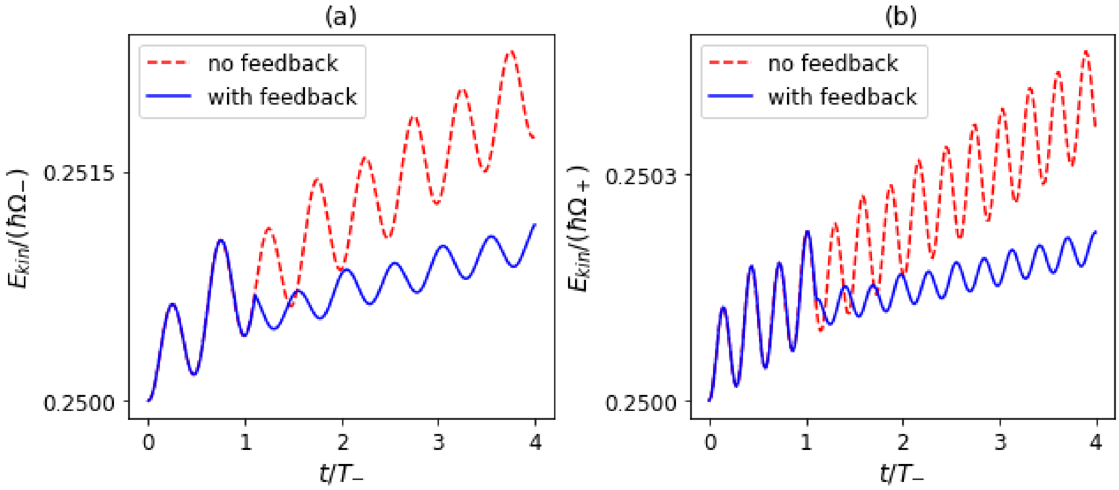

Figure 5. Kinetic energy of the normal modes over time for Ornstein-Uhlenbeck noise when feedback is used. (a) is for the center-of-mass mode (−), and (b) for the stretch mode (+). The noise between the two ions is uncorrelated. The correlation time of the noise is $1.5T_-$, and the feedback delay $\tau = 1.1T_-$.

both cases, with or without feedback. We again compare the feedback coefficients with the ones without feedback, but this time as a function of time, for the feedback delay $\tau = 1.1T_-$. Figure 4 shows the coefficients $\Theta^{\pm}_{jk}(t,\tau)/C_{jk}$ (with feedback) and $\Theta^{\pm}_{jk}(t)/C_{jk}$ (no feedback) for the correlation time $1/\Gamma = 0.3T_-$. The coefficients are smaller with the feedback, so it reduces the heating. Figure 5 shows that the heating is reduced for the Ornstein-Uhlenbeck noise, even with a long correlation time $1/\Gamma = 1.5T_-$, and the feedback also reduces the amplitude of the oscillations in the kinetic energy.

4 Application to a geometric phase gate

Now that we know that the feedback can reduce the heating of the normal modes, we apply the master equation (18) to a geometric phase gate to see what kind of impact the noise and feedback have on the performance of the gate. Geometric gates were first introduced in [6, 7], and here we study a particular implementation of the geometric phase gate, proposed in [14]. These gates are more resistant to errors, because they make use of the geometric phase, which depends only on some geometric properties of the evolution. For this gate, the phase change depends on the area enclosed by the trajectory of the normal modes in phase space, in the interaction picture. The phase gate implements the following operation on a pair of qubits

$$\begin{aligned} |00\rangle &\rightarrow e^{i\phi_{00}}\,|00\rangle\,, \quad |11\rangle \rightarrow e^{i\phi_{11}}\,|11\rangle\,, \\ |01\rangle &\rightarrow e^{i\phi_{01}}\,|01\rangle\,, \quad |10\rangle \rightarrow e^{i\phi_{10}}\,|10\rangle\,, \end{aligned} \tag{31}$$

where $|0\rangle$ and $|1\rangle$ are the eigenstates of the Pauli Z operator, and they represent the internal states of the ions. The phase gate is implemented by using lasers that induce forces on the ions that depend on these internal states, as was done in [24]. The forces are designed in a way where the ions return to the initial motional state at the end of the operation, and the qubit state acquires a phase according to (31), where the differential phase $\Delta\phi = \phi_{01} + \phi_{10} - \phi_{00} - \phi_{11} = -\pi$. This can be used to create entangled states between the two qubits.

The effects of dissipation and thermal fluctuations on this specific implementation of the geometric phase gate have been studied in [25]. We study how the heating of the ions caused by classical electric field fluctuations affect the performance of this gate, and whether it can be improved by using feedback.

To apply the master equation (18) to this phase gate, we need to include the internal qubit states of the ions in the total state of the system, and use the qubit-dependent forces $f_j(t) = \sigma_j^z F(t)$, where σ_j^z is the Pauli Z operator for ion j, and force $F(t)$ is the force constructed for this phase gate in [14]. This makes the ions experience different forces depending on their internal states, which leads to the ions having different trajectories, and therefore different phases. To see the impact of the noise and feedback on the performance of the phase gate, we calculate numerically the fidelity [26]

$$\mathcal{F} = \sqrt{\langle\psi_{\rm id}|\overline{\rho}(t)|\psi_{\rm id}\rangle}, \tag{32}$$

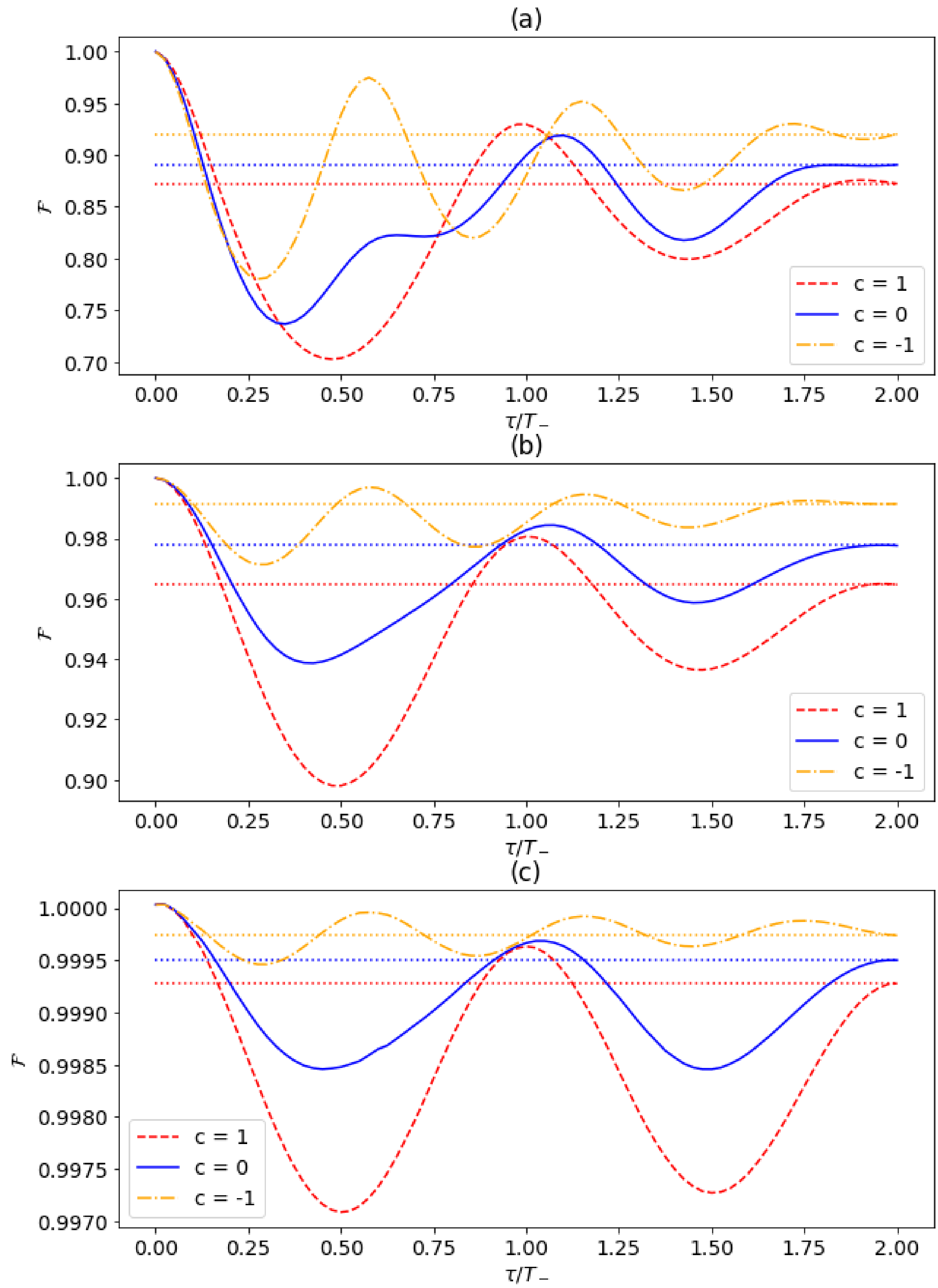

Figure 6. Fidelity of the phase gate as a function of the feedback delay τ for different values of the noise correlation coefficient c. The operation time of the phase gate is $2T_{-}$. For each value of c, there is a horizontal line for the fidelity without feedback, and a curved line for the fidelity when feedback is included. In (a), the ions are affected by white noise. In (b) and (c), they are affected by Ornstein-Uhlenbeck noise with the correlation times $0.3T_{-}$ and $3T_{-}$, respectively.

between the ideal final state $|\psi_{\rm id}\rangle$, where the system is not affected by noise, and the final state $\overline{\rho}(t)$ as the system evolves according to the master equation (18). The fidelity measures how close the two states are, and the value of the fidelity is between 0 and 1, where 1 means that the states are equal. We choose the operation time $2T_{-}$ for the phase gate, and the initial qubit state $|10\rangle\langle 10|$. This qubit state commutes with the operators σ_j^z in the forces $f_j(t)$, and the qubit state is not changed during the evolution, so all the commutators containing $P_{\pm}$ in the master equation (18) vanish.

Figure 6 shows how the fidelity $\mathcal{F}$ depends on the feedback delay τ for different noise correlation coefficients c. The horizontal lines show the fidelity when the ions are affected by the noise without the feedback. The curved lines show the fidelity when the feedback is included. Figure 6(a) shows the fidelity of the phase gate for white noise. We can see that for correlated noise ($c = 1$), the peaks in the fidelity match with integer multiples of T_{-}. This kind of noise only affects the center-of-mass mode, and we found out in the last section that we can completely cancel the noise in this mode by having a delay $\tau = nT_{-}$, where n is an integer. For $n > 0$, even though the noise is completely canceled, it still affects the system at times $t < \tau$, which results in loss of fidelity. The same happens for anticorrelated noise ($c = -1$) and the stretch mode, with peaks at $\tau = kT_{+} = kT_{-}/\sqrt{3} \approx k \cdot 0.58T_{-}$ for integer k. The fidelity is higher when the noise only affects the stretch mode, because the heating rate is lower. For uncorrelated noise ($c = 0$), we can see an improvement where these peaks overlap, at around $\tau = 1.1T_{-}$, which is expected based on the results of the last section.

The fidelity can be improved at similar values of the delay τ for the Ornstein-Uhlenbeck noise. Figure 6(b) shows the fidelity for the correlation time $1/\Gamma = 0.3T_{-}$. The overall fidelity is better for the Ornstein-Uhlenbeck noise, because the heating of the normal modes is slower compared to the white noise. Figure 6(c) shows the

Noise	c	τ/T_-	$\mathcal{F}$	$\mathcal{F}_{fb}$
White	-1	0.57	0.921	0.975
	0	1.09	0.890	0.919
	1	1.00	0.872	0.930
Ornstein-Uhlenbeck $1/\Gamma = 0.3T_-$	-1	0.58	0.9914	0.9970
	0	1.06	0.9776	0.9845
	1	1.01	0.9647	0.9806
Ornstein-Uhlenbeck $1/\Gamma = 3T_-$	-1	0.57	0.99974	0.99996
	0	1.04	0.99950	0.99969
	1	1.00	0.99928	0.99963

Table I. Fidelities of the phase gate with ($\mathcal{F}_{fb}$) and without ($\mathcal{F}$) feedback for the white noise, and for the Ornstein-Uhlenbeck noise for two different values of the correlation time $1/\Gamma$. The feedback delay τ for each correlation coefficient c was chosen so that the improvement in fidelity is the greatest, excluding very short delays. T_- is the period of the center-of-mass mode.

fidelity for a correlation time that is longer than the phase gate operation time, $1/\Gamma = 3T_-$. Table I shows the best possible fidelities achieved in the figure 6 for the different correlation times, and different correlation coefficients. It also shows the value of the feedback delay τ where this highest fidelity was achieved. Small values of the feedback delays were excluded, because at $\tau = 0$, the fidelity would always be 1, and it is also more realistic to assume that the measurement and feedback process takes some amount of time.

5 Conclusions

Trapped ions have good properties to be used in quantum information processing. Qubits can be encoded in their internal states, and multiple ions can be placed in a single trap, where the motion of the ions is coupled by the Coulomb repulsion, and this coupled motion can be used to implement multi-qubit gates to create entangled states. The ions couple to the electromagnetic field, which makes electric field noise a problem, because it causes decoherence, dissipation and heating, turning quantum states into classical ones. We want to use the quantum features of the system in information processing, and that is why it is important to mitigate the effects of the noise as much as possible.

We have derived a master equation for the average state for two ions in an ion trap with a fluctuating electric field that is modeled as a Gaussian stochastic process. We describe the motion of the ions using normal modes. This model leads to increasing kinetic energy in the normal modes, and the rate of increase depends on the correlation in the noise between the positions of the two ions. For perfectly correlated noise, only the center-of-mass mode heats up, and for perfectly anticorrelated noise, only the stretch mode heats up. For uncorrelated noise, both normal modes heat up at a slower rate. The heating is slower in the stretch mode, because it has a higher frequency. The longer the correlation time of the noise is, the slower the heating is.

Then we added a feedback process to the system, where the noise is measured and fed back to the system after some delay τ, and we derived another master equation with this process. This feedback affects the rate at which the normal modes heat up. It can either increase or decrease the rate, depending on the delay τ. If the delay is equal to an integer multiple of the oscillation period of a normal mode, the heating is completely canceled for that normal mode. If the delay is close but not equal to that, the heating can still be reduced. Because perfectly correlated or anticorrelated noise affects only one of the normal modes, the heating in that normal mode can be canceled with feedback, and as a result neither of the normal modes experience heating.

Finally, we applied these models of noise and feedback to a geometric phase gate.

We investigated how the noise and feedback affect the performance of the gate by calculating numerically the fidelities of the final states from the master equations. The fidelity can be improved with a specific choice of the feedback delay τ that again depends on the noise correlation. If the noise only affects one of the normal modes, the delay has to be close to an integer multiple of the period of that normal mode. If the noise affects both normal modes, the delay has to be chosen to be close to an integer multiple of both periods at the same time. The fidelity is improved for any correlation time of the noise.

In section 3, we assumed that we can somehow measure the stochastic processes $\xi_j(t)$. This would mean measuring the fluctuations in the electric field at the positions of the two ions. In [27], a feedback process was used to cool an ion by continuously measuring the position of the ion. This could be an indirect way of measuring the fluctuations because they of course affect the motion of the ion. However, it would not be applicable to the phase gate, because the internal electronic states are used as a qubit for the gate, and the fluorescence used to measure the position of the ion also requires using the electronic states. Measuring the position would also destroy the entanglement that may happen between the position and the internal states during the gate operation. So it is unclear whether the noise can actually be measured using this technique, but if it can be, then this could be used to reduce the heating of trapped ions.

More work could still be done on this topic. We have not considered superpositions of the qubit states, which would be needed to create entangled states. For a superposition of qubit states, the terms in the master equation (18) with the functions $P_{\pm}(t)$ do not vanish, so it would be good to study how they affect the system. One could also consider how different values of the noise strength D affect the fidelity of the geometric phase gate, and also what happens if the noise strength is different for the two ions, because then the mixed terms in the master equation (18) that have operators for both normal modes are nonzero.

A The functional derivative of the state

To calculate the functional derivative of $\rho(t)$ with respect to $\xi_j(s)$, we can use the expression (10) and the product rule. We then have to calculate the functional derivative of each of the unitary operators $U_{q\pm}(t)$ and $U_{p\pm}(t)$ and their Hermitian conjugates. For example, $U_{q+}(t)$ fully written out is

$$\begin{aligned} U_{q+}(t) &= \exp\left(\frac{1}{i\hbar}P_+(t)q_+\right) \\ &= \exp\left(\frac{1}{i\hbar\sqrt{2m}}\int_0^t (f_1(u)+\xi_1(u)-(f_2(u)+\xi_2(u)))\cos(\Omega_+u)\,\mathrm{d}u\,q_+\right), \end{aligned} \tag{33}$$

and we can take the functional derivative with respect to $\xi_1(s)$ by using the chain rule

$$\begin{aligned} \frac{\delta U_{q+}(t)}{\delta\xi_1(s)} &= \frac{1}{i\hbar\sqrt{2m}}\int_0^t \frac{\delta\xi_1(u)}{\delta\xi_1(s)}\cos(\Omega_+u)\,\mathrm{d}u\,q_+U_{q+}(t) \\ &= \frac{1}{i\hbar\sqrt{2m}}\cos(\Omega_+s)q_+U_{q+}(t), \end{aligned} \tag{34}$$

where we have used the fact that $\delta\xi_1(u)/\delta\xi_1(s) = \delta(u-s)$. Now one of the terms after using the product rule is

$$\begin{aligned} &U_{0+}\frac{\delta U_{q+}(t)}{\delta\xi_1(s)}U_{p+}U_{0-}U_{q-}U_{p-}U_f\rho(0)U_f^\dagger U_{p-}^\dagger U_{q-}^\dagger U_{0-}^\dagger U_{p+}^\dagger U_{q+}^\dagger U_{0+}^\dagger \\ =&\frac{1}{i\hbar\sqrt{2m}}\cos(\Omega_+s)U_{0+}q_+U_{q+}U_{p+}U_{0-}U_{q-}U_{p-}U_f\rho(0)U_f^\dagger U_{p-}^\dagger U_{q-}^\dagger U_{0-}^\dagger U_{p+}^\dagger U_{q+}^\dagger U_{0+}^\dagger \\ =&\frac{1}{i\hbar\sqrt{2m}}\cos(\Omega_+s)U_{0+}q_+U_{0+}^\dagger\rho(t), \end{aligned} \tag{35}$$

where we have inserted $I = U_{0+}^\dagger U_{0+}$ between q_+ and U_{q+} on the second line. The functional derivatives of the other unitary operators can be calculated in the same way, and this results in the expression

$$\begin{aligned} \frac{\delta\rho(t)}{\delta\xi_1(s)} =&\frac{1}{i\hbar\sqrt{2m}}\Big(\cos(\Omega_+s)\left[U_{0+}q_+U_{0+}^\dagger,\rho(t)\right]+\cos(\Omega_-s)\left[U_{0-}q_-U_{0-}^\dagger,\rho(t)\right] \\ &+\frac{\sin(\Omega_+s)}{\Omega_+}\left[U_{0+}U_{q+}p_+U_{q+}^\dagger U_{0+}^\dagger,\rho(t)\right]+\frac{\sin(\Omega_-s)}{\Omega_-}\left[U_{0-}U_{q-}p_-U_{q-}^\dagger U_{0-}^\dagger,\rho(t)\right]\Big). \end{aligned} \tag{36}$$

The operators in the commutators can be calculated by writing the position and momentum operators using the creation and annihilation operators

$$
\begin{aligned}
q_\pm &= \sqrt{\frac{\hbar}{2\Omega_\pm}}(a_\pm^\dagger + a_\pm), \\
p_\pm &= i\sqrt{\frac{\hbar\Omega_\pm}{2}}(a_\pm^\dagger - a_\pm),
\end{aligned}
\tag{37}
$$

and using the commutation relations $[a_\pm, a_\pm^\dagger] = I$:

$$
\begin{aligned}
U_{0\pm} q_\pm U_{0\pm}^\dagger &= q_\pm \cos(\Omega_\pm t) - \frac{p_\pm}{\Omega_\pm}\sin(\Omega_\pm t), \\
U_{0\pm} U_{q\pm} p_\pm U_{q\pm}^\dagger U_{0\pm}^\dagger &= U_{0\pm}(p_\pm + P_\pm(t))U_{0\pm}^\dagger \\
&= p_\pm \cos(\Omega_\pm t) + q_\pm \Omega_\pm \sin(\Omega_\pm t) + P_\pm(t).
\end{aligned}
\tag{38}
$$

Now using these expressions and some trigonometric identities in the equation (36), we get the final expression (15) for the functional derivative. The same calculation can be repeated when taking the functional derivative with repect to $\xi_2(s)$, with the difference that the stretch mode (+) terms have a minus sign because of how the forces $F_\pm(t)$ depend on ξ_2 according to the equations (6) and (8).

B The functional derivative with feedback

The process of calculating the functional derivative with feedback is mostly the same as without feedback. We just need to pay attention to the integration limits and delta functions. Now the unitary operator $U_{q_+}(t)$ written out is

$$
\begin{aligned}
U_{q_+}(t) =& \exp\left(\frac{1}{i\hbar}P_+(t)q_+\right) \\
=& \exp\left(\frac{1}{i\hbar}\int_0^t (f_+(u) + \frac{1}{\sqrt{2m}}(\xi_1(u) - \xi_2(u)))\cos(\Omega_+ u)\,\mathrm{d}u\, q_+\right. \\
&\left. - \frac{1}{i\hbar\sqrt{2m}}\int_\tau^t (\xi_1(u-\tau) - \xi_2(u-\tau))\cos(\Omega_+ u)\,\mathrm{d}u\, q_+\right),
\end{aligned}
\tag{39}
$$

where the lower limit for the integral in the feedback terms is τ, because that is when the feedback starts. The functional derivative of this is

$$\begin{aligned}
\frac{\delta U_{q+}(t)}{\delta \xi_1(s)} &= \frac{1}{i\hbar\sqrt{2m}}\left(\int_0^t \frac{\delta\xi_1(u)}{\delta\xi_1(s)}\cos(\Omega_+ u)\,\mathrm{d}u - \int_\tau^t \frac{\delta\xi_1(u-\tau)}{\delta\xi_1(s)}\cos(\Omega_+ u)\,\mathrm{d}u\right) q_+ U_{q+}(t)\\
&= \frac{1}{i\hbar\sqrt{2m}}\left(\cos(\Omega_+ s) - \int_\tau^t \delta(u-\tau-s)\cos(\Omega_+ u)\,\mathrm{d}u\right) q_+ U_{q+}(t)\\
&= \begin{cases} \frac{1}{i\hbar\sqrt{2m}}\cos(\Omega_+ s) q_+ U_{q+}(t) & s > t-\tau\\ \frac{1}{i\hbar\sqrt{2m}}(\cos(\Omega_+ s) - \cos(\Omega_+(s+\tau))) q_+ U_{q+}(t) & s \le t-\tau, \end{cases}
\end{aligned} \tag{40}$$

which now depends on s, which has to be taken into account when integrating the functional derivative over s in (25). For $s > t-\tau$, the functional derivative is the same as without feedback. For $s \le t-\tau$, the full functional derivative is

$$\begin{aligned}
\frac{\delta\rho(t)}{\delta\xi_1(s)} = \frac{1}{i\hbar\sqrt{2m}}\Bigg(&(\cos(\Omega_+ s) - \cos(\Omega_+(s+\tau)))\left[U_{0+} q_+ U_{0+}^\dagger, \rho(t)\right]\\
&+ (\cos(\Omega_- s) - \cos(\Omega_-(s+\tau)))\left[U_{0-} q_- U_{0-}^\dagger, \rho(t)\right]\\
&+ \frac{\sin(\Omega_+ s) - \sin(\Omega_+(s+\tau))}{\Omega_+}\left[U_{0+} U_{q+} p_+ U_{q+}^\dagger U_{0+}^\dagger, \rho(t)\right]\\
&+ \frac{\sin(\Omega_- s) - \sin(\Omega_-(s+\tau))}{\Omega_-}\left[U_{0-} U_{q-} p_- U_{q-}^\dagger U_{0-}^\dagger, \rho(t)\right]\Bigg).
\end{aligned} \tag{41}$$

We can use the equations (38) again for the commutators, and we will get the final expression (26) for the functional derivative. The functional derivative with respect to $\xi_2(s)$ again has the opposite sign for the stretch mode.

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