TARGET VOLATILITY STRATEGIES for GROUP SELF-ANNUITY PORTFOLIOS

Target Volatility Strategies for Group Self-Annuity Portfolios

Annamaria Olivieri* Samuel Thirurajah† Jonathan Ziveyi‡

10th March 2022

Abstract

While the current pandemic is causing mortality shocks globally, the management of longevity risk remains a major challenge for both individuals and institutions. It is high time there be private market solutions designed for efficient longevity risk transfer among various stakeholders such as individuals, pension funds and annuity providers. From individuals' point of view, appealing features of post-retirement solutions include stable and satisfactory benefit levels, flexibility, meeting bequest preferences and low fees. This paper proposes a dynamic target volatility strategy for Group Self-Annuitization (GSA) schemes aimed at enhancing living benefits for pool participants. More specifically, we suggest investing GSA funds in a portfolio consisting of equity and cash, continuously rebalanced to maintain a target volatility level. The performance of a dynamic target volatility strategy is assessed against the static case which does not involve portfolio rebalancing. Benefit profiles are assessed by analysing quantiles and alternative strategies involving varying equity compositions. The case of death benefits is included, and the fund dynamics analysed by assessing resulting investment returns and the mortality credits. Overall, higher living benefit profiles are obtained under a dynamic target volatility strategy. From the analysis performed, a trade-off between the equity proportion and the impact on the lower quantile of the living benefit amount emerges, suggesting an optimal proportion of equity composition.

Keywords: Group self-annuitization, Longevity-linked annuity benefits, Investment-linked annuity benefits, Mortality credits, Target volatility strategy, Exponentially weighted moving average, Heston stochastic volatility model.

JEL Classifications: G22.

*annamaria.olivieri@unipr.it; Department of Economics and Management, University of Parma, Italy.
†KPMG Sydney, Australia.
‡j.ziveyi@unsw.edu.au; School of Risk and Actuarial Studies, CEPAR, UNSW, Sydney, NSW 2052, Australia.

1 Introduction

Net of the effect of mortality shocks induced by COVID-19, significant improvements have been gained in the mortality of older cohorts in many areas around the world. Together with structural changes in many populations, the mortality dynamics affected Pillar I and II pension benefits, which have been revised downward in many countries. In particular, the shift from defined benefit (DB) to defined contribution (DC) pension schemes implies that both individual longevity and investment risk have to be self-managed by the individual, possibly transferring them to a private provider.

It is well documented that, among the private post-retirement solutions, standard annuity products offer the optimal decumulation strategy (Yaari, 1965), in particular thanks to the longevity and investment guarantees they embed. However, the annuity market remains low due to a host of factors contributing to the annuity puzzle (Modigliani, 1986). Such factors include, among the others, bequest motives, but also loadings that cause annuities to be perceived as unfairly priced (Brown, 2009). Further, standard annuities are considered to be an inflexible and illiquid asset by many individuals, as they imply an irreversible decision (Pitacco, 2016).

There is a growing need for arrangements capable of providing a suitable, sustainable, stable and flexible post-retirement income. The challenge is magnified by higher rates of baby boomers moving into retirement. Innovation is required in designing customized post-retirement income products, capable of mitigating longevity risk whilst preserving a stable income after retirement. A number of pooling structures, where a group of individuals create a fund which can be invested in the capital markets whilst periodically drawing down depending on survival, have been proposed in literature. Such products include group self-annuitization (GSA) schemes (Piggott et al., 2005, Valdez et al., 2006, Qiao and Sherris, 2013), pooled annuity funds (Stamos, 2008, Donnelly et al., 2013, Donnelly, 2015), tontines (Milevsky, 2014, Milevsky and Salisbury, 2015, Chen et al., 2019, Weinert and Gründl, 2020) among others. The design of these pooled products has mainly been analysed considering simple investment strategies.

In reality, the returns achievable with a simple investment strategy may prove insufficient to maintain the sustainability of the fund due to the ever-changing economic environment and mortality uncertainty. Ideally, the funds will have to be invested in the capital markets on various asset classes, such as equities, fixed income securities and (whenever available) longevity-linked securities, developing a strategy aimed at enhancing the fund performance. Unlike in the case of standard annuities, where the annuity provider takes charge (unless default) of the overall longevity and investment risk, under pooling structures risks are fully retained by the participants. It is thus critical for there to be innovative approaches for neutralizing the risks impacting the pool. One dimension would be enhancement of GSAs, as they have potential for providing low cost retirement income payouts; however, they may prove to be lacking stability of the stream of individual payments. In particular, current GSA designs are vulnerable to potential decrease in survival benefits in the event of mortality improvements (Qiao and Sherris, 2013), as well as of poor investment performance. While we do not model longevity risk, we propose a target volatility investment strategy, aimed at enhancing investment returns of the fund, capable of overcoming adverse conditions.

Target volatility strategies are premised on the empirical relationship of negative correlation between asset returns and conditional volatility (Bollerslev et al., 2006), which asserts that low volatility regimes are characterized by high equity returns, and vice-versa. Targeting a certain volatility level on a portfolio results in more predictable and attainable returns over a given investment horizon. Such a volatility level is achieved by dynamically rebalancing the portfolio composition. Pioneering literature on target volatility strategies has been empirical studies on portfolios consisting of several equities whose overall volatilities are calculated by estimating the corresponding variance-covariance matrices (see Fleming et al. (2001), Kirby and Ostdiek (2012) among others). Volatility forecasting has mainly been facilitated with the aid of time series based techniques, such as the generalized autoregressive conditional heteroskedasticity (GARCH) framework (Bollerslev, 1986).

Doan et al. (2018) devise target volatility forecasting strategies and assess their performance relative to benchmark indices in the Australian, German, UK and US markets. The authors create portfolios consisting of well diversified stock holdings and stock index futures. Volatility forecasts are generated by implementing a reduced form of the GARCH(1,1) model. The devised strategies are shown to outperform traditional benchmark indices in the analyzed markets.

While Doan et al. (2018) and majority of prior literature use time series based techniques such as the generalized autoregressive conditional heteroskedasticity (GARCH) framework, this work devises a continuous time framework by simulating the Heston (1993) stochastic volatility model. In implementing a target volatility strategy for a GSA pool, we compare GSA benefit profiles emerging under a static fund composition and when the fund is rebalanced dynamically. We consider investment strategies involving a combination of deterministic cash account and equity which evolves according to the Heston (1993) dynamics. Benefit profiles are assessed by analysing various quantiles and alternative strategies involving varying equity compositions are presented. The case of death benefits is included, and the fund dynamics analysed by assessing resulting investment returns and the mortality credits. We find that higher living benefit profiles are obtained under a dynamic target volatility strategy. From the analysis performed, a trade-off between the equity proportion and the impact on the lower quantile of the living benefit amount emerges, which suggests an optimal proportion of equity composition.

The remainder of the paper develops as follows. In Section 2, we set up the GSA arrangement, in particular describing the benefits provided and how the GSA fund builds up in time, also disclosing the main components explaining its dynamics. Section 3 provides the model framework; we define, in particular, the (static and dynamic) target volatility strategy. In Section 4, we analyse the results obtained by implementing a target volatility strategy for the GSA, exploring in particular the benefit profiles and quantities explaining the fund dynamics. Section 5 concludes, while summarised versions of the implementation algorithms are presented in Appendices A and B.

2 The GSA design

2.1 Benefits and GSA fund dynamics

We consider a homogeneous GSA pool consisting of n individuals aged x joining the fund at time 0, each of them providing an initial capital amount c. The total pool fund then amounts to $F_0 = c \cdot n$ at time 0.

At anytime, the GSA fund value, F_t, evolves according to the return on investment and the benefits paid out, the latter depending on the mortality experienced by the pool and realized investment returns. In what follows, we define the GSA fund dynamics in continuous time.

The number of surviving members at time t is N_t, and follows a pure death process mediated by a stochastic (or deterministic) transition intensity. More specifically, the transition rate from i to $i-1$ policyholders at time t is $i\mu_{x+t}$ where μ_{x+t} is the force of mortality for a life aged $x+t$. Clearly, $N_0 = n$. If a death occurs at time t, then $dN_t = -1$, otherwise $dN_t = 0$. Since dN_t can be viewed as a Poisson decrement process, the probability of multiple deaths is of order $o(dt)$ (Ross, 2014), and can be neglected.

The GSA scheme pays living benefits to the surviving participants; we further incorporate the possibility of paying out death benefits upon member's death in our framework. We assume that living benefits are continuously paid[1]; consistently with the GSA rationale, they are not guaranteed, but their amount is assessed at the time of payment, so as to keep the actuarial balance in respect of the current GSA fund value F_t. Let $\bar{a}_{x+t}$ denote the actuarial value at time t of a unitary annuity. Then, the total living benefit amount paid by the GSA scheme at time t is

$$B_t = \frac{F_t}{\bar{a}_{x+t}}, \tag{2.1}$$

and

$$L_t = \frac{B_t}{N_t} \tag{2.2}$$

represents the amount cashed by each survivor at time t.

[1]Note that in later sections when performing numerical illustrations, as is the practice, we discretise the time domain into discrete timesteps and apply the Euler scheme.

The actuarial value of the annuity $\overline{a}_{x+t}$ is computed taking deterministic assumptions about the discount rate and the force of mortality. Then $\overline{a}_{x+t} = e^{-\int_t^\infty (r+\mu_{x+s})\, ds}$, where r is the (flat) short rate and μ_{x+s} is assumed to be deterministic. We point out that this definition of $\overline{a}_{x+t}$ implies that no future risk is accounted for in the living benefits assessed at time t; this is a natural choice within a GSA arrangement, where risks are retained by the surviving participants.

As suggested in many studies (see, for example, (Modigliani, 1986, Brown, 2009)), individuals have bequest preferences, at least up to some (old) age. We then incorporate death benefits paid by the GSA scheme as members die. Upon one member's death at time t, an amount

$$D_t = \beta \frac{F_t}{N_{t^-}} \tag{2.3}$$

is paid to the member's beneficiaries. Here, β is a proportion of the fund value, and $N_{t^-} = N_t - dN_t$ is the number of survivors an instant before time t; then $\frac{F_t}{N_{t^-}}$ can be referred to as the notional individual's share of the GSA fund at time t, prior to the deaths reported at that time. We note that, similar to living benefits, the amount of death benefits is not guaranteed either. It is also important to note that while death benefits meet bequest preferences, they reduce the possibility for the GSA arrangement in achieving a satisfactory pooling effect. For instance, if $\beta = 1$ no mortality credits are left and the GSA scheme becomes a purely financial arrangement, where individual members fully retain their respective longevity risk. Conversely, if $\beta = 0$ all the money is retained by the GSA fund upon a member's death, and the individual longevity risk is pooled within the fund. While $\beta = 0$ is acceptable, $\beta = 1$ is not realistic in the logic of a GSA arrangement. In view of practical applications, we will access scenarios involving low values of β in our numerical illustrations.

In this paper, we assume that the GSA fund is proportionally invested in equity and riskless cash. We denote with S_t the equity price process, r the risk-free return of a cash account and w_t the proportion of the fund invested into equity (while $(1 - w_t)$ is the proportion invested in the cash account).

Under this setting, the GSA fund dynamics can be described as follows:

$$dF_t = \left[w_t \frac{dS_t}{S_t} + (1 - w_t) r dt\right] F_t - B_t dt + D_t dN_t, \tag{2.4}$$

where dS_t denotes the instantaneous change of the equity process over a time increment, dt. For a precise interpretation of Equation (2.4), and related expressions below, recall that dN_t either takes a 0 or a negative value; then, $D_t\, dN_t$ denotes an outflow for the GSA fund.

2.2 The components of the GSA fund

As is well known, annuity benefits are funded by the initial capital (and this is trivial), investment returns and mortality credits. In a GSA arrangement, neither investment returns nor mortality credits are guaranteed. The latter, in particular, depend on the realized mortality rate and the death benefit paid out by the scheme. In order to better understand the dynamics of the living benefits, it is convenient to split the GSA fund into three components as follows:

1. principal, that is, consumption of the fund (denoted as F_t^1);
2. interest (F_t^2);
3. mortality credit (F_t^3),

so that we have the fund value at time t as

$$F_t = F_t^1 + F_t^2 + F_t^3, \tag{2.5}$$

apart from time $t = 0$, when $F_0^1 = F_0 = n \cdot c$, while $F_0^2 = F_0^3 = 0$.

A decomposition similar to (2.5) can be developed for benefits, living benefits in particular; for example (see Equations (2.1) and (2.2)), $L_t = L_t^1 + L_t^2 + L_t^3$, where $L_t^i = \frac{F_t^i}{\bar{a}_{x+t} N_t}$. Understanding the dynamics of components of the GSA fund thus provides information about the benefit dynamics.

Before describing the dynamics of each of these components, we note that all three contribute to the funding of the living and death benefits at time t. We assume, in particular, that they contribute proportionally to both. Then, from Equation (2.4) we derive that $\frac{B_t\, dt}{F_t} \cdot F_t^i$ is the living benefit covered by the fund component i ($i = 1, 2, 3$), while $\frac{D_t\, dN_t}{F_t} \cdot F_t^i$ is the death benefit covered by the same fund component.

We assume that the principal, F_t^1, is simply the part of the fund value that can be attributed to the initial capital, without taking investment returns into account. Therefore, as already noted, at $t = 0$ we have: $F_0^1 = F_0$. As mentioned above, the fund component F_t^1 contributes to the funding of the living and death benefits. Further, it must contribute to the mortality credits. The dynamics of F_t^1 can be defined as follows:

$$\begin{aligned} dF_t^1 &= -\frac{B_t\, dt}{F_t} \cdot F_t^1 + \frac{D_t\, dN_t}{F_t} \cdot F_t^1 + (1-\beta) \cdot \frac{dN_t}{N_t} \cdot F_t^1 \\ &= -\frac{F_t^1}{\bar{a}_{x+t}}\, dt + \frac{dN_t}{N_t}\, F_t^1. \end{aligned} \tag{2.6}$$

In order to interpret $(1-\beta) \cdot \frac{dN_t}{N_t} \cdot F_t^1$, recall that $dN_t = -1$ if a death occurs, otherwise $dN_t = 0$. This means that $(1-\beta) \cdot \frac{dN_t}{N_t} \cdot F_t^1$ is either 0 or an outflow for F_t^1. We can interpret it as the contribution of F_t^1 to the mortality credits.

The interest component, F_t^2, is the cumulative amount of investment gains and losses incurred by the fund. This component changes by $(w_t \frac{dS_t}{S_t} + (1-w_t) r_t) dt) \cdot F_t$ in each time-interval dt. However, part of the benefits must be covered with the interest gained up to time

t, so that the absolute amount of accumulated investment returns decreases as living and death benefits are paid out. The dynamics of F_t^2 can be described as follows:

$$\begin{aligned} dF_t^2 &= \left(w_t \frac{dS_t}{S_t} + (1-w_t) r_t dt \right) F_t - \frac{B_t\, dt}{F_t} \cdot F_t^2 + \frac{D_t\, dN_t}{F_t} \cdot F_t^2 + (1-\beta) \cdot \frac{dN_t}{N_t} \cdot F_t^2 \\ &= \left(w_t \frac{dS_t}{S_t} + (1-w_t) r_t dt \right) F_t - \frac{F_t^2}{\bar{a}_{x+t}}\, dt + \frac{dN_t}{N_t}\, F_t^2. \end{aligned} \tag{2.7}$$

Similarly to the case for F_t^1, $(1-\beta) \cdot \frac{dN_t}{N_t} \cdot F_t^2$ can be interpreted as the contribution of F_t^2 to the mortality credits.

Mortality credits, F_t^3, accrue when an annuitant dies and correspond to the part of the fund (including interest) notionally belonging to the individual which is not paid back as a death benefit. As discussed above, when an annuitant dies at time t, an amount is transferred from the principal (F_t^1) and interest components (F_t^2) into mortality credits (F_t^3). The dynamics of F_t^3 can be described as follows:

$$\begin{aligned} dF_t^3 &= -\frac{B_t\, dt}{F_t} \cdot F_t^3 + \frac{D_t\, dN_t}{F_t} \cdot F_t^3 - (1-\beta) \cdot \frac{dN_t}{N_t} \cdot (F_t^1 + F_t^2) \\ &= -\frac{F_t^3}{\bar{a}_{x+t}}\, dt - \frac{dN_t}{N_t}\, F_t^3 + (1-\beta) \cdot \frac{dN_t}{N_t} \cdot (F_t^3 - F_t^1 - F_t^2). \end{aligned} \tag{2.8}$$

It can easily be checked that $dF_t^1 + dF_t^2 + dF_t^3 = dF_t$.

3 Modelling framework

3.1 The equity model

We assume that the equity process, S_t, evolves according to Heston (1993) stochastic volatility model

$$dS_t = \mu S_t dt + \rho \sqrt{v_t} S_t dW_t^1 + \sqrt{1-\rho^2} \sqrt{v_t} S_t dW_t^2, \tag{3.1}$$

$$dv_t = \kappa(\theta - v_t) dt + \sigma_u \sqrt{v_t} dW_t^1, \tag{3.2}$$

where μ is the instantaneous return of the equity process [2], v_t is the instantaneous variance of S_t which is a mean reverting process whose long-term average, speed of mean reversion and standard deviation are θ, κ and σ_u respectively. As presented in Feller (1951), for Equation (3.2) to be a positive process, the condition $2\kappa\theta \geq \sigma_u^2$ has to be satisfied. Here, dW_t^1 and dW_t^2 are correlated Brownian motion increments for the equity and variance processes whose correlation is denoted as ρ. An Euler scheme is adopted in implementing Equations (3.1) and

[2] Although the same letter is used to denote the force of mortality, μ_{x+t}, and the instantaneous return of the equity process, μ, we prefer to stick to the traditional notation for both, considering that in what follows any misunderstanding is practically negligible.

(3.2). The fund allocates a proportion w_t into equity and $(1 - w_t)$ into a cash account which evolves deterministically according to $dC_t = r\, C_t$. The parameters for the Heston model have been adapted from Andersen et al. (2002) for all numerical illustrations presented in this paper[3].

3.2 The target volatility strategy

At any instant during the life of the fund and as benefits are paid, the weights of the equity and cash holdings are dynamically rebalanced so as to maintain a target volatility level of the fund. The target volatility strategy is a self-insurance strategy which facilitates more equity holdings during low volatility periods, and vice-versa during high volatility periods. Constraining volatility of the fund around a given target enhances investment return targets to be attainable. We rebalance the weights of the equity by setting

$$w'_t = \min\left(\frac{TV}{\sqrt{\hat{\sigma}^2(t)}}, 1\right),$$

where TV is the exogenously specified (annualized) target volatility and $\hat{\sigma}(t)$ is an estimated volatility level of equity returns between any two adjacent rebalancing points, see Morrison and Tadrowski (2013). In all the numerical illustrations which follow, we will implement an Euler scheme for the fund dynamics where the time domain is discretized into incremental time steps. We will jointly assume that the fund is rebalanced weekly and benefits (to surviving members or as death benefits) paid out immediately before rebalancing. That is, the proportional investment in equity and cash will be dynamically adjusted every week in line with changing market conditions, so as to maintain or achieve a target volatility level of the fund[4]. Based on historical observations, the fund manager keeps track of the exponentially weighted moving average (EWMA) of the volatility such that

$$\hat{\sigma}^2(t + \Delta t) = \lambda \cdot \hat{\sigma}^2(t) + \frac{(1 - \lambda)}{\Delta t}\left(\ln\left(\frac{S_{t+\Delta t}}{S_t}\right)\right)^2, \tag{3.3}$$

where $\Delta t = \frac{1}{52}$ which corresponds to weekly rebalancing (Engle, 1982). The EWMA incorporates all prior observations, but with exponentially declining weights through time, whose rate of decay are detected by the parameter λ. A higher value of λ implies that the estimate

[3] We adopt the Heston model for equity dynamics to illustrate the flexibility of our approach and this can easily be adapted to any equity modelling framework.

[4] In this study we assume no transaction costs associated with buying and selling of equities. In reality, such costs can be significant depending on the volume of transactions and rebalancing frequency. While we acknowledge that the target volatility strategy will require active management with implications on rebalancing costs, we believe that this may also be the case for the fixed equity strategy for which the fund manager may actively select individual stocks while the overall asset allocation remained constant. We thus deferred this line of inquiry to future research.

volatility reacts slower to recent changes in the equity returns thus putting more weight on past observations.

For all our numerical illustrations, we adopted the parameter set for the Heston stochastic volatility model presented in Andersen et al. (2002) fitted using S&P 500 index returns. These parameters have been reproduced in Table 1 for completeness.

r	μ	κ	θ	ρ	σ_v	λ
0.01	0.0849	2	0.0299	-0.4480	0.2	0.80

Table 1: Parameters of the Heston stochastic volatility model and the decay parameter for the EWMA. These parameters are for illustrative purposes and one can fit either the Heston model or adapt any equity modelling process to dataset of interest using standard techniques like that presented in Andersen et al. (2002).

3.3 Illustrative performance of a target volatility strategy

In assessing the performance of the target volatility strategy, we compare the dynamic strategy with the static case which involves preassigning fixed weights to the equity-cash holding at initial time where the equity dynamics is governed by the Heston stochastic volatility process. The weights for the static case are chosen such that the initial fund volatility equates to the corresponding target volatility of the dynamic strategy. The static strategy works as follows: at $t = 0$, the fund manager estimates the long run volatility to be $\hat{\theta}$ and elects a constant proportion strategy in which $w_t = \frac{TV}{\sqrt{\hat{\theta}}}$ is allocated to equity and the remainder to cash. This strategy is static in the sense that the proportion w_t is determined in advance and is not rebalanced. A detailed algorithm for implementing the static strategy is illustrated in Algorithm 1 of Appendix A.

In contrast, the target volatility strategy continuously updates w_t based on observed returns. To be more precise, when the static weight allocation is w^{static}, we will compare it to a dynamic strategy with target volatility $w^{static} \times \sqrt{\theta}$, where θ is the long run variance of the equity market. This facilitates a fair comparison between the 'rewards' of two strategies by ensuring that they have a similar level of 'risk'. The dynamic target volatility strategy can be implemented by replicating the pseudo code outlined in Algorithms 1 and 2 of Appendix A. Table 2 shows the relationships between w^{static} and the equivalent volatility based on our selected value of θ.

w^{static}	90%	70%	50%	30%	20%	0%
Target Volatility	16%	12%	9%	5%	3%	0%

Table 2: Relationship between static equity allocation and 'equivalent' dynamic target volatility.

An illustrative example with sample paths for dynamic target volatility strategy and static Heston stochastic volatility case is depicted in Figure 1. The horizontal axis of all subplots of Figure 1 are in years. Sample paths in Figure 1(a) shows that the dynamic rebalancing strategy offers improved returns relative to the static case in the long run. This is consistent with existing literature such as Morrison and Tadrowski (2013), Doan et al. (2018) and Li et al. (2019) who note that target volatility strategies limit losses by disinvesting from equities during high volatility periods and vice-versa. Figure 1(b) shows a typical sample on how the weights of the dynamic trading strategy changes through time due to changing market conditions. As reflected on Figure 1(c), dynamic rebalancing the equity-cash holdings ensures that the fund volatility is always constrained around a targeted level.

Figure 1(d) shows a simulated path of the volatility process in blue depicting a typical static case. Applying an exponential weighting used here for constraining volatility towards the target level yields the red plot. The red plot has less variability relative to the unconstrained case.

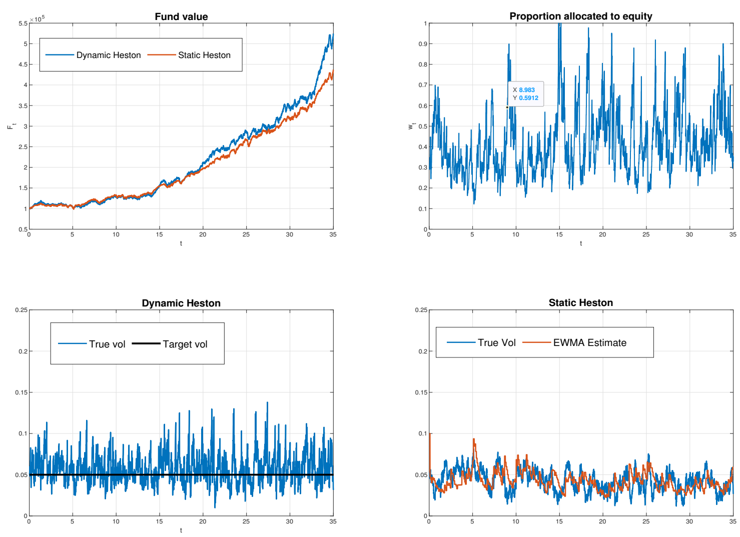


Figure 1: Illustrative simulation: Monthly target volatility rebalancing ('dynamic') is supposed to ensure that the model volatility remains close to the target. Rebalancing decisions are based on EWMA estimates of the volatility, since the true volatility is not observable in the market. The dynamic strategy offers some improvements in returns, but is limited in actually constraining the volatility of the fund.

3.4 The mortality model

For all illustrations presented in this paper, we adopt the Gompertz-Makeham mortality law which expresses the force of mortality as

$$\mu_x = a + e^{b_1 + b_2 x}, \tag{3.4}$$

where a, b_1 and b_2 are constant parameters which are determined through calibrating to mortality data. In this paper, we calibrate the model to the US male mortality data for ages 50 to 110 for the cohort born in 1915 with the data extracted from the Human Mortality Database[5]. By fitting the Gompertz-Makeham mortality law using the steps presented in Appendix B, the resulting parameters are presented in Table 3. The corresponding mortality and survival curves are shown in Figure 2. From the left panel of this figure we note that the fitted mortality rates tracks the realized mortality rate very well. As with the financial model, the Gompertz-Makeham law is adopted here for illustrative purposes. Any mortality model (either deterministic or stochastic) can be adapted to the approach presented in this paper.

a	b_1	b_2
0.0051	-9.5831	0.0889

Table 3: Gompertz-Makeham estimated parameters for US male mortality, ages 50 to 110.

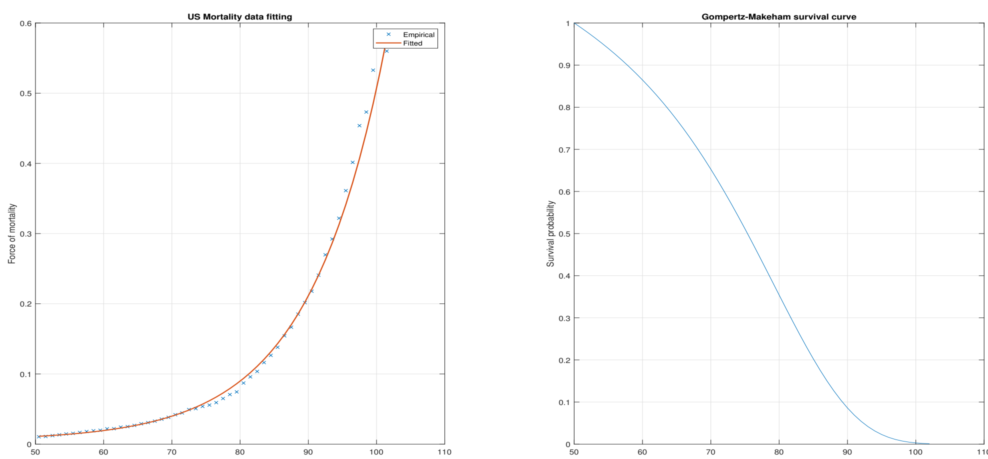


Figure 2: Plots for the fitted force of mortality and survival function for the US male data for ages 50 - 110 for the cohort born in 1915.

4 Implementation of the GSA target volatility strategy

This section presents numerical experiments analysing the performance of the two investment strategies presented in Subsection 3.2, applied to a GSA fund, as described in Section 2. The experiments are aimed at drawing insights on the interactions of the equity-cash holdings of the underlying fund and implications on living benefit payouts. We will assess scenarios where a fraction of mortality credits is paid out as bequest to beneficiaries of those leaving the pool. Evolution of each GSA fund component will be analysed through time during the

[5] https://www.mortality.org/

tenure of the pool, quantifying their respective contributions to living and death benefits. All results in this section have been generated by implementing relevant routines of Algorithms 1 - 4 presented in Appendix A derived from the methodology outlined in Sections 2 and 3 above. For all illustrations in this section, we have assumed a homogenous cohort consisting of $n = 1,000$ participants aged 65 at initial time with each participant contributing $100 as initial capital. The horizontal axis of all figures presented in this section are expressed in years from inception of the GSA fund.

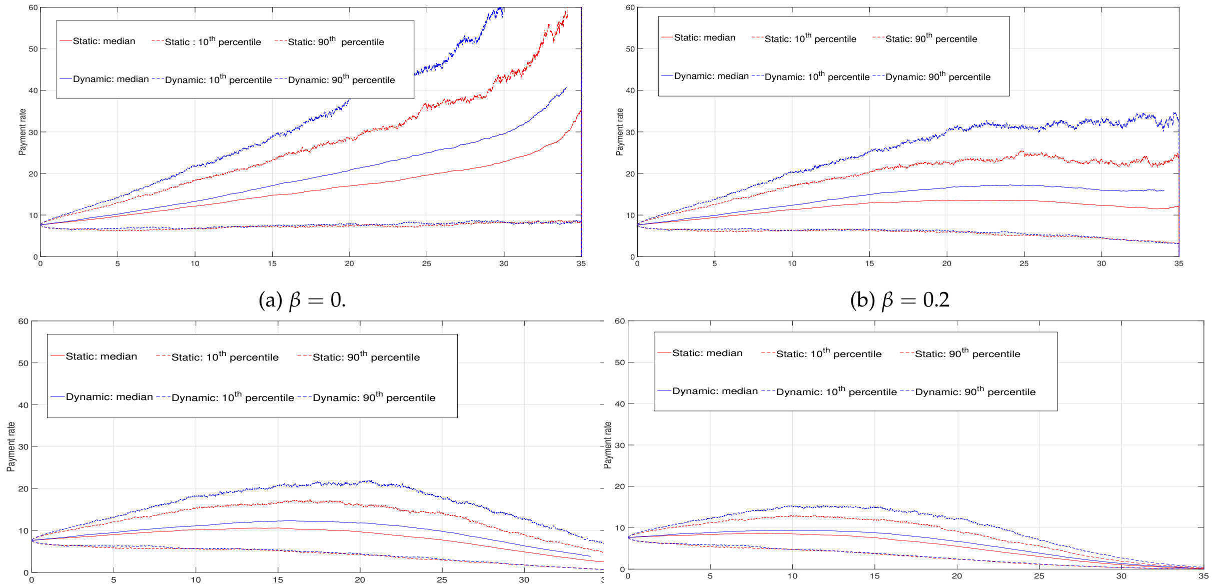
4.1 Benchmark case with 70% equity allocation at initial time

This subsection presents the benchmark case with equity allocation of 70% at initial time corresponding to a target volatility of 12% per annum. Figure 3 shows trajectories of living benefits for varying death benefit payments. In each subplot, summary statistics of living benefits (namely, the median, the 10^{th} and the 90^{th} percentile) are plotted in the case of a static (red) and a dynamic (blue) target volatility strategy. All subplots reveal that the dynamic target volatility strategy consistently pays out higher living benefits compared to the static volatility strategy, which does not adjust the equity and cash composition due to changing market conditions. In particular, we point out that while the lower quantiles of living benefits do not differ that much in respect of the investment strategy (with the dynamic strategy usually providing higher benefit amounts), the median and the upper quantiles show much more favourable paths in the case of a dynamic strategy.

In respect of the death benefit (see Figures 3(a) - 3(d)), whose size is defined by the proportion β, clearly higher death benefit payments result in reduced living benefit payments through time, as the fund value will be proportionally reduced due to lower mortality credits. This is reflected in Figure 3 where living benefits are decreasing as the benefit payments increase. When $\beta \geq 50\%$, living benefit payments may fall below initial payments as revealed in Figures 3(c) and 3(d). Having adopted a static or a dynamic target volatility strategy does not seem to be significant in terms of the relative reduction of the living benefit amount when death benefits are included.

4.2 Analysis of different components of the GSA

The framework adopted in the paper makes it readily possible to analyse the fund dynamics at anytime in terms of the principal component, investment returns and mortality credits, as presented in Equation (2.6) - (2.8). Figures 4 - 7 present typical trajectories of these respective components through time. From Figure 4 we note that, on average, the principal repayments are consistent and exponentially decreases through time in a predictable fashion, and independent of the investment strategy. This is also apparent in Equation (2.6), as the stochastic differential equation for the principal component of the fund depends only on the mortality



(a) $\beta = 0$. (b) $\beta = 0.2$

(c) $\beta = 0.5$ (d) $\beta = 1$

Figure 3: Individual living benefit payment (L_t) quantiles for various death benefit proportions when the static equity weight = 70% and target volatility = 12%. Quantiles for the static case are in red and those for the dynamic case are in blue.

experience (that we are assuming to be deterministic; this is why summary statistics coincide in Figure 4). Minor differences visible in Figure 4 are due to discretisation errors from Monte Carlo Simulation.

The interest component dominates during the initial stages of the pool with strategies for cases involving lower death benefit payments dominating those with higher as depicted in Figures 5(a) - 5(d). From these subplots, it is worth noting that regardless of how much is paid out as death benefits, interest accumulates aggressively until the 15^{th} year upon which the pool size starts to deteriorate and thus impacting more for cases within high death benefit payouts. Consistent with findings presented in Figure 3, the dynamic strategy returns dominate those obtained under a static strategy.

Mortality credits increase with increasing number of deaths as reflected in Figures 6(a) - 6(d). Again, the dynamic strategy yields higher mortality credits due to enhanced performance from the dynamic target volatility strategy. We note that as age increases, the variability in mortality credits returned also increases. This is reflective of a smaller pool size resulting in higher exposure to random fluctuations in mortality. As β increases we observe that the variability of mortality credits reduces. This is because the payment of death benefits are negatively correlated with survival benefits. Furthermore, these results provide comfort that mortality credits offer a useful qualitative measure of the variability in annuity payments that is attributable in part to the survival experience of the pool.

Figure 7 summarizes the contribution of each of these components to living benefits. During the initial phase of the pool, principal contributions make up the greater portion of benefit payouts with mortality credits gradually increasing as pool participants die through time. Investment returns exhibit a concave pattern, increasing to a climax before gradually decreasing as the pool size and principal depletes.

Figure 7 also reveals that interest accumulate at a faster rate on a target volatility strategy compared to associated static trading strategy cases. We note from Figures 7(a) that principal and interest equally contribute to living benefit payments much earlier compared to Figure 7(b) where interest match principal components after the 10^{th} anniversary under the current setting. In addition to this, the interest component proportion attains a higher value under the dynamic case compared to the static case. We note that higher investment returns could help mitigating the risk of lower mortality credits arising in a scenario with unanticipated longevity improvements (which is a situation not included in our assessments).

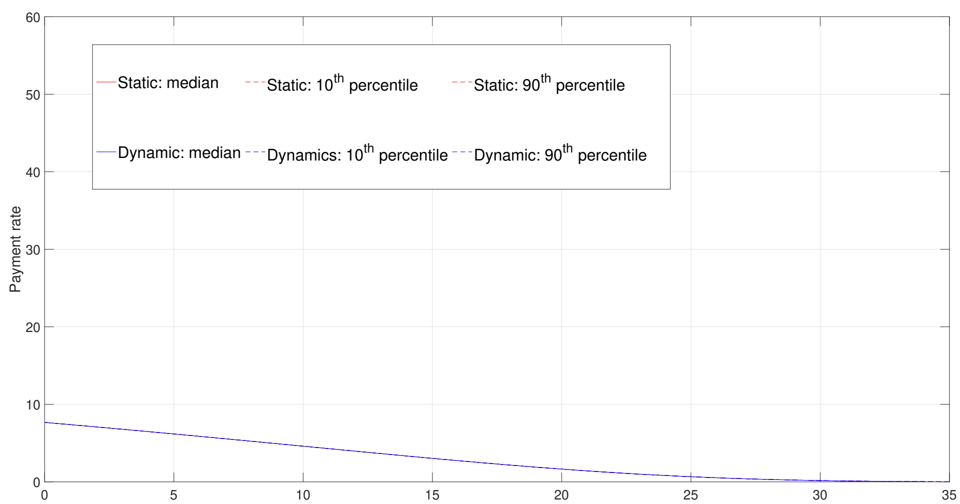
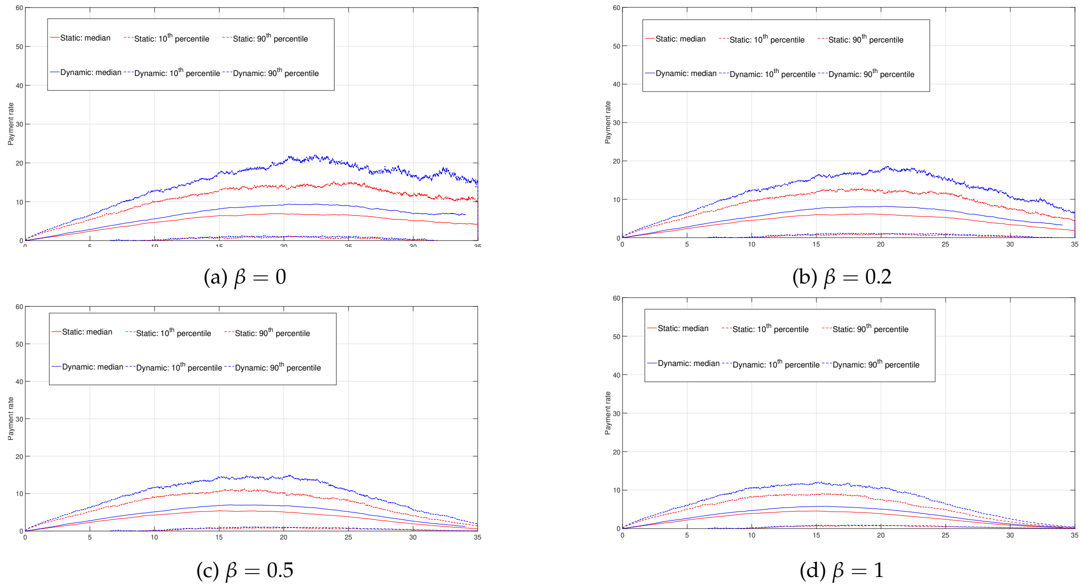


Figure 4: Quantiles for the principal component when the static equity weight = 70% and target volatility = 12%.

4.3 Comparison of living benefits for varying equity compositions

Having presented the benchmark case with an initial strategy consisting of 70% equity and 30% cash investments, we now perform sensitivity analysis aimed at revealing the impact of various investment strategies on living and death benefits. In what follows, all analysis will be performed relative to the 70% equity strategy whose corresponding target volatility is 12% per annum. Table 4 presents living benefit quantiles for the benchmark case across various ages for the target volatility and static investment strategies. As noted from Figure 3, the target volatility strategy dominates the static case across all ages implying that a target volatility strategy enhances living benefits for pool participants.

Tables 5-7 present comparison results for living benefit quantiles across various ages and equity compositions relative to the results presented in Table 4. From these tables, we note



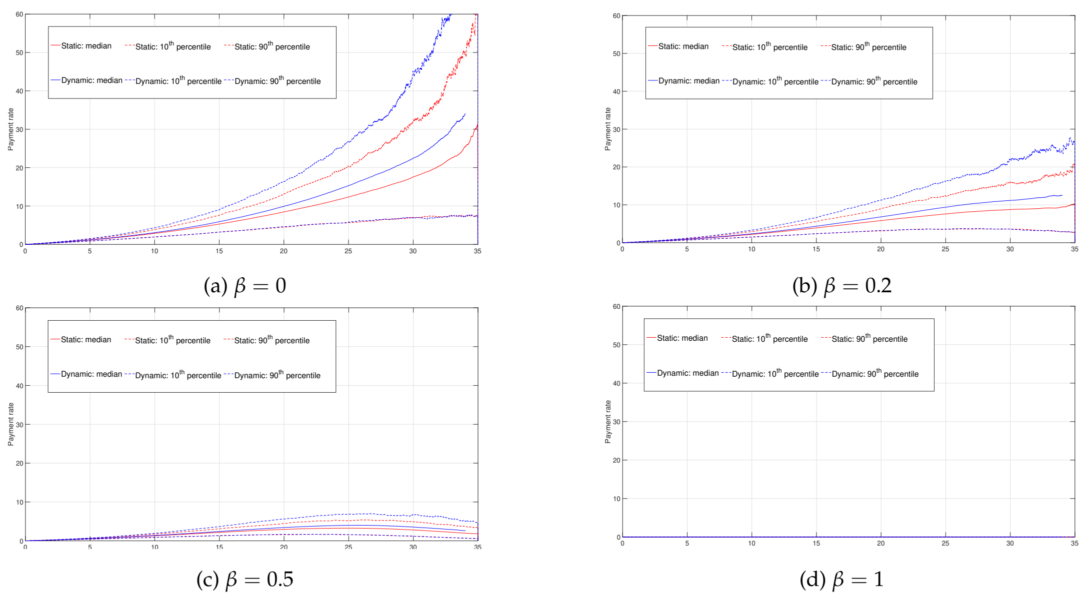
(a) $\beta = 0$

(b) $\beta = 0.2$

(c) $\beta = 0.5$

(d) $\beta = 1$

Figure 5: Interest component quantiles for varying death benefit proportions when the static equity weight = 70% and target volatility = 12%.



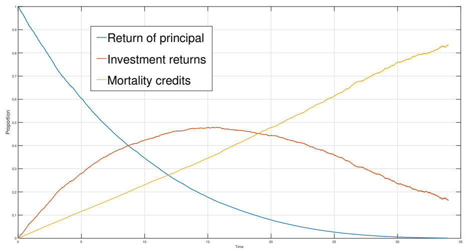
(a) $\beta = 0$

(b) $\beta = 0.2$

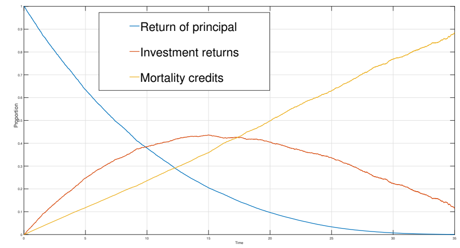
(c) $\beta = 0.5$

(d) $\beta = 1$

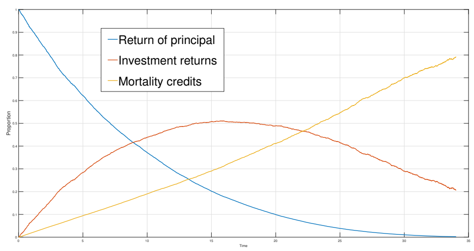
Figure 6: Mortality credit quantiles for varying death benefit proportions when the static equity weight = 70% and target volatility = 12%.



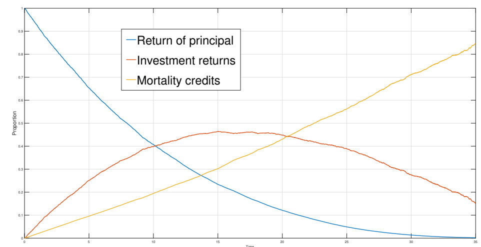
(a) Dynamic strategy when $\beta = 0$



(b) Static Strategy when $\beta = 0$



(c) Dynamic strategy when $\beta = 0.2$



(d) Static strategy when $\beta = 0.2$

Figure 7: Proportion of living benefit contributions from the principal, interest and mortality credits component for varying death benefits and investment strategies.

that when the initial equity composition is 90% of the fund, the median and 90^{th} percentile are superior to those for the 70% initial equity strategy, while the 10^{th} percentile is lower than in the benchmark case. Note that corresponding to the 90% initial equity composition is an initial volatility of 16% for both the static and target volatility strategy, compared to the initial volatility of 12% for the benchmark case (see Table 2 for initial volatilities associated with initial equity compositions). When the strategy is less aggressive with more fund holdings invested in cash and less in equity, the living benefits decrease as reflected on columns corresponding to less equity holding in Tables 5-7.

As would be expected, across all ages presented in these tables, lower equity proportion implies lower expected value of living benefits and lower upper quantile. For the lower quantile, it turns out to be initially increasing, and then decreasing with respect to the reduction of the equity weight. The switch of the path occurs at different ages, depending on the equity weight. There is a trade-off between equity proportion and impact on the lower quantile, which suggest an optimal proportion of the equity.

Strategies with less equity holdings have low payout structures due to the limited performance of the underlying fund. In all cases (that is, across all w^{static} weights presented in Table 2), the expected value of the living benefits is higher in the dynamic setting, as well as the higher quantile, whereas the lower quantile is approximately the same in the dynamic and static case. When $w^{static} = 0$, that is when all the money is invested in cash, the living benefits tend to decrease, due to longevity cost not being adequately compensated by investment returns. In general, when there is less equity, there is less volatility in benefit amounts, but they are lower, even decreasing.

Age	75		80		85	
Quantile	Dynamic	Static	Dynamic	Static	Dynamic	Static
0.1	6.8725	6.6352	7.4434	7.3153	7.8694	7.3393
0.5	12.1365	11.5346	15.3813	13.8841	17.3999	15.2872
0.9	21.936	18.1925	28.646	23.1723	37.8914	28.2494

Table 4: Individual living benefits for the base case with 70% equity allocation and $\beta = 0$.

Initial Equity	0%		20%		50%		90%	
Quantile	Dynamic	Static	Dynamic	Static	Dynamic	Static	Dynamic	Static
0.1	1.0334	1.0673	1.053	1.0783	1.0214	1.0414	0.98089	0.95122
0.5	0.59768	0.62887	0.71392	0.7285	0.89837	0.89141	1.0858	1.1172
0.9	0.3384	0.40804	0.47322	0.52612	0.78174	0.78109	1.1376	1.2684

Table 5: Relative individual living benefits at Age 75 for varying initial allocations and $\beta = 0$. In this table, quantile comparisons are performed relative to the benchmark case presented in Table 4.

Initial Equity	0%		20%		50%		90%	
Quantile	Dynamic	Static	Dynamic	Static	Dynamic	Static	Dynamic	Static
0.1	0.88752	1.0673	0.95677	0.97531	1.0032	1.0074	0.97647	0.98261
0.5	0.44493	0.62887	0.58978	0.61395	0.84934	0.83602	1.1179	1.182
0.9	0.24707	0.40804	0.38745	0.42875	0.73913	0.72292	1.2126	1.3751

Table 6: Relative individual living benefits at Age 80 for varying initial allocations and $\beta = 0$. In this table, quantile comparisons are performed relative to the benchmark case presented in Table 4.

4.4 Conservative strategy from Age 85

We now assess the behaviour of living benefits when the investment strategy becomes conservative with the entire fund invested in cash from Age 85 and beyond. Figure 8 presents subplots for varying death benefit payments. Benefits for both target and static volatility strategies are increasing till Age 85 and then flatten out beyond this age.

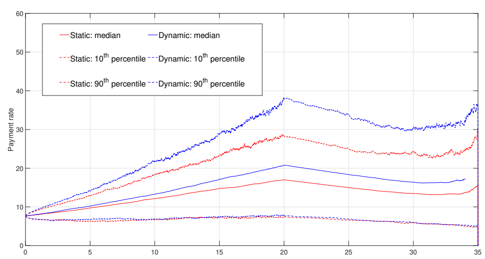
A switch to cash yields a smooth and more stable living benefit payout pattern at the expense of potentially enhanced benefits from equity participation. Comparing Figure 3(a) with Figure 8(a), one notes that pool participants surviving to advanced ages tend to receive higher living benefits enhanced by equity investment returns from the unconstrained case. Limiting the fund to cash only investment strategy compromises its potential as reflected in Figure 8. This is also revealed in Figure 9 showing the investment return contributions to the

Initial Equity Quantile	0% Dynamic	0% Static	20% Dynamic	20% Static	50% Dynamic	50% Static	90% Dynamic	90% Static
0.1	0.74861	1.0673	0.86961	0.92482	0.97553	0.98871	0.96387	0.99949
0.5	0.35686	0.62887	0.50849	0.54311	0.79279	0.80119	1.1327	1.2269
0.9	0.17379	0.40804	0.30016	0.35292	0.67854	0.67	1.257	1.4726

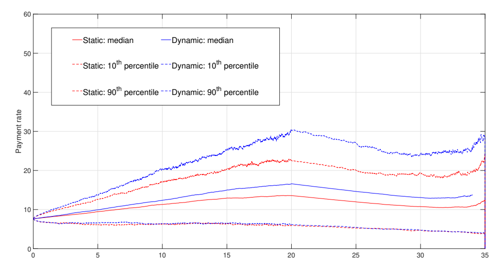
Table 7: Relative individual living benefits at Age 85 for varying initial allocations and $\beta = 0$. In this table, quantile comparisons are performed relative to the benchmark case presented in Table 4.

fund for varying death benefits. Due to depleting fund value and low returns associated with cash investments, we note that all investment return contributions exponentially decrease beyond Age 85.

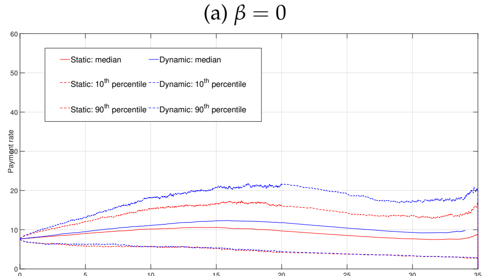
Comparing Figures 6 and 10 one easily notes that the contribution of mortality credits to living benefits under both scenarios is equivalent as the two cases involve the same pool participants with similar mortality developments.



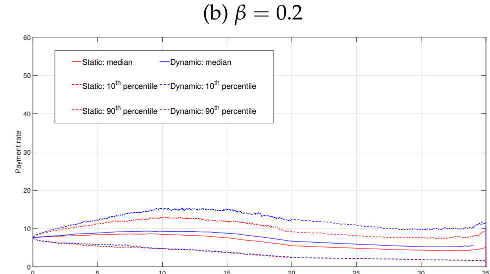
(a) $\beta = 0$



(b) $\beta = 0.2$



(c) $\beta = 0.5$

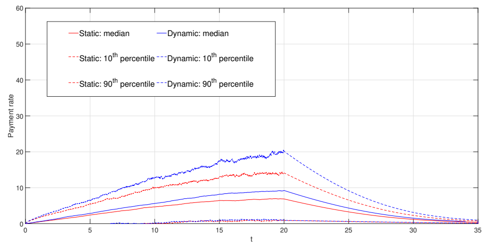


(d) $\beta = 1$

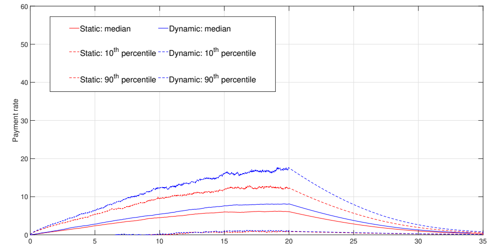
Figure 8: Living benefit payment quantiles for varying death benefit proportions when switching to all cash from Age 85 when static strategy equity weight = 70% and target volatility = 12%.

4.5 Comparison between different payout policies

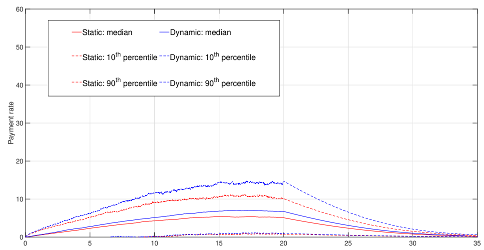
To have a better perspective of the superiority of the target volatility strategy, we present Figure 11 showing comparisons of the contribution of investment returns to living benefits for the case where there are no death benefits. We assess trajectories of investment returns for the case where all funds are switched to cash from Age 85 and the target volatility strategy for



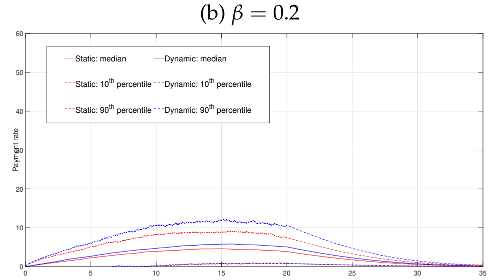
(a) $\beta = 0$



(b) $\beta = 0.2$

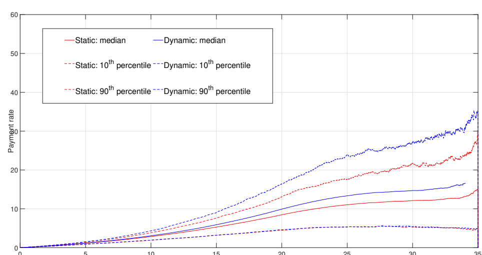


(c) $\beta = 0.5$

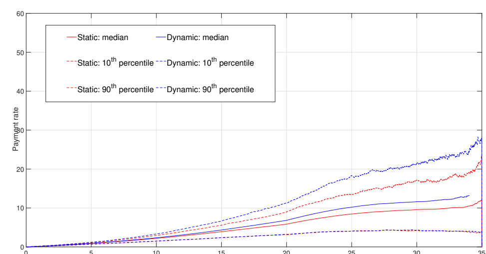


(d) $\beta = 1$

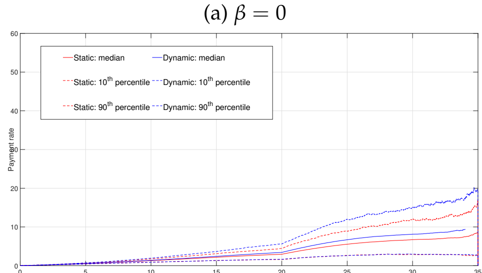
Figure 9: Interest component quantiles for varying death benefit proportions when switching to all cash from Age 85 when static strategy equity weight = 70% and target volatility = 12%.



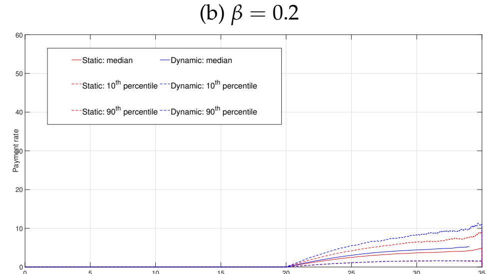
(a) $\beta = 0$



(b) $\beta = 0.2$



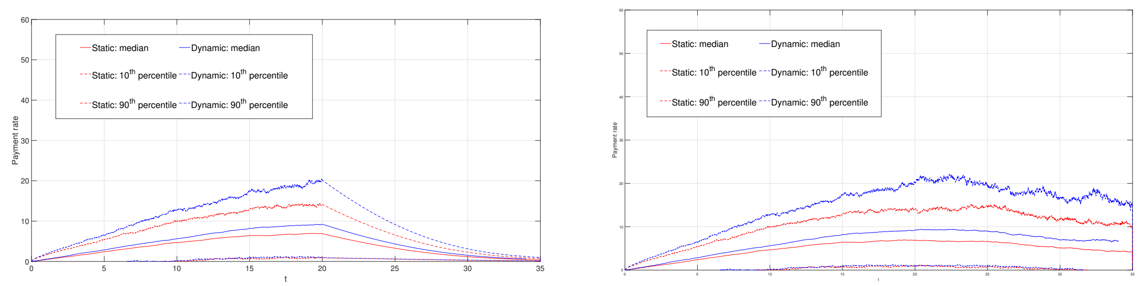
(c) $\beta = 0.5$



(d) $\beta = 1$

Figure 10: Mortality credit quantiles for varying death benefit proportions when switching to all cash from Age 85 when static strategy equity weight = 70% and target volatility = 12%.

the whole duration[6]. Figures 11(a) and 11(b) are the same up to age 85, when in Figure 11(a) all the equity is disinvested. While returns are less dispersed after the switch, the investment return contribution significantly drop, due to forgoing the potential high returns of equities. This is a striking insight, particularly in respect of a more traditional lifecycle investment approach, suggesting to disinvest equity as individuals age.



(a) Investment returns when $\beta = 0$ and all funds switched to cash post Age 85

(b) Investment returns when $\beta = 0$ with a target volatility strategy during the tenure of the GSA pool (see Figure 5(a)).

Figure 11: Comparison of investment returns between different payout policies during the tenure of the GSA plan when static strategy equity weight = 70% and target volatility = 12%.

4.6 Effects of mortality dynamics on living benefits

In this paper, we have assumed the Gompertz-Makeham mortality law which is a deterministic mortality model, implying a systematic way in which participants leave the GSA pool. In reality, mortality rates evolve in an unpredictable fashion with a great deal of research having been done on stochastic mortality modelling in both discrete and continuous time settings[7]. In as much our focus is on illustrating the effectiveness of the target volatility strategy in enhancing living benefit throughout the life of the fund, we can as well assess the impact of varying mortality rates on living benefit profiles. We accomplish this by shocking fitted parameters for the mortality process in Equation (3.4) so as to realise mortality rates which are $\pm$ 10% than those used in the preceding subsections.

Table 8 presents relative individual benefit comparisons between the benchmark case of Table 4 and the case where realised mortality is 10% lower across all ages. As reflected in Table 8, we note that irrespective of the investment strategy, mortality improvements result in lower living benefit payments across all ages. When realised mortality is higher than expected, we note higher living benefits across all ages irrespective of the investment strategy as reflected in Table 9 with all quantiles dominating the benchmark case.

[6]Note that Figure 11(b) is the same as Figure 5(a). We have reproduced this figure here for easy of comparison.

[7]See Lee and Carter (1992) for seminal work on stochastic mortality modelling and Cairns et al. (2008) for a review of extrapolative and time-continuous stochastic mortality models. Future research may accommodate such realistic mortality frameworks and incorporate mortality-linked instruments in the investment strategy.

For completeness, we present Appendix C which illustrates cases involving non-zero death benefit payouts for the 70% initial equity investment strategies with $\beta = 20\%$ as presented in Tables 10 - 12. In generating Tables 11 and 12, we have assumed corresponding quantiles in Table 10 as reference and computed the relative differences to facilitate comparisons. Table 11 corresponds to the case where realised mortality is 10% lower than expected with Table 12 being the case with realised mortality being 10% higher than expected. As highlighted in Subsection 4.2, death benefit payouts result in lower living benefits across all ages[8].

Age	75		80		85	
Quantile	Dynamic	Static	Dynamic	Static	Dynamic	Static
0.1	0.96689	0.97566	0.94885	0.95058	0.88838	1.0039
0.5	0.96437	0.99339	0.90713	0.97039	0.87426	0.95234
0.9	0.94113	1.0083	0.95306	0.99494	0.90941	0.98354

Table 8: Relative individual living benefits for 70% initial equity allocation and $\beta = 0$ when realised mortality is 10% lower across all ages. In this table, quantile comparisons are performed relative to the benchmark case presented in Table 4.

Age	75		80		85	
Quantile	Dynamic	Static	Dynamic	Static	Dynamic	Static
0.1	1.0113	1.0536	1.0301	1.0705	1.0624	1.1495
0.5	1.0069	1.0548	1.0065	1.0921	1.0806	1.1485
0.9	1.0001	1.0355	1.0538	1.0948	1.0872	1.1841

Table 9: Relative individual living benefits for 70% initial equity allocation and $\beta = 0$ when realised mortality is 10% higher across all ages. In this table, quantile comparisons are performed relative to the benchmark case presented in Table 4.

5 Concluding remarks

In this paper we have presented a target volatility investment strategy for enhancing the performance of a group self-annuitization (GSA) scheme whose funds are strategically invested in a combination of equity and cash. For illustrative purposes, we have adopted the Heston (1993) stochastic volatility process for modelling the dynamics of the equity process and the Gompertz-Makeham mortality law fitted to the US male mortality profile for the cohort born in 1915. The GSA fund maintains a certain volatility level by dynamically rebalancing the equity and cash holding whenever there are significant market movements. During high

[8] This can easily be inferred from comparing Table 10 with 4.

volatility periods, funds are switch from equity to cash whereas when volatility is low, more weights will be assigned to equity. By comparing benefit profiles emerging under the dynamic investment strategy with the static case which does not involve rebalancing, we have demonstrated how the dynamic case enhances the fund performance and hence improved living benefit payments.

Benefit profiles have been assessed by analysing various quantiles and alternative strategies involving varying equity compositions. A trade-off between the equity proportion and the impact on the lower quantiles of the living benefit amount emerges, which suggests an optimal proportion of equity composition. As potential members of a GSA pool may have bequest motives, we have assessed cases incorporating death benefits. Death benefits clearly reduce mortality credits for surviving members leading to lower living benefits. The presence of death benefits does not affect our conclusions about the better performance of a dynamic in respect of a static investment strategy.

We finally note that, while the numerical outputs obviously depend on all the modelling choices that we have introduced, the Heston model for the equity dynamics and the Gompertz-Makeham model for the force of mortality are not necessary choices in our approach. Alternative models could be adopted instead. In particular, implementing a stochastic mortality model could allow to assess how larger profits expected from a dynamic target volatility strategy could help mitigating a possible longevity risk emerging because of unanticipated mortality improvements.

Acknowledgements

The authors acknowledge financial support from the Australian Research Council Discovery Grant Project DP170102275 Retirement Income Product Innovation and and the Australian Research Council Centre of Excellence in Population Ageing Research (CEPAR) project number CE110001029. Helpful comments from two anonymous reviewers greatly improved the paper.

References

Andersen, T. G., Benzoni, L., Lund, J., 2002. An empirical investigation of continuous-time equity return models. The Journal of Finance 57 (3), 1239–1284.

Bollerslev, T., 1986. Generalized autoregressive conditional heteroskedasticity. Journal of econometrics 31 (3), 307–327.

Bollerslev, T., Litvinova, J., Tauchen, G., 2006. Leverage and volatility feedback effects in high-frequency data. Journal of Financial Econometrics 4 (3), 353–384.

Brown, J. R., 2009. Understanding the role of annuities in retirement planning. Overcoming the savings slump: How to increase the effectiveness of financial education and saving programs, 178–206.

Cairns, A. J., Blake, D., Dowd, K., 2008. Modelling and management of mortality risk: a review. Scandinavian Actuarial Journal 2008 (2-3), 79–113.

Chen, A., Hieber, P., Klein, J. K., 2019. Tonuity: A novel individual-oriented retirement plan. ASTIN Bulletin 49 (1), 5–30.

Doan, B., Papageorgiou, N., Reeves, J. J., Sherris, M., 2018. Portfolio management with targeted constant market volatility. Insurance: Mathematics and Economics 83, 134–147.

Donnelly, C., 2015. Actuarial fairness and solidarity in pooled annuity funds. ASTIN Bulletin: The Journal of the IAA 45 (1), 49–74.

Donnelly, C., Guillén, M., Nielsen, J. P., 2013. Exchanging uncertain mortality for a cost. Insurance: Mathematics and Economics 52 (1), 65–76.

Engle, R. F., 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. Econometrica: Journal of the econometric society, 987–1007.

Feller, W., 1951. Two singular diffusion problems. Annals of mathematics, 173–182.

Fleming, J., Kirby, C., Ostdiek, B., 2001. The economic value of volatility timing. The Journal of Finance 56 (1), 329–352.

Heston, S. L., 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. The review of financial studies 6 (2), 327–343.

Kirby, C., Ostdiek, B., 2012. It's all in the timing: Simple active portfolio strategies that outperform naïve diversification. Journal of Financial and Quantitative Analysis, 437–467.

Lee, R. D., Carter, L. R., 1992. Modeling and forecasting us mortality. Journal of the American statistical association 87 (419), 659–671.

Li, S., Labit Hardy, H., Sherris, M., Villegas, A., 2019. A managed volatility investment strategy for pooled annuity products. Available at SSRN 3455806.

Milevsky, M. A., 2014. Portfolio choice and longevity risk in the late seventeenth century. a re-examination of the first english tontine. Financial History Review 21 (3), 225–258.

Milevsky, M. A., Salisbury, T. S., 2015. Optimal retirement income tontines. Insurance: Mathematics and economics 64, 91–105.

Modigliani, F., 1986. Life cycle, individual thrift, and the wealth of nations. Science 234 (4777), 704–712.

Morrison, S., Tadrowski, L., 2013. Guarantees and target volatility funds. Moody's Analytics, 1–12.

Piggott, J., Valdez, E. A., Detzel, B., 2005. The simple analytics of a pooled annuity fund. Journal of Risk and Insurance 72 (3), 497–520.

Pitacco, E., 2016. Guarantee structures in life annuities: A comparative analysis. The Geneva Papers on Risk and Insurance - Issues and Practice 41 (1), 78–97.

Qiao, C., Sherris, M., 2013. Managing systematic mortality risk with group self-pooling and annuitization schemes. Journal of Risk and Insurance 80 (4), 949–974.

Ross, S. M., 2014. Introduction to probability models. Academic press.

Stamos, M. Z., 2008. Optimal consumption and portfolio choice for pooled annuity funds. Insurance: Mathematics and Economics 43 (1), 56–68.

Valdez, E. A., Piggott, J., Wang, L., 2006. Demand and adverse selection in a pooled annuity fund. Insurance: Mathematics and Economics 39 (2), 251–266.

Weinert, J.-H., Gründl, H., 2020. The modern tontine. European Actuarial Journal, 1–38.

Yaari, M. E., 1965. Uncertain lifetime, life insurance, and the theory of the consumer. The Review of Economic Studies 32 (2), 137–150.

A Algorithms for implementing the GSA structure

Algorithm 1: Simulate stock returns

input : $\mu, \rho, \kappa, \theta, \sigma, T, N$
output: Equity values $(S_t)_{t=1}^N$, latent volatilities $(v_t)_{t=1}^N$
Initialize $S \leftarrow 1, v \leftarrow \theta, dt \leftarrow \frac{T}{N}$;
for $i = 1$ **to** N **do**
  Simulate two i.i.d $dW_1, dW_2 \sim N(0, dt)$;
  $dv \leftarrow \kappa(\theta - v)dt + \sigma\sqrt{v}dW_1$;
  $dS \leftarrow \mu S dt + \sqrt{v}S(\rho dW_1 + \sqrt{1-\rho^2}dW_2)$;
  $S \leftarrow S + dS$;
  $v \leftarrow \max(v + dv, 0)$;
  Store $S_i, v_i \leftarrow S, v$;
end

If a dynamic strategy is used, then the weights are dynamically adjusted. If a static strategy is used, then the weights are set as $w_i \leftarrow \sqrt{\frac{v_{tgt}}{\theta}}$

Algorithm 2: Get target volatility weights

---

input : EWMA parameter, λ
target volatility, v_{tgt}
annualised returns per period, $(\frac{\Delta S_t}{S})_{t=1}^{n}$
initial volatility estimate, v_{init}
output: weights $(w_t)_{t=1}^{n}$
Initialize $v_{sq} \leftarrow v_{init}$;
for $i = 1$ **to** n **do**
  $v_{sq} \leftarrow \lambda v_{sq} + (1-\lambda)\left(\frac{\Delta S_i}{S}\right)^2$;
  $w_i \leftarrow \min(1, \frac{v_{tgt}}{\sqrt{v_{sq}}})$
end

Algorithm 3: Compute annuity factor

---

input : $(\mu_{x+idt})_{i=1}^{N}, r, dt$
output: $(\bar{a}_{x+idt})_{i=1}^{N}$
Initialize $val \leftarrow 0$;
for $i = N-1$ **to** 1 **do**
  $dval \leftarrow ((\mu_i + r)val - 1)dt$;
  $val \leftarrow val - dval$;
  $\bar{a}_{x+idt} \leftarrow val$;
end

Algorithm 4: Simulate fund path

input : $(\bar{a}_{x+idt})_{i=1}^{N}, (S_t)_{t=1}^{N}, (n_t)_{t=1}^{N}, \beta$
output: $(F_i)_{i=1}^{N}, (L_i)_{i=1}^{N}$
initialization;
for $i \leftarrow 2$ **to** N **do**
  $dF \leftarrow (w_i \frac{dS_i}{S_i} + (1 - w_i) r dt) F$;
  if *living benefits paid* **then**
    $dL \leftarrow \frac{F dt}{\bar{a}_{x+idt}}$;
  else
    $dL = 0$;
  end
  if *death benefits paid and deaths occurred* **then**
    $dB \leftarrow \frac{\beta F dt (n_{i-1} - n_i)}{n_i}$;
  else
    $dB = 0$;
  end
  $dF \leftarrow dF - dL - dB$;
  $F, L \leftarrow F + dF, L + dL$;
  Store $F_i, L_i \leftarrow F, L$;
end

B Standard procedure for fitting the Gompertz-Makeham mortality law

The procedure for fitting is the model is as follows:

1. Define $y(a)_i = log(\mu_i - a)$ where μ_i is the empirical force of mortality for age group i.

2. Regressing $y(a)$ against x gives the estimates

$$\hat{y}(a)_i = \hat{b}_1(a) x_i + \hat{b}_0(a).$$

3. The parameter estimates for a, b_1 and b_0 are respectively represented as

$$\hat{a} = argmin_a (\sum_i (\hat{y}(a)_i - y(a)_i)^2), \quad \hat{b}_1(\hat{a}) \quad \text{and} \quad \hat{b}_0(\hat{a}).$$

C Sensitivity analysis for varying mortality

Age	75		80		85	
Quantile	Dynamic	Static	Dynamic	Static	Dynamic	Static
0.1	6.5528	6.1921	6.6664	6.4414	6.5935	5.9092
0.5	11.631	10.8352	13.5925	12.1353	14.0938	12.3384
0.9	20.4146	17.0716	25.6844	20.7538	29.6217	23.5658

Table 10: Individual living benefits for 70% initial equity allocation and $\beta = 20\%$.

Age	75		80		85	
Quantile	Dynamic	Static	Dynamic	Static	Dynamic	Static
0.1	0.97064	0.98032	0.95999	0.95756	0.90472	1.0115
0.5	0.97202	1.0003	0.9206	0.98286	0.89284	0.97986
0.9	0.94809	1.0109	0.96192	1.0118	0.93159	0.99777

Table 11: Relative individual living benefits for 70% initial equity allocation and $\beta = 20\%$ when realised mortality is 10% lower across all ages. In this table, quantile comparisons are performed relative to the results presented in Table 10.

Age	75		80		85	
Quantile	Dynamic	Static	Dynamic	Static	Dynamic	Static
0.1	1.0034	1.0453	1.0145	1.0513	1.0341	1.115
0.5	1.001	1.0496	0.99755	1.0833	1.0567	1.1389
0.9	0.97724	1.0229	1.0345	1.0852	1.0763	1.1531

Table 12: Relative individual living benefits for 70% initial equity allocation and $\beta = 20\%$ when realised mortality is 10% higher across all ages. In this table, quantile comparisons are performed relative to the results presented in Table 10.