Cyclic projections in Hadamard spaces

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Abstract

We show that cyclic products of projections onto convex subsets of Hadamard spaces can behave in a more complicated way than in Hilbert spaces, resolving a problem formulated by Miroslav Bačák. Namely, we construct an example of convex subsets in a Hadamard space such that the corresponding cyclic product of projections is not asymptotically regular.

1 Introduction

The method of cyclic projections is a classical algorithm seeking an intersection point of a finite family $C_1, \dots, C_k$ of closed convex subsets in a Hilbert space X. Denote by P_i the closest-point projection $X \to C_i$; it sends a point $x \in X$ to the (necessarily unique) point $P_i(x)$ in C_i that minimizes the distance to x. Given a point $x \in X$ consider the sequence $x_n = P^n(x)$, where P is the cyclic composition of projections $P = P_1 \circ \dots \circ P_k$. The method of cyclic projections analyzes the sequence (x_n), tries to find a limit point x_∞, to show $x_\infty \in C_1 \cap \dots$ $\dots \cap C_k$, and to understand the rate of convergence.

Let us list some results in the area. *If the intersection $C_1 \cap \dots \cap C_k$ is non-empty, then (x_n) always converges weakly to some point in $C_1 \cap \dots \cap C_k$* [14]. However, this convergence does not need to be strong [19]. *If, in addition, C_i are linear subspaces, then the convergence is strong* [18, 23]. If the intersection $C_1 \cap \dots \cap C_k$ is not assumed to be non-empty, the analysis of the sequence (x_n) is more complicated. However, in [11] it has been established that the cyclic product $P = P_1 \circ \dots \circ P_k$ is *asymptotically regular*; by definition, this means, that for any starting point $x \in X$, we have $|x_n - x_{n+1}| \to 0$ as $n \to \infty$. The rates of convergence, respectively, the rates of asymptotic regularity have been investigated in several works, see, for instance [12, 20]. For further reference, see [4, 7, 8, 12, 17].

More recently, the method of cyclic projections has been investigated beyond the setting of Hilbert spaces in so-called Hadamard spaces (also known as CAT(0) spaces, or globally non-positively curved spaces in the sense of Alexandrov). This class of metric spaces includes hyperbolic spaces, metric trees, as well as complete simply-connected Riemannian manifolds of non-positive curvature; it has played an important role in many areas of mathematics in the last decades. We assume some familiarity with Hadamard spaces, refer the reader to [2, 3, 9, 10, 15, 16] as general references on this subject. For the introduction and applications of the method of cyclic projections in Hadamard spaces, see [6], [7, Section 6.8], and the references therein.

Hadamard spaces are defined (loosely speaking) by the property that their distance function is at least as convex as the distance function on a Hilbert space.

In particular, Hadamard spaces contain a huge variety of convex subsets; closest point projections to closed convex subsets are well-defined and 1-Lipschitz, and the questions discussed above about cyclic projections are absolutely meaningful in a Hadamard space X.

Many results discussed above have been transferred from the linear setting of Hilbert spaces to general Hadamard spaces. For instance, *if the subsets C_i have a non-empty intersection, then the cyclic product of projection P is asymptotically regular and, for any initial point $x \in X$, the sequence $x_n = P^n(x)$ converges* weakly *to a point $x_\infty \in C_1 \cap \dots \cap C_k$* [5, 8]. (The weak topology on Hadamard spaces is discussed in [6, 7, 22]). The rate of convergence in this setting has been studied in [21].

Therefore it is somewhat surprising, that the fundamental result of Heinz Bauschke [11] for (possibly) non-intersecting convex subsets C_i does not admit a generalization to the setting of Hadamard spaces. The following main result of this paper provides a negative answer to the question of Miroslav Bačák [7, Problem 6.13].

1.1. Theorem. *There exist a Hadamard space X and compact convex subsets $C_1, \dots, C_k$ in X such that the composition of the closest-point projections $P = P_1 \circ \dots \circ P_k$ is not asymptotically regular.*

We provide an explicit example with X being a product of two trees, proving the theorem for $k = 3$. Setting $C_3 = \dots = C_k$ defines examples for any $k \geqslant 3$.

In this example, all subsets C_i are isometric to the unit interval, the projections P_i map all of these segments isometrically onto C_i and the composition $P = P_1 \circ P_2 \circ P_3$ maps C_1 to itself isometrically but exchanges the endpoints of this interval. A stronger version of the theorem is proved in the appendix; it requires a somewhat deeper understanding of the geometry of Hadamard spaces. It seems possible, but would require some non-trivial technical work, to adapt the example from the appendix so that the Hadamard space becomes a smooth Hadamard manifold.

On the other hand, in the case $k = 2$, the result of Heinz Bauschke [11] does admit a generalization; in this case, the algorithm is sufficiently simple to be controlled explicitly, even providing an optimal rate of asymptotic regularity. As it was pointed out by an anonymous referee, the following statement follows from [5, Theorem 3.3], under the additional assumption of the existence of a fixed point of the composition P.

1.2. Proposition. *Let C_1, C_2 be two closed convex subsets of a Hadamard space X. Then the composition $P = P_1 \circ P_2$ is asymptotically regular.*

Moreover, $|x_n - x_{n+1}| = o(\frac{1}{\sqrt{n}})$ for any $x \in X$ and $x_n = P^n(x)$.

Here and further we denote by $|x - y|$ the distance between points x and y in any metric space, even without linear structure.

Examples given by the real axis $C_1 \subset \mathbb{R}^2$ and the set

$$C_2 = \{\, (x, y) : x > 0, y \geqslant 1 + x^{-\varepsilon} \,\}$$

reveal that the convergence rate in Proposition 1.2 cannot be improved to $O(n^{-\frac{1}{2}-\varepsilon})$ for any $\varepsilon > 0$.

This also shows that the optimal rate of asymptotic regularity for cyclic product of projections on two convex subsets is the same for the Euclidean plane and general Hadamard spaces.

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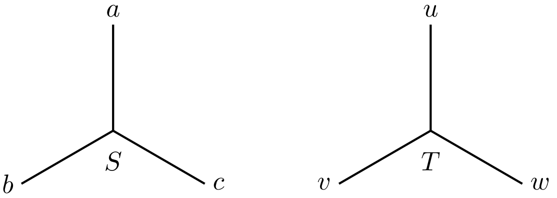
2 Three segments in a product of two tripods

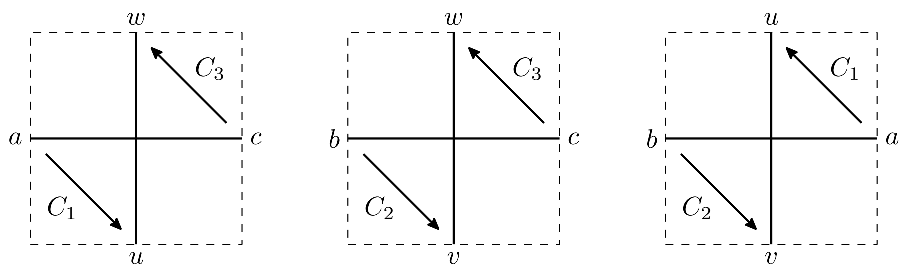
Proof of 1.1. A union of three unit segments that share one endpoint with the induced length metric will be called a *tripod*. Consider two tripods S and T and the product space $X = S \times T$. Our space X is a product of two trees, thus of



two Hadamard spaces. Hence X is a Hadamard space.

Denote by a, b, c and u, v, w the sides of S and T respectively.

The following diagram shows 3 isometric copies of 2×2-square in X; they are obtained as the products of two pairs of sides in S and T as labeled.



Consider the segments C_1, C_2, and C_3 shown on the diagram; they all have slope -1 and project to each other isometrically. Note that each projection P_i reverses the shown orientation. It follows that the composition $P = P_1 \circ P_2 \circ P_3$ sends the segment C_1 to itself isometrically and changes the orientation of the segment. In particular, P exchanges the ends of the segment, hence P is not asymptotically regular. (In fact, for an end e of C_1, and any n, we have $|P^n(e) - P^{n+1}(e)| = 1$.)

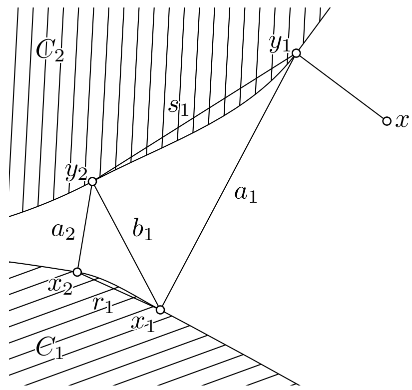
Finally, setting $C_3 = \dots = C_k$ defines examples for any $k \geqslant 3$. ☐

3 Two sets

Proof of 1.2. By definition, $x_n \in C_1$ for all n. Set $y_{n+1} = P_2 \circ P^n(x)$, so $y_1 = P_2(x)$, $x_1 = P_1(y_1)$, $y_2 = P_2(x_1)$, and so on. Further set

$$r_n := |x_n - x_{n+1}|,$$
$$s_n := |y_n - y_{n+1}|.$$

Since the closest-point projection is nonexpanding, we get



$$s_1 \geqslant r_1 \geqslant s_2 \geqslant r_2 \geqslant \dots \qquad \text{❶}$$

Set

$$\begin{aligned} a_n &:= |x_n - y_n| = \operatorname{dist}_{C_1} y_n, \\ b_n &:= |y_{n+1} - x_n| = \operatorname{dist}_{C_2} x_n. \end{aligned}$$

Note that

$$a_1 \geqslant b_1 \geqslant a_2 \geqslant b_2 \geqslant \dots \qquad \text{❷}$$

Since C_1 is convex and $x_n \in C_1$ lies at the minimal distance from y_n, we have $\measuredangle[x_n\,{}^{x_{n-1}}_{y_n}] \geqslant \frac{\pi}{2}$. Since X is a Hadamard space,

$$r_n^2 \leqslant b_n^2 - a_{n+1}^2.$$

Therefore, ❷ implies that

$$\sum_n r_n^2 \leqslant b_1^2.$$

By ❶, r_n is non-increasing. Therefore, $r_n = o(\frac{1}{\sqrt{n}})$. □

Appendix: Three discs

While the cyclic product of projections P constructed in Section 2 is not asymptotically regular, its square P^2 is the identity on C_1, in particular, P^2 *is* asymptotically regular. The construction in Section 2 produces a Möbius band B divided into three rectangles and a map from B to a Hadamard space that is distance-preserving on each rectangle.

In this appendix, we produce a Hadamard space that contains an embedding of a twisted solid torus with arbitrary twisting angle, such that the solid torus consists of 3 isometrically embedded flat cylinders. In this case, we obtain again 3 projections onto convex sets, each of them isometric to a Euclidean disc, the bases of the cylinders. Then the cyclic product of these projections is the rotation of a disc by the prescribed twisting angle α. In particular, if $\frac{\alpha}{\pi}$ is irrational, then any power of this cyclic product of projections may not be asymptotically regular.

A.1. Theorem. *There is a cyclic projection P as in Theorem 1.1 such that any of its power P^m is not asymptotically regular.*

Proof of A.1. Fix an angle α and a small $\varepsilon > 0$. Consider the closed ε-neighborhood A of a closed geodesic γ in the unit sphere $\mathbb{S}^3$. Note that the boundary of A is a saddle surface in $\mathbb{S}^3$; hence it has curvature bounded from above by 1. Thus, A is a compact Riemannian manifold with boundary, such that the curvature of the interior and of the boundary is bounded from above by 1. Therefore, by the result of Stephanie Alexander, David Berg, and Richard Bishop [1], *A equipped with the induced intrinsic metric is locally* CAT(1). The universal cover $\tilde{A}$ of A with its induced metric is locally CAT(1) as well. Since $\tilde{A}$ does not contain closed geodesics, it is CAT(1), by the generalized Hadamard–Cartan theorem [3, 8.13.3], [10, 6.8+6.9], [13].

Denote by E the inverse image of γ in $\tilde{A}$. The isometry group of $\tilde{A}$ contains the group of translations along E and the rotations that fix E. Let T be the composition of translation along E of length $2{\cdot}\pi+10{\cdot}\varepsilon$ and the rotation by angle α. The element T generates a discrete subgroup Γ in the group of isometries of $\tilde{A}$ that acts freely and discretely on $\tilde{A}$.

Set $Y = \tilde{A}/\Gamma$. Since ε is small, any nontrivial element of Γ moves every point of $\tilde{A}$ by more than $2{\cdot}\pi$. Therefore, Y is a compact locally CAT(1) space that does not contain closed geodesics of length less than $2{\cdot}\pi$. Hence, by the generalized Hadamard–Cartan theorem [3], Y is CAT(1). By construction, Y is locally isometric to $\mathbb{S}^3$ outside its boundary B. The projection of E to Y is a closed geodesic G of length $2{\cdot}\pi + 10{\cdot}\varepsilon$.

Denote by X the Euclidean cone over Y; since Y is CAT(1), we get that X is CAT(0); see [3]. Moreover, X is locally Euclidean outside its *boundary* — the cone over B.

The cone Z over the closed geodesic G is the Euclidean cone over a circle of length $2{\cdot}\pi+10{\cdot}\varepsilon$. By construction, Z is a locally convex subset of X. Hence, Z is a convex subset of X [2, 2.2.12]. Let us consider a geodesic triangle $[q_1q_2q_3]$ in Z that surrounds the origin o of the cone Z.

By construction, the sides of the triangle $[q_1q_2q_3]$ lie in the flat part of X. Thus, we can find a small $r > 0$ such that the $2{\cdot}r$-neighborhood U_1 of the geodesic $[q_1q_2]$ is isometric to a convex subset of the Euclidean space. We can assume that $2{\cdot}r$-neighborhoods U_2 of $[q_2q_3]$ and U_3 of $[q_3q_1]$ have the same property.

Denote by C_i the disc of radius r centered at q_i and being orthogonal to Z. By construction, C_i and C_{i+1}, for $i = 1, 2, 3 \pmod 3$ are contained in U_i. Since Z is convex, C_i and C_{i+1} are *parallel* inside U_i, thus their convex hull Q_i is isometric to the cylinder $C_i \times [q_i, q_{i+1}]$ with bottom and top C_i and C_{i+1}. In particular, the projection P_i defines an isometry $C_{i+1} \to C_i$.

By construction, the composition $P = P_1 \circ P_2 \circ P_3 \colon C_1 \to C_1$ rotates C_1 by angle α. If $\frac{\alpha}{\pi}$ is irrational, then P, as well as all its powers, are *not* asymptotically regular.

As before, setting $C_3 = \dots = C_k$ defines examples for any $k \geqslant 3$. ☐

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