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Brief paper

Stochastic optimal control over unreliable communication links ☆

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ABSTRACT

In this paper LQG control over unreliable communication links is derived. That is to say, the communication channels between the controller and the actuators and between the sensors and the controller are unreliable. This is of growing importance as networked control systems and use of wireless communication in control are becoming increasingly common. The problem of how to optimize LQG control in this case is examined in the situation where communication between the components is done with acknowledgments. Previous solutions to finite horizon discrete time hold-input LQG control for this case do not fully utilize the available information. Here a new solution is presented which resolves this limitation. The solution is linear and covers communication channels subject to both packet losses and delays. The new control scheme is compared with previous solutions for LQG control in simulations, which demonstrates that a significant improvement in the cost can be achieved by fully utilizing the available information.

1. Introduction

The use of wireless communication between sensors, actuators and controllers can create large savings by avoiding costly wiring and hardware. Depending on locations and external conditions, however, it can be difficult to ensure reliable communication. This raises numerous issues for control and estimation, as discussed in Hespanha, Naghshtabrizi, and Xu (2007).

LQG control (optimized state feedback control for linear systems with Gaussian disturbance and quadratic cost) generally used in conjunction with Kalman filters, is a well established method applicable to MIMO systems (Kwakernaak & Sivan, 1972). It was introduced already in the sixties and remains one of the most implemented type of controllers. Consequently, it is important to examine how to determine an LQG controller when communication channels are unreliable.

When dealing with LQ control over unreliable links there are several different aspects to consider (see Table 1), each implying different solutions.

First of all there are two basic ways a channel can be unreliable, one being *packet drops* and the other *packet delays*. LQG control in the case of packet losses has been examined in Hadjicostis and Touri (2002), Imer, Yüksel, and Basar (2006), Moayedi, Foo, and Soh (2010) and Schenato, Sinopoli, Franceschetti, Poolla, and Sastry (2007). Then the cost can be over a finite or infinite time horizon. Infinite horizon LQG control for the delayed case has been examined in Lincoln and Bernhardsson (2000) and Shousong and Qixin (2003), where it is shown that the optimal solution will not only depend on the states but also on the previous control signals. In Lincoln and Bernhardsson (2000) and Wang et al. (2018) infinite horizon control for the case of a system with both random time delays and packet dropouts is investigated. However, no explicit way to derive the solution is presented. In Liang, Xu, and Zhang (2016) and Ma, Qi, and Zhang (2017) finite horizon LQ control is examined for the case of packet losses and constant delays.

When there is an unreliable channel between the controller and the actuator this means that the latest signal sent might not yet have arrived when the actuator needs to execute a new control action. When this occurs there are two basic strategies the actuator can adopt (Schenato, 2009; Yu & Fu, 2013). One is to apply the last input received until a more recent one arrives, which is known as *hold-input*. In the other one, the input is set

Table 1
LQG control over unreliable links — problem formulations (cases covered here in white).

Type	Packet loss	Packet delays
Control	Finite horizon	Infinite horizon
Strategy	Zero-input	Hold input
Signal	No acknowledgments (UDP)	Acknowledgments (TCP)

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to zero if the latest control signal sent is delayed or lost, which is known as *zero-input*. Zero-input treats delayed and lost signals the same way and optimizing control for such systems has been derived in Imer et al. (2006). A third alternative is to at each time instant send a series of signals together with a specification when they should be applied, assuming no later signal has arrived (Fischer, Hekler, Dolgov, & Hanebeck, 2013; Moayedi, Foo, & Soh, 2011). Although this has obvious potential advantages it implies increased communication with a consequent risk of additional delays.

Finally, there are two basic types of unreliable communication links. In one the sender does not know if the sent packet has arrived, which is known as the *UDP-like case*. In the other one there is a system of acknowledgment that ensures that the sender knows if the sent packet has arrived. This is referred to as the *TCP-like case*. As indicated in Table 1, we will focus on the TCP-like case, designing a hold-input control strategy for a system subject to a random unbounded delay and to packet losses. Furthermore, we will assume the acknowledgments arrive without losses or delays. This may seem contradictory, but uplink and downlink communication are often handled differently and at different rates.

Most previous works have focused on infinite horizon control, though without presenting an explicit solution, see Lincoln and Bernhardsson (2000) and Wang et al. (2018). Finite horizon and a variant of zero-input has been examined in a recent work proposing a solution where the channels are modeled as Markov chains (Xu, Gu, Tang, & Qian, 2022). As a consequence, though, the complexity grows exponentially with the number of delays, thus requiring a bound on the delays. To be optimal the method is also non-causal in the sense that the controller must at each time know the delay on beforehand. We will also focus on finite horizon control, but for hold-input, and present an explicit solution fully utilizing all available information and taking into account all interdependencies. A similar case was examined in Bengtsson, Hassibi, and Wik (2016), but only for a specific probability function for the delay. Moreover, the knowledge whether the sent packet has arrived or not was then only used to facilitate the estimation of the states. Here, this knowledge is also used to optimize the control scheme, yielding a truly optimal solution. This optimality, though, comes at the price of increased complexity of the solution and increased computational cost.

Finally, to derive the explicit optimal control we assume the probability functions of the delays and packet losses are known. For cases when the probability functions are unknown most conventional methods cannot be used, though Q-learning can then be a way to derive solutions as discussed in Xu, Jagannathan, and Lewis (2012).

The structure of this article is as follows: First the problem is presented in Section 2, and then the solution is presented in Section 3. In Section 4 the optimal state estimation to use for feedback is presented. How to best implement the solution to lower the computational burden is discussed in Section 5, and in Section 6 the method is evaluated in simulations. A summary of the solution is included in Appendix, written for ease of implementation.

2. Problem formulation

The plant considered is assumed to be an LTI system,

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

where $x \in R^n$ is the state vector, $w \in R^n$ is white Gaussian noise, and $u_k \in R^m$ is the control signal applied by the actuator at time k (for hold-input control this will be the latest control signal that has arrived). As the noise is white, and the LQG cost criterion is with respect to expectation, the noise will not impact the optimal control scheme and therefore we will disregard it from now on. For now the state x_k is assumed to be known at time k, though in Section 6 it is shown how x_k can be replaced by its estimate.

The communication between the controller and the actuator is subject to a random delay, assumed to follow a known probability function $p(d)$, where d is the number of samples the packet is delayed. Note that there are no requirements on $p(d)$; it can be of both finite or infinite length, and it can be used to describe systems with both delays and packet losses. In the latter case $\{p(d)\}_{d=0}^{\infty}$ will simply sum up to less than one.

The delays between consecutive steps are assumed to be independent. From this assumption the probability that the latest control signal that has arrived is the signal sent i time units before can be derived as

$$p_d(i) = P(i)\prod_{j=0}^{i-1}\bar{P}(j), \tag{2}$$

where $P(i) = \sum_{k=0}^{i} p(k)$ denotes the cumulative probability and $\bar{P}(i)$ denotes the complementary probability to $P(i)$, i.e. $\bar{P}(i) = 1-P(i)$, which is the probability that none of the latest i signals have arrived.

Another way to express $p_d(i)$ is

$$p_d(i) = P(i)\bar{P}_d(i-1), \tag{3}$$

where $\bar{P}_d(i-1)$ is the complementary probability of the cumulative probability $P_d(i-1) = \sum_{k=0}^{i-1} p_d(k)$.

Now, the goal is to design a controller that determines the control signals v, sent from the controller, which minimizes the quadratic criterion

$$J_N = E\left[\sum_{i=0}^{N} x_i^T Q x_i + \sum_{i=0}^{N} u_i^T R u_i + x_{N+1}^T S_{N+1} x_{N+1}\right], \tag{4}$$

where R is a positive definite symmetric matrix and Q and S_{N+1} are positive semi-definite symmetric matrices.

As previously mentioned, we will examine the TCP-like case, where the controller knows if a signal has reached the actuator or not. To handle this we use a variable size controller state ζ_k holding x_{k+1} and all issued control signals v_i since the last one acknowledged, i.e.

$$\zeta_k = \begin{bmatrix} v_k^T & \cdots & v_{\tau_k}^T & x_{k+1}^T \end{bmatrix}^T$$

where τ_k, is the time the latest control signal arrived was sent, i.e. $u_k = v_{\tau_k}$. The update ζ_{k+1} of the state ζ_k follows from the update of τ_k, i.e.

$$\tau_{k+1} = \begin{cases} \tau, & \text{if } v_\tau,\ \tau > \tau_k, \text{ is the most recently} \\ & \text{acknowledged signal} \\ \tau_k, & \text{if no, more recent, } v \text{ is acknowledged} \end{cases} \tag{5}$$

Note that in the update of τ_k, obsolete acknowledgments are always discarded.

3. Optimal LQG control

In this section we will present the solution and describe how it is derived. Unfortunately, the derivations are lengthy and therefore the full detailed derivations are presented in Bengtsson and Wik (2021). Similar to the derivation of the standard LQG solution we will use dynamic programming. This means that we will start by finding the last optimal control signal v_N that minimizes the cost function (4), expressed in terms of states and previous control signals available at that time, i.e. ζ_{N-1}. For the remaining cost we will then find the control signal v_{N-1} that minimizes this cost expressed in signals available at that time, ζ_{N-2}. After this, v_{N-2} is found to minimize the now remaining cost. Repeating this once more reveals a pattern of induction such that all the remaining v_k can be calculated.

We start by noting that

$$E\left[u_i^T R u_i\right] = \sum_{j=0}^{i} p_d(i-j) v_j R v_j. \tag{6}$$

From this and from the fact that the probability that u_k is applied by the actuator at time k is unaffected by the delays of earlier control signals, the second term in (4) can be expressed as

$$\begin{aligned} E\left[\sum_{i=0}^{N} u_i^T R u_i\right] =& E\left[\sum_{i=0}^{N}\sum_{j=0}^{i} p_d(i-j) v_j^T R v_j\right] \\ =& E\left[\sum_{i=0}^{N}\sum_{j=0}^{N-i} p_d(j) v_i^T R v_i\right] \\ =& E\left[\sum_{i=0}^{N} P_d(N-i) v_i^T R v_i\right] \\ =& E\left[\sum_{i=0}^{N} v_i^T R_i v_i\right], \end{aligned} \tag{7}$$

where $R_i \triangleq P_d(N-i)R$.

The term $x_{N+1}^T S_{N+1} x_{N+1}$ we wish to express in terms of signals available at time N. Using (1) and disregarding the noise, since it is independent of u_N and x_N and thus does not affect the optimal solution, we have

$$\begin{aligned} E\left[x_{N+1}^T S_{N+1} x_{N+1}\right] =& E\left[u_N^T B^T S_{N+1} B u_N + 2u_N^T B^T S_{N+1} A x_N \right. \\ & \left. + x_N^T A^T S_{N+1} A x_N\right], \end{aligned} \tag{8}$$

where

$$E\left[u_N^T B^T S_{N+1} B u_N\right] = \sum_{i=0}^{N} p_d(N-i) v_i^T B^T S_{N+1} B v_i$$

$$\begin{aligned} E\left[u_N^T B^T S_{N+1} A x_N\right] =& E\left[p_d(0) v_N^T B^T S_{N+1} A x_N \right. \\ & \left. + \bar{P}_d(0) u_{N|u_N \neq v_N}^T B^T S_{N+1} A x_N\right] \end{aligned}$$

and $E\left[u_{N|u_N \neq v_N}\right]$ is the expected value of the actuated control signal at time N given that the control signal that was sent at time N has not yet arrived. Thus

$$\begin{aligned} E\left[x_{N+1}^T S_{N+1} x_{N+1}\right] =& E\left[\sum_{i=0}^{N} p_d(N-i) v_i^T B^T S_{N+1} B v_i \right. \\ & + 2p_d(0) v_N^T B^T S_{N+1} A x_N \\ & + 2\bar{P}_d(0) u_{N|u_N \neq v_N}^T B^T S_{N+1} A x_N \\ & \left. + \; x_N^T A^T S_{N+1} A x_N\right]. \end{aligned} \tag{9}$$

Now, define

$$\begin{aligned} T(0,b) \triangleq & \sum_{i=0}^{N+1-b} v_i^T R_i v_i + \sum_{i=0}^{N+1-b} p_d(N-i) v_i^T B^T S_{N+1} B v_i \\ & + x_N^T A^T S_{N+1} A x_N, \end{aligned} \tag{10}$$

where b is a counter. Increasing it by one removes the latest control signal from the expression. The first argument (0), is used in conjunction with b to specify which is the latest control signal contained in the expression. For example, in this case the latest control signal $T(0,b)$ contains is at $N+1-b-0$. The criterion (4) can then be written as

$$\begin{aligned} J_N =& E\left[\sum_{i=0}^{N} x_i^T Q x_i + T(0,1) + 2p_d(0) v_N^T B^T S_{N+1} A x_N \right. \\ & \left. + \; 2\bar{P}_d(0) u_{N|u_N \neq v_N}^T B^T S_{N+1} A x_N\right]. \end{aligned} \tag{11}$$

To extract the parts of the cost that depend on the latest control signal we introduce

$$T_c(0,b) \triangleq R_{N+1-b} + p_d(b-1) B^T S_{N+1} B, \tag{12}$$

such that

$$T(0,b) = T(0,b+1) + v_{N+1-b}^T T_c(0,b) v_{N+1-b}. \tag{13}$$

The cost (11) can then be expressed as

$$\begin{aligned} J_N =& E\left[\sum_{i=0}^{N} x_i^T Q x_i + v_N^T A_{11}(N) v_N + 2v_N^T A_{12}(N) x_N \right. \\ & \left. + \; 2\bar{P}_d(0) u_{N|u_N \neq v_N}^T B^T S_{N+1} A x_N + T(0,2)\right] \end{aligned} \tag{14}$$

$$A_{11}(N) = T_c(0,1) \tag{15}$$

$$A_{12}(N) = p_d(0) B^T S_{N+1} A. \tag{16}$$

Only two terms in (14) depend on v_N and as A_{11} is positive definite the optimal v_N that minimizes (14) is

$$v_N = -A_{11}^{-1}(N) A_{12}(N) x_N, \tag{17}$$

which results in a minimum cost

$$\begin{aligned} J_N^* =& E\left[\sum_{i=0}^{N} x_i^T Q x_i - x_N^T (A_{12}^T(N) A_{11}^{-1}(N) A_{12}(N)) x_N \right. \\ & \left. + 2\bar{P}_d(0) u_{N|u_N \neq v_N}^T B^T S_{N+1} A x_N + T(0,2)\right]. \end{aligned} \tag{18}$$

The next step is to find the control signal v_{N-1} that minimizes (18). Now taking into consideration (10), (18) is similar to (4) and can be handled in a similar way, except for the term $E\left[2\bar{P}_d(0) u_{N|u_N \neq v_N}^T B^T S_{N+1} A x_N\right]$. The main difficulty then is that x_N depends on u_{N-1} and because we are using hold input there is a co-dependency between $u_{N|u_N \neq v_N}^T$ and u_{N-1}. We have

$$\begin{aligned} & E\left[\bar{P}_d(0) u_{N|u_N \neq v_N}^T B^T S_{N+1} A x_N\right] \\ =& E\left[\bar{P}_d(0) u_{N|u_N \neq v_N}^T B^T S_{N+1} A (A x_{N-1} + B u_{N-1})\right] \\ =& E\left[\bar{P}_d(0) u_{N|u_N \neq v_N}^T B^T S_{N+1} A B u_{N-1} + \bar{P}_d(0) u_{N|u_N \neq v_N}^T B^T S_{N+1} A^2 x_{N-1}\right] \end{aligned} \tag{19}$$

Now $p(u_{N|u_N \neq v_N} = u_{N-1})$ is simply the probability that v_{N-1} has arrived at time N, i.e. $P(1)$, and thus the expression can be expressed as

$$\begin{aligned} =& E\left[\bar{P}_d(0) u_{N|u_N \neq v_N}^T B^T S_{N+1} A B u_{N-1} + P(1)\bar{P}_d(0) v_{N-1}^T B^T S_{N+1} A^2 x_{N-1} \right. \\ & \left. + \bar{P}(1)\bar{P}_d(0) u_{N|u_N \neq v_N, v_{N-1}}^T B^T S_{N+1} A^2 x_{N-1}\right] \\ =& E\left[\bar{P}_d(0) u_{N|u_N \neq v_N}^T B^T S_{N+1} A B u_{N-1} + p_d(1) v_{N-1}^T B^T S_{N+1} A^2 x_{N-1} \right. \\ & \left. + \bar{P}_d(1) u_{N|u_N \neq v_N, v_{N-1}}^T B^T S_{N+1} A^2 x_{N-1}\right]. \end{aligned} \tag{20}$$

Now to evaluate $E\left[\bar{P}_d(0) u_{N|u_N \neq v_N}^T B^T S_{N+1} A B u_{N-1}\right]$ we need to take into account that the sample time of the actuator signal applied at time $N-2$ (referred to as τ_{N-2}) is known. In Bengtsson and Wik (2021) it is shown that it can be expressed as

$$E\left[\bar{P}_d(0) P(0) v_{N-1} B^T S_{N+1} A B v_{N-1} + p(1) v_{N-1} B^T S_{N+1} A B\right.$$

$$\times\left(\sum_{i=1}^{N-\tau_{N-2}-2} p(i)v_{N-i-1}^T + \bar{P}(N-\tau_{N-2}-2)v_{\tau_{N-2}}^T\right)$$
$$+\bar{P}_d(1)u^T_{N|u_N\neq v_N, v_{N-1}}B^TS_{N+1}ABu_{N-1|u_{N-1}\neq v_{N-1}}\Big]. \quad (21)$$

From this and with calculations similar to those in (9)–(10), it can be shown that (18) can be expressed as (Bengtsson & Wik, 2021)

$$\begin{aligned}J_{N-1} =& E\Bigg[\sum_{i=0}^{N-1} x_i^TQx_i + v_{N-1}^TA_{11}(N-1)v_{N-1}.\\ &+ 2v_{N-1}^TA_{12}(N-1,\tau_{N-2})\zeta_{N-2} + T(1,2)\\ &+ 2\bar{P}_d(0)u^T_{N-1|u_{N-1}\neq v_{N-1}}B^TS_NAx_{N-1}^T\\ &+ 2\bar{P}_d(1)u^T_{N|u_N\neq v_N,v_{N-1}}B^TS_{N+1}A^2x_{N-1}\\ &+2\bar{P}_d(1)u^T_{N|u_N\neq v_N,v_{N-1}}B^TS_{N+1}ABu_{N-1|u_{N-1}\neq v_{N-1}}\Bigg],\end{aligned} \quad (22)$$

where

$$A_{11}(N-1) = 2\bar{P}_d(0)P(0)B^TS_{N+1}AB + T_c(1,1) \quad (23)$$

$$\begin{aligned}A_{12}(N-1,\tau_{N-2})\zeta_{N-2} =& M(N-1)x_{N-1}\\ &+ p(1)B^TS_{N+1}AB\left(\sum_{i=1}^{N-\tau_{N-2}-2}p(i)v_{N-i-1}^T\right.\\ &\left.+\bar{P}(N-\tau_{N-2}-2)v_{\tau_{N-2}}^T\right)\end{aligned} \quad (24)$$

$$\zeta_{N-2} = \begin{bmatrix} v_{N-2}^T & v_{N-3}^T & \cdots & v_{\tau_{N-2}}^T & x_{N-1}^T\end{bmatrix}^T$$

$$M(N-k) = \sum_{i=N-k}^{N} p_d(i-(N-k))B^TS_{i+1}(A)^{i-(N-k)+1}$$

Once more, the sought control signal (v_{N-1}) only contributes to two terms of the cost expressed in control signals and delays known at time $N-1$. The cost (22) can therefore be minimized by

$$v_{N-1} = -A_{11}^{-1}(N-1)A_{12}(N-1,\tau_{N-2})\zeta_{N-2}, \quad (25)$$

which gives the optimal cost

$$\begin{aligned}J^*_{N-1} =& E\Bigg[-\zeta_{N-2}^TA_{12}^T(N-1,\tau_{N-2})A_{11}^{-1}(N-1)\\ &\times A_{12}(N-1,\tau_{N-2})\zeta_{N-2} + 2\bar{P}_d(0)u^T_{N-1|u_{N-1}\neq v_{N-1}}B^TS_NAx_{N-1}^T\\ &+ 2\bar{P}_d(1)u^T_{N|u_N\neq v_N,v_{N-1}}B^TS_{N+1}A^2x_{N-1}\\ &+ 2\bar{P}_d(1)u^T_{N|u_N\neq v_N,v_{N-1}}B^TS_{N+1}ABu_{N-1|\neq v_{N-1}}\\ &+T(1,2) + \sum_{i=0}^{N-1}x_i^TQx_i\Bigg].\end{aligned} \quad (26)$$

The next step is to express this cost in terms available at time $N-2$. To resolve this for the first term we start by splitting the expression into parts that depend on the states and parts that depend on the control signal:

$$\begin{aligned}&E\left[\zeta_{N-2}^TA_{12}^T(N-1,\tau_{N-2})A_{11}^{-1}(N-1)A_{12}(N-1,\tau_{N-2})\zeta_{N-2}\right] =\\ &E\left[K_\eta^T(1,\tau_{N-2},1)A_{11}^{-1}(N-1)K_\eta(1,\tau_{N-2},1)\right.\\ &+ x_{N-1}^TM^T(N-1)A_{11}^{-1}(N-1)M(N-1)x_{N-1}\\ &\left.+2K_\theta(1,\tau_{N-2},1)x_{N-1}\right],\end{aligned} \quad (27)$$

where we have introduced

$$\begin{aligned}K_\eta(1,\tau_{N-2},1) =& p(1)B^TS_{N+1}AB\left(\sum_{i=1}^{N-\tau_{N-2}-2}p(i)v_{N-i-1}^T\right.\\ &\left.+\bar{P}(N-\tau_{N-2}-2)v_{\tau_{N-2}}^T\right)\\ K_\theta(1,\tau_{N-2},b) =& K_\eta^T(1,\tau_{N-2},b)A_{11}^{-1}(N-1)M(N-1).\end{aligned}$$

Note that this expression depends on the information of what signal was applied at time $N-2$, i.e. τ_{N-2}. However, this is not known when calculating v_{N-2} and therefore we need to use the probabilities of the possible values of τ_{N-2}. Doing this, using $x_{N-1} = Ax_{N-2}+Bu_{N-2}$ and then splitting the expression into parts which contain v_{N-2} and parts which do not give the following expression (for full proof see Bengtsson and Wik (2021))

$$\begin{aligned}&E\left[\zeta_{N-2}^TA_{12}^T(N-1,\tau_{N-2})A_{11}^{-1}(N-1)A_{12}(N-1,\tau_{N-2})\zeta_{N-2}\right] =\\ &E\big[x_{N-1}^TM^T(N-1)A_{11}^{-1}(N-1)M(N-1)x_{N-1}\\ &+2K_{ux}(2,\tau_{N-3},1)x_{N-2} + 2K_{uu}(2,\tau_{N-3},1)\\ &+2v_{N-2}^TK_{gx}(2,1)x_{N-2} + 2v_{N-2}^TK_{gu}(2,\tau_{N-3},1,1)\\ &+K_\alpha(2,\tau_{N-3},1) + v_{N-2}^TK_{gg}(2)v_{N-2}\\ &+2p(0)v_{N-2}^TK_{\theta RL}(1,1)Bv_{N-2}\big].\end{aligned} \quad (28)$$

The functions $K_{(\cdot)}$ and M are defined in Appendix.

The remaining terms in (26) can be resolved in the same ways as shown previously, with calculations similar to those shown in (20) and (9)–(14), to yield

$$\begin{aligned}J_{N-2} = E\Bigg[&-v_{N-2}^TK_{gg}(2)v_{N-2} - 2p(0)v_{N-2}^TK_{\theta RL}(1,1)B\\ &+2v_{N-2}T_c(2,1)v_{N-2} + 2v_{N-2}^TK_e(2)v_{N-2}\\ &+2K_\zeta(2,\tau_{N-3},1)v_{N-2} + 2v_{N-2}^TM(N-2)x_{N-2}\\ &-2v_{N-2}^TK_{gx}(2,1)x_{N-2} - 2v_{N-2}^TK_{gu}(2,\tau_{N-3},1,1)\\ &-2K_{ux}(2,\tau_{N-3},1)x_{N-2} - 2K_{uu}(2,\tau_{N-3},1)\\ &-K_\alpha(2,\tau_{N-3},1) + 2F(N-2) + T(2,2) + 2K_a(2)\\ &+\sum_{i=0}^{N-2}x_i^TQx_i\Bigg].\end{aligned} \quad (29)$$

The functions $K_{(\cdot)}$, T_c and M, are all presented in Appendix ($F(N)$ is an intermediary function not part of the final solution). None of the functions depend on v_{N-2} and, therefore, this cost is minimized by

$$v_{N-2} = -A_{11}^{-1}(N-2)A_{12}(N-2,\tau_{N-3})\zeta_{N-3}, \quad (30)$$

where

$$\begin{aligned}A_{11}(N-2) =& -K_{gg}(2) - 2p(0)K_{\theta RL}(1,1)B\\ &+T_c(2,1) + 2K_e(2)\end{aligned} \quad (31)$$

$$\begin{aligned}A_{12}(N-2,\tau_{N-3})\zeta_{N-3} =& K_\zeta(2,\tau_{N-3},1) - K_{gx}(2,1)x_{N-2}\\ &+M(N-2)x_{N-2}\\ &-K_{gu}(2,\tau_{N-3},1,1).\end{aligned} \quad (32)$$

Now, denote the cost corresponding to the optimal cost J^*_{N-j+1} by J_{N-j}, which is minimized by v_{N-j}. Using similar calculations as before (see Bengtsson and Wik (2021)) it can be shown that

$$\begin{aligned}J_{N-j} = E\Big[&v_{N-j}^TA_{11}(N-j)v_{N-j} + 2v_{N-j}^TA_{12}(N-j,\tau_{N-j-1})\zeta_{N-j-1}\\ &-2K_{ux}(j,\tau_{N-j-1},1,1)x_{N-j} - 2K_{uu}(j,\tau_{N-j-1},1)\\ &-K_r(j,\tau_{j-1},1) + 2F(N-j) + T(j,2) + 2K_a(j)\end{aligned}$$

$$+E\sum_{i=0}^{N-j} x_i^T Q x_i \Bigg], \quad (33)$$

where expressions for the functions $K_{(\cdot)}$ can be found in Appendix and

$$A_{11}(N-j) = -K_{gg}(j) - 2p(0)K_{\theta RL}(j-1,1)B + T_c(j,1) + 2K_e(j) \quad (34)$$

$$\begin{aligned} A_{12}(N-j,\tau_{N-j-1})\zeta_{N-j-1} =& K_\zeta(j,\tau_{N-j-1},1) - K_{gx}(j,1)x_{N-j} \\ &+ M(N-j)x_{N-j} - K_{gu}(j,\tau_{N-j-1},1,1), \end{aligned} \quad (35)$$

which are on exactly the same form as (31) and (32). By induction we have thus proven that the optimal control is given by

$$\begin{aligned} v_N &= A_{11}^{-1}(N)A_{12}(N)x_N \\ v_{N-k} &= A_{11}^{-1}(N-k)A_{12}(N-k,\tau_{N-k-1})\zeta_{N-k-1}, \;\; k \geq 1. \end{aligned}$$

where A_{11} is given by (15), (23), (31) and (34), $A_{12}(N)$ is given by (16), and $A_{12}(N-j,\tau_{N-j-1})\zeta_{N-j-1}$ is given by (24), (32), and (35).

4. Optimal estimation

In the previous section we have assumed that the state at all times is known. However, in many cases the states are not available and we must rely on a measured output

$$y_k = Cx_k + e_k,$$

where we assume e is white Gaussian noise. In addition, there may be communication delays also between the sensor and the controller. We will here discuss how the states can be estimated, and show that the separation principle holds such that the optimal control solution is the same but with estimated states replacing the true states. As acknowledgments ensure that it is known which inputs are applied at each time instance, the states can be estimated optimally using a traditional Kalman filter (Lincoln & Bernhardsson, 2000), i.e.

$$\hat{x}_{k|k} = A\hat{x}_{k-1|k-1} + Bu_{k-1} + K_kCA\tilde{x}_{k-1|k-1} + K_kCw_k + K_ke_k, \quad (36)$$

where $\tilde{x}_{k-1|k-1} = x_{k-1} - \hat{x}_{k-1|k-1}$ is the estimation error and K_k is the Kalman gain, set to zero for measurements y_k that are not available because of delays or losses. We can express this as

$$\hat{x}_{k|k} = A\hat{x}_{k-1|k-1} + Bu_{k-1} + w_k^*, \quad (37)$$

where $w_k^* = K_kCA\tilde{x}_{k-1|k-1} + K_kCw_k + \bar{K}_ke_k$ is white noise as $\tilde{x}_{k-1|k-1}$, w_k and e_k are all independent white noises. Furthermore, by substituting $x_i = \hat{x}_{i|i} + \tilde{x}_{i|i}$ in the criterion (4) and using that $\hat{x}_{i|i}$ and $\tilde{x}_{i|i}$ are uncorrelated (shown for $\hat{x}_{i+1|i}$ and $\tilde{x}_{i+1|i}$ in Åström (1970), which can trivially be expanded for $\hat{x}_{i|i}$ and $\tilde{x}_{i|i}$), we get

$$\begin{aligned} J_N =& \min_u E\sum_{i=0}^{N} \hat{x}_i^T Q\hat{x}_i + \sum_{i=0}^{N} u_i^T R u_i + \hat{x}_{N+1}^T S_{N+1}\hat{x}_{N+1} \\ &+ \sum_{i=0}^{N} \tilde{x}_i^T Q\tilde{x}_i + \tilde{x}_{N+1}^T S_{N+1}\tilde{x}_{N+1}, \end{aligned} \quad (38)$$

where $\hat{x}_i$ denotes $\hat{x}_{i|i}$ and $\tilde{x}_i = \tilde{x}_{i|i}$. Since $\tilde{x}_i$ is white (Åström, 1970), $\sum_{i=0}^{N} \tilde{x}_N^T Q\tilde{x}_N + \tilde{x}_{N+1}^T S_{N+1}\tilde{x}_{N+1}$ does not influence the choice of control signal. As (37) is also on the same form as (1) the optimal control problem for estimated states must be the same as for known states. Therefore the solution will be the same as for the full information case and thus the separation principle holds.

5. Computational implementation

The optimal solution as derived here is computationally demanding as it requires a great deal of function evaluations. However, the same functions are evaluated many times with the same inputs, so by saving the result of each function evaluation and reusing the result whenever the function is called with the same inputs the computational burden can be reduced considerably. Doing so is nearly a necessity for implementing the solution, as otherwise horizons longer than $N = 10$ become so computationally demanding that real time implementation is unrealistic today.

Another computational issue is that parts of the expression for A_{12} depend on the current delay and hence should in principle be calculated online. However, if the delay is bounded, A_{12} can be calculated for all possible delays offline, which significantly reduces the amount of online computations required.

6. Results and evaluation

To evaluate our results we first use the fact that the solution can be used to generate an expected value of the cost, namely $x_0^T S_0 x_0$, where x_0 is the initial value. From simulation of different systems and with different probabilities of delays and packet loss we could confirm that the mean cost of the simulations indeed converges to this value.

Most other similar methods are designed for infinite horizon control and optimized for short delays which makes direct comparisons difficult. In previous work (Bengtsson et al., 2016) we have presented the solution for optimal LQG control when the knowledge of whether the packet has arrived or not is only used for the state estimation, which we will henceforth refer to as the simplified solution. This solution is computationally easier but should of course be inferior to the one derived here. To illustrate we compare it with the solution proposed here, using the specific delay probability function assumed in Bengtsson et al. (2016), i.e.,

$$p(d) = (1-\alpha)\alpha^d.$$

Two systems were tested, a simple system on the form of

$$x(t+1) = ax(t) + u(t) \quad (39)$$

and a more complex second order system

$$\begin{aligned} x(t+1) &= \begin{bmatrix} a-0.95 & 1 \\ 0.95a & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \omega(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned} \quad (40)$$

where $\omega(t)$ is white Gaussian of variance 0.01. This system has poles in a and -0.95, and a zero in 2.

The comparison was made by comparing the simulated total costs (as given by (4)) for different values of a and α. The other settings were $R = Q = S_{N+1} = I$, and $N = 100$. To see the advantage of considering the uncertainty at all we also simulated a standard LQG control of the systems. For each set of parameters the simulation was repeated 500 times with different random seed to achieve reliable mean costs.

In Figs. 1 and 2 we show both the average and average normalized costs of the different methods on the systems. As can be seen the new method yields considerably better results than the simplified TCP method, especially in cases where the system is highly unstable (large a) and/or the probability of delay is high (large α). It can also be noted that the standard LQG solution without any modifications, due to the delays, performed very poorly on the delayed systems, in many cases exhibiting unstable behavior.

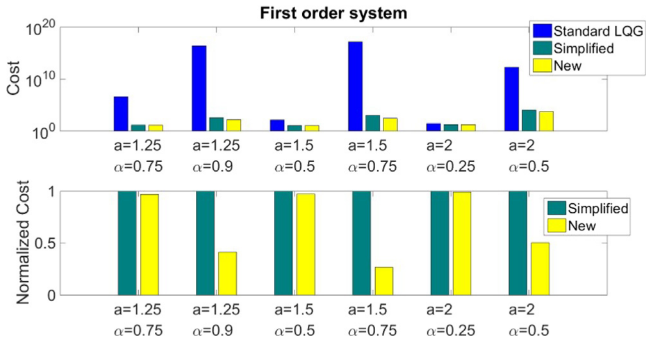

Fig. 1. A comparison between the solution proposed here (yellow) and the solution described in Bengtsson et al. (2016) (green) for different a and α for the first order system (39). Note that the upper diagram is in logarithmic scale. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

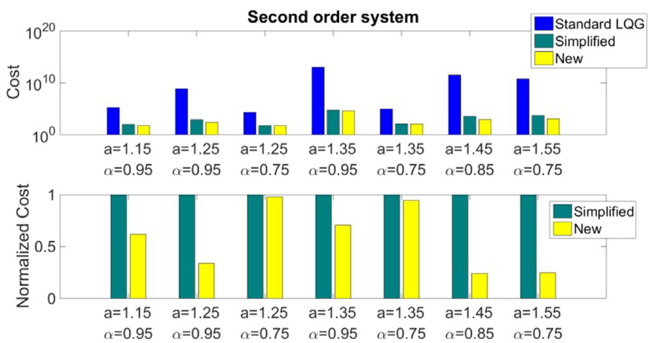

Fig. 2. A comparison between the solution proposed here (yellow) and the solution described in Bengtsson et al. (2016) (green) for different a and α for the second order system (40). Note that the upper diagram is in logarithmic scale. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

7. Conclusion and further work

We have examined LQG control in the case when the channels between the controller and the actuator and the sensor and the controller are unreliable, being subject to packet delays and packet losses. For this case, with the assumption that a system of acknowledgments informs the controller when a signal has reached the actuator, an optimal hold-input LQG controller was derived, yielding a control output which is a linear combination of not only the states but also the previous control signals. This solution was compared to a solution which did not utilize the knowledge of which control signal that has reached the actuator other than for estimation, and it was found that considerable improvements can be achieved.

The resulting controller can be computationally demanding, although this can be alleviated by precomputations. For many applications it would still be too demanding, but then, since the solution is analytical, it can serve as a benchmark for other approximate methods.

Acknowledgements

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Appendix. Implementation summary

The formulas needed to calculate the optimal control are here summarized to facilitate an easy implementation. Note that the only information not known prior to implementation are the previous delays $\tau_{(\cdot)}$. Thus all calculations not including $\tau_{(\cdot)}$ can be done on before hand.

$$\begin{aligned}
T(0,b) &= \sum_{i=0}^{N+1-b} v_i^T R_i v_i + x_N^T A^T S_{N+1} A x_N \\
&\quad + \sum_{i=0}^{N+1-b} p_d(N-i) v_i^T B^T S_{N+1} B v_i \\
R_i &= P_d(N-i)R \\
T(k,b) &= T_{noX}(k,b) + x_{N-k}^T T_X(k) x_{N-k} \\
T(k,b) &= T(k,b+1) + v_{N+1-b}^T T_c(k,b) v_{N+1-b} \\
T_c(0,b) &= R_{N+1-b} + p_d(b-1) B^T S_{N+1} B \\
T_{noX}(0,b) &= \sum_{i=0}^{N+1-b} v_i^T R_i v_i + \sum_{i=0}^{N+1-b} p_d(N-i) v_i^T B^T S_{N+1} B v_i \\
T_X(0) &= A^T S_{N+1} A
\end{aligned}$$

For $k > 0$:

$$\begin{aligned}
T(k,b) &= T_{noX}(k-1,b+1) \\
&\quad + \sum_{i=0}^{N-k+1-b} p_d(N-k-i) v_i^T B^T S_{N-k+1} B v_i \\
&\quad + x_{N-k}^T A^T S_{N-k+1} A x_{N-k} \\
T_c(k,b) &= T_c(k-1,b+1) + p_d(b-1) B^T S_{N-k+1} B \\
T_{noX}(k,b) &= T_{noX}(k-1,b+1) \\
&\quad + \sum_{i=0}^{N-k+1-b} p_d(N-k-i) v_i^T B^T S_{N-k+1} B v_i \\
T_X(k) &= A^T S_{N-k+1} A
\end{aligned}$$

If $k \leq 1$:

$$S_{N-k} = T_X(k) - M(N-k)^T A_{11}^{-1}(N-k) M(N-k) + Q$$

If $k > 1$:

$$\begin{aligned}
S_{N-k} &= -(M(N-k) - K_{gx}(k,\tau_{N-k-1}))^T A_{11}^{-1}(N-k) \\
&\quad \times (M(N-k) - K_{gx}(k,\tau_{N-k-1})) + T_X(k) + Q
\end{aligned}$$

$$M(N-k) = \sum_{i=N-k}^{N} p_d(i-(N-k)) B^T S_{i+1} (A)^{i-(N-k)+1}$$

$$\begin{aligned}
K_e(k) &= \sum_{i=N-k+1}^{N} \sum_{j=N-k+1}^{i} \bar{P}_d(i-(N-k+1)) \\
&\quad \times P(k+j-N-1) B^T S_{i+1} A^{i-j+1}
\end{aligned}$$

For $k < 3$:

$$\begin{aligned}
K_{gg}(k) &= p(0) K_{\eta RL}^T(k-1,1) A_{11}^{-1}(N-k+1) K_{\eta RL}(k-1,1) \\
&\quad + \bar{p}(0) K_{\eta C}^T(k-1,1) A_{11}^{-1}(N-k+1) K_{\eta C}(k-1,1)
\end{aligned}$$

For $k \geq 3$:

$$\begin{aligned}
K_{gg}(k) &= p(0) K_{\eta RL}^T(k-1,1) A_{11}^{-1}(N-k+1) K_{\eta RL}(k-1,1) \\
&\quad + \bar{p}(0) K_{\eta C}^T(k-1,1) A_{11}^{-1}(N-k+1) K_{\eta C}(k-1,1) \\
&\quad + 2\bar{p}(0) K_{uuCs}(k-1,1) + \bar{p}(0) K_{rCs}(k-1,1) \\
&\quad + p(0)\left(2K_{uuRL}(k-1,1) + K_{rRL}(k-1,1)\right)
\end{aligned}$$

$$K_\alpha(k,\tau,b)=\sum_{j=\tau+1}^{N-k-b} p(N-k-j)K_\eta^T(k-1,j,b+1) \times A_{11}^{-1}(N-k+1)K_\eta(k-1,j,b+1) + \bar{P}(N-k-1-\tau)K_\eta^T(k-1,\tau,b+1) \times A_{11}^{-1}(N-k+1)K_\eta(k-1,\tau,b+1)$$

In the case when $\tau < N-k-b$ then

$$K_\alpha(k,\tau,b)=v_{N-k-b}^T K_{\alpha Cs}(k,b)v_{N-k-b}+K_\alpha(k,\tau,b+1) + K_{\alpha Cd}^T(k,\tau,b+1,b)v_{N-k-b} + v_{N-k-b}^T K_{\alpha Cd}(k,\tau,b+1,b).$$

The subfunctions above are given by

$$K_{\alpha Cs}(k,b)=p(b)K_{\eta RL}^T(k-1,b+1)A_{11}^{-1}(N-k+1) \times K_{\eta RL}(k-1,b+1) + \bar{P}(b)K_{\eta C}^T(k-1,b+1)A_{11}^{-1}(N-k+1) \times K_{\eta C}(k-1,b+1)$$

$$K_{\alpha RL}(k,b)=\bar{P}(b-1)K_{\eta RL}^T(k-1,b+1) \times A_{11}^{-1}(N-k+1)K_{\eta RL}(k-1,b+1)$$

$$K_{\alpha Cd}(k,\tau,b,h)=\sum_{j=\tau+1}^{N-k-b} p(N-k-j)K_{\eta C}^T(k-1,h+1) \times A_{11}^{-1}(N-k+1)K_\eta(k-1,j,b+1) + \bar{P}(N-k-1-\tau)K_{\eta C}^T(k-1,h+1) \times A_{11}^{-1}(N-k+1)K_\eta(k-1,\tau,b+1)$$

$$K_{\alpha CdC}(k,b,h)=\bar{P}(b)K_{\eta C}^T(k-1,h+1) \times A_{11}^{-1}(N-k+1)K_{\eta C}(k-1,b+1) + p(b)K_{\eta C}^T(k-1,h+1) \times A_{11}^{-1}(N-k+1)K_{\eta RL}(k-1,b+1)$$

$$K_{\alpha CdRL}(k,b,h)=\bar{P}(b-1)K_{\eta C}^T(k-1,h+1) \times A_{11}^{-1}(N-k+1)K_{\eta RL}(k-1,b+1)$$

For $\tau < N-k-b$:

$$K_{\alpha Cd}(k,\tau,b,h)=K_{\alpha Cd}(k,\tau,b+1,h) + K_{\alpha CdC}(k,b,h)v_{N-k-b}$$

$$K_\beta(k,\tau,b)=\left(\sum_{j=\tau+1}^{N-k-b} p(N-k-j)K_\eta(k-1,j,b+1) + \bar{P}(N-k-1-\tau_{N-3})K_\eta(k-1,\tau,b+1)\right)^T \times A_{11}^{-1}(N-k+1)K_{\eta C}(k-1,1)$$

$$K_{\beta C}(k,b)=\big(p(b)K_{\eta RL}(k-1,b+1) + \bar{P}(b)K_{\eta C}(k-1,b+1)\big)^T \times A_{11}^{-1}(N-k+1)K_{\eta C}(k-1,1)$$

$$K_{\beta RL}(k,b)=\bar{P}(b-1)K_{\eta RL}^T(k-1,b+1) \times A_{11}^{-1}(N-k+1)K_{\eta C}(k-1,1)$$

For $\tau < N-k-b$:

$$K_\beta(k,\tau,b)=v_{N-k-b}^T K_{\beta C}(k,b)+K_\beta(k,\tau,b+1)$$

$$K_r(2,\tau,1)=K_\alpha(2,\tau,1)$$

For $k>2$:

$$K_r(k,\tau,b)=\sum_{j=\tau+1}^{N-k-b} p(N-k-j)K_r(k-1,j,b+1) + \bar{P}(N-k-1-\tau)K_r(k-1,\tau,b+1) + K_\alpha(k,\tau,b)$$

For $\tau < N-k-b$:

$$K_r(k,\tau,b)=K_r(k,\tau,b+1)+v_{N-k-b}^T K_{rCs}(k,b)v_{N-k-b} + 2v_{N-k-b}^T K_{rCd}(k,\tau,b+1,b)$$

Moreover,

$$K_{rCs}(k,b)=\bar{P}(b)K_{rCs}(k-1,b+1)+K_{\alpha Cs}(k,b) + p(b)K_{rRL}(k-1,b+1)$$

$$K_{rRL}(k,b)=\bar{P}(b-1)K_{rRL}(k-1,b+1)+K_{\alpha RL}(k,b)$$

$$K_{rCd}(k,\tau,b,h)=\sum_{j=\tau+1}^{N-k-b} p(N-k-j) \times K_{rCd}(k-1,j,b+1,h+1) + \bar{P}(N-k-1-\tau_{N-4}) \times K_{rCd}(k-1,\tau,b+1,h+1) + K_{\alpha Cd}(k,\tau,b,h)$$

$$K_{rCdC}(k,b,h)=\bar{P}(b)K_{rCdC}(k-1,b+1,h) + K_{\alpha CdC}(k,b,h) + p(b)K_{rCdRL}(k-1,b+1,h)$$

$$K_{rCdRL}(k,b,h)=\bar{P}(b-1)K_{rCdRL}(k-1,b+1,h) + K_{\alpha CdRL}(k,b,h)$$

For $\tau < N-k-b$:

$$K_{rCd}(k,\tau,b,h)=K_{rCd}(k,\tau,b+1,h) + K_{rCdC}(k,b,h)v_{N-k-b}$$

$$K_\eta(1,\tau,b)=K_\zeta(1,\tau,b)$$
$$K_{\eta C}(1,b)=K_{\zeta C}(1,b)$$
$$K_{\eta RL}(1,b)=K_{\zeta RL}(1,b)$$

For $k>1$:

$$K_\eta(k,\tau,b)=K_\zeta(k,\tau,b)-K_{gu}(k,\tau,b,1)$$
$$K_{\eta C}(k,b)=K_{\zeta C}(k,b)-K_{guC}(k,b)$$
$$K_{\eta RL}(k,b)=K_{\zeta RL}(k,b)-K_{guRL}(k,b)$$

For $\tau < N-k-b$:

$$K_\eta(k,\tau,b)=K_\eta(k,\tau,b+1)+K_{\eta C}(k,b)v_{N-k-b}$$

For $k=1$:

$$K_\theta(1,\tau,b)=K_\eta^T(1,\tau,b)A_{11}^{-1}(N-1)M(N-1)$$
$$K_{\theta C}(1,b)=K_{\eta C}^T(1,b)A_{11}^{-1}(N-1)M(N-1)$$
$$K_{\theta RL}(1,b)=K_{\eta RL}^T(1,b)A_{11}^{-1}(N-1)M(N-1)$$

If $\tau < N-k-b$:

$$K_\theta(1,\tau,b)=K_\theta(1,\tau,b+1)+v_{N-k-b}^T K_{\theta C}(1,b)$$

For $k>1$:

$$K_\theta(k,\tau,b)=K_\eta^T(k,\tau,b)A_{11}^{-1}(N-k) \times \big(-K_{gx}(k,1)+M(N-k)\big) + K_{ux}(k,\tau,b)$$

$$K_{\theta C}(k,b)=K_{\eta C}^T(k,b)A_{11}^{-1}(N-k)$$

$$\times \ \left((-K_{gx}(k,1)+M(N-k)\right)$$
$$+ \ K_{uxC}(k,b)$$
$$K_{\theta RL}(k,b) = K_{\eta RL}^{T}(k,b)A_{11}^{-1}(N-k) \times \ \left((-K_{gx}(k,1)+M(N-2)\right) + \ K_{uxRL}(k,b)$$

If $\tau < N-k-b$:

$$K_{\theta}(k,\tau,b) = K_{\theta}(k,\tau,b+1) + v_{N-k-b}^{T}K_{\theta C}(k,b)$$

$$K_{\zeta}(k,\tau,b) = \sum_{i=N-k+1}^{N}\sum_{j=N-k+1}^{i}\bar{P}_d(i-(N-k+1))B^TS_{i+1} \times A^{i-j+1}B(P(i-N+k)-P(k+j-N-1)) \times \left(\sum_{t=b}^{N-k-1-\tau}\frac{P(k+j-N-1+t)-P(t-1)}{\bar{P}(t-1)} \times \prod_{h=2}^{t}\left(\frac{\bar{P}(k+j-N+h-2)}{\bar{P}(h-2)}\right)v_{N-k-t} + \prod_{h=1}^{N-k-1-\tau}\left(\frac{\bar{P}(k+j-N+h-1)}{\bar{P}(h-1)}\right)v_{\tau}\right)$$

$$K_{\zeta C}(k,b) = \sum_{i=N-k+1}^{N}\sum_{j=N-k+1}^{i}\bar{P}_d(i-(N-k+1)) \times B^TS_{i+1}A^{i-j+1}B \times (P(i-N+k)-P(k+j-N-1)) \times \left(\frac{P(k+j-N-1+b)-P(b-1)}{\bar{P}(b-1)} \times \prod_{h=2}^{b}\left(\frac{\bar{P}(k+j-N+h-2)}{\bar{P}(h-2)}\right)v_{N-k-b}\right)$$

$$K_{\zeta RL}(k,b) = \sum_{i=N-k+1}^{N}\sum_{j=N-k+1}^{i}\bar{P}_d(i-(N-k+1)) \times B^TS_{i+1}A^{i-j+1}B \times (P(i-N+k)-P(k+j-N-1)) \times \prod_{h=1}^{b-1}\left(\frac{\bar{P}(k+j-N+h-1)}{\bar{P}(h-1)}\right)$$

If $\tau < N-k-b$:

$$K_{\zeta}(k,\tau,b) = K_{\zeta}(k,\tau,b+1) + K_{\zeta C}(k,b)v_{N-k-b}$$

$$K_{ux}(k,\tau,b) = \sum_{j=\tau+1}^{N-k-b}p(N-k-j)K_{\theta}(k-1,j,b+1)A + \bar{P}(N-k-1-\tau_{N-3})K_{\theta}(k-1,\tau,b+1)A$$
$$K_{uxC}(k,b) = \bar{P}(b)K_{\theta C}(k-1,b+1)A + p(b)K_{\theta RL}(k-1,b+1)A$$
$$K_{uxC}(k,b) = \bar{P}(b-1)K_{\theta RL}(k-1,b+1)A$$

If $\tau < N-k-b$:

$$K_{ux}(k,\tau,b) = K_{ux}(k,\tau,b+1) + v_{N-k-b}^{T}K_{uxC}(k,b)$$

$$K_{gx}(k,b) = \bar{p}(0)K_{\theta C}(k-1,b)A + p(0)K_{\theta RL}(k-1,b)A$$

$$K_{uu}(2,\tau,b) = \sum_{j=\tau+1}^{N-2-b}p(N-2-j)K_{\theta}(1,j,b+1)Bv_j + \bar{P}(N-3-\tau)K_{\theta}(1,\tau,b+1)Bv_{\tau}$$
$$K_{uuCs}(2,\tau,b) = p(b)K_{\theta RL}(1,b+1)B$$
$$K_{uuRL}(2,b) = \bar{P}(b-1)K_{\theta RL}(1,b+1)B$$
$$K_{uuCd}(2,\tau,b,h) = \sum_{j=\tau+1}^{N-2-b}p(N-2-j)K_{\theta C}(1,h)Bv_j + \bar{P}(N-3-\tau)K_{\theta C}(1,h)Bv_{\tau}$$
$$K_{uuCdC}(2,\tau,b,h) = p(b)K_{\theta C}(1,h)B$$
$$K_{uuCdRL}(2,b,h) = \bar{P}(b-1)K_{\theta C}(1,h)B$$

If $\tau < N-k-b$:

$$K_{uuCd}(2,\tau,b,h) = K_{uuCd}(2,\tau,b+1,h) + K_{uuCdC}(2,\tau,b,h)v_{N-k-b}$$
$$K_{uu}(2,\tau,b) = K_{uu}(2,\tau,b+1) + v_{N-k-b}^{T}K_{uuCd}(2,\tau,b+1,b+1) + v_{N-k-b}^{T}K_{uuCs}(2,b)v_{N-k-b}$$

For $k > 2$:

$$K_{uu}(k,\tau,b) = \sum_{j=\tau+1}^{N-k-b}p(N-k-j)K_{\theta}(k-1,j,b+1)Bv_j + \bar{P}(N-k-1-\tau)K_{\theta}(k-1,\tau,b+1)Bv_{\tau} + \sum_{j=\tau+1}^{N-k-b}p(N-k-j)K_{uu}(k-1,j,b+1) + \bar{P}(N-k-1-\tau)K_{uu}(k-1,\tau,b+1)$$
$$K_{uuRL}(k,b) = \bar{P}(b-1)K_{\theta RL}(k-1,b+1)B + \bar{P}(b-1)K_{uuRL}(k-1,b+1)$$
$$K_{uuCs}(k,b) = p(b)K_{\theta RL}(k-1,b+1)B + p(b)K_{uuRL}(k-1,b+1) + \bar{P}(b)K_{uuCs}(k-1,b+1)$$
$$K_{uuCd}(k,\tau,b,h) = \sum_{j=\tau+1}^{N-k-b}p(N-k-j)K_{\theta C}(k-1,h)Bv_j + \bar{P}(N-k-1-\tau)K_{\theta C}(k-1,h)Bv_{\tau} + \sum_{j=\tau+1}^{N-k-b}p(N-k-j) \times K_{uuCd}(k-1,j,b+1,b+1) + \bar{P}(N-k-1-\tau) \times K_{uuCd}(k-1,\tau,b+1,b+1)$$
$$K_{uuCd}(k,\tau,b,h) = K_{uuCd}(k,\tau,b+1,h) + K_{uuCdC}(k,\tau,b,h)v_{N-k-b}$$

$$
\begin{aligned}
K_{uuCdRL}(k,b,b) =& \bar{P}(b-1)K_{\theta C}(k-1,h)B \\
&+ \bar{P}(b-1)K_{uuCdRL}(k-1,b+1,b+1)
\end{aligned}
$$

If $\tau < N-k-b$:

$$
\begin{aligned}
K_{uu}(k,\tau,b) =& K_{uu}(k,\tau,b+1) \\
&+ v_{N-k-b}^T K_{uuCs}(k,b)v_{N-k-b} \\
&+ v_{N-k-b}^T K_{uuCd}(k,\tau,b+1,b+1) \\
K_{uuCdC}(k,b,h) =& p(b)K_{\theta C}(k-1,h)B \\
&+ \bar{P}(b)K_{uuCdC}(k-1,b+1,b+1) \\
&+ p(b)K_{uuCdRL}(k-1,b+1,b+1)
\end{aligned}
$$

$$
\begin{aligned}
K_{gu}(2,\tau,b,h) =& \sum_{j=\tau+1}^{N-2-b} p(N-2-j)K_{\theta C}(1,h)Bv_j \\
&+ \bar{P}(N-3-\tau)K_{\theta C}(1,h)Bv_{\tau_{N-3}} \\
&+ K_{\beta}(2,\tau,b) \\
K_{guC}(2,b,h) =& p(b)K_{\theta C}(1,h)B + K_{\beta C}(2,b) \\
K_{guRL}(2,b,h) =& \bar{P}(b-1)K_{\theta C}(1,h)B + K_{\beta RL}(2,b)
\end{aligned}
$$

If $\tau < N-k-b$:

$$
K_{gu}(2,\tau,b,h) = K_{guC}(2,b,h)v_{N-k-b} + K_{gu}(2,\tau,b+1,h)
$$

For $k > 2$:

$$
\begin{aligned}
K_{gu}(k,\tau,b,h) =& \sum_{j=\tau+1}^{N-k-b} p(N-k-j)K_{\theta C}(k-1,h)Bv_j \\
&+ \bar{P}(N-k-1-\tau)K_{\theta C}(k-1,h)Bv_{\tau} \\
&+ K_{\beta}(k,\tau,b) \\
&+ \bar{P}(N-k-1-\tau) \\
&\times K_{uuCd}(k-1,\tau,b+1,h+1) \\
&+ \sum_{j=\tau+1}^{N-k-b} p(N-k-j) \\
&\times K_{uuCd}(k-1,j,b+1,h+1) \\
&+ \sum_{j=\tau+1}^{N-k-b} p(N-k-j)K_{rCd}(k-1,j,b+1) \\
&+ \bar{P}(N-k-1-\tau) \\
&\times K_{rCd}(k-1,\tau,b+1,h) \\
K_{guC}(k,b,h) =& p(b)K_{\theta C}(k-1,h)B + K_{\beta C}(k-1,b) \\
&+ \bar{P}(b)K_{uuCdC}(k-1,b+1,h+1) \\
&+ \bar{P}(b)K_{rCdC}(k-1,b+1,h+1) \\
&+ p(b)K_{uuCdRL}(k-1,b+1) \\
&+ p(b)K_{rCdRL}(k-1,b+1) \\
K_{guRL}(k,b) =& \bar{P}(b-1)K_{\theta C}(k-1,h)B + K_{\beta RL}(k,b) \\
&+ \bar{P}(b-1)K_{uuCdRL}(k-1,b+1) \\
&+ \bar{P}(b-1)K_{rCdRL}(k-1,b+1)
\end{aligned}
$$

If $\tau < N-k-b$:

$$
\begin{aligned}
K_{gu}(k,\tau,b,h) =& K_{gu}(k,\tau,b+1,h) \\
&+ K_{guC}(k,b,h)v_{N-k-b}
\end{aligned}
$$

References

Åström, Karl Johan (1970). *Mathematics in science and engineering: vol. 70, Introduction to stochastic control theory*. Academic Press.

Bengtsson, Fredrik, Hassibi, Babak, & Wik, Torsten (2016). LQG control for systems with random unbounded communication delay. In *Decision and control (CDC), 2016 IEEE 55th conference on* (pp. 1048–1055). IEEE.

Bengtsson, Fredrik, & Wik, Torsten (2021). LQG control over unreliable channels-full proof. arXiv preprint arXiv:2103.04394.

Fischer, Jörg, Hekler, Achim, Dolgov, Maxim, & Hanebeck, Uwe D. (2013). Optimal sequence-based LQG control over TCP-like networks subject to random transmission delays and packet losses. In *American control conference* (pp. 1543–1549). IEEE.

Hadjicostis, Christoforos N., & Touri, Rouzbeh (2002). Feedback control utilizing packet dropping network links. In *Proceedings of the 41st IEEE conference on decision and control, Vol. 2* (pp. 1205–1210). IEEE.

Hespanha, Joo P., Naghshtabrizi, Payam, & Xu, Yonggang (2007). A survey of recent results in networked control systems. *Proceedings of the IEEE*, *95*(1), 138–162.

Imer, Orhan C., Yüksel, Serdar, & Basar, Tamer (2006). Optimal control of LTI systems over unreliable communication links. *Automatica*, *42*(9), 1429–1439.

Kwakernaak, Huibert, & Sivan, Raphael (1972). *Linear optimal control systems*. John Wiley and Sons.

Liang, Xiao, Xu, Juanjuan, & Zhang, Huanshui (2016). Optimal control and stabilization for networked control systems with packet dropout and input delay. *IEEE Transactions on Circuits and Systems II: Express Briefs*, *64*(9), 1087–1091.

Lincoln, Bo, & Bernhardsson, Bo (2000). Optimal control over networks with long random delays. In *Proceedings of the international symposium on mathematical theory of networks and systems, Vol. 7*. Perpignan.

Ma, Xiao, Qi, Qingyuan, & Zhang, Huanshui (2017). Optimal LQ control for NCSs with packet dropout and delay. In *2017 Chinese automation congress (CAC)* (pp. 5726–5729). IEEE.

Moayedi, M., Foo, Y. K., & Soh, Y. C. (2010). Networked LQG control over unreliable channels. In *49th IEEE conference on decision and control* (pp. 5851–5856).

Moayedi, Maryam, Foo, Yung K., & Soh, Yeng C. (2011). LQG control for networked control systems with random packet delays and dropouts via multiple predictive-input control packets. *IFAC Proceedings Volumes*, *44*(1), 72–77.

Schenato, Luca (2009). To zero or to hold control inputs with lossy links? *IEEE Transactions on Automatic Control*, *54*(5), 1093–1099.

Schenato, Luca, Sinopoli, Bruno, Franceschetti, Massimo, Poolla, Kameshwar, & Sastry, S. Shankar (2007). Foundations of control and estimation over lossy networks. *Proceedings of the IEEE*, *95*(1), 163–187.

Shousong, Hu, & Qixin, Zhu (2003). Stochastic optimal control and analysis of stability of networked control systems with long delay. *Automatica*, *39*(11), 1877–1884.

Wang, Zhuwei, Liu, Lihan, Fang, Chao, Wang, Xiaodong, Si, Pengbo, & Wu, Hong (2018). Optimal linear quadratic control for wireless sensor and actuator networks with random delays and packet dropouts. *International Journal of Distributed Sensor Networks*, *14*(6), Article 1550147718779560.

Xu, Jiapeng, Gu, Guoxiang, Tang, Yang, & Qian, Feng (2022). Channel modeling and LQG control in the presence of random delays and packet drops. *Automatica*, *135*, Article 109967.

Xu, Hao, Jagannathan, Sarangapani, & Lewis, Frank L. (2012). Stochastic optimal control of unknown linear networked control system in the presence of random delays and packet losses. *Automatica*, *48*(6), 1017–1030.

Yu, Jen-te, & Fu, Li-Chen (2013). A new compensation framework for LQ control over lossy networks. In *2013 IEEE 52nd annual conference on decision and control (CDC)* (pp. 6610–6614).

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