Stability of Lamellar Configurations in a Nonlocal Sharp Interface Model

STABILITY OF LAMELLAR CONFIGURATIONS IN A NONLOCAL SHARP INTERFACE MODEL*

EMILIO ACERBI †, CHAO-NIEN CHEN ‡, AND YUNG-SZE CHOI §

Abstract. Equilibrium models based on a free energy functional deserve special interest in recent investigations, as their critical points exhibit various pattern structures. These systems are characterized by the presence of coexisting phases, whose distribution results from the competition between short and long-range interactions. This article deals with an energy-driven sharp interface model with long-range interaction being governed by a screened Coulomb kernel. We investigate a number of criteria for the stability of lamellar configurations to ensure that they are indeed strict local minimizers. We also give a sufficient condition to ensure a nontrivial periodic 2D minimal energy configuration.

Key words. Nonlocal geometric variational problem, Sharp interface model, Stability, Lamella

AMS subject classifications. 49J20 49K20 49Q10 92C15 35K57

1. Introduction. The mechanisms responsible for pattern formation have been extensively studied in a number of fields of science [5, 6, 21, 23, 24, 25, 28, 30, 31, 32, 36]; for instance, ferroelectric and ferromagnetic films, diblock copolymers and degenerate ferromagnetic semiconductors. Equilibrium models based on a free energy functional deserve special interest in recent investigations, see e.g. [4, 15, 16, 17, 26, 33, 34] and the references therein. A typical form of this free energy functional is

$$\mathcal{J}_\epsilon(u) = \int_\Omega \left(\frac{\epsilon}{2}|\nabla u|^2 + \epsilon^{-1}F(u)\right)dx + \frac{\sigma}{2}\int_\Omega\int_\Omega \psi(u(x))G(x,\xi)\psi(u(\xi))d\xi dx\,, \tag{1.1}$$

where u is a scalar function, F is a double-well potential, G is a positive kernel, ψ is a given smooth function, ϵ is a small parameter and $\Omega \subset \mathbb{R}^N$ is a given bounded domain. These systems are characterized by the presence of coexisting phases induced by the two wells; the resulting structure of sharp transition interfaces defines the pattern. A well-known example of G is the Green's function associated with a uniformly elliptic operator. This turns (1.1) into a competition between short and long-range interactions; who is winning depends on the precise tuning of the control parameters. The short-range ramification, represented by the term with single integral, leads to congregation, favoring large domains of pure phases with boundary shape that minimizes surface area. The long-range effect, depicted by the double integral term, is repulsive in nature biasing towards small domains.

A diblock copolymer is a linear-chain molecule consisting of two subchains joined covalently to each other. Depending on the material properties of the diblock macromolecules, the observed mesoscopic domains are highly regular periodic structures

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that include lamellae, spheres, cylindrical tubes, and double-gyroids [6]. It is a common belief that these patterns are metastable in certain ranges of the parameters and that they can undergo morphological instabilities leading to the formation of more complex patterns. In a model of microphase separation for diblock copolymer melts [32], it was proposed to study the critical points of a functional like (1.1) with G being the Green function for the Laplace operator subject to the homogeneous Neumann boundary conditions or periodic boundary conditions. By setting $\psi(u) = u - \frac{1}{|\Omega|}\int_\Omega u\,dx$ and $F(u) = \frac{u^2(u-1)^2}{4}$ (or choosing $F(u) = \frac{(u+1)^2(u-1)^2}{4}$ in some articles) in (1.1), several authors [4, 15, 17, 18, 22, 31, 33, 34] investigated the patterns generated by

$$\int_\Omega \left(\frac{\epsilon}{2}|\nabla u|^2 + \frac{u^2(u-1)^2}{4\epsilon})\right) dx + \frac{\sigma}{2}\int_\Omega\int_\Omega (u(x)-m)G(x,\xi)(u(\xi)-m)d\xi dx \tag{1.2}$$

with prescribed mass constraint $\frac{1}{|\Omega|}\int_\Omega u\,dx = m$ and small ϵ. A derivation of (1.2) based on the statistical physics of interacting block copolymers can be found in [18]. We refer to a pioneer work of Nishiura and Ohnishi [31] for earlier results of this model.

As $\epsilon \to 0$ the L^1 norm Γ-limit of the functional (1.2) goes to (except for a multiplicative constant)

$$\int_\Omega \left(|\nabla\chi| + \frac{\sigma}{2}|\nabla v|^2\right) dx, \tag{1.3}$$

where χ is a characteristic function and

$$v(x) = \int_\Omega G(x,\xi)(\chi(\xi)-m)d\xi\,. \tag{1.4}$$

When Ω is a very large domain, one expects that the effect of boundary condition on v diminishes in its interior and the minimizer may settle down into a natural minimal energy periodic configuration. Indeed in one space dimension, minimizers of (1.2) and (1.3) are periodic [15, 34]. To address the fundamental questions, namely to what extent periodicity holds in higher space dimensions and what effect the nonlocal term has on the stability of such periodic patterns, Alberti, Choksi and Otto [4] studied the sharp interface model (1.3)-(1.4) when Ω was a N-dimensional square box $\mathbb{T} = [-T/2, T/2]^N \subset \mathbb{R}^N$ with homogeneous Neumann boundary condition. Using a direct method in the calculus of variations, they showed uniform energy distribution for the minimizers in the interior of a large torus; indeed the boundary condition influence did diminish as far as energy was concerned. On the other hand one still could not tell if a genuine multi-dimensional periodic minimal energy periodic configuration existed and if so, what its structure was.

From now on in this paper we regard $\mathbb{T}$ as a torus by imposing periodic boundary condition. We recall a local stability result: Acerbi, Fusco and Morini [3] proved that any critical configuration of (1.3)-(1.4) in $\mathbb{T}$, with positive definite second variation is a strict local minimizer with respect to small L^1-perturbations. In [19, 33, 34, 35] the authors constructed several examples of lamellar, spherical and cylindrical critical configurations and found related conditions under which they are stable. On the other hand, it remains open if the global minimizers of (1.3)-(1.4) are one dimensional lamellar configurations. We study this last question for the model (1.5) below.

There are spatial patterns resulting from the competition between thermodynamic forces operating on different length scales. In the derivation of the energy-driven model, the Green's function G associated with $-\Delta+\kappa^2$ represents a screened Coulomb kernel, while it is called unscreened Coulomb kernel when $\kappa = 0$. The constant κ has the physical meaning of the inverse of the Debye screening length [28, 29].

In this paper we are interested in the following energy-driven model:

$$\int_{\mathbb{T}} \Big(\frac{\epsilon}{2}|\nabla u|^2 + F(u)\Big)\, dx + \frac{\sigma}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} u(x)(-\Delta+1)^{-1}u(\xi)d\xi dx\ ; \tag{1.5}$$

With a screened Coulomb kernel, we seek the critical points of (1.5) with no volume (or mass) constraint. Instead, the appearance of a volume term gets into the competition process if the potential wells are slightly imbalanced; for instance

$$F(u) = \frac{u^2(u-1)^2}{4\epsilon} + \frac{\alpha}{\sqrt{2}}\Big(\frac{u^3}{3} - \frac{u^2}{2}\Big) \tag{1.6}$$

for small ϵ. Through the Γ-convergence the *sharp interface model* associated with (1.5) is

$$J(E) = \mathcal{P}_{\mathbb{T}}(E) - \alpha|E| + \frac{\sigma}{2}\int_E \mathcal{N}_E\, dx\ . \tag{1.7}$$

Here $|E|$ is the Lebesgue measure of E and $\mathcal{N}$ is an operator that assigns a measurable subset E of $\mathbb{T}$ the solution of the following modified Helmholtz equation:

$$-\Delta\mathcal{N}_E + \mathcal{N}_E = \chi_E \text{ in } \mathbb{T}, \qquad \mathcal{N}_E \text{ is periodic in } \mathbb{T}\ ; \tag{1.8}$$

as known to be the unique $\mathbb{T}$-periodic minimizer of

$$v \mapsto \int_{\mathbb{T}}\Big(\frac{|Dv|^2}{2} + \frac{v^2}{2} - v\chi_E\Big)\, dx\ . \tag{1.9}$$

The admissible set of J is

$$\mathcal{A} = \big\{E \subset \mathbb{T} :\ E \text{ is Lebesgue measurable}\big\}. \tag{1.10}$$

The (possibly infinite) perimeter of E in $\mathbb{T}$ is denoted by $\mathcal{P}_{\mathbb{T}}(E)$. If E is of class C^1, $\mathcal{P}_{\mathbb{T}}(E)$ is the surface measure of the boundary of $\partial E \cap \mathbb{T}$. A classical stationary set of J has a C^2 interface that satisfies the Euler-Lagrange equation

$$\mathcal{K}(\partial E \cap \mathbb{T}) - \alpha + \sigma\mathcal{N}_E = 0 \text{ on } \partial E \cap \mathbb{T}, \tag{1.11}$$

where as known in [12, 13], $\mathcal{K}$ denotes the sum of principal curvatures, which equals $(N-1)$ times the mean curvature.

In recent years (1.5) has been extensively studied as a paradigmatic activator-inhibitor system, like the FitzHugh-Nagumo equations, for patterns generated from homogeneous media destabilized by a spatial modulation. Not only serving as a prototype model for patterns like stripes and spots, variants of (1.5) preserve rich structures in systems exhibiting dissipative soliton phenomena [8, 9, 10, 14, 15, 25, 41]. Following similar asymptotic analysis on the Ohta-Kawasaki model [16, 17, 32, 33] as a certain physical parameter going to zero, a Γ-convergence treatment leads to the geometric variational functional (1.7) as a sharp interface model, which provides an effective setting for studying localized patterns and waves. The extra volumetric term $\alpha|E|$ is a result of the imbalance in energy wells due to the nonlinearity F. Depending on the system parameters, the competitions among the perimeter, the volume and the nonlocal interactions in this functional give rise to localized structures which may stay at rest or propagate with a dynamically stabilized velocity. See [12, 13] for studying pattern formation and [11] in dealing with traveling waves.

Our goal in this paper is to investigate the stability of lamellar configurations of (1.7). The structure of global and local minimizers of (1.7) has recently been investigated [2]. By minimality one sees that necessarily $\mathcal{N}_E \geq 0$, and since $\mathcal{N}_{\mathbb{T}\setminus E} = 1 - \mathcal{N}_E$ also that $\mathcal{N}_E \leq 1$. From (1.8), by the divergence theorem one gets

$$\int_{\mathbb{T}} \mathcal{N}_E \, dx = |E| \, .$$

Writing E' for the complement $\mathbb{T} \setminus E$ of E, we thus have

$$\int_E \mathcal{N}_E \, dx = \int_{\mathbb{T}} \mathcal{N}_E \, dx - \int_{E'} \mathcal{N}_E \, dx = |E| - \int_{E'} (1 - \mathcal{N}_{E'}) \, dx = |E| - |E'| + \int_{E'} \mathcal{N}_{E'} \, dx \, . \tag{1.12}$$

This implies

$$J(E) = J(E') + \Big(\frac{\sigma}{2} - \alpha\Big)(|E| - |E'|) \, . \tag{1.13}$$

The nonlocal interaction term of (1.7) containing a positive parameter σ. Its effect favors an identically zero solution as a minimizer. On the other hand the positive parameter α measures the driving force towards a non-zero state.

Partially motivated by (1.13), we introduce a parameter

$$c = c(\alpha, \sigma) := 1 - \frac{2\alpha}{\sigma}. \tag{1.14}$$

Clearly the empty state $E = \emptyset$ and the full state $E = \mathbb{T}$ satisfy

$$J(\emptyset) = 0 \, , \qquad J(\mathbb{T}) = \frac{\sigma}{2} c T^N \, ; \tag{1.15}$$

the sign of the "fullness parameter" c determines whether the empty torus is more (when $c > 0$) or less ($c < 0$) energetically favorable than the full torus, and not only that, as when $c > 0$ global minimizers of J all have measure less than $|\mathbb{T}|/2$, and the reverse is true if $c < 0$, see [2, Remark 1.3]. It is also true [2, Corollaries 1.6 and 1.7] that the empty (resp. full) state is a global minimizer iff $0 \leq \alpha \leq \alpha_\emptyset$ (resp. iff $\alpha_{\mathbb{T}} \leq \alpha \leq \sigma$) for some $0 < \alpha_\emptyset < \alpha_{\mathbb{T}} < \sigma$. As a remark, of the three terms composing $J(E)$, only the volumetric term is nonpositive. Since both the empty state and the full state have no phase boundary, their competitive advantages depend only on the volumetric and the nonlocal terms, which is determined by the ratio α/σ.

As been demonstrated in [2], there can be multiple laminar configurations in a fixed torus with the same physical parameters. Among these configurations there is a lamella with the lowest energy. For this new concept of minimal lamella we showed that with suitable parameters α, σ in a large torus, a lamella has a lower energy than both the empty set and the full torus (thus in particular there can be global minimizers other than both trivial states). Under this circumstance a periodic extension of the minimal lamella is a global minimizer in one space dimension; we will address the question if global minimizers in a two dimensional torus have lamella structures. The main results of [2] together with some relevant properties will be given in Section 2 (see Remark 2.3); we will need them in such an investigation.

The central issue of this paper is the stability of lamellar configurations of (1.11), that is, sets E which beside being $\mathbb{T}$-periodic are also invariant by translations orthogonal to a certain direction $\mathbf{v}$. Without loss of generality, we take $\mathbf{v}$ as the first axis, and use $(x, x') \in [0, T] \times [0, T]^{N-1}$ as coordinates. Next we fix the notation for a single lamella and a k-lamella. Let $0 < x_0 < T$ and let $E = L_{x_0} = [0, x_0] \times [0, T]^{N-1}$ be a single lamella with a thickness x_0 in the torus $\mathbb{T}$. A k-lamellar configuration $\mathbb{L}$ is composed of k "vertical" lamellae (where $\chi_{\mathbb{L}} = 1$) separated by wedges (where $\chi_{\mathbb{L}} = 0$) with the first lamella beginning at the left side of $\mathbb{T}$, i.e. at $x = 0$, and the total widths of all k lamellae being x_0. It has been shown [2] that, in every stationary k-lamellar configuration, all lamellae have the same width x_0/k and are equally spaced; so this configuration is not only T-periodic, but has a smaller period T/k. Moreover for fixed T and k, x_0/k is determined by the ratios α/σ and T/k only (see (2.5) for the precise formula). This observation helps our investigation later on. In what follows, k will be referred to as the (lamellar) tightness.

In general it is (relatively) easy to check that a candidate E satisfies the Euler-Lagrange equation of J, i.e., $J'(E) = 0$; much, much harder is the task of proving that the candidate is a local minimizer of J. As an intermediate step to eliminate translation modes, one may prove that in some suitable sense $J''(E) > 0$, a property which we call stability (see Definition 3.4 for the precise meaning of stability), and then proceed to prove that all stable critical points are local minimizers indeed.

It is not difficult to show that for every given α, σ, T, the global minimizer of (1.7) always exists. Below is a general result for the stability of lamellar configurations on a N-dimensional torus.

THEOREM 1.1. *Let $\mathbb{L}$ be a lamellar configuration of (1.11).*
(i) Stable lamellae are isolated local minimizers of (1.7).
(ii) Given σ and α, $\mathbb{L}$ is a stable solution on a N-dimensional torus $[-T/2, T/2]^N$ if T is sufficiently small.

To dig into more delicate stability results, we focus on the case $\mathbb{T} = [-T/2, T/2] \times [-T/2, T/2]$ in the investigation of the dependence of J on the parameter c defined by (1.14). Although we are confident that some of the results hold in the general cases, the delicate techniques employed here do not seem to extend for free to more than two dimensions. The next theorem indicates how stability of lamellar configurations is affected by the physical parameters α and σ, and the disturbance Fourier modes $m \in \mathbb{N} \cup \{0\}$ on each individual lamellar interface; in particular we work out good comparison associated with the value c, the tightness k and the disturbance mode m. It turns out that the mode $m = 0$ is always stable.

THEOREM 1.2. *Let $\mathbb{T} = [-T/2, T/2] \times [-T/2, T/2]$ and $\mathbb{L}_k(c)$ denote a k-lamellar stationary point of (1.11) with c being the measure of physical parameter.*
(i) $\mathbb{L}_k(c)$ is stable if and only if the disturbance mode $m = 1$ is stable. In addition, if $\mathbb{L}_k(c_1)$ is stable and $|c_2| \geq |c_1|$, then $\mathbb{L}_k(c_2)$ is stable.
(ii) If $\mathbb{L}_k(0)$ is stable then $\mathbb{L}_j(c)$ is stable for all $j \geq k$ and $|c| < 1$.

(iii) A necessary and sufficient condition for all stationary k-lamellae to be stable for every value of c and k, is that

$$\sigma < 8\pi^2 \left[T^3 \left(\frac{\tanh(T/4)}{T} - \frac{\tanh(\sqrt{T^2+4\pi^2}/4)}{\sqrt{T^2+4\pi^2}} \right) \right]^{-1} .$$

Without loss of generality, we only carry out the proof for the case $c \geq 0$; in this case $x_0 \in (0, T/2k]$. In the whole space $\mathbb{R}^N$ (an infinite torus), stationary 1-lamella will occupy the whole space as $c \to 0^+$, see [13, equations (1.18) and (1.19)]. This lamellar solution disappears for $c \leq 0$. Thus bifurcation from infinity occurs at $c = 0$ in $\mathbb{R}^N$. It is interesting to note that for radially symmetric solutions in infinite domains, it has also been demonstrated that the line $\sigma = 2\alpha$ in the (α, σ) plane, equivalently $c = 0$, is a boundary where bifurcation occurs; see [13, Figure 2], [12, Figure 2]. In this case an infinitely large bubble disappears once c turns negative. A further study in this regard is underway.

From (2.5) it is observed that a stationary k-lamella in a torus of size T is a stationary 1-lamella in a torus of size T/k; by Proposition 2.2 the corresponding v_0 and d_0 stay the same. They therefore possess the same stability properties with respect to (T/k)-periodic perturbations. Since T-periodic disturbance is allowed in the T-torus but not in the (T/k)-torus, the extra modes may induce instability in the larger torus. In other words, in a torus a 1-lamella is always unstable whenever k-lamellae are unstable.

As a further exploration, we introduced a function

$$\Gamma(c) = |c| - 1 - |c| \log |c| , \qquad |c| \leq 1, \tag{1.16}$$

extended by continuity at $c = 0$ by $\Gamma(0) = -1$. This function is a term derived [2] from an asymptotic formula of the energy for extremely large tori; i.e. as $T \to \infty$. More detailed properties of $\Gamma(c)$ will be given in Section 2, in particular see Remark 2.3. Not only $\Gamma(c)$ provides a guide to select out a lamellar configuration with least energy (density), it points out a threshold of stability exchange as follows.

THEOREM 1.3. *The following stability results hold:*

(i) When $4 + \sigma\Gamma(c) > 0$, stationary lamellae are stable for all T.

(ii) If $4 + \sigma\Gamma(c) < 0$, stationary lamellae are unstable when T is sufficiently large. Moreover the global minimizer of (1.7) has a genuine (non-lamellar) 2D structure if $0 < c < 1$.

(iii) In particular if $c = 0$ and $\sigma > 4$, there exists a $T_k = T_k(\sigma)$ such that the k-lamella is stable if $T < T_k$ and unstable if $T > T_k$.

Even though x_0 (i.e. the lamellar configuration) is completely determined by c, we note that σ can change its stability while keeping a fixed c. As a consequence of statement (ii), if periodicity were to hold in 2D, the mesoscopic structure has to be a genuine 2D finite size minimal energy configuration when $4 + \sigma\Gamma(c) < 0$. Though not the subject in this paper, knowing its structure will be extremely interesting. For (iii), the same result may still be valid for any c, but the calculation complexity prevents us from drawing a concrete conclusion. Numerical validation [38, 39, 40] has been successfully worked out in certain problems of pattern formation (e.g. the original Ohta-Kawasaki model). It should be equally interesting to have analogous development for studying the geometric variational functional.

Section 2 begins with a list of known facts for minimal lamellae. Section 3 works on first and second variation, as the preliminary for studying the stability of lamellar configurations. Theorem 1.1(i) follows from Theorem 3.5, which ensures that stable critical points of (1.7) are isolated local minimizers. That the situation is not trivial is made evident by the instablility result in Proposition 5.5 in some parameter regimes. The proof of Theorem 3.5 is lengthy, and since it is similar to that of [3, Theorem 1.1], we highlight the relevant differences only (see Appendix A). Theorem 1.1(ii) is an immediate cosequence of Poincaré inequality as to be seen in Theorem 5.1.

For a critical point E of (1.7), its local stability can be investigated through the second variation calculated by imposing various flows generated by (smooth) velocity vector fields X, detailed at the beginning of Section 3. The idea is that the critical set is stable if the functional increases under the perturbation through every such vector field over a short time interval. If E is a critical lamellar configuration $\mathbb{L}$, only the normal component $\eta := X \cdot \nu$ matters, where ν is the unit outward normal to $\mathbb{L}$. We decompose $\eta = \mu + \zeta$ where on each connected component of $\mathbb{L}$ the term μ is a constant and the integral of ζ is zero; μ and ζ are called the mean part and the zero-average part, respectively. One motivation for this decomposition is that the rigid body translation mode resides only in the mean part; moreover both parts are independent of one another in stability analysis as will be seen in expressions (3.12) and (3.13), which make up the second variation formula. As a by-product, our analysis on the mean part indicates that all stationary lamellae are stable with respect to 1D perturbation, see Corollary 4.5.

The proof of stability naturally divides into two steps: the mean value part in Section 4 for checking the stability against 1D periodic perturbations, and then the zero-average part in Section 5 to draw complete conclusion. We recall that this approach was also used in a recent paper of Morini and Sternberg [27] who dealt with the stability of lamellar configurations of the Ohta-Kawasaki model (or a nonlocal isoperimetric problem) in a thin domain $[0, \epsilon] \times [0, 1]$. There the long-range interaction is governed by the Green function associated with the Laplace operator, so a k-lamellar can be constructed by multiple repeated reflection of a single lamellar in small interval. In our case the length rescaling argument does not work when the Helmholtz operator replaces the Laplace operator, even the existence of minimal lamella is not a simple process in the calculation of energy density. When ϵ is small enough, the 1D stable periodic configuration remains stable on $[0, \epsilon] \times [0, 1]$ because the stabilizing effect resulted from the Poincaré inequality on the zero-average part dominates anything else.

Our stability analysis quantitatively calculates for the first time the energy contribution of the nonlocal term, without which an instability result cannot be formulated. In addition to making extensive use of non-trivial properties of convex functions, we rely on the explicit computation of the eigenvalues of symmetric block circulant Hermitian matrices in the investigation of the mean value part. Examining the similarity of the structures of the stability matrices, we obtain a simple criterion (5.12) for stability of zero-average part. The bulk of the paper is devoted to proving that in dimension $N = 2$ the worst case for stability is when $c = 0$, depicted in Theorem 5.13, and that stability is most delicate for 1-lamellae, Theorem 5.15. These give rise to the main consequence, Corollary 5.20, that precisely describes the stability range as been summarized in Theorem 1.2.

Stability of lamellar solutions in a Ohta-Kawasaki model has been studied in [35].

Computing the spectrum of the linearized governing equation, the authors obtained good estimates for the eigenvalues with the help of a Γ-limit as $\epsilon \to 0$. This calculation determines the sign of all eigenvalues if the number of interfaces is large. As a conclusion [35, p.26], 1D local minimizers with higher lamellar tightness are likely to be stable while those with lower tightness are likely to be unstable in three dimensions. Similar phenomena happen in our study as laid out in Theorem 1.2(ii). On the other hand, our results indicate a sharp threshold governed by the sign of $4 + \sigma\Gamma(c)$. The calculation of spectrum in both studies employed the technique of separation of variables.

2. Known facts on minimal lamellae.

In this section we first prove the existence of global minimizer of (1.7) and then state certain properties of minimal lamellae for the convenience of readers.

THEOREM 2.1. *There always exists a global minimizer of* (1.7) *for all positive* α, σ, T.

Proof. First we recall that for a $\mathbb{T}$-periodic set E

$$\mathcal{P}_{\mathbb{T}}(E) = \|D\chi_E\|_{\text{per}} =: \sup\{\int_{\mathbb{T}} \chi_E \, \text{div}\varphi \, dz : \varphi \in C^1(\mathbb{T}), \varphi \text{ is } \mathbb{T}\text{-periodic}, |\varphi| \leq 1\}$$

which represents the variation measure of χ_E in a periodic setting. As $J(E) \geq -\alpha T^N$ for any measurable $E \subset \mathbb{T}$, there exists a minimizing sequence $\{E_j\}_{j=1}^{\infty}$ such that $1 + \inf J \geq J(E_j) \to \inf J$, which leads to a uniform upper bound

$$\mathcal{P}_{\mathbb{T}}(E_j) \leq 1 + \inf J + \alpha T^N.$$

By compactness there exists a $\mathbb{T}$-periodic $E_0 \subset \mathbb{T}$ and a subsequence, still designated by $\{E_j\}$, such that $\chi_{E_j} \to \chi_{E_0}$ in $L^1(\mathbb{T})$ and pointwise a.e.; moreover $\liminf \mathcal{P}_{\mathbb{T}}(E_j) \geq \mathcal{P}_{\mathbb{T}}(E_0)$. As the L^∞ norm of characteristic functions are 1, it follows that $\chi_{E_j} \to \chi_{E_0}$ in $L^2(\mathbb{T})$; this immediately gives $\mathcal{N}_{E_j} \to \mathcal{N}_{E_0}$ in $H^1_{\text{per}}(\mathbb{T})$ so that $\int_{E_j} \mathcal{N}_{E_j} dx \to \int_{E_0} \mathcal{N}_{E_0} dx$. Hence E_0 is a global minimzier. ∎

For a while we denote by L the projection of a lamella $\mathbb{L}$ on the x-axis; we also denote the total thickness of the k-lamella by $x_0 := |L|$. The function $\mathcal{N}_{\mathbb{L}}$ appearing in the nonlocal term of (1.7) is the unique $\mathbb{T}$-periodic minimizer of the strictly convex energy (1.9). But replacing $\mathcal{N}_{\mathbb{L}}$ with its average in the x' directions, by strict convexity we deduce that $\mathcal{N}_{\mathbb{L}}$ depends only on x. Since not only $\mathbb{L}$, but also $\mathcal{N}_{\mathbb{L}}$ has a one-dimensional structure, it will be sometimes useful to drop all but the first variable and work in one dimension; using the simpler notation $u(x)$ in place of $\mathcal{N}_{\mathbb{L}}(x, x')$, it is useful to introduce the one-dimensional analogues of (1.8) and(1.9), that is, equation

$$-v'' + v = \chi_L \tag{2.1}$$

(with periodic boundary conditions in $[0, T]$) and energy

$$\frac{1}{2}\int_0^T \left(|v'(x)|^2 + |v(x)|^2\right) dx - \int_L v(x)\, dx\,, \qquad v \text{ is } T\text{-periodic}\,. \tag{2.2}$$

We collect some facts which will be useful in our stability analysis, all references being to [2].

PROPOSITION 2.2. *Suppose that the k-lamella $\mathbb{L}$ is a stationary point of the energy* (1.7) *and let v be the 1-dimensional function introduced above. Set $v_0 = v(0)$ and $d_0 = v'(0)$. Then (Proposition 2.6) all lamellae have the same size and are equally spaced; (Lemma 2.4) the function v is symmetric inside each lamella and inside each wedge, and in particular v takes the value v_0 at all sides of the lamellae, whereas v' takes value $+d_0$ (resp. $-d_0$) at each left (resp. right) side of the lamellae. If x_0 is the total width of the lamellae then (equations 2.6 and 2.7)*

$$v_0 = \frac{1}{\sinh\frac{T}{2k}}\cosh\frac{T-x_0}{2k}\sinh\frac{x_0}{2k} = \frac{1}{2\sinh\frac{T}{2k}}\Big(\sinh\frac{T}{2k} - \sinh\frac{T-2x_0}{2k}\Big)\,, \tag{2.3}$$

$$d_0 = \frac{1}{\sinh\frac{T}{2k}}\sinh\frac{T-x_0}{2k}\sinh\frac{x_0}{2k}\,. \tag{2.4}$$

Moreover (Theorem 2.9) necessarily $\alpha \le \sigma$ (which is equivalent to $|c| \le 1$), the total thickness x_0 satisfies

$$\frac{x_0}{k} = \frac{T}{2k} - \operatorname{arcsinh}\Big(c\sinh\frac{T}{2k}\Big) \tag{2.5}$$

and the corresponding energy is

$$\begin{aligned} J(\mathbb{L}) \quad &= \quad kT^{N-1}\Big\{2 + c\frac{\sigma}{2}\Big[\frac{T}{2k} - \operatorname{arcsinh}\Big(c\sinh\frac{T}{2k}\Big)\Big] \\ &\qquad -\frac{\sigma}{2\sinh\frac{T}{2k}}\left(\cosh\frac{T}{2k} - \sqrt{1 + c^2\sinh^2\frac{T}{2k}}\right)\Big\}\,. \end{aligned} \tag{2.6}$$

Equation (2.5) concretely justifies the name given to the fullness parameter c: for stationary k-lamellae, when $c > 0$ lamellae are thinner than wedges, and the opposite is true when $c < 0$.

We now specialize to minimal lamellae, i.e., k-lamellae in a torus which are optimal among all multi-lamellar configurations (the focus is on the best choice of k). Given (2.6) it is convenient to set

$$\mathcal{A}(c,t) = \operatorname{arcsinh}\big(c\sinh(t)\big)\,, \qquad \mathcal{B}(c,t) = \frac{\cosh t - \sqrt{1 + c^2\sinh^2 t}}{\sinh t}\,,$$

$$\mathcal{L}(c,t) = c\big(t - \mathcal{A}(c,t)\big) - \mathcal{B}(c,t)$$

and

$$\mathcal{E}(\sigma,c,t) = \frac{1}{t}\Big(2 + \frac{\sigma}{2}\mathcal{L}(c,t)\Big)\,,$$

so that (2.6) reads

$$J_{\mathbb{T}}(\mathbb{L}) = \frac{T^N}{2}\mathcal{E}\Big(\sigma, c, \frac{T}{2k}\Big)\,.$$

Many properties of these functions are investigated in [2, Section 3], but here we will only need to know that

$$t - \mathcal{A}(c,t) = \begin{cases} -\log c + \omega_t & \text{if } c > 0\,, \\ 2t + \log|c| + \omega_t & \text{if } c < 0\,, \end{cases} \tag{2.7}$$

where ω_t designates a function that vanishes as $t \to \infty$.

A relevant property of the function $t\mathcal{E}(\sigma, c, t) = 2 + (\sigma/2)\mathcal{L}$, see [2, Proposition 3.4], is that if $c > 0$ its limit as $t \to +\infty$ is

$$2 + \frac{\sigma}{2}(c - 1 - c\log c) \ ;$$

whereas if $c < 0$ it has as an asymptote as $t \to +\infty$ the function

$$\sigma ct + \left[2 + \frac{\sigma}{2}(|c| - 1 - |c|\log|c|)\right] .$$

REMARK 2.3. *The threshold function $\Gamma(c)$ plays a crucial role to distinguish the best lamellar configuration. In particular from [2, Theorem 3.5, Remark 3.7] when $2 + \sigma\Gamma(c)/2 \geq 0$, a finer lamella partition of the torus results in a higher energy configuration; thus 1-lamella is the best, but this configuration is always beaten by either trivial state); but if $2+\sigma\Gamma(c)/2 < 0$ then there is a unique point $t_0 = t_0(c, \sigma) > 0$ such that $\mathcal{E}(\sigma, c, t)$ is strictly decreasing for $0 < t \leq t_0$ and strictly increasing afterwards, thus the best lamellar configuration divides the torus in approximately $T/2t_0$ bands, i.e. when $T/2t_0$ is not an integer, then the optimal number of bands is either the integer just above or just below $T/2t_0$.*

3. First and second variation, and preliminaries to stability. For the rest of this paper, all functions defined on $\mathbb{T}$ are understood to be $\mathbb{T}$-periodic, and those defined on a face S of a lamella are S-periodic.

We first recall the definition of the variations of our functional J at a set $E \subset \mathbb{T}$ of class $\mathcal{C}^2$. Let $X : \mathbb{T} \to \mathbb{R}^N$ be a $\mathcal{C}^2$ vector field and consider the associated flow $\Psi : \mathbb{T} \times (-1, 1) \to \mathbb{T}$ defined by $\Psi_t = X(\Psi)$, $\Psi(x, 0) = x$ and set

$$E_t := \Psi(E, t) \ .$$

The first and second variations of J at E with respect to the flow associated with the field X are defined as the first and second derivatives at $t = 0$ of $J(E_t)$. Computing the first and second variation of the energy (1.7) is a lengthy exercise, already carried out in similar settings, see for example [20, Theorem 2.6], [7, Theorem 3.6], [3, Theorem 3.1]. We highlight only the major differences as follows:

1. these papers use characteristic functions, denoted by u or U, with values in $\{-1, 1\}$ instead of our $\{0, 1\}$-valued χ. Some factors of 2's will disappear, in particular each time when a boundary integral appears in the derivation; also, with respect to [20] which contains the bulk of the computation one may dismiss the integrals on the complementary set (where $U = -1$), which cause all the 2's;
2. in place of a volumetric constraint on E, we have an extra term which is proportional to the volume of E;
3. our potential function $\mathcal{N}_E$ (as opposed to the notation v or V in the other papers) is governed by the (modified) Helmholtz operator instead of the Laplacian.

The only likely dangerous point seems to be the last remark; but if G_H and G_L denote the Green's functions for the modified Helmholtz and the Laplacian operators, respectively, in both instances one has

$$\mathcal{N}_E(x) = \int G_H(x, y)\chi_E(y)\, dy \ , \qquad v(x) = \int G_L(x, y)u(y)\, dy$$

and the nonlocal terms in their governing functionals are given by

$$\int \mathcal{N}_E(x)\chi_E(x)\,dx\;, \qquad \int v(x)u(x)\,dx\;,$$

respectively. Then throughout the derivation all calculations are the same, since the derivation in [20] uses this form as a starting point. Thus the variations coming from the nonlocal term can be directly taken from [20], not forgetting to drop the extra 2's and stopping at formula (2.67) since after this the authors deal with the necessary corrections due to the volume constraint.

The second variation of volume may be found in [20, formula (2.30)], and the second variation of the perimeter is computed at every regular set E and not only at critical points in [3, Theorem 3.1]. Neither in the derivation of the nonlocal term nor in that of the perimeter term the infinitesimal volume preservation condition $\int_{\partial E}(X\cdot\nu)\,d\mathcal{H}^{N-1}=0$ is used, thus in the end one has the following result.

Proposition 3.1. *The first variation of* (1.7) *with respect to the flow associated with any (regular) vector field* $X:\mathbb{T}\to\mathbb{R}^N$ *defined near the boundary of a regular set* E*, of class* $\mathcal{C}^2$ *in a torus* $\mathbb{T}$*, is*

$$dJ(E)X=\int_{\partial E}\Big(\mathcal{K}(\partial E)-\alpha+\sigma\mathcal{N}_E\Big)(X\cdot\nu)\,d\mathcal{H}^{N-1}$$

and the second variation is

$$\begin{aligned}
d^2J(E)[X]=&\int_{\partial E}\Big(|\nabla_\tau(X\cdot\nu)|^2-\|B_{\partial E}\|^2(X\cdot\nu)^2\Big)\,d\mathcal{H}^{N-1}\\
&+\sigma\int_{\partial E}\int_{\partial E}G(x,y)(X\cdot\nu)(x)(X\cdot\nu)(y)\,d\mathcal{H}^{N-1}_x\,d\mathcal{H}^{N-1}_y\\
&+\sigma\int_{\partial E}(\nabla\mathcal{N}_E\cdot\nu)(X\cdot\nu)^2\,d\mathcal{H}^{N-1}\\
&+\int_{\partial E}\Big(\mathcal{K}(\partial E)-\alpha+\sigma\mathcal{N}_E\Big)(\operatorname{div}X)(X\cdot\nu)\,d\mathcal{H}^{N-1}\\
&-\int_{\partial E}\Big(\mathcal{K}(\partial E)+\sigma\mathcal{N}_E\Big)\operatorname{div}_\tau\big(X_\tau(X\cdot\nu)\big)\,d\mathcal{H}^{N-1}\;.
\end{aligned}$$

Here $\|B_{\partial E}\|^2$ *is the sum of the squares of the principal curvatures of* ∂E*;* G *is the Green's function for the Helmholtz operator in* $\mathbb{T}$ *with periodic boundary conditions;* ν *is the unit outward normal on* ∂E*;* $\mathcal{K}(\partial E)$ *is the sum of principal curvatures of* ∂E*;* ∇_τ *is the gradient on* ∂E*; and* X_τ *is the tangential component of* X*.*

Definition 3.2. *A regular subset* E *of* $\mathbb{T}$ *is a stationary (or critical) point for* (1.7) *if*

$$\mathcal{K}(\partial E)-\alpha+\sigma\mathcal{N}_E=0\quad \text{on } \partial E\;.$$

Remark 3.3. *Since* $\mathcal{N}_E$ *is of class* $W^{2,p}$ *for any* $p>1$*, standard regularity theory and Schauder estimates imply that any regular critical set is of class* $\mathcal{C}^{3,\alpha}(\mathbb{T})$ *for any* $0<\alpha<1$.

We remark that we may add to the last integral in Proposition 3.1 a harmless

$$\int_{\partial E}-\alpha\operatorname{div}_\tau\big(X_\tau(X\cdot\nu)\big)\,d\mathcal{H}^{N-1}$$

(which vanishes by the tangential divergence theorem) so that the last two integrals may be grouped into

$$\int_{\partial E}\Big(\mathcal{K}(\partial E)-\alpha+\sigma\mathcal{N}_E\Big)\cdot(\cdots)\,d\mathcal{H}^{N-1}$$

which vanishes if E was stationary. As all other terms for $d^2J(E)[X]$ only depend on the normal component of X, it is convenient to introduce a function defined on all $\eta\in H^1(\partial E)$ as

$$J''(E)[\eta]=\int_{\partial E}\Big(|\nabla_\tau\eta|^2-\|B_{\partial E}\|^2\eta^2\Big)d\mathcal{H}^{N-1}$$
$$+\sigma\int_{\partial E}\int_{\partial E}G(x,y)\eta(x)\eta(y)\,d\mathcal{H}^{N-1}_x\,d\mathcal{H}^{N-1}_y+\sigma\int_{\partial E}(\nabla\mathcal{N}_E\cdot\nu)\eta^2\,d\mathcal{H}^{N-1}\,.$$

Since $J(E)=J(E+\tau)$ for any translation τ, no set (beside the empty and full states) may be a strict minimum point, so following [3, formula (1.3)] we consider as equivalent any two sets one of which is a translation of the other and define a distance between sets modulo translations as

$$\delta(E,F):=\min_\tau|E\triangle(F+\tau)|\,.$$

Invariance by translation implies that the second derivative of $J(E+t\tau)$ always vanishes. In particular on a critical point E the second variation is zero for every constant vector field $X=e_i$ along the coordinate axes resulting in $\eta=X\cdot\nu=\nu_E^i$ for $i=1,2,\ldots,N$ (the i-th component of the normal ν). There is thus a linear subspace of $H^1(\partial E)$, spanned by the components of the normal, on which $J''(E)$ vanishes. We remark that this subspace can have a dimension less than N, as in the case for lamellar sets. Using $\mathcal{L}\{\ldots\}$ to denote the vector space spanned by the functions inside the brackets and $W^{1,2}_{\mathrm{per}}(\partial E)$ for periodic $W^{1,2}(\partial E)$ functions, we set

$$\mathcal{T}(\partial E)=\mathcal{L}\{\nu_E^1,\ldots,\nu_E^N\}$$
$$\mathcal{T}^\perp(\partial E)=\{\eta\in W^{1,2}_{\mathrm{per}}(\partial E):\textstyle\int_{\partial E}\eta\nu_E^i d\mathcal{H}^{N-1}=0,\ i=1,\ldots,N\}\,.$$

DEFINITION 3.4. *A regular critical point E of J is stable if*

$$J''(E)[\eta]>0\qquad\text{for all }\eta\in\mathcal{T}^\perp(\partial E)\setminus\{0\}\,. \tag{3.1}$$

The notion of stability of a stationary point E is crucial in the applications, since as we will see it implies that E is a strict local minimizer of J, isolated in the δ distance sense (which measures the norm in L^1 modulo translations). In the spirit of [3, Theorem 1.1] we have

THEOREM 3.5. *Let $E\subset\mathbb{T}$ be a regular critical set of J such that*

$$J''(E)[\eta]>0\qquad\textit{for all }\eta\in\mathcal{T}^\perp(\partial E)\setminus\{0\}\,.$$

Then there exist $\varepsilon,C>0$ such that

$$J(F)\ge J(E)+C\delta^2(E,F)$$

for all $F\subset\mathbb{T}$ with $\delta(E,F)<\varepsilon$.

The proof closely follows that of [3, Theorem 1.1], which takes up 25 pages, so we only highlight the relevant differences in the Appendix; the crucial estimate of [3, Lemma 2.6] is replaced by an easier readable version for the Helmholtz operator:

LEMMA 3.6. *If $E, F \subset \mathbb{T}$ are measurable then*

$$\Big|\int_{\mathbb{T}}(|D\mathcal{N}_E|^2+\mathcal{N}_E^2)\,dx-\int_{\mathbb{T}}(|D\mathcal{N}_F|^2+\mathcal{N}_F^2)\,dx\Big|\le 2|E\triangle F|\ .$$

Proof. We write

$$\begin{aligned}
&\int_{\mathbb{T}}(|D\mathcal{N}_E|^2+\mathcal{N}_E^2)\,dx-\int_{\mathbb{T}}(|D\mathcal{N}_F|^2+\mathcal{N}_F^2)\,dx\\
&\qquad=\int_{\mathbb{T}}\big[(D\mathcal{N}_E+D\mathcal{N}_F)(D\mathcal{N}_E-D\mathcal{N}_F)+(\mathcal{N}_E+\mathcal{N}_F)(\mathcal{N}_E-\mathcal{N}_F)\big]\,dx\\
&\qquad=\int_{\mathbb{T}}(\mathcal{N}_E+\mathcal{N}_F)\big[(-\Delta\mathcal{N}_E+\mathcal{N}_E)-(-\Delta\mathcal{N}_F+\mathcal{N}_F)\big]\,dx\\
&\qquad=\int_{\mathbb{T}}(\mathcal{N}_E+\mathcal{N}_F)(\chi_E-\chi_F)\,dx
\end{aligned}$$

by (1.8), and the result follows since $\|\chi_E-\chi_F\|_{L^1}=|E\triangle F|$ and $0\le\mathcal{N}_{E,F}\le 1$. ∎

Stationary points for the area functional have constant mean curvature; they are more or less easily classified. A nonlocal perturbation of the area functional has been studied in the Ohta-Kawasaki model; it gives rise to a series of interesting stationary surfaces (the boundaries of lamellae and, in the Neumann case, also of cylinders, spheres and some 3D-structures called gyroids) which have been proven to be stable under certain assumptions on the parameters. Their shapes are easy to handle, the Laplacian scales well and is well understood, so the proof of their stability requires some effort but is quite general. Equation (1.11), which is another nonlocal perturbation, is less neat, and the only known solution in the periodic setting is given by lamellae [2] (in the entire space there are bubble solutions, see [12, 13]).

We now examine k-lamellar stationary points, in order to establish their stability in certain parameter regimes. The second variation for stationary lamellae $\mathbb{L}$ takes a simplified form and reads

$$\begin{aligned}
J''(\mathbb{L})[\eta]=&\int_{\partial\mathbb{L}}|\nabla\eta|^2\,d\mathcal{H}^{N-1}\\
&+\sigma\int_{\partial\mathbb{L}}\int_{\partial\mathbb{L}}G(x,y)\eta(x)\eta(y)\,d\mathcal{H}^{N-1}_x\,d\mathcal{H}^{N-1}_y\\
&+\sigma\int_{\partial\mathbb{L}}(\nabla\mathcal{N}_{\mathbb{L}}\cdot\nu)\eta^2\,d\mathcal{H}^{N-1}\ .
\end{aligned}\tag{3.2}$$

We recall that by Proposition 2.2 k-lamellae which are stationary points of J are of equal size and spacing, and that the outward normal derivative of the function $\mathcal{N}_{\mathbb{L}}$ takes value $-d_0$ on both sides of each lamella, with d_0 given by (2.4), since the outward normal points backwards on left sides of lamellae.

We now fix some notation, some of which we already employed. As a coordinate system we use $z:=(x,x')\in\mathbb{T}$, where $x\in[0,T]$; we consider a stationary k-lamella $\mathbb{L}$ with all k lamellae having a total width $0<x_0<T$, orthogonal to the x-axis, with the first lamella starting at $x=0$, and we sequentially label ℓ_i, with $i=1,\dots,2k+1$,

the x coordinates of the sides of the lamellae (the last is a duplicate of the first side, but is included for convenience), so that

$$\ell_1 = 0, \quad \ell_2 = \frac{x_0}{k}, \quad \ell_3 = \frac{T}{k}, \quad \ell_4 = \frac{T}{k} + \frac{x_0}{k}, \quad \ell_5 = 2\frac{T}{k}, \tag{3.3}$$

$$\dots \quad \ell_{2k} = (k-1)\frac{T}{k} + \frac{x_0}{k} = T - \frac{T - x_0}{k}\ , \quad \ell_{2k+1} = T\ .$$

We also name the corresponding faces, which are $(N-1)$-dimensional squares orthogonal to the x axis, as $L_1, \dots, L_{2k+1}$. We easily identify the space $\mathcal{T}^\perp(\partial\mathbb{L})$: since the only non-zero component of the outward normal field to $\mathbb{L}$ is the first one, and it takes value -1 on odd sides (i.e., on L_i with i odd) and $+1$ on even sides, a periodic function $\eta \in W^{1,2}_{\mathrm{per}}(\partial\mathbb{L})$ belongs to $\mathcal{T}^\perp$ if

$$\sum_{j=1}^{k} \int_{L_{2j}} \eta \, d\mathcal{H}^{N-1} - \sum_{j=1}^{k} \int_{L_{2j-1}} \eta \, d\mathcal{H}^{N-1} = 0\ .$$

Following a reduction method introduced in [27, section 4], for any $\eta \in W^{1,2}_{\mathrm{per}}(\partial\mathbb{L})$ we call η_i the function which coincides with η on L_i and vanishes on all other L_j and we further split η_i as its mean value μ_i on L_i plus a zero-average term ζ_i:

$$\mu_i = \frac{1}{T^{N-1}} \int_{L_i} \eta_i(z) \, d\mathcal{H}^{N-1}\ , \qquad \zeta_i(z) = \eta_i(z) - \mu_i\ ,$$

so in particular $\int_{L_i} \zeta_i \, d\mathcal{H}^{N-1} = 0$. We remark that

$$\eta \in \mathcal{T}^\perp(\partial\mathbb{L}) \iff \sum_{j=1}^{k} \mu_{2j} - \sum_{j=1}^{k} \mu_{2j-1} = 0 \tag{3.4}$$

which is independent of ζ. For subsequent use we denote $\mu := \sum_{j=1}^{2k} \mu_j$ and $\zeta := \sum_{j=1}^{2k} \zeta_j$ so that $\eta = \mu + \zeta$. We now examine the various components of $J''(\mathbb{L})$; for the first we immediately have

$$\int_{\partial\mathbb{L}} |\nabla\eta|^2 \, d\mathcal{H}^{N-1} = \sum_{i=1}^{2k} \int_{L_i} |\nabla\zeta_i|^2 \, d\mathcal{H}^{N-1}\ . \tag{3.5}$$

We have for all i

$$\int_{L_i} |\eta^2| \, d\mathcal{H}^{N-1} = \int_{L_i} |\eta_i^2| \, d\mathcal{H}^{N-1} = T^{N-1}\mu_i^2 + \int_{L_i} |\zeta_i^2| \, d\mathcal{H}^{N-1} + 2\mu_i \int_{L_i} \zeta_i \, d\mathcal{H}^{N-1}$$

$$= T^{N-1}\mu_i^2 + \int_{L_i} |\zeta_i^2| \, d\mathcal{H}^{N-1}\ .$$

At the same time $\nabla\mathcal{N}_{\mathbb{L}} \cdot \nu = -d_0$ at all L_i so that the last term in (3.2) becomes

$$-\sigma d_0 \int_{\partial\mathbb{L}} \eta^2 \, d\mathcal{H}^{N-1} = -\sigma d_0 T^{N-1} \sum_{i=1}^{2k} \mu_i^2 - \sigma d_0 \sum_{i=1}^{2k} \int_{L_i} |\zeta_i^2| \, d\mathcal{H}^{N-1}\ . \tag{3.6}$$

Next comes the Green's function term which, upon setting aside the factor σ, we copy as

$$\int_{\partial\mathbb{L}} \int_{\partial\mathbb{L}} G(X,Y)\eta(X)\eta(Y) \, d\mathcal{H}_X^{N-1} \, d\mathcal{H}_Y^{N-1}$$

$$= \int_{\partial\mathbb{L}} \int_{\partial\mathbb{L}} G(X,Y)\big((\mu+\zeta)(X)\big)\big((\mu+\zeta)(Y)\big) \, d\mathcal{H}_X^{N-1} \, d\mathcal{H}_Y^{N-1}$$

where $X = (x, x')$ and $Y = (y, y')$. We define two (arrays of) measures on $\mathbb{T}$ and one on $[0, T]$ as

$$M^i = \mu_i \mathcal{H}^{N-1} \llcorner L_i\ , \qquad Z^i = \zeta_i \mathcal{H}^{N-1} \llcorner L_i\ , \qquad m^i = \mu_i \delta_{\ell_i}\ , \tag{3.7}$$

and we solve Helmholtz equation (twice in $\mathbb{T}$ and once in $[0, T]$), thus defining V_M^i, V_Z^i and v_m^i as the weak solutions of

$$-\Delta V_M^i + V_M^i = M^i\ , \qquad -\Delta V_Z^i + V_Z^i = Z^i\ , \qquad -(v_m^i)'' + v_m^i = m^i \tag{3.8}$$

with periodic boundary conditions. Notice that if we extend each of the functions $v_m^i(x)$ to $\mathbb{T}$ as $\tilde{v}_m^i(x, x') = v_m^i(x)$, then $\tilde{v}_m^i$ is $\mathbb{T}$-periodic and satisfies the same Helmholtz equation as V_M^i, thus it coincides with V_M^i, which means that each V_M^i only depends on x but not on x'. In particular this implies

$$\int_{L_i} G(X, Y)\, d\mathcal{H}_Y^{N-1} = G_{1D}(x, \ell_i) \tag{3.9}$$

where $G_{1D} : [0, T] \times [0, T] \to \mathbb{R}$ is the Green's function of $-\frac{d^2}{dx^2} + 1$ in 1D with periodic boundary condition on $[0, T]$. For latter purpose we explicitly compute it: to begin with, if $\mathcal{G}(x)$ is the $[0, T]$-periodic solution of

$$-\mathcal{G}'' + \mathcal{G} = \delta_0\ ,$$

a direct computation yields

$$\mathcal{G}(x) = \frac{1}{2\sinh(T/2)} \cosh\Big(x - \frac{T}{2}\Big) \qquad \text{in } [0, T]\ , \tag{3.10}$$

and we view it as periodically repeated on $\mathbb{R}$. It is readily checked that $G_{1D}(x, y) = \mathcal{G}(|x - y|_T)$ where $|x - y|_T \le T/2$ represents the closest distance of $x, y \in [0, T]$ in the torus, i.e. $|x - y|_T = \min_{m \in \mathbb{Z}} |x + mT - y|$. Other easy properties are

$$\mathcal{G}(x) = \mathcal{G}(|x|) = \mathcal{G}(x + T) = \mathcal{G}(T - x)$$

and from these we deduce

$$\begin{aligned} &0 \le x \le y \le T \Rightarrow G_{1D}(x, y) = \frac{1}{2\sinh(T/2)} \cosh\Big(y - x - \frac{T}{2}\Big) \\ &0 \le y < x \le T \Rightarrow G_{1D}(x, y) = \frac{1}{2\sinh(T/2)} \cosh\Big((y + T) - x - \frac{T}{2}\Big)\ , \end{aligned} \tag{3.11}$$

which will be useful since in general $x, y \in [0, T]$.

By linearity when setting

$$V_M = \sum_{i=1}^{2k} V_M^i\ , \qquad V_Z = \sum_{i=1}^{2k} V_Z^i\ , \qquad v_m = \sum_{i=1}^{2k} v_m^i\ ,$$

these functions solve with periodic boundary conditions the Helmholtz equations

$$-\Delta V_M + V_M = M\ , \qquad -\Delta V_Z + V_Z = Z\ , \qquad -v_m'' + v_m = m\ ,$$

and V_M only depends on x. Now

$$\begin{aligned}
&\int_{\partial\mathbb{L}}\int_{\partial\mathbb{L}} G(X,Y)\mu(X)\zeta(Y)\,d\mathcal{H}^{N-1}_X\,d\mathcal{H}^{N-1}_Y\\
&\qquad=\int_{\mathbb{T}}\int_{\mathbb{T}} G(X,Y)\,dM(X)\,dZ(Y)=\int_{\mathbb{T}} v_m(y)\,dZ(y,y')\\
&\qquad=\sum_{i=1}^{2k} v_m(\ell_i)\int_{L_i}\zeta_i(\ell_i,y')\,d\mathcal{H}^{N-1}_{y'}=0
\end{aligned}$$

so using (3.9)

$$\begin{aligned}
&\int_{\partial\mathbb{L}}\int_{\partial\mathbb{L}} G(X,Y)\eta(X)\eta(Y)\,d\mathcal{H}^{N-1}_X\,d\mathcal{H}^{N-1}_Y\\
&\quad=\int_{\partial\mathbb{L}}\int_{\partial\mathbb{L}} G(X,Y)\mu(X)\mu(Y)\,d\mathcal{H}^{N-1}_X\,d\mathcal{H}^{N-1}_Y\\
&\qquad+\int_{\partial\mathbb{L}}\int_{\partial\mathbb{L}} G(X,Y)\zeta(X)\zeta(Y)\,d\mathcal{H}^{N-1}_X\,d\mathcal{H}^{N-1}_Y\\
&\quad=\sum_{i,j=1}^{2k}\mu_i\mu_j\int_{\partial L_i}\int_{\partial L_j} G(X,Y)\,d\mathcal{H}^{N-1}_X\,d\mathcal{H}^{N-1}_Y\\
&\qquad+\int_{\partial\mathbb{L}}\int_{\partial\mathbb{L}} G(X,Y)\zeta(X)\zeta(Y)\,d\mathcal{H}^{N-1}_X\,d\mathcal{H}^{N-1}_Y\\
&\quad=\sum_{i,j=1}^{2k}\mu_i\mu_j\int_{\partial L_i} G_{1D}(x,\ell_j)\,d\mathcal{H}^{N-1}_X\ +\int_{\mathbb{T}}\big(|\nabla V_Z|^2+|V_Z|^2\big)\,dX\\
&\quad=T^{N-1}\sum_{i,j=1}^{2k}\mu_i\mu_j\mathcal{G}(|\ell_i-\ell_j|_T)+\int_{\mathbb{T}}\big(|\nabla V_Z|^2+|V_Z|^2\big)\,dX\ .
\end{aligned}$$

This equality, together with (3.5) and (3.6), may be put into the expression (3.2) for J'', thus obtaining for any stationary lamella

$$J''(\mathbb{L})[\mu+\zeta]$$

$$=\sigma T^{N-1}\Big(\sum_{i,j=1}^{2k}\mu_i\mu_j\mathcal{G}(|\ell_i-\ell_j|_T)-d_0\sum_{i=1}^{2k}\mu_i^2\Big) \tag{3.12}$$

$$+\sum_{i=1}^{2k}\Big(\int_{L_i}\big(|\nabla\zeta_i|^2-\sigma d_0|\zeta_i^2|\big)\,d\mathcal{H}^{N-1}\Big)+\sigma\int_{\mathbb{T}}\big(|\nabla V_Z|^2+|V_Z|^2\big)\,dX\ . \tag{3.13}$$

As a reminder, we impose only translation-free perturbation $\eta=\mu+\zeta$ for stability consideration; this amounts to requiring that $\mu\in\mathbb{R}^{2k}$ satisfies (3.4). Remark that the two lines on the right hand side of the above equation are entirely independent: then it is easy to see that a necessary and sufficient condition for a stationary k-lamella to be stable is to establish that the first line (3.12) on the right hand side is positive for all $\mu\in\mathbb{R}^{2k}\backslash\{0\}$ satisfying (3.4), for ζ may well be zero; and that the second line (3.13) is positive for all not identically vanishing ζ such that each ζ_i is periodic and with zero average on L_i, because positivity of J'' must be attained also at $0\mu+\zeta$.

4. Stability, mean value part. We study the mean part μ in (3.12). To prove positivity of (3.12) it suffices to show

$$\sum_{i,j=1}^{2k} \mu_i \mu_j \mathcal{G}(|\ell_j - \ell_i|_T) - d_0 \sum_{i=1}^{2k} \mu_i^2 > 0$$

for all non-zero $\mu \in \mathbb{R}^{2k}$ satisfying (3.4). Defining the (symmetric) matrix

$$\mathcal{A}_{i,j} = \mathcal{G}(|\ell_j - \ell_i|_T) \tag{4.1}$$

and considering the vector in $\mathbb{R}^{2k}$

$$E = (-1, 1, -1, 1, \cdots)$$

(so that (3.4) reads $\mu \cdot E = 0$) the above may be rewritten as

$$\langle (\mathcal{A} - d_0 \mathcal{I})\mu, \mu \rangle > 0 \qquad \text{for all } \mu \perp E\ ,\ \mu \neq 0 \tag{4.2}$$

where $\mathcal{I}$ is the identity matrix. We prove in this section the following

THEOREM 4.1. *The matrix $\mathcal{A}$ has one simple eigenvalue d_0, corresponding to the eigenvector E, and all other eigenvalues are strictly larger than d_0. In particular* (4.2) *holds, so* (3.12) *is positive for all $\mu \in \mathbb{R}^{2k}$ satisfying* (3.4).

We highlight some properties of $\mathcal{A}$. The matrix $\mathcal{A}$ is symmetric because $\mathcal{G}$ is even. Next, since the distance from L_i to L_j is the same as the distance of the sides we get by shifting both in the same direction by T/k, i.e. $|\ell_i - \ell_j|_T = |\ell_{i+2} - \ell_{j+2}|_T$, we have

$$\mathcal{A}_{i+2,j+2} = \mathcal{A}_{i,j}\ ,$$

thus all entries in $\mathcal{A}$ repeat themselves if we shift (modulo $2k$) by 2 columns right and 2 rows down. It is convenient to think of $\mathcal{A}$ as made of 2×2 blocks $B_0, B_1, \ldots, B_{k-1}$ for a k-lamella: the structure of $\mathcal{A}$ is then

$$\mathcal{A} = \begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_{k-1} \\ B_{k-1} & B_0 & B_1 & \cdots & B_{k-2} \\ B_{k-2} & B_{k-1} & B_0 & \cdots & B_{k-3} \\ & \vdots & & \ddots & \vdots \\ B_1 & B_2 & B_3 & \cdots & B_0 \end{pmatrix}.$$

Due to symmetry of $\mathcal{A}$, we have $B_j = B_{k-j}^T$ for $j = 0, 1, 2, \ldots, k-1$. This means that $\mathcal{A}$ is a *block circulant symmetric matrix*, which has interesting properties regarding its eigenvalues: let all k distinct complex roots of the unity be denoted by

$$\rho_p = e^{i\phi_p}\ , \quad \phi_p = p\frac{2\pi}{k}\ , \qquad p = 0, \ldots, k-1\ . \tag{4.3}$$

With $p = 0, \ldots, k-1$ define the 2×2 matrices

$$H_p = B_0 \rho_p^0 + B_1 \rho_p^1 + \cdots + B_{k-1} \rho_p^{k-1}\ : \tag{4.4}$$

each has two (if we count multiplicity) eigenvalues λ_p', λ_p'', and we have, see [37, Section 3.1]:

PROPOSITION 4.2. *The eigenvalues of $\mathcal{A}$ are all the numbers λ'_p and λ''_p for $p = 0, 1, \ldots, k-1$.*

Now recall $B_{k-j} = B_j^T$ for $j = 0, 1, 2, \ldots, k-1$. In particular B_0 is symmetric and, if k is even, also the middle one $B_{k/2}$ is symmetric. We remark that ρ_p^j is the conjugate of ρ_p^{k-j}. Therefore in the sum (4.4) we may group terms $\rho^j B_j$ in pairs, excluding the first one and also $B_{k/2}$ if k was even, to get

$$\rho_p^j B_j + \rho_p^{k-j} B_{k-j} = \rho_p^j B_j + \bar{\rho}_p^j B_j^T$$

for $j = 1, 2, \ldots, k/2 - 1$ when k is even or for $j = 1, 2, \ldots, (k-1)/2$ when j is odd. Each pair forms a Hermitian matrix, thus H_p in (4.4) is a Hermitian matrix since the first term B_0 is real symmetric and so is the middle term $(-1)^p B_{k/2}$ for even k.

Finally we remark that for every p, the entry $[B_p]_{1,1}$ comes from evaluating $\tilde{v}$ with an input equal to the distance between the left sides of some two lamellae, and $[B_p]_{2,2}$ relates to the distance between the right sides of the same lamellae. Since these two distances are the same, the diagonal elements in each matrix on the right hand side in (4.4) equal one another, thus the same is true for H_p. We combine the above facts to obtain that each matrix H_p has the form

$$H_p = \begin{pmatrix} a_p & b_p \\ \bar{b}_p & a_p \end{pmatrix} \tag{4.5}$$

for some a_p, b_p. As H_p is Hermitian, a_p has to be real. Its eigenvalues are

$$\lambda'_p = a_p - |b_p| \,, \qquad \lambda''_p = a_p + |b_p| \,.$$

Since $\lambda''_p \geq \lambda'_p$, to prove Theorem 4.1, in view of Proposition 4.2 we will show that

PROPOSITION 4.3. *The number b_0 is not zero. Moreover $\lambda'_0 = d_0$ and $\lambda'_p > d_0$ for all $p > 0$.*

Proof. In the course of the proof we will also see that E is the eigenvector corresponding to d_0. We are about to compute a_p and b_p. Only the first row of the matrix $\mathcal{A}$ needs to be considered in computing H_p. We write it in full using (3.11),(4.1): since $\ell_1 = 0$, the odd elements are for $p = 0, \ldots, k-1$

$$a_{1,2p+1} = \frac{1}{2\sinh(T/2)} \cosh\Big(\frac{T}{2} - p\frac{T}{k}\Big) \,, \tag{4.6}$$

(so for e.g. $p = 3$ we get $[B_3]_{1,1}$) whereas the even elements are

$$a_{1,2p+2} = \frac{1}{2\sinh(T/2)} \cosh\Big(\frac{T}{2} - \frac{x_0}{k} - p\frac{T}{k}\Big) \,. \tag{4.7}$$

To proceed further we first establish the following lemma.

LEMMA 4.4. *If $e^{i\phi}$ is any k-th root of 1 and $\delta \in \mathbb{R}$ then*

$$\sum_{n=0}^{k-1} e^{in\phi} \cosh\Big(\frac{T}{2} + \delta - n\frac{T}{k}\Big) = \frac{\sinh(\delta + T/k) - e^{-i\phi}\sinh\delta}{\cosh(T/k) - \cos\phi} \sinh\frac{T}{2} \,.$$

Proof. We expand the hyperbolic cosine so that

$$e^{in\phi} \cosh\Big(\frac{T}{2} + \delta - n\frac{T}{k}\Big) = \frac{1}{2}\big(e^{\delta + T/2}\, e^{n(i\phi - T/k)} + e^{-\delta - T/2}\, e^{n(i\phi + T/k)}\big)$$

and thus (recalling in the second equality below that $k\phi$ is a multiple of 2π)

$$\begin{aligned}
\sum_{n=0}^{k-1} e^{in\phi}\cosh\Big(\frac{T}{2}+\delta-n\frac{T}{k}\Big) &= \frac{1}{2}e^{\delta+T/2}\,\frac{1-e^{-T+ik\phi}}{1-e^{i\phi-T/k}}+\frac{1}{2}e^{-\delta-T/2}\,\frac{1-e^{T+ik\phi}}{1-e^{i\phi+T/k}}\\
&= \frac{1}{2}e^{\delta}\,\frac{e^{T/2}-e^{-T/2}}{1-e^{i\phi-T/k}}-\frac{1}{2}e^{-\delta}\,\frac{e^{T/2}-e^{-T/2}}{1-e^{i\phi+T/k}}\\
&= \sinh\frac{T}{2}\cdot\Big(\frac{e^{\delta}}{1-e^{i\phi-T/k}}-\frac{e^{-\delta}}{1-e^{i\phi+T/k}}\Big)\\
&= \sinh\frac{T}{2}\cdot\frac{e^{\delta}-e^{-\delta}-e^{\delta+i\phi+T/k}+e^{-\delta+i\phi-T/k}}{(1-e^{i\phi-T/k})(1-e^{i\phi+T/k})}\,.
\end{aligned}$$

But

$$\begin{aligned}
1-e^{i\phi-T/k} &= \big(e^{(T/2k)-i(\phi/2)}-e^{-(T/2k)+i(\phi/2)}\big)e^{-T/2k}e^{i\phi/2}\\
1-e^{i\phi+T/k} &= \big(e^{-(T/2k)-i(\phi/2)}-e^{(T/2k)+i(\phi/2)}\big)e^{T/2k}e^{i\phi/2}
\end{aligned}$$

so using hyperbolic function identities

$$\begin{aligned}
(1-e^{i\phi-T/k})(1-e^{i\phi+T/k}) &= -4e^{i\phi}\sinh\Big(\frac{T}{2k}-i\frac{\phi}{2}\Big)\sinh\Big(\frac{T}{2k}+i\frac{\phi}{2}\Big)\\
&= -2e^{i\phi}\Big(\cosh\frac{T}{k}-\cosh i\phi\Big) = -2e^{i\phi}\Big(\cosh\frac{T}{k}-\cos\phi\Big)
\end{aligned}$$

since $\cos z = \cosh(iz)$, as well as $i\sin z = \sinh(iz)$. We may thus resume by writing

$$\begin{aligned}
\sum_{n=0}^{k-1} e^{in\phi}\cosh\Big(\frac{T}{2}+\delta-n\frac{T}{k}\Big) &= -\sinh\frac{T}{2}\cdot\frac{e^{\delta}-e^{-\delta}-e^{\delta+i\phi+T/k}+e^{-\delta+i\phi-T/k}}{2e^{i\phi}\big(\cosh(T/k)-\cos\phi\big)}\\
&= \sinh\frac{T}{2}\cdot\frac{-e^{-i\phi}\sinh\delta+\sinh(\delta+T/k)}{\cosh(T/k)-\cos\phi}
\end{aligned}$$

which concludes the proof. ∎

Returning now to the proof of Proposition 4.3, we apply this formula to compute the coefficients in the matrices H_p: let $\rho_p = e^{i\phi_p}$, recall (3.10),(4.1),(4.5),(4.6) and we have

$$a_p = [H_p]_{1,1} = \sum_{n=0}^{k-1}\rho_p^n\,\frac{1}{2\sinh(T/2)}\cosh\Big(\frac{T}{2}-n\frac{T}{k}\Big)\,. \tag{4.8}$$

Analogously

$$b_p = [H_p]_{1,2} = \sum_{n=0}^{k-1}\rho_p^n\,\frac{1}{2\sinh(T/2)}\cosh\Big(\frac{T}{2}-\frac{x_0}{k}-n\frac{T}{k}\Big)\,. \tag{4.9}$$

Lemma 4.4 then implies

$$a_p = \frac{\sinh(T/k)}{2\big(\cosh(T/k)-\cos\phi_p\big)}\,,\qquad b_p = \frac{\sinh(T/k-x_0/k)+e^{-i\phi_p}\sinh(x_0/k)}{2\big(\cosh(T/k)-\cos\phi_p\big)}\,.$$

We are now ready to conclude the proof of Proposition 4.3:
Case $p = 0$: we first consider the case $p = 0$ in (4.3). This gives

$$a_0 = \frac{\sinh(T/k)}{2\big(\cosh(T/k) - 1\big)}\ , \qquad b_0 = \frac{\sinh(T/k - x_0/k) + \sinh(x_0/k)}{2\big(\cosh(T/k) - 1\big)}\ ,$$

the number b_0 is strictly positive so the two eigenvalues of H_0 are distinct, and the lower one is $\lambda'_0 = a_0 - b_0$. We now

$$\text{claim:} \qquad a_0 - b_0 = d_0\ , \tag{4.10}$$

thus d_0 will be a simple eigenvalue of H_0 and therefore also an eigenvalue of $\mathcal{A}$. We remark that a_0 is the sum of the odd elements in the first row of $\mathcal{A}$ and b_0 is the sum of even elements, so our claim, when proved, will show that their difference is d_0.

Assume the validity of the claim for the time being; by the symmetry of all matrices $B_j,\ j = 0, 1, \ldots, k-1$, for $p = 0$, the second row of $\mathcal{A} - d_0\mathcal{I}$ has the same entries as the first, only interchanging the pair of consecutive odd and even places starting from the first entry; thus the difference between the sum-of-odd and the sum-of-even entries of the second row is also zero. These facts may be rewritten as: the first two entries of $(\mathcal{A} - d_0\mathcal{I})E$ are zero. But as all subsequent rows of $\mathcal{A} - d_0\mathcal{I}$ are just shifted copies of the first two, we get

$$(\mathcal{A} - d_0\mathcal{I})E = 0\ ,$$

so E will be an eigenvector corresponding to the eigenvalue d_0. All we have to do is to prove our claim which we rewrite as

$$a_0 - b_0 = d_0 \qquad \Leftrightarrow \qquad \sinh\frac{T}{k} - \sinh\frac{x_0}{k} - \sinh\frac{T - x_0}{k} = 2d_0\big(\cosh(T/k) - 1\big)\ . \tag{4.11}$$

We now make extensive use of identities associated with hyperbolic functions [1, Chapter 4, section 5] in this paper without further reference. At the left hand side

$$\sinh\frac{T}{k} - \sinh\frac{x_0}{k} = 2\cosh\frac{T + x_0}{2k}\sinh\frac{T - x_0}{2k}$$

and observe

$$\sinh\frac{T - x_0}{k} = 2\sinh\frac{T - x_0}{2k}\cosh\frac{T - x_0}{2k}\ ,$$

so the left hand side of (4.11) is equal to

$$2\sinh\frac{T - x_0}{2k}\Big(\cosh\frac{T + x_0}{2k} - \cosh\frac{T - x_0}{2k}\Big) = 4\sinh\frac{T - x_0}{2k}\sinh\frac{T}{2k}\sinh\frac{x_0}{2k}\ .$$

On the other hand, using the expression (2.4) of d_0 and applying hyperbolic function identity to $\big[\cosh(T/k) - \cosh 0\big]$ the right hand side of (4.11) is equal to

$$2\frac{\sinh(x_0/2k)\sinh\big((T - x_0)/2k\big)}{\sinh(T/2k)} \cdot 2\sinh^2\frac{T}{2k} = 4\sinh\frac{T - x_0}{2k}\sinh\frac{T}{2k}\sinh\frac{x_0}{2k}$$

and claim (4.10) is proved.

Case $p \neq 0$: the eigenvalues of H_p are now $a_p \pm |b_p|$, and we

$$\text{claim:} \qquad \lambda'_p = a_p - |b_p| > d_0\ , \tag{4.12}$$

which would conclude the proof of Proposition 4.3 and therefore also of Theorem 4.1. We write the inequality as

$$\sinh\frac{T}{k} - \sqrt{\Big(\sinh\frac{T-x_0}{k} + \sinh\frac{x_0}{k}\cos\phi_p\Big)^2 + \Big(\sinh\frac{x_0}{k}\sin\phi_p\Big)^2} > 2d_0\Big(\cosh\frac{T}{k} - \cos\phi_p\Big)\ .$$

We make use of what we proved in the case $p = 0$ by subtracting

$$\sinh\frac{T}{k} - \sinh\frac{x_0}{k} - \sinh\frac{T-x_0}{k}$$

from the left hand side and $2d_0\big(\cosh(T/k) - 1\big)$, which is the same by (4.11), from the right hand side. The claim now reads

$$\sinh\frac{T-x_0}{k} + \sinh\frac{x_0}{k} - \sqrt{\Big(\sinh\frac{T-x_0}{k} + \sinh\frac{x_0}{k}\cos\phi_p\Big)^2 + \Big(\sinh\frac{x_0}{k}\sin\phi_p\Big)^2} > 2d_0(1 - \cos\phi_p)\ . \tag{4.13}$$

We rewrite the argument of the square root, which is

$$\begin{aligned}
&\sinh^2\frac{T-x_0}{k} + \sinh^2\frac{x_0}{k}\cos^2\phi_p + 2\sinh\frac{T-x_0}{k}\sinh\frac{x_0}{k}\cos\phi_p + \sinh^2\frac{x_0}{k}\sin^2\phi_p\\
&\quad = \sinh^2\frac{T-x_0}{k} + \sinh^2\frac{x_0}{k} + 2\sinh\frac{T-x_0}{k}\sinh\frac{x_0}{k}\cos\phi_p\\
&\quad = \Big(\sinh\frac{T-x_0}{k} + \sinh\frac{x_0}{k}\Big)^2 - 2\sinh\frac{T-x_0}{k}\sinh\frac{x_0}{k}\big(1 - \cos\phi_p\big)\ .
\end{aligned}$$

Now (4.13) may be rewritten

$$a - \sqrt{a^2 - 2bt} > 2d_0 t$$

where we have put

$$a = \sinh\frac{T-x_0}{k} + \sinh\frac{x_0}{k}\ , \qquad b = \sinh\frac{T-x_0}{k}\sinh\frac{x_0}{k}\ , \qquad t = 1 - \cos\phi_p\ .$$

We remark that $a, b, d_0 > 0$ and that $0 < t \leq 2$ because $\cos\phi_p$ is not equal to 1 in the case $p \neq 0$. Set

$$f(t) = a - \sqrt{a^2 - 2bt} - 2d_0 t$$

so that $f(0) = 0$; all we have to prove is that $f(t) > 0$ for $0 < t \leq 2$. We first remark that

$$a^2 - 2bt \geq a^2 - 4b = \Big(\sinh\frac{T-x_0}{k} - \sinh\frac{x_0}{k}\Big)^2 \geq 0$$

and we note that for $0 \leq t < 2$

$$f'(t) = \frac{b}{\sqrt{a^2 - 2bt}} - 2d_0$$

which is a strictly increasing function of t, so f is strictly convex in $[0, 2]$. We now prove that $f'(0) \geq 0$: we have $f'(0) = (b/a) - 2d_0$ so we have to prove that $b/a \geq 2d_0$.

We use hyperbolic function identities at both the numerator and the denominator to write

$$\begin{aligned}\frac{b}{a} &= \frac{\sinh\big((T-x_0)/k\big)\sinh(x_0/k)}{\sinh\big((T-x_0)/k\big)+\sinh(x_0/k)}\\ &= \frac{4\sinh\big((T-x_0)/2k\big)\sinh(x_0/2k)\cosh\big((T-x_0)/2k\big)\cosh(x_0/2k)}{2\sinh(T/2k)\cosh\big((T-2x_0)/2k\big)}\\ &= 2d_0\frac{\cosh\big((T-x_0)/2k\big)\cosh(x_0/2k)}{\cosh\big((T-2x_0)/2k\big)}\ ,\end{aligned}$$

so $b/a \geq 2d_0$ provided

$$\frac{\cosh\big((T-x_0)/2k\big)\cosh(x_0/2k)}{\cosh\big((T-2x_0)/2k\big)} \geq 1 \quad\Leftrightarrow\quad \cosh\frac{T-x_0}{2k}\cosh\frac{x_0}{2k} \geq \cosh\frac{T-2x_0}{2k}\ . \tag{4.14}$$

By hyperbolic function identity

$$\cosh\frac{T-x_0}{2k}\cosh\frac{x_0}{2k} = \frac{1}{2}\Big(\cosh\frac{T-2x_0}{2k}+\cosh\frac{T}{2k}\Big)$$

so (4.14) becomes

$$\cosh\frac{T}{2k} \geq \cosh\frac{T-2x_0}{2k} = \cosh\frac{|T-2x_0|}{2k}\ ,$$

which is true because from $0 \leq x_0 \leq T$ we deduce that $|T-2x_0| \leq T$. This concludes the proof that $f'(0) \geq 0$, consequently the convex function f is strictly increasing in $[0,2]$. As $f(0) = 0$ this impies that $f(t) > 0$ for $t > 0$, as desired, and the proof of (4.12) is concluded, thus ending the proof of of Proposition 4.3, and also of Theorem 4.1. ∎

Global minimizers in 1D (which may be the empty set, the full torus or the minimal lamella) are stable when subjecting to 1D perturbation. The above Theorem 4.1 yields a related strong result.

Corollary 4.5. *All stationary periodic lamellae are stable with respect to 1D periodic perturbations.*

For use in the next section, we need an important

Remark 4.6. *Throughout this section we did not use the explicit value* (2.5) *of* x_0 *for minimal lamellae, but only the fact that* $0 \leq x_0 \leq T$ *and the expression* (2.4) *of* d_0 *in terms of the numbers* T *and* x_0*, so in particular Propositions 4.2 and 4.3 hold for any numbers* $0 \leq x_0 \leq T$ *and* d_0 *linked by* (2.4)*, provided the coefficients of the matrix* $\mathcal{A}$ *are defined through* (4.1) *and* (3.10).

5. Stability, zero-average part and conclusion. To conclude the stability analysis for stationary lamellar configurations we have to prove that the sum of the two terms appearing in (3.13) is non-negative for periodic functions defined on all sides of the lamellae, with zero average on each side. We begin with a general (easy) result, then we specialize to a k-lamella in dimension 2, to get some results which to our knowledge are in an entirely new spirit.

Let $C_{P,N-1}$ denote the Poincaré constant in the unit torus $\mathbb{T}_1$ of $\mathbb{R}^{N-1}$ with periodic boundary conditions (and zero mean), i.e.,

$$\int_{\mathbb{T}_1} |\nabla\zeta|^2 \, d\mathcal{H}^{N-1} \geq C_{P,N-1} \int_{\mathbb{T}_1} \zeta^2 \, d\mathcal{H}^{N-1} \qquad \forall \zeta \in H^1_{\mathrm{per}}(\mathbb{T}_1) \text{ s.t. } \int_{\mathbb{T}_1} \zeta \, d\mathcal{H}^{N-1} = 0 \, ;$$

then

$$\int_{L_i} |\nabla\zeta_i|^2 \, d\mathcal{H}^{N-1} - \sigma d_0 \int_{L_i} |\zeta_i^2| \, d\mathcal{H}^{N-1} \geq \Big(\frac{C_{P,N-1}}{T^2} - \sigma d_0 \Big) \int_{L_i} |\zeta_i^2| \, d\mathcal{H}^{N-1} \, .$$

THEOREM 5.1. *Let $\mathbb{L}$ be a stationary k-lamella, and assume*

$$\frac{C_{P,N-1}}{T^2} - \sigma d_0 > 0 \, . \tag{5.1}$$

Then $\mathbb{L}$ is stable in the sense of (3.1).

The proof is just a check: the first part of (3.13) is non-negative due to assumption (5.1), whereas the last part of (3.13), which contains the contribution of Green's function term, is obviously non-negative.

REMARK 5.2. *If the original torus $\mathbb{T}$ was not a cube but had length T in the x direction and sides of length T' in the orthogonal direction, the factor T^2 appearing in* (5.1) *should be $(T')^2$ instead. Thus, the smaller is T' the easier it is to obtain stability, as e.g. in [27].*

We now focus only on a stationary k-lamella in a two dimensional torus $\mathbb{T}$, so that $N = 2$, and let $X = (x, x') \in \mathbb{T}$. First we recall

$$J''(\mathbb{L})[\zeta] = \sum_{i=1}^{2k} \Big(\int_{L_i} |\zeta_i'(x')|^2 \, dx' - \sigma d_0 \int_{L_i} |\zeta_i(x')|^2 \, dx' \Big) + \sigma \int_{\mathbb{T}} \big(|\nabla V_Z|^2 + |V_Z|^2 \big) \, dX \tag{5.2}$$

on zero-average functions ζ. For $r = 1, 2, \ldots$, define $\rho_{2r-1} = \rho_{2r} := (\frac{2\pi r}{T})^2$ and

$$\varphi_{2r-1}(x') := \sin \frac{2\pi r x'}{T} \, , \qquad \varphi_{2r}(x') := \cos \frac{2\pi r x'}{T} \, .$$

The eigenvalues for the operator $-d^2/dx'^2$ for zero-average functions with periodic boundary condition on each L_i are then the numbers ρ_m with corresponding eigenfunctions φ_m, for $m = 1, 2, \ldots$. Moreover

$$\int_{L_i} \varphi_m(z) \varphi_r(z) \, dz = \begin{cases} 0, & \text{if } m \neq r, \\ T/2, & \text{if } m = r, \end{cases} \tag{5.3}$$

$$\int_{L_i} \varphi_m'(z) \varphi_r'(z) \, dz = \begin{cases} 0, & \text{if } m \neq r, \\ \rho_m T/2, & \text{if } m = r. \end{cases} \tag{5.4}$$

We keep the notation in Section 3, and in particular we label ℓ_i the x coordinates of the sides of lamellae as in (3.3) where x_0/k is the thickness of each lamella. Suppose $\zeta_i(x') = \sum_m \alpha_m^i \varphi_m(x')$, where henceforth all sums run for $m \geq 1$ unless otherwise noted; then

$$\sum_{i=1}^{2k} \int_{L_i} |\zeta_i|^2 \, dx' = \frac{T}{2} \sum_{i=1}^{2k} \sum_m (\alpha_m^i)^2 \, , \qquad \sum_{i=1}^{2k} \int_{L_i} |\zeta_i'|^2 \, dx' = \frac{T}{2} \sum_{i=1}^{2k} \sum_m \rho_m (\alpha_m^i)^2 \, . \tag{5.5}$$

With slight abuse, regard $\zeta_i(x,x') = \sum_m \alpha_m^i \varphi_m(x')\,dx' \llcorner L_i$ as a measure in the equation $-\Delta V_i + V_i = \zeta_i$ on the torus $\mathbb{T}$, analogously to what we did in (3.7); it is easily verified that $V_i(x,x') = \sum_m u_m^i(x)\varphi_m(x')$ is the unique solution provided u_m^i satisfies

$$-(u_m^i)''(x) + (1+\rho_m)u_m^i(x) = \alpha_m^i \delta_{\ell_i}(x)$$

and the periodic boundary condition on $[0,T]$. This yields

$$u_m^i(x) = \alpha_m^i C_m \cosh\big(\sqrt{1+\rho_m}(|x-\ell_i|_T - T/2)\big)$$

for $0 \le x \le T$ when we set

$$C_m = \frac{1}{2\sqrt{1+\rho_m}\sinh\Big(\frac{T}{2}\sqrt{1+\rho_m}\Big)}\,. \tag{5.6}$$

In other words

$$V_i(x,x') = \sum_m \alpha_m^i C_m \cosh\big(\sqrt{1+\rho_m}(|x-\ell_i|_T - T/2)\big)\ \varphi_m(x')\,. \tag{5.7}$$

As the functions φ_m are orthogonal to one another, we obtain (again we treat the functions ζ_i as measures)

$$\begin{aligned}\sigma\int_{\mathbb{T}} V_i\,d\zeta_j &= \sigma\sum_m \int_{L_j} V_i(\ell_j,x')\,\alpha_m^j\varphi_m(x')dx' \\ &= \sigma\sum_m \alpha_m^i\alpha_m^j C_m\cosh\big(\sqrt{1+\rho_m}(|\ell_j-\ell_i|_T - T/2)\big)\int_{L_j}\varphi_m^2(x')\,dx' \\ &= \frac{\sigma T}{2}\sum_m \alpha_m^i\alpha_m^j C_m\cosh\big(\sqrt{1+\rho_m}(|\ell_j-\ell_i|_T - T/2)\big)\,.\end{aligned}$$

Making use of self-adjointness of the Green's function G, and grouping terms by oscillation mode m, the last term in (5.2) becomes

$$\begin{aligned}&\sigma\int_{\mathbb{T}}\big(|\nabla V_Z|^2 + |V_Z|^2\big)\,dX \\ &= \sigma\int_{\mathbb{T}}\Big(\sum_{i=1}^{2k} V_i\Big)\,d\Big(\sum_{j=1}^{2k}\zeta_j\Big) = \sigma\sum_{i,j=1}^{2k}\int_{\mathbb{T}} V_i\,d\zeta_j \\ &= \frac{\sigma T}{2}\sum_m C_m\sum_{i,j=1}^{2k}\alpha_m^i\alpha_m^j\cosh\big(\sqrt{1+\rho_m}(|\ell_j-\ell_i|_T - T/2)\big)\,.\end{aligned} \tag{5.8}$$

Putting (5.5), (5.8) to (5.2), we obtain

$$\begin{aligned}\frac{2}{T}J''(\mathbb{L})[\zeta] = \sum_m\Big\{&(\rho_m - \sigma d_0)\sum_{i=1}^{2k}(\alpha_m^i)^2 \\ &+\sigma C_m\sum_{i,j=1}^{2k}\alpha_m^i\alpha_m^j\cosh\big(\sqrt{1+\rho_m}(|\ell_j-\ell_i|_T - T/2)\big)\Big\}\,.\end{aligned} \tag{5.9}$$

Since the function ζ may well exhibit just one mode, for the k-lamella to be stable it is necessary and sufficient to show that

$$(\rho_m - \sigma d_0)\sum_{i=1}^{2k}(\alpha_m^i)^2 + \sigma C_m \sum_{i,j=1}^{2k} \alpha_m^i \alpha_m^j \cosh\bigl(\sqrt{1+\rho_m}(|\ell_j - \ell_i|_T - T/2)\bigr) > 0$$

$$\forall (\alpha_m^1, \dots, \alpha_m^{2k}) \neq 0 \tag{5.10}$$

for each m. We study the last term and we rewrite it as

$$\sigma C_m \sum_{i,j=1}^{2k} \alpha_m^i \alpha_m^j \cosh\bigl(\sqrt{1+\rho_m}(|\ell_j - \ell_i|_T - T/2)\bigr) = \frac{\sigma}{\sqrt{1+\rho_m}} \sum_{i,j=1}^{2k} \bigl(\mathcal{A}^{(m)}\bigr)_{i,j} \alpha_m^i \alpha_m^j$$

where we set, according to (5.6),

$$\bigl(\mathcal{A}^{(m)}\bigr)_{i,j} := \frac{1}{2\sinh\Bigl(\frac{T}{2}\sqrt{1+\rho_m}\Bigr)} \cosh\bigl(\sqrt{1+\rho_m}(|\ell_j - \ell_i|_T - T/2)\bigr) .$$

We now define

$$T^{(m)} := T\sqrt{1+\rho_m} , \qquad x_0^{(m)} := x_0\sqrt{1+\rho_m}$$

so that the numbers

$$\ell_i^{(m)} := \ell_i \sqrt{1+\rho_m}$$

have the same definition in terms of $T^{(m)}$ and $x_0^{(m)}$ as the numbers ℓ_i had in terms of T and x_0 in (3.3); it is convenient to put

$$a := x_0/T \tag{5.11}$$

(and we remark in particular that $a = x_0^{(m)}/T^{(m)}$), and finally

$$\begin{aligned} d_0^{(m)} &= \frac{1}{\sinh(T^{(m)}/2k)} \sinh\frac{T^{(m)} - x_0^{(m)}}{2k} \sinh\frac{x_0^{(m)}}{2k} \\ &= \frac{1}{\sinh(T^{(m)}/2k)} \sinh\frac{(1-a)T^{(m)}}{2k} \sinh\frac{aT^{(m)}}{2k} . \end{aligned}$$

Then we may rewrite

$$\bigl(\mathcal{A}^{(m)}\bigr)_{i,j} = \frac{1}{2\sinh(T^{(m)}/2)} \cosh\bigl(|\ell_j^{(m)} - \ell_i^{(m)}|_{T^{(m)}} - T^{(m)}/2)\bigr) .$$

Comparing this with (4.1),(3.10), by Remark 4.6 we may apply the first part of Theorem 4.1 and obtain that the least eigenvalue of $\mathcal{A}^{(m)}$ is $d_0^{(m)}$. Recalling the coefficient in front of $\mathcal{A}^{(m)}$, we have that (5.10) is equivalent to proving that

$$\rho_m - \sigma d_0 + \frac{\sigma}{\sqrt{1+\rho_m}} d_0^{(m)} > 0 \tag{5.12}$$

for each $m \geq 1$.

We may now precise the result of Theorem 5.1 to obtain a somewhat generic stability result: as the Poincaré constant on the segment $[0,1]$ with periodic boundary conditions is $C_{P,1} = 4\pi^2$, equation (5.1) turns into

$$\sigma < \frac{4\pi^2}{d_0 T^2} ;$$

if we want to get a result which is independent of the fullness parameter $c = 1 - 2\alpha/\sigma$ of (1.14), and therefore of the ratio of x_0 to T as seen from (2.5), we may remark that

$$d_0 \le \max_{0 \le y \le T} \frac{\sinh\big((T-y)/2k\big)\,\sinh y/2k}{\sinh(T/2k)}\ ,$$

which is attained at $y = T/2$. Hence

$$d_0 \le \frac{\sinh^2 T/4k}{\sinh(T/2k)} = \frac{1}{2}\tanh(T/4k)\ , \tag{5.13}$$

thus a sufficient condition for any (i.e. for any fullness parameter c) stationary k-lamella to be stable is the following.

Corollary 5.3. *When $\sigma < 8\pi^2/[T^2 \tanh(T/4k)]$, the stationary k-lamella is stable for any value of the fullness parameter c.*

Remark that the most delicate case (thus the worst for stability) is $k = 1$, and small values of T contribute to stability; also remark that the worst (i.e., the maximum) value of d_0 was obtained for $x_0 = T/2$, thus for $c = 0$. Theorem 5.1 was obtained by disregarding the positive contribution of the Green's function term, and we may now show that when instead we take it into account this corollary becomes much stronger, see Corollary 5.17.

Recall that $\rho_{2r-1} = \rho_{2r}$ so the same happens with all the various quantities depending on the eigenvalues, such as $C_m, T^{(m)}, x_0^{(m)}, d_0^{(m)}$; it therefore suffices to prove (5.12) only for even m. With slight abuse, we redefine

$$\rho_m = \frac{4\pi^2 m^2}{T^2} \quad \text{for } m = 0, 1, \ldots$$

and study (5.12) for all $m = 1, 2, \ldots$. We set for $m = 0, 1, \ldots$

$$\theta_m := \frac{T^2}{4k^2} + \frac{\pi^2 m^2}{k^2} = \frac{T^2}{4k^2}(1 + \rho_m) = \Big(\frac{T^{(m)}}{2k}\Big)^2 \tag{5.14}$$

so that

$$\rho_m = \frac{4k^2}{T^2}\theta_m - 1 = \frac{4k^2}{T^2}(\theta_m - \theta_0)$$

as $\theta_0 = (T/2k)^2$; we remark that (since $x_0^{(m)}/T^{(m)} = x_0/T = a$)

$$d_0^{(m)} = \frac{1}{\sinh\sqrt{\theta_m}} \sinh\big((1-a)\sqrt{\theta_m}\,\big)\sinh\big(a\sqrt{\theta_m}\,\big)$$

thus

$$\frac{d_0^{(m)}}{\sqrt{1+\rho_m}} = \frac{T}{2k}\,\frac{\sinh\big((1-a)\sqrt{\theta_m}\,\big)\sinh\big(a\sqrt{\theta_m}\,\big)}{\sqrt{\theta_m}\sinh\sqrt{\theta_m}}\ .$$

Since $d_0^{(0)} = d_0$, it is useful to introduce the function

$$h(x) := \frac{4k^2}{T^2}x + \frac{\sigma T}{2k}\,\frac{\sinh\big((1-a)\sqrt{x}\,\big)\sinh\big(a\sqrt{x}\,\big)}{\sqrt{x}\sinh\sqrt{x}} \tag{5.15}$$

so that we may rewrite (5.12) as

$$h(\theta_m) - h(\theta_0) > 0 \qquad \forall m \ge 1\ . \tag{5.16}$$

Now define for every real $x \geq 0$

$$\theta(x) := \frac{T^2}{4k^2} + \frac{\pi^2}{k^2} x \ ,$$

so that $\theta_m = \theta(m^2)$. A sufficient condition for (5.16) is to check that

$$h(\theta(x)) - h(\theta(0)) > 0 \qquad \forall x \geq 1 \ . \tag{5.17}$$

Remark 5.4. *Although we did not stress dependence on the various quantities involved, not to overburden the notation, from (2.5) and (2.4), both x_0 and d_0 depend only on T and c, but not on σ. In addition, changing sign of c converts x_0 into $T - x_0$; this in turn converts a to $1 - a$. However this change will not affect h, so it suffices to study stability only for $c \geq 0$. The cases $x_0 = 0$ (empty set) and $x_0 = T$ (full torus) corresponding to $c = \pm 1$ are trivially stable, we therefore focus only on $0 < x_0 < T$, equivalently $0 < a < 1$, so that the last term in the definition (5.15) of h is positive.*

We see that an unrestricted stability statement, such as Theorem 4.1, cannot be attained, through the negative result underneath with Γ as defined in (1.16).

Proposition 5.5. *If $0 < |c| < 1$ and $\sigma > -4/\Gamma(c)$, then for any sufficiently large T the stationary 1-lamella is unstable.*

Proof. From Remark 5.4 and the observation $\Gamma(c) = \Gamma(-c)$, it suffices to study the case $0 < c < 1$. We will show $m = 1$ is an unstable mode for (5.12). Let $T >> 1$ and denote by ω_T all terms which are exponentially small in T (we need to keep track of algebraic small quantities). Then (2.7) still holds, so $x_0 = -\log c + \omega_T$ from (2.5) (see [2, Proposition 3.2 (vi)]) and $d_0 = \frac{1-c}{2} + \omega_T$ from (2.4); moreover

$$\begin{aligned}
\sqrt{\theta_1} &= \sqrt{\frac{T^2}{4} + \pi^2} = \frac{T}{2}\left(1 + \frac{2\pi^2}{T^2} + O(\frac{1}{T^4})\right) \ , \\
\sqrt{1+\rho_1} &= 2\sqrt{\theta_1}/T \ , \\
T^{(1)} &= 2\sqrt{\theta_1} \ , \qquad x_0^{(1)} = 2x_0\sqrt{\theta_1}/T \ .
\end{aligned}$$

Thus computing directly from the left side of (5.12), we obtain

$$\begin{aligned}
h(\theta_1) - h(\theta_0) =& \frac{4\pi^2}{T^2} - \sigma\left(\frac{1-c}{2} + \omega_T\right) + \frac{\sigma T}{4\sqrt{\theta_1}}\left(1 - c + \frac{2\pi^2 c x_0}{T^2} + O(\frac{1}{T^4})\right) \\
=& \frac{4\pi^2}{T^2} - \sigma\left(\frac{1-c}{2} + \omega_T\right) \\
&+ \frac{\sigma}{2}\left(1 - \frac{2\pi^2}{T^2} + O(\frac{1}{T^4})\right)\left(1 - c + \frac{2\pi^2 c x_0}{T^2} + O(\frac{1}{T^4})\right) \\
=& \frac{\pi^2}{T^2}(4 + \sigma\Gamma(c)) + O(\frac{1}{T^4}) \\
<& \, 0
\end{aligned}$$

for T large. ∎

Remark 5.6. *The condition $\sigma > 4/|\Gamma(c)|$ imposed in Proposition 5.5 turns out to be both necessary and sufficient for instability of all k-lamellae in a sufficiently large torus. Indeed in the above proof we only treat the mode $m = 1$; but by Theorem 5.11* *(which will be proved later) we do not discard any generality for stability studies. Second, if we carry out the above proof on a k-lamella, then* $x_0/k = -\log c + \omega_T$*; however the same final condition, which is independent of* k*, results.*

In view of Remark 2.3, when $c > 0$ and we pick a large square torus with side $T = 2t_0$, then $J(\mathbb{L}) < J(\emptyset) = 0 < J(\mathbb{T})$. This gives

COROLLARY 5.7. *Let* $0 < c < 1$ *and* $4 + \sigma\Gamma(c) < 0$*. Then for some sufficiently large torus there exists an unstable minimal lamella* $\mathbb{L}$ *such that* $J(\mathbb{L}) < J(\emptyset) = 0 < J(\mathbb{T})$*. Hence in this parameter regime global minimizers (which always exist by Theorem 2.1), being neither the trivial states nor the lamellae, has to have a genuine 2D structure.*

We now collect the necessary preliminaries to prove the main results. We begin with easy properties of convex functions.

LEMMA 5.8. *If* f *is (strictly) convex then so is* e^f*; if* f *is (strictly) convex, so is* $f(a + bx)$ *for* $b \neq 0$*; if* f *is convex on* $[0, +\infty)$*, then for* $0 < a < 1$

$$f(a) + f(1 - a) \leq f(0) + f(1) ,$$

and the inequality is strict if f *is strictly convex.*

Proof. We only care about the last assertion; convexity of f implies $f(a) \leq (1 - a)f(0) + af(1)$. Replace a by $1 - a$ to obtain a similar inequality and sum the two inequalities. ∎

LEMMA 5.9. *The function* $P(t) := \frac{t}{\tanh t} + \frac{t^2}{\sinh^2 t} - 2$*, continuously extended by* $P(0) = 0$*, is increasing and strictly convex on* $[0, \infty)$*, thus positive for* $t > 0$*.*

Proof. We have

$$\begin{aligned} P' &= \frac{1}{\tanh t} + \frac{t}{\sinh^2 t} - \frac{2t^2 \cosh t}{\sinh^3 t} = \frac{1}{\sinh^2 t}\Big(\sinh t\, \cosh t + t - 2t^2\, \coth t\Big) \\ &= \frac{1}{\sinh^2 t}\Big(\frac{\sinh 2t}{2} + t - 2t^2\, \coth t\Big) =: \frac{1}{\sinh^2 t}\, g(t) . \end{aligned}$$

It is clear that $g(0) = 0$. A direct calculation gives

$$\begin{aligned} g'(t) &= \cosh 2t + 1 + \frac{2t^2}{\sinh^2 t} - \frac{4t \cosh t}{\sinh t} = 2\cosh^2 t + \frac{2t^2}{\sinh^2 t} - \frac{4t \cosh t}{\sinh t} \\ &= 2\Big(\cosh t - \frac{t}{\sinh t}\Big)^2 > 0 \end{aligned}$$

for $t \in (0, \infty)$. Hence $g > 0$ and we conclude that P is strictly increasing. Moreover

$$\begin{aligned}
P'' &= \frac{1}{\sinh^2 t} g'(t) - \frac{2}{\sinh^3 t} \cosh t \; g(t) \\
&= \frac{2}{\sinh^2 t} \Big((\cosh t - \frac{t}{\sinh t})^2 - \frac{\cosh t}{\sinh t} (\sinh t \, \cosh t + t - 2t^2 \, \coth t) \Big) \\
&= \frac{2}{\sinh^2 t} \Big(\frac{t^2}{\sinh^2 t} - \frac{3t \, \cosh t}{\sinh t} + \frac{2t^2 \, \cosh^2 t}{\sinh^2 t} \Big) \\
&= \frac{2t}{\sinh^4 t} \, (t - 3 \cosh t \, \sinh t + 2t \cosh^2 t) = \frac{2t}{\sinh^4 t} \, (2t - \frac{3}{2} \sinh 2t + t \cosh 2t) \\
&= \frac{2t}{\sinh^4 t} \left(-\frac{3}{2} \sum_{n=1}^{\infty} \frac{(2t)^{2n+1}}{(2n+1)!} + t \sum_{n=1}^{\infty} \frac{(2t)^{2n}}{(2n)!} \right) \\
&= \frac{t}{\sinh^4 t} \sum_{n=1}^{\infty} \frac{(2t)^{2n+1}}{(2n+1)!} (2n + 1 - 3) \\
&> 0 \, .
\end{aligned}$$

∎

The key tool is the following result.

Lemma 5.10. *The functions h and $h \circ \theta$ are strictly convex.*

Proof. By Lemma 5.8, since θ is an affine function of x it is enough to prove h is strictly convex, which we will do for $x > 0$ or, extending h at 0 by continuity, for $x \geq 0$; we remark that this precision will not be needed, since $\theta(0) = \theta_0 = T^2/4k^2$ will be the least value of the argument of h we will be interested in. As the first term in the definition (5.15) of h is linear, we are only concerned with the second (which, we recall, is positive), and in view of Lemma 5.8 again we may just prove that its logarithm is strictly convex. Disregarding the coefficient $\sigma T/2k$ we set

$$u(x) := \log \frac{\sinh\big((1-a)\sqrt{x}\,\big) \sinh\big(a\sqrt{x}\,\big)}{\sqrt{x} \sinh \sqrt{x}} \; ;$$

then

$$2u'(x) = \frac{a}{\sqrt{x} \tanh\big(a\sqrt{x}\,\big)} + \frac{1-a}{\sqrt{x} \tanh\big((1-a)\sqrt{x}\,\big)} - \frac{1}{\sqrt{x} \tanh \sqrt{x}} - \frac{1}{x} \, , \tag{5.18}$$

$$\begin{aligned}
4x^2 u''(x) = &- \frac{a\sqrt{x}}{\tanh\big(a\sqrt{x}\,\big)} - \frac{a^2 x}{\sinh^2\big(a\sqrt{x}\,\big)} - \frac{(1-a)\sqrt{x}}{\tanh\big((1-a)\sqrt{x}\,\big)} - \frac{(1-a)^2 x}{\sinh^2\big((1-a)\sqrt{x}\,\big)} \\
&+ \frac{\sqrt{x}}{\tanh \sqrt{x}} + \frac{x}{\sinh^2 \sqrt{x}} + 2 \, .
\end{aligned} \tag{5.19}$$

Now let x be fixed and define

$$Q(t) := \frac{t\sqrt{x}}{\tanh\big(t\sqrt{x}\,\big)} + \frac{t^2 x}{\sinh^2\big(t\sqrt{x}\,\big)} - 2 \, .$$

With P as denoted in Lemma 5.9, it is clear that $Q(t) = P\big(t\sqrt{x}\,\big)$; moreover

$$4x^2 u''(x) = Q(1) - Q(a) - Q(1-a) \, .$$

Using Lemma 5.9 we see that Q is non-negative and vanishing at 0, strictly convex and increasing, and applying the last part of Lemma 5.8 we obtain $4x^2 u''(x) > 0$. ∎

We may now examine the function h and the necessary and sufficient condition (5.16).

THEOREM 5.11. *It is necessary and sufficient for the k-lamella to be stable that the first mode is stable, that is, $h(\theta_1) > h(\theta_0)$.*

Proof. The necessity of a stable first mode is clear. On the other hand suppose $h(\theta_1) > h(\theta_0)$. From the strict convexity of h and the fact that θ_m is strictly increasing with respect to m,

$$\frac{h(\theta_m) - h(\theta_1)}{\theta_m - \theta_1} > \frac{h(\theta_1) - h(\theta_0)}{\theta_1 - \theta_0} > 0 \, ,$$

hence $h(\theta_m) > h(\theta_1) > h(\theta_0)$ for all $m = 2, 3, \ldots$; this immediately gives (5.16). ◻

REMARK 5.12. *Whenever $h(\theta_1) > h(\theta_0)$, a slight modification of the above argument gives $h(\theta_{m+1}) > h(\theta_m)$ for $m = 0, 1, 2, \ldots$.*

We saw right after Corollary 5.3 that $c = 0$ and $k = 1$ seemed the most delicate cases; we are now going to substantiate the claim.

THEOREM 5.13. *Stability is increasing with $|c|$, in the sense that if the stationary k-lamella with $|c| = c_0 < 1$ is stable, then it is stable also for $c_0 < |c| \le 1$.*

COROLLARY 5.14. *A necessary and sufficient condition for the stationary k-lamella to be stable for all values of c is that it is stable for $c = 0$.*

Proof. By Remark 5.4 we may confine ourselves to the case $c \ge 0$, that is $0 \le a \le 1/2$ keeping the notation introduced in (5.11). By (5.15) and hyperbolic function identities we may rewrite

$$\begin{aligned} h(x) &= \frac{4k^2}{T^2} x + \frac{\sigma T}{2k} \frac{\sinh\big((1-a)\sqrt{x}\,\big) \sinh\big(a\sqrt{x}\,\big)}{\sqrt{x} \sinh \sqrt{x}} \\ &= \frac{4k^2}{T^2} x + \frac{\sigma T}{4k} \frac{\cosh \sqrt{x} - \cosh\big((1-2a)\sqrt{x}\,\big)}{\sqrt{x} \sinh \sqrt{x}} \, , \end{aligned}$$

so it is convenient to set $\lambda = 1 - 2a$ and remark that, as x_0 is decreasing with c, the parameter λ is increasing with c. We will prove that the function

$$\begin{aligned} h(\theta_1) - h(\theta_0) = \frac{4k^2}{T^2}(\theta_1 - \theta_0) &+ \frac{\sigma T}{4k} \left(\frac{\cosh \sqrt{\theta_1}}{\sqrt{\theta_1} \sinh \sqrt{\theta_1}} - \frac{\cosh \sqrt{\theta_0}}{\sqrt{\theta_0} \sinh \sqrt{\theta_0}} \right) \\ &- \frac{\sigma T}{4k} \left(\frac{\cosh\big(\lambda\sqrt{\theta_1}\,\big)}{\sqrt{\theta_1} \sinh \sqrt{\theta_1}} - \frac{\cosh\big(\lambda\sqrt{\theta_0}\,\big)}{\sqrt{\theta_0} \sinh \sqrt{\theta_0}} \right) \end{aligned}$$

is increasing with respect to λ, and therefore to c, thus if it is non-negative for a certain value of $c \ge 0$ (which by Theorem 5.11 is equivalent to stability) it is positive for all larger values of c: this claim would prove the result. We set

$$\phi(\lambda) = \frac{\cosh\big(\lambda\sqrt{\theta_1}\,\big)}{\sqrt{\theta_1} \sinh \sqrt{\theta_1}} - \frac{\cosh\big(\lambda\sqrt{\theta_0}\,\big)}{\sqrt{\theta_0} \sinh \sqrt{\theta_0}} \, ;$$

it suffices to show that ϕ is decreasing. Indeed (writing for simplicity $A = \sqrt{\theta_0}$ and $B = \sqrt{\theta_1}$ and remarking that $A < B$)

$$\phi'(\lambda) = \frac{\sinh(\lambda B)}{\sinh B} - \frac{\sinh(\lambda A)}{\sinh A}$$

and to prove that $\phi' < 0$ for $0 < \lambda < 1$ (which is enough) we establish that

$$\psi(x) = \frac{\sinh(\lambda x)}{\sinh x}$$

is decreasing for $x > 0$:

$$\psi'(x) = \frac{\lambda \cosh(\lambda x) \sinh x - \sinh(\lambda x) \cosh x}{\sinh^2 x} = \frac{\cosh(\lambda x) \cosh x}{\sinh^2 x} \big(\lambda \tanh x - \tanh(\lambda x)\big) \ .$$

The function

$$\omega(x) = \lambda \tanh x - \tanh(\lambda x)$$

vanishes at $x = 0$ and its derivative is

$$\omega'(x) = \frac{\lambda}{\cosh^2 x} - \frac{\lambda}{\cosh^2(\lambda x)} < 0$$

because $0 < \lambda < 1$, therefore $\omega < 0$ which concludes the proof. ∎

Now that we proved the worst case for stability is $c = 0$ we turn our attention to k.

THEOREM 5.15. *In the case $c = 0$, stability is increasing with k, in the sense that if the stationary k_0-lamella with $c = 0$ is stable, then all k-lamellae with $k \geq k_0$ and $c = 0$ are stable, which implies they are stable also for every c.*

COROLLARY 5.16. *A necessary and sufficient condition for the stationary k-lamella to be stable for all values of c and all values of k is that the stationary 1-lamella is stable for $c = 0$.*

Proof. We take $c = 0$ (correspondingly $a = 1/2$); recalling the definition (5.14) of the numbers θ_m, we introduce the quantities

$$\vartheta_1 := k^2 \theta_1 = \frac{T^2}{4} + \pi^2 \ , \qquad \vartheta_0 := k^2 \theta_0 = \frac{T^2}{4}$$

so they are independent of k, and we rewrite the left hand side of the stability inequality $h(\theta_1) - h(\theta_0) \geq 0$ as

$$\begin{aligned}
h(\theta_1) - h(\theta_0) &= \frac{4k^2}{T^2}(\theta_1 - \theta_0) + \frac{\sigma T}{4k}\left(\frac{\cosh\sqrt{\theta_1} - 1}{\sqrt{\theta_1}\sinh\sqrt{\theta_1}} - \frac{\cosh\sqrt{\theta_0} - 1}{\sqrt{\theta_0}\sinh\sqrt{\theta_0}}\right) \\
&= \frac{4}{T^2}(\vartheta_1 - \vartheta_0) + \frac{\sigma T}{4}\left(\frac{\cosh\sqrt{\theta_1} - 1}{\sqrt{\vartheta_1}\sinh\sqrt{\theta_1}} - \frac{\cosh\sqrt{\theta_0} - 1}{\sqrt{\vartheta_0}\sinh\sqrt{\theta_0}}\right) \\
&= \frac{4}{T^2}(\vartheta_1 - \vartheta_0) + \frac{\sigma T}{4}\left(\frac{\tanh(\sqrt{\theta_1}/2)}{\sqrt{\vartheta_1}} - \frac{\tanh(\sqrt{\theta_0}/2)}{\sqrt{\vartheta_0}}\right) \\
&= \frac{4}{T^2}(\vartheta_1 - \vartheta_0) + \frac{\sigma T}{8}\left(\frac{\tanh(\sqrt{\vartheta_1}/2k)}{\sqrt{\vartheta_1}/2} - \frac{\tanh(\sqrt{\vartheta_0}/2k)}{\sqrt{\vartheta_0}/2}\right) \ . \qquad (5.20)
\end{aligned}$$

∎

The first term is independent of k, and to prove the assertion we will show that the second term is increasing with respect to k. We set

$$A = \sqrt{\vartheta_0}/2 \ , \qquad B = \sqrt{\vartheta_1}/2 \ , \qquad x = 1/k$$

so we have to show that if $A < B$ the function

$$\phi(x) = \frac{\tanh(Bx)}{B} - \frac{\tanh(Ax)}{A}$$

is decreasing. But

$$\phi'(x) = \frac{1}{\cosh^2(Bx)} - \frac{1}{\cosh^2(Ax)} < 0 \ .$$

We now see how taking the Green's function term into consideration dramatically improves the estimate of Corollary 5.3. According to Theorem 5.11, the worst case of all, that is, $c = 0$ and $k = 1$, is stable if and only if

$$0 < h(\theta_1) - h(\theta_0) = \frac{4}{T^2}(\vartheta_1 - \vartheta_0) + \frac{\sigma T}{4}\left(\frac{\tanh(\sqrt{\vartheta_1}/2)}{\sqrt{\vartheta_1}} - \frac{\tanh(\sqrt{\vartheta_0}/2)}{\sqrt{\vartheta_0}}\right)$$

$$= \frac{4\pi^2}{T^2} - \frac{\sigma T}{2}\left(\frac{\tanh(T/4)}{T} - \frac{\tanh(\sqrt{T^2+4\pi^2}/4)}{\sqrt{T^2+4\pi^2}}\right).$$

Immediately we deduce that

Corollary 5.17. *A necessary and sufficient condition for all stationary k-lamellae to be stable, for every value of c and k, is that*

$$\sigma < 8\pi^2 \Big/ \left[T^3\left(\frac{\tanh(T/4)}{T} - \frac{\tanh(\sqrt{T^2+4\pi^2}/4)}{\sqrt{T^2+4\pi^2}}\right)\right].$$

To compare this result (which is a sharp condition) with Corollary 5.3 we recall that $\tanh t = 1 - O(e^{-2T})$ as $T \to +\infty$, so that

$$\frac{\tanh(T/4)}{T} \qquad \begin{cases} \to \frac{1}{4} = 0.25 & \text{as } T \to 0 \\ \sim \frac{1}{T} & \text{as } T \to +\infty \end{cases}$$

whereas

$$\frac{\tanh(T/4)}{T} - \frac{\tanh(\sqrt{T^2+4\pi^2}/4)}{\sqrt{T^2+4\pi^2}} \qquad \begin{cases} \to \frac{1}{4} - \frac{1}{2\pi}\tanh\frac{\pi}{2} \sim 0.10 & \text{as } T \to 0 \\ \sim 2\pi^2/T^3 & \text{as } T \to +\infty\,. \end{cases}$$

To leading order accuracy, the estimate of the easier Corollary 5.3 reads

$$\sigma \le \frac{32\pi^2}{T^3} \quad \text{as } T \to 0\,, \qquad \sigma \le \frac{8\pi^2}{T^2} \quad \text{as } T \to +\infty$$

whereas Corollary 5.17 gives (the numerical figure at 0 is an approximation only)

$$\sigma < \frac{77\pi^2}{T^3} \quad \text{as } T \to 0\,, \qquad \sigma \le 4 \quad \text{as } T \to +\infty\,.$$

We do an independent check for the case $T \to \infty$. By Remark 5.6 all lamellae are stable when $\sigma < 4/|\Gamma(c)|$ and the torus is large. If we insist on stability for all $|c| < 1$, then $\sigma < \inf_c \frac{4}{|\Gamma(c)|} = 4$.

In the sequel we set

$$\eta(x) = \frac{\tanh x}{x}\,, \qquad G(x) = \frac{\tanh\sqrt{x}}{\sqrt{x}}\,.$$

Referring to the calculation in the proof of Corollary 5.16, for $c = 0$ the condition $h(\theta_1) - h(\theta_0) > 0$ may be rewritten as

$$\frac{4\pi^2}{T^2} + \frac{\sigma T}{8k}\left[G(\vartheta_1/4k^2) - G(\vartheta_0/4k^2)\right] > 0\,,$$

so we investigate some properties of G.

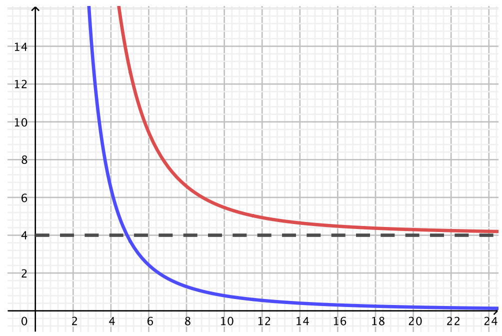

FIG. 5.1. *The critical σ as a function of T for Corollaries 5.3 (lower) and 5.17 (upper).*

LEMMA 5.18. *The function G, continuously extended by $G(0) = 1$, is decreasing and strictly convex for $x \geq 0$. Moreover*

$$G'(x) \sim -\frac{1}{2x^{3/2}} \quad \text{for large } x. \tag{5.21}$$

Finally as $x \to +\infty$, for any $\alpha > 0$

$$G(x+\alpha) - G(x) = -\frac{\alpha}{2x^{3/2}} + o(x^{-3/2}) \,. \tag{5.22}$$

Before proving the result, we note that instead, η is not convex near the origin.

Proof. Taking logarithmic differentiation for $x > 0$ we see that

$$\frac{G'}{G} = \frac{1}{2\sqrt{x}} \frac{(\cosh^2\sqrt{x} - \sinh^2\sqrt{x})}{\sinh\sqrt{x}\,\cosh\sqrt{x}} - \frac{1}{2x}$$

leading by hyperbolic function identity to

$$G' = \left(\frac{1}{\sqrt{x}\sinh 2\sqrt{x}} - \frac{1}{2x}\right) G := p(x)G(x) \,, \tag{5.23}$$

which immediately gives monotonicity of G and (5.21). Taking another derivative and replacing G' with pG we have $G'' = (p' + p^2)G$. It is clear now

$$\begin{aligned} G'' > 0 &\iff p' + p^2 > 0 \iff 1 - (\frac{1}{p})' > 0 \\ &\iff 1 - \frac{d}{dx}\left(-2x + \frac{2x}{1 - \frac{\sinh 2\sqrt{x}}{2\sqrt{x}}}\right) > 0 \iff 3 - 2\frac{d}{dx}\left(\frac{x}{1 - \frac{\sinh 2\sqrt{x}}{2\sqrt{x}}}\right) > 0 \\ &\iff 3 + 2\frac{d}{dx}\left(\left(\sum_{n=1}^{\infty} \frac{2^{2n}x^{n-1}}{(2n+1)!}\right)^{-1}\right) > 0 \,. \end{aligned}$$

A direct computation yields

$$\Big|\frac{d}{dx}\Big(\Big(\sum_{n=1}^{\infty}\frac{2^{2n}x^{n-1}}{(2n+1)!}\Big)^{-1}\Big)\Big|$$

$$= \left|-\left(\sum_{n=1}^{\infty}\frac{2^{2n}x^{n-1}}{(2n+1)!}\right)^{-2}\sum_{m=1}^{\infty}\frac{4m}{(2m+2)(2m+3)}\,\frac{2^{2m}x^{m-1}}{(2m+1)!}\right|$$

$$< \left|\left(\sum_{n=1}^{\infty}\frac{2^{2n}x^{n-1}}{(2n+1)!}\right)^{-1}\right|$$

$$< 3/2 \quad \text{by taking only the first term,}$$

thus $G'' > 0$. Next, the behavior at infinity is an exercise, since $1 - \tanh x$ decays exponentially fast. ◻

Now we set

$$H_k(T,\sigma) = h(\theta_1) - h(\theta_0) = \frac{4\pi^2}{T^2} + \frac{\sigma T}{8k}\big[G(\vartheta_1/4k^2) - G(\vartheta_0/4k^2)\big] \ ;$$

to begin with, since G is decreasing the difference enclosed by the brackets is negative, so for any $T > 0$ there exists a unique

$$\sigma_k(T) = \frac{32k\pi^2}{T^3}\Big/\big[G(\vartheta_0/4k^2) - G(\vartheta_1/4k^2)\big]$$

at which $H_k(T,\sigma) = 0$ with H_k being positive for $\sigma < \sigma_k(T)$. We remark that

$$\frac{\vartheta_0}{4k^2} = \frac{T^2}{16k^2} \ , \qquad \frac{\vartheta_1}{4k^2} = \frac{\vartheta_0}{4k^2} + \frac{\pi^2}{4k^2}$$

and using (5.22) we see that

$$\lim_{T\to+\infty}\sigma_k(T) = 4$$

whereas $\sigma_k(T) \to +\infty$ as $T \to 0^+$. We will now prove

PROPOSITION 5.19. *The function $\sigma_k(T)$ is injective, thus strictly decreasing from* $]0,+\infty[$ *to* $]4,+\infty[$.

Proof. We begin by remarking that by (5.22)

$$\lim_{T\to 0^+} H_k(T,\sigma) = +\infty \ , \qquad \lim_{T\to+\infty} T^2 H_k(T,\sigma) = (4-\sigma)\pi^2 < 0$$

for any $\sigma > 4$, thus for any $\hat\sigma > 4$ there is at least one value $\hat T$ of T such that $H_k(\hat T,\hat\sigma) = 0$, i.e. $\sigma_k(\hat T) = \hat\sigma$. The result will be proved if we show that such $\hat T$ is unique; to this aim, we remark that

$$H_k(T,\hat\sigma) = 0 \iff \frac{4\pi^2}{T^2} + \frac{\hat\sigma T}{8k}\big[G(\vartheta_1/4k^2) - G(\vartheta_0/4k^2)\big] = 0$$

$$\iff \Big(\frac{T}{k}\Big)^3\big[G(\vartheta_1/4k^2) - G(\vartheta_0/4k^2)\big] = -\frac{32\pi^2}{\hat\sigma k^2} \ ,$$

and uniqueness of $\hat T$ will be proved if we show that the function at the left hand side in the last line is strictly decreasing with respect to T. Now we rewrite this function as

$$64\cdot(T/4k)^3\big[G\big((T/4k)^2 + \pi^2/4k^2\big) - G\big((T/4k)^2\big)\big]$$

and we prove that

$$x \mapsto x^3\left[G(x^2+\pi^2/4k^2)-G(x^2)\right]$$

is strictly decreasing. We have

$$x^3\left[G(x^2+\pi^2/4k^2)-G(x^2)\right]=\int_0^{\pi^2/4k^2} x^3 G'(x^2+s)\,ds$$

and the claim will be proved if we show that

$$\frac{\partial}{\partial x}[x^3G'(x^2+s)]<0 \qquad \text{for all } s>0\,.$$

But

$$\begin{aligned} &\frac{\partial}{\partial x}[x^3G'(x^2+s)] = x^2[3G'(x^2+s)+2x^2G''(x^2+s)]<0\\ \iff\ & 3G'(x^2+s)+2x^2G''(x^2+s)<0\\ \iff\ & 3G'(x^2+s)+2(x^2+s)G''(x^2+s)<2sG''(x^2+s)\,. \end{aligned}$$

We prove the left hand side is strictly negative, so the conclusion follows by the convexity of G proved in Lemma 5.18: it is enough to show that for any $X>0$

$$3G'(X)+2XG''(X)<0\,, \tag{5.24}$$ ∎

but recalling that $G(X)=\eta(\sqrt{X})$ we compute

$$G'(X)=\eta'(\sqrt{X})\cdot\frac{1}{2\sqrt{X}}\,, \qquad G''(X)=\eta''(\sqrt{X})\cdot\frac{1}{4X}-\frac{1}{4X\sqrt{X}}\eta'(\sqrt{X})$$

so that

$$3G'(X)+2XG''(X)=\frac{\eta'(\sqrt{X})}{\sqrt{X}}+\frac{1}{2}\eta''(\sqrt{X})$$

and (5.24) is equivalent to

$$\frac{\eta'(t)}{t}+\frac{1}{2}\eta''(t)<0 \qquad \forall t>0\,.$$

A direct computation yields

$$\frac{\eta'(t)}{t}+\frac{1}{2}\eta''(t)=-\frac{(\tanh t)(1-\tanh^2 t)}{t}<0\,.$$

We call $T_k(\sigma)$ the inverse function of $\sigma_k(T)$.

Corollary 5.20. *In the case $c=0$, for every $\sigma>4$ the k-lamella is stable for $T<T_k(\sigma)$ and unstable for $T\geq T_k(\sigma)$.*

Appendix A. Road map to prove Theorem 3.5. Throughout this Appendix we refer to statements, formulas and pages of [3], and highlight the changes and focal points needed to adapt the proof of [3, Theorem 1.1] for our Theorem 3.5 in this paper. The proof in [3] needs to resolve a major technicality: the volumetric constraint. Addressing this issue requires lots of efforts to reduce the problem to an unconstrained one, to keep track of the inequalities needed, then to tackle the Lagrange multiplier (and a sequence of them, too).

1. The Euler-Lagrange equation [3, formula (2.8)], which contains a Lagrange multiplier, takes a new form (see Proposition 3.1)
$$\mathcal{K}(\partial E) - \alpha + \sigma \mathcal{N}_E = 0\ .$$
The corresponding new weak formulation drops the volumetric constraint in [3, Definition 2.2], but adds a term $-\alpha\zeta \cdot \nu$ in its integrands.
2. The key [3, Lemma 2.6] for the Laplacian is replaced by the (stronger) Lemma 3.6 for the Hemholtz operator.
3. We do not need [3, Proposition 2.7], which is used to weaken volume constraint.
4. The slight changes to the derivation of the second variation formula [3, Theorem 3.1] have already been summarized at the beginning of Section 3.
5. The definition [3, formula (3.4)] of $\partial^2 J$, which is our J'', acts on all of $H^1(\partial E)$; there is no need to only specify volume preserving vector field X, see (3.2).
6. The very convenient equality [3, formula (3.5)] regarding Green's function for the Laplacian (and zero average) is replaced by the equally versatile
$$\int_{\partial E}\int_{\partial E} G(x,y)\phi(x)\phi(y)\, d\mathcal{H}^{N-1}(x)\, d\mathcal{H}^{N-1}(y) = \int_{\mathbb{T}} (|\nabla V|^2 + |V|^2)\, dx$$
where V is the unique weak solution to the equation $-\Delta V + V = \phi \mathcal{H}^{N-1} \llcorner \partial E$ with periodic boundary conditions on $\mathbb{T}$; we use this e.g. in Lemma 3.6.
7. The field X in [3, Corollary 3.4] is to be chosen as the gradient of the solution u of
$$-\Delta u = \frac{1}{|\partial E|} \int_{\partial E} \phi\, d\mathcal{H}^{N-1}\ ;$$
ours has no such restriction.
8. The function spaces and vector fields with tilde, introduced on page 528 of [3] and afterwards, are not needed. Our ambient space is all of H^1.
Both [3, formula (3.9) and Lemma 3.6] still hold, whereas in [3, Theorem 3.7] the last assertion does not, but is not needed in our case (again, it relates to volume preservation).
9. The proof of the tricky [3, Lemma 3.8], used to control and later remove the translation part, is not related to energies or equations, so it still holds.
10. The trouble after [3, formula (3.39)] to keep track of the zero average condition is not necessary, thus a_h is not needed and $\tilde{\phi}_h$ is simply $\phi_h \circ \Phi_h$, that is ϕ_h acts on ∂E.
In [3, formula (3.40)] we use that the full H^1 product of $(v_h - v)$ and ϕ is $\leq c_\epsilon \|\phi\|$.
In [3, formula (3.43)] we also have the difference of $z_h^2 - \tilde{z}_h^2$, but next equation contains the Helmholz operator and not only the Laplacian, so convergence of $\mu_h - \tilde{\mu}_h$ to zero is preserved.
After [3, formula (3.46)] we also have the volume term α and another term appears, but it is not dangerous because the full (not only tangential) divergence of X is zero.
11. The volume penalization after [3, formula (4.2)] is not needed; on the other hand in the chain of inequalities after [3, formula (4.7)] we also have a $-\alpha(|F| - |K_h|) \geq -\alpha|F \triangle K_h|$ so the number Λ chosen in [3, formula (4.6)] must be increased by α.

12. We have no Lagrange multipliers, so the choice of f_h in [3, formula (4.9)] is

$$f_h := \begin{cases} \alpha - \sigma v_{F_h} \\ \alpha - \sigma v_E + \rho_h \end{cases}$$

and the rest of the proof becomes silly.

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