Generalized Schur-Weyl dualities for quantum affine symmetric pairs and orientifold Khovanov-Lauda-Rouquier algebras

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Mini-Workshop: Three Facets of R-Matrices (hybrid meeting)

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Abstract. By definition, an R–matrix with spectral parameter is a solution to the Yang–Baxter equation, introduced in the 1970's by C.N. Yang and R.J. Baxter. Such a matrix encodes the Boltzmann weights of a lattice model of statistical mechanics, and the Yang–Baxter equation appears naturally as a sufficient condition for its solvability.

In the last decade, several mathematical and physical theories have led to seemingly different constructions of R–matrices. The theme of this workshop was the interaction of three such approaches, each of which has independently proven to be valuable: the *geometric*, *analytic* and *gauge–theoretic* constructions of R-matrices. Its aim was to bring together leading experts and researchers from each school of thought, whose recent works have given novel interpretations to this nearly classical topic.

Mathematics Subject Classification (2010): 16T25 (primary), 81R10, 81T13, 17B37,14D20 (secondary).

Introduction by the Organizers

The mini-workshop *Three facets of R–matrices* took place in a hybrid format: with 14 participants in-person and 7 virtual. The central objects of interest to this workshop were R–matrices; solutions of the celebrated *Yang–Baxter equation*

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \tag{YBE}$$

Here, $R(u)$ is an operator on $V \otimes V$ depending on a parameter u (called the spectral parameter), where V is a fixed finite-dimensional vector space over $\mathbb{C}$.

The equation takes place in $\mathrm{End}(V^{\otimes 3})$ and the subscripts indicate which tensor factors R acts upon. This equation emerged in the works of C.N. Yang and R.J. Baxter in thermodynamics and statistical physics, around 1975.

The workshop consisted of introductory talks, research talks and discussion sessions. Its goal was to understand three seemingly diverse theories which give constructive approaches towards R–matrices. In addition, the participants shared ideas about open problems in each theory, as well as possible *a priori* connections among them.

Introductory talks. On the first day, there were three introductory talks. Brian Williams and Masahito Yamazaki gave an introduction to Costello's approach towards quantum field theories and its explicit incarnation in the setting of a 4–dimensional Chern-Simons type gauge theory defined by Costello-Witten-Yamazaki. David Hernandez talked about the geometric construction of R–matrices via stable maps à la Maulik-Okounkov. David also discussed the algebraic interpretation of R–matrices as intertwiners, as well as the related transfer operators. The analytic construction of R–matrices of Yangians via difference-differential equations was explained in the fourth introductory talk by Sachin Gautam on the second day of the workshop.

Research talks. In addition, the workshop had 9 talks by participants regarding their research work related to R–matrices. These talks formed the bases for the discussion sessions and provided us with more concrete questions and research directions.

Discussion sessions. We had one discussion session at the end of the first day, lead by Masahito Yamazaki and Sachin Gautam. It started with a summary of the talks of the day and an explanation of the organizers' vision of the main theme of the workshop. The session continued with questions from the participants (both in-person and online) about the talks. Masahito presented details of the Feynman calculus mentioned in his talk. David explained normalization of the universal R–matrix and their significance for the study of homological properties of the category of finite-dimensional representations. The discussion between David Hernandez and Valerio Toledano Laredo (online) led to the following open problem:

Open problem. Classify indecomposable, finite-dimensional representations of the quantum loop algebras. In other words, what can we say about the *split Grothendieck group* of this category? A recent work of Chari-Moura-Young regarding self-extensions of irreducible representations was mentioned as a starting point for this problem.

We had two more discussion sessions on Thursday. In the morning we talked about concrete relations between the three facets of R–matrices. Brian Williams, Masahito Yamazaki and Meer Ashwinkumar (online) proposed a method of constructing *abelianized* R–matrix akin to the one from Sachin's talk, which could unite the gauge–theoretic and analytic approaches. We discussed a conjecture due to David Hernandez and Andrei Okounkov relating analytic approach with the geometric one. Additionally we learnt about the corresponding *three facets of the*

K–matrix, solution to the reflection equation (type B analogue of the YBE), as explained in Bart Vlaar's talk. The discussion session on Thursday evening was split into smaller groups aiming at more focused potential collaborations.

Mini-Workshop (hybrid meeting): Three Facets of R-Matrices

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The Yangian and four-dimensional Chern–Simons theory

Brian Williams

The Yangian $Y\mathfrak{g}$ of a complex semisimple Lie algebra $\mathfrak{g}$ is a Hopf algebra which deforms the enveloping algebra of the current algebra $\mathfrak{g}[z]$, of $\mathfrak{g}$ valued polynomials in one variable. It was introduced by Drinfeld in the mid–80's as one of the algebraic structures underpinning the study of integrable one and two-dimensional models in statistical mechanics.

The purpose of this talk is to explain a relationship, pioneered by Kevin Costello and collaborators, between a type of Chern–Simons gauge theory defined on four-manifolds and the Yangian quantum group. Mathematically, a gauge theory involves studying connections defined on bundles over a smooth manifold. In fact, there is an interpolation of Chern–Simons type theories which exist in dimensions three, four, five, and six. In dimension three, this is usual Chern–Simons theory, or the study of flat connections on three-manifolds. On the other side, in dimension six, this is so-called holomorphic Chern–Simons theory which describes the moduli space of holomorphic G-bundles. (This theory is known to describe space-filling open strings in topological string theory on $\mathbb{R}^6$, but will not play a role in this talk.)

We are interested in the four-dimensional theory[1]. This gauge theory is defined on manifolds of the form $\Sigma \times S$ where Σ is Riemann surface and S is a real two-dimensional manifold. Mathematically, the moduli space associated to this theory is the space of connections which are *holomorphic* in the direction of Σ and *flat* in the direction of S. Most of this talk concerns the case $\Sigma = \mathbb{C}$ and $S = \mathbb{R}^2$. The connection between four-dimensional gauge theory and quantum groups is similar to the more familiar relationship between three-dimensional Chern-Simons theory and quantum groups.

The key idea of Costello [1] is that the Yangian is controlled by the algebra of local operators of the theory on $\mathbb{C} \times \mathbb{R}^2$. The key to this result relies on the formalism of Costello-Gwilliam [2] that the algebra of operators of a quantum field theory form a *factorization algebra*. This is a vast generalization of the description of algebras of operators in conformal field theory as vertex algebras. Factorization algebras simultaneously generalize the notion of a vertex algebra and algebras over more familiar operads, such as the operad of little disks.

Using the topological $\mathbb{R}^2$-direction, the local operators $\mathcal{A}$ are endowed with the structure of an algebra over the operad of (framed) little two-disks; in other words, and $\mathbb{E}_2$-algebra. Moreover, there is a map of $\mathbb{E}_1$-algebras $\mathcal{A} \to \mathbb{C}$, where $\mathbb{C}$ is the trivial $\mathbb{E}_1$-algebra. By a generalization of a result of Tamarkin [4], the endomorphisms of the resulting fiber functor $\mathrm{Alg}_{\mathcal{A}} \to \mathrm{Mod}_{\mathbb{C}}$ endows the Koszul dual $\mathcal{A}^!$ of $\mathcal{A}$ with the structure of a Hopf algebra. Very roughly, the original

[1] The five-dimensional theory also appears in physics as a limit of M-theory in the so-called Ω-background [3].

algebra $\mathcal{A}$ has two topological directions and so two compatible multiplications. Koszul duality with respect to one of these algebra structures results in a Hopf algebra. It is one the main results of [1] that this Hopf algebra is equivalent to the Yangian $Y\mathfrak{g}$.

The proof of this result relies on a uniqueness result of Drinfeld. Up to a proper normalization, the coalgebra structure on the Yangian quantum group is completely determined by its first-order behavior; semi-classically this is encoded by a certain Poisson bracket. One can explicitly compute this bracket in the QFT language and find a direct match, and this is done in [1]. There another quantization that uses the 'holomorphic gauge' which one can use to compute this one-loop effect in a different way [5].

Very important to the theory of the Yangian is the universal R-matrix. An explicit characterization of the R-matrix in terms of QFT is given in the works of Costello, Witten, and Yamazaki [6, 7]. In terms of factorization algebras, this R-matrix is encoded by the operator product expansion (OPE) in the holomorphic $\Sigma = \mathbb{C}$ direction. Using this, one sees that $\mathcal{A}$ enhances to the structure of a vertex algebra in the category of Hopf algebras. The resulting vertex algebra OPE completely pins down the R-matrix.

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Four-Dimensional Chern-Simons Theory and the Yangian, II

Masahito Yamazaki

Integrable models have long been studied extensively both in mathematics and physics. One of the reasons for this is that there are so many different faces of integrability, all of which are nicely and intricately tided together into a coherent research topic of "integrable models".

One of the goals of this mini-workshop has been to highlight three different facets of integrability, namely 'geometric,' 'analytic,' and 'gauge-theoretic,' and to explore the relations between them. The goal of this talk, which can be regarded as a continuation of the talk of the same title (Part I) given by Brian Williams, is to introduce the 'gauge-theoretic' side of the story, along the lines of Ref. [1] (in collaboration with Kevin Costello and Edward Witten).

The gauge-theoretical explanation of [1] is perturbative in nature—we have an order-by-order analysis with respect to the deformation parameter $\hbar$. In integrable model language this is a parameter deforming the universal enveloping algebra $\mathcal{U}(\mathfrak{g}[[z]])$ into the Yangian $Y_\hbar(\mathfrak{g})$ (here $\mathfrak{g}$ is a semi-simple Lie algebra).

The perturbative expansion with respect to this parameter $\hbar$ is given by the Feynman diagram analysis of the four-dimensional quantum field theory known as the four-dimensional Chern-Simons theory. Its action functional is given by (for the simplest case, namely the rational case with the spectral parameter is given by a point of $\mathbb{C}$)

$$S = \int_{\mathbb{R}^2\times\mathbb{C}} \mathrm{d}z \wedge \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right) , \tag{1}$$

where A is a $\mathfrak{g}$-valued connection and Tr denotes the $\mathfrak{g}$-invariant Killing form. The lattice of integrable model is then created by considering Wilson lines

$$W_\gamma = P \exp\left(\int_\gamma A\right) , \tag{2}$$

where γ is a straight line in $\mathbb{R}^2$, and P denotes the ordering along the path γ. When we have multiple such Wilson lines we obtain the statistical lattice in $\mathbb{R}^2$, so that the correlation function of the Wilson lines give the partition function of the associated integrable model (by evaluating the Wilson lines in different representations we obtain different integrable models).

One of the main insights (which goes back to [2]) is that the topological invariance of the theory along the $\mathbb{R}^2$-direction naturally explains the Yang-Baxter equation, and hence the integrability of the model. This is very close to the explanation of the knots invariants from three-dimensional Chern-Simons theory [3], except now the four-dimensional theory in question has topological invariance only along $\mathbb{R}^2$.

While the general definition of the correlation function requires the path-integral and hence is ill-defined mathematically, the perturbative expansion in terms of Feynman diagrams can be formulated precisely, and this gives a well-defined perturbative expansion in the expansion parameter $\hbar$. In this talk some examples of Feynman diagram computation have been explained in detail. The statement is that such an expansion reproduces the perturbative expansion of the integrable models. For example, it is known that the adjoint representation of $\mathfrak{g} = \mathfrak{so}_N$ does not lift to a representation of the Yangian $Y(\mathfrak{g}_N)$ for $N > 4$, and this is reproduced in gauge theory from the quantum gauge anomaly of the associated Wilson line.

While gauge-theoretic approach has been successful, there are obviously many remaining questions to be asked. In the context of this mini-workshop, one obvious set of questions is to understand/clarify/learn from the relations between the gauge-theoretic approach and geometric/analytic approaches. During the workshop there have been many discussions along these lines, and I am hoping to see many future works motivated by this mini-workshop.

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3-facets of R-matrices: stable maps, intertwiners and transfer-matrices

David Hernandez

Our first lecture was devoted to present some of the important applications of R-matrices. Recall an R-matrix $\mathcal{R}(z) \in \mathcal{A}^{\otimes 2}((z))$ is a solution of the Yang-Baxter equation

$$\mathcal{R}_{12}(z)\mathcal{R}_{13}(zw)\mathcal{R}_{23}(w) = \mathcal{R}_{23}(w)\mathcal{R}_{13}(zw)\mathcal{R}_{12}(z)$$

for $\mathcal{A}$ an algebra, z, w formal variables,

$$\mathcal{R}_{12} = \mathcal{R} \otimes 1 \ , \ \mathcal{R}_{23} = 1 \otimes \mathcal{R} \ , \ \mathcal{R}_{13} = (\tau \otimes Id)(\mathcal{R}_{23}),$$

where τ is the flip. We recalled three of the various contexts in which R-matrices appear (see [H1]) :

(i) Intertwiners for quantum groups (representation theory). The specialization of the universal R-matrix $\mathcal{R}(z)$ of a quantum affine algebra to a tensor product of representations $U \otimes V$ gives rise, after composition by the flip, to an isomorphism of (deformed) representations :

$$\mathcal{R}_{U,V}(z) : U \otimes V(z) \to V(z) \otimes U.$$

The Yang-Baxter equation guarantees the hexagonal identity is satisfied :

(1)
$$\begin{tikzcd}
& V \otimes U \otimes W \arrow[r, "\mathrm{Id}\otimes P\mathcal{R}_{U,W}(zw)"] & V \otimes W \otimes U \arrow[dr, "P\mathcal{R}_{V,W}(w)\otimes \mathrm{Id}"] & \\
U \otimes V \otimes W \arrow[ur, "P\mathcal{R}_{U,V}(z)\otimes \mathrm{Id}"] \arrow[dr, "\mathrm{Id}\otimes P\mathcal{R}_{V,W}(w)"'] & & & W \otimes V \otimes U \\
& U \otimes W \otimes V \arrow[r, "P\mathcal{R}_{U,W}(zw)\otimes \mathrm{Id}"'] & W \otimes U \otimes V \arrow[ur, "\mathrm{Id}\otimes P\mathcal{R}_{U,V}(z)"'] &
\end{tikzcd} .$$

For U, V simple representations, after a proper renormalization and by taking the limit $z \to 1$, it gives rise to a (non necessarily invertible) morphism

$$R_{U,V} : U \otimes V \to V \otimes U.$$

(ii) Construction of transfer-matrices for quantum integrable models (mathematical physics). For V a finite-dimensional representation of a quantum affine algebra, we have the associated transfer-matrix :

$$T_V(z) = (Tr_V \otimes \mathrm{Id})\mathcal{R}(z).$$

The Yang-Baxter equation implies the integrability condition

$$[T_V(z), T_{V'}(w)] = 0,$$

for two such representations V, V'. Then, for another representation W, we obtain a commuting families of operators on W whose spectra constitute the spectrum of the corresponding quantum integrable model.

(iii) Maulik-Okounkov construction of stable maps (symplectic geometry). Maulik-Okounkov [MO] have presented a very general construction of R-matrices from the action of a pair of tori $A \subset T$ on a symplectic variety X. In its cohomological version, the construction gives the stable maps, morphisms of $H_T^\bullet(pt)$-modules

$$\mathrm{Stab}_{\mathcal{C}} : H_T^\bullet(X^A) \to H_T^\bullet(X)$$

depending in particular on a certain chamber $\mathcal{C}$. In good situations, for two chambers $\mathcal{C}$ and $\mathcal{C}'$, we obtain two stable maps:

$$\begin{tikzcd} & H_T^\bullet(X) & \\ H_T^\bullet(X^A) \arrow[ur, "\mathrm{Stab}_{\mathcal{C}}"'] \arrow[rr, dashed, "\mathcal{R}_{\mathcal{C}',\mathcal{C}}"'] & & H_T^\bullet(X^A) \arrow[ul, "\mathrm{Stab}_{\mathcal{C}'}"] \end{tikzcd},$$

invertible after a proper localization, so that $\mathrm{Stab}_{\mathcal{C}'}^{-1}\mathrm{Stab}_{\mathcal{C}}$ gives an R-matrix.

Our second lecture was entitled,

Baxter polynomials and truncated shifted quantum affine algebras

We explained the application of polynomiality of Q-operators to representations of truncated shifted quantum affine algebras (quantized K-theoretical Coulomb branches). Shifted quantum affine algebras and their truncations arose [FT] in the study of quantized K-theoretic Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories in the sense of Braverman-Finkelberg-Nakajima which are at the center of current important developments. The Q-operators are transfer-matrices associated to prefundamental representations of the Borel subalgebra of a quantum affine algebra, via the standard R-matrix construction. In a joint work [FH] with E. Frenkel, we have proved that, up to a scalar multiple, they act polynomially on simple finite-dimensional representations of a quantum affine algebra. This establishes the existence of Baxter polynomial in a general setting.

We propose a conjectural parameterization of simple modules of a non simply-laced truncation in terms of the Langlands dual quantum affine Lie algebra (this has various motivations, including the symplectic duality relating Coulomb branches and quiver varieties). We prove [H2] that a simple finite-dimensional representation of a shifted quantum affine algebra descends to a truncation as predicted by this conjecture. This is derived from the existence of Baxter polynomials.

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R-matrix of Yangians

Sachin Gautam

(joint work with V. Toledano Laredo, C. Wendlandt)

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, $\hbar \in \mathbb{C}^\times$ be a deformation parameter. Let $Y = Y_\hbar(\mathfrak{g})$ be the Yangian - a Hopf algebra defined by Drinfeld. Let $\Delta : Y \to Y \otimes Y$ denote its coproduct. The Yangian comes together with a 1-parameter group of Hopf algebra automorphisms $\tau_s : Y \to Y$, $s \in \mathbb{C}$. For a representation V of Y, we define $V(s) = \tau_s^*(V)$. Moreover, we set $\Delta_s = (\tau_s \otimes 1) \circ \Delta$ and $\Delta_s^{\mathrm{op}} = (\tau_s \otimes 1) \circ \Delta^{\mathrm{op}}$. The following theorem is due to Drinfeld [1].

Theorem 1. *There is a unique $R(s) \in (Y \otimes Y)[\![s^{-1}]\!]$ subject to the following conditions: (1) $R(s) = 1^{\otimes 2} + O(s^{-1})$, (2) For every $x \in Y$, we have $\Delta_s^{\mathrm{op}}(x) = R(s)\Delta_s(x)R(s)^{-1}$, and (3) the following cabling identities are satisfied:*

$$(\Delta \otimes 1)(R(s)) = R_{13}(s)R_{23}(s), \qquad (1 \otimes \Delta)(R(s)) = R_{13}(s)R_{12}(s).$$

It is well-known that conditions (1) and (2) imply the Yang-Baxter equation:

$$R_{12}(s)R_{13}(s+t)R_{23}(t) = R_{23}(t)R_{13}(s+t)R_{12}(s). \tag{YBE}$$

Furthermore, Drinfeld's R–matrix has the following properties: (4) $R(s)^{-1} = R_{21}(-s)$, (5) $(\tau_a \otimes \tau_b)(R(s)) = R(s + a - b)$.

If V, W are two *irreducible*, finite-dimensional representations of Y, and $R_{V,W}(s)$ is the evaluation of $R(s)$ on $V \otimes W$, then it was also shown by Drinfeld that $R_{V,W}(s) = \chi_{V,W}(s)R^{\mathrm{rat}}(s)$. Here $\chi_{V,W}(s) \in \mathbb{C}[\![s^{-1}]\!]$ and $R^{\mathrm{rat}}(s)$ is a rational function of s (rational solution to (YBE)) uniquely determined by the condition that it maps the tensor product of highest weight vectors to itself. The formal series $\chi_{V,W}(s)$ is known to be divergent.

Remark. The (unpublished) proof of the existence part of Theorem 1 relies on certain cohomological arguments which are *a priori* non-constructive. This is the main point of departure from the analogous case of the quantum loop algebras, where the existence is obtained via *Drinfeld double method*. The uniqueness part is easy, based on the determination of primitive elements in Y.

Questions. How to interpret $R_{V,W}(s)$ as a function of one complex variable? Can we get an explicit construction of $R_{V,W}(s)$?

The answers to both these questions are obtained in [2, 3]:

Theorem 2. *Given two finite-dimensionl representations V, W, there exist two meromorphic functions $R^{\uparrow/\downarrow}_{V,W}(s) : \mathbb{C} \to \mathrm{End}(V \otimes W)$, explicitly given, natural in V and W, and satisfying the following conditions.*

(1) *$R^{\uparrow/\downarrow}_{V,W}(s)$ is holomorphic and invertible in $\Sigma^{\uparrow/\downarrow} \subset \mathbb{C}$, where $\Sigma^{\uparrow} = \{\Re(s/\hbar) \gg 0\} = -\Sigma^{\downarrow}$. Moreover, $R^{\uparrow/\downarrow}_{V,W}(s) \to \mathrm{Id}_{V\otimes W}$ as $s \to \infty$, $s \in \Sigma^{\uparrow/\downarrow}$.*

(2) *For $\eta \in \{\uparrow, \downarrow\}$, $\sigma \circ R^{\eta}_{V,W}(s) : V(s) \otimes W \to W \otimes V(s)$ is Y–linear. Here σ is the flip of tensor factors.*

(3) *For $\eta \in \{\uparrow, \downarrow\}$, and V_1, V_2, V_3 finite-dimensional representations of Y, we have:*

$$R^{\eta}_{V_1(s_1)\otimes V_2, V_3}(s_2) = R^{\eta}_{V_1,V_3}(s_1 + s_2) R^{\eta}_{V_2,V_3}(s_2),$$
$$R^{\eta}_{V_1, V_2(s_2)\otimes W}(s_1 + s_2) = R^{\eta}_{V_1,V_3}(s_1 + s_2) R^{\eta}_{V_1,V_2}(s_1).$$

(4) *$R^{\uparrow}_{V,W}(s)^{-1} = \sigma \circ R^{\downarrow}_{W,V}(-s) \circ \sigma$.*

(5) *$R^{\eta}_{V(a),W(b)}(s) = R^{\eta}_{V,W}(s + a - b)$.*

(6) *$R^{\eta}_{V,W}(s)$ is asymptotic to $R_{V,W}(s)$ as $s \to \infty$ in Σ^{η}.*

(7) *$R^{\eta}_{V,W}(s) = R^{\mathrm{rat}}_{V,W}(s)\mathfrak{X}^{\eta}_{V,W}(s)$, where $R^{\mathrm{rat}}_{V,W}(s)$ is rational in s, and $\mathfrak{X}^{\eta}_{V,W}(s)$: $V(s) \otimes W \to V(s) \otimes W$ is Y–linear.*

Our proof of this theorem is based on the Gaussian decomposition of the R–matrix. Namely, if we write $R(s) = R^{+}(s)R^{0}(s)R^{-}(s)$, where $R^{\pm}(s)$ are upper/lower triangular unipotent matrices and $R^{0}(s)$ is diagonal, then we have the following.

- $R^{0}(s)$ is a solution of an additive, abelian, regular difference equation. Such equations admit two meromorphic solutions satisfying condition (1) of Theorem 2.
- $R^{+}(s) = R^{-}_{21}(-s)^{-1}$, and $R^{-}(s)$ is a rational function of s which is built in a recursive fashion from an intertwining equation.

In my talk I presented the aforementioned difference equation and the recursive relation in detail and how to solve them in the exmaple of $\mathfrak{g} = \mathfrak{sl}_2$ when $V = W = \mathbb{C}^2$. I ended my exposition with the following two questions central to the theme of the workshop.

Question 1. How can we obtain $R^{0}(s)$ from 4-dimensional Chern-Simons theory of Costello-Witten-Yamazaki? Some possible approaches towards this were suggested by B. Williams, M. Yamazaki and M. Ashwinkumar during the discussion sessions.

Question 2. What is the interpretation of $R^{-}(s)$ in the geometric setting of Maulik-Okounkov? A conjectural answer to this question was explained by David Hernandez in his talk.

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Deformed Cartan matrices and generalized preprojective algebras of finite type

Ryo Fujita

(joint work with Kota Murakami)

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra. In order to define the deformed $\mathcal{W}$-algebra associated with $\mathfrak{g}$, E. Frenkel and Reshetikhin [2] introduced a two parameters deformation $C(q,t)$ of the Cartan matrix $C = (c_{ij})_{i,j\in I}$ of $\mathfrak{g}$. Letting $D = \mathrm{diag}(d_i \mid i \in I)$ be the minimal left symmetrizer of C, its (i,j)-entry is defined by

$$C_{ij}(q,t) = \begin{cases} q^{d_i}t^{-1} + q^{-d_i}t & \text{if } i = j, \\ [c_{ij}]_q & \text{if } i \neq j. \end{cases}$$

for each $i, j \in I$, where $[k]_q := \frac{q^k - q^{-k}}{q - q^{-1}}$. In the specialization $(q,t) = (1,1)$, it certainly recovers the Cartan matrix C.

Its specialization $C(q) = C(q,1)$ at $t = 1$ is sometimes called the q-Cartan matrix, or the quantum Cartan matrix. It plays an important role in the representation theory of affine quantum groups (e.g. in the description of the diagonal part of the universal R-matrix of the Yangian/quantum loop algebra). Consider the inverse $\widetilde{C}(q) := C(q)^{-1}$ and let $\widetilde{C}_{ij}(q) = \sum_{u\geq 0} \widetilde{c}_{ij}(u)q^u$ denote the Taylor expansion at $q = 0$ of its (i,j)-entry. These Taylor coefficients satisfy some interesting properties: for any $i, j \in I$, we have

(P1) periodicity: $\widetilde{c}_{ij}(u + rh^\vee) = -\widetilde{c}_{ij^*}(u)$ for $u \geq 0$,
(P2) positivity: $\widetilde{c}_{ij}(u) \geq 0$ for $0 \leq u \leq rh^\vee$,
(P3) palindromicity: $\widetilde{c}_{ij}(rh^\vee - u) = \widetilde{c}_{ij^*}(u)$ for $0 \leq u \leq rh^\vee$,

where $r := \max(d_i \mid i \in I)$, $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, and $i \mapsto i^*$ is the involution of I induced by the longest element of the Weyl group.

We give a representation-theoretic interpretation for these properties in terms of the generalized preprojective algebra Π. It is introduced by Geiß, Leclerc, and Schröer [5] to generalize the ordinary preprojective algebra associated with the simply-laced Dynkin diagrams to the case of non-simply-laced Dynkin diagrams. The algebra Π is a finite-dimensional self-injective algebra (over a field) defined as a quotient of the Jacobian algebra $\widetilde{Pi}$ associated with a certain quiver with potential. The algebra $\widetilde{Pi}$ also appeared in the context of theoretical physics [1] and in the representation theory of the quantum loop algebras [6].

We can equip the algebras Π and $\widetilde{Pi}$ with a $\mathbb{Z}$-grading following [6]. Associated with each vertex $i \in I$, we have the simple Π-module S_i. In addition, there is the maximal indecomposable iterated self-extension E_i of S_i in the category of graded Π-modules. We can show that the graded Euler-Poincaré paring $\langle E_i, S_j \rangle_q$ is well-defined as a formal Laurent series in q and it can be expressed in terms of the q-Cartan matrix $C(q)$. As its dual statement, we have the following result. Let $\bar{I}_i$ be the graded submodule of the i-th injective Π-module satisfying $\langle E_j, \bar{I}_i \rangle_q = \delta_{ij}$, and $\dim_q \bar{I}_i \in \mathbb{Z}_{\geq 0}[q^{\pm 1}]$ its graded dimension.

Theorem 1 ([3, Theorem A]). *For any $i, j \in I$, we have*

$$\widetilde{C}_{ij}(q) = \frac{q^{d_j}}{1 - q^{2rh^\vee}} \left(\dim_q e_i \bar{I}_j - q^{rh^\vee} \dim_q e_i \bar{I}_{j^*} \right),$$

$$\dim_q e_i \bar{I}_j = q^{-d_j} \sum_{u=0}^{rh^\vee} \widetilde{c}_{ij}(u) q^u.$$

This result gives a simple explanation for the above properties (P1) and (P2). Moreover, it enables us to understand the other property (P3) as an incarnation of the self-injectivity of the algebra Π.

As an application, we can compute the graded dimensions of the first extension groups between the generic kernels introduced by Hernandez and Leclerc in [6]. The generic kernels are certain graded $\widetilde{Pi}$-modules, which can be seen as the additive counterparts of the Kirillov-Reshetikhin (KR) modules over the quantum loop algebra $U_q(L\mathfrak{g})$. More precisely, for each KR module V, the cluster character of the corresponding generic kernel $K_V \in \widetilde{Pi}$-gmod coincides with the q-character of V after a monomial transformation. Comparing our computations of Ext^1 with the computations of the denominators of the normalized R-matrices between the KR modules due to Oh and Scrimshaw [7] (see also [4]), we are led to the following conjecture. Let $\mathfrak{o}(V, W)$ denote the pole order of the normalized R-matrix $R_{V,W}(z)$ at $z = 1$ for simple $U_q(L\mathfrak{g})$-modules V and W.

Conjecture 2 ([3, Conjecture B]). *For any KR modules V and W, we have*

$$\mathfrak{o}(V, W) = \dim \mathrm{Ext}^1_{\widetilde{Pi}}(K_V, K_W).$$

At this moment, we can check that this conjecture is true as long as the left hand side is known.

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Shifted Yangians, prefundamental representations and polynomial R-matrices

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(joint work with David Hernandez)

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. The Yangian $Y(\mathfrak{g})$ of $\mathfrak{g}$, introduced by Drinfeld, is one of the first examples of quantum groups. It is a deformation of the universal enveloping algebra of the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$.

Depending on an integral coweight μ of $\mathfrak{g}$, the shifted Yangian $Y_\mu(\mathfrak{g})$ as an associative algebra is obtained from the Drinfeld presentation of the ordinary Yangian $Y(\mathfrak{g})$ by re-indexing the Drinfeld–Cartan generators while keeping all the relations. These algebras appeared first for type A in the context of representation theory of finite W-algebras by Brundan–Kleshchev [2], then for arbitrary types in the study of affine Grassmannian slices by Kamnitzer–Webster–Weekes–Yacobi [6] and in the study of quantized Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories by Braverman–Finkelberg–Nakajima [1]. In these works, certain quotients of shifted Yangians, the truncated shifted Yangians, arise naturally.

Essentially all the basic algebraic properties of the ordinary Yangian, including the one-parameter family of algebra automorphisms τ_a for $a \in \mathbb{C}$, triangular decomposition, PBW basis and coproduct, have been generalized to shifted Yangians by Finkelberg–Kamnitzer–Pham–Rybnikov–Weekes [3].

Fix I the set of Dynkin nodes of $\mathfrak{g}$. By a highest weight we mean an I-tuple $(f_i(u))_{i\in I}$ of Laurent series in u^{-1} with leading term being a power of u. Let μ be the coweight whose coefficient of the ith fundamental coweight is the degree of $f_i(u)$. Then the I-tuple can be viewed as a character of the affine Cartan subalgebra of the shifted Yangian $Y_\mu(\mathfrak{g})$, and the triangular decomposition leads to the highest weight irreducible module.

Call the irreducible module positive prefundamental if precisely one component of the I-tuple is $u - a$ for $a \in \mathbb{C}$ and the other components are 1; such a module is one-dimensional and it appeared first in the work of Brundan–Kleshchev [2]. Replacing $u - a$ with $\frac{1}{u-a}$, we get a negative prefundamental module.

In [4] we establish cyclicity and co-cyclicity properties for these modules.

Theorem 1. [4] *Let N be a tensor product of negative prefundamental module and V be an arbitrary highest weight irreducible module. Then $V \otimes N$ is generated by* *a tensor product of highest weight vectors, and $N \otimes V$ is co-generated by a tensor product of highest weight vectors. In particular, N is irreducible.*

Since twisting V by the algebra automorphism τ_a still preserves irreducibility, we obtain the normalized R-matrix, which is a module morphism sending a tensor product of highest weight vectors to the opposite tensor product

$$\check{R}_{V,N}(a) : \tau_a^* V \otimes N \longrightarrow N \otimes \tau_a^* V \quad \text{for } a \in \mathbb{C}.$$

We show that this R-matrix as an operator-valued function of a is polynomial. In particular cases, we recover the truncation series in the definition of truncated shifted Yangians from a diagonal entry of the R-matrix. As a consequence of polynomiality, we produce modules over truncated shifted Yangians.

Theorem 2. [4] *If P is a tensor product of positive prefundamental modules and N is a tensor product of negative prefundamental modules, then $P \otimes N$ factorizes through a truncated shifted Yangian.*

In simply-laced types Kamnitzer–Tingley–Webster–Weekes–Yacobi [5] obtained an explicit classification of highest weight irreducible modules over truncated shifted Yangians, from which follows the above theorem. Our proof via R-matrices works uniformly in general types.

Notice that $P \otimes N$ in the above theorem is a highest weight module whose highest weight is rational, namely, each component is the Taylor expansion at $u = \infty$ of a rational function of u. There is a category $\mathcal{O}^{sh}$ of modules over shifted Yangians, introduced in [2, 5], whose simple objects are the irreducible modules of rational highest weights. It is an analog of the ordinary category $\mathcal{O}$ of modules over the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$. Our last result is the Jordan–Holder property for category $\mathcal{O}^{sh}$ which fails for the ordinary category $\mathcal{O}$. Its proof relies on the cyclicity and co-cyclicity properties and on a geometric result [5] that each truncated shifted Yangian admits a finite number of highest weight irreducible modules.

Theorem 3. [4] *The tensor product of two irreducible modules in category $\mathcal{O}^{sh}$ is of finite representation length.*

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Chern-Simons Theory and the R-Matrix

Nanna Aamand

It was shown by Costello, Witten and Yamazaki in [1],[2] that solutions to the Yang-Baxter equation can be obtained as the expectation value of crossing Wilson lines in a 4-dimensional Chern-Simons theory. As discussed in [2], their setup can be translated into the setting of the usual 3-dimension Chern-Simons theory on $\mathbb{R}^2 \times [0,1]$ with the following boundary conditions on the gauge field: Let $\mathfrak{g}$ be the (simple) lie algebra of the gauge group G of the Chern-Simons theory, and assume that $\mathfrak{g}$ admits a Manin triple: $(\mathfrak{g}, \mathfrak{l}_-, \mathfrak{l}_+)$. Then we require that the gauge field takes value in $\mathfrak{l}_-$ on the boundary $\mathbb{R}^2 \times \{0\}$ and in $\mathfrak{l}_+$ on $\mathbb{R}^2 \times \{1\}$.

It was shown in [3], by carrying out computations at leading order in perturbation theory, that one can indeed deduce constant solutions to the classical Yang-Baxter equation (without spectral parameter) from this setup. I present my work on showing that the result of [3] holds to all orders in perturbation theory, thus producing a full R-matrix.

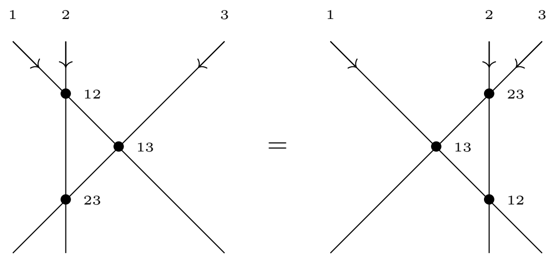

Figure 1. Graphical representation of the Yang-Baxter equation.

More concretely, consider the usual graphical representation of the Yang-Baxter equation shown in Figure 1. Each incoming line carries some vector space V and at each crossing between two lines the incoming vector spaces are transformed by an endomorphism $R \in \mathrm{End}(V)^{\otimes 2}$. Then the equality of the left and right side of the figure reproduces the Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where $R_{ij} \in \mathrm{End}(V)^{\otimes 3}$ is the transformation obtained by acting with R on the i'th and j'th copy of V and with the identity map on the remaining copy. In the context of gauge theory, we think of the lines as representing Wilson line operators in $\mathbb{R}^2 \times [0,1]$ with each Wilson line supported at a different point in $[0,1]$ and carrying some representation V of $\mathfrak{g}$. The expectation value of the Wilson line configurations on either side of the figure is an element of $\mathrm{End}(V)^{\otimes 3}$ given by a perturbative sum over weighted graphs (Feynman diagrams). Showing that the expectation value of a pair of crossing Wilson lines is an R-matrix then amounts to showing that the expectation value of the configuration in Figure 1 is invariant when moving the middle Wilson line from left to right. In order to accomplish this, I use of a method due to Bott and Taubes (see [4]), considering a certain compactification of the configuration space of Feynman diagram vertices. By applying Stokes' theorem to the Feynman integrals on the compactified configuration space, the problem of showing invariance of the expectation value under deformations of the Wilson lines becomes a problem of showing that the Feynman integrals vanish on the boundary of the configuration space.

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Bethe/gauge correspondence for superspin chains from string theory

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(joint work with Seyed F. Moosavian, Surya Raghavendran, Junya Yagi)

We provide a brane construction of rational $\mathfrak{gl}(m|n)$ spin chains with spins valued in Verma modules and show that string dualities map the Bethe eigenstates to massive vacua of certain 2d $\mathcal{N} = (2,2)$ quiver gauge theories. This extends the dictionary of Bethe/gauge correspondence for rational compact spin chains proposed by Nekrasov and Shatashvili [1, 2, 3] to include non-compact spin chains as well. More importantly, this provides a simple explanation for the Bethe/gauge correspondence by relating it to string dualities.

The brane construction of spin chains relies on the construction of 4d Chern-Simons theory from the world-volume theory of D5 branes. The theory on a stack of m D5 branes is the 6d $\mathcal{N} = (1,1)$ super Yang-Mills theory with gauge group $\mathrm{U}(m)$. Turning on a certain closed string background, which includes a particular Ramond-Ramond 2-form field, induces a topological-holomorphic twist and Ω-deformation of this theory reducing it to 4d Chern-Simons theory with complexified gauge group $\mathrm{GL}(m)$ [4]. We get 4d Chern-Simons theory with gauge supergroup $\mathrm{GL}(m|n)$ by introducing another stack of n D5 branes that are rotated with respect to the previous D5-branes so that they share only four directions. The open strings stretched between these two stacks of D5 branes provide 4d $\mathcal{N} = 2$ hypermultiplets at the intersection of the two stacks of D5 branes and they couple the two 6d theories with $\mathrm{U}(m)$ and $\mathrm{U}(n)$ gauge symmetry. The topological-holomorphic twist and Ω-deformation now reduces the D-brane theories to 4d Chern-Simons theory with gauge group $\mathrm{GL}(m|n)$. The bosonic components of the super connection come from the connections on the two stacks of D5 branes and the fermionic components are remnants of the 4d $\mathcal{N} = 2$ hypermultiplets.

Integrable spin chains are represented by line operators in 4d Chern-Simons theory [5]. We create line operators in the $\mathrm{GL}(m|n)$ Chern-Simons theory by putting semi-infinite D3 branes ending on the D5 branes. This introduces 3d defects in the 6d super Yang-Mills theories which reduce to line operators valued in Verma modules after Ω-deformation. Highest weights of these modules are determined by locations of the D3 branes along the D5 branes. Supersymmetric configurations of fundamental strings (F1 branes) stretched between the D-branes provide states in the spin chains represented by the line operators. A composition of S and T-duality converts the D5-D3-F1 brane configurations into NS5-D4-D2 configurations. Supersymmetric F1 branes are mapped to supersymmetric D2 branes which correspond to the massive vacua of the world-volume theories of the D2 branes. These world-volume theories are 2d quiver gauge theories with $\mathcal{N} = (2,2)$ supersymmetry. Thus we find a one-to-one map between states in the integrable non-compact superspin chains and vacua of 2d supersymmetric gauge theories.

The spin chains carry actions of the Yangian algebra $Y(\mathfrak{gl}(m|n))$. The Bethe/gauge correspondence therefore implies an action of the Yangian on the vector space spanned by the massive vacua of the 2d gauge theories. These vacua can be characterized by the equivariant cohomology of certain Kähler varieties associated to the gauge theories called the Higgs branches. We thus conjecture a geometric construction of infinite dimensional highest weight representations of $Y(\mathfrak{gl}(m|n))$ in terms of equivariant cohomology of certain handsaw-type quiver varieties.

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Universal k-matrices for quantum symmetric Kac-Moody pairs

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(joint work with Andrea Appel)

Let $(A, \mathcal{R})$ be a quasitriangular bialgebra. Consider, following [Da07], a *twist pair* $(\psi, \mathcal{F})$. Here $\mathcal{F} \in (A \otimes A)^{\times}$ is a Drinfeld twist controlling the failure of $\psi \in \mathsf{Aut}_{\mathsf{alg}}(A)$ to be a quasitriangular isomorphism from $(A, \mathcal{R})$ to $(A^{\mathsf{cop}}, \mathcal{R}_{21})$. Hence $(\psi, \mathcal{F})$ induces a braided tensor equivalence between Mod_A and $\mathsf{Mod}_A^{\mathsf{op}}$.

Definition 1 ([AV20]). *Given a quasitriangular bialgebra* $(A, \mathcal{R})$ *and twist pair* $(\psi, \mathcal{F})$*, we say that* $(A, \mathcal{R})$ *is* $(\psi, \mathcal{F})$-cylindrical *if there exists* $k \in A^\times$ *such that*

$$\Delta(k) = \mathcal{F}^{-1} \cdot (1 \otimes k) \cdot (\psi \otimes \mathsf{id})(\mathcal{R}) \cdot (k \otimes 1). \tag{1}$$

Owing to (1), the twist pair property $(\psi \otimes \psi)(\mathcal{R}) = \mathcal{F} \cdot \mathcal{R}_{21} \cdot \mathcal{F}_{21}^{-1}$ and the intertwining property of $\mathcal{R}$, the *ψ-twisted reflection equation* in $A \otimes A$ holds:

$$(k \otimes 1) \cdot (\psi \otimes \mathsf{id})(\mathcal{R})_{21} \cdot (1 \otimes k) \cdot \mathcal{R} = (\psi \otimes \psi)(\mathcal{R})_{21} \cdot (1 \otimes k) \cdot (\psi \otimes \mathsf{id})(\mathcal{R}) \cdot (k \otimes 1). \tag{2}$$

Given a suitable right coideal subalgebra $B \subseteq A$, i.e. $\Delta(B) \subseteq B \otimes A$, and twist pair $(\psi, \mathcal{F})$, we discuss a construction of such k, based on the condition

$$k \cdot b = \psi(b) \cdot k \qquad \text{for all } b \in B. \tag{3}$$

Namely, let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a symmetrizable Kac-Moody algebra, defined over an algebraically closed field $\mathbb{F}$ of characteristic 0. Consider the q-deformed universal enveloping algebra $A = U_q\mathfrak{g}$, defined over $\overline{\mathbb{F}(q)}$. Its "universal R-matrix" $\mathcal{R}$, up to a Cartan factor, resides in the $\mathcal{O}$-completion[1] of, say, $U_q\mathfrak{n}^- \otimes U_q\mathfrak{n}^+$.

An automorphism θ of $\mathfrak{g}$ is said to be *of the second kind* if $\theta(\mathfrak{n}^+) \cap \mathfrak{n}^+$ is finite-dimensional; for instance, the Chevalley involution ω. Denote by $\mathsf{Aut}^{\mathsf{II}}_{\mathsf{inv}}(\mathfrak{g})$ the set of *involutions* of the second kind and by I the set of Dynkin nodes of $\mathfrak{g}$. Given any $\zeta \in \mathsf{Aut}^{\mathsf{II}}_{\mathsf{inv}}(\mathfrak{g})$, the parabolic Weyl group of the subset $X(\zeta) := \{i \in I \,|\, \zeta(h_i) = h_i\}$ is finite. Associated to its longest element is a Cartan modification T_ζ of Lusztig's braid group action [Lu94] on the subcategory $\mathcal{O}_{\mathsf{int}} \subset \mathcal{O}$ of integrable modules with the following properties.

- The universal R-matrix of the diagrammatic subbialgebra $U_q\mathfrak{g}_{X(\zeta)} \subseteq U_q\mathfrak{g}$ possesses a Cartan modification $\mathcal{R}_\zeta := (T_\zeta^{-1} \otimes T_\zeta^{-1}) \cdot \Delta(T_\zeta)$.
- For a particular $\zeta_q \in \mathsf{Aut}(U_q\mathfrak{g})$ which q-deforms ζ and Cartan-modifies a construction in [Ko14], $(\zeta_q, \mathcal{R}_\zeta)$ is a twist pair "up to completion", inducing a braided tensor equivalence between $\mathcal{O}_{\mathsf{int}}$ and $\omega^*(\mathcal{O}_{\mathsf{int}}^{\mathsf{op}})$.

Given $\theta \in \mathsf{Aut}^{\mathsf{II}}_{\mathsf{inv}}(\mathfrak{g})$, the *Letzter-Kolb coideal subalgebra* [Le99, Ko14] is the unique maximal right coideal subalgebra $B = U_q(\mathfrak{g}^\theta) \subseteq U_q\mathfrak{g}$ which q-deforms $U(\mathfrak{g}^\theta)$. Now set $(\psi, \mathcal{F}) = (\theta_q, \mathcal{R}_\theta)$. A recursive argument using Lusztig's skew derivations [Lu94] yields the following extension of results from [BK19].

- Up to a group-like Cartan modification, there is a unique $k = k_\theta$ in the $\mathcal{O}$-completion of $U_q\mathfrak{n}^+$ such that $\epsilon(k) = 1$ and condition (3) is satisfied;
- By a direct computation, the coproduct of $k = k_\theta$ is given by (1).

The two roles of $\mathsf{Aut}^{\mathsf{II}}_{\mathsf{inv}}(\mathfrak{g})$ come together in the following key result.

Theorem 2 ([AV20]). *Let* $A = U_q\mathfrak{g}$ *with universal R-matrix* $\mathcal{R}$ *and let* $B = U_q(\mathfrak{g}^\theta)$ *a Letzter-Kolb coideal subalgebra in terms of* $\theta \in \mathsf{Aut}^{\mathsf{II}}_{\mathsf{inv}}(\mathfrak{g})$*. Let* $\zeta \in \mathsf{Aut}^{\mathsf{II}}_{\mathsf{inv}}(\mathfrak{g})$ *be arbitrary. There exists a quasitriangular automorphism* β *of* $(U_q\mathfrak{g}, \mathcal{R})$ *such that*

- $(\psi, \mathcal{F}) := (\zeta_q \circ \beta, \mathcal{R}_\zeta)$ *is a twist pair;*
- *the element* $k_\zeta := T_\zeta^{-1} \cdot T_\theta \cdot k_\theta$ *of the* $\mathcal{O}_{\mathsf{int}}$*-completion of* $U_q\mathfrak{g}$ *satisfies* (3)*;*
- *the coproduct formula* (1) *is satisfied.*

[1]In particular, $\mathcal{R}$ can be evaluated in tensor products of modules in the category $\mathcal{O}$.

Thus given a pair of involutions of the second kind one obtains a solution of the universal reflection equation (2) in a completion of $U_q\mathfrak{g}^{\otimes 2}$. Evaluating (2) on a tensor product of $\mathcal{O}_{\mathsf{int}}$-modules, we obtain a representation of a defining relation of the *type* B (or "cylindrical") *ribbon Artin braid groupoid*. Here the two sides of any ribbon are coloured by $\mathcal{O}_{\mathsf{int}}$ and $\omega^*(\mathcal{O}_{\mathsf{int}}^{\mathsf{op}})$, respectively.

If $\mathfrak{g}$ is finite-dimensional and $\zeta = \mathsf{id}$ we recover the construction from [BK19]. If $\mathfrak{g}$ is of affine type we can also let k act on (finite-dimensional) evaluation modules $V(z)$, resulting in matrix-valued formal Laurent series. Under a mild assumption on ζ, one observes that $\zeta_q^*(V(z)) = (\zeta_q^* V)(z^{-1})$, so that (2) results in Cherednik's generalized parameter-dependent reflection equation [Ch92].

If the following conjecture holds (it can be verified for $U_q\widehat{\mathfrak{sl}_2}$) we can recover the original (untwisted or transpose-twisted) parameter-dependent reflection equation, playing a key role in quantum integrability near a boundary, see e.g. [Ch84, Sk88].

Conjecture 3. *Let $V(z)$ be an evaluation module. There exists $\zeta \in \mathsf{Aut}_{\mathsf{inv}}^{\mathsf{II}}(\mathfrak{g})$ so that $\zeta_q^* V = V$.*

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Generalized Schur-Weyl dualities for quantum affine symmetric pairs and orientifold Khovanov-Lauda-Rouquier algebras

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(joint work with Tomasz Przezdziecki)

The classical Schur-Weyl duality is a fundamental symmetry, which allows to identify the category of finite-dimensional representations of the symmetric group $\mathfrak{S}_\ell$ with the subcategory of finite-dimensional representations of $\mathfrak{sl}_N$ appearing in the decomposition of the ℓ-tensor power of the fundamental representation $\mathbb{V} := \mathbb{C}^N$ of $\mathfrak{sl}_N$. The identity stems from the simple observation that $\mathbb{V}^{\otimes \ell}$ is $(U\mathfrak{sl}_N, \mathfrak{S}_\ell)$-bimodule, thus yielding a functor $\mathbb{V}^{\otimes \ell} \otimes_{\mathfrak{S}_\ell} \bullet : \mathrm{Rep}_{\mathrm{fd}}(\mathfrak{S}_\ell) \to \mathrm{Rep}_{\mathrm{fd}}(U\mathfrak{sl}_N)$.

The quantum analogue of this construction appears as a duality between quantum groups and Hecke algebras. In particular, in [4], Chari and Pressley provide a functor $\mathrm{Rep}_{\mathrm{fd}}(\hat{H}_{\ell,q^2}) \to \mathrm{Rep}_{\mathrm{fd}}(U_qL\mathfrak{sl}_N)$ between the affine Hecke algebra $\hat{H}_{\ell,q^2}$ and the quantum loop algebra $U_qL\mathfrak{sl}_N$, arising from their joint action on the ℓ-tensor product of the affinized fundamental $U_qL\mathfrak{sl}_N$-representation $\mathbb{V} := \mathbb{C}(q)^N[z, z^{-1}]$.

More recently, in the series of papers [6, 7, 8, 9], Kang, Kashiwara, Kim, and Oh obtain a *generalized* version of Chari-Pressley Schur-Weyl duality, which goes beyond type A and is expressed in terms of Khovanov-Lauda-Rouquier (KLR) algebras, also known as quiver Hecke algebras. More precisely, let Q be a quiver of finite-type, $\mathfrak{g}_Q$ the corresponding complex simple Lie algebra, and, for any dimension vector β, let $\mathfrak{R}_Q^\beta$ be the corresponding KLR algebra. Then, there is a functor $\mathrm{Rep}_{\mathrm{gr,fd}}(\mathfrak{R}_Q^\beta) \to \mathrm{Rep}_{\mathrm{fd}}(U_qL\mathfrak{g}_Q)$, induced as before by a bimodule $\mathbb{V}^\beta$, given by a direct sum of tensor products of certain affinized $U_qL\mathfrak{g}_Q$-modules. As in the case of the affine Hecke algebra, the action of $\mathfrak{R}_Q^\beta$ on $\mathbb{V}^\beta$ is given in terms of certain normalized R–matrices of $U_qL\mathfrak{g}_Q$. The sum over all possible dimension vector yields a functor

$$\mathcal{F} : \bigoplus_\beta \mathrm{Rep}_{\mathrm{gr,fd}}(\mathfrak{R}_Q^\beta) \to \mathrm{Rep}_{\mathrm{fd}}(U_qL\mathfrak{g}_Q)$$

which is proved to be monoidal with respect to the tensor product induced on the KLR side by the convolution product $\mathfrak{R}_Q^\beta \otimes \mathfrak{R}_Q^{\beta'} \to \mathfrak{R}_Q^{\beta+\beta'}$. Moreover, up to a suitable localized quotient of the KLR category, $\mathcal{F}$ restricts to an equivalence with (a generalization of) the Hernandez-Leclerc category $\mathcal{C}_Q \subset \mathrm{Rep}_{\mathrm{fd}}(U_qL\mathfrak{g}_Q)$ [5, 10].

In fact, the original construction of the functor is more general and depends upon a combinatorial datum consisting of a (possibly infinite) set of finite-dimensional $U_qL\mathfrak{g}$-representations V_i, each decorated with a non-zero scalar $f_i \in \mathbb{C}(q)$. By comparing the poles of the normalized R-matrices on $V_i \otimes V_j$ with f_j/f_i, one obtains a quiver Q, which determines the KLR algebra. It is then proved that there exists a combinatorial datum for which Q coincides with the Dynkin type of $\mathfrak{g}$. In view of these results, the representation theory of KLR algebras can be thought of as a powerful tool to study the representation theory of quantum affine algebras.

A *quantum affine symmetric pair (QSP) subalgebra* is a distinguished coideal subalgebra $U_q\mathfrak{k} \subset U_qL\mathfrak{g}$ (also known as Letzter-Kolb subalgebras or affine $\imath$quantum groups. In [1], the author and B. Vlaar prove that QSP subalgebras (of arbitrary Kac–Moody type) give rise to *universal k-matrices*, i.e. solutions of the reflection equation (cf. the previous abstract by B. Vlaar). The latter can be thought of as a boundary analogue of the Yang-Baxter equation, since it arises as a consistency condition in the case of particles moving on a half-line. It will be proved in [2] that, as in the case of the universal R-matrix, such universal k-matrices descend to finite-dimensional $U_qL\mathfrak{g}$-representations and provide a conceptual approach to the trigonometric solutions of the spectral reflection equation. Moreover, as pointed out by V. Toledano Laredo during the talk, it is clear that the existence of a universal solution implies that of a *rational k-matrix* on any finite-dimensional $U_qL\mathfrak{g}$-representation.

The understanding of the category $\mathrm{Rep}_{\mathrm{fd}}(U_qL\mathfrak{g})$, which is naturally acted upon by $\mathrm{Rep}_{\mathrm{fd}}(U_qL\mathfrak{g})$, is however extremely limited. It is therefore desirable to produce a boundary analogue of the functor $\mathcal{F}$, establishing a generalized Schur-Weyl duality between the QSP algebra $U_q\mathfrak{k}$ with a suitable notion of KLR algebra. The key idea is to enhance the combinatorial model (V_i, f_i), developed by Kang-Kashiwara-Kim, by taking into account the poles of the rational k-matrix on the representations V_i. Such enhanced combinatorial datum yields a *framed* quiver Q. Note that the QSP subalgebra $U_q\mathfrak{k}$ depends on two families of parameters, which appear in the poles of the k-matrix and therefore determine the framing. In particular, one can observe that, for *generic* parameters, the framing will be trivial.

We further assume that there exists a contravariant involution θ on Q, which preserves the framing. In [11], Varagnolo and Vasserot introduced a new KLR algebra $\mathfrak{R}^{\beta}_{Q,\theta}$, naturally associated to the datum (Q,θ), which we refer to as the *orientifold* KLR algebra. We then get the following

Theorem 1.

(1) There exists a $(U_q\mathfrak{k}, \mathfrak{R}^{\beta}_{Q,\theta})$-bimodule $\mathbb{V}^{\beta}$, which induces a functor

$$\mathcal{F}_\theta : \bigoplus_{\beta} \mathrm{Rep}_{\mathrm{gr,fd}}(\mathfrak{R}^{\beta}_{Q,\theta}) \to \mathrm{Rep}_{\mathrm{fd}}(U_q\mathfrak{k})$$

(2) The functors $(\mathcal{F}_\theta, \mathcal{F})$ intertwine the natural categorical actions

$$\begin{array}{ccc} \bigoplus_{\beta} \mathrm{Rep}_{\mathrm{gr,fd}}(\mathfrak{R}^{\beta}_{Q}) & \xrightarrow{\mathcal{F}} & \mathrm{Rep}_{\mathrm{fd}}(U_qL\mathfrak{g}) \\ \downarrow & & \downarrow \\ \bigoplus_{\beta} \mathrm{Rep}_{\mathrm{gr,fd}}(\mathfrak{R}^{\beta}_{Q,\theta}) & \xrightarrow[\mathcal{F}_\theta]{} & \mathrm{Rep}_{\mathrm{fd}}(U_q\mathfrak{k}) \end{array}$$

where the vertical arrow on the KLR side is given by induction.

Many follow-up questions remain open and are currently being investigated. In particular, we expect the functor $\mathcal{F}_\theta$ to give rise to an equivalence. This would require a much deeper understanding of the structural properties of the orientifold KLR representation theory. In turn, it would yield a QSP analogue of the Herndez-Leclerc category, whose Grothendieck ring would be the best candidate for a boundary analogue of cluster algebras.

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Presentation of generic K-theoretical Hall algebras of quivers via Shuffle algebras

Olivier Schiffmann
(joint work with A. Negut, F. Sala)

Let Q be a finite quiver, with vertex set I and edge set E (edge loops and multiple edges are allowed). Cohomological Hall algebras associated to Q for a Borel-Moore homology theory (including K-theory) were recently introduced, in relation to Donaldson-Thomas theory on the one hand and Nakajima quiver varieties on the other hand (see [KS11, SV13, YZ18]). More precisely, the *K-theoretic Hall algebra* of Q is the vector space:

$$K_Q := \bigoplus_{\boldsymbol{n}\in\mathbb{N}^I} K^T(\mathrm{Rep}_{\boldsymbol{n}}\,\Pi_Q)$$

where Π_Q is the preprojective algebra of Q, and $\mathrm{Rep}_{\boldsymbol{n}}\,\Pi_Q$ is the stack of complex $\boldsymbol{n}$-dimensional representations of Π_Q. The vector space K_Q is equipped with a natural Hall multiplication making it into an associative algebra. Here T is a torus acting in a Hamiltonian way on $\mathrm{Rep}_{\boldsymbol{n}}\,\Pi_Q$ by appropriately rescaling the maps attached to the arrows $e \in E$. The algebra K_Q acts on the T-equivariant K-theory of Nakajima quiver varieties :

$$\mathcal{N}_{\boldsymbol{w}} = \bigsqcup_{\boldsymbol{v}\in\mathbb{N}^I} \mathcal{N}_{\boldsymbol{v},\boldsymbol{w}}$$

and is in some sense the largest algebra thus acting via Hecke correspondences. When Q is a finite type quiver, K_Q is identified with the positive half of the quantum loop algebra (in Drinfeld's sense) of $\mathfrak{g}_Q$; the situation of an affine quiver, in which case K_Q is a quantum toroidal algebra, is studied in detail (in type A) in [Neg15] (see also [VV20] for other affine quivers). More generally, for a quiver without edge loops and a specific one-dimensional torus T, there is an algebra morphism:

$$U_q^+(L\mathfrak{g}_Q) \longrightarrow K_Q \tag{1}$$

which recovers Nakajima's construction of representations of quantum affinizations of Kac-Moody algebras on the equivariant K-theory of quiver varieties. The map (1) is surjective (under some mild conditions on the torus action on $\mathrm{Rep}_{\boldsymbol{n}}\,\Pi_Q$), but it is not known to be injective in general (see [VV20]). Beyond these cases, however, very little is known. Moreover, the structure of K_Q as an algebra depends in a rather subtle way on T. Note that there is a natural gauge action of the group $(\mathbb{C}^*)^I$ on T, but as soon as Q contains edge loops or multiple edges the quotient of T by this gauge group is nontrivial.

In the present paper, we consider the case when the torus $T = (\mathbb{C}^*)^E \times \mathbb{C}^*$ is as large as possible and we work over the fraction field: $\mathbb{F} = \mathrm{Frac}(K^T(\mathrm{pt})) = \mathbb{Q}(q, t_e)_{e\in E}$. Our main result provides an explicit description of:

$$K_{Q,\mathrm{loc}} := K_Q \bigotimes_{K^T(\mathrm{pt})} \mathbb{F}$$

by generators and relations which we will now summarize. Let $\overline{E}$ be the "double" of the edge set E, i.e. there are two edges $e = \overrightarrow{ij}$ and $e^* = \overrightarrow{ji}$ in $\overline{E}$ for every edge $e = \overrightarrow{ij} \in E$. The set $\overline{E}$ is equipped with a canonical involution $e \leftrightarrow e^*$. We extend the notation t_e to an arbitrary $e \in \overline{E}$ by the formula: $t_{e^*} = \frac{q}{t_e}$. For any $i, j \in I$, consider the rational function

$$\zeta_{ij}(x) = \left(\frac{1 - xq^{-1}}{1 - x}\right)^{\delta_j^i} \prod_{e\in\overrightarrow{ij}} \left(\frac{1}{t_e} - x\right) \prod_{e\in\overrightarrow{ji}} \left(1 - \frac{t_e}{qx}\right) \tag{2}$$

and set $\widetilde{\zeta}_{ij}(x) = \zeta_{ij}(x) \cdot (1-x)^{\delta_j^i}$. Let $\mathbf{U}_Q^+$ be the algebra generated by elements $e_{i,d}$ for $i \in I, d \in \mathbb{Z}$ subject to the following set of quadratic and cubic relations, where we have set $e_i(z) = \sum_{d\in\mathbb{Z}} e_{i,d} z^{-d}$:

- For any pair $(i, j) \in I^2$, the **quadratic** relation:

$$e_i(z)e_j(w)\widetilde{\zeta}_{ji}\left(\frac{w}{z}\right) z^{\delta_j^i} = e_j(w)e_i(z)\widetilde{\zeta}_{ij}\left(\frac{z}{w}\right)(-w)^{\delta_j^i} \tag{3}$$

- For any edge $\overline{E} \ni e = \overrightarrow{ij}$, the **cubic** relation:

$$
\begin{aligned}
(4) \quad & \frac{\widetilde{\zeta}_{ii}\left(\frac{x_2}{x_1}\right)\widetilde{\zeta}_{ji}\left(\frac{y}{x_1}\right)\widetilde{\zeta}_{ji}\left(\frac{y}{x_2}\right)}{\left(1-\frac{x_2}{x_1 q}\right)\left(1-\frac{yq}{x_2 t_e}\right)} \cdot e_i(x_1)e_i(x_2)e_j(y) \\
& + \frac{\widetilde{\zeta}_{ii}\left(\frac{x_1}{x_2}\right)\widetilde{\zeta}_{ji}\left(\frac{y}{x_2}\right)\widetilde{\zeta}_{ij}\left(\frac{x_1}{y}\right)\left(-\frac{x_2 t_e}{y}\right)\left(-\frac{y}{x_1}\right)^{\delta_j^i}}{\left(1-\frac{yq}{x_2 t_e}\right)\left(1-\frac{x_1 t_e}{y}\right)} \cdot e_i(x_2)e_j(y)e_i(x_1) \\
& + \frac{\widetilde{\zeta}_{ii}\left(\frac{x_2}{x_1}\right)\widetilde{\zeta}_{ij}\left(\frac{x_1}{y}\right)\widetilde{\zeta}_{ij}\left(\frac{x_2}{y}\right)\left(\frac{x_2 t_e}{yq}\right)\left(\frac{y^2}{x_1 x_2}\right)^{\delta_j^i}}{\left(1-\frac{x_2}{x_1 q}\right)\left(1-\frac{x_1 t_e}{y}\right)} \cdot e_j(y)e_i(x_1)e_i(x_2) = 0
\end{aligned}
$$

Theorem 1. *There is an algebra isomorphism* $K_{Q,\mathrm{loc}} \simeq \mathbf{U}_Q^+$.

When Q is a tree, the quotient of $(\mathbb{C}^*)^E \times \mathbb{C}^*$ by the action of the gauge group $(\mathbb{C}^*)^I$ is one-dimensional. In addition, one can check that in the case of an A_2 quiver, the cubic relations (4) are equivalent to the standard q-Serre relations. With this in mind, our results imply:

Theorem 2. *Suppose that Q is a tree, and that T scales the symplectic form on* $\mathrm{Rep}_{\boldsymbol{n}} \Pi_Q$ *nontrivially. Then the localization:*

$$U_q^+(L\mathfrak{g}_Q) \bigotimes_{K^T(\mathrm{pt})} \mathbb{F} \longrightarrow K_{Q,\mathrm{loc}}$$

of the map (1) *is an isomorphism.*

Cohomological Hall algebras of quivers are known (at least in the case of Borel-Moore homology and K-theory) to embed in a suitable (big) *shuffle algebra* $\mathcal{V}_Q$, whose multiplication encodes the structure of Q (see [SV20, VV20, YZ18]). In the K-theoretic case and for maximal T, recent work ([Neg21, Zha19]) identified the image of this embedding as the subspace $\mathcal{S}_Q$ determined by the so-called *3-variable wheel conditions.* Theorem 1 is thus a direct corollary of the following theorem, which is the main result of the present paper.

Theorem 3. *There is an isomorphism* $\mathbf{U}_Q^+ \cong \mathcal{S}_Q$.

For a general Q and a general choice of T (which satisfies some mild conditions), there is a chain of algebra homomorphisms: $\mathbf{U}_Q^+ \longrightarrow K_{Q,\mathrm{loc}} \longrightarrow \mathcal{S}_Q$. The content of Theorem 3 is that these maps are all isomorphisms for T maximal.

Let us mention one other application of Theorem 3. When Q is the quiver with one vertex and g loops, it is known by combining [SV12] and [Neg21] that the *spherical Hall algebra* $\mathbf{H}_X^{sph}$ of the category of coherent sheaves on a genus g curve X defined over $\mathbb{F}_{q^{-1}}$ is isomorphic to K_Q. Here the equivariant parameters $t_1, \dots, t_g$ are set to be the inverses of the Weil numbers $\sigma_1, \ldots, \sigma_g$ of X, hence:

$$\zeta_g(x) = \frac{1 - xq^{-1}}{1 - x} \prod_{e=1}^{g} (\sigma_e - x)(1 - \overline{\sigma_e} x^{-1})$$

For any $e = 1, \ldots, g$, set:

$$Q_e(z_1, z_2, z_3) = \prod_{1 \leq i < j \leq 3} \prod_{f \neq e} \left(\sigma_f - \frac{z_j}{z_i} \right) \left(1 - \overline{\sigma_f} \frac{z_i}{z_j} \right)$$

Then we have the following result :

Theorem 4. *The (generic) genus g spherical Hall algebra $\mathbf{H}_g^{sph}$ is generated over $\mathbb{F}$ by elements $\kappa_{1,0}^{\pm 1}$, $\theta_{0,l}, 1_d^{vec}$ for $l \geq 0, d \in \mathbb{Z}$ subject to the following set of relations:*

$$H^+(z)H^+(w) = H^+(w)H^+(z) \tag{5}$$

$$E(z)H^+(w) = H^+(w)E(z)\frac{\zeta_g\left(\frac{z}{w}\right)}{\zeta_g\left(\frac{w}{z}\right)} \qquad (\text{for } |w| \gg |z|) \tag{6}$$

$$E(z)E(w)\zeta_g\left(\frac{w}{z}\right) = E(w)E(z)\zeta_g\left(\frac{z}{w}\right) \tag{7}$$

and for all $e = 1, \ldots, g$ and $m \in \mathbb{Z}$ the relation:

$$\Big[(xyz)^m (x+z)(xz - y^2) Q_e(x,y,z) E(x)E(y)E(z)\Big]_{\mathrm{ct}} = 0. \tag{8}$$

In the formulas above, we set:

$$E(z) = \sum_{d \in \mathbb{Z}} 1_d^{vec} z^{-d}, \qquad H^+(z) = \kappa_{1,0} \left(1 + \sum_{l \geq 1} \theta_{0,l} z^{-l} \right)$$

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Three-dimensional WZW model and the R-matrix of the Yangian

Meer Ashwinkumar

In this talk, based on [1], I describe a three-dimensional quantum field theory dual to the four-dimensional Chern-Simons theory reviewed by Masahito Yamazaki earlier in the week. To be precise I analyze four-dimensional Chern-Simons theory on a product of a disk, D, and the complex plane $\mathbb{C}$ (which is understood to describe rational solutions of the Yang-Baxter equation from the work of Costello, Witten and Yamazaki [2]), with the action

$$S = \frac{1}{2\pi\hbar}\int_{D\times\mathbb{C}} \omega \wedge \mathrm{Tr}\Big(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big), \tag{1}$$

and complex gauge group, G, where $\omega = dz$. Here, $\mathcal{A}$ can be understood to be the partial connection $\mathcal{A} = \mathcal{A}_r dr + \mathcal{A}_\varphi d\varphi + \mathcal{A}_{\bar z} d\bar z$, where (r,φ) are polar coordinates on D and $(z,\bar z)$ are complex coordinates on $\mathbb{C}$. With the boundary condition $\mathcal{A}_{\bar z} = 0$ at ∂D, this theory can be shown to be equivalent to a three-dimensional analogue of the two-dimensional chiral WZW model, with the action where the field g is valued in G.

This three-dimensional WZW model admits a symmetry under the transformation $g(\varphi, z, \bar z) \to \tilde\Omega(\varphi, z) g$ with corresponding conserved current $J_\varphi = -\frac{1}{\pi\hbar}\partial_\varphi g g^{-1}$. This current can be shown, via quantization of Poisson brackets, to satisfy the following current algebra:

$$\begin{aligned}\big[\mathrm{Tr}AJ^n_\varphi(z), \mathrm{Tr}BJ^m_\varphi(z')\big] =& i\mathrm{Tr}[A,B]J^{n+m}_\varphi(z)\delta(z-z') + \frac{2}{\hbar}(n\delta_{m+n,0})\delta(z-z')\mathrm{Tr}AB\\ &+\dots,\end{aligned} \tag{2}$$

where the Fourier mode expansion $J_\varphi(\varphi, z) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} J^n_\varphi(z) e^{in\varphi}$ has been employed, where A and B are arbitrary elements of the Lie algebra $\mathfrak{g}$, and where the ellipsis indicates total derivative terms multiplied by positive powers of $\hbar$. This can be interpreted as an "analytically-continued" toroidal Lie algebra, since writing $z = \epsilon t + i\theta$, compactifying the θ direction to be valued in $[0, 2\pi]$, and subsequently taking $\epsilon \to 0$ reduces it to a two-toroidal Lie algebra.

In addition, bulk correlation functions of Wilson lines in the four-dimensional Chern-Simons theory can be captured by boundary correlation functions of local operators in the three-dimensional WZW model, at least up to order $\hbar^2$ in perturbation theory. The relevant operators are depicted in Figure 1, where R_1 and R_2 denote representations of G. In particular, I reproduced the leading nontrivial contribution to the rational R-matrix purely from the boundary theory, i.e., by computing the four-point function

$$\langle g^{-1}_{R_1}(0, z_1) g_{R_1}(\pi, z_1) \otimes g^{-1}_{R_2}(\pi/2, z_2) g_{R_2}(3\pi/2, z_2)\rangle = \mathbb{1} + \frac{\hbar}{z_1 - z_2} T^a_{R_1} \otimes T_{R_2 a} + O(\hbar^2). \tag{3}$$

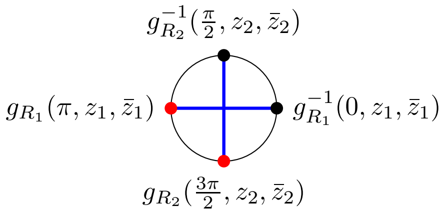
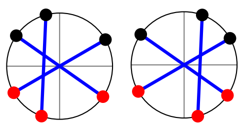


FIGURE 1

Moreover, the bulk topological symmetry on D is manifest at the boundary at order $\hbar$, and the result (3) holds for Wilson lines that are crossed in any configuration, not just the perpendicular one. Agreement up to order $\hbar^2$ holds as well, modulo the framing anomaly. The six-point functions corresponding to configurations of three simultaneously crossed Wilson lines depicted in Figure 2 were also computed up to order $\hbar^2$, modulo the framing anomaly, and were shown to agree with the bulk computations.



(A) (B)

FIGURE 2

An open question is whether these results can be generalized to derive trigonometric and elliptic R-matrices (note that 3d WZW models for other choices of ω can be obtained by generalizing the methods of [3]). It might also be possible to study the 2d chiral WZW model analogously, and obtain the R-matrix of $U_q(\mathfrak{g})$ (cf. Nanna Aamand's talk). Finally, the 3d WZW model might be able to realize known factorizations of the rational R-matrix described in Sachin Gautam's talk.

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A geometric R-matrix for the Hilbert scheme of points on a general surface

Noah Arbesfeld

In [4], it is explained how to use the representation theory of quantum groups to describe structures in the enumerative geometry of Nakajima quiver varieties. A key ingredient is an R-matrix acting in equivariant cohomology, constructed geometrically using correspondences called stable envelopes. The R-matrix gives rise to a quantum group called the Yangian; it is shown in [4] that this Yangian's Baxter subalgebras can be identified with the operators of quantum multiplication by tautological divisors in the corresponding quiver variety.

In this project, we extend a portion of this package beyond the setting of Nakajima quiver varieties. The stable envelope construction of the R-matrix in [4] requires that the underlying variety carries both a symplectic form and a large automorphism group, which need not be the case for a general variety. However, for the Hilbert scheme of points on $\mathbb{C}^2$, an alternative construction of the R-matrix is given in [4, Ch. 13] using the language of conformal field theory. We adapt this construction to produce R-matrices for the Hilbert scheme of points on a general surface. We also show that classical multiplication by the tautological divisor in the Hilbert scheme coincides with a Baxter subalgebra of the associated Yangian.

1. Construction

Let S be a surface that is either proper, or admits an action of a non-trivial torus T such that the fixed locus S^T is proper. We then equip $H_T^*(S)$ with the structure of a Frobenius algebra over $\mathrm{Frac}(H_T^*(\mathrm{pt}))$, with trace given by $\epsilon(\gamma) := -\int_S \gamma$.

Let $\mathrm{Hilb}(S) = \sqcup_{n\geq 0}\mathrm{Hilb}_n(S)$ and let $\alpha_m(\gamma)$, for $m \in \mathbb{Z}$ and $\gamma \in H_T^*(S)$, be the generators of the Heisenberg algebra of [2] and [5] acting on $H_T^*(\mathrm{Hilb}(S))$. These operators are given by correspondences and satisfy the supercommutator relations

$$[\alpha_n(\gamma), \alpha_{n'}(\gamma')] = \delta_{n+n'} n\epsilon(\gamma\gamma').$$

Let F_S denote the Heisenberg module $H_T^*(\mathrm{Hilb}(S))$. We regard F_S a Fock space with lowest weight vector $|\varnothing\rangle \in H^0(\mathrm{Hilb}_0 S)$.

As defined, the Heisenberg action leaves the action of the "zero modes" $\alpha_0(\gamma)$ ambiguous. Taking advantage of this ambiguity, we introduce a formal parameter u, we let $F_S(u)$ denote $F_S \otimes \mathbb{C}(u)$ where $\alpha_0(\gamma)$ scales F_S by $-u\epsilon(\gamma)$.

The desired R-matrix will be defined in terms of the modified generators

$$\alpha_n^-(\gamma) := \frac{1}{\sqrt{2}}(\alpha_n(\gamma) \otimes 1 - 1 \otimes \alpha_n(\gamma))$$

acting in $F_S(u_1) \otimes F_S(u_2) := F(S)^{\otimes 2} \otimes \mathbb{C}(u_1, u_2)$ and will be a function of $u_1 - u_2$.

We use the Feigin-Fuchs construction to obtain a Virasoro algebra action on $F_S(u_1) \otimes F_S(u_2)$. Adjoin a parameter κ to the ground field. Then, if $\Delta\gamma =$

$\sum_i \gamma_i^{(1)} \otimes \gamma_i^{(2)}$, set

$$L_n(\gamma,\kappa) = \frac{1}{2}\sum_{m\in\mathbb{Z}}\sum_i :\alpha_m^-(\gamma_i^{(1)})\alpha_{n-m}^-(\gamma_i^{(2)}): -\frac{\kappa}{\sqrt{2}} n\alpha_n^-(\gamma) - \delta_n \frac{\kappa^2}{4}\epsilon(\gamma).$$

By [3, Thm 3.3], the operators $L_n(\gamma,\kappa)$ satisfy the following Virasoro relation

(1)
$$[L_n(\gamma,\kappa), L_{n'}(\gamma',\kappa)] = (n-n')L_{n+n'}(\gamma\gamma',\kappa) + \delta_{n+n'}\frac{n^3-n}{12}\epsilon(\gamma\gamma'(c_2(S)-6\kappa^2)).$$

For generic κ and $u_1 - u_2$, the Virasoro operators applied to $|\varnothing\rangle\otimes|\varnothing\rangle \in F_S(u_1)\otimes F_S(u_2)$ generate an irreducible lowest weight module with

$$L_0(\gamma,\kappa)|\varnothing\rangle\otimes|\varnothing\rangle = \frac{1}{4}(2(u_2-u_1)^2-\kappa^2)\epsilon(\gamma)|\varnothing\rangle\otimes|\varnothing\rangle. \tag{2}$$

The argument of ϵ in (1) plays the role of the central charge, while the scalar in (2) plays the role of lowest weight. Observe that both of these quantities are even functions of κ. We may therefore define $R(u_1-u_2)$ to be the unique operator in $F_S(u_1)\otimes F_S(u_2)$ which fixes the vacuum $|\varnothing\rangle\otimes|\varnothing\rangle$, and satisfies

$$R(u_1-u_2)L_n(\gamma,\kappa) = L_n(\gamma,-\kappa)R(u_1-u_2),$$
$$R(u_1-u_2)(\alpha_n(\gamma)\otimes 1 + 1\otimes\alpha_n(\gamma)) = (\alpha_n(\gamma)\otimes 1 + 1\otimes\alpha_n(\gamma))R(u_1-u_2)$$

for all n and γ.

2. Results

Theorem 1. *The operator $R(u)$ satisfies the Yang-Baxter equation with spectral parameter.*

This result is proved for $S=\mathbb{C}^2$ in [4, Thm. 14.3.1]; the proof for general S uses this special case. The quantum inverse scattering method then produces a Yangian Y_S with an action on $\oplus_i F_S^{\otimes i}$.

The matrix elements of $R(u)$ also encode multiplication by Chern classes of the tautological bundle.

Theorem 2. *For $n\geq 0$, let $x_1,\dots,x_n$ be the Chern roots of the tautological bundle $\mathcal{O}^{[n]}$ on $\mathrm{Hilb}_n(S)$. Then, the vacuum matrix element*

$$|\varnothing\rangle\otimes H_T^*(\mathrm{Hilb}_n(S)) \to |\varnothing\rangle\otimes H_T^*(\mathrm{Hilb}_n(S))$$

of the normalized operator $R(u/\sqrt{2})$ is equal to multiplication by

$$\prod_{i=1}^n \frac{u-x_i}{u-\kappa-x_i}.$$

Given a non-trivial line bundle $\mathcal{L}$ on S, one can modify the action of the zero modes in the construction of $R(u)$ to a produce a new operator which does not solve the Yang-Baxter equation, but does satisfy an analog of Theorem 2 where $\mathcal{O}^{[n]}$ is replaced by $\mathcal{L}^{[n]}$.

3. Open questions

(1) What portion of the quantum cohomology of $\mathrm{Hilb}(S)$ is controlled by the Yangian Y_S? This question seems most tractable when S is a K3 surface.

(2) Taking inspiration from the level of generality of [1], note that the constructions in Section 1.1 can still be carried out if $H^*_T(S)$ is replaced by the cohomology of a higher-dimensional variety or, more generally, a graded supercommutative Frobenius algebra A. Does Theorem 1.1 still hold in this more general setting? Either an affirmative or a negative answer would be interesting.

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