A point process theoretic study of Gibbs measures in abstract state spaces

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First of all, I am deeply grateful to my supervisor Günter Last. His timing in offering advice precisely when I needed it, and leaving me be when I was onto something, was simply impeccable. His integrity as a researcher is inspiring and I look forward to the joint work we are yet to undertake.

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I am indebted to my former math teacher Reiner, who sparked my enthusiasm for mathematics and eventually put me onto this journey.

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Prior Publications

Substantial parts of this thesis are based on a prior publication and a preprint which is available on arXiv and submitted to a peer reviewed journal. More specifically, Sections 2.2.2, 2.3 – 2.6, and 2.8 in Chapter 2 as well as Sections 3.1, 3.2, 3.3.1 – 3.3.4, and 3.4 in Chapter 3 largely consist of the preprint

- Betsch, S. (2022). Structural properties of Gibbsian point processes in abstract spaces. Preprint available at arXiv 2201.11415.

The same is true for most of the content of the appendix sections A.1 – A.3 and A.5 – A.6. Moreover, the results in Section 4.4 are stated in this preprint, but not proven in detail. The state of the preprint refers to the date of submission of this thesis, namely June 3, 2022. As the work was under review during the preparation of the thesis, future versions of the publication might differ and include more or less of the results presented here.

Chapter 5 is based on the paper

- Betsch, S. and Last, G. (2022). On the uniqueness of Gibbs distributions with a non-negative and subcritical pair potential. To appear in *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*. Preprint available at arXiv 2108.06303.

New to this thesis, as compared to the previous publications, are Sections 2.7, 3.3.5, and 3.3.6 as well as large parts of the technical details in Chapter 4, and the appendix sections A.4 and A.7 – A.9.

Note that the allocation of the results from the two papers which appear in this thesis, as described above, cannot claim to be accurate to the last quote. This is due to the fact that the papers share similar preliminaries and discuss related topics, which, for the sake of better readability of this thesis, have been merged, extended, or repackaged. As such, preliminary results of Betsch and Last (2022) also appear in earlier chapters while discussions or comments of Betsch (2022) might appear in Chapter 5. Mingling these papers with additional new results makes for a more consistent and streamlined thesis, but hardly allows for a completely concise classification of which results come from which of the author's papers. With this in mind, it should be stated explicitly that direct quotes of the preprint and publication above appear throughout the thesis. The author and his coauthor, and supervisor, Prof. Dr. Günter Last, have contributed equally to all parts of their publication in *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*. The presentation of these results in the thesis at hand had been agree to by Prof. Last.

Contents

Chapter 1

Introduction

The mathematical understanding of random point patterns is essential in spatial statistics, stochastic geometry, and statistical mechanics. Interest in a study of such patterns arises immediately from the many applied scientific fields where they appear, such as astronomy (locations of celestial objects, cf. Gabrielli et al., 2005), ecology (locations of plants or species, cf. Thompson, 1955), epidemiology (occurrence of disease hotspots, cf. Quesada et al., 2017, and their references), geology (earthquake sites, cf. Bray and Schoenberg, 2013), and physics (interacting particles, cf. Ruelle, 1969), to name just a few.

For a probabilistic study of these random point patterns and the development of theoretically founded tools for statistical analysis, it is crucial to scrutinize the mechanism which generates this type of data. In modern probability theory and statistics, any data-generating mechanism is formalized by a probability measure on some suitable state space. While these distributions contain the full information on the associated random quantities, it is usually difficult to cope with the entire structure. Rather than that, one is often content with the study of specific characteristics of the random object in question. For instance, instead of investigating the full distribution of a point pattern, one may focus on the number of points in a given part of the space or on the occurrence of clusters. This requires random elements, which describe a data-generating mechanism, to be amenable to manipulations that reduce the difficulty of the investigation.

The question is, which mathematical objects lead to a neat and practical description of random spatial point patterns? The focus nowadays lies on point processes, which are random counting measures whose atoms are interpreted as points in the space on which the measure lives. This approach leads to models that allow for a clear intuition but also for an extensive mathematical analysis as each realization of a point process is a measure, hence itself a highly structured object.

The simplest way of constructing a point process is by taking independent random points in the state space and combining them to a point pattern. If the number of points thus constructed is itself random and follows a Poisson distribution, the resulting random point pattern is a Poisson process. The central property of a Poisson point process is the strong inherent stochastic independence between its points. From the perspective of statistical mechanics this is usually interpreted as the lack of interactions between points and the Poisson process is called an ideal gas, which is nicely explained by Jansen (2017). In this context it is rather evident why the Poisson process is an idealization, as the fact that particles of a gas should not interact is hardly a realistic assumption. The same can be said for all applications listed above, where the existence of certain interactions between points is intuitively obvious. Once we leave the independence property behind, models might become more realistic, but their mathematical analysis is more involved. A popular model which allows for a wide variety of interactions and dependencies among points is broadly known as a Gibbs process and can be understood, in several ways, as an extension of the Poisson process. As such, Gibbs processes inherit structural properties that pave the way for a probabilistic understanding.

The goal of this thesis is to contribute to the body of theoretical results available for Gibbs processes. The setting and level of the following introductory part are chosen such that they enable a mathematical reader who is not familiar with Gibbs processes to get a very rough impression of these objects. For a transition to the actual framework of the thesis we conclude the introduction with a few words that relate to the much more general setting. While most of the relevant literature is provided in the main body of the thesis, where references can be classified properly, the literature which is central to the topics of the introduction is mentioned right away.

A first rigorous treatment of point processes in the wider sense is due to Wiener and Wintner (1943) and the notion of point processes as random counting measures goes back to Moyal (1962). To have this notion at hand for the remainder of the introduction, consider as a state space $\mathbb{R}^d$, endowed with the Lebesgue measure $\mathcal{L}^d$, and let $\mathbf{N}^d$ be the collection of all *counting measures* $\sum_j \delta_{x_j}$ on $\mathbb{R}^d$ whose points do not accumulate. Note that the points x_j are in $\mathbb{R}^d$ and that $\delta_{x_j} = \mathbb{1}\{x_j \in \cdot\}$ is the Dirac point measure in x_j. Without addressing issues of measurability right now, we call a random element η of $\mathbf{N}^d$ a *point process* in $\mathbb{R}^d$. That these objects provide a valid model for the structures described before might get more intuitive when noting that each point process η may be written as

$$\eta = \sum_{j=1}^{\eta(\mathbb{R}^d)} \delta_{X_j}$$

for some random elements $X_1, X_2, \ldots$ of $\mathbb{R}^d$ and, conversely, any collection of random points in $\mathbb{R}^d$ (which do not accumulate) can be combined to a point process accordingly.

As mentioned previously, the simplest such process arises from taking a finite number of independent random points $X_1, \ldots, X_m$, distributed uniformly over a bounded subset of $\mathbb{R}^d$, and considering the *binomial process* $\Phi = \sum_{j=1}^m \delta_{X_j}$. If m is not a fixed integer but a Poisson random variable, the resulting *mixed binomial process* is also called a *Poisson process* or *continuous chaos*, where our presentation follows Last and Penrose (2017) but the latter term was coined by Wiener (1938). Such a process has strong independence properties, namely for any disjoint Borel sets $B_1, B_2 \subset \mathbb{R}^d$, the number of Poisson points in B_1 and B_2, that is, the random variables $\Phi(B_1)$ and $\Phi(B_2)$, are stochastically independent. A (homogeneous) Poisson process with intensity $\gamma \geq 0$ on the full space is a point process Φ such that, for any Borel set $B \subset \mathbb{R}^d$, $\Phi(B)$ is a Poisson random variable with parameter $\gamma \cdot \mathcal{L}^d(B)$, and such that $\Phi(B_1)$ and $\Phi(B_2)$ are independent for all disjoint Borel sets $B_1, B_2 \subset \mathbb{R}^d$. Part of a realization of a Poisson process is shown in Figure 1.1.

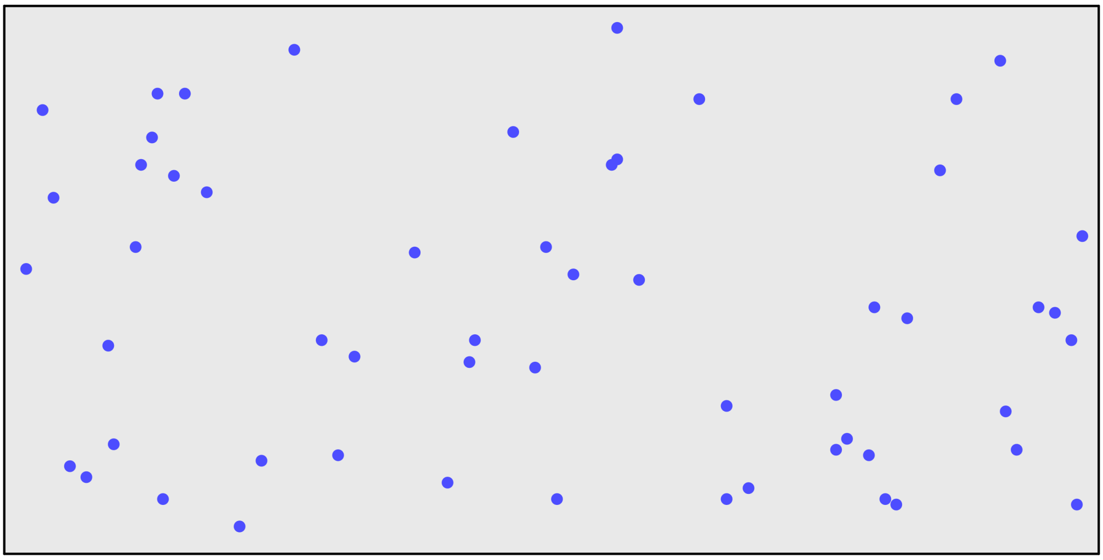


Figure 1.1.: Realization of a homogeneous Poisson process in $[0, 2] \times [0, 1]$ with intensity $\gamma = 30$.

In line with the independence property inherent in continuous chaos, a Poisson process on the full space, also called *continuous homogeneous chaos*, is constructed by taking a (Poisson-)mixed binomial process on each portion of a partition of $\mathbb{R}^d$ into bounded windows, with all of these processes stochastically independent, and taking their sum. This construction goes back to Moyal (1962) and guarantees that Poisson processes on the full space exist.

An essential feature of Poisson processes is that we may reshuffle its points in any part of the space without changing the distribution. More specifically, if Φ and Φ' are two independent Poisson processes with intensity γ and $B \subset \mathbb{R}^d$ is a Borel set, then by the superposition principle (writing B^{c} for the complement of B)

$$\Phi_{B^{\mathsf{c}}} + \Phi'_B = \Phi(\,\cdot \cap B^{\mathsf{c}}) + \Phi'(\,\cdot \cap B)$$

is again a Poisson process with intensity γ. This construction corresponds to removing the points of Φ in B and replacing them with new Poisson points given through Φ', as illustrated in Figure 1.2. In terms of the distribution of a Poisson process with intensity γ restricted to B, which we denote by $\Pi_{B,\gamma}$ (as a probability measure on $\mathbf{N}^d$), this property may be rewritten as

$$\mathbb{P}(\Phi \in \cdot\,) = \mathbb{E}\Big[\int_{\mathbf{N}^d} \mathbb{1}\{\Phi_{B^{\mathsf{c}}} + \mu \in \cdot\,\} \,\mathrm{d}\Pi_{B,\gamma}(\mu) \Big]. \tag{1.1}$$

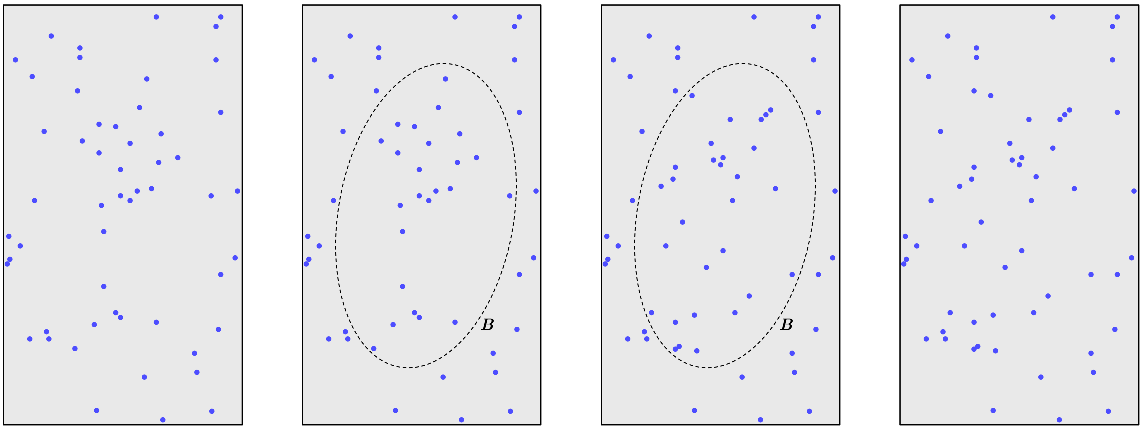


Figure 1.2.: Illustration of the reshuffling. From left to right: the Poisson points inside B are simulated anew while the points outside B remain.

A second characteristic of Poisson processes is the formula due to Mecke (1967), stating that a homogeneous Poisson process Φ with intensity γ satisfies, and is in fact characterized by,

$$\mathbb{E}\Big[\int_{\mathbb{R}^d} f(x, \Phi) \,\mathrm{d}\Phi(x) \Big] = \gamma \int_{\mathbb{R}^d} \mathbb{E}\big[f(x, \Phi + \delta_x) \big] \,\mathrm{d}\mathcal{L}^d(x)$$

for all measurable functions $f : \mathbb{R}^d \times \mathbf{N}^d \to [0, \infty)$. Heuristically, the Mecke equation contains the independence of $\Phi_{\{x\}^{\mathsf{c}}}$ and $\Phi(\mathrm{d}x)$, together with the fact that $\Phi(\{x\}) = 0$ almost surely. The formula is a useful tool in calculations involving Poisson integrals and is one reason why probabilistic models which are based on spatial Poisson processes allow for the derivation of many explicit results and properties. Both the reshuffling property and a Mecke type formula also play a huge role in the study of Gibbs processes.

As the strong independence assumptions inherent in a Poisson process are too restrictive for practical models, more complicated processes which allow for dependencies or interactions between points need to be considered. For a motivation of the initial definition of such models, recall first that a standard procedure for the construction of continuous probability distributions is to consider measures which have a probability density function with respect to the Lebesgue measure. Intuitively, the Lebesgue measure corresponds to a uniform distribution, disregarding any specific structure in the random quantity, and the density function corresponds to a reallocation of probability mass in the sense that certain outcomes become more or less probable, or are excluded entirely. In line with these constructions, we take the distribution of a Poisson process as a reference measure on $\mathbf{N}^d$ and reallocate probability mass according to a density function. As soon as these density functions are given in terms of a so-called energy function, or Hamiltonian, we call the resulting distribution a *Gibbs measure*, named after the mathematical physicist Josiah Willard Gibbs (1884, 1902). Even tough we later study general interactions between points in arbitrary state spaces, let us stick for now to Euclidean space and limit ourselves to repulsive pair interactions. To specify the construction, assume that the entirety of interactions in a point configuration (called the energy) is obtained by only taking into account every pair of two points in the configuration, with their interaction being determined by a symmetric function $v : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$. Thus, the energy of a configuration consisting of points $x_1, x_2, \ldots \in \mathbb{R}^d$ is given by

$$H\Big(\sum_j \delta_{x_j}\Big) = \sum_{i<j} v(x_i, x_j),$$

following Ruelle (1969). The function H is called the *energy function*. An extension thereof, the so-called *Hamiltonian*, takes into account preexisting boundary points $y_1, y_2, \ldots$ with which each of the x_j may also interact in the sense that the overall energy is

$$H\Big(\sum_j \delta_{x_j}, \sum_k \delta_{y_k}\Big) = \sum_{i<j} v(x_i, x_j) + \sum_{j,k} v(x_j, y_k).$$

On a fixed window $B \subset \mathbb{R}^d$, and for some boundary configuration $\psi \in \mathbf{N}^d$, we follow Georgii (2011) in considering the measure

$$\int_{\mathbf{N}^d} \mathbb{1}\{\mu \in \cdot\,\} \, \mathrm{e}^{-H(\mu,\,\psi)} \, \mathrm{d}\Pi_{B,\gamma}(\mu).$$

Whenever this measure assigns finite mass to the full space $\mathbf{N}^d$, that is,

$$Z_{B,\gamma}(\psi) = \int_{\mathbf{N}^d} \mathrm{e}^{-H(\mu,\,\psi)} \, \mathrm{d}\Pi_{B,\gamma}(\mu) < \infty,$$

we normalize the measure and call the resulting probability law a Gibbs measure on B with boundary configuration ψ and activity $\gamma \geq 0$. A point process which is distributed according to the normalized measure is then called a Gibbs process in B with boundary condition ψ and activity γ. To get a feeling for this construction, consider the *hard core* pair interaction

$$v(x, y) = \begin{cases} \infty, & |x - y| \leq R, \\ 0, & |x - y| > R, \end{cases}$$

where $R > 0$. If $\mu = \sum_j \delta_{x_j}$ and $\psi = \sum_k \delta_{y_k}$, then $H(\mu, \psi) = \infty$ as soon as two points x_i and x_j have distance less than R or one of the x_j has distance less than R from one of the y_k. If no such instance occurs, then $H(\mu, \psi) = 0$. In the first case, the corresponding density $\mathrm{e}^{-H(\mu,\,\psi)}$ is 0, so these configurations cannot be realized in the Gibbs process. In the second case, the density function of the Gibbs measure is 1 and we are back in the Poisson world. In other words, no two points of the Gibbs process can be closer to one another than distance R and none of the Gibbs points can lie closer than R to any of the boundary points. Other than these, no interactions are taken into account. We can also interpret such a process as a Poisson process conditioned on all points having at least distance R from one another, as suggested by Figure 1.3. Hence, the reallocation of probability mass is realized by excluding configurations and redistributing the corresponding mass under a Poisson process evenly over all remaining Poisson configurations.

For most relevant choices of v, in particular for the previous example, the normalization of the measure $\mathrm{e}^{-H(\mu,\,\psi)}\mathrm{d}\Pi_{B,\gamma}(\mu)$ is not possible when considered on the full space, as the normalization constant is 0. In the example above this is due to the fact that a Poisson process on the full space will have two points which lie closer to each other than distance R with probability one. Hence the approach of considering density functions with respect to a Poisson process is impractical for the definition of infinite volume Gibbs processes. However, with the above understanding of finite Gibbs processes, it is apparent that certain structural properties of the underlying Poisson process are inherited. In fact, any pair interaction Gibbs process η on some bounded set B

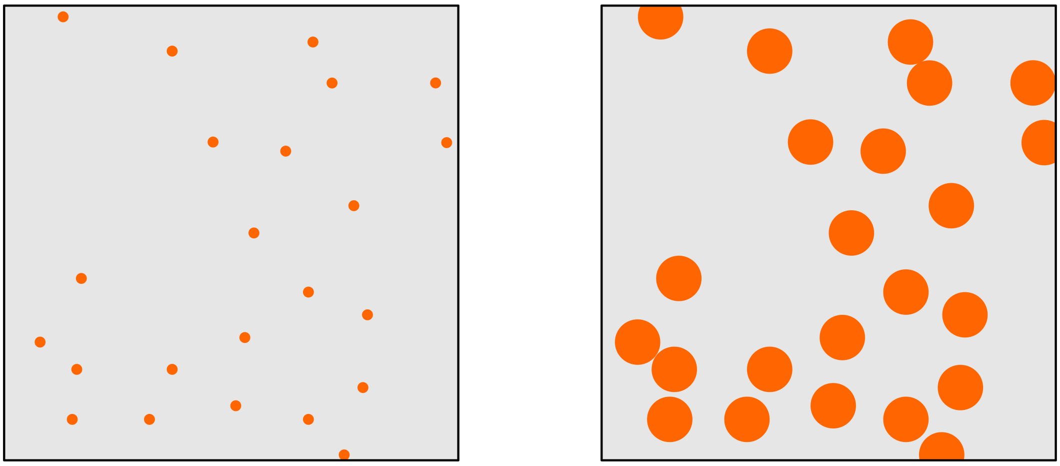


Figure 1.3.: Realization of a hard core process in $[0,1] \times [0,1]$ with intensity $\gamma = 100$ and $R = 0.1$ (left). The impenetrable core of radius $R/2$ of each point is printed for better visualization (right).

with empty boundary condition and activity γ satisfies

$$\mathbb{E}\Big[\int_{\mathbb{R}^d} f(x,\eta)\,\mathrm{d}\eta(x)\Big] = \gamma \int_{\mathbb{R}^d} \mathbb{E}\big[f(x,\eta+\delta_x)\,\kappa(x,\eta)\big]\,\mathrm{d}\mathcal{L}^d(x)$$

for all measurable functions $f : \mathbb{R}^d \times \mathbf{N}^d \to [0,\infty)$, where

$$\kappa(x,\psi) = \mathrm{e}^{-H(\delta_x,\psi)} = \exp\Big(-\int_{\mathbb{R}^d} v(x,y)\,\mathrm{d}\psi(y)\Big)$$

is called the *Papangelou intensity* of η, named after Papangelou (1974). This generalization of the Mecke equation is attributed to Georgii (1976) and Nguyen and Zessin (1979) and called the *GNZ equation*. For a Poisson process, where we have no interactions ($v \equiv 0$), the κ-term disappears ($\kappa \equiv 1$) in accordance with the Mecke formula. Once interactions are allowed, the definition of κ leads to the heuristic understanding that $\kappa(x,\eta)$ provides an indication of how likely it is to find a Gibbs point in x given the points of η. In a sense, both the density function $\mathrm{e}^{-H(\,\cdot\,,\psi)}$ in the definition of the finite Gibbs process as well as the function κ in the GNZ equation capture the deviation from a Poisson process. Note that the GNZ equation formally makes sense for any point process η without reference to some bounded window B and without any issues concerning the normalization of a measure on $\mathbf{N}^d$. We thus actually choose the GNZ equation as a definition of a Gibbs process with Papangelou intensity κ. With this definition, however, it is still not clear if infinite volume Gibbs measures exist or how they can be constructed, as the GNZ equation in itself is only an integral equation without an obvious solution.

The first impulse for constructing an infinite volume Gibbs process, namely proceeding as in the Poisson case, is bound to fail. Since points interact, the occurrence or missing of points in some part of the space influences the occurrence or missing of points in separate parts of the space, so we cannot simply produce finite Gibbs processes on each set of a partition of the space and glue them together, at least not to the effect that the resulting process is again a Gibbs process. However, another intuitive approach leads to success. Namely, as illustrated in Figure 1.4, one takes a sequence of finite volume Gibbs process $\xi_1, \xi_2, \dots$ on bounded windows

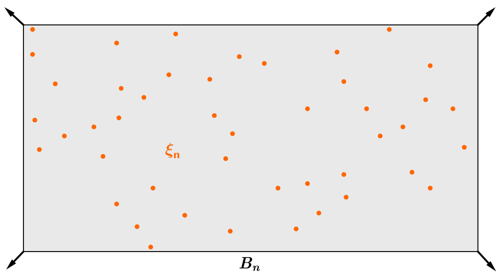


Figure 1.4.: Realizations of finite volume Gibbs processes ξ_n on suitable sets B_n.

$B_1 \subset B_2 \subset \ldots$ with $\bigcup_{n=1}^{\infty} B_n = \mathbb{R}^d$, extracts a subsequence which converges in an appropriate sense, and shows that the limit process still satisfies the GNZ equation and is therefore a Gibbs process on the full space. In considering such a limit construction on a formal level, the strength of the interactions has to decay in a controlled way. For the descriptive setting of repulsive pair interactions chosen in this introduction, a rather weak integrability condition on v is sufficient. Indeed, if

$$\int_{\mathbb{R}^d} \left(1 - \mathrm{e}^{-v(x,y)}\right) \mathrm{d}\mathcal{L}^d(y) < \infty$$

for $\mathcal{L}^d$-almost every $x \in \mathbb{R}^d$, then a pair interaction process on the full space exists according to Ruelle (1970). Note that this integrability property is satisfied by any pair potential with a finite interaction range, as in the explicit example above.

With general Gibbs processes defined via an extension of the Mecke equation and their existence guaranteed by a construction via finite volume Gibbs processes based on an underlying Poisson process, it is natural to ask what other properties a Gibbs process inherits or generalizes from a Poisson process. One insightful and graphic property is related to the reshuffling described earlier. Again, it should be clear that it is not to be expected that such a property holds without changes. If we remove the points of a Gibbs process in some part B of the space, the remaining Gibbs points still exert influence on what can possibly happen inside B. Intuitively, a reshuffling of the points in B has to take into account the points which exist outside B. With the previously introduced notion of boundary conditions this can be formalized to the statement that any Gibbs process η with pair interaction v and activity γ is characterized by the identity

$$\mathbb{P}(\eta \in \cdot\,) = \mathbb{E}\left[\frac{1}{Z_{B,\gamma}(\eta_{B^c})} \int_{\mathbf{N}^d} \mathbb{1}\{\mu + \eta_{B^c} \in \cdot\,\}\, \mathrm{e}^{-H(\mu,\, \eta_{B^c})}\, \mathrm{d}\Pi_{B,\gamma}(\mu)\right]$$

being satisfied for all bounded windows $B \subset \mathbb{R}^d$. The normalized integral on the right hand side is essentially a Gibbs process in B with pair interaction v, activity γ, and boundary condition η_{B^c}, which is added to the points of η_{B^c}. This corresponds to a reshuffling of the points in B, taking the (expected) influence of the Gibbs points outside of B into account, which is graphically shown in Figure 1.5. If there are no interactions ($H \equiv 0$) we end up with the Poisson setting in (1.1), as anticipated. The formula in the previous display is of extraordinary interest. In statistical physics it is one of the central characteristics of Gibbs measures and, in appreciation of Dobrushin (1968b,c, 1969), Lanford and Ruelle (1969), and Ruelle (1970), called the DLR equation.

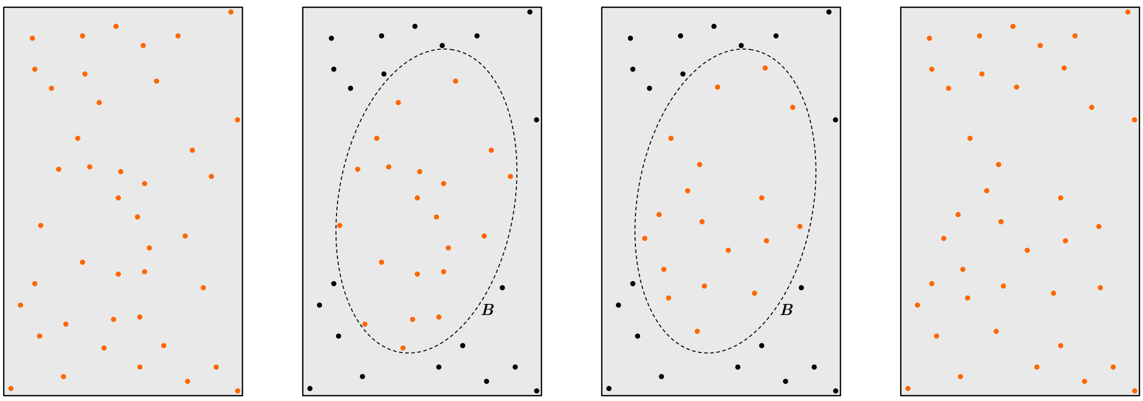


Figure 1.5.: Illustration of the DLR equation. From left to right: the Gibbs points inside B are simulated anew, but the existing points outside B are taken into account as a boundary condition.

Once the question of existence of solutions to the GNZ equation, hence the existence of Gibbs processes, is settled, the natural mathematical follow-up question is whether the solution is unique. Even in the rather simple setting we consider right now, this question is notoriously difficult. The general belief is that Gibbs measures are unique for sufficiently small activities, but a transition occurs as the activity increases. It is possible to conjecture these results in many specific models as they correspond to physical phenomena like liquid-vapor phase transitions or magnetization, as summarized by Georgii (2011). While rigorous mathematical results for the phase transition are sparse and exist only for very few models, previous years have at least seen an improvement of lower bounds on the activity at which a phase transition may occur as well as a better understanding of the uniqueness problem for Gibbs measures, cf. Jansen (2019), Houdebert and Zass (2022), and Michelen and Perkins (2021a) to mention just the most recent ones. The main result of the thesis at hand is a further contribution to this endeavor which comes with a probabilistically intuitive interpretation. Namely, we consider the function $\varphi_v = 1 - \mathrm{e}^{-v}$, which takes values in $[0, 1]$ as v is non-negative, and we connect any two points $x, y \in \Phi$ of a Poisson process Φ with probability $\varphi_v(x, y)$, independently for all pairs of points. The resulting random graph is called the *random connection model* based on Φ and φ_v. A graphical illustration is provided in Figure 1.6. Regarding the construction of the random connection model in dependence on the intensity of the Poisson process, it is known from Penrose (1991) that there exists a critical intensity γ_c such that for any intensity below γ_c the random connection model has no infinite connected component (with probability one). In this case, the random connection model is said to be subcritical. Our main result states that the Gibbs measure with pair interaction v is unique as soon as the random connection model based on φ_v is subcritical. The proof of the result is based on a coupling construction and technically involved, but the result itself allows for an interpretation. Taking the pair interaction process as being embedded into the Poisson process used to construct the random connection model, an existing interaction between points may establish a mutual connection depending on the strength of the interaction, with stronger interactions leading to higher connection probabilities. Then, in order for the Gibbs measure to be unique, the range of the interaction that any given point can exhibit to other points via its clusters has to be controlled suitably.

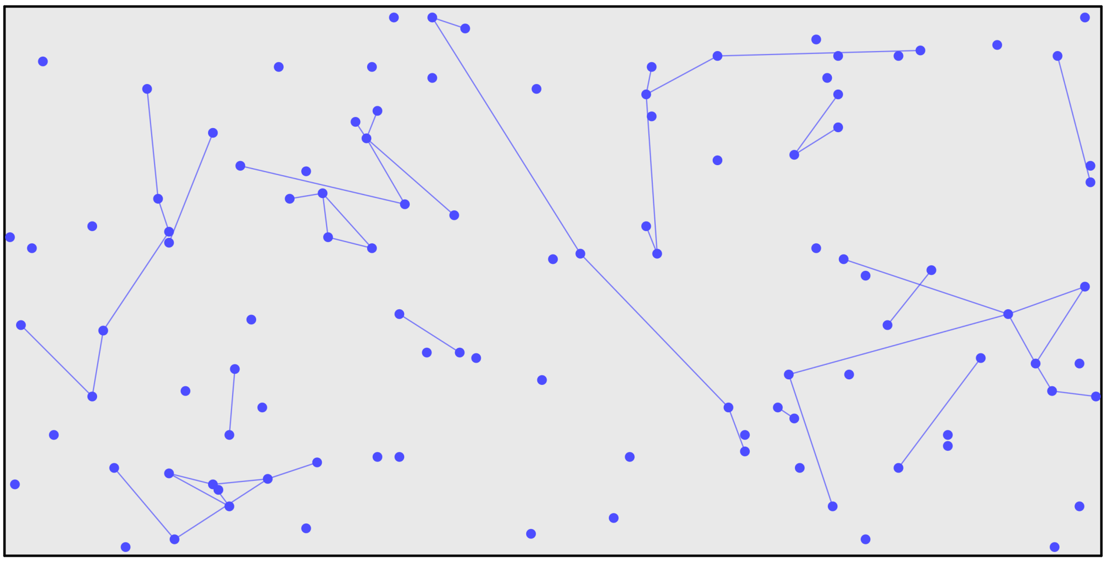


Figure 1.6.: Excerpt of a realization of a random connection model based on a Poisson process with intensity $\gamma = 50$ and the connection function $\varphi(x, y) = \exp(-10 \cdot |x - y|)/3$. Depicted are only the Poisson points within $[0, 2] \times [0, 1]$ and their connections among each other.

The selected references which appear above already indicate that some general understanding of Gibbs measures compiled during the 60s and 70s. Despite this fact, Chapter 2 contains a self-contained study of the basic concepts regarding Gibbs processes, the novelty being that these notions are considered in arbitrary measurable spaces, disregarding topological and geometric structures of the underlying state space completely. This allows to understand how some properties of Gibbs measures are purely structural and generic. In Chapter 3 general existence and uniqueness results, as well as decorrelation and mixing properties, for Gibbs processes are established, still focusing on abstract state spaces. In particular, both Chapters 2 and 3 include higher order potentials which can be attractive, not just repulsive pair interactions, and the results require no stationarity assumptions. Chapter 4 contains essentials on, and an existence result for, pair interaction processes which are derived from the general considerations in Chapter 3 and provide an overlying structure for results from the existing literature. Similar to Chapter 2, Chapter 4 emphasizes the fact that the whole study is undertaken in very abstract state spaces, thus untying the above concept of pair interaction processes from $\mathbb{R}^d$. This should come as no surprise, since the given definition of pair potential Gibbs measures is generic and does not refer to the specific geometry of Euclidean space, or to any topological properties (as assumed in the existing literature) for that matter. Lastly, but most importantly, Chapter 5 makes precise the uniqueness result as summarized above. While the restriction to repulsive pair interactions is essential to even make sense of the required random connection model, we can again detach completely from any geometric assumptions on the state space and work on arbitrary separable metric spaces endowed with a reference measure which may include both continuous and discrete parts. As such, the uniqueness result includes both the discrete and continuous setting, it allows for pair potentials which are not invariant under translations, and does not require the interactions to be finite in their range. More detailed introductions to each of the four parts are given at the beginning of the respective chapter.

Note that all pictures which appear in the thesis were generated by the author. The depicted point processes were simulated according to Section 3.2.3 and Algorithm 7.4 of Møller and Waagepetersen (2004) in Python 3.7 (referring to van Rossum and Drake, 2009, for a manual). The resulting points were then transferred into the graphics tool of Geogebra (cf. Hohenwarter, 2002) to produce nicer pictures.

CHAPTER 2

General Gibbs processes in abstract state spaces

The chapter is structured as follows. We first summarize, in Section 2.1, which mathematical preliminaries are required for the thorough understanding of the thesis. In Section 2.2 we recall basics from point process theory. It is not yet standard to consider these objects in arbitrary measurable spaces but the book by Last and Penrose (2017) provides a valuable reference. We also discuss some properties of Poisson processes which are required throughout the manuscript. These preliminaries are mostly provided as a reminder and to fix notation, but not to discuss full proofs. A little more emphasize is given to Janossy and factorial moment measures, where older results are generalized and presented with state of the art proofs.

In Section 2.3 we define Gibbs processes using the so-called GNZ equations, thus specifying such a process in terms of a Papangelou intensity which, intuitively, determines the probability of finding a Gibbs point in some specific location given all other points. Moreover, we discuss immediate consequences of this definition. Section 2.4 is devoted to a brief derivation of the Janossy densities and correlation functions of Gibbs processes and in Section 2.5 we study finite Gibbs processes, a setting which allows for an alternative and explicit representation of Gibbs measures. In Section 2.6 we tackle the most pressing issue which arises when considering Gibbs processes in general measurable spaces, where counting measures cannot be written as sums of Dirac measures and hence the interacting points of the process cannot be identified within the state space. Namely, we answer the question how Hamiltonians can be defined when the points of a configuration cannot be specified explicitly. Section 2.7 contains a characterization of finite Gibbs processes in terms of absolute continuity with respect to a Poisson process distribution. This last interpretation of finite Gibbs processes is usually stated as their definition in spatial statistics. Finally, we prove, in Section 2.8, the DLR equations that provide another characterization of Gibbs processes which usually serves as their definition in statistical mechanics. Formally we generalize the DLR characterization to arbitrary measurable state spaces but there are no novel techniques involved.

2.1. Required preliminaries

Naturally, this thesis cannot start from zero in every respect but requires some previous knowledge in measure and probability theory as well as in topology. We use this section as a short primer to make the reader aware of what is required for a thorough understanding of the thesis.

First, and foremost, the reader requires a sound knowledge of the basics from measure theory. This includes the concepts of families of sets, like Dynkin systems, rings, fields, and σ-fields as well as their mutual relations and operations involving these objects. Throughout, we need to work with measure spaces, measurable functions, product spaces, product measures, and absolute continuity. We also work with the extended reals and recall the handling of "∞" in Section 2.2. This last point is essential as we rely extensively on the integration of non-negative measurable functions with respect to some measure, particularly allowing for the value ∞ in both the function and the integral. Needless to say, understanding the construction of these integrals, defining them first for indicator functions, then for simple functions, and extending the notion to measurable functions by monotone approximation with simple functions, is essential. We also put to use major tools of integration theory like Fubini's theorem, Fatou's lemma, monotone and dominated convergence, and we work with L^p-spaces. Our standard references for measure theory are the books by Bogachev (2007a) and Cohn (2013).

The second major topic, where basic previous knowledge is essential, is measure theoretic probability. In particular, the reader should be familiar with the notions of probability spaces, random variables, probability distributions, expectations, stochastic independence, conditional expectations and conditional distributions, and weak convergence. For these preliminaries we refer to Chapters 3, 4, and 6 of Kallenberg (2002).

The thesis also requires very basic knowledge of topology, especially metric topology. Key concepts for us are topological and metric spaces and their properties. We also use that in metric spaces compactness and sequential compactness are equivalent, as well as several similar facts. Moreover, we rely, at some points, on properties of continuous functions and constructions involving the product topology. Also, the reader should be familiar with the fundamentals of the interplay between topology and measure theory.

Based on these preliminaries, which are not repeated here, the thesis starts with a short reminder of the basics from point process theory and, from thereon, tries to be as self-contained as possible (and reasonable). Since the point process theoretic perspective of Gibbs processes is presented in uncommon (and often unprecedented) abstraction, it is natural to start from the very definition and be precise and comprehensive in every detail. We try to do the same with all technical results from measure and probability theory or functional analysis that cannot be assumed to be in every reader's repertoire. Such results are either detailed completely in the appendix, in cases where they are adapted or generalized, or thoroughly referenced.

Of course the above list of preliminaries is not comprehensive, both in the sense that measure theory, probability theory, and topology are also based on even more basic mathematical concepts and in the sense that we sometimes use results which are not stated explicitly in the list. It is certainly difficult to determine where to draw the line between required knowledge and the entry point of the thesis, but the section at hand should enable the reader to roughly classify where this thesis starts in terms of required previous knowledge.

2.2. General notation and basics from point process theory

In this whole chapter we consider the most general setting for a state space, namely we let $(\mathbb{X}, \mathcal{X})$ be a measurable space which is assumed to be *localized*, meaning there exists a nested sequence of sets, $B_1 \subset B_2 \subset \dots$, in $\mathcal{X}$ such that $\bigcup_{j=1}^{\infty} B_j = \mathbb{X}$. We define the *bounded* sets $\mathcal{X}_b$ as the collection of all sets in $\mathcal{X}$ which are contained in one of the B_j. The collection $\mathcal{X}_b$ constitutes a ring over $\mathbb{X}$ with $\sigma(\mathcal{X}_b) = \mathcal{X}$. A measure λ on $\mathbb{X}$ is called *locally finite* if $\lambda(B_j) < \infty$ for all $j \in \mathbb{N}$. We write $\mathcal{X}^{\otimes m}$ for the m-fold product σ-field on $\mathbb{X}^m$ and λ^m for the m-fold product measure, where $m \in \mathbb{N}$.

We denote by $\mathbf{N} = \mathbf{N}(\mathbb{X})$ the set of all counting measures, that is, the set of all measures μ on $\mathbb{X}$ for which $\mu(B) \in \mathbb{N}_0$ for each $B \in \mathcal{X}_b$. The space $\mathbf{N}$ is endowed with the σ-field $\mathcal{N}$ generated by the evaluation maps $\pi_B : \mu \mapsto \mu(B)$, $B \in \mathcal{X}$. In other words, $\mathcal{N}$ is the σ-field generated by the sets

$$\big\{\mu \in \mathbf{N} : \mu(B) = k\big\}, \quad B \in \mathcal{X},\, k \in \mathbb{N}_0.$$

We denote by $\mathbf{N}_f = \mathbf{N}_f(\mathbb{X})$ the set of all finite counting measures and by $\mathbf{N}_B = \mathbf{N}_B(\mathbb{X})$ the set of all $\mu \in \mathbf{N}$ with $\mu(B^c) = 0$, where $B^c = \mathbb{X} \setminus B$ denotes the complement of $B \in \mathcal{X}$. Moreover, we denote by $\nu_B = \nu(\,\cdot \cap B)$ the restriction of a measure ν on $\mathbb{X}$ to the set $B \in \mathcal{X}$.

A *point process* in $\mathbb{X}$ is a random element η of $\mathbf{N}$ defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By definition of $\mathbf{N}$, any realization $\eta(\omega)$, with $\omega \in \Omega$, constitutes a locally finite measure on $\mathbb{X}$. For a measurable function $f : \mathbb{X} \to [0, \infty]$ we denote by

$$\int_{\mathbb{X}} f(x)\, \mathrm{d}\eta(x) \tag{2.1}$$

the $[0, \infty]$-valued random variable whose realizations arise from integrating f with respect to $\eta(\omega)$. Measurability of the random variable in (2.1) is guaranteed, for instance, by Lemma A.4. If $f = \mathbb{1}_D$, the indicator function of a set $D \in \mathcal{X}$, we write $\eta(D)$ for the corresponding $(\mathbb{N}_0 \cup \{\infty\})$-valued random variable. Throughout this work we adopt the usual ∞-conventions, that is, $\infty + c = c + \infty = \infty$ for all $c \in (-\infty, \infty]$ and $\infty \cdot c = c \cdot \infty = \infty$ for all $c \in (0, \infty]$, as well as $\infty \cdot c = c \cdot \infty = -\infty$ for all $c \in [-\infty, 0)$. We also use the standard measure theory convention $\infty \cdot 0 = 0 \cdot \infty = 0$, and put $\mathrm{e}^{-\infty} = 0$ as well as $\log(0) = -\infty$. Other than these, we do not define any terms involving $+\infty$ or $-\infty$.

By Proposition 2.10 of Last and Penrose (2017), two point processes η and η' have the same distribution if, and only if, the vectors

$$\big(\eta(D_1), \dots, \eta(D_m)\big) \quad \text{and} \quad \big(\eta'(D_1), \dots, \eta'(D_m)\big)$$

have the same distribution for all $D_1, \dots, D_m \in \mathcal{X}$ and $m \in \mathbb{N}$. In fact, it suffices to consider pairwise disjoint sets.

The intensity measure of a point process η is the measure

$$\mathbb{E}[\eta](B) = \mathbb{E}\big[\eta(B)\big], \quad B \in \mathcal{X},$$

and Campbell's theorem yields

$$\mathbb{E}\bigg[\int_{\mathbb{X}} f(x)\,\mathrm{d}\eta(x)\bigg] = \int_{\mathbb{X}} f(x)\,\mathrm{d}\mathbb{E}[\eta](x)$$

for all measurable functions $f : \mathbb{X} \to [0, \infty]$. For a reference, see Proposition 2.7 of Last and Penrose (2017).

A point process η is called *proper* if there exist random elements $X_1, X_2, \dots$ of $\mathbb{X}$ such that

$$\eta = \sum_{j=1}^{\eta(\mathbb{X})} \delta_{X_j} \quad \mathbb{P}\text{-almost surely (a.s.)},$$

where $\delta_x = \mathbb{1}\{x \in \cdot\,\}$ denotes the Dirac measure in $x \in \mathbb{X}$, using that $x \mapsto \delta_x \in \mathbf{N}$ is trivially measurable. If $\eta(\mathbb{X}) = 0$ we interpret the sum as the zero-measure $\mathbf{0} \in \mathbf{N}$ on $\mathbb{X}$. Note that in measurable state spaces, point processes are generally not proper. This is in line with the fact that a measure $\mu \in \mathbf{N}$ can, in general, not be written as a sum of Dirac measures (cf. Exercise 2.5 of Last and Penrose, 2017). If, however, $(\mathbb{X}, \mathcal{X})$ is a Borel space, in the sense that there exists a bi-measurable bijection ι from $\mathbb{X}$ onto a Borel subset of $[0, 1]$, then there exist measurable maps $\zeta_1, \zeta_2, \dots : \mathbf{N} \to \mathbb{X}$ such that

$$\mu = \sum_{j=1}^{\mu(\mathbb{X})} \delta_{\zeta_j(\mu)}, \quad \mu \in \mathbf{N}.$$

This is virtually proven in Chapter 6.1 of Last and Penrose (2017) or in Lemma 1.6 of Kallenberg (2017). Therefore, point processes in localized Borel spaces are always proper.

Whenever $(\mathbb{M}, \mathcal{M}, Q)$ is a finite measure space, we endow the product space $(\mathbb{X} \times \mathbb{M}, \mathcal{X} \otimes \mathcal{M})$ with the localizing structure $B_1 \times \mathbb{M}, B_2 \times \mathbb{M}, \dots$. We then define the space $\mathbf{N}(\mathbb{X} \times \mathbb{M})$ analogously to $\mathbf{N} = \mathbf{N}(\mathbb{X})$ and consider point processes in $\mathbb{X} \times \mathbb{M}$ as random elements of $\mathbf{N}(\mathbb{X} \times \mathbb{M})$. As usual, these processes can be interpreted as marked point processes in $\mathbb{X}$ with mark space $\mathbb{M}$. In Chapters 2 – 4 we always use the shorthand notation $\mathbf{N}$ for $\mathbf{N}(\mathbb{X})$. In Chapter 5, however, we have to distinguish very carefully between point processes in $\mathbb{X}$ and $\mathbb{X} \times \mathbb{M}$, so we will use the more precise notation $\mathbf{N}(\mathbb{X})$.

2.2.1. The Poisson process

Fix any locally finite measure λ on $\mathbb{X}$. A point process Φ in $\mathbb{X}$ is called a Poisson process with intensity measure λ if $\Phi(B)$ is Poisson-distributed with parameter $\lambda(B)$, for every $B \in \mathcal{X}$, and $\Phi(D_1), \dots, \Phi(D_m)$ are independent for all pairwise disjoint sets $D_1, \dots, D_m \in \mathcal{X}$ and every $m \in \mathbb{N}$. Note that if $\lambda(B) = 0$ (or $\lambda(B) = \infty$) then $\Phi(B)$ being Poisson-distributed with parameter $\lambda(B)$ means $\Phi(B) = 0$ (or $\Phi(B) = \infty$) almost surely. It follows immediately that λ is indeed the intensity measure of Φ and that, for any $B \in \mathcal{X}$, we have $\Phi(B) < \infty$ almost surely if, and only if, $\lambda(B) < \infty$.

If Φ is a Poisson process with intensity measure λ and $D_1, D_2, \ldots \in \mathcal{X}$ are pairwise disjoint, then $\Phi_{D_1}, \Phi_{D_2}, \ldots$ are independent Poisson processes with intensity measures $\lambda_{D_1}, \lambda_{D_2}, \ldots$, respectively, by Theorem 5.2 of Last and Penrose (2017).

We write Π_λ for the distribution of a Poisson process with intensity measure λ, with Π_λ thus being a probability measure on $\mathbf{N}$ with

$$\Pi_\lambda = \mathbb{P}^\Phi = \mathbb{P}(\Phi \in \cdot\,).$$

Recall from Theorem 3.6 and Proposition 3.2 of Last and Penrose (2017) that a Poisson process in $\mathbb{X}$ with intensity measure λ exists and that the distribution of a Poisson process is uniquely determined by its intensity measure. The following result due to Mecke (1967) generalizes Campbell's theorem and characterizes the Poisson process. We refer to Section 4.1 of Last and Penrose (2017) for a proof.

Proposition 2.1. *A point process Φ in $\mathbb{X}$ is a Poisson process with intensity measure λ if, and only if,*

$$\mathbb{E}\bigg[\int_{\mathbb{X}} f(x, \Phi)\, \mathrm{d}\Phi(x)\bigg] = \mathbb{E}\bigg[\int_{\mathbb{X}} f(x, \Phi + \delta_x)\, \mathrm{d}\lambda(x)\bigg]$$

for all measurable functions $f : \mathbb{X} \times \mathbf{N} \to [0, \infty]$.

If $(\mathbb{M}, \mathcal{M}, Q)$ is a probability space and Ψ a Poisson process in $\mathbb{X} \times \mathbb{M}$ with intensity measure $\lambda \otimes Q$, then the projection of Ψ onto the space $\mathbb{X}$, namely $\Psi(\,\cdot \times \mathbb{M})$, is a Poisson process in $\mathbb{X}$ with intensity measure λ. This follows immediately from the definition of a Poisson process. Whenever the processes involved are proper, Ψ can be interpreted as an independent Q-marking of a Poisson process in $\mathbb{X}$, as in Section 5.2 of Last and Penrose (2017).

2.2.2. Janossy and factorial moment measures

We first recall the definition of factorial moment and Janossy measures, following Chapter 4 of Last and Penrose (2017), and then discuss basic properties and their mutual relations (most of which are stated in weaker versions in Chapter 5.4 of Daley and Vere-Jones, 2005), with up-to-date notation of point process theory and some variations and extensions.

Let η be a point process in $\mathbb{X}$ and let $m \in \mathbb{N}$. The m-th *factorial moment measure* of η is

$$\alpha_{\eta,m}(\cdot) = \mathbb{E}\big[\eta^{(m)}(\cdot)\big],$$

which is a symmetric measure on $(\mathbb{X}^m, \mathcal{X}^{\otimes m})$. As for the notation in this definition, we write $\nu^{(m)}$ for the m-th factorial measure of $\nu \in \mathbf{N}$, the existence of which is guaranteed in this full generality by Proposition 4.3 of Last and Penrose (2017), see Appendix A.1. If the factorial moment measure $\alpha_{\eta,m}$ of a point process η is absolutely continuous with respect to λ^m with Radon-Nikodym density $\rho_{\eta,m}$, then $\rho_{\eta,m}$ is called *correlation function* of order m (of η with respect to λ).

Fix $B \in \mathcal{X}$. The *Janossy measure* $J_{\eta,B,m}$ of order $m \in \mathbb{N}$ of η restricted to B is the measure on the product space $(\mathbb{X}^m, \mathcal{X}^{\otimes m})$ defined as

$$J_{\eta,B,m}(\cdot) = \frac{1}{m!}\,\mathbb{E}\Big[\mathbb{1}\big\{\eta(B) = m\big\}\,\eta_B^{(m)}(\cdot)\Big].$$

The Janossy measures are symmetric and (by Proposition A.2) satisfy

$$J_{\eta,B,m}(\mathbb{X}^m) = \mathbb{P}\big(\eta(B) = m\big).$$

In line with this last observation, we put $J_{\eta,B,0} = \mathbb{P}\big(\eta(B) = 0\big)$. With our choice of the space $\mathbf{N}$, Theorem 4.7 of Last and Penrose (2017) states that if η and η' are two point processes with $J_{\eta,B,m} = J_{\eta',B,m}$ for all $m \in \mathbb{N}_0$

and some set $B \in \mathcal{X}_b$, then η_B and η'_B have the same distribution. By Example 4.8 of Last and Penrose (2017), the Janossy measures of a Poisson process with intensity measure λ are

$$J_{B,m} = \frac{\mathrm{e}^{-\lambda(B)}}{m!}\, \lambda_B^m(\cdot). \tag{2.2}$$

If, for fixed $B \in \mathcal{X}$ and $m \in \mathbb{N}$, the Janossy measure $J_{\eta,B,m}$ of some point process η is absolutely continuous with respect to λ_B^m, then the corresponding density function $j_{\eta,B,m}$ is called *Janossy density.*

Remark 2.2. Note that, by definition, $J_{\eta,B,m} \leq \frac{1}{m!} \cdot \alpha_{\eta,m}$ for all $B \in \mathcal{X}$ and $m \in \mathbb{N}$, where for two measures μ, ν on some measurable space $(\mathbb{Y}, \mathcal{Y})$ we write $\mu \leq \nu$ if $\mu(D) \leq \nu(D)$ for every $D \in \mathcal{Y}$. Note that a similar relation passes on to the corresponding density functions, if existent. More precisely, we have

$$j_{\eta,B,m}(x_1, \dots, x_m) \leq \frac{1}{m!} \cdot \rho_{\eta,m}(x_1, \dots, x_m) \tag{2.3}$$

for λ^m-almost every (a.e.) $(x_1, \dots, x_m) \in \mathbb{X}^m$, every $B \in \mathcal{X}$, and each $m \in \mathbb{N}$. Indeed, if this were not the case, we could find $m \in \mathbb{N}$, $B \in \mathcal{X}$, and a set $D \in \mathcal{X}^{\otimes m}$ with $\lambda^m(D) > 0$ such that (2.3) is violated in every point of D. This choice, however, would imply

$$\begin{aligned} J_{\eta,B,m}(D) = \int_D j_{\eta,B,m}(x_1, \dots, x_m)\, \mathrm{d}\lambda^m(x_1, \dots, x_m) &> \frac{1}{m!} \int_D \rho_{\eta,m}(x_1, \dots, x_m)\, \mathrm{d}\lambda^m(x_1, \dots, x_m) \\ &= \frac{1}{m!} \cdot \alpha_{\eta,m}(D), \end{aligned}$$

which contradicts the first part of this remark. ○

It follows immediately from Proposition A.5 that expectations of functionals of η_B can be expressed via the Janossy measures.

Lemma 2.3. *Let η be a point process in $\mathbb{X}$ and fix $B \in \mathcal{X}$. Then*

$$\mathbb{E}\Big[F(\eta_B) \cdot \mathbb{1}\{\eta(B) < \infty\}\Big] = F(\mathbf{0}) \cdot \mathbb{P}\big(\eta(B) = 0\big) + \sum_{m=1}^{\infty} \int_{\mathbb{X}^m} F\Big(\sum_{j=1}^m \delta_{x_j}\Big)\, \mathrm{d}J_{\eta,B,m}(x_1, \dots, x_m)$$

for every measurable function $F : \mathbf{N} \to [0, \infty]$.

Lemma 2.3 and Equation (2.2) imply the claim of Exercise 3.7 of Last and Penrose (2017), a statement which is used regularly throughout this work and reads as follows.

Corollary 2.4. *Let Φ be a Poisson process in $\mathbb{X}$ with intensity measure λ. Then, for any set $B \in \mathcal{X}$ with $\lambda(B) < \infty$ and every measurable map $F : \mathbf{N} \to [0, \infty]$,*

$$\mathbb{E}\big[F(\Phi_B)\big] = \mathrm{e}^{-\lambda(B)} F(\mathbf{0}) + \mathrm{e}^{-\lambda(B)} \sum_{m=1}^{\infty} \frac{1}{m!} \int_{B^m} F\Big(\sum_{j=1}^m \delta_{x_j}\Big)\, \mathrm{d}\lambda^m(x_1, \dots, x_m).$$

With the formalism of local events and functions from Appendix A.2, a converse of Lemma 2.3 can be formulated. To this end, recall from Appendix A.2 that $\mathcal{N}_B = \sigma(p_B)$, where $p_B(\mu) = \mu_B$ is the restriction mapping onto $B \in \mathcal{X}$.

Lemma 2.5. *Let η be a point process in $\mathbb{X}$ and let $B \in \mathcal{X}$. Assume there exist symmetric measures $J_{B,m}$ on $(\mathbb{X}^m, \mathcal{X}^{\otimes m})$ which vanish outside B^m, for each $m \in \mathbb{N}$, and $J_{B,0} \in [0, \infty)$ such that, for all $A \in \mathcal{N}_B$,*

$$\mathbb{P}(\eta \in A) = \mathbb{1}_A(\mathbf{0}) \cdot J_{B,0} + \sum_{m=1}^{\infty} \int_{B^m} \mathbb{1}_A\Big(\sum_{j=1}^m \delta_{x_j}\Big)\, \mathrm{d}J_{B,m}(x_1, \dots, x_m).$$

Then the $J_{B,m}$ $(m \in \mathbb{N}_0)$ are the Janossy measures of η restricted to B, and $\mathbb{P}\big(\eta(B) < \infty\big) = 1$.

Proof. First, choose $A = \big\{\mu \in \mathbf{N} : \mu(B) = 0\big\} = \pi_B^{-1}(\{0\}) \in \mathcal{N}_B$ and notice that

$$\mathbb{P}\big(\eta(B) = 0\big) = \mathbb{P}(\eta \in A) = J_{B,0},$$

so $J_{B,0}$ is the Janossy measure of order 0 of η_B. Now, let $F : \mathbf{N} \to [0,\infty]$ be a measurable function. By part (iv) of Lemma A.10, the map $F \circ p_B$ is $\mathcal{N}_B$-measurable. Hence, monotone convergence extends the assumption to

$$\mathbb{E}\big[F(\eta_B)\big] = F(\mathbf{0}) \cdot J_{B,0} + \sum_{m=1}^{\infty} \int_{B^m} F\Big(\sum_{j=1}^{m} \delta_{x_j}\Big) \, \mathrm{d}J_{B,m}(x_1, \ldots, x_m).$$

Let $k \in \mathbb{N}$ and $D \in \mathcal{X}^{\otimes k}$. With the specific choice $F(\mu) = \frac{1}{k!} \mathbb{1}\big\{\mu(B) = k\big\}\, \mu_B^{(k)}(D)$ we obtain

$$\begin{aligned}\frac{1}{k!}\, \mathbb{E}\Big[\mathbb{1}\big\{\eta(B) = k\big\}\, \eta_B^{(k)}(D)\Big] = \mathbb{E}\big[F(\eta_B)\big] &= \frac{1}{k!} \int_{B^k} \mathbb{1}\Big\{\sum_{j=1}^{k} \delta_{x_j}(B) = k\Big\} \Big(\sum_{j=1}^{k} \delta_{x_j}\Big)_B^{(k)}(D) \, \mathrm{d}J_{B,k}(x_1, \ldots, x_k) \\ &= \frac{1}{k!} \sum_{j_1,\ldots,j_k \in [k]}^{\neq} \int_{B^k} \mathbb{1}_D(x_{j_1}, \ldots, x_{j_k}) \, \mathrm{d}J_{B,k}(x_1, \ldots, x_k).\end{aligned}$$

By the symmetry of $J_{B,m}$, the right hand side equals

$$\frac{1}{k!} \sum_{j_1,\ldots,j_k \in [k]}^{\neq} J_{B,k}(D \cap B^k) = J_{B,k}(D \cap B^k) = J_{B,k}(D).$$

As $D \in \mathcal{X}^{\otimes k}$ was arbitrary, we have verified that $J_{B,k}$ is the Janossy measure of order k of η_B. Finally, we choose $A = \big\{\mu \in \mathbf{N} : \mu(B) = \infty\big\} \in \mathcal{N}_B$ to conclude that $\mathbb{P}\big(\eta(B) = \infty\big) = \mathbb{P}(\eta \in A) = 0$. $\square$

Next, we discuss the connections between factorial moment and Janossy measures. The following two theorems are also stated in principle in Chapter 5.4 of Daley and Vere-Jones (2005), but we generalize the results to arbitrary measurable spaces. We also translate them into up-to-date notation and give more elegant proofs (in our subjective opinion) which include the case of infinite point processes right away. We start by showing that the factorial moment measures can be expressed locally via the Janossy measures without any additional assumptions.

Theorem 2.6. *Let η be a point process in $\mathbb{X}$. Fix $B \in \mathcal{X}$ with $\mathbb{P}\big(\eta(B) < \infty\big) = 1$. For each $m \in \mathbb{N}$ and every measurable function $f : \mathbb{X}^m \to [0,\infty]$ with $f = 0$ on $\mathbb{X}^m \setminus B^m$, it holds that*

$$\int_{\mathbb{X}^m} f \, \mathrm{d}\alpha_{\eta,m} = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} \int_{\mathbb{X}^m} f(x_1, \ldots, x_m)\, J_{\eta,B,k}\big(\mathrm{d}(x_1, \ldots, x_m) \times B^{k-m}\big).$$

Proof. Fix $m \in \mathbb{N}$. For $D \in \mathcal{X}^{\otimes m}$ with $D \subset B^m$, we have

$$\alpha_{\eta,m}(D) = \mathbb{E}\big[\eta^{(m)}(D)\big] = \mathbb{E}\big[\eta_B^{(m)}(D)\big] = \sum_{k=0}^{\infty} \mathbb{E}\Big[\mathbb{1}\big\{\eta(B) = k\big\}\, \eta_B^{(m)}(D)\Big], \tag{2.4}$$

using that $(\mu^{(m)})_{B^m} = \mu_B^{(m)}$ as recalled in Proposition A.2. For $\mu \in \mathbf{N}$ with $\mu(B) = k < m$, we have $\mu_B^{(m)}(D) \leq \mu_B^{(m)}(B^m) = 0$ by Proposition A.2, so the right hand side of (2.4) equals

$$\sum_{k=m}^{\infty} \mathbb{E}\Big[\mathbb{1}\big\{\eta(B) = k\big\}\, \eta_B^{(m)}(D)\Big],$$

which, by Lemma A.3, equals

$$\sum_{k=m}^{\infty} \frac{1}{(k-m)!} \mathbb{E}\Big[\mathbb{1}\{\eta(B)=k\}\, \eta_B^{(k)}(D \times B^{k-m})\Big] = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} J_{\eta,B,k}(D \times B^{k-m}).$$

Thus, the result holds for $f = \mathbb{1}_D$ and monotone approximation extends this to general functions. $\square$

Corollary 2.7. *Let η be a point process in $\mathbb{X}$. Assume that the Janossy measures $J_{\eta,B,k}$ of η admit density functions $j_{\eta,B,k}$ with respect to λ^k, for each $k \in \mathbb{N}$ and $B \in \mathcal{X}_b$. Then the correlation functions of η exist and satisfy*

$$\rho_{\eta,m}(x_1,\dots,x_m) = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} \int_{B^{k-m}} j_{\eta,B,k}(x_1,\dots,x_k)\, \mathrm{d}\lambda^{k-m}(x_{m+1},\dots,x_k)$$

for λ^m-a.e. $(x_1,\dots,x_m) \in B^m$, all $m \in \mathbb{N}$, and each $B \in \mathcal{X}_b$.

Proof. Fix $m \in \mathbb{N}$. Let $B \in \mathcal{X}_b$ and $f : \mathbb{X}^m \to [0,\infty]$ a measurable function such that $f = 0$ on $\mathbb{X}^m \setminus B^m$. By Theorem 2.6 we have

$$\int_{\mathbb{X}^m} f \,\mathrm{d}\alpha_{\eta,m} = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} \int_{\mathbb{X}^k} f(x_1,\dots,x_m)\, \mathbb{1}_{B^{k-m}}(x_{m+1},\dots,x_k)\, j_{\eta,B,k}(x_1,\dots,x_k)\, \mathrm{d}\lambda^k(x_1,\dots,x_k)$$

and the right hand side, rewritten with the help of the Fubini–Tonelli theorem, equals

$$\int_{\mathbb{X}^m} f(x_1,\dots,x_m) \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} \int_{B^{k-m}} j_{\eta,B,k}(x_1,\dots,x_k)\, \mathrm{d}\lambda^{k-m}(x_{m+1},\dots,x_k)\, \mathrm{d}\lambda^m(x_1,\dots,x_m).$$

We conclude that $(\alpha_{\eta,m})_{B^m}$ is absolutely continuous with respect to λ_B^m for all $B \in \mathcal{X}_b$. Now, let $D \in \mathcal{X}^{\otimes m}$ with $\lambda^m(D) = 0$. Then $\lambda_{B_\ell}^m(D) = \lambda^m(D \cap B_\ell^m) = 0$, for each $\ell \in \mathbb{N}$, and therefore

$$\alpha_{\eta,m}(D) = \alpha_{\eta,m}\left(D \cap \bigcup_{\ell=1}^{\infty} B_\ell^m\right) = \lim_{\ell\to\infty} \alpha_{\eta,m}(D \cap B_\ell^m) = \lim_{\ell\to\infty} (\alpha_{\eta,m})_{B_\ell^m}(D) = 0.$$

Thus, $\alpha_{\eta,m}$ is absolutely continuous with respect to λ^m, so the correlation functions of η exist and they satisfy the claim. $\square$

We proceed by providing a converse of Theorem 2.6. Notice that this time we need to impose an additional assumption.

Theorem 2.8. *Let η be a point process in $\mathbb{X}$. Let $B \in \mathcal{X}$ and $m \in \mathbb{N}$ be such that*

$$\sum_{k=0}^{\infty} \frac{\alpha_{\eta,m+k}(B^{m+k})}{k!} < \infty.$$

Then, for all measurable and bounded functions $f : \mathbb{X}^m \to [0,\infty)$ with $f = 0$ on $\mathbb{X}^m \setminus B^m$,

$$\int_{\mathbb{X}^m} f \,\mathrm{d}J_{\eta,B,m} = \frac{1}{m!} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{X}^m} f(x_1,\dots,x_m)\, \alpha_{\eta,m+k}\big(\mathrm{d}(x_1,\dots,x_m) \times B^k\big).$$

Furthermore, for each $B \in \mathcal{X}$ with $\mathbb{E}\big[2^{\eta(B)}\big] < \infty$, the Janossy measure of order 0 is given as

$$J_{\eta,B,0} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \alpha_{\eta,k}(B^k).$$

Proof. First note that the assumption gives $\mathbb{E}\big[\eta^{(m)}(B^m)\big] = \alpha_{\eta,m}(B^m) < \infty$ which implies $\mathbb{P}\big(\eta(B) < \infty\big) = 1$, by Proposition A.2. For $D \in \mathcal{X}^{\otimes m}$ with $D \subset B^m$, Theorem 2.6 and an index shift yield

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\, \alpha_{\eta,m+k}(D \times B^k) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{\ell=m+k}^{\infty} \frac{\ell!}{(\ell-m-k)!}\, J_{\eta,B,\ell}(D \times B^{\ell-m})$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{\ell=k}^{\infty} \frac{(\ell+m)!}{(\ell-k)!}\, J_{\eta,B,m+\ell}(D \times B^{\ell}). \tag{2.5}$$

An identical calculation and the assumption give

$$\sum_{k=0}^{\infty}\sum_{\ell=k}^{\infty} \frac{(\ell+m)!}{k!\,(\ell-k)!}\, J_{\eta,B,m+\ell}(D \times B^{\ell}) \leq \sum_{k=0}^{\infty} \frac{\alpha_{\eta,m+k}(B^{m+k})}{k!} < \infty,$$

so Fubini's theorem is applicable and yields that the right hand side of (2.5) is equal to

$$\sum_{\ell=0}^{\infty} (\ell+m)!\, J_{\eta,B,m+\ell}(D \times B^{\ell}) \sum_{k=0}^{\ell} \frac{(-1)^k}{k!\,(\ell-k)!}.$$

Using

$$\sum_{k=0}^{\ell} \frac{(-1)^k}{k!\,(\ell-k)!} = \frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k = \mathbb{1}\{\ell = 0\},$$

we conclude that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\, \alpha_{\eta,m+k}(D \times B^k) = m!\, J_{\eta,B,m}(D),$$

which is the claim for $f = \mathbb{1}_D$. A standard monotone approximation, using dominated convergence (plus the assumption and the boundedness of the considered functions) to exchange limits with the infinite sum, proves the general claim.

Finally, consider the case $m = 0$. As in the assumption, let $B \in \mathcal{X}$ satisfy $\mathbb{E}\big[2^{\eta(B)}\big] < \infty$. It follows that $\mathbb{P}\big(\eta(B) < \infty\big) = 1$ and the Fubini–Tonelli theorem as well as part (ii) of Proposition A.2 yield

$$1 + \sum_{k=1}^{\infty} \frac{\alpha_{\eta,k}(B^k)}{k!} = \mathbb{E}\left[1 + \sum_{k=1}^{\infty} \frac{1}{k!}\, \eta^{(k)}(B^k)\right] = \mathbb{E}\left[\sum_{k=0}^{\eta(B)} \binom{\eta(B)}{k}\right] = \mathbb{E}\big[2^{\eta(B)}\big] < \infty. \tag{2.6}$$

In particular, we may use Fubini's theorem to conclude that

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\, \alpha_{\eta,k}(B^k) = \mathbb{E}\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\, \eta^{(k)}(B^k)\right] = \mathbb{E}\left[\sum_{k=0}^{\eta(B)} \binom{\eta(B)}{k} (-1)^k\right] = \mathbb{E}\Big[\mathbb{1}\big\{\eta(B) = 0\big\}\Big].$$

As the right hand side equals $\mathbb{P}\big(\eta(B) = 0\big) = J_{\eta,B,0}$, the proof is complete. □

In terms of correlation functions and Janossy densities, Theorem 2.8 immediately gives the following corollary.

Corollary 2.9. *Let η be a point process in $\mathbb{X}$. Let $B \in \mathcal{X}$ and $m \in \mathbb{N}$ be such that*

$$\sum_{k=0}^{\infty} \frac{\alpha_{\eta,m+k}(B^{m+k})}{k!} < \infty.$$

Assume that the correlation functions $\rho_{\eta,m+k}$ of η with respect to λ^{m+k} exist for all $k \in \mathbb{N}_0$. Then $J_{\eta,B,m}$ is absolutely continuous with respect to λ^m and the corresponding Janossy density is given by

$$j_{\eta,B,m}(x_1,\dots,x_m) = \frac{1}{m!} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{B^k} \rho_{\eta,m+k}(x_1,\dots,x_{m+k})\, \mathrm{d}\lambda^k(x_{m+1},\dots,x_{m+k})$$

for λ^m-a.e. $(x_1, \dots, x_m) \in B^m$. For $B \in \mathcal{X}$ with $\mathbb{E}\big[2^{\eta(B)}\big] < \infty$ the Janossy measure of order 0 is

$$J_{\eta,B,0} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{B^k} \rho_{\eta,k}(x_1, \dots, x_k) \, \mathrm{d}\lambda^k(x_1, \dots, x_k).$$

We conclude the discussion with a remark on sufficient conditions for the assumption in Theorem 2.8 and Corollary 2.9.

Remark 2.10. Fix a set $B \in \mathcal{X}$ with $\mathbb{P}\big(\eta(B) < \infty\big) = 1$. Notice that when considering constructions on the full space $\mathbb{X}$, the bounding constants and maps appearing in this remark may all depend on B.

1. The obvious assumption is to require the existence of a constant $c \geq 0$ such that $\alpha_{\eta,\ell}(B^\ell) \leq c^\ell$, for each $\ell \in \mathbb{N}$. Then we have
$$\sum_{k=0}^{\infty} \frac{\alpha_{\eta,m+k}(B^{m+k})}{k!} \leq c^m \sum_{k=0}^{\infty} \frac{c^k}{k!} = c^m \cdot \mathrm{e}^c < \infty$$
for every $m \in \mathbb{N}$. Similarly,
$$\mathbb{E}\big[2^{\eta(B)}\big] = 1 + \sum_{k=1}^{\infty} \frac{\alpha_{\eta,k}(B^k)}{k!} \leq 1 + \sum_{k=1}^{\infty} \frac{c^k}{k!} = \mathrm{e}^c < \infty.$$

2. For any $\varepsilon > 0$ and $m \in \mathbb{N}$, we find $k_{\varepsilon,m} \in \mathbb{N}$, $k_{\varepsilon,m} > m$, such that for every $k \in \mathbb{N}$ with $k \geq k_{\varepsilon,m}$ we have $\big(\frac{k}{2}\big)^m \leq \big(1 + \frac{\varepsilon}{2}\big)^k$. This we use to calculate
$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\alpha_{\eta,m+k}(B^{m+k})}{k!} &= \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{1}{k!} \, \eta(B)\big(\eta(B) - 1\big) \cdot \ldots \cdot \big(\eta(B) - k - m + 1\big)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\eta(B)-m} \frac{1}{k!} \cdot \frac{\eta(B)!}{\big(\eta(B) - k - m\big)!} \cdot \mathbb{1}\big\{\eta(B) \geq m\big\}\right] \\
&= \mathbb{E}\left[\frac{\eta(B)!}{\big(\eta(B) - m\big)!} \cdot \mathbb{1}\big\{\eta(B) \geq m\big\} \sum_{k=0}^{\eta(B)-m} \binom{\eta(B) - m}{k}\right] \\
&= \mathbb{E}\left[\frac{\eta(B)!}{\big(\eta(B) - m\big)!} \cdot 2^{\eta(B)-m} \cdot \mathbb{1}\big\{\eta(B) \geq m\big\}\right] \\
&\leq \mathbb{E}\left[\left(\frac{\eta(B)}{2}\right)^m \cdot 2^{\eta(B)} \cdot \mathbb{1}\big\{\eta(B) \geq m\big\}\right] \\
&= \sum_{k=m}^{\infty} \left(\frac{k}{2}\right)^m \cdot 2^k \cdot \mathbb{P}\big(\eta(B) = k\big) \\
&\leq \sum_{k=m}^{k_{\varepsilon,m}-1} \left(\frac{k}{2}\right)^m \cdot 2^k \cdot \mathbb{P}\big(\eta(B) = k\big) + \sum_{k=k_{\varepsilon,m}}^{\infty} (2 + \varepsilon)^k \cdot \mathbb{P}\big(\eta(B) = k\big) \\
&\leq 2^{k_{\varepsilon,m}} \cdot \left(\frac{k_{\varepsilon,m}}{2}\right)^m + \mathbb{E}\Big[(2 + \varepsilon)^{\eta(B)}\Big].
\end{aligned}$$
Thus, if there exists some $b \in (2, \infty)$ such that $\mathbb{E}\big[b^{\eta(B)}\big] < \infty$, the series converges for every $m \in \mathbb{N}$, and clearly also for $m = 0$.

3. Assume there exists a map $c : \mathbb{N} \to [0, \infty)$ and a constant $b \in (2, \infty)$ such that $\sum_{k=1}^{\infty} b^k \cdot c(k) < \infty$ as well as
$$J_{\eta,B,k}(B^k) \leq c(k)$$

for all $k \in \mathbb{N}$. Then

$$\mathbb{E}\big[b^{\eta(B)}\big] = \sum_{k=0}^{\infty} b^k \, \mathbb{P}\big(\eta(B) = k\big) = \sum_{k=0}^{\infty} b^k J_{\eta,B,k}(B^k) \le \sum_{k=1}^{\infty} b^k \cdot c(k) < \infty.$$

By part 2, the assumption of Theorem 2.8 is satisfied for every $m \in \mathbb{N}_0$. One natural choice for c is

$$c(k) = \frac{\vartheta^k}{k!}$$

for some $\vartheta \ge 0$.

4. As a special case of part 3, we consider the case where Janossy densities exist. These allow to write

$$J_{\eta,B,k}(B^k) = \int_{B^k} j_{\eta,B,k}(x_1, \dots, x_k) \, \mathrm{d}\lambda^k(x_1, \dots, x_k)$$

for each $k \in \mathbb{N}$. Thus, a sufficient assumption is that

$$j_{\eta,B,k}(x_1, \dots, x_k) \le \frac{\vartheta(x_1) \cdot \ldots \cdot \vartheta(x_k)}{k!}$$

for λ^k-a.e. $(x_1, \dots, x_k) \in B^k$, all $k \in \mathbb{N}$, and some λ_B-integrable function $\vartheta : \mathbb{X} \to [0, \infty]$. ○

We end this subsection with a final observation about Janossy measures, namely that there exists a close relation between absolute continuity of the distributions of two point processes restricted to some bounded set and absolute continuity of their Janossy measures. For two point processes η, η' in $\mathbb{X}$ and some set $B \in \mathcal{X}_b$, denote by $\mathbb{P}^{\eta_B}, \mathbb{P}^{\eta'_B}$ the images of $\mathbb{P}^{\eta}, \mathbb{P}^{\eta'}$ under the restriction map $p_B(\mu) = \mu_B$, which correspond to the distributions of η_B, η'_B. Following Jacod and Shiryaev (2003), if

$$\mathbb{P}^{\eta_B} \ll \mathbb{P}^{\eta'_B}$$

for each set $B \in \mathcal{X}_b$ then $\mathbb{P}^{\eta}$ is called locally absolutely continuous with respect to $\mathbb{P}^{\eta'}$. The following lemma establishes that local absolute continuity is equivalent to absolute continuity of Janossy measures. The result is essentially stated as Proposition 7.1.III by Daley and Vere-Jones (2005), but again we give a different proof, adapted to our more general setting.

Lemma 2.11. *Let η, η' be point processes in $\mathbb{X}$ and fix a set $B \in \mathcal{X}_b$ such that $\mathbb{P}\big(\eta'(B) = 0\big) > 0$. Then $\mathbb{P}^{\eta_B} \ll \mathbb{P}^{\eta'_B}$ if, and only if, $J_{\eta,B,m} \ll J_{\eta',B,m}$ for all $m \in \mathbb{N}$.*

Proof. Let $B \in \mathcal{X}$ be such that $\mathbb{P}\big(\eta(B) < \infty\big) = 1$ and $\mathbb{P}\big(\eta'(B) < \infty\big) = 1$, as well as $\mathbb{P}\big(\eta'(B) = 0\big) > 0$. First, suppose that $\mathbb{P}^{\eta_B} \ll \mathbb{P}^{\eta'_B}$ and let $m \in \mathbb{N}$. Consider a set $D \in \mathcal{X}^{\otimes m}$ such that $J_{\eta',B,m}(D) = 0$. Define the measurable map $F : \mathbf{N} \to [0, 1]$ via

$$F(\mu) = \frac{1}{m!} \mathbb{1}\big\{\mu(B) = m\big\} \, \mu_B^{(m)}(D).$$

By definition of the Janossy measure, we have

$$0 = J_{\eta',B,m}(D) = \mathbb{E}\big[F(\eta'_B)\big] = \int_{\mathbf{N}} F(\mu) \, \mathrm{d}\mathbb{P}^{\eta'_B}(\mu).$$

We conclude that $F = 0$ $\mathbb{P}^{\eta'_B}$-a.s., which implies $F = 0$ $\mathbb{P}^{\eta_B}$-a.s., and thus

$$J_{\eta,B,m}(D) = \mathbb{E}\big[F(\eta_B)\big] = \int_{\mathbf{N}} F(\mu) \, \mathrm{d}\mathbb{P}^{\eta_B}(\mu) = 0.$$

For the proof of the converse statement, assume that $J_{\eta,B,m} \ll J_{\eta',B,m}$ for all $m \in \mathbb{N}$. Let $A \in \mathcal{N}$ with $\mathbb{P}^{\eta'_B}(A) = 0$. This implies $\mathbf{0} \notin A$ since otherwise

$$\mathbb{P}(\eta'_B \in A) \geq \mathbb{P}\big(\eta'_B \in \{\mathbf{0}\}\big) = \mathbb{P}\big(\eta'(B) = 0\big) > 0,$$

noting that $\{\mathbf{0}\} = \pi_{\mathbb{X}}^{-1}(\{0\}) \in \mathcal{N}$. Therefore,

$$\mathbb{P}^{\eta_B}\big(A \cap \big\{\mu \in \mathbf{N} : \mu(B) = 0\big\}\big) = \mathbb{P}^{\eta_B}\big(A \cap \{\mathbf{0}\}\big) = \mathbb{P}^{\eta_B}(\varnothing) = 0.$$

For each $m \in \mathbb{N}$, the fact that $\mathbb{P}^{\eta'_B}(A) = 0$ implies

$$0 = \mathbb{P}^{\eta'_B}\big(A \cap \big\{\mu \in \mathbf{N} : \mu(B) = m\big\}\big) = \mathbb{E}\Big[\mathbb{1}_A(\eta'_B) \cdot \mathbb{1}\big\{\eta'(B) = m\big\}\Big],$$

so by Lemma 2.3

$$\int_{\mathbb{X}^m} \mathbb{1}_A\Big(\sum_{j=1}^m \delta_{x_j}\Big) \, \mathrm{d}J_{\eta',B,m}(x_1, \ldots, x_m) = 0.$$

Hence, $\mathbb{1}_A(\delta_{x_1} + \ldots + \delta_{x_m}) = 0$ for $J_{\eta',B,m}$-a.e. $(x_1, \ldots, x_m) \in \mathbb{X}^m$. By the absolute continuity assumption on the Janossy measures, we get $\mathbb{1}_A(\delta_{x_1} + \ldots + \delta_{x_m}) = 0$ for $J_{\eta,B,m}$-a.e. $(x_1, \ldots, x_m) \in \mathbb{X}^m$, and thus obtain

$$\mathbb{P}^{\eta_B}\big(A \cap \big\{\mu \in \mathbf{N} : \mu(B) = m\big\}\big) = \int_{\mathbb{X}^m} \mathbb{1}_A\Big(\sum_{j=1}^m \delta_{x_j}\Big) \, \mathrm{d}J_{\eta,B,m}(x_1, \ldots, x_m) = 0.$$

With the choice of the set B and the previous observations, we conclude that

$$\mathbb{P}^{\eta_B}(A) = \sum_{m=0}^{\infty} \mathbb{P}^{\eta_B}\big(A \cap \big\{\mu \in \mathbf{N} : \mu(B) = m\big\}\big) = 0,$$

which finishes the proof. □

If the reference process is a Poisson process, the previous lemma becomes more explicit. Indeed, recall that the Janossy measures of a Poisson process Φ in $\mathbb{X}$ with intensity measure λ are given by (2.2) and that $\mathbb{P}\big(\Phi(B) < \infty\big) = 1$ whenever $\lambda(B) < \infty$. The latter condition also ensures $\mathbb{P}\big(\Phi(B) = 0\big) = \mathrm{e}^{-\lambda(B)} > 0$. Thus, Lemma 2.11 reduces to the following result, stating that the distribution of a point process η is locally absolutely continuous with respect to Π_λ precisely when η admits Janossy densities.

Corollary 2.12. *Let η be a point process in $\mathbb{X}$ and $B \in \mathcal{X}$ with $\lambda(B) < \infty$ and $\mathbb{P}\big(\eta(B) < \infty\big) = 1$. Then $\mathbb{P}^{\eta_B} \ll \Pi_{\lambda_B}$ if, and only if, $J_{\eta,B,m} \ll \lambda_B^m$ for each $m \in \mathbb{N}$.*

Remark 2.13. With the help of Corollary 2.4 and Lemma 2.3 it is easy to specify the density functions corresponding to the absolute continuity statements in Corollary 2.12. More precisely, if $\mathbb{P}^{\eta_B} \ll \Pi_{\lambda_B}$ then the Janossy densities can be calculated in terms of the Π_{λ_B}-density of $\mathbb{P}^{\eta_B}$ and, similarly, if $J_{\eta,B,m} \ll \lambda_B^m$ for each $m \in \mathbb{N}$, then the Π_{λ_B}-density of $\mathbb{P}^{\eta_B}$ can be calculated in terms of the Janossy densities. ○

2.3. The definition of Gibbs processes via GNZ equations

Let $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ be a measurable function and fix a locally finite measure λ on $\mathbb{X}$. A point process η in $\mathbb{X}$ is called *Gibbs process* with *Papangelou (conditional) intensity* (PI) κ (and reference measure λ) if

$$\mathbb{E}\bigg[\int_{\mathbb{X}} f(x, \eta) \, \mathrm{d}\eta(x)\bigg] = \mathbb{E}\bigg[\int_{\mathbb{X}} f(x, \eta + \delta_x) \, \kappa(x, \eta) \, \mathrm{d}\lambda(x)\bigg], \tag{2.7}$$

for all measurable maps $f : \mathbb{X} \times \mathbf{N} \to [0, \infty)$. These defining equations are the *GNZ equations* named after Georgii (1976), Nguyen and Zessin (1979). For $\kappa \equiv 1$ they reduce to Mecke's formula in Proposition 2.1 and describe a Poisson process. The notion of conditional intensities goes back to Papangelou (1974).

The terms in Equation (2.7) and in the following lemma are well defined by Lemma A.4. For $m \in \mathbb{N}$ define the function $\kappa_m : \mathbb{X}^m \times \mathbf{N} \to [0, \infty)$ by

$$\kappa_m(x_1, \ldots, x_m, \mu) = \kappa(x_m, \mu) \cdot \kappa(x_{m-1}, \mu + \delta_{x_m}) \cdot \ldots \cdot \kappa(x_1, \mu + \delta_{x_2} + \ldots + \delta_{x_m}).$$

These functions allow for an iterated version of the GNZ equations.

Lemma 2.14 (Multivariate GNZ equations). *Let η be a Gibbs process with PI κ and $m \in \mathbb{N}$. Then, for any measurable function $f : \mathbb{X}^m \times \mathbf{N} \to [0, \infty]$,*

$$\begin{aligned}&\mathbb{E}\bigg[\int_{\mathbb{X}^m} f(x_1, \ldots, x_m, \eta) \, \mathrm{d}\eta^{(m)}(x_1, \ldots, x_m)\bigg] \\ &\quad = \mathbb{E}\bigg[\int_{\mathbb{X}^m} f(x_1, \ldots, x_m, \eta + \delta_{x_1} + \ldots + \delta_{x_m}) \, \kappa_m(x_1, \ldots, x_m, \eta) \, \mathrm{d}\lambda^m(x_1, \ldots, x_m)\bigg].\end{aligned}$$

Proof. We prove the assertion by induction over m. For $m = 1$ the lemma corresponds to the GNZ equations. We proceed to prove that if the claim is true for some fixed m, it also holds for $m + 1$. Let $D \in \mathcal{X}^{\otimes(m+1)}$ and $A \in \mathcal{N}$. Observe that, for any $\mu \in \mathbf{N}$, the proof of Proposition A.18 of Last and Penrose (2017) gives

$$\sum_{j=1}^{m} \mathbb{1}_D(x_1, \ldots, x_m, x_j) \leq \int_{\mathbb{X}} \mathbb{1}_D(x_1, \ldots, x_m, y) \, \mathrm{d}\mu(y)$$

for $\mu^{(m)}$-a.e. $(x_1, \ldots, x_m) \in \mathbb{X}^m$. We define the measurable map $F : \mathbb{X}^m \times \mathbf{N} \to (-\infty, \infty]$,

$$F(x_1, \ldots, x_m, \mu) = \mathbb{1}_A(\mu) \cdot \bigg(\int_{\mathbb{X}} \mathbb{1}_D(x_1, \ldots, x_m, y) \, \mathrm{d}\mu(y) - \sum_{j=1}^{m} \mathbb{1}_D(x_1, \ldots, x_m, x_j)\bigg).$$

By the above observation, we have $F^-(x_1, \ldots, x_m, \mu) = 0$ for $\mu^{(m)}$-a.e. $(x_1, \ldots, x_m) \in \mathbb{X}^m$ and all $\mu \in \mathbf{N}$, where F^- denotes the negative part of F. Therefore, we have

$$\int_{\mathbb{X}^m} F(x_1, \ldots, x_m, \mu) \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m) = \int_{\mathbb{X}^m} F^+(x_1, \ldots, x_m, \mu) \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m)$$

for all $\mu \in \mathbf{N}$, with F^+ the positive part of F. By the characterizing equation (A.2) of the factorial measure, we obtain

$$\mathbb{E}\bigg[\int_D \mathbb{1}_A(\eta) \, \mathrm{d}\eta^{(m+1)}(x_1, \ldots, x_{m+1})\bigg] = \mathbb{E}\Big[\mathbb{1}_A(\eta) \cdot \eta^{(m+1)}(D)\Big] = \mathbb{E}\bigg[\int_{\mathbb{X}^m} F^+(x_1, \ldots, x_m, \eta) \, \mathrm{d}\eta^{(m)}(x_1, \ldots, x_m)\bigg].$$

By the induction hypothesis, the multivariate GNZ equation holds for all measurable maps $\mathbb{X}^m \times \mathbf{N} \to [0, \infty]$, so we can apply it to F^+ to see that the right hand side of the previous display equals

$$\begin{aligned}&\mathbb{E}\bigg[\int_{\mathbb{X}^m} F^+\Big(x_1, \ldots, x_m, \eta + \sum_{i=1}^{m} \delta_{x_i}\Big) \, \kappa_m(x_1, \ldots, x_m, \eta) \, \mathrm{d}\lambda^m(x_1, \ldots, x_m)\bigg] \\ &\quad = \mathbb{E}\bigg[\int_{\mathbb{X}^m} \mathbb{1}_A\Big(\eta + \sum_{i=1}^{m} \delta_{x_i}\Big) \Big(\int_{\mathbb{X}} \mathbb{1}_D(x_1, \ldots, x_{m+1}) \, \mathrm{d}\eta(x_{m+1})\Big) \, \kappa_m(x_1, \ldots, x_m, \eta) \, \mathrm{d}\lambda^m(x_1, \ldots, x_m)\bigg].\end{aligned}$$

An application of the (univariate) GNZ equation and the definition of κ_{m+1} show that this term equals

$$\mathbb{E}\Big[\int_{\mathbb{X}^{m+1}} \mathbb{1}_D(x_1,\dots,x_{m+1})\, \mathbb{1}_A\Big(\eta+\sum_{i=1}^{m+1}\delta_{x_i}\Big)\, \kappa_m(x_1,\dots,x_m,\eta+\delta_{x_{m+1}})\, \kappa(x_{m+1},\eta)\, \mathrm{d}\lambda^{m+1}(x_1,\dots,x_{m+1})\Big]$$
$$= \mathbb{E}\Big[\int_D \mathbb{1}_A\Big(\eta+\sum_{i=1}^{m+1}\delta_{x_i}\Big)\, \kappa_{m+1}(x_1,\dots,x_{m+1},\eta)\, \mathrm{d}\lambda^{m+1}(x_1,\dots,x_{m+1})\Big].$$

Define on $\mathbb{X}^{m+1}\times\mathbf{N}$ the measures

$$\mathrm{C}(E) = \mathbb{E}\Big[\int_{\mathbb{X}^{m+1}} \mathbb{1}_E(x_1,\dots,x_{m+1},\eta)\, \mathrm{d}\eta^{(m+1)}(x_1,\dots,x_{m+1})\Big]$$

and

$$\tilde{\mathrm{C}}(E) = \mathbb{E}\Big[\int_{\mathbb{X}^{m+1}} \mathbb{1}_E\Big(x_1,\dots,x_{m+1},\eta+\sum_{i=1}^{m+1}\delta_{x_i}\Big)\, \kappa_{m+1}(x_1,\dots,x_{m+1},\eta)\, \mathrm{d}\lambda^{m+1}(x_1,\dots,x_{m+1})\Big],$$

for $E\in\mathcal{X}^{\otimes(m+1)}\otimes\mathcal{N}$. By the calculation above, these measures are equal on the π-system

$$\big\{D\times A : D\in\mathcal{X}^{\otimes(m+1)},\ A\in\mathcal{N}\big\}$$

which generates $\mathcal{X}^{\otimes(m+1)}\otimes\mathcal{N}$. Moreover, the measures are σ-finite as

$$\begin{aligned}\tilde{\mathrm{C}}\Big(B_\ell^{m+1}\times\big\{\mu\in\mathbf{N}:\mu(B_\ell)\le n\big\}\Big) &= \mathrm{C}\Big(B_\ell^{m+1}\times\big\{\mu\in\mathbf{N}:\mu(B_\ell)\le n\big\}\Big)\\ &= \mathbb{E}\Big[\mathbb{1}\big\{\eta(B_\ell)\le n\big\}\cdot\eta^{(m+1)}(B_\ell^{m+1})\Big]\\ &\le n!\end{aligned}$$

for all $\ell,n\in\mathbb{N}$, and

$$\bigcup_{\ell,n=1}^{\infty}\Big(B_\ell^{m+1}\times\big\{\mu\in\mathbf{N}:\mu(B_\ell)\le n\big\}\Big) = \mathbb{X}^{m+1}\times\mathbf{N}.$$

Thus, the uniqueness theorem for measures yields $\mathrm{C}=\tilde{\mathrm{C}}$. Monotone approximation with simple functions implies the claim for general measurable functions $f:\mathbb{X}^{m+1}\times\mathbf{N}\to[0,\infty]$, and the induction is complete. □

If the underlying measurable space has a measurable diagonal, that is, $\{(x,x):x\in\mathbb{X}\}\in\mathcal{X}^{\otimes 2}$, then the map $\mathbb{X}\times\mathbf{N}\ni(x,\mu)\mapsto\mu\setminus\delta_x\in\mathbf{N}$ is measurable, where

$$\mu\setminus\delta_x = \mu-\delta_x\,\mathbb{1}\big\{\mu(\{x\})>0\big\}.$$

This fact is exhaustively discussed in Appendix A.3. In particular, a proof is given that if $(\mathbb{X},\mathcal{X})$ is separable (i.e. $\{x\}\in\mathcal{X}$ for each $x\in\mathbb{X}$) and (the σ-field $\mathcal{X}$ is) countably generated then it has a measurable diagonal. With this general notion of the removal of points from a counting measure we can restate Lemma 2.14.

Corollary 2.15. *Let $(\mathbb{X},\mathcal{X})$ have a measurable diagonal. Further, let η be a Gibbs process with PI κ, and fix $m\in\mathbb{N}$. Then, for any measurable function $f:\mathbb{X}^m\times\mathbf{N}\to[0,\infty]$,*

$$\mathbb{E}\Big[\int_{\mathbb{X}^m} f(x_1,\dots,x_m,\eta\setminus\delta_{x_1}\setminus\dots\setminus\delta_{x_m})\, \mathrm{d}\eta^{(m)}(x_1,\dots,x_m)\Big]$$
$$= \mathbb{E}\Big[\int_{\mathbb{X}^m} f(x_1,\dots,x_m,\eta)\, \kappa_m(x_1,\dots,x_m,\eta)\, \mathrm{d}\lambda^m(x_1,\dots,x_m)\Big].$$

The proof is immediate upon applying Lemma 2.14 to the map

$$(x_1, \dots, x_m, \mu) \mapsto f(x_1, \dots, x_m, \mu \setminus \delta_{x_1} \setminus \dots \setminus \delta_{x_m}),$$

which is measurable by Lemma A.13, and using that

$$(\mu + \delta_{x_1} + \dots + \delta_{x_m}) \setminus \delta_{x_1} \setminus \dots \setminus \delta_{x_m} = \mu$$

for any $\mu \in \mathbf{N}$ and $x_1, \dots, x_m \in \mathbb{X}$ with $m \in \mathbb{N}$.

We apply Corollary 2.15 to provide a first observation about κ_m.

Lemma 2.16. *Assume that $(\mathbb{X}, \mathcal{X})$ has a measurable diagonal. Let η be a Gibbs process with PI κ, and fix $m \in \mathbb{N}$. Then, for any permutation τ of $\{1, \dots, m\}$,*

$$\kappa_m(x_1, \dots, x_m, \mu) = \kappa_m\big(x_{\tau(1)}, \dots, x_{\tau(m)}, \mu\big)$$

for $(\lambda^m \otimes \mathbb{P}^\eta)$-a.e. $(x_1, \dots, x_m, \mu) \in \mathbb{X}^m \times \mathbf{N}$.

Proof. Let $f : \mathbb{X}^m \times \mathbf{N} \to [0, \infty]$ be measurable. Then, by Corollary 2.15 and the symmetry of product and factorial measures, we have

$$\begin{aligned}
&\int_{\mathbf{N}} \int_{\mathbb{X}^m} f(x_1, \dots, x_m, \mu)\, \kappa_m(x_{\tau(1)}, \dots, x_{\tau(m)}, \mu) \,\mathrm{d}\lambda^m(x_1, \dots, x_m) \,\mathrm{d}\mathbb{P}^\eta(\mu) \\
&\quad = \mathbb{E}\bigg[\int_{\mathbb{X}^m} f\big(x_{\tau^{-1}(1)}, \dots, x_{\tau^{-1}(m)}, \eta\big)\big)\, \kappa_m(x_1, \dots, x_m, \eta) \,\mathrm{d}\lambda^m(x_1, \dots, x_m) \bigg] \\
&\quad = \mathbb{E}\bigg[\int_{\mathbb{X}^m} f\big(x_{\tau^{-1}(1)}, \dots, x_{\tau^{-1}(m)}, \eta \setminus \delta_{x_1} \setminus \dots \setminus \delta_{x_m}\big) \,\mathrm{d}\eta^{(m)}(x_1, \dots, x_m) \bigg] \\
&\quad = \mathbb{E}\bigg[\int_{\mathbb{X}^m} f\big(x_1, \dots, x_m, \eta \setminus \delta_{x_{\tau(1)}} \setminus \dots \setminus \delta_{x_{\tau(m)}}\big) \,\mathrm{d}\eta^{(m)}(x_1, \dots, x_m) \bigg] \\
&\quad = \mathbb{E}\bigg[\int_{\mathbb{X}^m} f\big(x_1, \dots, x_m, \eta \setminus \delta_{x_1} \setminus \dots \setminus \delta_{x_m}\big) \,\mathrm{d}\eta^{(m)}(x_1, \dots, x_m) \bigg] \\
&\quad = \int_{\mathbf{N}} \int_{\mathbb{X}^m} f(x_1, \dots, x_m, \mu)\, \kappa_m(x_1, \dots, x_m, \mu) \,\mathrm{d}\lambda^m(x_1, \dots, x_m) \,\mathrm{d}\mathbb{P}^\eta(\mu),
\end{aligned}$$

which implies the claim. $\square$

As Borel spaces have a measurable diagonal, Lemma 2.16 essentially states that, in just about any reasonable state space, the existence of a Gibbs process with PI κ implies that the κ_m corresponding to κ are necessarily symmetric. This assumption we adopt in everything that follows. More specifically, we assume that κ satisfies the *cocycle relation*

$$\kappa(x, \mu) \cdot \kappa(y, \mu + \delta_x) = \kappa(y, \mu) \cdot \kappa(x, \mu + \delta_y) \tag{2.8}$$

for all $\mu \in \mathbf{N}$ and $x, y \in \mathbb{X}$. Indeed, if κ has this property, then the function κ_m is, by its definition, symmetric in the first m components. Conversely, if κ_m is symmetric in the first m components, then

$$\kappa(x, \mu) \cdot \kappa(y, \mu + \delta_x) = \kappa_2(x, y, \mu) = \kappa_2(y, x, \mu) = \kappa(y, \mu) \cdot \kappa(x, \mu + \delta_y), \quad x, y \in \mathbb{X},\ \mu \in \mathbf{N}.$$

For $B \in \mathcal{X}$ we define the *partition function* $Z_B : \mathbf{N} \to [0, \infty]$ as

$$Z_B(\psi) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{B^m} \kappa_m(x_1, \dots, x_m, \psi) \,\mathrm{d}\lambda^m(x_1, \dots, x_m).$$

The function Z_B is measurable and satisfies $Z_B(\psi) \geq 1$ for every $\psi \in \mathbf{N}$. In statistical physics, the finiteness of the partition function usually has to be stated as an explicit assumption. The point process theoretic definition of the Gibbs process via the GNZ equations already contains essential information implicitly, as the following lemma shows. For less general state spaces this is known from Nguyen and Zessin (1979) and Matthes et al. (1979), but their techniques are generic and transfer to the abstract setting.

Lemma 2.17. *If η is a Gibbs process with PI κ, and $B \in \mathcal{X}$ with $\mathbb{P}\big(\eta(B) < \infty\big) = 1$, then*

$$\mathbb{P}\big(Z_B(\eta_{B^c}) < \infty\big) = 1$$

as well as

$$\mathbb{P}\big(\eta(B) = 0 \mid \eta_{B^c}\big) = \frac{1}{Z_B(\eta_{B^c})} \quad \mathbb{P}\text{-}a.s.$$

Proof. For any measurable function $g : \mathbf{N} \to [0, \infty]$ and $D \in \mathcal{X}$ we have (using (ii) of Proposition A.2)

$$\begin{aligned}\mathbb{E}\Big[g(\eta)\,\mathbb{1}\{\eta(D) < \infty\}\Big] &= \sum_{m=0}^{\infty} \mathbb{E}\Big[g(\eta)\,\mathbb{1}\{\eta(D) = m\}\Big] \\ &= \mathbb{E}\Big[g(\eta)\,\mathbb{1}\{\eta(D) = 0\}\Big] + \sum_{m=1}^{\infty} \frac{1}{m!}\,\mathbb{E}\bigg[\int_{D^m} g(\eta)\,\mathbb{1}\{\eta(D) = m\}\,\mathrm{d}\eta^{(m)}(x_1, \ldots, x_m)\bigg]\end{aligned}$$

which, by Lemma 2.14, equals

$$\mathbb{E}\Bigg[\mathbb{1}\{\eta(D) = 0\}\bigg(g(\eta) + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{D^m} g\Big(\eta + \sum_{j=1}^{m} \delta_{x_j}\Big)\,\kappa_m(x_1, \ldots, x_m, \eta)\,\mathrm{d}\lambda^m(x_1, \ldots, x_m)\bigg)\Bigg].$$

Applied to $D = B$ and $g(\mu) = \mathbb{1}_A(\mu_{B^c})$, for $A \in \mathcal{N}$, this observation yields

$$\mathbb{P}(\eta_{B^c} \in A) = \mathbb{E}\Big[\mathbb{1}\{\eta(B) = 0\} \cdot \mathbb{1}_A(\eta_{B^c}) \cdot Z_B(\eta_{B^c})\Big] = \mathbb{E}\Big[\mathbb{1}_A(\eta_{B^c}) \cdot Z_B(\eta_{B^c}) \cdot \mathbb{P}\big(\eta(B) = 0 \mid \eta_{B^c}\big)\Big].$$

As $A \in \mathcal{N}$ was arbitrary, we find that

$$Z_B(\eta_{B^c}) \cdot \mathbb{P}\big(\eta(B) = 0 \mid \eta_{B^c}\big) = 1 \quad \mathbb{P}\text{-a.s.}$$

As $\infty \cdot 0 = 0$, by convention, we conclude that

$$\mathbb{P}\Big(\mathbb{P}\big(\eta(B) = 0 \mid \eta_{B^c}\big) > 0\Big) = 1$$

as well as $\mathbb{P}\big(Z_B(\eta_{B^c}) < \infty\big) = 1$. □

By construction of the set $\mathbf{N}$, every set $B \in \mathcal{X}_b$ qualifies for Lemma 2.17, as point processes are almost surely finite on sets in $\mathcal{X}_b$.

2.4. Janossy densities and correlation functions of Gibbs processes

In this section we calculate the Janossy densities and correlation functions of a general Gibbs process. Within the abstract setting of this chapter, suppose that $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ is measurable and satisfies the cocycle assumption (2.8).

Lemma 2.18. *Let η be a Gibbs process in $\mathbb{X}$ with PI κ. Then η has the Janossy densities*

$$j_{\eta,B,m}(x_1, \ldots, x_m) = \frac{1}{m!}\,\mathbb{E}\Big[\mathbb{1}\big\{\eta(B) = 0\big\}\,\kappa_m(x_1, \ldots, x_m, \eta)\Big]\,\mathbb{1}_{B^m}(x_1, \ldots, x_m)$$

for $(x_1, \ldots, x_m) \in \mathbb{X}^m$, $m \in \mathbb{N}$, *and* $B \in \mathcal{X}$.

Proof. Fix $B \in \mathcal{X}$ and $m \in \mathbb{N}$. For any $D \in \mathcal{X}^{\otimes m}$ we have, by definition of the Janossy measure,

$$J_{\eta,B,m}(D) = \frac{1}{m!}\,\mathbb{E}\Big[\mathbb{1}\big\{\eta(B) = m\big\}\,\eta_B^{(m)}(D)\Big] = \frac{1}{m!}\,\mathbb{E}\bigg[\int_{B^m} \mathbb{1}\big\{\eta(B) = m\big\}\,\mathbb{1}_D(x_1, \ldots, x_m)\,\mathrm{d}\eta^{(m)}(x_1, \ldots, x_m)\bigg].$$

By Lemma 2.14 this term equals

$$\begin{aligned}
&\frac{1}{m!}\int_{B^m} \mathbb{E}\bigg[\mathbb{1}\bigg\{\Big(\eta + \sum_{j=1}^{m} \delta_{x_j}\Big)(B) = m\bigg\}\,\mathbb{1}_D(x_1, \ldots, x_m)\,\kappa_m(x_1, \ldots, x_m, \eta)\bigg]\,\mathrm{d}\lambda^m(x_1, \ldots, x_m)\\
&\quad= \int_{D \cap B^m} \frac{1}{m!}\,\mathbb{E}\Big[\mathbb{1}\big\{\eta(B) = 0\big\}\,\kappa_m(x_1, \ldots, x_m, \eta)\Big]\,\mathrm{d}\lambda^m(x_1, \ldots, x_m),
\end{aligned}$$

which concludes the proof. □

Recall that the existence of Janossy densities implies that the distribution of a Gibbs process is locally absolutely continuous with respect to the distribution of a Poisson process as detailed in Corollary 2.12.

Remark 2.19. For $B \in \mathcal{X}$ with $\mathbb{P}\big(\eta(B) < \infty\big) = 1$, we can use Lemma 2.17 to rewrite

$$j_{\eta,B,m}(x_1, \ldots, x_m) = \frac{1}{m!}\,\mathbb{E}\bigg[\frac{\kappa_m(x_1, \ldots, x_m, \eta_{B^c})}{Z_B(\eta_{B^c})}\bigg]\,\mathbb{1}_{B^m}(x_1, \ldots, x_m).$$

We now turn to the correlation functions of a Gibbs process.

Lemma 2.20. *Let* η *be a Gibbs process in* $\mathbb{X}$ *with PI* κ. *Then* η *has the correlation functions*

$$\rho_{\eta,m}(x_1, \ldots, x_m) = \mathbb{E}\big[\kappa_m(x_1, \ldots, x_m, \eta)\big]$$

for $(x_1, \ldots, x_m) \in \mathbb{X}^m$ *and* $m \in \mathbb{N}$.

Proof. Fix $m \in \mathbb{N}$ and $D \in \mathcal{X}^{\otimes m}$. Lemma 2.14 implies

$$\begin{aligned}
\alpha_{\eta,m}(D) = \mathbb{E}\big[\eta^{(m)}(D)\big] &= \mathbb{E}\bigg[\int_{\mathbb{X}^m} \mathbb{1}_D(x_1, \ldots, x_m)\,\mathrm{d}\eta^{(m)}(x_1, \ldots, x_m)\bigg]\\
&= \mathbb{E}\bigg[\int_{\mathbb{X}^m} \mathbb{1}_D(x_1, \ldots, x_m)\,\kappa_m(x_1, \ldots, x_m, \eta)\,\mathrm{d}\lambda^m(x_1, \ldots, x_m)\bigg]\\
&= \int_{\mathbb{X}^m} \mathbb{1}_D(x_1, \ldots, x_m)\,\mathbb{E}\big[\kappa_m(x_1, \ldots, x_m, \eta)\big]\,\mathrm{d}\lambda^m(x_1, \ldots, x_m).
\end{aligned}$$

□

Remark 2.21. With the knowledge about correlation functions we supplement the result on the finiteness of the partition function in Lemma 2.17. Let η be a Gibbs process with PI κ and fix $B \in \mathcal{X}$ such that $\mathbb{P}\big(\eta(B) < \infty\big) = 1$. By definition of the partition function, we have

$$\mathbb{E}\big[Z_B(\eta)\big] = 1 + \sum_{m=1}^{\infty} \frac{1}{m!}\int_{B^m} \mathbb{E}\big[\kappa_m(x_1, \ldots, x_m, \eta)\big]\,\mathrm{d}\lambda^m(x_1, \ldots, x_m)$$

and, using Lemma 2.20, we obtain

$$\mathbb{E}\big[Z_B(\eta)\big] = 1 + \sum_{m=1}^{\infty} \frac{\alpha_{\eta,m}(B^m)}{m!} = \mathbb{E}\big[2^{\eta(B)}\big]$$

as in (2.6). Hence, $\mathbb{P}\big(Z_B(\eta) < \infty\big) = 1$ whenever $\mathbb{E}\big[2^{\eta(B)}\big] < \infty$. Remark 2.10 gives an overview as to when this property is satisfied. ○

Next, we consider intensity and Palm measures of Gibbs processes. We call a function $\vartheta : \mathbb{X} \to [0, \infty]$ locally integrable with respect to λ (or locally λ-integrable) if $\int_B \vartheta(x)\,\mathrm{d}\lambda(x) < \infty$ for each $B \in \mathcal{X}_b$.

Lemma 2.22. *Let η be a Gibbs process in $\mathbb{X}$ with PI κ. Assume that the function $x \mapsto \mathbb{E}\big[\kappa(x, \eta)\big]$ is locally integrable with respect to λ. Then*

$$\Theta(B) = \int_B \mathbb{E}\big[\kappa(x, \eta)\big]\,\mathrm{d}\lambda(x), \quad B \in \mathcal{X},$$

is the locally finite intensity measure of η. The Palm measures of η are given by

$$\mathrm{P}_x(A) = \frac{\mathbb{E}\big[\mathbb{1}_A(\eta + \delta_x)\,\kappa(x, \eta)\big]}{\mathbb{E}\big[\kappa(x, \eta)\big]}, \quad A \in \mathcal{N},$$

for Θ-a.e. $x \in \mathbb{X}$.

Proof. We first calculate the intensity measure. For $B \in \mathcal{X}$ the GNZ equation gives

$$\Theta(B) = \mathbb{E}\bigg[\int_{\mathbb{X}} \mathbb{1}_B(x)\,\mathrm{d}\eta(x)\bigg] = \mathbb{E}\bigg[\int_{\mathbb{X}} \mathbb{1}_B(x)\,\kappa(x, \eta)\,\mathrm{d}\lambda(x)\bigg] = \int_B \mathbb{E}\big[\kappa(x, \eta)\big]\,\mathrm{d}\lambda(x).$$

By the integrability assumption, the integral on the right hand side is finite whenever $B \in \mathcal{X}_b$, so Θ is locally finite. We put

$$D = \big\{x \in \mathbb{X} : \mathbb{E}\big[\kappa(x, \eta)\big] = 0\big\}.$$

Apparently, $D \in \mathcal{X}$ with $\Theta(D) = 0$. For $x \in D^{\mathrm{c}}$, let P_x be given as in the statement of the lemma and, for $x \in D$, let $\mathrm{P}_x = \mathrm{P}$ for some fixed probability measure P on $(\mathbf{N}, \mathcal{N})$. Then P_x is a probability measure on $(\mathbf{N}, \mathcal{N})$ for each $x \in \mathbb{X}$ and, for any measurable function $f : \mathbb{X} \times \mathbf{N} \to [0, \infty]$, we have

$$\begin{aligned}\int_{\mathbb{X}} \int_{\mathbf{N}} f(x, \mu)\,\mathrm{d}\mathrm{P}_x(\mu)\,\mathrm{d}\Theta(x) = \int_{\mathbb{X}} \frac{\mathbb{E}\big[f(x, \eta + \delta_x)\,\kappa(x, \eta)\big]}{\mathbb{E}\big[\kappa(x, \eta)\big]}\,\mathrm{d}\Theta(x) &= \int_{\mathbb{X}} \mathbb{E}\big[f(x, \eta + \delta_x)\,\kappa(x, \eta)\big]\,\mathrm{d}\lambda(x) \\ &= \mathbb{E}\bigg[\int_{\mathbb{X}} f(x, \eta)\,\mathrm{d}\eta(x)\bigg] \\ &= \int_{\mathbf{N}} \int_{\mathbb{X}} f(x, \mu)\,\mathrm{d}\mu(x)\,\mathrm{d}\mathbb{P}^{\eta}(\mu),\end{aligned}$$

which proves that the P_x are Palm measures of η. □

We conclude from Lemma 2.22 that the intensity measure of a Gibbs process is absolutely continuous with respect to its reference measure λ. If $(\mathbb{X}, \mathcal{X})$ is separable, in that $\{x\} \in \mathcal{X}$ for each $x \in \mathbb{X}$, we call a measure ν on $\mathbb{X}$ diffuse if $\nu(\{x\}) = 0$ for all $x \in \mathbb{X}$. Hence if, in such a space, λ is diffuse then so is the intensity measure of the Gibbs process. Let us recall that $\mu \in \mathbf{N}$ is called simple if $\mu(\{x\}) \leq 1$ for each $x \in \mathbb{X}$ and that the set containing all such measure is denoted by $\mathbf{N}_s$. A point process η is called simple if

$$\mathbb{P}(\eta \in \mathbf{N}_s) = 1,$$

subtly assuming that $\mathbf{N}_s \in \mathcal{N}$ which, by Remark A.14, holds if $(\mathbb{X}, \mathcal{X})$ has a measurable diagonal. Just as any Poisson process with diffuse intensity measure is simple, any Gibbs process with a diffuse reference measure is simple.

Lemma 2.23. *Let $(\mathbb{X}, \mathcal{X})$ have a measurable diagonal and let η be a Gibbs process in $\mathbb{X}$ with PI κ and diffuse reference measure λ. Then η is simple.*

Proof. By Remark A.14 it suffices to prove that $\eta^{(2)}(D_{\mathbb{X}}) = 0$ almost surely. Since λ is diffuse, we have

$$\lambda^2(D_{\mathbb{X}}) = \int_{\mathbb{X}^2} \mathbb{1}\{x = y\} \, \mathrm{d}\lambda(y) \, \mathrm{d}\lambda(x) = \int_{\mathbb{X}} \lambda(\{x\}) \, \mathrm{d}\lambda(x) = 0.$$

Lemma 2.20 implies

$$\mathbb{E}\big[\eta^{(2)}(D_{\mathbb{X}})\big] = \int_{D_{\mathbb{X}}} \mathbb{E}\big[\kappa_2(x, y, \eta)\big] \, \mathrm{d}\lambda^2(x, y) = 0,$$

which gives $\eta^{(2)}(D_{\mathbb{X}}) = 0$ $\mathbb{P}$-a.s. and finishes the proof. □

2.5. Finite Gibbs processes

Under the assumption that the partition function is finite we can explicitly specify a probability distribution which qualifies as that of a finite Gibbs process, even in abstract measurable spaces. In fact, we show in Lemma 2.27 that all finite Gibbs processes are necessarily of this form.

Let $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ be measurable and satisfy the cocycle property (2.8). For $C \in \mathcal{X}$ and $\psi \in \mathbf{N}$ define

$$\kappa^{(C,\psi)}(x, \mu) = \kappa(x, \psi + \mu) \, \mathbb{1}_C(x), \quad x \in \mathbb{X}, \; \mu \in \mathbf{N}. \tag{2.9}$$

The map $\kappa^{(C,\psi)} : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ is measurable and inherits the cocycle property from κ. As in Section 2.3 we define the symmetric functions $\kappa_m^{(C,\psi)}$ $(m \in \mathbb{N})$ and the partition function $Z_B^{(C,\psi)}$ $(B \in \mathcal{X})$ corresponding to $\kappa^{(C,\psi)}$. The partition functions of $\kappa^{(C,\psi)}$ and κ relate in the following way.

Lemma 2.24. *For $B, C \in \mathcal{X}$ and $\psi, \nu \in \mathbf{N}$ it holds true that $Z_B^{(C,\psi)}(\nu) = Z_{B \cap C}(\psi + \nu)$.*

Proof. By definition, we have

$$\begin{aligned} Z_B^{(C,\psi)}(\nu) &= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{B^m} \kappa_m^{(C,\psi)}(x_1, \ldots, x_m, \nu) \, \mathrm{d}\lambda^m(x_1, \ldots, x_m) \\ &= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{(B \cap C)^m} \kappa_m(x_1, \ldots, x_m, \psi + \nu) \, \mathrm{d}\lambda^m(x_1, \ldots, x_m) \\ &= Z_{B \cap C}(\psi + \nu). \end{aligned}$$ □

We now take the first step in specifying the distribution of finite Gibbs processes.

Lemma 2.25. *Let $C \in \mathcal{X}$ with $\lambda(C) < \infty$ and $\psi \in \mathbf{N}$ be such that $Z_C(\psi) < \infty$. Consider a point process ξ in $\mathbb{X}$ whose distribution is given by*

$$\mathbb{P}^{\xi} = \frac{1}{Z_C(\psi)} \left(\mathbb{1}\{\mathbf{0} \in \cdot\} + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{C^m} \mathbb{1}\left\{ \sum_{j=1}^{m} \delta_{x_j} \in \cdot \right\} \kappa_m(x_1, \ldots, x_m, \psi) \, \mathrm{d}\lambda^m(x_1, \ldots, x_m) \right). \tag{2.10}$$

Then ξ is a Gibbs process with PI $\kappa^{(C,\psi)}$ and reference measure λ. Moreover, ξ is almost surely finite and satisfies $\xi(C^{\mathrm{c}}) = 0$ $\mathbb{P}$-a.s.

Proof. By definition of $Z_C(\psi)$, $\mathbb{P}^{\xi}$ is a probability measure on $\mathbf{N}$, and we clearly have $\mathbb{P}^{\xi}(\mathbf{N}_C \cap \mathbf{N}_f) = 1$. Let the map $\mathfrak{D}_{C,\psi} : \mathbf{N} \to [0, \infty]$ be defined by

$$\mathfrak{D}_{C,\psi}(\mu) = \frac{\mathrm{e}^{\lambda(C)}}{Z_C(\psi)} \left(\mathbb{1}\{\mu(C) = 0\} + \sum_{m=1}^{\infty} \frac{1}{m!} \mathbb{1}\{\mu(C) = m\} \int_{C^m} \kappa_m(x_1, \ldots, x_m, \psi) \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m) \right).$$

The map $\mathfrak{D}_{C,\psi}$ is measurable by Lemma A.4 and we have

$$\mathfrak{D}_{C,\psi}\Big(\sum_{j=1}^{m}\delta_{x_j}\Big) = \frac{\mathrm{e}^{\lambda(C)}}{Z_C(\psi)} \cdot \kappa_m(x_1,\dots,x_m,\psi) \tag{2.11}$$

for all $x_1,\dots,x_m \in C$ and each $m \in \mathbb{N}$. For $\mu \in \mathbf{N}$ and $x \in C$, we have

$$\mathfrak{D}_{C,\psi}(\mu+\delta_x) = \mathfrak{D}_{C,\psi}(\mu)\cdot\kappa(x,\psi+\mu_C). \tag{2.12}$$

Indeed, this follows immediately if $\mu(C)=0$ or if $\mu(C)=\infty$. For $\mu\in\mathbf{N}$ with $\mu(C)=m\in\mathbb{N}$ observe that the definition of $\mathfrak{D}_{C,\psi}$ and Equation (A.4) give

$$\begin{aligned}\mathfrak{D}_{C,\psi}(\mu+\delta_x) &= \frac{\mathrm{e}^{\lambda(C)}}{Z_C(\psi)}\cdot\frac{1}{(m+1)!}\int_{C^{m+1}}\kappa_{m+1}(x_1,\dots,x_{m+1},\psi)\,\mathrm{d}(\mu+\delta_x)^{(m+1)}(x_1,\dots,x_{m+1})\\ &= \frac{\mathrm{e}^{\lambda(C)}}{Z_C(\psi)}\cdot\frac{1}{m!}\int_{C^m}\kappa_{m+1}(x,x_1,\dots,x_m,\psi)\,\mathrm{d}\mu^{(m)}(x_1,\dots,x_m)\end{aligned}$$

and, by definition of κ_{m+1}, Equation (2.11), and Proposition A.5, this term equals

$$\frac{1}{m!}\int_{C^m}\mathfrak{D}_{C,\psi}\Big(\sum_{j=1}^{m}\delta_{x_j}\Big)\cdot\kappa\Big(x,\psi+\sum_{j=1}^{m}\delta_{x_j}\Big)\,\mathrm{d}\mu^{(m)}(x_1,\dots,x_m) = \mathfrak{D}_{C,\psi}(\mu)\cdot\kappa(x,\psi+\mu_C).$$

Let Φ be a Poisson process in $\mathbb{X}$ with intensity measure λ. The definition of $\mathbb{P}^\xi$ and Corollary 2.4 imply that, for any measurable map $F:\mathbf{N}\to[0,\infty]$,

$$\begin{aligned}\mathbb{E}\big[F(\xi)\big] &= \frac{1}{Z_C(\psi)}\bigg(F(\mathbf{0})+\sum_{m=1}^{\infty}\frac{1}{m!}\int_{C^m}F\Big(\sum_{j=1}^{m}\delta_{x_j}\Big)\,\kappa_m(x_1,\dots,x_m,\psi)\,\mathrm{d}\lambda^m(x_1,\dots,x_m)\bigg)\\ &= \mathbb{E}\big[F(\Phi_C)\cdot\mathfrak{D}_{C,\psi}(\Phi)\big].\end{aligned} \tag{2.13}$$

Applied to $F(\mu)=\int_{\mathbb{X}}f(x,\mu)\,\mathrm{d}\mu(x)$, where $f:\mathbb{X}\times\mathbf{N}\to[0,\infty]$ is a measurable function, and combined with the Mecke equation (Proposition 2.1) as well as (2.12), this yields

$$\begin{aligned}\mathbb{E}\bigg[\int_{\mathbb{X}}f(x,\xi)\,\mathrm{d}\xi(x)\bigg] &= \mathbb{E}\bigg[\int_{\mathbb{X}}f(x,\Phi_C)\,\mathfrak{D}_{C,\psi}(\Phi)\,\mathrm{d}\Phi_C(x)\bigg]\\ &= \mathbb{E}\bigg[\int_C f(x,\Phi_C+\delta_x)\,\mathfrak{D}_{C,\psi}(\Phi+\delta_x)\,\mathrm{d}\lambda(x)\bigg]\\ &= \mathbb{E}\bigg[\int_C f(x,\Phi_C+\delta_x)\,\kappa(x,\psi+\Phi_C)\,\mathrm{d}\lambda(x)\cdot\mathfrak{D}_{C,\psi}(\Phi)\bigg].\end{aligned}$$

A second application of (2.13) to $F(\mu)=\int_C f(x,\mu+\delta_x)\,\kappa(x,\psi+\mu)\,\mathrm{d}\lambda(x)$ implies that the right hand side of the previous display equals

$$\mathbb{E}\bigg[\int_C f(x,\xi+\delta_x)\,\kappa(x,\psi+\xi)\,\mathrm{d}\lambda(x)\bigg] = \mathbb{E}\bigg[\int_{\mathbb{X}}f(x,\xi+\delta_x)\,\kappa^{(C,\psi)}(x,\xi)\,\mathrm{d}\lambda(x)\bigg],$$

which concludes the proof. □

It is obvious that the distribution defined in Lemma 2.25 makes sense as soon as $Z_C(\psi)<\infty$ even without the assumption that $\lambda(C)<\infty$. Indeed, the previous result easily generalizes.

Corollary 2.26. *Let $C\in\mathcal{X}$ and $\psi\in\mathbf{N}$ be such that $Z_C(\psi)<\infty$. A point process ξ in $\mathbb{X}$ with distribution given by (2.10) is a finite Gibbs process with PI $\kappa^{(C,\psi)}$ and reference measure λ.*

Proof. By the definition of partition functions, and monotone convergence, we have

$$\lim_{\ell \to \infty} Z_{C \cap B_\ell}(\psi) = Z_C(\psi),$$

where $1 \leq Z_{C \cap B_\ell}(\psi) \leq Z_C(\psi) < \infty$ for all $\ell \in \mathbb{N}$. For each $\ell \in \mathbb{N}$, let ξ_ℓ be a point process with distribution as in (2.10), but with C replaced by $C \cap B_\ell$. As such, each ξ_ℓ is a Gibbs process with PI $\kappa^{(C \cap B_\ell, \psi)}$. Let $F : \mathbf{N} \to [0, \infty]$ be an arbitrary measurable function and $F_\ell : \mathbf{N} \to [0, \infty]$ ($\ell \in \mathbb{N}$) any sequence of measurable functions with $F_\ell(\mu) \nearrow F(\mu)$ for each $\mu \in \mathbf{N}$ as $\ell \to \infty$. By construction of the processes, and monotone convergence, we have (as $\ell \to \infty$)

$$\begin{aligned}
\mathbb{E}\big[F_\ell(\xi_\ell)\big] &= \frac{1}{Z_{C \cap B_\ell}(\psi)} \bigg(F_\ell(\mathbf{0}) + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{(C \cap B_\ell)^m} F_\ell\Big(\sum_{j=1}^{m} \delta_{x_j} \Big) \, \kappa_m(x_1, \ldots, x_m, \psi) \, \mathrm{d}\lambda^m(x_1, \ldots, x_m) \bigg) \\
&\to \frac{1}{Z_C(\psi)} \bigg(F(\mathbf{0}) + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{C^m} F\Big(\sum_{j=1}^{m} \delta_{x_j} \Big) \, \kappa_m(x_1, \ldots, x_m, \psi) \, \mathrm{d}\lambda^m(x_1, \ldots, x_m) \bigg) \\
&= \mathbb{E}\big[F(\xi)\big].
\end{aligned}$$

Applied twice, this observation yields

$$\begin{aligned}
\mathbb{E}\bigg[\int_{\mathbb{X}} f(x, \xi) \, \mathrm{d}\xi(x) \bigg] = \lim_{\ell \to \infty} \mathbb{E}\bigg[\int_{\mathbb{X}} f(x, \xi_\ell) \, \mathrm{d}\xi_\ell(x) \bigg] &= \lim_{\ell \to \infty} \mathbb{E}\bigg[\int_{\mathbb{X}} f(x, \xi_\ell + \delta_x) \, \kappa^{(C \cap B_\ell, \psi)}(x, \xi_\ell) \, \mathrm{d}\lambda(x) \bigg] \\
&= \lim_{\ell \to \infty} \mathbb{E}\bigg[\int_{\mathbb{X}} f(x, \xi_\ell + \delta_x) \, \kappa^{(C, \psi)}(x, \xi_\ell) \, \mathbb{1}_{B_\ell}(x) \, \mathrm{d}\lambda(x) \bigg] \\
&= \mathbb{E}\bigg[\int_{\mathbb{X}} f(x, \xi + \delta_x) \, \kappa^{(C, \psi)}(x, \xi) \, \mathrm{d}\lambda(x) \bigg]
\end{aligned}$$

for every measurable map $f : \mathbb{X} \times \mathbf{N} \to [0, \infty]$, which concludes the proof. $\square$

The measure ψ in Lemma 2.25 or Corollary 2.26 is called *boundary condition*. If $\psi = \mathbf{0}$, the construction gives a Gibbs process whose PI is κ (restricted to the domain C). With the upcoming result we establish that any finite Gibbs process in a window $C \in \mathcal{X}$ has to have the distribution from Lemma 2.25. In particular, finite Gibbs processes are unique in distribution.

Lemma 2.27. *Let $C \in \mathcal{X}$ and $\psi \in \mathbf{N}$. A finite Gibbs process with PI $\kappa^{(C, \psi)}$ and reference measure λ exists if, and only if, $Z_C(\psi) < \infty$. If ξ is such a process, then the distribution of ξ is given by* (2.10).

While it was known that finite partition functions are essential to produce finite Gibbs processes, the equivalence seems not to have been stated in the literature until the preprint corresponding to this part of the thesis appeared. Before, finite Gibbs processes were constructed exclusively on bounded sets.

Proof. If $Z_C(\psi) < \infty$, a finite Gibbs process with PI $\kappa^{(C, \psi)}$ exists by Corollary 2.26. Now, let ξ be a finite Gibbs process with PI $\kappa^{(C, \psi)}$. The GNZ equation yields

$$\mathbb{E}\big[\xi(C^c)\big] = \mathbb{E}\bigg[\int_{\mathbb{X}} \mathbb{1}_{C^c}(x) \, \mathrm{d}\xi(x) \bigg] = \mathbb{E}\bigg[\int_{\mathbb{X}} \mathbb{1}_{C^c}(x) \, \kappa(x, \psi + \xi) \, \mathbb{1}_C(x) \, \mathrm{d}\lambda(x) \bigg] = 0,$$

so $\mathbb{P}^\xi(\mathbf{N}_C) = 1$. By Lemmata 2.24 and 2.17 we get

$$Z_C(\psi) = Z_C^{(C, \psi)}(\mathbf{0}) = Z_C^{(C, \psi)}(\xi_{C^c}) < \infty,$$

where the statements involving ξ_{C^c} hold almost surely. By Lemma 2.17 and Remark 2.19, the Janossy densities of ξ on the full domain C are $J_{\xi,C,0} = Z_C(\psi)^{-1}$ and

$$j_{\xi,C,m}(x_1,\dots,x_m) = \frac{1}{m!}\cdot\frac{\kappa_m(x_1,\dots,x_m,\psi)}{Z_C(\psi)}\,\mathbb{1}_{C^m}(x_1,\dots,x_m),\quad x_1,\dots,x_m\in\mathbb{X},$$

so Lemma 2.3 concludes the proof. □

In the proof of the previous lemma we have used a corollary to Lemma 2.17 and Remark 2.19 concerning the Janossy densities of a Gibbs process on its full domain. This we state explicitly at this point and we add a further conclusion on the absolute continuity of the finite Gibbs process distribution with respect to the distribution of a Poisson process. The density function corresponding to the latter observation is $\mathfrak{D}_{C,\psi}$ from the proof of Lemma 2.25.

Corollary 2.28. *Let $C\in\mathcal{X}$ and $\psi\in\mathbf{N}$. If ξ is a finite Gibbs process with PI $\kappa^{(C,\psi)}$, then*

$$J_{\xi,C,0} = \mathbb{P}\big(\xi(C)=0\big) = \frac{1}{Z_C(\psi)}.$$

Moreover, the Janossy densities of ξ on the full domain C are

$$j_{\xi,C,m}(x_1,\dots,x_m) = \frac{1}{m!}\cdot\frac{\kappa_m(x_1,\dots,x_m,\psi)}{Z_C(\psi)}\,\mathbb{1}_{C^m}(x_1,\dots,x_m)$$

for $x_1,\dots,x_m\in\mathbb{X}$ and $m\in\mathbb{N}$. If $\lambda(C)<\infty$, then $\mathbb{P}^\xi$ is absolutely continuous with respect to Π_{λ_C} with density function $\mathfrak{D}_{C,\psi}$.

Remark 2.29. When considering a Poisson process Φ in $\mathbb{X}$ with intensity measure λ, knowing that a set $B\in\mathcal{X}$ satisfies $\lambda(B)<\infty$ is enough to conclude that $\Phi(B)<\infty$ almost surely. For a Gibbs process η this is not necessarily so, and we have to state the assumptions $\lambda(B)<\infty$ and $\eta(B)<\infty$ (almost surely) separately if any of them is needed. One situation in which this separation is not necessary is when κ is uniformly bounded by some constant $\vartheta\ge 0$, as this gives

$$\mathbb{E}\big[\eta(B)\big] = \int_{\mathbb{X}}\mathbb{E}\big[\kappa(x,\eta)\big]\,\mathrm{d}\lambda(x)\le\vartheta\cdot\lambda(B).$$

In Corollary 2.35 in Section 2.6 we discuss a classical alternative representation of the distribution in (2.10) given in terms of the Poisson process distribution. However, that representation requires the assumption $\lambda(C)<\infty$ in addition to the finiteness of the partition function. Thus, the results discussed in this section are slightly more general. ○

2.6. Hamiltonians and energy functions

Let $\kappa:\mathbb{X}\times\mathbf{N}\to[0,\infty)$ be measurable and satisfy the cocycle assumption (2.8). Based on κ we define the *Hamiltonian* $H:\mathbf{N}\times\mathbf{N}\to(-\infty,\infty]$ as

$$H(\mu,\psi) = \infty\cdot\mathbb{1}\big\{\mu(\mathbb{X})=\infty\big\} - \sum_{m=1}^{\infty}\mathbb{1}\big\{\mu(\mathbb{X})=m\big\}\cdot\log\left(\frac{1}{m!}\int_{\mathbb{X}^m}\kappa_m(x_1,\dots,x_m,\psi)\,\mathrm{d}\mu^{(m)}(x_1,\dots,x_m)\right).$$

Lemma 2.30. *The Hamiltonian is well-defined and measurable.*

Proof. For any $m\in\mathbb{N}$ and $\psi\in\mathbf{N}$, the map

$$G(\mu) = \int_{\mathbb{X}^m}\kappa_m(x_1,\dots,x_m,\psi)\,\mathrm{d}\mu^{(m)}(x_1,\dots,x_m)$$

is measurable by Lemma A.4, and Lemma A.6 implies that $G(\mu) < \infty$ for all $\mu \in \mathbf{N}_f$. Hence, taking indicator functions into account, the Hamiltonian cannot attain the value $-\infty$ and is well-defined. Using that $[0,\infty) \ni s \mapsto -\log(s) \in (-\infty,\infty]$ is measurable, Lemma A.4 implies that H is measurable. □

The following properties of the Hamiltonian are immediate from its definition.

Corollary 2.31. *The Hamiltonian satisfies $H(\mathbf{0},\psi) = 0$ for each $\psi \in \mathbf{N}$, and $H(\mu,\psi) = \infty$ if $\mu(\mathbb{X}) = \infty$. Moreover, if $\mu = \sum_{j=1}^m \delta_{x_j}$ for $x_1,\dots,x_m \in \mathbb{X}$ and $m \in \mathbb{N}$, then*

$$H(\mu,\psi) = -\log\big(\kappa_m(x_1,\dots,x_m,\psi)\big)$$

or, equivalently, $\kappa_m(x_1,\dots,x_m,\psi) = \mathrm{e}^{-H(\mu,\psi)}$.

Remark 2.32. Let $\mu,\psi \in \mathbf{N}$ with $\mu(\mathbb{X}) = m$. Proposition A.5 gives

$$\begin{aligned}
&-\log\left(\frac{1}{m!}\int_{\mathbb{X}^m} \kappa_m(x_1,\dots,x_m,\psi)\,\mathrm{d}\mu^{(m)}(x_1,\dots,x_m)\right) \\
&\quad= \frac{1}{m!}\int_{\mathbb{X}^m} -\log\left(\frac{1}{m!}\int_{\mathbb{X}^m} \kappa_m(y_1,\dots,y_m,\psi)\,\mathrm{d}\Big(\sum_{j=1}^m \delta_{x_j}\Big)^{(m)}(y_1,\dots,y_m)\right)\mathrm{d}\mu^{(m)}(x_1,\dots,x_m) \\
&\quad= \frac{1}{m!}\int_{\mathbb{X}^m} -\log\big(\kappa_m(x_1,\dots,x_m,\psi)\big)\,\mathrm{d}\mu^{(m)}(x_1,\dots,x_m),
\end{aligned}$$

keeping in mind that the integral on the left hand side is finite by the proof of Lemma 2.30. Thus, the logarithm in the definition of H may be drawn into the integral accordingly. ○

For $\mu = \sum_{i=1}^k \delta_{x_i}$, $\nu = \sum_{j=1}^m \delta_{y_j}$ (with $x_1,\dots,x_k,y_1,\dots,y_m \in \mathbb{X}$ and $k,m \in \mathbb{N}$) and $\psi \in \mathbf{N}$, Corollary 2.31, as well as the definition and symmetry of κ_{k+m}, imply that the Hamiltonian satisfies

$$\begin{aligned}
H(\mu+\nu,\psi) &= -\log\Big(\kappa_{k+m}(x_1,\dots,x_k,y_1,\dots,y_m,\psi)\Big) \\
&= -\log\Big(\kappa_k(x_1,\dots,x_k,\psi)\cdot\kappa_m(y_1,\dots,y_m,\psi+\delta_{x_1}+\ldots+\delta_{x_k})\Big) \\
&= H(\mu,\psi) + H(\nu,\psi+\mu), \qquad (2.14)
\end{aligned}$$

which is the *energy function property.* The property gives rise to the interpretation that $H(\mu,\psi)$ captures the interactions between points in the configuration μ as well as between points in μ and points in the boundary condition ψ. In fact, the following lemma shows, with the help of Proposition A.5, that the general definition of H in measurable spaces keeps the energy function property more generally and not only for sums of Dirac measures as in (2.14).

Lemma 2.33 (Energy function property). *Let $\mu,\nu,\psi \in \mathbf{N}$. The Hamiltonian satisfies*

$$H(\mu+\nu,\psi) = H(\mu,\psi) + H(\nu,\psi+\mu).$$

Proof. If $\mu(\mathbb{X}) = 0$ or $\nu(\mathbb{X}) = 0$, the claim is trivially true. The same can be said if $\mu(\mathbb{X}) = \infty$ or $\nu(\mathbb{X}) = \infty$. Thus, we assume that $\mu(\mathbb{X}) = k \in \mathbb{N}$ and $\nu(\mathbb{X}) = m \in \mathbb{N}$. Applying Proposition A.5 to the map $\mathrm{e}^{-H(\,\cdot\,,\psi)} : \mathbf{N} \to [0,\infty)$ gives

$$\mathrm{e}^{-H(\mu+\nu,\psi)} = \frac{1}{(k+m)!}\int_{\mathbb{X}^{k+m}} \exp\left(-H\Big(\sum_{j=1}^{k+m}\delta_{z_j},\,\psi\Big)\right)\mathrm{d}(\mu+\nu)^{(k+m)}(z_1,\dots,z_{k+m}).$$

By Lemma A.8 and Equation (2.14), the right hand side is equal to

$$\begin{aligned}
&\frac{1}{k!\cdot m!}\int_{\mathbb{X}^k}\int_{\mathbb{X}^m}\exp\Big(-H\Big(\sum_{i=1}^{k}\delta_{x_i}+\sum_{j=1}^{m}\delta_{y_j},\,\psi\Big)\Big)\,\mathrm{d}\nu^{(m)}(y_1,\dots,y_m)\,\mathrm{d}\mu^{(k)}(x_1,\dots,x_k)\\
&\quad=\frac{1}{k!}\int_{\mathbb{X}^k}\exp\Big(-H\Big(\sum_{i=1}^{k}\delta_{x_i},\,\psi\Big)\Big)\\
&\qquad\qquad\cdot\Big[\frac{1}{m!}\int_{\mathbb{X}^m}\exp\Big(-H\Big(\sum_{j=1}^{m}\delta_{y_j},\,\psi+\sum_{i=1}^{k}\delta_{x_i}\Big)\Big)\,\mathrm{d}\nu^{(m)}(y_1,\dots,y_m)\Big]\,\mathrm{d}\mu^{(k)}(x_1,\dots,x_k).
\end{aligned}$$

Proposition A.5 applied to the map $\mathrm{e}^{-H(\,\cdot\,,\,\psi+\delta_{x_1}+\ldots+\delta_{x_k})}:\mathbf{N}\to[0,\infty)$ yields

$$\mathrm{e}^{-H(\mu+\nu,\,\psi)}=\frac{1}{k!}\int_{\mathbb{X}^k}\exp\Big(-H\Big(\sum_{i=1}^{k}\delta_{x_i},\,\psi\Big)\Big)\exp\Big(-H\Big(\nu,\,\psi+\sum_{i=1}^{k}\delta_{x_i}\Big)\Big)\,\mathrm{d}\mu^{(k)}(x_1,\dots,x_k)$$

and a final application of Proposition A.5 to the map $\mathrm{e}^{-H(\,\cdot\,,\,\psi)}\,\mathrm{e}^{-H(\nu,\,\psi+\,\cdot\,)}:\mathbf{N}\to[0,\infty)$ gives

$$\mathrm{e}^{-H(\mu+\nu,\,\psi)}=\mathrm{e}^{-H(\mu,\,\psi)}\cdot\mathrm{e}^{-H(\nu,\,\psi+\mu)}=\mathrm{e}^{-H(\mu,\,\psi)-H(\nu,\,\psi+\mu)}.$$

Taking logarithms concludes the proof. □

With a slight ambiguity in notation, which will not lead to any confusion, we define $H:\mathbf{N}\to(-\infty,\infty]$ by $H(\mu)=H(\mu,\mathbf{0})$. This map is usually called *energy function*.

We now use the Hamiltonian to rewrite several of our previous observations about Gibbs processes. The formulae we obtain are perhaps more familiar to some readers. First of all, we write the partition function as an integral with respect to the distribution of a Poisson process. Note that while the partition function is defined on general measurable sets, the following representation only holds for sets of finite λ-measure.

Lemma 2.34. *Fix a set $B\in\mathcal{X}$ with $\lambda(B)<\infty$. For each $\psi\in\mathbf{N}$ the partition function $Z_B(\psi)$ can be written as*

$$Z_B(\psi)=\mathrm{e}^{\lambda(B)}\int_{\mathbf{N}}\mathrm{e}^{-H(\mu,\,\psi)}\,\mathrm{d}\Pi_{\lambda_B}(\mu).$$

Proof. Corollaries 2.4 and 2.31 imply that, for every $\psi\in\mathbf{N}$,

$$\begin{aligned}
\mathrm{e}^{\lambda(B)}\int_{\mathbf{N}}\mathrm{e}^{-H(\mu,\,\psi)}\,\mathrm{d}\Pi_{\lambda_B}(\mu)&=\mathrm{e}^{-H(\mathbf{0},\,\psi)}+\sum_{m=1}^{\infty}\frac{1}{m!}\int_{\mathbb{X}^m}\exp\Big(-H\Big(\sum_{j=1}^{m}\delta_{x_j},\,\psi\Big)\Big)\mathrm{d}\lambda_B^m(x_1,\dots,x_m)\\
&=1+\sum_{m=1}^{\infty}\frac{1}{m!}\int_{B^m}\kappa_m(x_1,\dots,x_m,\psi)\,\mathrm{d}\lambda^m(x_1,\dots,x_m)\\
&=Z_B(\psi).
\end{aligned}$$

□

In Lemma 2.27 we learned that the existence of finite Gibbs processes is invariably linked to the finiteness of partition functions and that the distribution of such a finite Gibbs process can be given explicitly. On sets of finite λ-measure this distribution can be rewritten in terms of the distribution of a Poisson process, similar to the observation in the previous lemma on partition functions. The following result is a consequence of Lemmata 2.25 and 2.27, and Corollary 2.4.

Corollary 2.35. *Let $C\in\mathcal{X}$ with $\lambda(C)<\infty$ and $\psi\in\mathbf{N}$ be such that $Z_C(\psi)<\infty$. A point process ξ in $\mathbb{X}$ is a finite Gibbs process with PI $\kappa^{(C,\psi)}$ as in (2.9) if, and only if, the distribution of ξ is given by*

$$\mathbb{P}^{\xi}=\frac{\mathrm{e}^{\lambda(C)}}{Z_C(\psi)}\int_{\mathbf{N}}\mathbb{1}\{\mu\in\cdot\,\}\,\mathrm{e}^{-H(\mu,\,\psi)}\,\mathrm{d}\Pi_{\lambda_C}(\mu).$$

It is reasonable to construct the Hamiltonian $H^{(C,\psi)}$ corresponding to $\kappa^{(C,\psi)}$ as

$$H^{(C,\psi)}(\mu,\nu) = -\sum_{m=1}^{\infty} \mathbb{1}\{\mu(C)=m\} \cdot \log\bigg(\frac{1}{m!}\int_{C^m} \kappa_m(x_1,\dots,x_m,\psi+\nu)\,\mathrm{d}\mu^{(m)}(x_1,\dots,x_m)\bigg) + \infty \cdot \mathbb{1}\{\mu(C)=\infty\},$$

for $\mu,\nu \in \mathbf{N}$. This choice guarantees the following intuitive relation to H, which follows immediately from the definitions.

Corollary 2.36. *Let $C \in \mathcal{X}$ and $\psi \in \mathbf{N}$. Then $H^{(C,\psi)}(\mu,\nu) = H(\mu_C,\psi+\nu)$ for all $\mu,\nu \in \mathbf{N}$.*

Both Corollaries 2.28 and 2.35 express the distribution of a finite Gibbs process in terms of a density function with respect to the Poisson process distribution. Of course these density functions have to agree, so at this point we should study their relation.

Remark 2.37. Let $C \in \mathcal{X}$ with $\lambda(C) < \infty$ and $\psi \in \mathbf{N}$ be such that $Z_C(\psi) < \infty$. If ξ is a finite Gibbs process with PI $\kappa^{(C,\psi)}$, then Corollary 2.28 implies that $\mathbb{P}^\xi$ is absolutely continuous with respect to Π_{λ_C} with density function $\mathfrak{D}_{C,\psi}$. For $\mu \in \mathbf{N}$ with $\mu(C) = m \in \mathbb{N}$, Proposition A.5 yields

$$\begin{aligned}
\mathfrak{D}_{C,\psi}(\mu) &= \frac{\mathrm{e}^{\lambda(C)}}{Z_C(\psi)} \cdot \frac{1}{m!}\int_{C^m} \kappa_m(x_1,\dots,x_m,\psi)\,\mathrm{d}\mu^{(m)}(x_1,\dots,x_m) \\
&= \frac{\mathrm{e}^{\lambda(C)}}{Z_C(\psi)} \cdot \frac{1}{m!}\int_{\mathbb{X}^m} \exp\Big(-H\Big(\sum_{j=1}^{m}\delta_{x_j},\psi\Big)\Big)\,\mathrm{d}\mu_C^{(m)}(x_1,\dots,x_m) \\
&= \frac{\mathrm{e}^{\lambda(C)}}{Z_C(\psi)} \cdot \exp\big(-H(\mu_C,\psi)\big),
\end{aligned}$$

and the left and right hand side are also equal if $\mu(C) \in \{0,\infty\}$. Since $\Pi_{\lambda_C}(\mathbf{N}_C) = 1$, this is consistent with Corollary 2.35. ○

We conclude this section by discussing another property of the Hamiltonian, namely hereditarity. There are several definitions of this notion. The choice we make with regard to the following section is a combination of the definitions from Møller and Waagepetersen (2004) and Jansen (2017).

Definition 2.38 (Hereditarity). A map $\mathfrak{D} : \mathbf{N} \to [0,\infty]$ is called *hereditary* if $\mathfrak{D}(\mu+\delta_x) > 0$ implies $\mathfrak{D}(\mu) > 0$ for all $\mu \in \mathbf{N}$ and $x \in \mathbb{X}$.

The Hamiltonian has strong hereditarity properties as the following lemma shows. In particular, the map $\mu \mapsto \mathrm{e}^{-H(\mu,\psi)}$, and hence the density function of the finite Gibbs process, is hereditary in terms of the previous definition for any $\psi \in \mathbf{N}$.

Lemma 2.39. *Let $\psi \in \mathbf{N}$. If $\mu \in \mathbf{N}$ is such that $H(\mu,\psi) = \infty$, then $H(\mu+\nu,\psi) = \infty$ for every $\nu \in \mathbf{N}$.*

Proof. If $\mu,\psi \in \mathbf{N}$ are such that $H(\mu,\psi) = \infty$, then Lemma 2.33 implies

$$H(\mu+\nu,\psi) = H(\mu,\psi) + H(\nu,\mu+\psi) = \infty$$

for every $\nu \in \mathbf{N}$. □

2.7. A MEASURE THEORETIC CHARACTERIZATION OF FINITE GIBBS PROCESSES

While the (arguably) most elegant definition of Gibbs processes in the point process theoretic setting is via the GNZ equations, definitions of Gibbs measures in statistical physics usually involve some version of the

DLR equations we discuss in the subsequent section. In spatial statistics, however, yet another definition is frequently employed in the context of finite Gibbs processes. In the corresponding literature, Gibbs measure are introduced, simply, as those probability measures on $\mathbf{N}_f$ which are absolutely continuous with respect to a Poisson process distribution, cf. Møller and Waagepetersen (2004) and the introductory part of this thesis. That perspective is perfectly compatible with our interpretation of finite Gibbs measures as solutions to the GNZ equations and the equivalence can be established in full generality.

We know from Corollary 2.35 that, for any finite Gibbs process ξ concentrated on $C \in \mathcal{X}$ with $\lambda(C) < \infty$, we have $\mathbb{P}^\xi \ll \Pi_{\lambda_C}$. The corresponding density function is given in terms of the Hamiltonian and it is hereditary by Lemma 2.39. The following characterization provides a converse to this statement. More specifically, any point process whose distribution has a hereditary density with respect to Π_{λ_C} is a Gibbs process with some suitable Papangelou intensity.

Theorem 2.40. *Let $C \in \mathcal{X}$. If ξ is a point process in $\mathbb{X}$ such that $\mathbb{P}^\xi$ has a hereditary Π_{λ_C}-density, then ξ is a Gibbs process.*

Note that if $\lambda(C) < \infty$, then $\Pi_{\lambda_C}(\mathbf{N}_f) = 1$ and $\mathbb{P}^\xi \ll \Pi_{\lambda_C}$ implies $\xi(\mathbb{X}) < \infty$ almost surely, so the theorem gives a finite Gibbs process in this case.

Proof. Denote by $\mathfrak{D}^*$ the (hereditary) density function of $\mathbb{P}^\xi$ with respect to Π_{λ_C}. Following Definition 6.1 of Møller and Waagepetersen (2004), we put

$$\kappa^*(x, \mu) = \begin{cases} \dfrac{\mathfrak{D}^*(\mu + \delta_x)}{\mathfrak{D}^*(\mu)}, & \text{if } x \in C \text{ and } \mathfrak{D}^*(\mu) > 0, \\ 0, & \text{else.} \end{cases} \tag{2.15}$$

Then κ^* is a measurable mapping $\mathbb{X} \times \mathbf{N} \to [0, \infty)$. For any measurable function $f : \mathbb{X} \times \mathbf{N} \to [0, \infty]$, the assumption of the theorem and Mecke's formula (Proposition 2.1) yield

$$\begin{aligned} \mathbb{E}\bigg[\int_{\mathbb{X}} f(x, \xi)\, \mathrm{d}\xi(x)\bigg] = \int_{\mathbf{N}} \int_{\mathbb{X}} f(x, \mu)\, \mathrm{d}\mu(x)\, \mathrm{d}\mathbb{P}^\xi(\mu) &= \int_{\mathbf{N}} \int_{\mathbb{X}} f(x, \mu)\, \mathfrak{D}^*(\mu)\, \mathrm{d}\mu(x)\, \mathrm{d}\Pi_{\lambda_C}(\mu) \\ &= \int_{\mathbf{N}} \int_C f(x, \mu + \delta_x)\, \mathfrak{D}^*(\mu + \delta_x)\, \mathrm{d}\lambda(x)\, \mathrm{d}\Pi_{\lambda_C}(\mu). \end{aligned}$$

As $\mathfrak{D}^*(\mu + \delta_x) > 0$ implies $\mathfrak{D}^*(\mu) > 0$ (by hereditarity of $\mathfrak{D}^*$), the choice of κ^* ensures that the right hand side of the above equation equals

$$\begin{aligned} \int_{\mathbf{N}} \int_C f(x, \mu + \delta_x)\, \kappa^*(x, \mu)\, \mathfrak{D}^*(\mu)\, \mathrm{d}\lambda(x)\, \mathrm{d}\Pi_{\lambda_C}(\mu) &= \int_{\mathbf{N}} \int_{\mathbb{X}} f(x, \mu + \delta_x)\, \kappa^*(x, \mu)\, \mathrm{d}\lambda(x)\, \mathrm{d}\mathbb{P}^\xi(\mu) \\ &= \mathbb{E}\bigg[\int_{\mathbb{X}} f(x, \xi + \delta_x)\, \kappa^*(x, \xi)\, \mathrm{d}\lambda(x)\bigg], \end{aligned}$$

which concludes the proof. □

Remark 2.41. Note that κ^* as defined in (2.15) satisfies the cocycle relation. To see this, let $x, y \in \mathbb{X}$ and $\mu \in \mathbf{N}$. If $\mathfrak{D}^*(\mu + \delta_x + \delta_y) > 0$, the hereditarity implies $\mathfrak{D}^*(\mu) > 0$, $\mathfrak{D}^*(\mu + \delta_x) > 0$, and $\mathfrak{D}^*(\mu + \delta_y) > 0$, so

$$\begin{aligned} \kappa^*(x, \mu) \cdot \kappa^*(y, \mu + \delta_x) = \frac{\mathfrak{D}^*(\mu + \delta_x)}{\mathfrak{D}^*(\mu)} \cdot \frac{\mathfrak{D}^*(\mu + \delta_x + \delta_y)}{\mathfrak{D}^*(\mu + \delta_x)} &= \frac{\mathfrak{D}^*(\mu + \delta_y)}{\mathfrak{D}^*(\mu)} \cdot \frac{\mathfrak{D}^*(\mu + \delta_x + \delta_y)}{\mathfrak{D}^*(\mu + \delta_y)} \\ &= \kappa^*(y, \mu) \cdot \kappa^*(x, \mu + \delta_y). \end{aligned}$$

If $\mathfrak{D}^*(\mu + \delta_x + \delta_y) = 0$, then $\kappa^*(y, \mu + \delta_x) = 0 = \kappa^*(x, \mu + \delta_y)$. In any case, the cocycle relation holds. ○

Remark 2.42. The take-away message of Theorem 2.40 is that finite Gibbs processes are precisely those point processes whose distribution is absolutely continuous with respect to the distribution of a Poisson process.

Focusing on the technical details shows that Gibbs processes admit slightly more structure, namely, their density function is hereditary.

A question suggested by Lemma 2.18 and Corollary 2.12 is whether a result similar to Theorem 2.40 holds for infinite Gibbs processes, in the sense that any point process whose distribution is locally absolutely continuous with respect to Π_λ is already a Gibbs process. When considering the proof of Theorem 2.40, however, the following problem poses itself immediately. Once density functions are only available locally, but some Papangelou intensity has to be defined globally in order to obtain a Gibbs process, the density functions have to satisfy consistency assumptions to allow for a well-defined PI. This problem is related to the identification of admissible kernels in the context of the definition of Gibbs processes based on specifications, cf. Preston (1976). The notion of specifications is discussed briefly in the context of the DLR equations in the following section. The DLR equations will also provide additional insight on the question of local absolute continuity. ○

2.8. The DLR equations and the notion of specifications

In this section, we prove the so-called DLR equations, named after Dobrushin, Lanford, and Ruelle, referring to Dobrushin (1968b,c, 1969), Lanford and Ruelle (1969), and Ruelle (1970). Our presentation is loosely based on Nguyen and Zessin (1979) and Matthes et al. (1979) who established the equivalence of the DLR and GNZ equations in a less general framework. Conceptually their techniques carry over and also work in greater generality. The DLR equations are of extraordinary importance in statistical physics and are often used as a definition of the Gibbs process (instead of the GNZ equations). The fact that no topological structure is required on the state space shows how the characterization captures essential structural features of Gibbs processes.

Theorem 2.43 (**DLR equations**). *A point process η in $\mathbb{X}$ is a Gibbs process with PI κ if, and only if, for all $B \in \mathcal{X}_b$, the process satisfies $\mathbb{P}\big(Z_B(\eta_{B^c}) < \infty\big) = 1$ as well as*

$$\mathbb{E}\big[F(\eta_B) \mid \eta_{B^c}\big] = \frac{\mathrm{e}^{\lambda(B)}}{Z_B(\eta_{B^c})} \int_{\mathbf{N}} F(\mu)\, \mathrm{e}^{-H(\mu,\, \eta_{B^c})}\, \mathrm{d}\Pi_{\lambda_B}(\mu) \quad \mathbb{P}\text{-a.s.}$$

for each measurable function $F : \mathbf{N} \to [0, \infty]$.

Proof. We first show that a Gibbs process η satisfies the stated properties for any $B \in \mathcal{X}$ with $\lambda(B) < \infty$ and $\mathbb{P}\big(\eta(B) < \infty\big) = 1$. The almost sure finiteness of the partition function was proven in Lemma 2.17. Let $A, A' \in \mathcal{N}$. Applying the first part of the proof of Lemma 2.17 to the function $g(\mu) = \mathbb{1}\{\mu_B \in A,\, \mu_{B^c} \in A'\}$, we obtain

$$\mathbb{P}(\eta_B \in A,\, \eta_{B^c} \in A') = \mathbb{E}\Bigg[\mathbb{1}\{\eta_B = \mathbf{0}\}\Bigg(\mathbb{1}\{\eta_B \in A,\, \eta_{B^c} \in A'\} + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{B^m} \mathbb{1}\bigg\{\Big(\eta + \sum_{j=1}^{m} \delta_{x_j}\Big)_B \in A\bigg\}$$
$$\mathbb{1}\bigg\{\Big(\eta + \sum_{j=1}^{m} \delta_{x_j}\Big)_{B^c} \in A'\bigg\}\, \kappa_m(x_1, \dots, x_m, \eta)\, \mathrm{d}\lambda^m(x_1, \dots, x_m)\Bigg)\Bigg].$$

Sorting out the indicator functions, the right hand side equals

$$\mathbb{E}\Bigg[\mathbb{1}\{\eta_{B^c} \in A'\}\, \mathbb{1}\{\eta_B = \mathbf{0}\}\Bigg(\mathbb{1}_A(\mathbf{0}) + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{B^m} \mathbb{1}_A\Big(\sum_{j=1}^{m} \delta_{x_j}\Big)\, \kappa_m(x_1, \dots, x_m, \eta_{B^c})\, \mathrm{d}\lambda^m(x_1, \dots, x_m)\Bigg)\Bigg]$$

which, by Corollaries 2.31 and 2.4, equals

$$\mathbb{E}\bigg[\mathbb{1}\{\eta_{B^c} \in A'\}\, \mathbb{1}\{\eta_B = \mathbf{0}\} \cdot \mathrm{e}^{\lambda(B)} \int_{\mathbf{N}} \mathbb{1}_A(\mu)\, \mathrm{e}^{-H(\mu,\, \eta_{B^c})}\, \mathrm{d}\Pi_{\lambda_B}(\mu)\bigg]$$

$$= \mathbb{E}\Big[\mathbb{1}\{\eta_{B^c} \in A'\} \cdot \mathbb{P}\big(\eta(B) = 0 \mid \eta_{B^c}\big) \cdot \mathrm{e}^{\lambda(B)} \int_{\mathbf{N}} \mathbb{1}_A(\mu)\, \mathrm{e}^{-H(\mu,\, \eta_{B^c})}\, \mathrm{d}\Pi_{\lambda_B}(\mu)\Big].$$

Lemma 2.17 lets us conclude that

$$\mathbb{P}(\eta_B \in A,\, \eta_{B^c} \in A') = \mathbb{E}\Big[\mathbb{1}\{\eta_{B^c} \in A'\} \cdot \frac{\mathrm{e}^{\lambda(B)}}{Z_B(\eta_{B^c})} \int_{\mathbf{N}} \mathbb{1}_A(\mu)\, \mathrm{e}^{-H(\mu,\, \eta_{B^c})}\, \mathrm{d}\Pi_{\lambda_B}(\mu)\Big].$$

A standard monotone approximation extends this to any measurable map $F : \mathbf{N} \to [0, \infty]$, yielding

$$\mathbb{E}\Big[\mathbb{1}\{\eta_{B^c} \in A'\} \cdot F(\eta_B)\Big] = \mathbb{E}\Big[\mathbb{1}\{\eta_{B^c} \in A'\} \cdot \frac{\mathrm{e}^{\lambda(B)}}{Z_B(\eta_{B^c})} \int_{\mathbf{N}} F(\mu)\, \mathrm{e}^{-H(\mu,\, \eta_{B^c})}\, \mathrm{d}\Pi_{\lambda_B}(\mu)\Big].$$

As this relation holds for any $A' \in \mathcal{N}$, and since

$$\frac{\mathrm{e}^{\lambda(B)}}{Z_B(\eta_{B^c})} \int_{\mathbf{N}} F(\mu)\, \mathrm{e}^{-H(\mu,\, \eta_{B^c})}\, \mathrm{d}\Pi_{\lambda_B}(\mu)$$

is a $\sigma(\eta_{B^c})$-measurable map on Ω, the claim follows.

To prove the converse statement, assume that the process η has the stated properties. We must prove that η satisfies the GNZ equations (2.7). For now, fix $B \in \mathcal{X}_b$. Let $A' \in \mathcal{N}$ and notice that the assumptions on η imply

$$\mathbb{E}\Big[\mathbb{1}\big\{\eta_{B^c} \in A' \cap \tilde{A},\, \eta_B \in A\big\}\Big] = \mathbb{E}\Big[\mathbb{1}\{\eta_{B^c} \in A'\} \cdot \frac{\mathrm{e}^{\lambda(B)}}{Z_B(\eta_{B^c})} \int_{\mathbf{N}} \mathbb{1}\big\{\mu \in A,\, \eta_{B^c} \in \tilde{A}\big\}\, \mathrm{e}^{-H(\mu,\, \eta_{B^c})}\, \mathrm{d}\Pi_{\lambda_B}(\mu)\Big]$$

for all $A, \tilde{A} \in \mathcal{N}$. On $(\mathbf{N} \times \mathbf{N}, \mathcal{N} \otimes \mathcal{N})$ define the measures

$$\mathrm{P}(E) = \mathbb{E}\Big[\mathbb{1}\{\eta_{B^c} \in A'\} \cdot \mathbb{1}\big\{(\eta_B, \eta_{B^c}) \in E\big\}\Big], \quad E \in \mathcal{N} \otimes \mathcal{N},$$

and

$$\tilde{\mathrm{P}}(E) = \mathbb{E}\Big[\mathbb{1}\{\eta_{B^c} \in A'\} \cdot \frac{\mathrm{e}^{\lambda(B)}}{Z_B(\eta_{B^c})} \int_{\mathbf{N}} \mathbb{1}\big\{(\mu, \eta_{B^c}) \in E\big\}\, \mathrm{e}^{-H(\mu,\, \eta_{B^c})}\, \mathrm{d}\Pi_{\lambda_B}(\mu)\Big], \quad E \in \mathcal{N} \otimes \mathcal{N}.$$

By the previous observation, we have $\mathrm{P}(A \times \tilde{A}) = \tilde{\mathrm{P}}(A \times \tilde{A})$ for all $A, \tilde{A} \in \mathcal{N}$, so P and $\tilde{\mathrm{P}}$ agree on a π-system that generates $\mathcal{N} \otimes \mathcal{N}$. We obviously have $\mathrm{P}(\mathbf{N} \times \mathbf{N}) \leq 1$ and Lemma 2.34 ensures that $\tilde{\mathrm{P}}(\mathbf{N} \times \mathbf{N}) = \mathbb{P}\big(\eta_{B^c} \in A'\big) \leq 1$, so both measures are finite. Hence, $\mathrm{P} = \tilde{\mathrm{P}}$. Now, let $G : \mathbf{N} \times \mathbf{N} \to [0, \infty]$ be a measurable function. Monotone approximation extends the equality of the measures P and $\tilde{\mathrm{P}}$ to

$$\mathbb{E}\Big[\mathbb{1}\{\eta_{B^c} \in A'\} \cdot G(\eta_B, \eta_{B^c})\Big] = \mathbb{E}\Big[\mathbb{1}\{\eta_{B^c} \in A'\} \cdot \frac{\mathrm{e}^{\lambda(B)}}{Z_B(\eta_{B^c})} \int_{\mathbf{N}} G(\mu, \eta_{B^c})\, \mathrm{e}^{-H(\mu,\, \eta_{B^c})}\, \mathrm{d}\Pi_{\lambda_B}(\mu)\Big]. \tag{2.16}$$

Let $A \in \mathcal{N}$. Equation (2.16) applied with $A' = \mathbf{N}$ and $G(\mu, \nu) = \mathbb{1}_A(\mu + \nu) \cdot \mu(B)$ yields

$$\begin{aligned}
\mathbb{E}\Big[\int_B \mathbb{1}_A(\eta)\, \mathrm{d}\eta(x)\Big] &= \mathbb{E}\Big[\frac{\mathrm{e}^{\lambda(B)}}{Z_B(\eta_{B^c})} \int_{\mathbf{N}} \mathbb{1}_A(\mu + \eta_{B^c}) \cdot \mu(B)\, \mathrm{e}^{-H(\mu,\, \eta_{B^c})}\, \mathrm{d}\Pi_{\lambda_B}(\mu)\Big] \\
&= \mathbb{E}\Big[\frac{\mathrm{e}^{\lambda(B)}}{Z_B(\eta_{B^c})} \int_{\mathbf{N}} \int_B \mathbb{1}_A(\mu + \eta_{B^c})\, \mathrm{e}^{-H(\mu,\, \eta_{B^c})}\, \mathrm{d}\mu(x)\, \mathrm{d}\Pi_{\lambda_B}(\mu)\Big].
\end{aligned}$$

Applying Mecke's formula (Proposition 2.1) to the right hand side, the term equals

$$\mathbb{E}\Big[\frac{\mathrm{e}^{\lambda(B)}}{Z_B(\eta_{B^c})} \int_{\mathbf{N}} \int_B \mathbb{1}_A(\mu + \delta_x + \eta_{B^c})\, \mathrm{e}^{-H(\mu + \delta_x,\, \eta_{B^c})}\, \mathrm{d}\lambda(x)\, \mathrm{d}\Pi_{\lambda_B}(\mu)\Big].$$

By Lemma 2.33 and Corollary 2.31 we have

$$\mathrm{e}^{-H(\mu + \delta_x,\, \eta_{B^c})} = \mathrm{e}^{-H(\mu,\, \eta_{B^c})} \cdot \kappa(x, \eta_{B^c} + \mu),$$

so another application of Equation (2.16), with $A' = \mathbf{N}$ and $G(\mu,\nu) = \int_B \mathbb{1}_A(\mu+\delta_x+\nu)\cdot\kappa(x,\nu+\mu)\,\mathrm{d}\lambda(x)$, lets us conclude that

$$\mathbb{E}\Big[\int_B \mathbb{1}_A(\eta)\,\mathrm{d}\eta(x)\Big] = \mathbb{E}\Big[\int_B \mathbb{1}_A(\eta_B+\delta_x+\eta_{B^c})\,\kappa(x,\eta_{B^c}+\eta_B)\,\mathrm{d}\lambda(x)\Big] = \mathbb{E}\Big[\int_B \mathbb{1}_A(\eta+\delta_x)\,\kappa(x,\eta)\,\mathrm{d}\lambda(x)\Big],$$

so the GNZ equation holds for the map $(x,\mu)\mapsto \mathbb{1}_B(x)\cdot\mathbb{1}_A(\mu)$. As $A\in\mathcal{N}$ was arbitrary, and the whole construction can be carried out for any $B\in\mathcal{X}_b$, the GNZ equation holds for all indicator functions of product sets $B\times A$. A literal copy of the final part of the proof of Lemma 2.14 (for $m=0$) extends this to all maps $(x,\mu)\mapsto\mathbb{1}_E(x,\mu)$, $E\in\mathcal{X}\otimes\mathcal{N}$, and by monotone approximation we can allow for general measurable $f:\mathbb{X}\times\mathbf{N}\to[0,\infty]$. We conclude that the GNZ equations are satisfied and η is a Gibbs process. □

A slightly different and more straightforward argument for the sufficiency part, closer to the original proof of Nguyen and Zessin (1979) and the presentation by Jansen (2017), allows to shorten the proof of Theorem 2.43. The proof we chose here, however, allows for an immediate corollary which follows from Equation (2.16).

Corollary 2.44. *Let η be a Gibbs process in $\mathbb{X}$ with PI κ, and fix $B\in\mathcal{X}_b$. Then, for every measurable function $G:\mathbf{N}\times\mathbf{N}\to[0,\infty]$,*

$$\mathbb{E}\big[G(\eta_B,\eta_{B^c})\mid\eta_{B^c}\big] = \frac{\mathrm{e}^{\lambda(B)}}{Z_B(\eta_{B^c})}\int_{\mathbf{N}} G(\mu,\eta_{B^c})\,\mathrm{e}^{-H(\mu,\,\eta_{B^c})}\,\mathrm{d}\Pi_{\lambda_B}(\mu)\quad \mathbb{P}\text{-}a.s.$$

In particular, if $F:\mathbf{N}\to[0,\infty]$ is measurable, then

$$\mathbb{E}\big[F(\eta)\mid\eta_{B^c}\big] = \frac{\mathrm{e}^{\lambda(B)}}{Z_B(\eta_{B^c})}\int_{\mathbf{N}} F(\mu+\eta_{B^c})\,\mathrm{e}^{-H(\mu,\,\eta_{B^c})}\,\mathrm{d}\Pi_{\lambda_B}(\mu)\quad \mathbb{P}\text{-}a.s.$$

Recalling that we can express κ_m via the Hamiltonian H as in Corollary 2.31, a straightforward application of Corollary 2.4 allows one to rewrite the DLR equations in terms of κ_m, but we do without an explicit statement of this reformulation.

Theorem 2.43, together with Theorem 2.40, allows us to make more precise the comment after Lemma 2.18. While the existence of Janossy densities implies that any Gibbs process is locally absolutely continuous with respect to a Poisson process by Corollary 2.12, Theorem 2.43 provides the corresponding density functions. These density functions inherit the hereditarity property of the Hamiltonian, and Theorem 2.40 implies that restrictions of a Gibbs process to bounded subsets are themselves (finite) Gibbs processes. This sketchy observation is made precise in the following lemma.

Lemma 2.45. *Let η be a Gibbs process in $\mathbb{X}$ with PI κ, and let $B\in\mathcal{X}_b$. Then η_B is a Gibbs process with PI given by*

$$\kappa_B^*(x,\mu) = \begin{cases} \dfrac{\mathbb{E}\big[Z_B(\eta_{B^c})^{-1}\cdot\mathrm{e}^{-H(\mu+\delta_x,\,\eta_{B^c})}\big]}{\mathbb{E}\big[Z_B(\eta_{B^c})^{-1}\cdot\mathrm{e}^{-H(\mu,\,\eta_{B^c})}\big]}, & \text{if } x\in B \text{ and } \mathbb{E}\big[Z_B(\eta_{B^c})^{-1}\cdot\mathrm{e}^{-H(\mu,\,\eta_{B^c})}\big]>0,\\ 0, & \text{else.}\end{cases} \tag{2.17}$$

Proof. Taking expectations in Theorem 2.43 immediately shows that $\mathbb{P}^{\eta_B}\ll\Pi_{\lambda_B}$ with density function

$$\mathfrak{D}_B^*(\mu) = \mathrm{e}^{\lambda(B)}\cdot\mathbb{E}\big[Z_B(\eta_{B^c})^{-1}\cdot\mathrm{e}^{-H(\mu,\,\eta_{B^c})}\big],\quad \mu\in\mathbf{N}.$$

Let $x\in\mathbb{X}$ and $\mu\in\mathbf{N}$. If $\mathfrak{D}_B^*(\mu)=0$ then, as the term in the expectation is non-negative and $Z_B(\eta_{B^c})^{-1}>0$ $\mathbb{P}$-a.s., we have $\mathrm{e}^{-H(\mu,\,\eta_{B^c})}=0$ almost surely. Lemma 2.39 implies that $\mathrm{e}^{-H(\mu+\delta_x,\,\eta_{B^c})}=0$ almost surely and therefore $\mathfrak{D}_B^*(\mu+\delta_x)=0$. Hence, the function $\mathfrak{D}_B^*$ is hereditary and Theorem 2.40 shows that η_B is a Gibbs process. The formula for the Papangelou intensity follows from (2.15). □

We now provide an alternative formulation of the DLR equations. More specifically, we use the formalism for finite Gibbs processes to rewrite the equations, which brings us back to the equilibrium equation of Ruelle (1970). Let $B \in \mathcal{X}$ with $\lambda(B) < \infty$ and $\psi \in \mathbf{N}$ be such that $Z_B(\psi) < \infty$. Denote by $\mathrm{P}_{B,\psi}$ the distribution of a finite Gibbs process with PI $\kappa^{(B,\psi)}$ as a measure on $(\mathbf{N}, \mathcal{N})$. In particular, $\mathrm{P}_{B,\psi}(\mathbf{N}_B) = 1$ and (by Corollary 2.35) $\mathrm{P}_{B,\psi}$ is absolutely continuous with respect to Π_{λ_B} with density function

$$\frac{\mathrm{dP}_{B,\psi}}{\mathrm{d}\Pi_{\lambda_B}}(\mu) = \frac{\mathrm{e}^{\lambda(B)}}{Z_B(\psi)}\, \mathrm{e}^{-H(\mu,\,\psi)}.$$

If $Z_B(\psi) = \infty$, Lemma 2.27 implies that no finite Gibbs process with PI $\kappa^{(B,\psi)}$ exists, so we put $\mathrm{P}_{B,\psi} \equiv 0$.

Lemma 2.46. *Fix $B \in \mathcal{X}$ with $\lambda(B) < \infty$. The map*

$$\mathbf{N} \times \mathbf{N} \ni (\psi, \nu) \mapsto \int_{\mathbf{N}} G(\mu, \nu)\, \mathrm{dP}_{B,\psi_{B^c}}(\mu)$$

is $\mathcal{N}_{B^c} \otimes \mathcal{N}$-measurable (hence $\mathcal{N} \otimes \mathcal{N}$-measurable) for every measurable function $G : \mathbf{N} \times \mathbf{N} \to [0, \infty]$.

Proof. Since the partition function $Z_B : \mathbf{N} \to [1, \infty]$ is measurable, the map $\psi \mapsto Z_B(\psi_{B^c}) = Z_B \circ p_{B^c}(\psi)$ is $\mathcal{N}_{B^c}$-measurable. Thus, Fubini's theorem implies that

$$(\psi, \nu) \mapsto \int_{\mathbf{N}} G(\mu, \nu)\, \mathrm{dP}_{B,\psi_{B^c}}(\mu) = \mathbb{1}\big\{Z_B(\psi_{B^c}) < \infty\big\} \cdot \frac{\mathrm{e}^{\lambda(B)}}{Z_B(\psi_{B^c})} \int_{\mathbf{N}} G(\mu, \nu)\, \mathrm{e}^{-H(\mu,\,\psi_{B^c})}\, \mathrm{d}\Pi_{\lambda_B}(\mu)$$

is $\mathcal{N}_{B^c} \otimes \mathcal{N}$-measurable. □

With this formalism we reformulate the DLR equations in terms of finite Gibbs processes depending on the boundary condition. Notice that, while considerations via conditional probabilities go way back to Dobrushin (1968b), the following corollary comes somewhat closer to the equilibrium equations stated by Lanford and Ruelle (1969) and Ruelle (1970).

Lemma 2.47. *A point process η in $\mathbb{X}$ is a Gibbs process with PI κ if, and only if,*

$$\mathbb{E}\big[F(\eta)\big] = \mathbb{E}\bigg[\int_{\mathbf{N}} F(\mu + \eta_{B^c})\, \mathrm{dP}_{B,\eta_{B^c}}(\mu)\bigg]$$

for all measurable functions $F : \mathbf{N} \to [0, \infty]$ and each $B \in \mathcal{X}_b$.

Proof. First of all, note that by Lemma 2.46 the term in the lemma is well-defined. If η is a Gibbs process with PI κ, then taking expectations in the second equation of Corollary 2.44, and using the definition of $\mathrm{P}_{B,\eta_{B^c}}$, immediately yields the claim.

Now, assume that η satisfies the formula in the statement of the lemma. Then, choosing $F \equiv 1$, we get

$$1 = \mathbb{E}\big[\mathrm{P}_{B,\eta_{B^c}}(\mathbf{N})\big] = \mathbb{E}\Big[\mathbb{1}\big\{Z_B(\eta_{B^c}) < \infty\big\}\Big] = \mathbb{P}\big(Z_B(\eta_{B^c}) < \infty\big).$$

For $A, A' \in \mathcal{N}$, apply the formula to $F(\mu) = \mathbb{1}\{\mu_B \in A,\, \mu_{B^c} \in A'\}$ to obtain

$$\begin{aligned}\mathbb{P}(\eta_B \in A,\, \eta_{B^c} \in A') &= \mathbb{E}\bigg[\int_{\mathbf{N}} \mathbb{1}\Big\{(\mu + \eta_{B^c})_B \in A,\, (\mu + \eta_{B^c})_{B^c} \in A'\Big\}\, \mathrm{dP}_{B,\eta_{B^c}}(\mu)\bigg] \\ &= \mathbb{E}\bigg[\mathbb{1}\{\eta_{B^c} \in A'\} \cdot \frac{\mathrm{e}^{\lambda(B)}}{Z_B(\eta_{B^c})} \int_{\mathbf{N}} \mathbb{1}_A(\mu)\, \mathrm{e}^{-H(\mu,\,\eta_{B^c})}\, \mathrm{d}\Pi_{\lambda_B}(\mu)\bigg].\end{aligned}$$

Extending this relation from indicator functions $\mathbb{1}_A$ to general measurable functions F by monotone approximation, it follows that the conditional expectations $\mathbb{E}\big[F(\eta_B) \mid \eta_{B^c}\big]$ are as in Theorem 2.43, hence η is a Gibbs process with PI κ. □

Remark 2.48. Notice that in the necessity part of Lemma 2.47, where it is assumed that η is a Gibbs process, the stated formula actually holds, a little more generally, for each $B \in \mathcal{X}$ with $\lambda(B) < \infty$ and $\mathbb{P}\big(\eta(B) < \infty\big) = 1$. Other versions of the DLR equations, for instance from Theorem 2.43 or Corollary 2.44, can also be rewritten in terms of $\mathrm{P}_{B,\eta_{B^c}}$, but the version stated in Lemma 2.47 is what we need explicitly later on. ○

In the context of the DLR equations many authors use the notion of specifications, some even use it to define Gibbs measures, referring to Preston (1976) and Nguyen and Zessin (1979). We conclude this section by a discussion of specifications in our general setting. Even though we do not require the notion later on, the following thoughts might be interesting for readers who are familiar with the specification-based approach to Gibbs processes.

For each $B \in \mathcal{X}$ with $\lambda(B) < \infty$, define the kernel $\mathfrak{K}_B : \mathbf{N} \times \mathcal{N} \to [0, 1]$ by

$$\mathfrak{K}_B(\psi, A) = \mathrm{P}_{B,\psi_{B^c}}\big(\{\mu \in \mathbf{N} : \mu + \psi_{B^c} \in A\}\big).$$

Notice that $\mathfrak{K}_B(\psi, \cdot\,)$ is indeed a measure on $\mathbf{N}$ for any fixed $\psi \in \mathbf{N}$. More precisely, it is the push-forward measure of $\mathrm{P}_{B,\psi_{B^c}}$ under the map $\mu \mapsto \mu + \psi_{B^c}$. Moreover, for fixed $A \in \mathcal{N}$, the map

$$\psi \mapsto \mathfrak{K}_B(\psi, A) = \int_{\mathbf{N}} \mathbb{1}_A(\mu + \psi_{B^c})\, \mathrm{dP}_{B,\psi_{B^c}}(\mu)$$

is $\mathcal{N}_{B^c}$-measurable by Lemma 2.46. We have $\mathfrak{K}_B(\psi, \mathbf{N}) = \mathrm{P}_{B,\psi_{B^c}}(\mathbf{N}) \in \{0, 1\}$ for each $\psi \in \mathbf{N}$, where the value 0 occurs whenever $Z_B(\psi_{B^c}) = \infty$ and the value 1 occurs if $Z_B(\psi_{B^c}) < \infty$.

Remark 2.49. If $B \in \mathcal{X}$ is such that $\lambda(B) = 0$, then $Z_B(\nu) = 1$ for each $\nu \in \mathbf{N}$ as well as $\Pi_{\lambda_B} = \delta_{\mathbf{0}}$, so for any $\psi \in \mathbf{N}$ and $A \in \mathcal{N}$ we get $\mathrm{P}_{B,\psi_{B^c}}(A) = \mathbb{1}\{\mathbf{0} \in A\}$ and

$$\mathfrak{K}_B(\psi, A) = \mathbb{1}\{\psi_{B^c} \in A\}.$$

If $B = \varnothing$, we even have $\mathfrak{K}_B(\psi, A) = \mathbb{1}\{\psi \in A\}$. ○

Fix $B \in \mathcal{X}$ with $\lambda(B) < \infty$. For a measurable function $F : \mathbf{N} \to [0, \infty]$ we define the measurable map $\mathfrak{K}_B F : \mathbf{N} \to [0, \infty]$ by

$$\mathfrak{K}_B F(\psi) = \int_{\mathbf{N}} F(\mu)\, \mathfrak{K}_B(\psi, \mathrm{d}\mu) = \int_{\mathbf{N}} F(\mu + \psi_{B^c})\, \mathrm{dP}_{B,\psi_{B^c}}(\mu).$$

Furthermore, we consider for $C \in \mathcal{X}$ with $\lambda(C) < \infty$ the product $\mathfrak{K}_C\mathfrak{K}_B : \mathbf{N} \times \mathcal{N} \to [0, 1]$,

$$\mathfrak{K}_C\mathfrak{K}_B(\psi, A) = \int_{\mathbf{N}} \mathfrak{K}_B(\mu, A)\, \mathfrak{K}_C(\psi, \mathrm{d}\mu),$$

which is itself a kernel. The following corollary is a reformulation of Lemma 2.47 in terms of the kernels $\mathfrak{K}_B$.

Corollary 2.50. *A point process η in $\mathbb{X}$ is a Gibbs process with PI κ if, and only if,*

$$\mathbb{E}\big[\mathfrak{K}_B F(\eta)\big] = \mathbb{E}\big[F(\eta)\big]$$

for all measurable functions $F : \mathbf{N} \to [0, \infty]$ and each $B \in \mathcal{X}_b$.

Before we state another version of the DLR equations, we discuss properties of the kernels defined above. The following lemma can be found in Preston (1976) and is also stated by Nguyen and Zessin (1979).

Lemma 2.51. *The kernels $\{\mathfrak{K}_B : B \in \mathcal{X},\ \lambda(B) < \infty\}$ form a specification, that is, the following properties are satisfied. For any $B \in \mathcal{X}$ with $\lambda(B) < \infty$,*

(i) $\mathfrak{K}_B(\psi, \cdot\,)$ *is a probability measure on* $\mathbf{N}$ *for each* $\psi \in \mathbf{N}$ *with* $Z_B(\psi_{B^c}) < \infty$,

(ii) $\mathfrak{K}_B(\psi, \cdot\,) \equiv 0$ *for each* $\psi \in \mathbf{N}$ *with* $Z_B(\psi_{B^c}) = \infty$,

(iii) *the map* $\psi \mapsto \mathfrak{K}_B(\psi, A)$ *is* $\mathcal{N}_{B^c}$*-measurable for each* $A \in \mathcal{N}$,

(iv) $\mathfrak{K}_B(\psi, A) = \mathbb{1}_A(\psi) \cdot \mathbb{1}\big\{Z_B(\psi_{B^c}) < \infty\big\}$ *for all* $\psi \in \mathbf{N}$ *and* $A \in \mathcal{N}_{B^c}$, *and*

(v) $\mathfrak{K}_C \mathfrak{K}_B = \mathfrak{K}_C$ *for any* $C \in \mathcal{X}$ *such that* $C \supset B$ *and* $\lambda(C) < \infty$.

Proof. Properties (i) – (iii) were already discussed above. For the proof of (iv) and (v), fix $B \in \mathcal{X}$ with $\lambda(B) < \infty$. For $\psi \in \mathbf{N}$ and $A \in \mathcal{N}_{B^c}$ we have

$$\mathfrak{K}_B(\psi, A) = \mathrm{P}_{B,\psi_{B^c}}\big(\{\mu \in \mathbf{N}_B : \mu_{B^c} + \psi_{B^c} \in A\}\big) = \mathbb{1}\{\psi_{B^c} \in A\} \cdot \mathrm{P}_{B,\psi_{B^c}}(\mathbf{N}) = \mathbb{1}_A(\psi) \cdot \mathbb{1}\big\{Z_B(\psi_{B^c}) < \infty\big\}.$$

Now, consider $C \in \mathcal{X}$ with $C \supset B$ and $\lambda(C) < \infty$. Let $A \in \mathcal{N}$ be arbitrary. For $\psi \in \mathbf{N}$ with $Z_C(\psi_{C^c}) = \infty$, part (ii) implies $\mathfrak{K}_C(\psi, \cdot\,) \equiv 0$, so

$$\mathfrak{K}_C \mathfrak{K}_B(\psi, A) = \int_{\mathbf{N}} \mathfrak{K}_B(\mu, A)\, \mathfrak{K}_C(\psi, \mathrm{d}\mu) = 0 = \mathfrak{K}_C(\psi, A).$$

Next, assume that $\psi \in \mathbf{N}$ satisfies $Z_C(\psi_{C^c}) < \infty$. Then we calculate

$$\begin{aligned}
\mathfrak{K}_C \mathfrak{K}_B(\psi, A) &= \int_{\mathbf{N}} \mathfrak{K}_B(\mu, A)\, \mathfrak{K}_C(\psi, \mathrm{d}\mu) \\
&= \int_{\mathbf{N}} \mathfrak{K}_B(\mu + \psi_{C^c}, A)\, \mathrm{d}\mathrm{P}_{C,\psi_{C^c}}(\mu) \\
&= \int_{\mathbf{N}} \bigg(\int_{\mathbf{N}} \mathbb{1}_A(\nu + \mu_{B^c} + \psi_{C^c})\, \mathrm{d}\mathrm{P}_{B,\mu_{B^c}+\psi_{C^c}}(\nu) \bigg)\, \mathrm{d}\mathrm{P}_{C,\psi_{C^c}}(\mu) \\
&= \mathbb{E}\bigg[\int_{\mathbf{N}} \mathbb{1}_A(\nu + \xi_{B^c} + \psi_{C^c})\, \mathrm{d}\mathrm{P}_{B,\xi_{B^c}+\psi_{C^c}}(\nu) \bigg],
\end{aligned} \tag{2.18}$$

where ξ denotes a point process with distribution $\mathrm{P}_{C,\psi_{C^c}}$, or in other words, a finite Gibbs process with PI $\kappa^{(C,\psi_{C^c})}$ as in Lemma 2.25. By Lemma 2.24, we have

$$Z_B^{(C,\psi_{C^c})}(\xi_{B^c}) = Z_B(\xi_{B^c} + \psi_{C^c}),$$

where left hand side is finite $\mathbb{P}$-a.s. by Lemma 2.17. We conclude from (2.18) that

$$\begin{aligned}
\mathfrak{K}_C \mathfrak{K}_B(\psi, A) &= \mathbb{E}\bigg[\frac{e^{\lambda(B)}}{Z_B(\xi_{B^c} + \psi_{C^c})} \int_{\mathbf{N}} \mathbb{1}_A\big(\nu + \xi_{B^c} + \psi_{C^c}\big) \exp\big(- H(\nu, \xi_{B^c} + \psi_{C^c})\big)\, \mathrm{d}\Pi_{\lambda_B}(\nu) \bigg] \\
&= \mathbb{E}\bigg[\frac{e^{\lambda(B)}}{Z_B^{(C,\psi_{C^c})}(\xi_{B^c})} \int_{\mathbf{N}} \mathbb{1}_A\big(\nu + \xi_{B^c} + \psi_{C^c}\big) \exp\big(- H^{(C,\psi_{C^c})}(\nu, \xi_{B^c})\big)\, \mathrm{d}\Pi_{\lambda_B}(\nu) \bigg],
\end{aligned}$$

and Corollary 2.44, applied to ξ, gives

$$\mathfrak{K}_C \mathfrak{K}_B(\psi, A) = \mathbb{E}\big[\mathbb{1}_A\big(\xi + \psi_{C^c}\big)\big] = \int_{\mathbf{N}} \mathbb{1}_A(\mu + \psi_{C^c})\, \mathrm{d}\mathrm{P}_{C,\psi_{C^c}}(\mu) = \mathfrak{K}_C(\psi, A),$$

which concludes the proof. □

The following characterization of Gibbs processes is usually chosen as their definition in the context of specifications.

Lemma 2.52. *A point process η in $\mathbb{X}$ is a Gibbs process with PI κ if, and only if,*

$$\mathbb{P}(\eta \in A \mid \eta_{B^c}) = \mathfrak{K}_B(\eta, A) \quad \mathbb{P}\text{-}a.s.$$

for all $A \in \mathcal{N}$ and each $B \in \mathcal{X}_b$.

Proof. Assume first that η is a Gibbs process. Taking into account the definition of $\mathrm{P}_{B,\eta_{B^c}}$, Corollary 2.44 implies that, for any $A \in \mathcal{N}$ and $B \in \mathcal{X}_b$,

$$\mathbb{P}(\eta \in A \mid \eta_{B^c}) = \int_{\mathbf{N}} \mathbb{1}_A(\mu + \eta_{B^c}) \, \mathrm{dP}_{B,\eta_{B^c}}(\mu) = \mathfrak{K}_B(\eta, A)$$

almost surely. Now, assume that $\mathbb{P}(\eta \in A \mid \eta_{B^c}) = \mathfrak{K}_B(\eta, A)$ holds $\mathbb{P}$-a.s. for all $A \in \mathcal{N}$ and $B \in \mathcal{X}_b$. Then

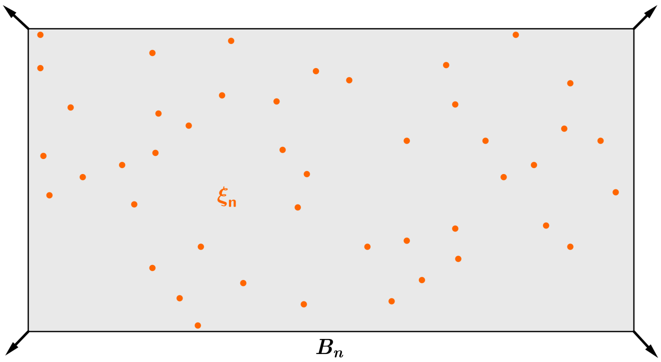
$$\mathbb{P}(\eta \in A) = \mathbb{E}\big[\mathfrak{K}_B(\eta, A)\big] = \mathbb{E}\left[\int_{\mathbf{N}} \mathbb{1}_A(\mu + \eta_{B^c}) \, \mathrm{dP}_{B,\eta_{B^c}}(\mu)\right].$$

Monotone approximation extends this relation to general measurable functions and Lemma 2.47 yields the claim. □

CHAPTER 3

Existence and uniqueness results for general Gibbs processes

The goal of this chapter is to provide existence and uniqueness results for Gibbs processes on a general level, trying to use as little prerequisites as possible with the chosen approach. Recall that the superposition principle for Poisson processes (Theorem 3.3 of Last and Penrose, 2017) allows for an elegant construction of infinite Poisson processes: Partitioning the underlying space into bounded subsets and taking a finite Poisson process on each of these parts of the space, one can simply combine these finite processes, by adding them up, to obtain a Poisson process on the full space. This uses heavily the spatial independence properties of Poisson processes and, keeping in mind that points of Gibbs processes interact and hence do not allow for such strong independence, it is rather intuitive that a simple superposition of Gibbs processes will not do the trick as regards existence of such processes. The approach we choose as a remedy is to construct suitable finite Gibbs processes $\xi_1, \xi_2, \dots$ on a sequence of bounded sets $B_1 \subset B_2 \subset \dots$ which exhausts the full space, to construct a limit process, where the limit is taken with respect to some appropriate topology, and to prove that the limit process is Gibbs. In Section 3.1 we discuss the local convergence topology, extend known results on the corresponding mode of convergence, and provide a new limit result specifically suited to the study of Gibbs processes. Section 3.2 contains a general existence result for Gibbs processes and some initial discussions of the necessary assumptions. As a first, and certainly not new, example we show that the existence result includes Gibbs processes with a finite interaction range. In Section 3.3 we introduce cluster-dependent interactions, provide an existence result for Gibbs processes with such an interaction, and prove uniqueness. More specifically, we show that if the size of the clusters that underlie the interaction is controlled in a suitable sense, then the distribution of the corresponding Gibbs process is unique. We also give insight into a decorrelation property which cluster-dependent Gibbs processes admit. In Section 3.4, which concludes this chapter, we apply the previous results to Gibbsian particle processes, thus giving some practical justification for drawing up the whole theory in general state spaces.



3.1. The local convergence topology and new convergence results

In this section we stick to the setting from the beginning of Section 2.2. In particular, we work, whenever it is not specified otherwise, on a measurable space $(\mathbb{X}, \mathcal{X})$ with localizing structure constructed via the sets $B_1, B_2, \dots$. We discuss the concept of local convergence introduced by Georgii and Zessin (1993) and provide connections to Janossy and factorial moment measures. In the definition of local convergence we follow the recent publications by Jansen (2019) and Rœlly and Zass (2020), where the concept includes the use of local and tame functions (as proposed by Georgii and Zessin, 1993), but other authors also use the term local convergence when only including local and bounded functions, see Dereudre and Vasseur (2020) for one instance.

A function $F : \mathbf{N} \to [-\infty, \infty]$ is called *local* if there exists a set $B \in \mathcal{X}_b$ such that $F(\mu) = F(\mu_B)$ for every $\mu \in \mathbf{N}$. We call F *tame* if there exists a constant $c \geq 0$ and a set $B \in \mathcal{X}_b$ such that $|F(\mu)| \leq c\big(1 + \mu(B)\big)$ for all $\mu \in \mathbf{N}$. In particular, every bounded function is tame. Many useful properties of local functions are discussed in Appendix A.2.

Definition 3.1 (Local convergence). Let $\eta, \eta_1, \eta_2, \ldots$ be point processes in $\mathbb{X}$. We say that $(\eta_n)_{n\in\mathbb{N}}$ converges locally to η if $\mathbb{E}\big[F(\eta_n)\big] \to \mathbb{E}\big[F(\eta)\big]$, as $n \to \infty$, for every measurable, local, and tame function $F : \mathbf{N} \to [0, \infty)$ for which the expectations are finite. For short, we write $\eta_n \xrightarrow{loc} \eta$.

If we denote by $\mathcal{P}$ the set of all probability measures P on $(\mathbf{N}, \mathcal{N})$ that satisfy $\int_{\mathbf{N}} \mu(B)\,\mathrm{dP}(\mu) < \infty$ for each $B \in \mathcal{X}_b$ (locally finite intensity measures), then the *local convergence topology* constitutes the smallest topology on $\mathcal{P}$ for which the maps $\mathcal{P} \to [0, \infty)$, $\mathrm{P} \mapsto \int_{\mathbf{N}} F(\mu)\,\mathrm{dP}(\mu)$ are continuous for every measurable, local, and tame $F : \mathbf{N} \to [0, \infty)$. Note that there is no need for the functions F to be continuous as in many other modes of convergence, thus no topological structure is needed on $\mathbf{N}$. The following lemma settles some initial issues one could raise given the definition at hand.

Lemma 3.2. *Let $\eta, \eta', \eta_1, \eta_2, \ldots$ be point processes in $\mathbb{X}$.*

(i) *The class of local and tame functions determines measures, that is, η and η' have the same distribution if, and only if, $\mathbb{E}\big[F(\eta)\big] = \mathbb{E}\big[F(\eta')\big]$ for all measurable, local, and tame functions $F : \mathbf{N} \to [0, \infty)$ for which the expectations are finite.*

(ii) *Local limits are unique in distribution.*

(iii) *If the underlying space $\mathbb{X}$ is a complete separable metric space, then local convergence is stronger than convergence in law, that is, if $\eta_n \xrightarrow{loc} \eta$ then $\eta_n \xrightarrow{d} \eta$, where $\xrightarrow{d}$ denotes convergence in distribution.*

Proof. (i) If η and η' have the same distribution, the equality of the expectations follows trivially for all measurable F for which the expectations are finite. For the converse, assume that $\mathbb{E}\big[F(\eta)\big] = \mathbb{E}\big[F(\eta')\big]$ for all measurable, local, and bounded F. Applied to $F(\mu) = \mathbb{1}\big\{\mu(D_1) = k_1, \ldots, \mu(D_m) = k_m\big\}$, for arbitrary $D_1, \ldots, D_m \in \mathcal{X}_b$, $k_1, \ldots, k_m \in \mathbb{N}_0$, and $m \in \mathbb{N}$, we obtain

$$\mathbb{P}\big(\eta(D_1) = k_1, \ldots, \eta(D_m) = k_m\big) = \mathbb{P}\big(\eta'(D_1) = k_1, \ldots, \eta'(D_m) = k_m\big),$$

which yields that η and η' have the same distribution. Notice that the considered function F is indeed D-local for $D = \bigcup_{j=1}^m D_j \in \mathcal{X}_b$.

(ii) Assume that $\eta_n \xrightarrow{loc} \eta$ as well as $\eta_n \xrightarrow{loc} \eta'$. It follows that $\mathbb{E}\big[F(\eta)\big] = \mathbb{E}\big[F(\eta')\big]$ for all measurable, local, and bounded F, so part (i) implies that η and η' have the same distribution.

(iii) Suppose that $\eta_n \xrightarrow{loc} \eta$. Let $D_1, \ldots, D_m \in \mathcal{X}_b$ be arbitrary and consider any continuous and bounded function $f : \mathbb{R}^m \to [0, \infty)$. Define the local and bounded map $F : \mathbf{N} \to [0, \infty)$ as $F(\mu) = f\big(\mu(D_1), \ldots, \mu(D_m)\big)$. By the local convergence assumption, we have, as $n \to \infty$,

$$\int_{\mathbb{R}^m} f\,\mathrm{d}\mathbb{P}^{(\eta_n(D_1),\ldots,\eta_n(D_m))} = \mathbb{E}\big[F(\eta_n)\big] \longrightarrow \mathbb{E}\big[F(\eta)\big] = \int_{\mathbb{R}^m} f\,\mathrm{d}\mathbb{P}^{(\eta(D_1),\ldots,\eta(D_m))},$$

that is, the vector $\big(\eta_n(D_1), \ldots, \eta_n(D_m)\big)$ converges in law to $\big(\eta(D_1), \ldots, \eta(D_m)\big)$ (in $\mathbb{R}^m$). For a point process in a complete separable metric space this convergence of the finite dimensional distributions is already enough to conclude that η_n converges in law to η, see Theorem 11.1.VII of Daley and Vere-Jones (2008).

Note that the metric structure on $\mathbb{X}$ in part (iii) ensures that there exists a metric structure on $\mathbf{N}$ (see Appendix A.7) so it makes sense to talk about convergence in law of point processes. □

It is well-known that local convergences is equivalent to a suitable weak* convergence (in the functional analytic sense) of correlations functions. We prove this result later on in this subsection, but first provide the connection between local convergence and convergence of the Janossy measures. This connection is used (albeit not explicitly) in the literature, for instance in the appendix of Jansen (2019), but to the author's knowledge this is the first time the results are stated formally. We state them in full abstraction and under weaker assumptions than used previously.

Lemma 3.3. *Let $\eta, \eta_1, \eta_2, \dots$ be point processes in $\mathbb{X}$ and fix a bounded set $B \in \mathcal{X}_b$. Assume there exists a map $c_B : \mathbb{N} \to [0, \infty)$ with $\sum_{m=1}^{\infty} c_B(m) < \infty$ such that the Janossy measures of η_n restricted to B satisfy*

$$\sup_{n \in \mathbb{N}} J_{\eta_n, B, m}(B^m) \leq c_B(m)$$

for each $m \in \mathbb{N}$. Further, suppose that, as $n \to \infty$,

$$\int_{\mathbb{X}^m} f \, \mathrm{d}J_{\eta_n, B, m} \longrightarrow \int_{\mathbb{X}^m} f \, \mathrm{d}J_{\eta, B, m}$$

for all measurable and bounded functions $f : \mathbb{X}^m \to [0, \infty)$ and all $m \in \mathbb{N}$. Then $\mathbb{E}\big[F(\eta_n)\big] \to \mathbb{E}\big[F(\eta)\big]$, as $n \to \infty$, for all measurable, B-local, and bounded maps $F : \mathbf{N} \to [0, \infty)$.

Proof. Let $F : \mathbf{N} \to [0, \infty)$ be measurable, B-local, and bounded by $c \geq 0$. For now, assume that $F(\mathbf{0}) = 0$. Lemma 2.3 implies

$$\mathbb{E}\big[F(\eta_n)\big] = \sum_{m=1}^{\infty} \int_{\mathbb{X}^m} F\Big(\sum_{j=1}^{m} \delta_{x_j}\Big) \, \mathrm{d}J_{\eta_n, B, m}(x_1, \dots, x_m).$$

For each $m \in \mathbb{N}$, define the measurable and symmetric function $f_m : \mathbb{X}^m \to [0, \infty)$ as

$$f_m(x_1, \dots, x_m) = F\Big(\sum_{j=1}^{m} \delta_{x_j}\Big).$$

Since f_m is bounded by c,

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{X}^m} f_m(x_1, \dots, x_m) \, \mathrm{d}J_{\eta_n, B, m}(x_1, \dots, x_m) \leq c \cdot c_B(m)$$

for all $m \in \mathbb{N}$ and the right hand side is summable over m. Thus, dominated convergence yields

$$\begin{aligned}
\lim_{n \to \infty} \mathbb{E}\big[F(\eta_n)\big] &= \lim_{n \to \infty} \sum_{m=1}^{\infty} \int_{\mathbb{X}^m} f_m(x_1, \dots, x_m) \, \mathrm{d}J_{\eta_n, B, m}(x_1, \dots, x_m) \\
&= \sum_{m=1}^{\infty} \int_{\mathbb{X}^m} f_m(x_1, \dots, x_m) \, \mathrm{d}J_{\eta, B, m}(x_1, \dots, x_m) \\
&= \mathbb{E}\big[F(\eta)\big].
\end{aligned}$$

It remains to handle the case where $F(\mathbf{0}) > 0$. Consider the decomposition

$$F(\mu) = \max\big\{F(\mu) - F(\mathbf{0}),\, 0\big\} - \max\big\{F(\mathbf{0}) - F(\mu),\, 0\big\} + F(\mathbf{0}).$$

The parts corresponding to the maxima are measurable, B-local functions $\mathbf{N} \to [0, \infty)$ which are bounded and have value 0 in $\mu = \mathbf{0}$. Applying the first part of the proof, we obtain

$$\begin{aligned}\mathbb{E}\big[F(\eta_n)\big] &= \mathbb{E}\big[\max\big\{F(\eta_n) - F(\mathbf{0}),\, 0\big\}\big] - \mathbb{E}\big[\max\big\{F(\mathbf{0}) - F(\eta_n),\, 0\big\}\big] + F(\mathbf{0}) \\ &\to \mathbb{E}\big[\max\big\{F(\eta) - F(\mathbf{0}),\, 0\big\}\big] - \mathbb{E}\big[\max\big\{F(\mathbf{0}) - F(\eta),\, 0\big\}\big] + F(\mathbf{0}) \\ &= \mathbb{E}\big[F(\eta)\big],\end{aligned}$$

as $n \to \infty$, where the expectations are finite. □

Remark 3.4. If the map c_B in Lemma 3.3 satisfies

$$\sum_{m=1}^{\infty} m \cdot c_B(m) < \infty,$$

the result holds for all measurable, B-local, and tame maps F. The slightly stronger assumption becomes necessary since the maps f_m defined in the proof only satisfy $f_m \leq c \cdot (1 + m)$ if F is tame with constant c. Note that the expectations are finite for such maps by the assumption on c_B, since Lemma 2.3 implies

$$\begin{aligned}\mathbb{E}\big[F(\eta)\big] = \mathbb{E}\big[F(\eta_B)\big] &= F(\mathbf{0}) \cdot \mathbb{P}\big(\eta(B) = 0\big) + \sum_{m=1}^{\infty} \int_{\mathbb{X}^m} F\Big(\sum_{j=1}^{m} \delta_{x_j}\Big)\, \mathrm{d}J_{\eta,B,m}(x_1, \dots, x_m) \\ &\leq c + \sum_{m=1}^{\infty} \int_{\mathbb{X}^m} c \cdot (1 + m)\, \mathrm{d}J_{\eta,B,m}(x_1, \dots, x_m) \\ &\leq c + c \sum_{m=1}^{\infty} (1 + m) \cdot c_B(m),\end{aligned}$$

and similarly for η_n ($n \in \mathbb{N}$). If the boundedness and convergence assumption on the Janossy measures holds for all $B \in \mathcal{X}_b$ (with the slightly stronger assumption on c_B indicated above), the expectations converge for all local and tame F, so η_n converges locally to η. ○

Example 3.5 (Local convergence of Poisson processes). Let $D_1, D_2, \ldots \in \mathcal{X}$ with $D_1 \subset D_2 \subset \ldots$ and $\bigcup_{n=1}^{\infty} D_n = \mathbb{X}$. For each $n \in \mathbb{N}$ let Φ_n be a Poisson process in $\mathbb{X}$ with intensity measure λ_{D_n} and let Φ be a Poisson process in $\mathbb{X}$ with intensity measure λ. Then $\Phi_n \xrightarrow{loc} \Phi$ as $n \to \infty$. Indeed, by (2.2) we have, for each $B \in \mathcal{X}_b$ and $m \in \mathbb{N}$,

$$\sup_{n \in \mathbb{N}} J_{\Phi_n,B,m}(B^m) = \sup_{n \in \mathbb{N}} \frac{\mathrm{e}^{-\lambda(B \cap D_n)}}{m!} \cdot \lambda(B \cap D_n)^m \leq \frac{\lambda(B)^m}{m!}.$$

Furthermore, for each measurable and bounded function $f : \mathbb{X}^m \to [0, \infty)$ we have (by monotone convergence)

$$\lim_{n \to \infty} \int_{\mathbb{X}^m} f\, \mathrm{d}J_{\Phi_n,B,m} = \lim_{n \to \infty} \frac{\mathrm{e}^{-\lambda(B \cap D_n)}}{m!} \int_{D_n^m} f\, \mathrm{d}\lambda_B^m = \frac{\mathrm{e}^{-\lambda(B)}}{m!} \int_{\mathbb{X}^m} f\, \mathrm{d}\lambda_B^m = \int_{\mathbb{X}^m} f\, \mathrm{d}J_{\Phi,B,m}$$

and the asserted convergence follows from Lemma 3.3 and Remark 3.4. ○

The converse to Lemma 3.3 is also true, even without any integrability assumptions.

Lemma 3.6. *Let $\eta, \eta_1, \eta_2, \ldots$ be point processes in $\mathbb{X}$ and fix a set $B \in \mathcal{X}_b$. Suppose that, as $n \to \infty$, $\mathbb{E}\big[F(\eta_n)\big] \to \mathbb{E}\big[F(\eta)\big]$ for all measurable, B-local, and bounded functions $F : \mathbf{N} \to [0, \infty)$. Then, as $n \to \infty$,*

$$\int_{\mathbb{X}^m} f\, \mathrm{d}J_{\eta_n,B,m} \longrightarrow \int_{\mathbb{X}^m} f\, \mathrm{d}J_{\eta,B,m}$$

for all measurable and bounded functions $f : \mathbb{X}^m \to [0, \infty)$ and all $m \in \mathbb{N}$.

Proof. Fix $m \in \mathbb{N}$ and let $f : \mathbb{X}^m \to [0, \infty)$ be measurable and bounded by $c \geq 0$. Define

$$F(\mu) = \frac{1}{m!} \cdot \mathbb{1}\{\mu(B) = m\} \int_{B^m} f(x_1, \ldots, x_m) \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m).$$

The map F is measurable by Lemma A.4, it is B-local since $(\mu^{(m)})_{B^m} = \mu_B^{(m)}$, and bounded as

$$F(\mu) \leq \frac{c}{m!} \cdot \mathbb{1}\{\mu(B) = m\} \cdot \mu^{(m)}(B^m) \leq c,$$

using that $\mu \in \mathbf{N}$ with $\mu(B) = m$ satisfies $\mu^{(m)}(B^m) = m!$ (see Proposition A.2). Thus, by assumption and the definition of the Janossy measures, we get

$$\int_{\mathbb{X}^m} f \, \mathrm{d}J_{\eta_n,B,m} = \frac{1}{m!} \mathbb{E}\Big[\mathbb{1}\{\eta_n(B) = m\} \int_{\mathbb{X}^m} f \, \mathrm{d}(\eta_n)_B^{(m)}\Big] = \mathbb{E}\big[F(\eta_n)\big] \longrightarrow \mathbb{E}\big[F(\eta)\big] = \int_{\mathbb{X}^m} f \, \mathrm{d}J_{\eta,B,m},$$

as $n \to \infty$. □

We now state a result which implies the well-known connection between local convergence and weak* convergence of correlation functions. We show that this convergence of the correlation functions is equivalent to the convergence of the Janossy measures in the previous lemmata, but only under a stronger assumption, namely a version of Ruelle's condition.

Definition 3.7 (Ruelle's condition). A point process η in $\mathbb{X}$ is said to satisfy Ruelle's condition if

$$\alpha_{\eta,m} \leq (\vartheta\lambda)^m$$

for each $m \in \mathbb{N}$ and some measurable function $\vartheta : \mathbb{X} \to [0, \infty)$. Here we denote by $\vartheta\lambda$ the measure on $(\mathbb{X}, \mathcal{X})$ given by

$$\vartheta\lambda = \int_{\mathbb{X}} \mathbb{1}\{x \in \cdot\,\} \, \vartheta(x) \, \mathrm{d}\lambda(x).$$

If a sequence $(\eta_n)_{n \in \mathbb{N}}$ of point processes satisfies the classical version of Ruelle's condition (where $\vartheta \equiv c$, $c \geq 0$) with the same constant, the corresponding correlation functions exist and give a bounded sequence in L^∞, and weak* convergence in L^∞ is precisely what occurs in the following lemma if one observes that the factorial moment measures are given by the corresponding λ^m-densities, that is, the correlation functions. Notice that our forthcoming existence proof for Gibbs processes works via Lemma 3.3 and allows for a way around the L^1 test functions which appear in the following lemma.

Lemma 3.8. *Let $\eta, \eta_1, \eta_2, \ldots$ be point processes in $\mathbb{X}$. Assume that there exists a measurable, locally λ-integrable map $\vartheta : \mathbb{X} \to [0, \infty)$ such that $\alpha_{\eta_n,m} \leq (\vartheta\lambda)^m$ for all $m \in \mathbb{N}$ and all $n \in \mathbb{N}$. Then, as $n \to \infty$,*

$$\int_{\mathbb{X}^m} f \, \mathrm{d}J_{\eta_n,B,m} \longrightarrow \int_{\mathbb{X}^m} f \, \mathrm{d}J_{\eta,B,m}$$

for all measurable and bounded functions $f : \mathbb{X}^m \to [0, \infty)$, all $B \in \mathcal{X}_b$, and each $m \in \mathbb{N}$, if, and only if,

$$\int_{\mathbb{X}^m} g \, \mathrm{d}\alpha_{\eta_n,m} \longrightarrow \int_{\mathbb{X}^m} g \, \mathrm{d}\alpha_{\eta,m},$$

as $n \to \infty$, for all $g \in L^1\big(\mathbb{X}^m, (\vartheta\lambda)^m\big)$ and each $m \in \mathbb{N}$.

It follows readily from Remark 2.2 that Ruelle's condition (with locally integrable ϑ) is a formally stronger assumption than what the Janossy measures must satisfy in Lemma 3.3. Lemmata 3.8 and 3.3 (together with Remark 3.4) imply that if the factorial moment measures (or correlation functions) converge in the weak* sense of Lemma 3.8, then η_n converges locally to η. The converse statement can be formulated via Lemma 3.6. Of course, if $\vartheta \equiv c$ for some $c \geq 0$, the functions g in the lemma are from $L^1(\mathbb{X}^m, \lambda^m)$.

Proof. For the necessity part, assume the convergence of the Janossy measures. Fix $m \in \mathbb{N}$. Let $B \in \mathcal{X}_b$, and let $g : \mathbb{X}^m \to [0, \infty)$ be measurable and bounded by $c \geq 0$ such that $g = 0$ on $\mathbb{X}^m \setminus B^m$. Then, by Theorem 2.6, we have

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{X}^m} g \, \mathrm{d}\alpha_{\eta_n,m} &= \lim_{n\to\infty} \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} \int_{\mathbb{X}^m} g(x_1,\dots,x_m)\, J_{\eta_n,B,k}\big(\mathrm{d}(x_1,\dots,x_m) \times B^{k-m}\big) \\
&= \lim_{n\to\infty} \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} \int_{\mathbb{X}^k} g(x_1,\dots,x_m)\, \mathbb{1}_{B^{k-m}}(x_{m+1},\dots,x_k)\, \mathrm{d}J_{\eta_n,B,k}(x_1,\dots,x_k) \\
&= \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} \int_{\mathbb{X}^k} g(x_1,\dots,x_m)\, \mathbb{1}_{B^{k-m}}(x_{m+1},\dots,x_k)\, \mathrm{d}J_{\eta,B,k}(x_1,\dots,x_k) \\
&= \int_{\mathbb{X}^m} g \, \mathrm{d}\alpha_{\eta,m},
\end{aligned}$$

where the use of dominated convergence is justified, as

$$\int_{\mathbb{X}^k} g(x_1,\dots,x_m)\, \mathrm{d}J_{\eta_n,B,k}(x_1,\dots,x_k) \leq c \cdot J_{\eta_n,B,k}(B^k) \leq \frac{c}{k!} \cdot \alpha_{\eta_n,k}(B^k) \leq \frac{c}{k!} \cdot \big(\vartheta\lambda(B)\big)^k$$

uniformly in n, and

$$\sum_{k=m}^{\infty} \frac{k!}{(k-m)!} \cdot \frac{\big(\vartheta\lambda(B)\big)^k}{k!} = \bigg(\int_B \vartheta(x)\, \mathrm{d}\lambda(x)\bigg)^m \exp\bigg(\int_B \vartheta(x)\, \mathrm{d}\lambda(x)\bigg) < \infty.$$

The above convergence (applied to $g = \mathbb{1}_D$) implies that, for any set $D \in \mathcal{X}^{\otimes m}$ with $D \subset B^m$ for some $B \in \mathcal{X}_b$,

$$\alpha_{\eta,m}(D) = \limsup_{n\to\infty} \alpha_{\eta_n,m}(D) \leq (\vartheta\lambda)^m(D).$$

Using the continuity (from below) of measures, we obtain that, for every $D \in \mathcal{X}^{\otimes m}$,

$$\alpha_{\eta,m}(D) = \limsup_{\ell\to\infty} \alpha_{\eta,m}(D \cap B_\ell^m) \leq \limsup_{\ell\to\infty} (\vartheta\lambda)^m(D \cap B_\ell^m) = (\vartheta\lambda)^m(D).$$

Next, consider $g \in L^1\big(\mathbb{X}^m, (\vartheta\lambda)^m\big)$. The following standard approximation argument is essentially an adaptation of the proof of Theorem 2.51 from Jansen (2017). Define the functions

$$g_\ell(x_1,\dots,x_m) = g(x_1,\dots,x_m)\, \mathbb{1}_{B_\ell^m}(x_1,\dots,x_m)\, \mathbb{1}\big\{|g(x_1,\dots,x_m)| \leq \ell\big\}, \quad \ell \in \mathbb{N}.$$

Each g_ℓ is a measurable and bounded function that vanishes outside of B_ℓ^m. Thus, the first part of the proof applies, and

$$\lim_{n\to\infty} \bigg| \int_{\mathbb{X}^m} g_\ell \, \mathrm{d}\alpha_{\eta_n,m} - \int_{\mathbb{X}^m} g_\ell \, \mathrm{d}\alpha_{\eta,m} \bigg| = 0$$

for each $\ell \in \mathbb{N}$. Furthermore, we have

$$\begin{aligned}
&\limsup_{\ell\to\infty} \bigg| \int_{\mathbb{X}^m} g \, \mathrm{d}\alpha_{\eta,m} - \int_{\mathbb{X}^m} g_\ell \, \mathrm{d}\alpha_{\eta,m} \bigg| \\
&\quad\leq \limsup_{\ell\to\infty} \int_{\mathbb{X}^m} \big|g(x_1,\dots,x_m)\big| \Big(\mathbb{1}_{\mathbb{X}^m \setminus B_\ell^m}(x_1,\dots,x_m) + \mathbb{1}\big\{|g(x_1,\dots,x_m)| > \ell\big\}\Big)\, \mathrm{d}\alpha_{\eta,m}(x_1,\dots,x_m) \\
&\quad\leq \limsup_{\ell\to\infty} \int_{\mathbb{X}^m} \big|g(x_1,\dots,x_m)\big| \Big(\mathbb{1}_{\mathbb{X}^m \setminus B_\ell^m}(x_1,\dots,x_m) + \mathbb{1}\big\{|g(x_1,\dots,x_m)| > \ell\big\}\Big)\, \mathrm{d}(\vartheta\lambda)^m(x_1,\dots,x_m)
\end{aligned}$$

$$= 0,$$

where dominated convergence applies, as

$$\big|g(x_1,\ldots,x_m)\big|\Big(\mathbb{1}_{\mathbb{X}^m\setminus B_\ell^m}(x_1,\ldots,x_m)+\mathbb{1}\big\{|g(x_1,\ldots,x_m)|>\ell\big\}\Big)\le 2\cdot\big|g(x_1,\ldots,x_m)\big|$$

uniformly in ℓ, and we have $g\in L^1\big(\mathbb{X}^m,(\vartheta\lambda)^m\big)$. Similarly, we have

$$\begin{aligned}
&\limsup_{\ell\to\infty}\sup_{n\in\mathbb{N}}\left|\int_{\mathbb{X}^m} g\,\mathrm{d}\alpha_{\eta_n,m}-\int_{\mathbb{X}^m} g_\ell\,\mathrm{d}\alpha_{\eta_n,m}\right|\\
&\quad\le\limsup_{\ell\to\infty}\sup_{n\in\mathbb{N}}\int_{\mathbb{X}^m}\big|g(x_1,\ldots,x_m)\big|\Big(\mathbb{1}_{\mathbb{X}^m\setminus B_\ell^m}(x_1,\ldots,x_m)+\mathbb{1}\big\{|g(x_1,\ldots,x_m)|>\ell\big\}\Big)\,\mathrm{d}\alpha_{\eta_n,m}(x_1,\ldots,x_m)\\
&\quad\le\limsup_{\ell\to\infty}\int_{\mathbb{X}^m}\big|g(x_1,\ldots,x_m)\big|\Big(\mathbb{1}_{\mathbb{X}^m\setminus B_\ell^m}(x_1,\ldots,x_m)+\mathbb{1}\big\{|g(x_1,\ldots,x_m)|>\ell\big\}\Big)\,\mathrm{d}(\vartheta\lambda)^m(x_1,\ldots,x_m)\\
&\quad=0.
\end{aligned}$$

Using the triangle inequality twice, the three limit relations establish that

$$\lim_{n\to\infty}\left|\int_{\mathbb{X}^m} g\,\mathrm{d}\alpha_{\eta_n,m}-\int_{\mathbb{X}^m} g\,\mathrm{d}\alpha_{\eta,m}\right|=0.$$

To prove the sufficiency part of the lemma, assume that the factorial moment measures converge in the stated sense. Exactly as above, it follows that η satisfies Ruelle's condition with the map ϑ. By part 1 of Remark 2.10, the assumed version of Ruelle's condition guarantees the validity of the assumption in Theorem 2.8 for each process in consideration and for each $B\in\mathcal{X}_b$. Thus, for any measurable and bounded function $f:\mathbb{X}^m\to[0,\infty)$ and any $B\in\mathcal{X}_b$, Theorem 2.8 and the assumption ensure that, as $n\to\infty$,

$$\begin{aligned}
\int_{\mathbb{X}^m} f\,\mathrm{d}J_{\eta_n,B,m}&=\int_{\mathbb{X}^m} f\,\mathbb{1}_{B^m}\,\mathrm{d}J_{\eta_n,B,m}\\
&=\frac{1}{m!}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{\mathbb{X}^m} f(x_1,\ldots,x_m)\,\mathbb{1}_{B^m}(x_1,\ldots,x_m)\,\alpha_{\eta_n,m+k}\big(\mathrm{d}(x_1,\ldots,x_m)\times B^k\big)\\
&=\frac{1}{m!}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{\mathbb{X}^{m+k}} f(x_1,\ldots,x_m)\,\mathbb{1}_{B^{m+k}}(x_1,\ldots,x_{m+k})\,\mathrm{d}\alpha_{\eta_n,m+k}(x_1,\ldots,x_{m+k})\\
&\to\frac{1}{m!}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{\mathbb{X}^{m+k}} f(x_1,\ldots,x_m)\,\mathbb{1}_{B^{m+k}}(x_1,\ldots,x_{m+k})\,\mathrm{d}\alpha_{\eta,m+k}(x_1,\ldots,x_{m+k})\\
&=\int_{\mathbb{X}^m} f\,\mathrm{d}J_{\eta,B,m},
\end{aligned}$$

where we used that $f\,\mathbb{1}_{B^{m+k}}\in L^1(\mathbb{X}^{m+k},(\vartheta\lambda)^{m+k})$, and dominated convergence is applicable since f is bounded and we have Ruelle's condition and the local integrability of ϑ. □

Remark 3.9 (**Previous existence proofs for Gibbs processes).** We give a short sketch of the proof of a previous version of an existence result for Gibbs processes. Assume that $\eta_1,\eta_2,\ldots$ are point processes in a complete separable metric space $(\mathbb{X},d)$ which satisfy Ruelle's condition for some universal constant $c\ge 0$. Then the corresponding correlation functions exist and $\rho_{n,m}=\rho_{\eta_n,m}\in L^\infty(\mathbb{X}^m,\lambda^m)=\big(L^1(\mathbb{X}^m,\lambda^m)\big)'$. With the Banach–Alaoglu theorem from functional analysis (see Appendix A.4) and a standard diagonal sequence construction, it is possible to extract a subsequence $\{n_k:k\in\mathbb{N}\}\subset\mathbb{N}$ such that

$$\lim_{k\to\infty}\int_{\mathbb{X}^m} g\,\rho_{n_k,m}\,\mathrm{d}\lambda^m=\int_{\mathbb{X}^m} g\,\rho_m\,\mathrm{d}\lambda^m$$

for all $g \in L^1(\mathbb{X}^m, \lambda^m)$, some function $\rho_m \in L^\infty(\mathbb{X}^m, \lambda^m)$, and each $m \in \mathbb{N}$. If one can show that the functions ρ_m are the correlation functions of a point process η, then Lemmata 3.8 and 3.3 imply that $\eta_{n_k} \xrightarrow{loc} \eta$, as $k \to \infty$. In order to show the existence of an infinite volume Gibbs process, the idea is to start with a suitable sequence of finite Gibbs processes, to guarantee that they satisfy Ruelle's bound, to construct the local limit as above, and to prove that the limit is itself Gibbs.

To obtain the limit process η in this construction, it is used that a family $\{\rho_m : m \in \mathbb{N}\}$ of symmetric functions $\rho_m : \mathbb{X}^m \to [0, \infty)$, which satisfy the Ruelle condition, are the correlation functions of some point process if, and only if, for all $B \in \mathcal{X}_b$, all $m \in \mathbb{N}$, and λ^m-a.e. $(x_1, \dots, x_m) \in B^m$,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{B^k} \rho_{m+k}(x_1, \dots, x_{m+k}) \, \mathrm{d}\lambda^k(x_{m+1}, \dots, x_{m+k}) \geq 0$$

as well as

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{B^k} \rho_k(x_1, \dots, x_k) \, \mathrm{d}\lambda^k(x_1, \dots, x_k) \geq 0.$$

This construction is carried out comprehensively for locally stable energy functions in the lectures notes by Jansen (2017) (and for pair interactions also by Jansen, 2019). There, the whole argument, including the proof of a less general version of Theorem 2.8, is given in terms of the so-called K-transform, as (implicitly) introduced by Lenard (1973) and (explicitly) used by Kondratiev and Kuna (2002). These proofs based on the K-transform are technical and the whole construction relies on Ruelle's bound.

Requiring a Ruelle type condition to assure that the correlation functions are a bounded sequence in L^∞ is somewhat unnatural. With Lemma 3.3 in mind, we do not have to make sure that the correlation functions are bounded, but can employ that Janossy densities naturally form a bounded sequence in L^1, as

$$\sup_{n \in \mathbb{N}} \|j_{\eta_n, B, m}\|_{L^1(\mathbb{X}^m, \lambda^m)} = \sup_{n \in \mathbb{N}} J_{\eta_n, B, m}(B^m) = \sup_{n \in \mathbb{N}} \mathbb{P}\big(\eta_n(B) = m\big) \leq 1$$

for all $m \in \mathbb{N}$ and $B \in \mathcal{X}_b$, given that the Janossy densities exist.

As we have shown in our proof of Theorem 2.8 and our other results, we focus on the point process theoretic perspective which often leads to very neat proofs. We use this perspective to provide a new result which allows for the extraction of locally convergent subsequences under weaker assumptions than the classical Ruelle condition. In the context of Gibbs processes, even Theorem 2.8 is often proven under the assumption that Ruelle's condition is satisfied, but in this situation the condition is also too strong as Remark 2.10 shows. However, it is not exactly clear of how much worth these formal improvements are in practical examples. ○

The following result forms the foundation of our existence proof of infinite volume Gibbs processes in Section 3.2. For the proof we need the Kolmogorov extension theorem for probability measures on $\mathbf{N}$ as recalled in Appendix A.6. To apply this theorem we have to restrict our attention to substandard Borel spaces (see Appendix A.6). Notice that no claim is made about the uniqueness of the constructed subsequence and the limit process, we merely provide an existence result.

Theorem 3.10. *Let $(\mathbb{X}, \mathcal{X})$ be a substandard Borel space. Let $\eta_1, \eta_2, \dots$ be point processes in $\mathbb{X}$ such that the Janossy densities $j_{n,B,m} = j_{\eta_n,B,m}$ corresponding to these processes exist and satisfy, for each $B \in \mathcal{X}_b$ and $m \in \mathbb{N}$,*

$$\lim_{c \to \infty} \sup_{n \in \mathbb{N}} \int_{\mathbb{X}^m} j_{n,B,m}(x_1, \dots, x_m) \, \mathbb{1}\big\{j_{n,B,m}(x_1, \dots, x_m) \geq c\big\} \, \mathrm{d}\lambda_B^m(x_1, \dots, x_m) = 0.$$

Also assume that, for each $B \in \mathcal{X}_b$, there exist maps $c_B : \mathbb{N} \to [0, \infty)$ with $\sum_{m=1}^\infty c_B(m) < \infty$ such that

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{X}^m} j_{n,B,m}(x_1, \dots, x_m) \, \mathrm{d}\lambda_B^m(x_1, \dots, x_m) \leq c_B(m)$$

for all $m \in \mathbb{N}$. Then there exists a point process η in $\mathbb{X}$ and a subsequence $(\eta_{n_k})_{k\in\mathbb{N}}$ of $(\eta_n)_{n\in\mathbb{N}}$ such that

$$\mathbb{E}\big[F(\eta_{n_k})\big] \longrightarrow \mathbb{E}\big[F(\eta)\big],$$

as $k \to \infty$, for all measurable, local, and bounded functions $F : \mathbf{N} \to [0,\infty)$. If the maps c_B are such that $\sum_{m=1}^{\infty} m \cdot c_B(m) < \infty$, then $\eta_{n_k} \xrightarrow{loc} \eta$ as $k \to \infty$.

Proof. We use the same notation, $J_{n,B,m} = J_{\eta_n,B,m}$, for Janossy measures as we do for Janossy densities. For the following construction, fix $B \in \mathcal{X}_b$. Since $J_{n,B,0} = \mathbb{P}\big(\eta_n(B) = 0\big)$ is a bounded sequence in $[0,1]$, there exists a strictly increasing map $r_0 : \mathbb{N} \to \mathbb{N}$, corresponding to the selection of a subsequence, such that

$$\lim_{k\to\infty} J_{r_0(k),B,0} = J_{B,0}$$

for some $J_{B,0} \in [0,1]$. Iteratively applying the Dunford–Pettis Lemma (Corollary A.17 in Appendix A.5) in the spaces $L^1(\mathbb{X}^m, \lambda_B^m)$ gives, for each $m \in \mathbb{N}$, a strictly increasing map $r_m : \mathbb{N} \to \mathbb{N}$ and a function $j_{B,m} \in L^1(\mathbb{X}^m, \lambda_B^m)$ which is set to 0 outside of B^m such that

$$\lim_{k\to\infty} \int_{\mathbb{X}^m} f\, j_{r_0\circ\ldots\circ r_m(k),B,m} \, \mathrm{d}\lambda_B^m = \int_{\mathbb{X}^m} f\, j_{B,m} \, \mathrm{d}\lambda_B^m$$

for all $f \in L^\infty(\mathbb{X}^m, \lambda_B^m)$. If we put $n_k = r_0 \circ \ldots \circ r_k(k)$, $k \in \mathbb{N}$, which corresponds to taking the diagonal sequence, then

$$\lim_{k\to\infty} J_{n_k,B,0} = J_{B,0} \quad \text{and} \quad \lim_{k\to\infty} \int_{\mathbb{X}^m} f\, j_{n_k,B,m} \, \mathrm{d}\lambda_B^m = \int_{\mathbb{X}^m} f\, j_{B,m} \, \mathrm{d}\lambda_B^m \tag{3.1}$$

for all $f \in L^\infty(\mathbb{X}^m, \lambda_B^m)$ and each $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, the limit function $j_{B,m}$ is non-negative λ^m-a.e. This is easy to see right away, but a short verification is provided in Appendix A.5 for the readers convenience. Moreover, the measures $J_{B,m}$, $m \in \mathbb{N}$, defined as

$$J_{B,m}(D) = \int_{\mathbb{X}^m} \mathbb{1}_D \cdot j_{B,m} \, \mathrm{d}\lambda^m, \quad D \in \mathcal{X}^{\otimes m},$$

are symmetric. Indeed, for any sets $D_1, \ldots, D_m \in \mathcal{X}$ and any permutation τ of $\{1, \ldots, m\}$, (3.1) yields

$$\begin{aligned} J_{B,m}\big(D_{\tau(1)} \times \ldots \times D_{\tau(m)}\big) = \int_{\mathbb{X}^m} \mathbb{1}_{D_{\tau(1)}\times\ldots\times D_{\tau(m)}}\, j_{B,m} \, \mathrm{d}\lambda^m &= \lim_{k\to\infty} \int_{\mathbb{X}^m} \mathbb{1}_{D_{\tau(1)}\times\ldots\times D_{\tau(m)}}\, j_{n_k,B,m} \, \mathrm{d}\lambda^m \\ &= \lim_{k\to\infty} J_{n_k,B,m}\big(D_{\tau(1)} \times \ldots \times D_{\tau(m)}\big) \\ &= \lim_{k\to\infty} J_{n_k,B,m}(D_1 \times \ldots \times D_m) \\ &= J_{B,m}(D_1 \times \ldots \times D_m). \end{aligned}$$

Notice that the so constructed subsequence depends on the chosen set B.

We now proceed to apply the above construction to the localizing sets B_ℓ, $\ell \in \mathbb{N}$. Applied to B_1, the previous arguments provide a subsequence $\{n_k^1 : k \in \mathbb{N}\} \subset \mathbb{N}$ as well as $J_{B_1,0} \in [0,1]$ and (almost everywhere) non-negative functions $j_{B_1,m} \in L^1(\mathbb{X}^m, \lambda^m)$, $m \in \mathbb{N}$, which vanish outside of B_1^m, such that

$$\lim_{k\to\infty} J_{n_k^1,B_1,0} = J_{B_1,0} \quad \text{and} \quad \lim_{k\to\infty} \int_{\mathbb{X}^m} f\, j_{n_k^1,B_1,m} \, \mathrm{d}\lambda^m = \int_{\mathbb{X}^m} f\, j_{B_1,m} \, \mathrm{d}\lambda^m$$

for all $f \in L^\infty(\mathbb{X}^m, \lambda^m)$ and each $m \in \mathbb{N}$. Iteratively applying this scheme gives (in the ℓ-th step) a subsequence $\{n_k^\ell : k \in \mathbb{N}\} \subset \{n_k^{\ell-1} : k \in \mathbb{N}\} \subset \ldots \subset \{n_k^1 : k \in \mathbb{N}\}$ as well as $J_{B_\ell,0} \in [0,1]$ and (almost everywhere) non-negative functions $j_{B_\ell,m} \in L^1(\mathbb{X}^m, \lambda^m)$, $m \in \mathbb{N}$, which vanish outside of B_ℓ^m, such that

$$\lim_{k\to\infty} J_{n_k^\ell,B_\ell,0} = J_{B_\ell,0} \quad \text{and} \quad \lim_{k\to\infty} \int_{\mathbb{X}^m} f\, j_{n_k^\ell,B_\ell,m}\, \mathrm{d}\lambda^m = \int_{\mathbb{X}^m} f\, j_{B_\ell,m}\, \mathrm{d}\lambda^m$$

for all $f \in L^\infty(\mathbb{X}^m, \lambda^m)$ and each $m \in \mathbb{N}$. Thus, choosing the diagonal sequence $n_k = n_k^k$, we have

$$\lim_{k\to\infty} J_{n_k,B_\ell,0} = J_{B_\ell,0} \quad \text{and} \quad \lim_{k\to\infty} \int_{\mathbb{X}^m} f\, j_{n_k,B_\ell,m}\, \mathrm{d}\lambda^m = \int_{\mathbb{X}^m} f\, j_{B_\ell,m}\, \mathrm{d}\lambda^m \tag{3.2}$$

for all $f \in L^\infty(\mathbb{X}^m, \lambda^m)$, each $m \in \mathbb{N}$, and every $\ell \in \mathbb{N}$. On each of the substandard Borel spaces $(\mathbf{N}, \mathcal{N}_{B_\ell})$, $\ell \in \mathbb{N}$, we define the measures

$$\mathrm{P}_\ell^{(n)}(A) = \mathbb{1}_A(\mathbf{0}) \cdot J_{n,B_\ell,0} + \sum_{m=1}^\infty \int_{B_\ell^m} \mathbb{1}_A\Big(\sum_{i=1}^m \delta_{x_i}\Big)\, j_{n,B_\ell,m}(x_1,\dots,x_m)\, \mathrm{d}\lambda^m(x_1,\dots,x_m), \quad A \in \mathcal{N}_{B_\ell},$$

for $n \in \mathbb{N}$, and

$$\mathrm{P}_\ell(A) = \mathbb{1}_A(\mathbf{0}) \cdot J_{B_\ell,0} + \sum_{m=1}^\infty \int_{B_\ell^m} \mathbb{1}_A\Big(\sum_{i=1}^m \delta_{x_i}\Big)\, j_{B_\ell,m}(x_1,\dots,x_m)\, \mathrm{d}\lambda^m(x_1,\dots,x_m), \quad A \in \mathcal{N}_{B_\ell}.$$

By assumption, we have

$$\sup_{k\in\mathbb{N}} \int_{B_\ell^m} \mathbb{1}_A\Big(\sum_{i=1}^m \delta_{x_i}\Big)\, j_{n_k,B_\ell,m}(x_1,\dots,x_m)\, \mathrm{d}\lambda^m(x_1,\dots,x_m) \le c_{B_\ell}(m),$$

where the right hand side constitutes an integrable bound with respect to summation over m. Hence, we can apply dominated convergence and the limit results from (3.2) to conclude that

$$\lim_{k\to\infty} \mathrm{P}_\ell^{(n_k)}(A) = \mathrm{P}_\ell(A)$$

for all $A \in \mathcal{N}_{B_\ell}$ and each $\ell \in \mathbb{N}$. Notice that Lemma 2.3 implies that, for $A \in \mathcal{N}_{B_\ell}$, we have

$$\mathrm{P}_\ell^{(n)}(A) = \mathbb{P}(\eta_n \in A).$$

It follows that $\mathrm{P}_\ell^{(n)}(\mathbf{N}) = 1$ and, therefore, $\mathrm{P}_\ell(\mathbf{N}) = \lim_{k\to\infty} \mathrm{P}_\ell^{(n_k)}(\mathbf{N}) = 1$, so $\mathrm{P}_\ell^{(n)}$ $(n \in \mathbb{N})$ and P_ℓ are probability measures on $(\mathbf{N}, \mathcal{N}_{B_\ell})$ for each $\ell \in \mathbb{N}$. Moreover, if we take indices $i < \ell$ and any set $A \in \mathcal{N}_{B_i} \subset \mathcal{N}_{B_\ell}$, we get

$$\mathrm{P}_\ell(A) = \lim_{k\to\infty} \mathrm{P}_\ell^{(n_k)}(A) = \lim_{k\to\infty} \mathbb{P}(\eta_{n_k} \in A) = \lim_{k\to\infty} \mathrm{P}_i^{(n_k)}(A) = \mathrm{P}_i(A).$$

Proposition A.18 implies that there exists a probability measure P on $(\mathbf{N}, \mathcal{N})$ such that $\mathrm{P}(A) = \mathrm{P}_\ell(A)$ for all $A \in \mathcal{N}_{B_\ell}$ and $\ell \in \mathbb{N}$. If we let η be a point process in $\mathbb{X}$ with distribution P, then by construction

$$\mathbb{P}(\eta \in A) = \mathrm{P}(A) = \mathrm{P}_\ell(A) = \mathbb{1}_A(\mathbf{0}) \cdot J_{B_\ell,0} + \sum_{m=1}^\infty \int_{B_\ell^m} \mathbb{1}_A\Big(\sum_{i=1}^m \delta_{x_i}\Big)\, j_{B_\ell,m}(x_1,\dots,x_m)\, \mathrm{d}\lambda^m(x_1,\dots,x_m),$$

for each $A \in \mathcal{N}_{B_\ell}$ and $\ell \in \mathbb{N}$, so Lemma 2.5 implies that $(J_{B_\ell,m})_{m\in\mathbb{N}_0}$ are the Janossy measures of η restricted to B_ℓ, for each $\ell \in \mathbb{N}$. The limit relations in (3.2) and Lemma 3.3 yield

$$\lim_{k\to\infty} \mathbb{E}\big[F(\eta_{n_k})\big] = \mathbb{E}\big[F(\eta)\big]$$

for all measurable, local, and bounded maps $F : \mathbf{N} \to [0, \infty)$, where we use that any local function is B_ℓ-local for some $\ell \in \mathbb{N}$. The additional claim concerning local convergence follows from Remark 3.4. □

It is clear from the proof that the assumptions on the Janossy densities in the theorem need only be satisfied on the sets B_ℓ, $\ell \in \mathbb{N}$.

Remark 3.11. We discuss some conditions which are sufficient to ensure the uniform integrability and summability conditions of Theorem 3.10. As in the theorem, let η_n, $n \in \mathbb{N}$, be point processes in $\mathbb{X}$ with Janossy densities $j_{\eta_n,B,m}$.

Similar to part 4 of Remark 2.10, assume that for each $B \in \mathcal{X}_b$ there exist $a_B \geq 0$ as well as a measurable and λ_B-integrable function $\vartheta_B : \mathbb{X} \to [0,\infty)$ such that

$$\sup_{n\in\mathbb{N}} j_{\eta_n,B,m}(x_1,\dots,x_m) \leq a_B \cdot \frac{\vartheta_B(x_1)\cdot\ldots\cdot\vartheta_B(x_m)}{m!} \tag{3.3}$$

for λ^m-a.e. $(x_1,\dots,x_m) \in \mathbb{X}^m$ and each $m \in \mathbb{N}$. Then, for all $B \in \mathcal{X}_b$ and $m \in \mathbb{N}$,

$$\begin{aligned}
&\lim_{c\to\infty} \sup_{n\in\mathbb{N}} \int_{\mathbb{X}^m} j_{\eta_n,B,m}(x_1,\dots,x_m)\,\mathbb{1}\big\{j_{\eta_n,B,m}(x_1,\dots,x_m) \geq c\big\}\,\mathrm{d}\lambda_B^m(x_1,\dots,x_m)\\
&\leq \frac{a_B}{m!}\lim_{c\to\infty}\int_{\mathbb{X}^m} \vartheta_B(x_1)\cdot\ldots\cdot\vartheta_B(x_m)\cdot\mathbb{1}\big\{a_B\cdot\vartheta_B(x_1)\cdot\ldots\cdot\vartheta_B(x_m) \geq c\cdot m!\big\}\,\mathrm{d}\lambda_B^m(x_1,\dots,x_m)\\
&= 0
\end{aligned}$$

as well as

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{X}^m} j_{\eta_n,B,m}(x_1,\dots,x_m)\,\mathrm{d}\lambda_B^m(x_1,\dots,x_m) \leq \frac{a_B}{m!}\bigg(\int_{\mathbb{X}} \vartheta_B(x)\,\mathrm{d}\lambda_B(x)\bigg)^m,$$

where m times the right hand side is clearly summable over m. These bounds can be interpreted as a local version of Ruelle's condition from Definition 3.7. Indeed, Corollary 2.7 gives

$$\sup_{n\in\mathbb{N}} \rho_{\eta_n,m}(x_1,\dots,x_m) \leq a_B\cdot\exp\bigg(\int_{\mathbb{X}}\vartheta_B(x)\,\mathrm{d}\lambda_B(x)\bigg)\cdot\vartheta_B(x_1)\cdot\ldots\cdot\vartheta_B(x_m)$$

for λ^m-a.e. $(x_1,\dots,x_m) \in B^m$, all $m \in \mathbb{N}$, and each $B \in \mathcal{X}_b$. On the other hand, any bound of the type

$$\sup_{n\in\mathbb{N}} \rho_{\eta_n,m}(x_1,\dots,x_m) \leq a_B\cdot\vartheta_B(x_1)\cdot\ldots\cdot\vartheta_B(x_m)$$

gives a bound on $\sup_{n\in\mathbb{N}} j_{\eta_n,B,m}(x_1,\dots,x_m)$ as $j_{\eta_n,B,m} \leq \frac{1}{m!}\cdot\rho_{\eta_n,m}$ almost everywhere by (2.3).

As the Janossy densities are derived from the localization of some overlying point process (and correlation functions are defined globally anyway), it is somewhat more intuitive for the bound in (3.3) to not depend on B. In this spirit, we mention the following two sufficient conditions for this bound.

1. If there exists a locally λ-integrable map $\vartheta : \mathbb{X} \to [0,\infty)$ such that

   $$\sup_{n\in\mathbb{N}} j_{\eta_n,B,m}(x_1,\dots,x_m) \leq \frac{\vartheta(x_1)\cdot\ldots\cdot\vartheta(x_m)}{m!}$$

   for λ^m-a.e. $(x_1,\dots,x_m) \in \mathbb{X}^m$, each $B \in \mathcal{X}_b$, and all $m \in \mathbb{N}$, then (3.3) follows. For instance, ϑ could be constant.

2. If there exists a locally λ-integrable map $\vartheta : \mathbb{X} \to [0,\infty)$ such that

   $$\sup_{n\in\mathbb{N}} \rho_{\eta_n,m}(x_1,\dots,x_m) \leq \vartheta(x_1)\cdot\ldots\cdot\vartheta(x_m)$$

   for λ^m-a.e. $(x_1,\dots,x_m) \in \mathbb{X}^m$ and all $m \in \mathbb{N}$, then (3.3) holds. Again, ϑ could simply be constant. ○

3.2. An existence result for general Gibbs point processes

The purpose of this section is to provide a general result which guarantees existence of infinite volume Gibbs processes. It is not at all clear for which Papangelou intensities such a process, as a solution of the GNZ integral equations, exists. As indicated in the introduction of this chapter, we intend to construct finite Gibbs processes as in Section 2.5 on a nested sequence of bounded sets in $\mathbb{X}$ and apply Theorem 3.10 to extract a limit process on the full space. It then remains to verify that the limit process solves the GNZ equations.

3.2.1. The abstract result

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space with localizing structure constructed via $B_1, B_2, \ldots$, as in Section 2.2. Let λ be a locally finite measure on $\mathbb{X}$ and consider a measurable function $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ which satisfies the cocycle relation (2.8). For each $n \in \mathbb{N}$ we consider a finite Gibbs processes ξ_n whose PI is given by

$$\kappa^{(B_n,\mathbf{0})}(x, \mu) = \kappa(x, \mu) \cdot \mathbb{1}_{B_n}(x), \quad x \in \mathbb{X}, \ \mu \in \mathbf{N}.$$

By Lemma 2.27, these finite Gibbs processes exist precisely when $Z_{B_n}(\mathbf{0}) < \infty$, $n \in \mathbb{N}$, and their distribution is given explicitly by (2.10).

In order to use Theorem 3.10 to extract from $(\xi_n)_{n\in\mathbb{N}}$ a locally convergent subsequence and a limit process η, we have to ensure that the required assumptions on the Janossy densities are satisfied. In this, we focus on an assumption related to Ruelle's condition from Definition 3.7 which seems adept to the study of Gibbs processes. Namely, we assume that

$$\sup_{n\in\mathbb{N}} j_{\xi_n,B,m}(x_1, \ldots, x_m) \leq \frac{\vartheta(x_1) \cdot \ldots \cdot \vartheta(x_m)}{m!}$$

for λ^m-a.e. $(x_1, \ldots, x_m) \in \mathbb{X}^m$, all $m \in \mathbb{N}$, every $B \in \mathcal{X}_b$, and some measurable, locally λ-integrable map $\vartheta : \mathbb{X} \to [0, \infty)$. In itself, this is an assumption on the whole construction of the finite Gibbs processes ξ_n and not a mere assumption on κ and λ. Such explicit assumptions will follow later, but the generality of the following theorem will be useful.

Theorem 3.12. *Let $(\mathbb{X}, \mathcal{X})$ be a substandard Borel space with localizing structure $B_1 \subset B_2 \subset \ldots$ and let λ be a locally finite measure on $\mathbb{X}$. Let $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ be a measurable map which satisfies the cocycle relation (2.8) and is such that*

$$Z_{B_n}(\mathbf{0}) < \infty, \quad n \in \mathbb{N}.$$

Moreover, suppose that, for λ-a.e. $x \in \mathbb{X}$,

$$\kappa(x, \mu) \leq \tilde{\vartheta}(x) \cdot c^{\mu(\mathbb{X})}, \quad \mu \in \mathbf{N},$$

for a constant $c \geq 0$ and a measurable, locally λ-integrable map $\tilde{\vartheta} : \mathbb{X} \to [0, \infty)$. Let ξ_n be finite Gibbs processes with PI $\kappa^{(B_n,\mathbf{0})}$ $(n \in \mathbb{N})$ and assume that

$$\sup_{n\in\mathbb{N}} j_{\xi_n,B,m}(x_1, \ldots, x_m) \leq \frac{\vartheta(x_1) \cdot \ldots \cdot \vartheta(x_m)}{m!}$$

for λ^m-a.e. $(x_1, \ldots, x_m) \in \mathbb{X}^m$, all $m \in \mathbb{N}$, every $B \in \mathcal{X}_b$, and some measurable, locally λ-integrable map $\vartheta : \mathbb{X} \to [0, \infty)$. Denote by η any one of the limit processes obtainable from Theorem 3.10 and assume that, for all $B \in \mathcal{X}_b^*$,

$$\limsup_{\ell\to\infty} \int_B \mathbb{E}\big|\kappa(x, \eta_{B_\ell}) - \kappa(x, \eta)\big| \, \mathrm{d}\lambda(x) = 0$$

as well as

$$\limsup_{\ell\to\infty} \sup_{k\in\mathbb{N}} \int_B \mathbb{E}\big|\kappa\big(x, (\xi_{n_k})_{B_\ell}\big) - \kappa(x, \xi_{n_k})\big| \, \mathrm{d}\lambda(x) = 0,$$

where $(\xi_{n_k})_{k\in\mathbb{N}}$ is the subsequence of $(\xi_n)_{n\in\mathbb{N}}$ which converges locally to η and where $\mathcal{X}_b^* \subset \mathcal{X}_b$ is a π-system which contains a nested sequence of sets that exhaust $\mathbb{X}$ and is such that

$$\sigma\big(\{B \times A : B \in \mathcal{X}_b^*,\, A \in \mathcal{Z}\}\big) = \mathcal{X} \otimes \mathcal{N},$$

with $\mathcal{Z}$ denoting the local events from Definition A.9. Then η is a Gibbs process with PI κ.

Proof. First of all, recall that by Lemma 2.27 the Gibbs processes ξ_n exist, as $Z_{B_n}(\mathbf{0}) < \infty$ for $n \in \mathbb{N}$. The bound on the Janossy densities covers the assumptions of Theorem 3.10 by Remark 3.11, so we have $\xi_{n_k} \xrightarrow{loc} \eta$, as $k \to \infty$, where the subsequence and the limit process η are assumed to be as in the statement of the theorem. It remains to prove that η is a Gibbs process with PI κ.

Notice that the Janossy densities of η satisfy the same bound as those of the processes ξ_n. Indeed, by Lemma 3.6, we have, for each $m \in \mathbb{N}$, every $B \in \mathcal{X}_b$, and any $D \in \mathcal{X}^{\otimes m}$,

$$\int_D j_{\eta,B,m}\,\mathrm{d}\lambda^m = \limsup_{k\to\infty} \int_D j_{\xi_{n_k},B,m}\,\mathrm{d}\lambda^m \le \frac{1}{m!} \int_{D\cap B^m} \vartheta(x_1)\cdot\ldots\cdot\vartheta(x_m)\,\mathrm{d}\lambda^m(x_1,\ldots,x_m),$$

which yields

$$j_{\eta,B,m}(x_1,\ldots,x_m) \le \frac{\vartheta(x_1)\cdot\ldots\cdot\vartheta(x_m)}{m!}$$

for λ^m-a.e. $(x_1,\ldots,x_m) \in \mathbb{X}^m$.

Fix $B \in \mathcal{X}_b^*$ and $A \in \mathcal{Z}$, and let $C \in \mathcal{X}_b$ be such that $A \in \mathcal{N}_C$. Define the following measurable maps from $\mathbf{N}$ to $[0,\infty]$,

$$\begin{aligned}
F(\mu) &= \int_{\mathbb{X}} \mathbb{1}_B(x)\,\mathbb{1}_A(\mu)\,\mathrm{d}\mu(x),\\
\tilde{F}(\mu) &= \int_{\mathbb{X}} \mathbb{1}_B(x)\,\mathbb{1}_A(\mu+\delta_x)\,\kappa(x,\mu)\,\mathrm{d}\lambda(x),\\
\tilde{F}_\ell(\mu) &= \int_{\mathbb{X}} \mathbb{1}_B(x)\,\mathbb{1}_A(\mu+\delta_x)\,\kappa(x,\mu_{B_\ell})\,\mathrm{d}\lambda(x), \quad \ell\in\mathbb{N}.
\end{aligned}$$

We collect in four steps the essential properties of these maps.

(1) The function F is $(B\cup C)$-local and tame, as $F(\mu) \le \mu(B)$, so the local convergence applies to F and

$$\lim_{k\to\infty} \mathbb{E}\big[F(\xi_{n_k})\big] = \mathbb{E}\big[F(\eta)\big],$$

where these expectations are bounded by

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \mathbb{E}\big[F(\xi_n)\big] \le \sup_{n\in\mathbb{N}} \mathbb{E}\big[\xi_n(B)\big] &= \sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} \int_{\mathbb{X}^m} \sum_{i=1}^{m} \delta_{x_i}(B)\cdot j_{\xi_n,B,m}(x_1,\ldots,x_m)\,\mathrm{d}\lambda^m(x_1,\ldots,x_m)\\
&\le \sum_{m=1}^{\infty} \frac{m}{m!}\bigg(\int_B \vartheta(x)\,\mathrm{d}\lambda(x)\bigg)^m\\
&= \int_B \vartheta(x)\,\mathrm{d}\lambda(x)\cdot\exp\bigg(\int_B \vartheta(x)\,\mathrm{d}\lambda(x)\bigg),
\end{aligned}$$

and similarly for $\mathbb{E}\big[F(\eta)\big]$ (see also Remark 3.4).

(2) Using that, for λ-a.e. $x \in \mathbb{X}$ and all $\ell \in \mathbb{N}$,

$$\mathbb{E}\big[\kappa(x,\eta_{B_\ell})\big] \le \tilde{\vartheta}(x)\cdot\mathbb{E}\big[c^{\eta(B_\ell)}\big] = \tilde{\vartheta}(x)\sum_{m=0}^{\infty} c^m\cdot\mathbb{P}\big(\eta(B_\ell)=m\big)$$

$$
\begin{aligned}
&= \tilde{\vartheta}(x) \cdot \left(J_{\eta, B_\ell, 0} + \sum_{m=1}^{\infty} c^m \cdot J_{\eta, B_\ell, m}(B_\ell^m) \right) \\
&\leq \tilde{\vartheta}(x) \cdot \exp\left(c \int_{B_\ell} \vartheta(x) \, \mathrm{d}\lambda(x) \right),
\end{aligned} \tag{3.4}
$$

we obtain

$$
\mathbb{E}\big[\tilde{F}_\ell(\eta)\big] \leq \int_B \mathbb{E}\big[\kappa(x, \eta_{B_\ell})\big] \, \mathrm{d}\lambda(x) \leq \int_B \tilde{\vartheta}(x) \, \mathrm{d}\lambda(x) \cdot \exp\left(c \int_{B_\ell} \vartheta(x) \, \mathrm{d}\lambda(x) \right) < \infty,
$$

and the very same term bounds $\sup_{n \in \mathbb{N}} \mathbb{E}\big[\tilde{F}_\ell(\xi_n)\big]$. By assumption, we have

$$
\limsup_{\ell \to \infty} \big| \mathbb{E}\big[\tilde{F}(\eta)\big] - \mathbb{E}\big[\tilde{F}_\ell(\eta)\big] \big| \leq \limsup_{\ell \to \infty} \int_B \mathbb{E}\big| \kappa(x, \eta_{B_\ell}) - \kappa(x, \eta) \big| \, \mathrm{d}\lambda(x) = 0,
$$

and

$$
\limsup_{\ell \to \infty} \sup_{k \in \mathbb{N}} \big| \mathbb{E}\big[\tilde{F}(\xi_{n_k})\big] - \mathbb{E}\big[\tilde{F}_\ell(\xi_{n_k})\big] \big| \leq \limsup_{\ell \to \infty} \sup_{k \in \mathbb{N}} \int_B \mathbb{E}\big| \kappa\big(x, (\xi_{n_k})_{B_\ell}\big) - \kappa(x, \xi_{n_k}) \big| \, \mathrm{d}\lambda(x) = 0.
$$

(3) We now show that, despite $\tilde{F}_\ell$ not being tame, we have $\lim_{k \to \infty} \mathbb{E}\big[\tilde{F}_\ell(\xi_{n_k})\big] = \mathbb{E}\big[\tilde{F}_\ell(\eta)\big]$ for each $\ell \in \mathbb{N}$. To this end, fix $\ell \in \mathbb{N}$ and define the measurable maps $\mathbf{N} \to [0, \infty)$,

$$
\tilde{F}_{\ell, j}(\mu) = \int_{\mathbb{X}} \mathbb{1}_B(x) \, \mathbb{1}_A(\mu + \delta_x) \, \kappa(x, \mu_{B_\ell}) \, \mathbb{1}\big\{ \kappa(x, \mu_{B_\ell}) \leq j \big\} \, \mathrm{d}\lambda(x), \quad j \in \mathbb{N}.
$$

For each $j \in \mathbb{N}$, the map $\tilde{F}_{\ell, j}$ is $(B \cup C \cup B_\ell)$-local and bounded by $j \cdot \lambda(B)$. Therefore, the local convergence of ξ_{n_k} to η gives

$$
\lim_{k \to \infty} \mathbb{E}\big[\tilde{F}_{\ell, j}(\xi_{n_k})\big] = \mathbb{E}\big[\tilde{F}_{\ell, j}(\eta)\big]
$$

for every $j \in \mathbb{N}$. Using (3.4) to justify the application of dominated convergence, we have

$$
\limsup_{j \to \infty} \big| \mathbb{E}\big[\tilde{F}_{\ell, j}(\eta)\big] - \mathbb{E}\big[\tilde{F}_\ell(\eta)\big] \big| \leq \limsup_{j \to \infty} \int_B \mathbb{E}\Big[\kappa(x, \eta_{B_\ell}) \, \mathbb{1}\big\{ \kappa(x, \eta_{B_\ell}) > j \big\} \Big] \, \mathrm{d}\lambda(x) = 0.
$$

Moreover, observe that, by Lemma 2.3 and the bounds on κ and the Janossy densities,

$$
\begin{aligned}
&\sup_{k \in \mathbb{N}} \big| \mathbb{E}\big[\tilde{F}_{\ell, j}(\xi_{n_k})\big] - \mathbb{E}\big[\tilde{F}_\ell(\xi_{n_k})\big] \big| \\
&\quad \leq \sup_{k \in \mathbb{N}} \int_B \mathbb{E}\Big[\kappa\big(x, (\xi_{n_k})_{B_\ell}\big) \, \mathbb{1}\big\{ \kappa\big(x, (\xi_{n_k})_{B_\ell}\big) > j \big\} \Big] \, \mathrm{d}\lambda(x) \\
&\quad = \sup_{k \in \mathbb{N}} \Bigg(\int_B \kappa(x, \mathbf{0}) \, \mathbb{1}\big\{ \kappa(x, \mathbf{0}) > j \big\} \, \mathrm{d}\lambda(x) \cdot \mathbb{P}\big(\xi_{n_k}(B_\ell) = 0 \big) \\
&\qquad\qquad + \sum_{m=1}^{\infty} \int_{\mathbb{X}^m} \int_B \kappa\Big(x, \sum_{i=1}^{m} (\delta_{x_i})_{B_\ell} \Big) \, \mathbb{1}\Big\{ \kappa\Big(x, \sum_{i=1}^{m} (\delta_{x_i})_{B_\ell} \Big) > j \Big\} \, \mathrm{d}\lambda(x) \, \mathrm{d}J_{\xi_{n_k}, B_\ell, m}(x_1, \ldots, x_m) \Bigg) \\
&\quad \leq \int_B \tilde{\vartheta}(x) \, \mathbb{1}\big\{ \tilde{\vartheta}(x) > j \big\} \, \mathrm{d}\lambda(x) + \sum_{m=1}^{\infty} \frac{c^m}{m!} \left(\int_{B_\ell} \vartheta(x) \, \mathrm{d}\lambda(x) \right)^m \int_B \tilde{\vartheta}(x) \, \mathbb{1}\big\{ \tilde{\vartheta}(x) \cdot c^m > j \big\} \, \mathrm{d}\lambda(x),
\end{aligned}
$$

and the right hand side converges to 0 as $j \to \infty$, by dominated convergence. Now, let $\varepsilon > 0$. Choose $j_0 \in \mathbb{N}$ such that

$$
\big| \mathbb{E}\big[\tilde{F}_{\ell, j_0}(\eta)\big] - \mathbb{E}\big[\tilde{F}_\ell(\eta)\big] \big| < \frac{\varepsilon}{3} \quad \text{and} \quad \sup_{k \in \mathbb{N}} \big| \mathbb{E}\big[\tilde{F}_{\ell, j_0}(\xi_{n_k})\big] - \mathbb{E}\big[\tilde{F}_\ell(\xi_{n_k})\big] \big| < \frac{\varepsilon}{3}.
$$

Choose $k_0 \in \mathbb{N}$ such that, for each $k \geq k_0$,

$$\big|\mathbb{E}\big[\tilde{F}_{\ell,j_0}(\xi_{n_k})\big] - \mathbb{E}\big[\tilde{F}_{\ell,j_0}(\eta)\big]\big| < \frac{\varepsilon}{3}.$$

Then, for each $k \geq k_0$, we have

$$\begin{aligned}\big|\mathbb{E}\big[\tilde{F}_{\ell}(\xi_{n_k})\big] - \mathbb{E}\big[\tilde{F}_{\ell}(\eta)\big]\big| &\leq \big|\mathbb{E}\big[\tilde{F}_{\ell}(\xi_{n_k})\big] - \mathbb{E}\big[\tilde{F}_{\ell,j_0}(\xi_{n_k})\big]\big| + \big|\mathbb{E}\big[\tilde{F}_{\ell,j_0}(\xi_{n_k})\big] - \mathbb{E}\big[\tilde{F}_{\ell,j_0}(\eta)\big]\big| \\ &\quad + \big|\mathbb{E}\big[\tilde{F}_{\ell,j_0}(\eta)\big] - \mathbb{E}\big[\tilde{F}_{\ell}(\eta)\big]\big| \\ &< \varepsilon.\end{aligned}$$

(4) We now use (2) and (3) to show that $\lim_{k\to\infty} \mathbb{E}\big[\tilde{F}(\xi_{n_k})\big] = \mathbb{E}\big[\tilde{F}(\eta)\big]$. Let $\varepsilon > 0$. By (2) we can choose $\ell_0 \in \mathbb{N}$ such that

$$\big|\mathbb{E}\big[\tilde{F}(\eta)\big] - \mathbb{E}\big[\tilde{F}_{\ell_0}(\eta)\big]\big| < \frac{\varepsilon}{3} \quad \text{and} \quad \sup_{k\in\mathbb{N}} \big|\mathbb{E}\big[\tilde{F}(\xi_{n_k})\big] - \mathbb{E}\big[\tilde{F}_{\ell_0}(\xi_{n_k})\big]\big| < \frac{\varepsilon}{3}.$$

By (3) we can then choose $k_0 \in \mathbb{N}$ such that, for each $k \geq k_0$,

$$\big|\mathbb{E}\big[\tilde{F}_{\ell_0}(\xi_{n_k})\big] - \mathbb{E}\big[\tilde{F}_{\ell_0}(\eta)\big]\big| < \frac{\varepsilon}{3}.$$

Then, for each $k \geq k_0$,

$$\begin{aligned}\big|\mathbb{E}\big[\tilde{F}(\xi_{n_k})\big] - \mathbb{E}\big[\tilde{F}(\eta)\big]\big| &\leq \big|\mathbb{E}\big[\tilde{F}(\xi_{n_k})\big] - \mathbb{E}\big[\tilde{F}_{\ell_0}(\xi_{n_k})\big]\big| + \big|\mathbb{E}\big[\tilde{F}_{\ell_0}(\xi_{n_k})\big] - \mathbb{E}\big[\tilde{F}_{\ell_0}(\eta)\big]\big| \\ &\quad + \big|\mathbb{E}\big[\tilde{F}_{\ell_0}(\eta)\big] - \mathbb{E}\big[\tilde{F}(\eta)\big]\big| \\ &< \varepsilon.\end{aligned}$$

Using the convergence results from (1) and (4), the GNZ equations for ξ_{n_k}, and that $B \subset B_{n_k}$ once k is large enough, we get

$$\begin{aligned}\mathbb{E}\bigg[\int_{\mathbb{X}} \mathbb{1}_B(x)\,\mathbb{1}_A(\eta)\,\mathrm{d}\eta(x)\bigg] = \mathbb{E}\big[F(\eta)\big] &= \lim_{k\to\infty} \mathbb{E}\big[F(\xi_{n_k})\big] \\ &= \lim_{k\to\infty} \mathbb{E}\bigg[\int_{\mathbb{X}} \mathbb{1}_B(x)\,\mathbb{1}_A(\xi_{n_k})\,\mathrm{d}\xi_{n_k}(x)\bigg] \\ &= \lim_{k\to\infty} \mathbb{E}\bigg[\int_{\mathbb{X}} \mathbb{1}_B(x)\,\mathbb{1}_A(\xi_{n_k} + \delta_x)\,\kappa(x,\xi_{n_k})\,\mathbb{1}_{B_{n_k}}(x)\,\mathrm{d}\lambda(x)\bigg] \\ &= \lim_{k\to\infty} \mathbb{E}\big[\tilde{F}(\xi_{n_k})\big] \\ &= \mathbb{E}\big[\tilde{F}(\eta)\big] \\ &= \mathbb{E}\bigg[\int_{\mathbb{X}} \mathbb{1}_B(x)\,\mathbb{1}_A(\eta + \delta_x)\,\kappa(x,\eta)\,\mathrm{d}\lambda(x)\bigg].\end{aligned}$$

Consequently, the GNZ equation holds for all functions $(x,\mu) \mapsto \mathbb{1}_{B\times A}(x,\mu)$ with $B \in \mathcal{X}_b^*$ and $A \in \mathcal{Z}$. A literal copy of the final step in the proof of Lemma 2.14 (for $m = 0$), using the properties of $\mathcal{X}_b^*$, extends the equality to all functions $(x,\mu) \mapsto \mathbb{1}_E(x,\mu)$, $E \in \mathcal{X} \otimes \mathcal{N}$, and monotone approximation allows for any measurable function $f : \mathbb{X} \times \mathbf{N} \to [0,\infty]$. We conclude that η is a Gibbs process with PI κ. □

Remark 3.13. We make some general remarks on the previous existence theorem.

- In order to construct Gibbs processes with a boundary condition ψ, one has to apply Theorem 3.12 to $\kappa^{(\mathbb{X},\psi)} = \kappa(\,\cdot\,,\psi + \cdot\,)$. However, apart from the cocycle assumption, it is not guaranteed that $\kappa^{(\mathbb{X},\psi)}$ inherits the necessary properties from κ. In the special case that κ is locally stable, a condition discussed below,

$\kappa^{(\mathbb{X},\psi)}$ inherits this property, and most of the assumptions in Theorem 3.12 are satisfied.

- Note that any Gibbs process whose Janossy densities satisfy the bound in Theorem 3.12 has a locally finite intensity measure. Indeed, we have shown this in item (1) of the proof of Theorem 3.12. ○

One particular assumption on κ, which covers the finiteness of the partition function, the bound on κ in Theorem 3.12, and the bound on the Janossy densities in that same theorem, is the local stability assumption which is frequent in stochastic geometry and spatial statistics as it is an essential assumption for many simulation algorithms for Gibbsian point processes, cf. Møller and Waagepetersen (2004).

Definition 3.14 (Local stability). A measurable map $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ is called (λ-)locally stable if

$$\sup_{\mu \in \mathbf{N}} \kappa(x, \mu) \leq \vartheta(x)$$

for λ-a.e. $x \in \mathbb{X}$ and some measurable, locally λ-integrable map $\vartheta : \mathbb{X} \to [0, \infty)$.

Remark 3.15. In the pair potential setting of Chapter 4 we will see that Theorem 3.12 can be used to show existence when κ is not locally stable. However, restricting as it is, the local stability assumption comes as a quite natural one. Just recall that, by Lemma 2.18, we have

$$j_{\xi_n,B,m}(x_1, \ldots, x_m) = \frac{1}{m!}\, \mathbb{E}\Big[\mathbb{1}\big\{\xi_n(B) = 0\big\}\, \kappa_m(x_1, \ldots, x_m, \xi_n)\Big]\, \mathbb{1}_{(B \cap B_n)^m}(x_1, \ldots, x_m),$$

where the processes ξ_n are as in Theorem 3.12. These Janossy densities are certainly bounded in the desired manner if we assume that

$$\sup_{\mu \in \mathbf{N}_f} \kappa_m(x_1, \ldots, x_m, \mu) \leq \vartheta(x_1) \cdot \ldots \cdot \vartheta(x_m)$$

for λ^m-a.e. $(x_1, \ldots, x_m) \in \mathbb{X}^m$ and some locally λ-integrable map $\vartheta : \mathbb{X} \to [0, \infty)$. However, this is essentially equivalent to κ being locally stable. Aside from the bound on the Janossy densities, the local stability assumption also guarantees that

$$Z_{B_n}(\mathbf{0}) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{B_n^m} \kappa_m(x_1, \ldots, x_m, \mathbf{0})\, \mathrm{d}\lambda^m(x_1, \ldots, x_m) \leq \exp\left(\int_{B_n} \vartheta(x)\, \mathrm{d}\lambda(x) \right) < \infty$$

for every $n \in \mathbb{N}$. Moreover, local stability implies

$$\kappa(x, \mu) \leq \vartheta(x) \cdot c^{\mu(\mathbb{X})}$$

for λ-a.e. $x \in \mathbb{X}$, all $\mu \in \mathbf{N}$, and any $c \geq 1$. Hence, local stability covers all assumptions in Theorem 3.12 except the two limit relations. In fact it suffices to assume the bound in Definition 3.14 with the supremum taken over all $\mu \in \mathbf{N}_f$.

Another possibility to obtain the bound on the Janossy densities in terms of an explicit assumption on κ is to study the correlation functions of the finite Gibbs processes a little more closely. By Corollary 2.28, the Janossy densities of these processes on their full domain are

$$j_{\xi_n,B_n,m}(x_1, \ldots, x_m) = \frac{1}{m!} \cdot \frac{\kappa_m(x_1, \ldots, x_m, \mathbf{0})}{Z_{B_n}(\mathbf{0})} \cdot \mathbb{1}_{B_n^m}(x_1, \ldots, x_m).$$

Corollary 2.7 implies that the correlation functions satisfy

$$\rho_{\xi_n,m}(x_1, \ldots, x_m) = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} \int_{B_n^{k-m}} j_{\xi_n,B_n,k}(x_1, \ldots, x_k)\, \mathrm{d}\lambda^{k-m}(x_{m+1}, \ldots, x_k)$$

$$= \frac{1}{Z_{B_n}(\mathbf{0})} \sum_{k=m}^{\infty} \frac{1}{(k-m)!} \int_{B_n^{k-m}} \kappa_k(x_1, \ldots, x_k, \mathbf{0}) \, \mathrm{d}\lambda^{k-m}(x_{m+1}, \ldots, x_k)$$

for λ^m-a.e. $(x_1, \ldots, x_m) \in B_n^m$ and all $m, n \in \mathbb{N}$. Hence, if the right hand side, which is an explicit quantity in terms of κ, is suitably bounded, we get

$$\sup_{n \in \mathbb{N}} \rho_{\xi_n, m}(x_1, \ldots, x_m) \leq \vartheta(x_1) \cdot \ldots \cdot \vartheta(x_m)$$

for λ^m-a.e. $(x_1, \ldots, x_m) \in \mathbb{X}^m$ and all $m \in \mathbb{N}$. By (2.3) this implies the bound on the Janossy densities. For pair potentials, the above representation of the correlation functions is studied by Ruelle (1970) and the required bound is derived under suitable assumptions. We come back to this condition later on. ○

Remark 3.16. In the definition of a Gibbs process via the GNZ equations in (2.7) we find the term $\kappa(x, \eta) \, \mathrm{d}\lambda(x)$ and this is the only place where κ and λ appear. Consequently, there is some freedom in the choice of these parameters. For instance, in arguments where the specific reference measure does not matter (which includes most of our arguments), it is no loss of generality to choose the local stability bound on κ as $\vartheta \equiv 1$. Indeed, if κ satisfies the local stability assumption with bounding function ϑ, we may consider

$$\tilde{\kappa}(x, \mu) = \begin{cases} \dfrac{\kappa(x, \mu)}{\vartheta(x)}, & \text{if } \vartheta(x) > 0, \\ 0, & \text{if } \vartheta(x) = 0. \end{cases}$$

With this choice, a point process η in $\mathbb{X}$ is a Gibbs process with PI κ and reference measure λ if, and only if, η is Gibbs process with PI $\tilde{\kappa}$ and reference measure $\vartheta\lambda$. A similar argument could be used to normalize bounds on the Janossy densities or correlation functions. However, we will not resort to this normalization and carry the bound for κ with us throughout the thesis.

Note that in allowing for κ to be bounded by general ϑ we follow Last and Otto (2021). Jansen (2019) also includes such an inhomogeneity but in terms of an intensity functions associated with the measure λ. ○

To put the new existence result into context, let us make a few more comments on the (mostly more recent) literature. For an existence proof of Gibbs processes in $\mathbb{R}^d$, Dobrushin (1969) uses a different compactness criterion. Moreover, Dereudre (2009), Dereudre et al. (2012), Dereudre and Vasseur (2020), and Rœlly and Zass (2020) use that Georgii and Zessin (1993) have established the compactness of level sets of the specific entropy in the local convergence topology. Approaching the existence question via these level sets of the specific entropy requires some kind of stationarity so the resulting existence results work only in $\mathbb{R}^d$ (or locally compact topological groups). In $\mathbb{R}^d$ this method seems favorable as it leads to stationary Gibbs processes and works under weak explicit assumptions on the energy function. Indeed, the paper by Dereudre and Vasseur (2020), in which the literature applying this approach culminates (in a sense), only requires the minimal assumption of stability as well as the so-called intensity regularity, an assumption that is made for similar technical reasons as the limit relations in Theorem 3.12. There also exists an analytical approach to the theory of Gibbs processes as laid out by Kondratiev et al. (2012) and Conache et al. (2018), but the focus is on pair potentials in $\mathbb{R}^d$ and the theory is quite technical and tough to follow. Another class of existence results comes from the specification point of view, which, like ours, are available in very abstract spaces, cf. Preston (1976). They often use some variant of the Dunford–Pettis property (see Chapter 5 of Preston, 2005b), just like we do. In our mind the main difference between this last class of existence results and ours lies in the general approach toward Gibbs process, namely between introducing the Gibbs process in terms of specifications related to the DLR equations and considering Gibbs processes as solutions to the GNZ equations in the modern point process theoretic framework that we adopt.

3.2.2. Gibbs processes with finite interaction range

As we have seen in Remark 3.15, local stability is one straightforward assumption to cover all of the prerequisites of Theorem 3.12 except the two limit relations. Probably the easiest way to ensure the validity of these limit requirements is to suppose that κ has finite range. Before we define this property, notice that, formally, it is enough to have that for each $x \in \mathbb{X}$ there exists a set $B_x \in \mathcal{X}_b$ with

$$\kappa(x, \mu) = \kappa(x, \mu_{B_x}), \quad \mu \in \mathbf{N},$$

such that $\bigcup_{x \in B} B_x \in \mathcal{X}_b$ for all $B \in \mathcal{X}_b$. Indeed, writing $B^\circ = \bigcup_{x \in B} B_x$, for fixed $B \in \mathcal{X}_b$, we have $\kappa(x, \mu) = \kappa(x, \mu_{B^\circ})$ for all $x \in B$ and $\mu \in \mathbf{N}$. Thus, if $\ell \in \mathbb{N}$ is large enough such that $B_\ell \supset B^\circ$, then

$$\kappa(x, \mu_{B_\ell}) = \kappa\big(x, (\mu_{B_\ell})_{B^\circ}\big) = \kappa(x, \mu_{B^\circ}) = \kappa(x, \mu)$$

for each $x \in B$ and $\mu \in \mathbf{N}$, and

$$\int_B \sup_{\mu \in \mathbf{N}} \big|\kappa(x, \mu_{B_\ell}) - \kappa(x, \mu)\big| \, \mathrm{d}\lambda(x) = 0.$$

Remark 3.17. We make an observation on the discussion above which influences our formal definition of finite range. Let $x, y \in \mathbb{X}$. Heuristically, B_y is chosen such that $x \notin B_y$ does not influence y, but $x \in B_y$ does influence y in the sense that, for some $\mu \in \mathbf{N}$,

$$\kappa(y, \mu + \delta_x) \neq \kappa(y, \mu).$$

It is thus intuitive that $x \in B_y$ precisely when $y \in B_x$. There are also formal reasons to assume this, relating to the cocycle assumption. If $x, y \in \mathbb{X}$ are such that

$$\kappa(y, \mu + \delta_x) \neq \kappa(y, \mu) \quad \text{for some } \mu \in \mathbf{N} \text{ with } \kappa(x, \mu) > 0, \tag{3.5}$$

then we must have $x \in B_y$ and $y \in B_x$. Otherwise, we would have (in case $x \notin B_y$)

$$\kappa(y, \mu + \delta_x) = \kappa\big(y, \mu_{B_y} + (\delta_x)_{B_y}\big) = \kappa(y, \mu_{B_y}) = \kappa(y, \mu)$$

or (in case $y \notin B_x$)

$$\kappa(x, \mu) \cdot \kappa(y, \mu + \delta_x) = \kappa(y, \mu) \cdot \kappa(x, \mu + \delta_y) = \kappa(y, \mu) \cdot \kappa\big(x, \mu_{B_x} + (\delta_y)_{B_x}\big) = \kappa(y, \mu) \cdot \kappa(x, \mu),$$

which gives $\kappa(y, \mu + \delta_x) = \kappa(y, \mu)$, leading to a contradiction in both cases. Also notice that if x influences y, in the sense that (3.5) holds, then

$$\kappa(x, \mu + \delta_y) \cdot \kappa(y, \mu) = \kappa(x, \mu) \cdot \kappa(y, \mu + \delta_x) \neq \kappa(x, \mu) \cdot \kappa(y, \mu)$$

and thus $\kappa(y, \mu) > 0$ as well as $\kappa(x, \mu + \delta_y) \neq \kappa(x, \mu)$, meaning that y influences x. For $x, y \in \mathbb{X}$ with $\kappa(y, \mu + \delta_x) = \kappa(y, \mu)$ for all $\mu \in \mathbf{N}$ there is, formally, no harm in letting $x \in B_y$ but $y \notin B_x$, however, this immediately implies $\kappa(x, \mu + \delta_y) = \kappa(x, \mu)$ for each $\mu \in \mathbf{N}$. All of these observations confirm the intuition about the strong symmetry in interactions between two points. ○

With the previous remark in mind, we choose the following definition.

Definition 3.18 (Finite range). A measurable map $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ is said to have *finite range* if for each $x \in \mathbb{X}$ there exists a set $B_x \in \mathcal{X}_b$ with

$$\kappa(x, \mu) = \kappa(x, \mu_{B_x}), \quad \mu \in \mathbf{N},$$

such that $\bigcup_{x \in B} B_x \in \mathcal{X}_b$ for each $B \in \mathcal{X}_b$ and such that $x \in B_y$ if, and only if, $y \in B_x$, for all $x, y \in \mathbb{X}$.

If $(\mathbb{X}, \mathcal{X})$ is separable, meaning that $\{x\} \in \mathcal{X}$ for each $x \in \mathbb{X}$, it is always possible to include x in B_x while maintaining all other properties. In metric spaces the classical finite range property known from the literature is sufficient to cover Definition 3.18. Note that in metric spaces we always choose as a localizing structure a sequence of balls with growing radius around a fixed origin in $\mathbb{X}$, so $\mathcal{X}_b$ corresponds to the sets that are bounded with respect to the metric.

Lemma 3.19. *Assume that $\mathbb{X}$ is a (Borel subset of a) metric space with metric denoted by d. A measurable map $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ has finite range if there exists a fixed radius of interaction $R > 0$ such that $\kappa(x, \mu) = \kappa\big(x, \mu_{B(x,R)}\big)$ for all $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$, where $B(x, R)$ denotes the closed ball of radius R around x with respect to d.*

Proof. If κ has interaction radius $R > 0$, we choose $B_x = B(x, R)$ for every $x \in \mathbb{X}$. Each B_x is a bounded set and, by assumption,

$$\kappa(x, \mu_{B_x}) = \kappa\big(x, \mu_{B(x,R)}\big) = \kappa(x, \mu), \quad x \in \mathbb{X}, \ \mu \in \mathbf{N}.$$

Moreover, we trivially have $x \in B_y$ if, and only if, $y \in B_x$. For $B \in \mathcal{X}_b \setminus \{\varnothing\}$, the set $B^\circ = \bigcup_{x \in B} B(x, R)$ corresponds to the R-envelope of B which is bounded and in $\mathcal{X}$ (cf. Appendices A.7 and A.9). $\square$

By the discussion at the beginning of this subsection, the finite range property is sufficient to ensure that the two limit properties in Theorem 3.12 are fulfilled. At this point, we could therefore state, as a corollary, another existence result just like Theorem 3.12 with the limit relations replaced by κ having finite range and all other properties unchanged. We do not state this result explicitly at this point, but focus on another setting: If we combine local stability and finite range, we can provide an existence result which also allows for boundary conditions without any problems concerning the formal prerequisites in the sense that $\kappa^{(\mathbb{X},\psi)} = \kappa(\,\cdot\,, \psi + \cdot)$ inherits the properties of κ. Note that the assumptions of the following result are purely on κ without any explicit reference to the processes involved.

Corollary 3.20. *Let $(\mathbb{X}, \mathcal{X})$ be a substandard Borel space with localizing structure $B_1 \subset B_2 \subset \ldots$, let $\psi \in \mathbf{N}$, and let λ be a locally finite measure on $\mathbb{X}$. Suppose that $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ is a measurable map such that the cocycle relation (2.8) holds. Furthermore, assume that κ is λ-locally stable as in Definition 3.14 and that κ has finite range. Then there exists a Gibbs process η with PI $\kappa^{(\mathbb{X},\psi)}$.*

Proof. The cocycle assumption as well as the local stability transfer to $\kappa^{(\mathbb{X},\psi)}$ trivially. By Remark 3.15, $\kappa^{(\mathbb{X},\psi)}$ satisfies all prerequisites from Theorem 3.12 except the two limit relations. To their effect, notice that, for fixed $B \in \mathcal{X}_b$, if we choose $\ell \in \mathbb{N}$ large enough such that $B_\ell \supset B^\circ = \bigcup_{x \in B} B_x$, then

$$\begin{aligned}
\int_B \sup_{\mu \in \mathbf{N}} \big| \kappa^{(\mathbb{X},\psi)}(x, \mu_{B_\ell}) - \kappa^{(\mathbb{X},\psi)}(x, \mu) \big| \, \mathrm{d}\lambda(x) &= \int_B \sup_{\mu \in \mathbf{N}} \big| \kappa(x, \psi + \mu_{B_\ell}) - \kappa(x, \psi + \mu) \big| \, \mathrm{d}\lambda(x) \\
&= \int_B \sup_{\mu \in \mathbf{N}} \big| \kappa(x, \psi_{B_x} + \mu_{B_x}) - \kappa(x, \psi_{B_x} + \mu_{B_x}) \big| \, \mathrm{d}\lambda(x) \\
&= 0.
\end{aligned}$$

The result follows by applying Theorem 3.12 with $\kappa^{(\mathbb{X},\psi)}$. $\square$

Example 3.21 (Strauss processes). Let $\mathbb{X}$ be a (Borel subset of a) complete separable metric space. Let $R > 0$, $c \in [0, 1]$, and $\vartheta : \mathbb{X} \to [0, \infty)$ a locally λ-integrable function. Following Strauss (1975), with the correction due to Kelly and Ripley (1976) in mind, consider

$$\kappa(x, \mu) = \vartheta(x) \cdot c^{\mu(B(x,R))}, \quad x \in \mathbb{X},\ \mu \in \mathbf{N},$$

which is a measurable mapping that satisfies the cocycle relation, as

$$\begin{aligned} \kappa(x, \mu) \cdot \kappa(y, \mu + \delta_x) &= \vartheta(x) \cdot \vartheta(y) \cdot c^{\mu(B(x,R))} \cdot c^{(\mu+\delta_x)(B(y,R))} \\ &= \vartheta(x) \cdot \vartheta(y) \cdot c^{(\mu+\delta_y)(B(x,R))} \cdot c^{\mu(B(y,R))} \\ &= \kappa(y, \mu) \cdot \kappa(x, \mu + \delta_y), \end{aligned}$$

for any $x, y \in \mathbb{X}$ and $\mu \in \mathbf{N}$. Moreover, $\sup_{\mu \in \mathbf{N}} \kappa(x, \mu) \leq \vartheta(x)$ for every $x \in \mathbb{X}$. Furthermore, κ obviously obeys $\kappa(x, \mu) = \kappa(x, \mu_{B(x,R)})$ for each $x \in \mathbb{X}$ and all $\mu \in \mathbf{N}$, that is, κ has finite range. Thus, Corollary 3.20 guarantees the existence of a Gibbs process in $\mathbb{X}$ with PI κ. A special case arises for $c = 0$, where

$$\kappa(x, \mu) = \vartheta(x) \cdot \mathbb{1}\big\{\mu\big(B(x, R)\big) = 0\big\}, \quad x \in \mathbb{X},\ \mu \in \mathbf{N}.$$

With the interpretation that $\kappa(x, \mu)$ gives an indication on how likely it is to find a point in x given the configuration μ, this special choice for κ does not allow for a point in x if μ already has a point which lies closer to x then the distance R. Figuratively speaking, each point in the Gibbs process is associated with a sphere of radius $R/2$ and the spheres may not overlap. The process represents *hard spheres*. Of course we can also consider Strauss processes with boundary conditions. ○

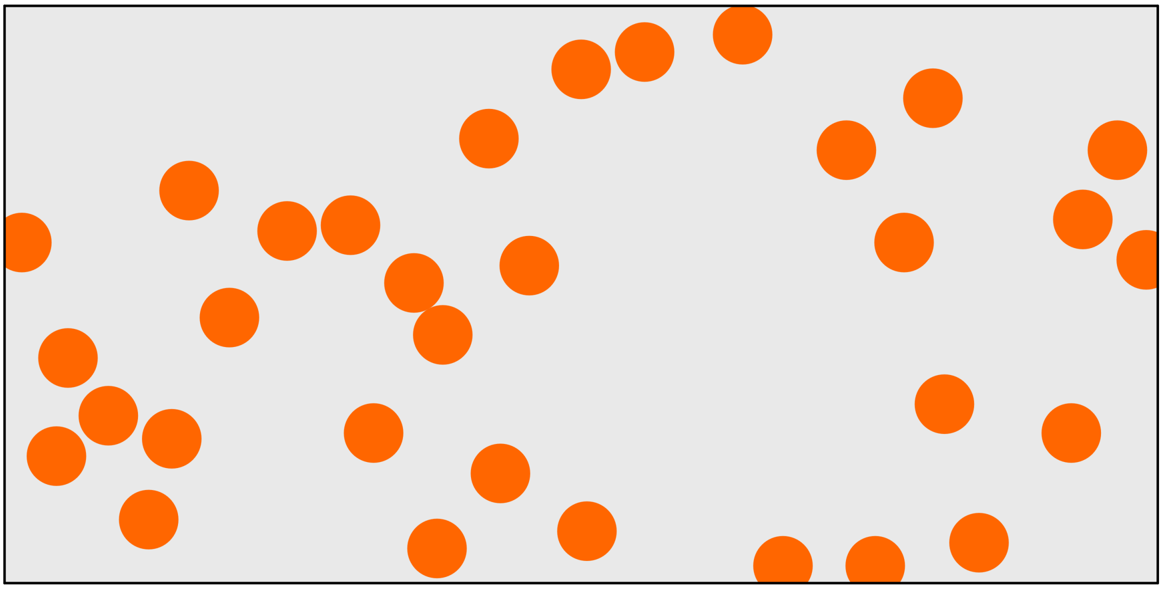


Figure 3.1.: A realization of a hard spheres process in $[0, 2] \times [0, 1]$ with intensity $\gamma = 50$ and $\vartheta \equiv 1$. The full disks with diameter $R = 0.1$ are printed for better visualization.

3.3. Cluster-dependent interactions and a general uniqueness result

Our study of Gibbs processes up to this point has shown that many structural properties as well as finite volume Gibbs processes can be discussed in full generality, referring to most of what happened in Chapter 2. In Section 3.2, however, we have also seen that in order to prove existence of infinite volume Gibbs processes, additional assumptions, beyond the cocycle relation and finite partition functions, are necessary. Apart from the restriction to substandard Borel spaces, which the Kolmogorov extension result requires, we have to provide some boundedness assumption to enable the extraction of convergent subsequences of a sequence of point processes. More precisely, we require that the Janossy densities are bounded in a suitable sense, referring to the assumptions of Theorem 3.10. Furthermore, in our general existence result, Theorem 3.12, we need κ to satisfy two limit relations in order to establish the GNZ equations for the infinite volume limit of a sequence of finite Gibbs processes. These limit relations are, in a sense, a compromise to avoid continuity assumptions on κ which are too restricting. One common assumption to ensure the validity of these limit relations is that κ has finite range, as discussed in Section 3.2.2. An issue with this last assumption is that many interesting interactions have an infinite range. While the pair potentials we discuss in Chapter 4, which allow for infinite range, provide a very specific structure for κ, the purpose of this section is to investigate another, rather general and abstract, class of interactions which also allow for an infinite range, but where the influence of points which are far away is controlled in a suitable sense. A bit more specifically, we introduce a notion of connectedness between points in the state space and consider interactions which only depend on the clusters that these connections form. In the so-called subcritical regime, where the clusters that the points of a Poisson process make up are assumed to be finite almost surely, the range of the interactions is sufficiently controlled to allow for existence and even uniqueness results for the corresponding Gibbs processes. This regime also allows for strong structural result like decorrelation and mixing properties.

In Section 3.3.1 we first recall that locally stable Gibbs processes are dominated by suitable Poisson processes and we generalize a randomization property which is very handy when extending properties of Gibbs processes with diffuse reference measures to the case of general reference measures. In Section 3.3.2 we introduce the concept of cluster-dependent interactions in the generality of Last and Otto (2021) and we prove that the corresponding Gibbs processes exist in the subcritical regime, thus providing a general overlying structure for the processes considered by Hofer-Temmel (2019), Hofer-Temmel and Houdebert (2019), Dereudre (2019), and Beneš et al. (2020). Section 3.3.3 contains a short summary of the disagreement coupling due to Last and Otto (2021) and a new approximation result which is the main ingredient of our uniqueness proof in Section 3.3.4. We conclude the study of abstract cluster-dependent interactions by proving decorrelation and mixing properties in Sections 3.3.5 and 3.3.6. Specific examples of cluster-dependent interactions, namely Gibbsian particle processes, are discussed later on in Section 3.4.

In this section on cluster-dependent interactions, let $(\mathbb{X}, d)$ be a complete separable metric space with localizing structure $B_n = B(x_0, n)$, $n \in \mathbb{N}$, for some fixed origin $x_0 \in \mathbb{X}$. Denote by $\mathcal{X}$ the Borel σ-field on $\mathbb{X}$ and by $\mathcal{X}_b$ the Borel sets which are bounded with respect to d. Let λ be a locally finite measure on $\mathbb{X}$. For some results in the section it is necessary that λ is diffuse. Whenever this assumption is needed, it is stated explicitly, if it not, then λ is assumed to be general. The restriction to metric spaces has technical reasons, namely that we need to deal with weak convergence of measures in Lemma 3.22. Many other results in the following could be derived in greater generality, namely on Borel spaces, but as Lemma 3.22 is central in the main results of this section, we fix the setting throughout. Recall that on the given state space it is possible to write any $\mu \in \mathbf{N} \setminus \{\mathbf{0}\}$ as $\mu = \sum_{j=1}^{m} \delta_{x_j}$, for $x_1, x_2, \ldots \in \mathbb{X}$ and $m \in \mathbb{N} \cup \{\infty\}$, by Proposition 6.3 of Last and Penrose (2017).

3.3.1. Stochastic domination and randomization techniques

For the proof of both the existence and uniqueness of cluster-dependent Gibbs processes, it is essential that any Gibbs process is dominated by a suitable Poisson process. For finite Gibbs processes this is known from Georgii and Küneth (1997), but the extension to infinite processes takes additional effort. Conceptually we follow Last and Otto (2021) in doing so. However, we need further properties related to the local convergence later on. A similar result could also be obtained by combining Lemma 5.3 of Last and Otto (2021) and Lemma 3.25 below.

Note that the space $\mathbf{N}$ is endowed with the metric defined in Appendix A.7 and that Lemma A.19 ensures

$$\big\{(\mu,\nu) \in \mathbf{N} \times \mathbf{N} : \mu \leq \nu\big\} \in \mathcal{N} \otimes \mathcal{N}.$$

In fact, this set is closed in $\mathbf{N} \times \mathbf{N}$.

Lemma 3.22. *Let $D_1, D_2, \ldots \in \mathcal{X}$ and let $\xi_1, \xi_2, \ldots$ be a sequence of finite Gibbs processes in $\mathbb{X}$, whose Papangelou intensities are locally stable as in Definition 3.14 with a uniform bound $\vartheta : \mathbb{X} \to [0,\infty)$, such that $\xi_n(D_n^c) = 0$ almost surely for each $n \in \mathbb{N}$. Assume that $\xi_n \xrightarrow{loc} \eta$ for a point process η in $\mathbb{X}$. Then (possibly after extending the underlying probability space) there exist Poisson processes Φ_n with intensity measures $\vartheta\lambda_{D_n}$ ($n \in \mathbb{N}$) and a Poisson process Φ with intensity measure $\vartheta\lambda$ as well as point processes $\tilde{\xi}_n$ ($n \in \mathbb{N}$) and $\tilde{\eta}$ so that*

$$\tilde{\xi}_n \overset{d}{=} \xi_n \quad \text{and} \quad \tilde{\xi}_n \leq \Phi_n \, (\mathbb{P}\text{-a.s.}) \quad \text{for all } n \in \mathbb{N}, \qquad \tilde{\eta} \overset{d}{=} \eta \quad \text{and} \quad \tilde{\eta} \leq \Phi \, (\mathbb{P}\text{-a.s.}),$$

and $\tilde{\xi}_n \xrightarrow{loc} \tilde{\eta}$ as $n \to \infty$. Here $\overset{d}{=}$ denotes equality in distribution.

Proof. Let $n \in \mathbb{N}$. By Example 2.1 of Georgii and Küneth (1997) and Strassen's theorem (cf. Lindvall, 1992, 1999), there exists a point process $\tilde{\xi}_n$ with $\tilde{\xi}_n \overset{d}{=} \xi_n$ and a Poisson process Φ_n with intensity measure $\vartheta\lambda_{D_n}$ such that $\tilde{\xi}_n \leq \Phi_n$ almost surely. By Lemma 3.2 we have $\tilde{\xi}_n \xrightarrow{d} \eta$. Theorem 16.3 of Kallenberg (2002) yields that $(\tilde{\xi}_n)_{n\in\mathbb{N}}$ is tight, meaning that for every $\varepsilon > 0$ there exists a compact set $K \subset \mathbf{N}$ such that

$$\liminf_{n\to\infty} \mathbb{P}\big(\tilde{\xi}_n \in K\big) \geq 1 - \varepsilon.$$

As $(\Phi_n)_{n\in\mathbb{N}}$ converges locally (hence in distribution) by Example 3.5, there also exists a compact set $K' \subset \mathbf{N}$ such that

$$\liminf_{n\to\infty} \mathbb{P}\big(\Phi_n \in K'\big) \geq 1 - \varepsilon.$$

The set $K \times K' \subset \mathbf{N} \times \mathbf{N}$ is compact in the product topology and

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{P}\big((\tilde{\xi}_n, \Phi_n) \in K \times K'\big) &= \liminf_{n\to\infty} \Big(\mathbb{P}\big(\tilde{\xi}_n \in K\big) + \mathbb{P}\big(\Phi_n \in K'\big) - \mathbb{P}\big(\{\tilde{\xi}_n \in K\} \cup \{\Phi_n \in K'\}\big)\Big) \\ &\geq 1 - 2\varepsilon,\end{aligned}$$

so $(\tilde{\xi}_n, \Phi_n)_{n\in\mathbb{N}}$ is a tight sequence. Thus, there exists a subsequence which converges in distribution to a limit element $(\tilde{\eta}, \Phi)$, where Φ is a Poisson process with intensity measure $\vartheta\lambda$ and $\tilde{\eta} \overset{d}{=} \eta$ (as weak limits are unique). By Lemma A.19 and the Portmanteau theorem (Theorem 4.25 of Kallenberg, 2002), we have

$$\mathbb{P}(\tilde{\eta} \leq \Phi) = 1.$$

Moreover, for each measurable, local, and tame function $F : \mathbf{N} \to [0,\infty)$ we obtain

$$\mathbb{E}\big[F(\tilde{\xi}_n)\big] = \mathbb{E}\big[F(\xi_n)\big] \longrightarrow \mathbb{E}\big[F(\eta)\big] = \mathbb{E}\big[F(\tilde{\eta})\big]$$

as $n \to \infty$, which implies the asserted local convergence. $\square$

Note that Lemma 3.22 does not claim that the limit process η in consideration is a Gibbs process. The limit relations in the assumptions of Theorem 3.12 hint at what additional requirements are needed in order for η to be Gibbs. However, if an infinite Gibbs process η can be written as a local limit of finite Gibbs processes whose PIs are locally stable with the same bound, then Lemma 3.22 implies that it is no loss of generality to assume that η is dominated by a suitable Poisson process. The question arises whether any Gibbs process with locally stable PI has this property. To address this question, we first show that any such Gibbs process can be obtained as a local limit of finite Gibbs processes.

Lemma 3.23. *Let $\kappa : \mathbb{X} \times \mathbf{N} \to [0,\infty)$ be a measurable function that satisfies the cocycle relation (2.8) and is locally stable as in Definition 3.14 with bound $\vartheta : \mathbb{X} \to [0,\infty)$. Suppose that η is a Gibbs process in $\mathbb{X}$ with PI κ. Then there exists a sequence of finite Gibbs processes $\xi_1, \xi_2, \dots$ in $\mathbb{X}$, whose PIs are locally stable with bound ϑ, such that $\xi_n \xrightarrow{loc} \eta$ as $n \to \infty$.*

Proof. For each $n \in \mathbb{N}$ choose $\xi_n = \eta_{B_n}$. Then ξ_n is a finite Gibbs process in $\mathbb{X}$ by Lemma 2.45 and for any measurable, B-local function $F : \mathbf{N} \to [0,\infty]$, with $B \in \mathcal{X}_b$, choosing $n \in \mathbb{N}$ large enough to have $B_n \supset B$ ensures that

$$\mathbb{E}\big[F(\xi_n)\big] = \mathbb{E}\big[F(\eta_{B_n})\big] = \mathbb{E}\big[F\big((\eta_{B_n})_B\big)\big] = \mathbb{E}\big[F(\eta_B)\big] = \mathbb{E}\big[F(\eta)\big].$$

Hence, we trivially have $\xi_n \xrightarrow{loc} \eta$. It remains to show that the PIs corresponding to the ξ_n, which are given by (2.17), inherit the local stability from κ. To this end, we fix any set $B \in \mathcal{X}_b$ and consider κ_B^* as in (2.17). If $x \in B^{\mathsf{c}}$ or $\mu(\mathbb{X}) = \infty$ then, by definition of κ_B^* and H, we have $\kappa_B^*(x,\mu) = 0$. If $x \in B$ and $\mu(\mathbb{X}) = m \in \mathbb{N}$ then, writing $\mu = \sum_{j=1}^m \delta_{x_j}$, Corollary 2.31 implies

$$\begin{aligned}
\mathrm{e}^{-H(\mu + \delta_x, \eta_{B^{\mathsf{c}}})} = \kappa_{m+1}(x, x_1, \dots, x_m, \eta_{B^{\mathsf{c}}}) &= \kappa_m(x_1, \dots, x_m, \eta_{B^{\mathsf{c}}}) \cdot \kappa(x, \eta_{B^{\mathsf{c}}} + \delta_{x_1} + \ldots + \delta_{x_m}) \\
&\le \kappa_m(x_1, \dots, x_m, \eta_{B^{\mathsf{c}}}) \cdot \vartheta(x) \\
&= \mathrm{e}^{-H(\mu, \eta_{B^{\mathsf{c}}})} \cdot \vartheta(x)
\end{aligned}$$

(which holds path-wise on Ω) and therefore

$$\mathbb{E}\big[Z_B(\eta_{B^{\mathsf{c}}})^{-1} \cdot \mathrm{e}^{-H(\mu + \delta_x, \eta_{B^{\mathsf{c}}})}\big] \le \vartheta(x) \cdot \mathbb{E}\big[Z_B(\eta_{B^{\mathsf{c}}})^{-1} \cdot \mathrm{e}^{-H(\mu, \eta_{B^{\mathsf{c}}})}\big].$$

This relation also holds if $\mu(\mathbb{X}) = 0$ since $\mathrm{e}^{-H(\delta_x, \eta_{B^{\mathsf{c}}})} = \kappa(x, \eta_{B^{\mathsf{c}}}) \le \vartheta(x)$ and $\mathrm{e}^{-H(\mathbf{0}, \eta_{B^{\mathsf{c}}})} = 1$. In any case, we have $\kappa_B^*(x,\mu) \le \vartheta(x)$, which concludes the proof. □

An immediate corollary of Lemmata 3.22 and 3.23 is the following result.

Corollary 3.24. *Let η be a Gibbs process in $\mathbb{X}$ whose PI is locally stable with bound $\vartheta : \mathbb{X} \to [0,\infty)$. Then (after extending the underlying probability space) there exists a Poisson process Φ with intensity measure $\vartheta\lambda$ such that $\eta \le \Phi$ almost surely.*

Some of the tools we use for uniqueness proofs later on, most prominently the disagreement coupling, are only available for diffuse reference measures. A simple randomization property, special cases of which were already used - though provided without proof - by Schuhmacher and Stucki (2014) and Betsch and Last (2022), serves as a remedy. We state and prove a slightly more general result. Note that the metric structure on $\mathbb{X}$ is not necessary for this result, it suffices that $(\mathbb{X}, \mathcal{X})$ is a localized Borel space. Also, recall from Section 2.2.1 the notion of point processes in $\mathbb{X} \times \mathbb{M}$.

Lemma 3.25. *Let $(\mathbb{M}, \mathcal{M}, Q)$ be a probability space. Let $\kappa : \mathbb{X} \times \mathbf{N} \to [0,\infty)$ be a measurable function that satisfies the cocycle relation (2.8). Put $\tilde{\kappa} : \mathbb{X} \times \mathbb{M} \times \mathbf{N}(\mathbb{X} \times \mathbb{M}) \to [0,\infty)$,*

$$\tilde{\kappa}(x, r, \mu) = \kappa\big(x, \mu(\,\cdot \times \mathbb{M})\big),$$

and let $\eta = \sum_{j=1}^{\eta(\mathbb{X})} \delta_{X_j}$ be a Gibbs process in $\mathbb{X}$ with PI κ and reference measure λ. Let $R_1, R_2, \dots$ be independent random variables in $\mathbb{M}$ distributed according to Q, with $(R_j)_{j\in\mathbb{N}}$ independent of η. Then the randomization $\tilde{\eta} = \sum_{j=1}^{\eta(\mathbb{X})} \delta_{(X_j, R_j)}$ of η is a Gibbs process in $\mathbb{X} \times \mathbb{M}$ with PI $\tilde{\kappa}$ and reference measure $\lambda \otimes Q$.

Proof. For $\mu = \sum_{j=1}^{\mu(\mathbb{X})} \delta_{x_j} \in \mathbf{N}$, with $x_1, x_2, \ldots \in \mathbb{X}$, and $s = (s_j)_{j\in\mathbb{N}} \in \mathbb{M}^\infty = \bigotimes_{j=1}^\infty \mathbb{M}$ define

$$T(\mu, s) = \sum_{j=1}^{\mu(\mathbb{X})} \delta_{(x_j, s_j)}$$

which constitutes a measurable map $\mathbf{N} \times \mathbb{M}^\infty \to \mathbf{N}(\mathbb{X} \times \mathbb{M})$. Write $R = (R_j)_{j \in \mathbb{N}}$ and $\mathbb{Q} = \bigotimes_{j=1}^\infty Q$. For any measurable function $f : \mathbb{X} \times \mathbb{M} \times \mathbf{N}(\mathbb{X} \times \mathbb{M}) \to [0, \infty]$ we have

$$\mathbb{E}\bigg[\int_{\mathbb{X}\times\mathbb{M}} f(x,r,\tilde{\eta})\,\mathrm{d}\tilde{\eta}(x,r)\bigg] = \mathbb{E}\bigg[\int_{\mathbb{X}\times\mathbb{M}} f\big(x,r,T(\eta,R)\big)\,\mathrm{d}\big(T(\eta,R)\big)(x,r)\bigg].$$

Due to the given independence properties, this term equals

$$\int_{\mathbb{M}^\infty} \mathbb{E}\bigg[\int_{\mathbb{X}\times\mathbb{M}} f\big(x,r,T(\eta,s)\big)\,\mathrm{d}\big(T(\eta,s)\big)(x,r)\bigg]\mathrm{d}\mathbb{Q}(s) = \mathbb{E}\Bigg[\sum_{j=1}^{\eta(\mathbb{X})} \int_{\mathbb{M}^\infty} f\big(X_j, s_j, T(\eta,s)\big)\,\mathrm{d}\mathbb{Q}(s)\Bigg]. \tag{3.6}$$

In order to write the term in the sum in (3.6) as $g(X_j, \eta)$ for a suitable map g, we have to ensure that g assigns the marks correctly in the sense that the fact that s_j is the mark of X_j is also relevant in $T(\eta, s)$. Upon defining the measurable map $g : \mathbb{X} \times \mathbf{N} \to [0, \infty]$ by

$$g(x,\mu) = \mathbb{1}\big\{\mu(\{x\}) > 0\big\} \int_{\mathbb{M}^\infty}\int_{\mathbb{M}} f\big(x,r,T(\mu\setminus\delta_x, s) + \delta_{(x,r)}\big)\,\mathrm{d}Q(r)\,\mathrm{d}\mathbb{Q}(s),$$

this is ensured as the infinite product structure implies (almost surely)

$$g(X_j,\eta) = \int_{\mathbb{M}^\infty}\int_{\mathbb{M}} f\big(X_j,r,T(\eta\setminus\delta_{X_j}, s) + \delta_{(X_j,r)}\big)\,\mathrm{d}Q(r)\,\mathrm{d}\mathbb{Q}(s) = \int_{\mathbb{M}^\infty} f\big(X_j, s_j, T(\eta,s)\big)\,\mathrm{d}\mathbb{Q}(s).$$

Hence, picking up the previous calculation from (3.6), we have

$$\mathbb{E}\bigg[\int_{\mathbb{X}\times\mathbb{M}} f(x,r,\tilde{\eta})\,\mathrm{d}\tilde{\eta}(x,r)\bigg] = \mathbb{E}\Bigg[\sum_{j=1}^{\eta(\mathbb{X})} g(X_j,\eta)\Bigg] = \mathbb{E}\bigg[\int_{\mathbb{X}} g(x,\eta)\,\mathrm{d}\eta(x)\bigg].$$

By the GNZ equation for η and the given independence properties, this term further equals

$$\begin{aligned}
\mathbb{E}\bigg[\int_{\mathbb{X}} g(x,\eta+\delta_x)\,\kappa(x,\eta)\,\mathrm{d}\lambda(x)\bigg] &= \mathbb{E}\bigg[\int_{\mathbb{X}}\bigg(\int_{\mathbb{M}^\infty}\int_{\mathbb{M}} f\big(x,r,T(\eta,s)+\delta_{(x,r)}\big)\,\mathrm{d}Q(r)\,\mathrm{d}\mathbb{Q}(s)\bigg)\kappa(x,\eta)\,\mathrm{d}\lambda(x)\bigg] \\
&= \mathbb{E}\bigg[\int_{\mathbb{X}}\int_{\mathbb{M}} f\big(x,r,T(\eta,R)+\delta_{(x,r)}\big)\,\kappa(x,\eta)\,\mathrm{d}Q(r)\,\mathrm{d}\lambda(x)\bigg] \\
&= \mathbb{E}\bigg[\int_{\mathbb{X}\times\mathbb{M}} f\big(x,r,\tilde{\eta}+\delta_{(x,r)}\big)\,\tilde{\kappa}(x,r,\tilde{\eta})\,\mathrm{d}(\lambda\otimes Q)(x,r)\bigg],
\end{aligned}$$

which concludes the proof. □

3.3.2. Cluster-dependent interactions

We now introduce the crucial concept of cluster-dependent interactions. Let us first specify what we understand by a cluster. Consider a symmetric binary relation $\sim$ on $\mathbb{X}$ such that

$$\big\{(x,y) \in \mathbb{X}^2 : x \sim y\big\} \in \mathcal{X}^{\otimes 2}.$$

We call $x, y \in \mathbb{X}$ connected via $D \subset \mathbb{X}$ if there exist $n \in \mathbb{N}_0$ and $z_1, \ldots, z_n \in D$ such that $z_j \sim z_{j+1}$ for each $j \in \{0, \ldots, n\}$, where we put $z_0 = x$ and $z_{n+1} = y$. This last terminology is also used for counting measures $\mu \in \mathbf{N}$ in the sense that $D = \mathrm{supp}(\mu) = \big\{x \in \mathbb{X} : \mu(\{x\}) > 0\big\}$. For formal completeness, observe that $\mathrm{supp}(\mu) \in \mathcal{X}$ by Lemma A.11. If x, y are connected via μ we write $x \overset{\mu}{\sim} y$. Moreover, we write $x \overset{\mu}{\sim} \psi$ if x is connected via μ to some point of $\psi \in \mathbf{N}$. We define

$$C(x, \mu) = \int_{\mathbb{X}} \mathbb{1}\{y \in \cdot\}\, \mathbb{1}\{x \overset{\mu}{\sim} y\}\, \mathrm{d}\mu(y) \tag{3.7}$$

and call $C(x, \mu)$ the (μ-)cluster of x, illustrated in Figure 3.2. The point x is usually not contained in $C(x, \mu)$ as the latter only contains points of μ, but even if $x \in \mathrm{supp}(\mu)$, the cluster $C(x, \mu)$ might still contain no point at all because $\sim$ need not be reflexive. However, if $x \in \mathrm{supp}(\mu)$ and x is connected to some point $y \in \mathrm{supp}(\mu)$ with $y \neq x$, then x is contained in $C(x, \mu)$. Note that these definitions, as well as the upcoming Lemma 3.26, do not require the explicit metric topology on $\mathbb{X}$, it suffices that $(\mathbb{X}, \mathcal{X})$ is a Borel space.

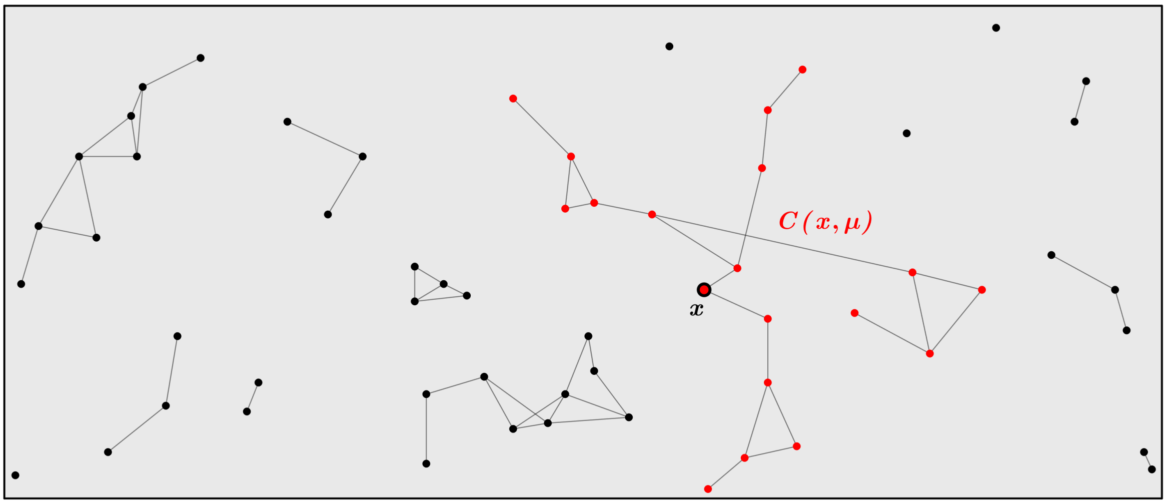


Figure 3.2.: Illustration of $C(x, \mu)$. All points in the picture, except possibly x, belong to μ. The black lines indicate which points interact via the relation $\sim$. The red points form the μ-cluster of x. Recall that x might not belong to $C(x, \mu)$, according to the discussion above.

Lemma 3.26. *For each $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$, it holds that $C(x, \mu) \in \mathbf{N}$. Moreover, the map $(x, \mu) \mapsto C(x, \mu)$ is $(\mathcal{X} \otimes \mathcal{N}, \mathcal{N})$-measurable.*

Proof. It is obvious that $C(x, \mu)$ is a measure on $\mathbb{X}$ and, since $C(x, \mu) \leq \mu$, it is locally finite. By Proposition 6.3 of Last and Penrose (2017) there exist measurable maps $\zeta_1, \zeta_2, \ldots : \mathbf{N} \to \mathbb{X}$ such that

$$\mu = \sum_{j=1}^{\mu(\mathbb{X})} \delta_{\zeta_j(\mu)}, \qquad \mu \in \mathbf{N}.$$

We can thus write $C(x, \mu) = \sum_{j=1}^{\mu(\mathbb{X})} \delta_{\zeta_j(\mu)} \mathbb{1}\{x \overset{\mu}{\sim} \zeta_j(\mu)\}$, which shows that $C(x, \mu)$ takes values in $\mathbb{N}_0 \cup \{\infty\}$, so $C(x, \mu) \in \mathbf{N}$. The same representation of $C(x, \mu)$ reduces the proof of measurability to showing that

$$(x, \mu) \mapsto \mathbb{1}\{x \overset{\mu}{\sim} \zeta_j(\mu)\}$$

is measurable for each $j \in \mathbb{N}$. By the definition of connectedness, the set $\{(x, \mu) \in \mathbb{X} \times \mathbf{N} : x \overset{\mu}{\sim} \zeta_j(\mu)\}$ can be rewritten as

$$\begin{aligned} \{(x, \mu) \in \mathbb{X} \times \mathbf{N} : x \sim \zeta_j(\mu)\} \cup \Bigg[\bigcup_{n=1}^{\infty} \bigcup_{\substack{i_1, \ldots, i_n \in \mathbb{N} \\ i_k \neq i_\ell}} \Big[&\{(x, \mu) \in \mathbb{X} \times \mathbf{N} : x \sim \zeta_{i_1}(\mu),\, i_1 \leq \mu(\mathbb{X})\} \\ &\cap \bigcap_{m=1}^{n-1} \Big(\mathbb{X} \times \{\mu \in \mathbf{N} : \zeta_{i_m}(\mu) \sim \zeta_{i_{m+1}}(\mu),\, i_m \leq \mu(\mathbb{X})\} \Big) \end{aligned}$$

$$\cap \Big(\mathbb{X} \times \big\{ \mu \in \mathbf{N} : \zeta_{i_n}(\mu) \sim \zeta_j(\mu),\, i_n \le \mu(\mathbb{X}) \big\} \Big) \Big] \Big].$$

Since $\big\{(x, y) \in \mathbb{X}^2 : x \sim y\big\} \in \mathcal{X}^{\otimes 2}$, the above set lies in $\mathcal{X} \otimes \mathcal{N}$, which finishes the proof. □

Corollary 3.27. *The mapping $\mathbb{X} \times \mathbf{N} \times \mathbf{N} \ni (x, \mu, \psi) \mapsto \mathbb{1}\{x \overset{\mu}{\sim} \psi\}$ is measurable.*

Proof. Note that, for any $x \in \mathbb{X}$ and $\mu, \psi \in \mathbf{N}$,

$$\mathbb{1}\{x \overset{\mu}{\sim} \psi\} = \mathbb{1}\{x \overset{\mu+\psi}{\sim} \psi\} = \mathbb{1}\Big\{C(x, \mu + \psi)\big(\operatorname{supp}(\psi)\big) > 0\Big\} = \mathbb{1}\bigg\{ \int_{\mathbb{X}} C(x, \mu + \psi)(\{y\})\, \mathrm{d}\psi(y) > 0 \bigg\}$$

and the claim follows from Lemmata 3.26, A.11, and A.4. □

We now introduce the notion of a cluster-dependent Gibbs process. Such a Gibbs process is one whose Papangelou intensity κ satisfies

$$\kappa(x, \mu) = \kappa\big(x, C(x, \mu)\big), \quad x \in \mathbb{X},\ \mu \in \mathbf{N}.$$

Intuitively, this means that points can only interact along their clusters. In particular, there is no interaction between points in disjoint clusters. As unbounded clusters are allowed, the finite range version of Theorem 3.12 in Corollary 3.20 will not guarantee the existence of such processes. However, a weaker assumption on the size of the clusters ensures that a corresponding cluster-dependent Gibbs process exists. The assumption we require is that the clusters associated with a suitable Poisson process are finite almost surely, in the sense that, for λ-a.e. $x \in \mathbb{X}$,

$$\Pi_{\vartheta\lambda}\big(\big\{\mu \in \mathbf{N} : C(x, \mu)(\mathbb{X}) < \infty\big\}\big) = 1,$$

where ϑ is the local stability bound on κ. In other words, if Φ is a Poisson process in $\mathbb{X}$ with intensity measure $\vartheta\lambda$, we assume that $C(x, \Phi)(\mathbb{X}) < \infty$ almost surely for λ-a.e. $x \in \mathbb{X}$. We then say that we are in the subcritical regime with respect to $\Pi_{\vartheta\lambda}$.

Note that the proof of the following result relies on Lemma 3.22 and hence on the metric topology of $\mathbb{X}$, but it applies to arbitrary reference measures λ.

Theorem 3.28. *Let $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ be a measurable map which satisfies the cocycle relation (2.8) and is locally stable as in Definition 3.14 with bound $\vartheta : \mathbb{X} \to [0, \infty)$. Also suppose that $\kappa(x, \mu) = \kappa\big(x, C(x, \mu)\big)$, for all $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$, and that, for λ-a.e. $x \in \mathbb{X}$,*

$$\Pi_{\vartheta\lambda}\big(\big\{\mu \in \mathbf{N} : C(x, \mu)(\mathbb{X}) < \infty\big\}\big) = 1.$$

Then a Gibbs process with PI κ and reference measure λ exists.

Proof. Remark 3.15 explains how the local stability assumption on κ covers all prerequisites of Theorem 3.12 except for the two limit relations. To show that these are also satisfied in the given setting, let ξ_n be a Gibbs process with PI $\kappa^{(B_n, \mathbf{0})}$, for each $n \in \mathbb{N}$, and denote by η a corresponding local limit obtained from Theorem 3.10. By Lemma 3.22 we can assume, without loss of generality, that $\xi_n \le \Phi_n$ ($n \in \mathbb{N}$) and $\eta \le \Phi$ almost surely, where Φ_n is a Poisson process with intensity measure $\vartheta\lambda_{B_n}$ and Φ is a Poisson process with intensity measure $\vartheta\lambda$. We denote the combined $\mathbb{P}$-null set, on which these relations are violated, by Ω_0. If, for $n, \ell \in \mathbb{N}$, $x \in \mathbb{X}$, and $\omega \in \Omega \setminus \Omega_0$, we have $C\big(x, \Phi_n(\omega)\big)(B_\ell^c) = 0$, then $C\big(x, \xi_n(\omega)\big)(B_\ell^c) = 0$ and therefore

$$C\big(x, (\xi_n)_{B_\ell}(\omega)\big) = C\big(x, \xi_n(\omega)\big)_{B_\ell} = C\big(x, \xi_n(\omega)\big). \tag{3.8}$$

For $n, \ell \in \mathbb{N}$ and λ-a.e. $x \in \mathbb{X}$ we have, by (3.8) and the properties of κ,

$$\begin{aligned}\mathbb{E}\big|\kappa\big(x,(\xi_n)_{B_\ell}\big)-\kappa(x,\xi_n)\big| &= \mathbb{E}\Big|\Big(\kappa\big(x,(\xi_n)_{B_\ell}\big)-\kappa(x,\xi_n)\Big)\mathbb{1}\big\{C(x,\Phi_n)(B_\ell^c)>0\big\}\Big| \\ &\le \vartheta(x)\cdot\mathbb{P}\big(C(x,\Phi_n)(B_\ell^c)>0\big) \\ &= \vartheta(x)\cdot\mathbb{P}\big(C(x,\Phi_{B_n})(B_\ell^c)>0\big) \\ &\le \vartheta(x)\cdot\mathbb{P}\big(C(x,\Phi)(B_\ell^c)>0\big).\end{aligned}$$

Combining this with the observation that, for λ-a.e. $x \in \mathbb{X}$,

$$\begin{aligned}\limsup_{\ell\to\infty}\mathbb{P}\big(C(x,\Phi)(B_\ell^c)>0\big) &= 1-\mathbb{P}\bigg(\bigcup_{\ell=1}^{\infty}\Big\{\omega\in\Omega : C\big(x,\Phi(\omega)\big)(B_\ell^c)=0\Big\}\bigg) \\ &= 1-\mathbb{P}\big(C(x,\Phi)(\mathbb{X})<\infty\big) \\ &= 0, \end{aligned} \tag{3.9}$$

dominated convergence yields

$$\limsup_{\ell\to\infty}\sup_{n\in\mathbb{N}}\int_B\mathbb{E}\big|\kappa\big(x,(\xi_n)_{B_\ell}\big)-\kappa(x,\xi_n)\big|\,\mathrm{d}\lambda(x)\le\limsup_{\ell\to\infty}\int_B\vartheta(x)\cdot\mathbb{P}\big(C(x,\Phi)(B_\ell^c)>0\big)\,\mathrm{d}\lambda(x)=0$$

for every $B \in \mathcal{X}_b$. A similar argument gives

$$\limsup_{\ell\to\infty}\int_B\mathbb{E}\big|\kappa(x,\eta_{B_\ell})-\kappa(x,\eta)\big|\,\mathrm{d}\lambda(x)=0$$

and Theorem 3.12 yields the claim. □

Remark 3.29. Notice that in Equation (3.8), in general, we might only have

$$C\big(x,(\xi_n)_{B_\ell}(\omega)\big)\le C\big(x,\xi_n(\omega)\big)_{B_\ell}$$

as there could be points in $(\xi_n)_{B_\ell}(\omega)$ which are connected to x via points in $(\xi_n)_{\mathbb{X}\setminus B_\ell}$. However, if we have $C\big(x,\xi_n(\omega)\big)(B_\ell^c)=0$ then such connections cannot exist. ○

3.3.3. The disagreement coupling and a bound on the difference of Gibbs measures

The main ingredient for our uniqueness proofs is the so-called disagreement coupling. The concept was first introduced by van den Berg and Maes (1994) in the discrete setting and transferred to the continuum by Hofer-Temmel (2019) and Hofer-Temmel and Houdebert (2019) though the latter works contain some technical gaps. Last and Otto (2021) provide a rigorous and very general version of the underlying technique.

Consider the setting from Section 3.3.2. For two finite counting measures $\mu, \mu' \in \mathbf{N}_f$ we denote by $|\mu-\mu'| \in \mathbf{N}_f$ the total variation measure of (the signed measure) $\mu-\mu'$ defined via the Jordan-decomposition, see Corollary 3.1.2 of Bogachev (2007a). More specifically, for $\nu=\mu-\mu'$ we consider the disjoint measurable sets $\mathbb{X}^+=\big\{x\in\mathbb{X} : \mu(\{x\})\ge\mu'(\{x\})\big\}$ and $\mathbb{X}^-=\big\{x\in\mathbb{X} : \mu(\{x\})<\mu'(\{x\})\big\}$ which yield a Hahn decomposition in the sense that $\mathbb{X}^+\cup\mathbb{X}^-=\mathbb{X}$ as well as

$$\nu(B\cap\mathbb{X}^+)\ge 0 \quad\text{and}\quad \nu(B\cap\mathbb{X}^-)\le 0$$

for all $B \in \mathcal{X}$. In accordance with Corollary 3.1.2 of Bogachev (2007a) we define on $\mathbb{X}$ the finite measures

$\nu^+ = \nu(\,\cdot \cap \mathbb{X}^+)$ and $\nu^- = -\nu(\,\cdot \cap \mathbb{X}^-)$, and we put $|\mu - \mu'| = |\nu| = \nu^+ + \nu^-$. By construction, we have

$$|\mu - \mu'|(B) = \int_B \big(\mu(\{x\}) - \mu'(\{x\})\big)\, \mathbb{1}\big\{\mu(\{x\}) > \mu'(\{x\})\big\}\, \mathrm{d}\mu(x) \\ + \int_B \big(\mu'(\{x\}) - \mu(\{x\})\big)\, \mathbb{1}\big\{\mu'(\{x\}) > \mu(\{x\})\big\}\, \mathrm{d}\mu'(x)$$

for all $B \in \mathcal{X}$. In particular, the map $\mathbf{N}_f \times \mathbf{N}_f \ni (\mu, \mu') \mapsto |\mu - \mu'| \in \mathbf{N}_f$ is measurable by Lemmata A.11 and A.4. Note that the previous construction, including the measurability result, does not require the metric topology on $\mathbb{X}$, it suffices that the measurable space $(\mathbb{X}, \mathcal{X})$ has a measurable diagonal.

The following result is a special case of Theorem 6.3 of Last and Otto (2021), restricting to locally stable Papangelou intensities. A domination property similar to Lemma 3.22 lies in the background, so the metric structure on $\mathbb{X}$ is required.

Proposition 3.30 (**Disagreement coupling due to Last and Otto**). *Assume that λ is diffuse. Let $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ be a measurable function that satisfies the cocycle relation (2.8) and is locally stable as in Definition 3.14 with bound $\vartheta : \mathbb{X} \to [0, \infty)$. Moreover, assume that, for all $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$,*

$$\kappa(x, \mu) = \kappa\big(x, C(x, \mu)\big).$$

Let $W \in \mathcal{X}_b$ and $\psi, \psi' \in \mathbf{N}_{W^c}$. There exists a Gibbs process ξ with PI $\kappa^{(W,\psi)}$ and a Gibbs process ξ' with PI $\kappa^{(W,\psi')}$ such that $\xi \leq \Phi$ and $\xi' \leq \Phi$ almost surely, where Φ is a Poisson process in $\mathbb{X}$ with intensity measure $\vartheta\lambda$, and such that every point in $|\xi - \xi'|$ is connected via $\xi + \xi'$ to some point in $\psi + \psi'$.

Note that if $\psi = \psi'$ the two so constructed Gibbs processes are identical (path-wise). The coupling from Proposition 3.30 is illustrated graphically in Figure 3.3.

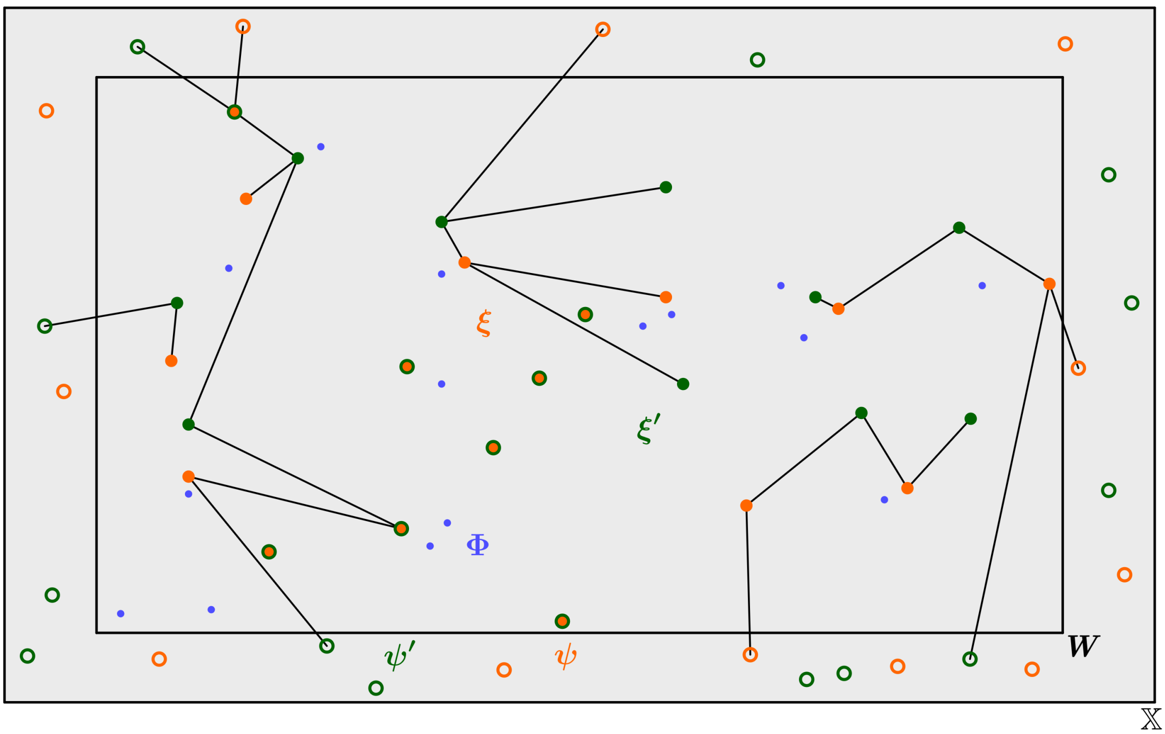


Figure 3.3.: Illustration of the disagreement coupling, with the notation from Proposition 3.30. Note that all points of ξ and ξ' are also points of Φ. Moreover, any point of $\xi + \xi'$ which is not connected to the boundary condition $\psi + \psi'$ necessarily belongs to both ξ and ξ'.

The disagreement coupling allows to bound the difference between the distributions of two finite Gibbs processes with different boundary conditions in terms of connection probabilities. The proof of the following theorem is conceptually similar to calculations from van den Berg and Maes (1994) and Hofer-Temmel and Houdebert (2019), but due to the much more general setting and for the readers convenience we provide full detail. The result, and in particular the unprecedented generality and rigor, is central to us. It will be applied to prove uniqueness in distribution of infinite volume Gibbs processes as well as for the study of decorrelation and mixing properties.

Recall from Section 2.8 that $\mathrm{P}_{W,\psi}$ denotes the distribution of a finite Gibbs process in $W \in \mathcal{X}_b$ with boundary condition $\psi \in \mathbf{N}$.

Theorem 3.31. *Assume that λ is diffuse. Let $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ be a measurable function that satisfies the cocycle relation (2.8) and is locally stable as in Definition 3.14 with bound $\vartheta : \mathbb{X} \to [0, \infty)$. Moreover, assume that, for all $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$,*

$$\kappa(x, \mu) = \kappa\big(x, C(x, \mu)\big).$$

Let Φ be a Poisson process in $\mathbb{X}$ with intensity measure $\vartheta\lambda$, let $W \in \mathcal{X}_b$, and $\psi, \psi' \in \mathbf{N}_{W^c}$. Moreover, let $B \in \mathcal{X}$ with $B \subset W$. Then, for any $E \in \mathcal{N}_B$,

$$\big|\mathrm{P}_{W,\psi}(E) - \mathrm{P}_{W,\psi'}(E)\big| \leq \int_B \mathbb{P}\Big(x \overset{\Phi_{W \setminus B}}{\sim} (\psi + \psi')\Big)\, \vartheta(x)\, \mathrm{d}\lambda(x).$$

If $F : \mathbf{N} \to [0, \infty]$ is measurable, B-local, $\Pi_{\vartheta\lambda}$-integrable, and satisfies $F(\nu) \leq F(\nu')$ whenever $\nu \leq \nu'$, then

$$\left| \int_{\mathbf{N}} F(\mu)\, \mathrm{d}\mathrm{P}_{W,\psi}(\mu) - \int_{\mathbf{N}} F(\mu')\, \mathrm{d}\mathrm{P}_{W,\psi'}(\mu') \right| \leq \int_B \mathbb{E}\big[F(\Phi_B + \delta_x)\big] \cdot \mathbb{P}\Big(x \overset{\Phi_{W \setminus B}}{\sim} (\psi + \psi')\Big)\, \vartheta(x)\, \mathrm{d}\lambda(x).$$

Proof. First note that all issues of measurability with regard to $\sim$ are settled by Corollary 3.27. In the given setting, let ξ, ξ', and Φ be exactly as in the disagreement coupling in Proposition 3.30. For any measurable and B-local map $F : \mathbf{N} \to [0, \infty]$, for which $\mathbb{E}\big[F(\xi)\big] < \infty$ or $\mathbb{E}\big[F(\xi')\big] < \infty$, we have

$$\left| \int_{\mathbf{N}} F(\mu)\, \mathrm{d}\mathrm{P}_{W,\psi}(\mu) - \int_{\mathbf{N}} F(\mu')\, \mathrm{d}\mathrm{P}_{W,\psi'}(\mu') \right| = \big|\mathbb{E}\big[F(\xi)\big] - \mathbb{E}\big[F(\xi')\big]\big| \leq \mathbb{E}\big|F(\xi_B) - F(\xi'_B)\big|.$$

In order for $F(\xi_B)$ and $F(\xi'_B)$ to differ, we must have $\xi_B \neq \xi'_B$, so the previous term is bounded by

$$\mathbb{E}\Big[\big|F(\xi_B) - F(\xi'_B)\big| \cdot \mathbb{1}\big\{|\xi - \xi'|(B) > 0\big\}\Big].$$

Whenever $|\xi - \xi'|(B) > 0$ then, by the properties of the disagreement coupling, there exists a point of Φ_B which is connected via Φ_W to $\psi + \psi'$. Considering the corresponding cluster and disregarding the connections which run entirely within B, we find a point in Φ_B which is connected via $\Phi_{W \setminus B}$ to $\psi + \psi'$. Combining these thoughts, we arrive at

$$\left| \int_{\mathbf{N}} F(\mu)\, \mathrm{d}\mathrm{P}_{W,\psi}(\mu) - \int_{\mathbf{N}} F(\mu')\, \mathrm{d}\mathrm{P}_{W,\psi'}(\mu') \right| \leq \mathbb{E}\bigg[\int_B \big|F(\xi_B) - F(\xi'_B)\big| \cdot \mathbb{1}\Big\{x \overset{\Phi_{W \setminus B}}{\sim} (\psi + \psi')\Big\}\, \mathrm{d}\Phi(x)\bigg]. \quad (3.10)$$

If $F = \mathbb{1}_E$ for $E \in \mathcal{N}_B$, then (3.10) gives

$$\big|\mathrm{P}_{W,\psi}(E) - \mathrm{P}_{W,\psi'}(E)\big| \leq \mathbb{E}\bigg[\int_B \mathbb{1}\Big\{x \overset{\Phi_{W \setminus B}}{\sim} (\psi + \psi')\Big\}\, \mathrm{d}\Phi(x)\bigg],$$

which, by Mecke's formula (Proposition 2.1), equals

$$\int_B \mathbb{P}\Big(x \overset{\Phi_{W \setminus B}}{\sim} (\psi + \psi')\Big)\, \vartheta(x)\, \mathrm{d}\lambda(x).$$

If F is as in the statement of the theorem, then $\mathbb{E}\big[F(\xi)\big] \le \mathbb{E}\big[F(\Phi)\big] < \infty$ and similarly $\mathbb{E}\big[F(\xi')\big] < \infty$, as $\xi \le \Phi$ and $\xi' \le \Phi$ almost surely. The bound from (3.10) yields

$$\left| \int_{\mathbf{N}} F(\mu)\, \mathrm{dP}_{W,\psi}(\mu) - \int_{\mathbf{N}} F(\mu')\, \mathrm{dP}_{W,\psi'}(\mu') \right| \le \mathbb{E}\bigg[\int_B F(\Phi_B) \cdot \mathbb{1}\Big\{ x \overset{\Phi_{W\setminus B}}{\sim} (\psi + \psi') \Big\}\, \mathrm{d}\Phi(x) \bigg].$$

By Mecke's formula and the independence properties of the Poisson process, the right hand side equals

$$\mathbb{E}\bigg[\int_B F(\Phi_B + \delta_x) \cdot \mathbb{1}\Big\{ x \overset{\Phi_{W\setminus B}}{\sim} (\psi + \psi') \Big\}\, \vartheta(x)\, \mathrm{d}\lambda(x) \bigg] = \int_B \mathbb{E}\big[F(\Phi_B + \delta_x)\big] \cdot \mathbb{P}\Big(x \overset{\Phi_{W\setminus B}}{\sim} (\psi + \psi') \Big)\, \vartheta(x)\, \mathrm{d}\lambda(x)$$

which completes the proof. $\square$

3.3.4. A uniqueness result in the subcritical regime

Apart from the question of existence of Gibbs processes, which is equivalent to asking whether solutions to the GNZ equations exist, it is natural to ask whether these solutions are unique. In other words, can we find manageable assumptions on κ which guarantee that the Gibbs process with PI κ is unique in distribution? It turns out that with the disagreement coupling from Proposition 3.30 at hand, the setting of Theorem 3.28 is perfectly suited to study this question. In fact, the exact same assumptions which guarantee the existence of a cluster-dependent Gibbs process in Theorem 3.28 also suffice to prove uniqueness.

We refrain from discussing the literature on uniqueness of Gibbs measures at this point, since most of the existing contributions refer to pair potentials and their discussion is more appropriate in Chapter 5, where uniqueness results for pair interaction processes are discussed.

Theorem 3.32. *Let $\kappa : \mathbb{X} \times \mathbf{N} \to [0,\infty)$ be a measurable map which satisfies the cocycle relation (2.8) and is locally stable as in Definition 3.14 with bound $\vartheta : \mathbb{X} \to [0,\infty)$. Also suppose that $\kappa(x,\mu) = \kappa\big(x, C(x,\mu)\big)$ for all $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$, and that, for λ-a.e. $x \in \mathbb{X}$,*

$$\Pi_{\vartheta\lambda}\big(\{\mu \in \mathbf{N} : C(x,\mu)(\mathbb{X}) < \infty\}\big) = 1.$$

Then, up to equality in distribution, there exists exactly one Gibbs process with PI κ and reference measure λ.

Proof. By Theorem 3.28 there exists at least one Gibbs process with PI κ, so we only need to prove that this process is unique in distribution. We first argue that we can assume, without loss of generality, that λ is diffuse. Consider $\tilde{\mathbb{X}} = \mathbb{X} \times [0,1]$ equipped with some complete metric that induces the product topology and let $\tilde{\lambda} = \lambda \otimes \mathcal{L}^1_{[0,1]}$. For $x \in \mathbb{X}$, $r \in [0,1]$, and $\mu \in \mathbf{N}(\tilde{\mathbb{X}})$ let $\tilde{C}(x,r,\mu)$ be the μ-cluster of (x,r), where $(x,r),(y,s) \in \tilde{\mathbb{X}}$ are connected simply when $x \sim y$. We have $\tilde{C}(x,r,\mu)(\tilde{\mathbb{X}}) = C\big(x, \mu(\,\cdot \times [0,1])\big)(\mathbb{X})$, so if Φ is a Poisson process with intensity measure $\vartheta\lambda$ and $\tilde{\Phi}$ is a uniform randomization of Φ (as in Lemma 3.25), which is a Poisson process in $\tilde{\mathbb{X}}$ with intensity measure $\vartheta\lambda \otimes \mathcal{L}^1_{[0,1]}$ by the marking theorem (Theorem 5.6 of Last and Penrose, 2017), then the subcriticality assumption immediately implies

$$\mathbb{P}\big(\tilde{C}(x,r,\tilde{\Phi})(\tilde{\mathbb{X}}) < \infty\big) = 1$$

for λ-a.e. $x \in \mathbb{X}$ and any $r \in [0,1]$. For $x \in \mathbb{X}$, $r \in [0,1]$, and $\mu \in \mathbf{N}(\tilde{\mathbb{X}})$ we also define

$$\tilde{\kappa}(x,r,\mu) = \kappa\big(x, \mu(\,\cdot \times [0,1])\big),$$

which inherits the cocycle and local stability property of κ. Moreover, $\tilde{\kappa}\big(x,r,\tilde{C}(x,r,\mu)\big) = \tilde{\kappa}(x,r,\mu)$. If η,η' are Gibbs processes with PI κ and reference measure λ, then by Lemma 3.25 the uniform randomizations $\tilde{\eta},\tilde{\eta}'$ of η and η' are Gibbs processes with PI $\tilde{\kappa}$ and reference measure $\tilde{\lambda}$. Thus, if the uniqueness result holds for diffuse reference measures, then $\tilde{\eta} \overset{d}{=} \tilde{\eta}'$ as $\tilde{\lambda}$ is diffuse, and we obtain

$$\eta \, = \, \tilde{\eta}(\, \cdot \times [0,1]) \overset{d}{=} \tilde{\eta}'(\, \cdot \times [0,1]) \, = \, \eta'.$$

We conclude that the general result holds if it is proven for diffuse reference measures. Hence, let us assume that λ is diffuse.

Now, let η, η' be two Gibbs processes with PI κ. By Corollary 3.24 we can assume, without loss of generality, that there exist Poisson processes Φ, Ψ, Ψ' with intensity measure $\vartheta\lambda$ such that $\eta \leq \Psi$ and $\eta' \leq \Psi'$ almost surely and such that $(\eta, \eta', \Psi, \Psi')$ is independent of Φ. Let $B \in \mathcal{X}_b$ be arbitrary and choose ℓ large enough so that $B \subset B_\ell$. Take $E \in \mathcal{N}_B$. With Lemma 2.47 (using that E is a B-local event) and Theorem 3.31 applied to $W = B_\ell$ we obtain

$$\big|\mathbb{P}(\eta \in E) - \mathbb{P}(\eta' \in E)\big| \leq \mathbb{E}\Big|\mathrm{P}_{B_\ell, \eta_{\mathbb{X}\setminus B_\ell}}(E) - \mathrm{P}_{B_\ell, \eta'_{\mathbb{X}\setminus B_\ell}}(E)\Big| \leq \int_B \mathbb{P}\Big(x \overset{\Phi_{B_\ell \setminus B}}{\sim} \big(\eta_{\mathbb{X}\setminus B_\ell} + \eta'_{\mathbb{X}\setminus B_\ell}\big)\Big) \, \vartheta(x) \, \mathrm{d}\lambda(x).$$

The right hand side is clearly bounded by

$$\int_B \mathbb{P}\Big(x \overset{\Phi_{B_\ell}}{\sim} \eta_{\mathbb{X}\setminus B_\ell}\Big) \, \vartheta(x) \, \mathrm{d}\lambda(x) + \int_B \mathbb{P}\Big(x \overset{\Phi_{B_\ell}}{\sim} \eta'_{\mathbb{X}\setminus B_\ell}\Big) \, \vartheta(x) \, \mathrm{d}\lambda(x).$$

As $\eta_{\mathbb{X}\setminus B_\ell} \leq \Psi_{\mathbb{X}\setminus B_\ell}$ and $\eta'_{\mathbb{X}\setminus B_\ell} \leq \Psi'_{\mathbb{X}\setminus B_\ell}$ almost surely, Ψ, Ψ' and Φ have the same distribution, and Φ_{B_ℓ}, $\Phi_{\mathbb{X}\setminus B_\ell}$ are independent, we conclude that

$$\big|\mathbb{P}(\eta \in E) - \mathbb{P}(\eta' \in E)\big| \leq 2 \int_B \mathbb{P}\Big(x \overset{\Phi_{B_\ell}}{\sim} \Phi_{\mathbb{X}\setminus B_\ell}\Big) \, \vartheta(x) \, \mathrm{d}\lambda(x) \leq 2 \int_B \mathbb{P}\big(C(x, \Phi)(B_\ell^c) > 0\big) \, \vartheta(x) \, \mathrm{d}\lambda(x).$$

It follows from (3.9) and dominated convergence that $\mathbb{P}^\eta(E) = \mathbb{P}^{\eta'}(E)$. As B and E were arbitrary and the algebra $\mathcal{Z}$ of local events generates $\mathcal{N}$, we conclude that $\mathbb{P}^\eta = \mathbb{P}^{\eta'}$. □

3.3.5. Decorrelation in the subcritical regime

We have discussed in several instances how the spatial independence properties of Poisson process are not available for general Gibbs processes, as points in one part of the space influence other regions. However, it is somewhat intuitive that if the interaction that a point exhibits onto others decays sufficiently fast, then suitable asymptotic independence properties might be established. In the setting of cluster dependent interactions we can indeed prove strong mixing properties as soon as the size of the cluster of any point is controlled. The strongest such result, namely decorrelation, is considered in the section at hand, while classical mixing properties related to stochastic independence are provided in the upcoming Section 3.3.6.

A point process η in $\mathbb{X}$ with existing correlation functions is said to decorrelate if there exists a function $\mathsf{w} : [0, \infty) \to [0, \infty)$ with $\lim_{r\to\infty} \mathsf{w}(r) = 0$ such that

$$\begin{aligned} &\big|\rho_{\eta,k+m}(x_1, \ldots, x_{k+m}) - \rho_{\eta,k}(x_1, \ldots, x_k) \cdot \rho_{\eta,m}(x_{k+1}, \ldots, x_{k+m})\big| \\ &\qquad \leq c(k,m) \cdot \mathsf{w}\big(\mathrm{dist}(\{x_1, \ldots, x_k\}, \{x_{k+1}, \ldots, x_{k+m}\})\big) \end{aligned} \tag{3.11}$$

for λ^{k+m}-a.e. $(x_1, \ldots, x_{k+m}) \in \mathbb{X}^{k+m}$ and all $k, m \in \mathbb{N}$, where $c : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ is some function and where $\mathrm{dist}(D, D') = \inf\big\{d(x,y) : x \in D, y \in D'\big\}$ denotes the distance between two non-empty sets $D, D' \subset \mathbb{X}$. Błaszczyszyn et al. (2019) also say that the correlation functions of η are w-mixing. If the decay of w outweighs the growth of any polynomial, then η is said to have weak exponential decrease of correlations (cf. Malyshev, 1975, who introduced the concept in the context of Gibbsian random fields). Other authors say that η has fast decay of correlations (cf. Błaszczyszyn et al., 2019) or that η decorrelates exponentially (cf. Beneš et al., 2020).

The following lemma on the factorization of factorial moment measures is central for the proof of our main result in the context of decorrelation. It also contains virtually all probabilistic arguments involved in the endeavor. Both the following lemma and Theorem 3.34 below contain substantial generalization of the results in Section 3.3 of Beneš et al. (2020). While our notation is neater and the calculations are more direct, the work of Beneš et al. (2020) served as a conceptual foundation for the results.

To ensure that the only dependence on $x_1, \ldots, x_{k+m}$ in the bound on the right hand side of (3.11) is in terms of the distance between $\{x_1, \ldots, x_k\}$ and $\{x_{k+1}, \ldots, x_{k+m}\}$ we assume that the local stability bound on the Papangelou density is constant. We already make this assumption in the preliminary lemma, even though it could be proven with a general local stability bound as well.

Lemma 3.33. *Assume that λ is diffuse and let $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ be a measurable function that satisfies the cocycle relation (2.8). Moreover, suppose that κ is locally stable as in Definition 3.14 with constant bound $\vartheta \geq 0$ and that*

$$\kappa(x, \mu) = \kappa\big(x, C(x, \mu)\big)$$

for all $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$. Let η be a Gibbs process with PI κ and reference measure λ and let Φ be a Poisson process with intensity measure $\vartheta \cdot \lambda$. Assume there exists a function $\mathsf{w} : [0, \infty) \to [0, \infty)$ such that, for any $B, W \in \mathcal{X}_b \setminus \{\varnothing\}$ with $B \subset W$,

$$\mathbb{P}\big(C(x, \Phi + \mu)(W^{\mathsf{c}}) > 0\big) \leq \mathsf{w}\big(\mathrm{dist}(B, W^{\mathsf{c}})\big)$$

for λ-a.e. $x \in B$ and all $\mu \in \mathbf{N}_f \cap \mathbf{N}_{W^{\mathsf{c}}}$. Then, for $k, m \in \mathbb{N}$ and any pairwise disjoint $D_1, \ldots, D_{k+m} \in \mathcal{X}_b \setminus \{\varnothing\}$,

$$\begin{aligned}
&\big|\alpha_{\eta,k+m}(D_1 \times \ldots \times D_{k+m}) - \alpha_{\eta,k}(D_1 \times \ldots \times D_k) \cdot \alpha_{\eta,m}(D_{k+1} \times \ldots \times D_{k+m})\big| \\
&\quad \leq 2\vartheta^{k+m} \prod_{i=1}^{k+m} \lambda(D_i) \cdot \min\left\{\vartheta \cdot \lambda\bigg(\bigcup_{i=1}^{k} D_i\bigg) + k,\ \vartheta \cdot \lambda\bigg(\bigcup_{j=k+1}^{k+m} D_j\bigg) + m\right\} \cdot \mathsf{w}\Bigg(\mathrm{dist}\bigg(\bigcup_{i=1}^{k} D_i, \bigcup_{j=k+1}^{k+m} D_j\bigg)\Bigg).
\end{aligned}$$

Proof. First of all, note that Corollary 3.24 ensures that it is no restriction to assume that $\eta \leq \Phi$ almost surely. Let $k, m \in \mathbb{N}$ and $D_1, \ldots, D_{k+m} \in \mathcal{X}_b \setminus \{\varnothing\}$ pairwise disjoint. Put $B = \bigcup_{i=1}^{k} D_i \in \mathcal{X}_b$ and let $\ell \in \mathbb{N}$ be large enough such that $B \subset B_\ell$ as well as

$$\mathrm{dist}\Bigg(B, \bigcup_{j=k+1}^{k+m} D_j \cup B_\ell^{\mathsf{c}}\Bigg) = \mathrm{dist}\Bigg(B, \bigcup_{j=k+1}^{k+m} D_j\Bigg).$$

Suppressing the dependence on ℓ, we define

$$W = B_\ell \setminus \Bigg(\bigcup_{j=k+1}^{k+m} D_j\Bigg).$$

Observe that $B \subset W$. Define the measurable and B-local function $F : \mathbf{N} \to [0, \infty)$, $F(\nu) = \prod_{i=1}^{k} \nu(D_i)$. We clearly have $F(\nu) \leq F(\nu')$ whenever $\nu \leq \nu'$ and, since $\Phi(D_1), \ldots, \Phi(D_k)$ are independent,

$$\int_{\mathbf{N}} F(\nu)\, \mathrm{d}\Pi_{\vartheta \cdot \lambda}(\nu) = \mathbb{E}\big[F(\Phi)\big] = \prod_{i=1}^{k} \mathbb{E}\big[\Phi(D_i)\big] = \vartheta^k \prod_{i=1}^{k} \lambda(D_i) < \infty.$$

Part (i) of Proposition A.2, the specific choice of F and W, and the linearity of the expectation give

$$\begin{aligned}
&\big|\alpha_{\eta,k+m}(D_1 \times \ldots \times D_{k+m}) - \alpha_{\eta,k}(D_1 \times \ldots \times D_k) \cdot \alpha_{\eta,m}(D_{k+1} \times \ldots \times D_{k+m})\big| \\
&= \left|\mathbb{E}\left[\prod_{i=1}^{k+m} \eta(D_i)\right] - \mathbb{E}\left[\prod_{i=1}^{k} \eta(D_i)\right] \cdot \mathbb{E}\left[\prod_{j=k+1}^{k+m} \eta(D_j)\right]\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \mathbb{E}\left[F(\eta_W) \cdot \prod_{i=k+1}^{k+m} \eta_{W^{\mathsf{c}}}(D_i) \right] - \mathbb{E}\big[F(\eta_W)\big] \cdot \mathbb{E}\left[\prod_{j=k+1}^{k+m} \eta_{W^{\mathsf{c}}}(D_j) \right] \right| \\
&= \left| \mathbb{E}\left[\Big(F(\eta_W) - \mathbb{E}\big[F(\eta_W)\big] \Big) \cdot \prod_{j=k+1}^{k+m} \eta_{W^{\mathsf{c}}}(D_j) \right] \right|. \qquad (3.12)
\end{aligned}
$$

Applying the DLR equation (in the form of Lemma 2.47) to each of the expectations and using the triangle inequality, the term in the previous display is seen to be bounded from above by

$$
\int_{\mathbf{N}} \int_{\mathbf{N}} \left| \int_{\mathbf{N}} F(\mu) \, \mathrm{dP}_{W,\psi}(\mu) - \int_{\mathbf{N}} F(\mu') \, \mathrm{dP}_{W,\psi'}(\mu') \right| \left(\prod_{j=k+1}^{k+m} \psi(D_j) \right) \mathrm{d}\mathbb{P}^{\eta_{W^{\mathsf{c}}}}(\psi') \, \mathrm{d}\mathbb{P}^{\eta_{W^{\mathsf{c}}}}(\psi).
$$

By Theorem 3.31, a further upper bound is given by

$$
\begin{aligned}
&\vartheta \int_{\mathbf{N}} \int_{\mathbf{N}} \int_B \mathbb{E}\big[F(\Phi_B + \delta_x)\big] \cdot \mathbb{P}\Big(x \overset{\Phi_{W \setminus B}}{\sim} (\psi + \psi') \Big) \, \mathrm{d}\lambda(x) \left(\prod_{j=k+1}^{k+m} \psi(D_j) \right) \mathrm{d}\mathbb{P}^{\eta_{W^{\mathsf{c}}}}(\psi') \, \mathrm{d}\mathbb{P}^{\eta_{W^{\mathsf{c}}}}(\psi) \\
&\leq \vartheta \int_{\mathbf{N}} \int_B \mathbb{E}\big[F(\Phi_B + \delta_x)\big] \cdot \mathbb{P}\Big(x \overset{\Phi_{W \setminus B}}{\sim} \psi \Big) \, \mathrm{d}\lambda(x) \left(\prod_{j=k+1}^{k+m} \psi(D_j) \right) \mathrm{d}\mathbb{P}^{\eta_{W^{\mathsf{c}}}}(\psi) \\
&\quad + \vartheta \int_{\mathbf{N}} \int_{\mathbf{N}} \int_B \mathbb{E}\big[F(\Phi_B + \delta_x)\big] \cdot \mathbb{P}\Big(x \overset{\Phi_{W \setminus B}}{\sim} \psi' \Big) \, \mathrm{d}\lambda(x) \left(\prod_{j=k+1}^{k+m} \psi(D_j) \right) \mathrm{d}\mathbb{P}^{\eta_{W^{\mathsf{c}}}}(\psi') \, \mathrm{d}\mathbb{P}^{\eta_{W^{\mathsf{c}}}}(\psi).
\end{aligned}
$$

Since $\eta \leq \Phi$ almost surely and $\Phi_B, \Phi_{W \setminus B}, \Phi_{W^{\mathsf{c}}}$ are independent, the previous term is bounded by

$$
\begin{aligned}
&\vartheta \int_B \mathbb{E}\big[F(\Phi_B + \delta_x)\big] \cdot \mathbb{E}\left[\mathbb{1}\Big\{ x \overset{\Phi_{W \setminus B}}{\sim} \Phi_{W^{\mathsf{c}}} \Big\} \cdot \prod_{j=k+1}^{k+m} \Phi(D_j) \right] \mathrm{d}\lambda(x) \\
&\quad + \vartheta \cdot \mathbb{E}\left[\prod_{j=k+1}^{k+m} \Phi(D_j) \right] \int_B \mathbb{E}\big[F(\Phi_B + \delta_x)\big] \cdot \mathbb{P}\Big(x \overset{\Phi_{W \setminus B}}{\sim} \Phi_{W^{\mathsf{c}}} \Big) \, \mathrm{d}\lambda(x) \\
&= \vartheta \int_B \mathbb{E}\big[F(\Phi_B + \delta_x)\big] \cdot \mathbb{E}\left[\int_{D_{k+1} \times \ldots \times D_{k+m}} \mathbb{1}\big\{ C(x, \Phi_{B^{\mathsf{c}}})(W^{\mathsf{c}}) > 0 \big\} \, \mathrm{d}\Phi^{(m)} \right] \mathrm{d}\lambda(x) \\
&\quad + \vartheta \cdot \mathbb{E}\left[\prod_{j=k+1}^{k+m} \Phi(D_j) \right] \int_B \mathbb{E}\big[F(\Phi_B + \delta_x)\big] \cdot \mathbb{P}\big(C(x, \Phi_{B^{\mathsf{c}}})(W^{\mathsf{c}}) > 0 \big) \, \mathrm{d}\lambda(x),
\end{aligned}
$$

which, by the multivariate Mecke formula (cf. Theorem 4.4 of Last and Penrose, 2017) and the fact that $\Phi(D_{k+1}), \ldots, \Phi(D_{k+m})$ are independent, equals

$$
\begin{aligned}
&\vartheta^{m+1} \int_B \mathbb{E}\big[F(\Phi_B + \delta_x)\big] \int_{D_{k+1} \times \ldots \times D_{k+m}} \mathbb{P}\left(C\Big(x, \Phi_{B^{\mathsf{c}}} + \sum_{i=1}^{m} \delta_{y_i} \Big)(W^{\mathsf{c}}) > 0 \right) \mathrm{d}\lambda^m(y_1, \ldots, y_m) \, \mathrm{d}\lambda(x) \\
&+ \vartheta^{m+1} \prod_{j=k+1}^{k+m} \lambda(D_j) \int_B \mathbb{E}\big[F(\Phi_B + \delta_x)\big] \cdot \mathbb{P}\big(C(x, \Phi_{B^{\mathsf{c}}})(W^{\mathsf{c}}) > 0 \big) \, \mathrm{d}\lambda(x).
\end{aligned}
$$

By assumption, this last quantity is bounded by

$$
2\vartheta^{m+1} \prod_{j=k+1}^{k+m} \lambda(D_j) \cdot \mathsf{w}\big(\mathrm{dist}(B, W^{\mathsf{c}})\big) \int_B \mathbb{E}\big[F(\Phi_B + \delta_x)\big] \, \mathrm{d}\lambda(x).
$$

Now, observe that

$$\begin{aligned}\int_B \mathbb{E}\big[F(\Phi_B+\delta_x)\big]\,\mathrm{d}\lambda(x) &= \sum_{i=1}^{k}\int_{D_i}\mathbb{E}\Bigg[\prod_{j=1}^{k}\big(\Phi(D_j)+\delta_x(D_j)\big)\Bigg]\,\mathrm{d}\lambda(x)\\ &= \sum_{i=1}^{k}\int_{D_i}\prod_{j=1}^{k}\big(\vartheta\cdot\lambda(D_j)+\mathbb{1}_{D_j}(x)\big)\,\mathrm{d}\lambda(x)\\ &= \vartheta^{k-1}\prod_{j=1}^{k}\lambda(D_j)\sum_{i=1}^{k}\big(\vartheta\cdot\lambda(D_i)+1\big)\\ &= \vartheta^{k-1}\prod_{j=1}^{k}\lambda(D_j)\cdot\big(\vartheta\cdot\lambda(B)+k\big).\end{aligned}$$

Combining the calculations up to this point, we have shown that

$$\begin{aligned}&\big|\alpha_{\eta,k+m}(D_1\times\ldots\times D_{k+m})-\alpha_{\eta,k}(D_1\times\ldots\times D_k)\cdot\alpha_{\eta,m}(D_{k+1}\times\ldots\times D_{k+m})\big|\\ &\quad\le 2\vartheta^{k+m}\prod_{i=1}^{k+m}\lambda(D_i)\cdot\big(\vartheta\cdot\lambda(B)+k\big)\cdot\mathsf{w}\big(\operatorname{dist}(B,W^{\mathsf{c}})\big)\\ &\quad= 2\vartheta^{k+m}\prod_{i=1}^{k+m}\lambda(D_i)\cdot\Bigg(\vartheta\cdot\lambda\Bigg(\bigcup_{i=1}^{k}D_i\Bigg)+k\Bigg)\cdot\mathsf{w}\Bigg(\operatorname{dist}\Bigg(\bigcup_{i=1}^{k}D_i,\bigcup_{j=k+1}^{k+m}D_j\Bigg)\Bigg).\end{aligned}$$

Changing the role of $(D_1,\ldots,D_k)$ and $(D_{k+1},\ldots,D_{k+m})$ in the definition of B, W and F, pulling out $\prod_{i=1}^{k}\eta(D_i)$ instead of $\prod_{j=k+1}^{k+m}\eta(D_j)$ in (3.12), and going through the same calculations as before gives

$$\begin{aligned}&\big|\alpha_{\eta,k+m}(D_1\times\ldots\times D_{k+m})-\alpha_{\eta,k}(D_1\times\ldots\times D_k)\cdot\alpha_{\eta,m}(D_{k+1}\times\ldots\times D_{k+m})\big|\\ &\quad\le 2\vartheta^{k+m}\prod_{i=1}^{k+m}\lambda(D_i)\cdot\Bigg(\vartheta\cdot\lambda\Bigg(\bigcup_{j=k+1}^{k+m}D_j\Bigg)+m\Bigg)\cdot\mathsf{w}\Bigg(\operatorname{dist}\Bigg(\bigcup_{i=1}^{k}D_i,\bigcup_{j=k+1}^{k+m}D_j\Bigg)\Bigg),\end{aligned}$$

which completes the proof. □

The following result formalizes the decorrelation property of cluster dependent Gibbs processes in the subcritical regime. More specifically, if the function which appears in the theorem satisfies $\lim_{r\to\infty}\mathsf{w}(r)=0$ then the corresponding Gibbs process decorrelates. The assumptions are similar to those in Lemma 3.33 but we can allow for general locally finite reference measures.

Theorem 3.34. *Let $\kappa:\mathbb{X}\times\mathbf{N}\to[0,\infty)$ be a measurable function that satisfies the cocycle relation (2.8). Moreover, suppose that κ is locally stable as in Definition 3.14 with constant bound $\vartheta\ge 0$ and that*

$$\kappa(x,\mu)=\kappa\big(x,C(x,\mu)\big)$$

for all $x\in\mathbb{X}$ and $\mu\in\mathbf{N}$. Let η be a Gibbs process with PI κ and reference measure λ. Assume there exists a continuous function $\mathsf{w}:[0,\infty)\to[0,\infty)$ such that, for any $B,W\in\mathcal{X}_b\setminus\{\varnothing\}$ with $B\subset W$,

$$\Pi_{\vartheta\cdot\lambda}\big(\{\nu\in\mathbf{N}:C(x,\nu+\mu)(W^{\mathsf{c}})>0\}\big)\le\mathsf{w}\big(\operatorname{dist}(B,W^{\mathsf{c}})\big)$$

for λ-a.e. $x\in B$ and every $\mu\in\mathbf{N}_f\cap\mathbf{N}_{W^{\mathsf{c}}}$. Then

$$\begin{aligned}&\big|\rho_{\eta,k+m}(x_1,\ldots,x_{k+m})-\rho_{\eta,k}(x_1,\ldots,x_k)\cdot\rho_{\eta,m}(x_{k+1},\ldots,x_{k+m})\big|\\ &\quad\le 2\vartheta^{k+m}\min\{k,m\}\cdot\mathsf{w}\big(\operatorname{dist}(\{x_1,\ldots,x_k\},\{x_{k+1},\ldots,x_{k+m}\})\big)\end{aligned}$$

for λ^{k+m}-a.e. $(x_1, \dots, x_{k+m}) \in \mathbb{X}^{k+m}$ and all $k, m \in \mathbb{N}$.

Proof. We first argue that we can assume, without loss of generality, that λ is diffuse. Indeed, as in the proof of Theorem 3.32 we consider $\tilde{\mathbb{X}} = \mathbb{X} \times [0,1]$, equipped with the complete metric

$$\big((x,s),(y,t)\big) \mapsto \max\big\{d(x,y), |t-s|\big\}$$

that induces the product topology, and $\tilde{\lambda} = \lambda \otimes \mathcal{L}^1_{[0,1]}$. For $x \in \mathbb{X}$, $s \in [0,1]$, and $\mu \in \mathbf{N}(\tilde{\mathbb{X}})$ let $\tilde{C}(x,s,\mu)$ be the μ-cluster of (x,s), where $(x,s),(y,t) \in \tilde{\mathbb{X}}$ are connected simply when $x \sim y$. Then, for $D \in \mathcal{X}$,

$$\tilde{C}(x,s,\mu)(D \times [0,1]) = C\big(x, \mu(\,\cdot \times [0,1])\big)(D).$$

If Φ is a Poisson process with intensity measure $\vartheta \cdot \lambda$ and $\tilde{\Phi}$ is a uniform randomization of Φ, which is a Poisson process in $\tilde{\mathbb{X}}$ with intensity measure $\vartheta \cdot \lambda \otimes \mathcal{L}^1_{[0,1]}$, then, for $B, W \in \mathcal{X}_b \setminus \{\varnothing\}$ with $B \subset W$,

$$\mathbb{P}\Big(\tilde{C}(x,s,\tilde{\Phi}+\mu)\big(\tilde{\mathbb{X}} \setminus (W \times [0,1])\big) > 0\Big) = \mathbb{P}\Big(C\big(x, \Phi + \mu(\,\cdot \times [0,1])\big)(W^{\mathsf{c}}) > 0\Big) \le \mathsf{w}\big(\mathrm{dist}(B, W^{\mathsf{c}})\big)$$

for λ-a.e. $x \in B$, all $s \in [0,1]$, and every $\mu \in \mathbf{N}_f(\tilde{\mathbb{X}})$ with $\mu(W \times [0,1]) = 0$. Moreover, we define

$$\tilde{\kappa}(x,s,\mu) = \kappa\big(x, \mu(\,\cdot \times [0,1])\big)$$

so that $\tilde{\kappa}$ inherits the cocycle and local stability property of κ and satisfies

$$\tilde{\kappa}\big(x,s,\tilde{C}(x,s,\mu)\big) = \tilde{\kappa}(x,s,\mu)$$

for all $x \in \mathbb{X}$, $s \in [0,1]$, and $\mu \in \mathbf{N}(\tilde{\mathbb{X}})$. If η is a Gibbs process with PI κ and reference measure λ, then by Lemma 3.25 the uniform randomization $\tilde{\eta}$ of η is a Gibbs process with PI $\tilde{\kappa}$ and reference measure $\tilde{\lambda}$. By Lemma 2.20 the correlation functions satisfy

$$\rho_{\tilde{\eta},m}(x_1,s_1,\dots,x_m,s_m) = \mathbb{E}\big[\tilde{\kappa}_m(x_1,s_1,\dots,x_m,s_m,\tilde{\eta})\big] = \mathbb{E}\big[\kappa_m(x_1,\dots,x_m,\eta)\big] = \rho_{\eta,m}(x_1,\dots,x_m)$$

for all $x_1,\dots,x_m \in \mathbb{X}$, $s_1,\dots,s_m \in [0,1]$, and each $m \in \mathbb{N}$. Hence, if the theorem holds for diffuse reference measures, then

$$\begin{aligned}
&\big|\rho_{\eta,k+m}(x_1,\dots,x_{k+m}) - \rho_{\eta,k}(x_1,\dots,x_k) \cdot \rho_{\eta,m}(x_{k+1},\dots,x_{k+m})\big| \\
&\quad = \big|\rho_{\tilde{\eta},k+m}(x_1,s_*,\dots,x_{k+m},s_*) - \rho_{\tilde{\eta},k}(x_1,s_*,\dots,x_k,s_*) \cdot \rho_{\tilde{\eta},m}(x_{k+1},s_*,\dots,x_{k+m},s_*)\big| \\
&\quad \le 2\vartheta^{k+m} \min\{k,m\} \cdot \mathsf{w}\Big(\mathrm{dist}\big(\{(x_1,s_*),\dots,(x_k,s_*)\}, \{(x_{k+1},s_*),\dots,(x_{k+m},s_*)\}\big)\Big) \\
&\quad = 2\vartheta^{k+m} \min\{k,m\} \cdot \mathsf{w}\big(\mathrm{dist}(\{x_1,\dots,x_k\}, \{x_{k+1},\dots,x_{k+m}\})\big)
\end{aligned}$$

for λ^{k+m}-a.e. $(x_1,\dots,x_{k+m}) \in \mathbb{X}^{k+m}$ and all $k,m \in \mathbb{N}$, where $s_* \in [0,1]$ is arbitrary. We conclude that the general result holds if it is proven for diffuse reference measures, so let us assume that λ is diffuse.

Let $\{D_{j,n} : j,n \in \mathbb{N}\} \subset \mathcal{X}_b \setminus \{\varnothing\}$ be a topological dissection system in $\mathbb{X}$, as defined in Appendix A.8. Let $k,m \in \mathbb{N}$ and note that the following reasoning is valid for λ^{k+m}-a.e. $(x_1,\dots,x_{k+m})$. Since correlation functions are the density functions of factorial moment measures with respect to the reference measure λ, Lemma A.20 and Proposition A.21 yield

$$\begin{aligned}
&\big|\rho_{\eta,k+m}(x_1,\dots,x_{k+m}) - \rho_{\eta,k}(x_1,\dots,x_k) \cdot \rho_{\eta,m}(x_{k+1},\dots,x_{k+m})\big| \\
&\quad = \limsup_{n\to\infty} \left| \frac{\alpha_{\eta,k+m}(D_{x_1,n} \times \ldots \times D_{x_{k+m},n})}{\lambda^{k+m}(D_{x_1,n} \times \ldots \times D_{x_{k+m},n})} - \frac{\alpha_{\eta,k}(D_{x_1,n} \times \ldots \times D_{x_k,n})}{\lambda^{k}(D_{x_1,n} \times \ldots \times D_{x_k,n})} \cdot \frac{\alpha_{\eta,m}(D_{x_{k+1},n} \times \ldots \times D_{x_{k+m},n})}{\lambda^{m}(D_{x_{k+1},n} \times \ldots \times D_{x_{k+m},n})} \right|
\end{aligned}$$

borrowing notation from Appendix A.8 where $D_{x,n}$ denotes the unique set in $\{D_{j,n} : j \in \mathbb{N}\}$ which contains $x \in \mathbb{X}$. Taking the factor

$$\left(\prod_{i=1}^{k+m} \lambda(D_{x_i,n}) \right)^{-1}$$

out of the modulus and applying Lemma 3.33 shows that the limit superior is bounded from above by

$$2\vartheta^{k+m} \limsup_{n\to\infty} \min\left\{ \vartheta \cdot \lambda\Big(\bigcup_{i=1}^{k} D_{x_i,n} \Big) + k,\ \vartheta \cdot \lambda\Big(\bigcup_{j=k+1}^{k+m} D_{x_j,n} \Big) + m \right\} \mathsf{w}\bigg(\operatorname{dist}\Big(\bigcup_{i=1}^{k} D_{x_i,n}, \bigcup_{j=k+1}^{k+m} D_{x_j,n} \Big) \bigg).$$

As $D_{x_i,n} \supset D_{x_i,n+1}$ for each $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} D_{x_i,n} = \{x_i\}$, for every $i \in \{1, \ldots, k+m\}$ (referring to Remark A.22), the fact that λ is a diffuse measure implies

$$0 \le \limsup_{n\to\infty} \lambda\Big(\bigcup_{i=1}^{k} D_{x_i,n} \Big) = \limsup_{n\to\infty} \sum_{i=1}^{k} \lambda(D_{x_i,n}) = \sum_{i=1}^{k} \lambda(\{x_i\}) = 0$$

and similarly

$$\lim_{n\to\infty} \lambda\Big(\bigcup_{j=k+1}^{k+m} D_{x_j,n} \Big) = 0.$$

Together with Lemma A.23 and the continuity of w we arrive at

$$\begin{aligned} &\big|\rho_{\eta,k+m}(x_1, \ldots, x_{k+m}) - \rho_{\eta,k}(x_1, \ldots, x_k) \cdot \rho_{\eta,m}(x_{k+1}, \ldots, x_{k+m})\big| \\ &\quad \le 2\vartheta^{k+m} \min\{k, m\} \cdot \mathsf{w}\big(\operatorname{dist}(\{x_1, \ldots, x_k\}, \{x_{k+1}, \ldots, x_{k+m}\})\big). \end{aligned}$$ □

We conclude this section with two remarks regarding the assumption on the clusters associated with the Poisson process in Lemma 3.33 and Theorem 3.34.

Remark 3.35. Recall from (3.9) that a Poisson process Φ with intensity measure $\vartheta \cdot \lambda$ satisfies

$$\lim_{\ell\to\infty} \mathbb{P}\big(C(x, \Phi)(B_\ell^{\mathsf{c}}) > 0\big) = 1 - \mathbb{P}\big(C(x, \Phi)(\mathbb{X}) < \infty\big)$$

for all $x \in \mathbb{X}$. Hence, the assumption in Lemma 3.33 and Theorem 3.34 implies

$$\mathbb{P}\big(C(x, \Phi)(\mathbb{X}) < \infty\big) = 1 - \limsup_{\ell\to\infty} \mathbb{P}\big(C(x, \Phi)(B_\ell^{\mathsf{c}}) > 0\big) \ge 1 - \limsup_{\ell\to\infty} \mathsf{w}\big(\operatorname{dist}(B, B_\ell^{\mathsf{c}})\big)$$

for λ-a.e. $x \in B$ and all $B \in \mathcal{X}_b$. As the bounded set B is contained in some large ball $B(x_0, R)$ for $R > 0$, and we chose $B_\ell = B(x_0, \ell)$, where $x_0 \in \mathbb{X}$ is a fixed origin, we have (for $\ell \ge R$)

$$\operatorname{dist}(B, B_\ell^{\mathsf{c}}) \ge \operatorname{dist}\big(B(x_0, R), B(x_0, \ell)^{\mathsf{c}}\big) \ge \ell - R.$$

Therefore, if w satisfies $\lim_{r\to\infty} \mathsf{w}(r) = 0$ then

$$\mathbb{P}\big(C(x, \Phi)(\mathbb{X}) < \infty\big) \ge 1 - \limsup_{\ell\to\infty} \mathsf{w}\big(\operatorname{dist}(B, B_\ell^{\mathsf{c}})\big) = 1,$$

so in this setting the assumption in Theorem 3.34 is stronger than the subcriticality assumed in the existence and uniqueness result, Theorem 3.32. ○

Remark 3.36. Let Φ be a Poisson process in $\mathbb{X}$ with intensity measure $\vartheta \cdot \lambda$ and denote by $\big\{x \overset{\Phi}{\sim} D\big\}$ the event that there exists a point $y \in D$ such that x is connected via Φ to y. If the relation $\sim$ is such that $\big\{x \overset{\Phi}{\sim} D\big\}$ is measurable for every $x \in \mathbb{X}$ and $D \in \mathcal{X}$, then, for each $\mu \in \mathbf{N}_{W^c}$,

$$\mathbb{P}\big(C(x, \Phi + \mu)(W^{\mathsf{c}}) > 0\big) \leq \mathbb{P}\big(x \overset{\Phi}{\sim} W^{\mathsf{c}}\big),$$

where $x \in B$ with $B \in \mathcal{X}_b \setminus \{\varnothing\}$ and $W \in \mathcal{X}_b$ with $W \supset B$. Hence the assumption in Lemma 3.33 and Theorem 3.34 is satisfied if

$$\mathbb{P}\big(x \overset{\Phi}{\sim} W^{\mathsf{c}}\big) \leq \mathsf{w}\big(\mathrm{dist}(B, W^{\mathsf{c}})\big)$$

for λ-a.e. $x \in B$. If w is decreasing, it even suffices to have

$$\mathbb{P}\big(x \overset{\Phi}{\sim} B(x,r)^{\mathsf{c}}\big) \leq \mathsf{w}(r) \tag{3.13}$$

for λ-a.e. $x \in B$ and every $r \geq 0$. Indeed, if this is true, then for any $W \in \mathcal{X}_b$ with $W \supset B$ and λ-a.e. $x \in B$ we choose $r_x = \mathrm{dist}(x, W^{\mathsf{c}})$ and obtain

$$\mathbb{P}\big(x \overset{\Phi}{\sim} W^{\mathsf{c}}\big) \leq \mathbb{P}\Big(x \overset{\Phi}{\sim} B(x,r_x)^{\mathsf{c}}\Big) \leq \mathsf{w}(r_x) = \mathsf{w}\big(\mathrm{dist}(x, W^{\mathsf{c}})\big) \leq \mathsf{w}\big(\mathrm{dist}(B, W^{\mathsf{c}})\big).$$

Note that instead of closed balls we can also consider open balls in (3.13). ○

3.3.6. Mixing properties in the subcritical regime

In this section we discuss mixing properties which can be understood, in a sense, as asymptotic independence properties. According to the general definitions in Section 1.6 of Rio (2017) (as well as Poinas, 2019, for the point process setting), we define the α-mixing coefficient of a point process η as

$$\mathfrak{m}_{s,t}^{(\alpha)}(r) = \sup\Big\{ \sup\big\{\big|\mathbb{P}^\eta(E_1 \cap E_2) - \mathbb{P}^\eta(E_1)\cdot\mathbb{P}^\eta(E_2)\big| : E_1 \in \mathcal{N}_B,\, E_2 \in \mathcal{N}_C\big\} : \\ B, C \in \mathcal{X}_b \text{ with } \lambda(B) \leq s,\, \lambda(C) \leq t, \text{ and } \mathrm{dist}(B,C) > r\Big\}$$

for $r, s, t \geq 0$. The process η is called α-mixing if $\lim_{r\to\infty} \mathfrak{m}_{s,t}^{(\alpha)}(r) = 0$ for all $s, t \geq 0$. For two probability measures P, Q on a measurable space $(\mathbb{Y}, \mathcal{Y})$ we consider their total variation distance

$$\|P - Q\|_{\mathrm{TV}} = \sup_{D\in\mathcal{Y}} \big|P(D) - Q(D)\big|.$$

With this notion, let the β-mixing coefficient of a point process η be

$$\mathfrak{m}_{s,t}^{(\beta)}(r) = \sup\Big\{\big\|\mathbb{P}^{(\eta_B,\eta_C)} - \mathbb{P}^{\eta_B} \otimes \mathbb{P}^{\eta_C}\big\|_{\mathrm{TV}} : B, C \in \mathcal{X}_b \text{ with } \lambda(B) \leq s,\, \lambda(C) \leq t, \text{ and } \mathrm{dist}(B,C) > r\Big\},$$

where $r, s, t \geq 0$. The process η is called β-mixing if $\lim_{r\to\infty} \mathfrak{m}_{s,t}^{(\beta)}(r) = 0$ for all $s, t \geq 0$.

Observe that, for any $B, C \in \mathcal{X}$,

$$\begin{aligned}
&\sup\big\{\big|\mathbb{P}^\eta(E_1 \cap E_2) - \mathbb{P}^\eta(E_1)\cdot\mathbb{P}^\eta(E_2)\big| : E_1 \in \mathcal{N}_B,\, E_2 \in \mathcal{N}_C\big\} \\
&\quad = \sup\big\{\big|\mathbb{P}^{(\eta_B,\eta_C)}(E_1 \times E_2) - \mathbb{P}^{\eta_B} \otimes \mathbb{P}^{\eta_C}(E_1 \times E_2)\big| : E_1 \in \mathcal{N}_B,\, E_2 \in \mathcal{N}_C\big\} \\
&\quad \leq \big\|\mathbb{P}^{(\eta_B,\eta_C)} - \mathbb{P}^{\eta_B} \otimes \mathbb{P}^{\eta_C}\big\|_{\mathrm{TV}},
\end{aligned}$$

so $\mathfrak{m}_{s,t}^{(\alpha)}(r) \leq \mathfrak{m}_{s,t}^{(\beta)}(r)$ for all $r, s, t \geq 0$. In particular, a β-mixing point process is always α-mixing as well.

Theorem 3.1 of Poinas (2019) provides a bound on the β-mixing coefficient of a point process in terms of the same difference of correlation functions on the left hand side of (3.11) in the previous section on decorrelation. This also justifies why Błaszczyszyn et al. (2019) refer to decorrelation as a mixing property. However, the result of Poinas (2019) is only available for simple point processes in $\mathbb{R}^d$. Before we can apply the bound in our setting, we thus first have to generalize it. Although the generalization is substantial, the proof is generic and similar to the one given by Poinas (2019).

Lemma 3.37. *Let η be a point process in $\mathbb{X}$ whose correlation functions with respect to λ exist. Assume that $\mathbb{E}\big[3^{\eta(D)}\big] < \infty$ for each $D \in \mathcal{X}_b$. Then, for any disjoint sets $B, C \in \mathcal{X}_b$,*

$$\big\|\mathbb{P}^{(\eta_B,\eta_C)} - \mathbb{P}^{\eta_B} \otimes \mathbb{P}^{\eta_C}\big\|_{\mathrm{TV}} \leq \sum_{k,m=1}^{\infty} \frac{2^{k+m}}{k! \cdot m!} \int_{B^k \times C^m} \big|\rho_{\eta,k}(x_1,\dots,x_k) \cdot \rho_{\eta,m}(x_{k+1},\dots,x_{k+m}) - \rho_{\eta,k+m}(x_1,\dots,x_{k+m})\big| \,\mathrm{d}\lambda^{k+m}(x_1,\dots,x_{k+m}).$$

Proof. Fix two disjoint sets $B, C \in \mathcal{X}_b$. If η' is an independent copy of η then

$$\big\|\mathbb{P}^{(\eta_B,\eta_C)} - \mathbb{P}^{\eta_B} \otimes \mathbb{P}^{\eta_C}\big\|_{\mathrm{TV}} = \sup_{E \in \mathcal{N}^{\otimes 2}} \big|\mathbb{P}\big((\eta_B,\eta_C) \in E\big) - \mathbb{P}\big((\eta_B,\eta'_C) \in E\big)\big|.$$

Fix $E \in \mathcal{N}^{\otimes 2}$. To lighten the notation, let $F : \mathbf{N} \to \{0,1\}$ and $G : \mathbf{N} \times \mathbf{N} \to \{0,1\}$ be given by

$$F(\mu) = \mathbb{1}\big\{(\mu_B,\mu_C) \in E\big\} \quad \text{and} \quad G(\mu,\nu) = \mathbb{1}\big\{(\mu_B,\nu_C) \in E\big\}.$$

Recall that for $m \in \mathbb{N}$ we set $[m] = \{1,\dots,m\}$, with $[0] = \varnothing$, and that $|J|$ denotes the cardinality of a set J. As in (2.6), and by assumption on η, we have

$$\begin{aligned}
\mathbb{E}\Bigg[\sum_{m=0}^{\infty} \frac{1}{m!} \int_{(B\cup C)^m} \sum_{J \subset [m]} F\Big(\sum_{j\in J} \delta_{x_j}\Big) \,\mathrm{d}\eta^{(m)}(x_1,\dots,x_m)\Bigg] &\leq \mathbb{E}\Bigg[\sum_{m=0}^{\infty} \frac{2^m}{m!} \cdot \eta^{(m)}\big((B\cup C)^m\big)\Bigg] \\
&= \mathbb{E}\Bigg[\sum_{m=0}^{\eta(B\cup C)} \binom{\eta(B\cup C)}{m} \cdot 2^m\Bigg] \\
&= \mathbb{E}\big[3^{\eta(B\cup C)}\big] < \infty, \qquad (3.14)
\end{aligned}$$

where for $m = 0$ the term in the sum on the left hand side is interpreted as $F(\mathbf{0})$. By Lemma A.7 and Fubini's theorem, and with (3.14) in mind, we have

$$\begin{aligned}
\mathbb{P}\big((\eta_B,\eta_C) \in E\big) &= \mathbb{E}\Bigg[\sum_{m=0}^{\infty} \frac{1}{m!} \int_{(B\cup C)^m} \sum_{J \subset [m]} (-1)^{m-|J|} F\Big(\sum_{j\in J} \delta_{x_j}\Big) \,\mathrm{d}\eta^{(m)}(x_1,\dots,x_m)\Bigg] \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \int_{(B\cup C)^m} \sum_{J \subset [m]} (-1)^{m-|J|} F\Big(\sum_{j\in J} \delta_{x_j}\Big) \,\rho_{\eta,m}(x_1,\dots,x_m) \,\mathrm{d}\lambda^m(x_1,\dots,x_m).
\end{aligned}$$

By symmetry of the integrated function, and since B and C are disjoint, the previous term equals

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} \int_{B^k \times C^{m-k}} \sum_{J \subset [m]} (-1)^{m-|J|} F\Big(\sum_{j\in J} \delta_{x_j}\Big) \,\rho_{\eta,m}(x_1,\dots,x_m) \,\mathrm{d}\lambda^m(x_1,\dots,x_m) \\
&= \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \frac{1}{k! \cdot (m-k)!} \int_{B^k \times C^{m-k}} \sum_{J \subset [m]} (-1)^{m-|J|} F\Big(\sum_{j\in J} \delta_{x_j}\Big) \,\rho_{\eta,m}(x_1,\dots,x_m) \,\mathrm{d}\lambda^m(x_1,\dots,x_m) \\
&= \sum_{k,m=0}^{\infty} \frac{1}{k! \cdot m!} \int_{B^k \times C^m} \sum_{J \subset [k+m]} (-1)^{k+m-|J|} F\Big(\sum_{j\in J} \delta_{x_j}\Big) \,\rho_{\eta,k+m}(x_1,\dots,x_{k+m}) \,\mathrm{d}\lambda^{k+m}(x_1,\dots,x_{k+m}),
\end{aligned}$$

using (3.14) to justify the change in the order of summation.

Next, observe that, for $\mu, \nu \in \mathbf{N}_f$, a twofold application of Lemma A.7 shows that $G(\mu, \nu)$ equals

$$\sum_{k,m=0}^{\infty} \frac{1}{k! \cdot m!} \int_{B^k} \int_{C^m} \sum_{I \subset [k]} \sum_{J \subset [m]} (-1)^{k+m-|I|-|J|} G\Big(\sum_{i \in I} \delta_{x_i}, \sum_{j \in J} \delta_{y_j}\Big) \, \mathrm{d}\nu^{(m)}(y_1, \dots, y_m) \, \mathrm{d}\mu^{(k)}(x_1, \dots, x_k),$$

which, by definition of F and G (using that B and C are disjoint), equals

$$\begin{aligned}
&\sum_{k,m=0}^{\infty} \frac{1}{k! \cdot m!} \int_{B^k} \int_{C^m} \sum_{I \subset [k]} \sum_{J \subset [m]} (-1)^{k+m-|I|-|J|} F\Big(\sum_{i \in I} \delta_{x_i} + \sum_{j \in J} \delta_{y_j}\Big) \, \mathrm{d}\nu^{(m)}(y_1, \dots, y_m) \, \mathrm{d}\mu^{(k)}(x_1, \dots, x_k) \\
&\quad = \sum_{k,m=0}^{\infty} \frac{1}{k! \cdot m!} \int_{B^k} \int_{C^m} \sum_{J \subset [k+m]} (-1)^{k+m-|J|} F\Big(\sum_{j \in J} \delta_{x_j}\Big) \, \mathrm{d}\nu^{(m)}(x_{k+1}, \dots, x_{k+m}) \, \mathrm{d}\mu^{(k)}(x_1, \dots, x_k).
\end{aligned}$$

Together with the independence of η and η', this calculation implies that

$$\begin{aligned}
&\mathbb{P}\big((\eta_B, \eta'_C) \in E\big) \\
&\quad = \mathbb{E}\bigg[\sum_{k,m=0}^{\infty} \frac{1}{k! \cdot m!} \int_{B^k} \int_{C^m} \sum_{J \subset [k+m]} (-1)^{k+m-|J|} F\Big(\sum_{j \in J} \delta_{x_j}\Big) \, \mathrm{d}\eta'^{(m)}(x_{k+1}, \dots, x_{k+m}) \, \mathrm{d}\eta^{(k)}(x_1, \dots, x_k)\bigg] \\
&\quad = \sum_{k,m=0}^{\infty} \frac{1}{k! \cdot m!} \int_{B^k \times C^m} \sum_{J \subset [k+m]} (-1)^{k+m-|J|} F\Big(\sum_{j \in J} \delta_{x_j}\Big) \, \rho_{\eta,k}(x_1, \dots, x_k) \\
&\qquad\qquad \cdot \rho_{\eta,m}(x_{k+1}, \dots, x_{k+m}) \, \mathrm{d}\lambda^{k+m}(x_1, \dots, x_{k+m}),
\end{aligned}$$

where the use of Fubini's theorem is justified similar to (3.14), in that

$$\begin{aligned}
&\mathbb{E}\bigg[\sum_{k,m=0}^{\infty} \frac{1}{k! \cdot m!} \int_{B^k} \int_{C^m} \sum_{J \subset [k+m]} F\Big(\sum_{j \in J} \delta_{x_j}\Big) \, \mathrm{d}\eta'^{(m)}(x_{k+1}, \dots, x_{k+m}) \, \mathrm{d}\eta^{(k)}(x_1, \dots, x_k)\bigg] \\
&\quad \leq \mathbb{E}\bigg[\sum_{k,m=0}^{\infty} \frac{2^{k+m}}{k! \cdot m!} \cdot \eta^{(k)}(B^k) \cdot \eta'^{(m)}(C^m)\bigg] = \mathbb{E}\big[3^{\eta(B)}\big] \cdot \mathbb{E}\big[3^{\eta(C)}\big] < \infty.
\end{aligned}$$

With the representations for $\mathbb{P}\big((\eta_B, \eta_C) \in E\big)$ and $\mathbb{P}\big((\eta_B, \eta'_C) \in E\big)$ we obtain

$$\begin{aligned}
&\Big|\mathbb{P}\big((\eta_B, \eta_C) \in E\big) - \mathbb{P}\big((\eta_B, \eta'_C) \in E\big)\Big| \\
&\quad \leq \sum_{k,m=1}^{\infty} \frac{1}{k! \cdot m!} \int_{B^k \times C^m} \sum_{J \subset [k+m]} F\Big(\sum_{j \in J} \delta_{x_j}\Big) \Big|\rho_{\eta,k}(x_1, \dots, x_k) \cdot \rho_{\eta,m}(x_{k+1}, \dots, x_{k+m}) \\
&\qquad\qquad - \rho_{\eta,k+m}(x_1, \dots, x_{k+m})\Big| \, \mathrm{d}\lambda^{k+m}(x_1, \dots, x_{k+m}),
\end{aligned}$$

where the terms in the sums for $k = 0$ or $m = 0$ vanish when taking the difference. As before, we use that $\sum_{J \subset [k+m]} F\big(\sum_{j \in J} \delta_{x_j}\big) \leq 2^{k+m}$ to complete the proof. □

Lemma 3.37 is perfectly valid in arbitrary measurable spaces. The metric structure on $\mathbb{X}$ enters only in the definition of $\mathfrak{m}_{s,t}^{(\beta)}(r)$. Also note that λ need not be diffuse (this is true for Theorem 3.38 as well).

We prove in the upcoming theorem that our decorrelation result implies that cluster-dependent Gibbs processes are β-mixing in the subcritical regime. Indeed, whenever the function w in the following theorem satisfies $\lim_{r \to \infty} \mathsf{w}(r) = 0$ then the Gibbs process in consideration is β-mixing (and hence also α-mixing).

Theorem 3.38. *Let $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ be a measurable function which satisfies the cocycle relation (2.8). Suppose that κ is locally stable as in Definition 3.14 with constant bound $\vartheta \geq 0$ and that*

$$\kappa(x, \mu) = \kappa\big(x, C(x, \mu)\big)$$

for all $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$. Let η be a Gibbs process with PI κ and reference measure λ. Assume there exists a continuous and decreasing function $\mathsf{w} : [0, \infty) \to [0, \infty)$ such that, for any $B, W \in \mathcal{X}_b \setminus \{\varnothing\}$ with $B \subset W$,

$$\Pi_{\vartheta \cdot \lambda}\big(\{\nu \in \mathbf{N} : C(x, \nu + \mu)(W^c) > 0\}\big) \leq \mathsf{w}\big(\mathrm{dist}(B, W^c)\big)$$

for λ-a.e. $x \in B$ and every $\mu \in \mathbf{N}_f \cap \mathbf{N}_{W^c}$. Then, for all $r, s, t \geq 0$,

$$\mathfrak{m}_{s,t}^{(\beta)}(r) \leq 4\vartheta \mathrm{e}^{2\vartheta(s+t)} \min\{s, t\} \cdot \mathsf{w}(r).$$

Proof. First note that η satisfies the assumption in Lemma 3.37 as the local stability bound and part 3 of Remark 2.10 ensure that $\mathbb{E}\big[b^{\eta(D)}\big] < \infty$ for all $D \in \mathcal{X}_b$ and every $b \geq 0$. Moreover, the prerequisites from Theorem 3.34 are satisfied. Let $B, C \in \mathcal{X}_b$ satisfy $\mathrm{dist}(B, C) > r \geq 0$ as well as $\lambda(B) \leq s$ and $\lambda(C) \leq t$ for some $s, t \geq 0$. By Lemma 3.37 and Theorem 3.34

$$\begin{aligned}
&\big\|\mathbb{P}^{(\eta_B, \eta_C)} - \mathbb{P}^{\eta_B} \otimes \mathbb{P}^{\eta_C}\big\|_{\mathrm{TV}} \\
&\quad \leq 2 \sum_{k,m=1}^{\infty} \frac{(2\vartheta)^{k+m}}{k! \cdot m!} \cdot \min\{k, m\} \int_{B^k \times C^m} \mathsf{w}\big(\mathrm{dist}(\{x_1, \ldots, x_k\}, \{x_{k+1}, \ldots, x_{k+m}\})\big) \, \mathrm{d}\lambda^{k+m}(x_1, \ldots, x_{k+m}).
\end{aligned}$$

Since w is decreasing, we have $\mathsf{w}\big(\mathrm{dist}(\{x_1, \ldots, x_k\}, \{x_{k+1}, \ldots, x_{k+m}\})\big) \leq \mathsf{w}(r)$ for all $x_1, \ldots, x_k \in B$ and $x_{k+1}, \ldots, x_{k+m} \in C$, so a further bound on the term in the previous display is given by

$$2 \cdot \mathsf{w}(r) \sum_{k,m=1}^{\infty} \frac{(2\vartheta)^{k+m}}{k! \cdot m!} \cdot \min\{k, m\} \cdot \lambda(B)^k \cdot \lambda(C)^m \leq 2 \cdot \mathsf{w}(r) \sum_{k,m=1}^{\infty} \frac{(2\vartheta)^{k+m}}{k! \cdot m!} \cdot \min\{k, m\} \cdot s^k \cdot t^m.$$

Now, observe that

$$\sum_{k,m=1}^{\infty} \frac{(2\vartheta)^{k+m}}{k! \cdot m!} \cdot \min\{k, m\} \cdot s^k \cdot t^m \leq \min\left\{\mathrm{e}^{2\vartheta t} \sum_{k=1}^{\infty} \frac{(2\vartheta s)^k}{(k-1)!}, \ \mathrm{e}^{2\vartheta s} \sum_{m=1}^{\infty} \frac{(2\vartheta t)^m}{(m-1)!}\right\} = 2\vartheta \mathrm{e}^{2\vartheta(s+t)} \min\{s, t\}.$$

Summarizing, we have shown that, for the given sets B and C,

$$\big\|\mathbb{P}^{(\eta_B, \eta_C)} - \mathbb{P}^{\eta_B} \otimes \mathbb{P}^{\eta_C}\big\|_{\mathrm{TV}} \leq 4\vartheta \mathrm{e}^{2\vartheta(s+t)} \min\{s, t\} \cdot \mathsf{w}(r).$$

Taking the supremum over all such sets, we arrive at

$$\mathfrak{m}_{s,t}^{(\beta)}(r) \leq 4\vartheta \mathrm{e}^{2\vartheta(s+t)} \min\{s, t\} \cdot \mathsf{w}(r). \qquad \square$$

For basics and historic references with regard to mixing properties, we refer to Rio (2017). As far as the point process setting is involved, let us mention that Poinas et al. (2019) have derived mixing properties for associated point processes, for instance that certain determinantal processes are α-mixing. Poinas (2019) collects several references for applications of mixing properties in spatial statistics.

Specifically for Gibbs processes there exist results regarding α-mixing within the Dobrushin uniqueness region (cf. Chapter 5), referring to Föllmer (1982) for one instance. A short review of mixing results for discrete Gibbs measures is contained in the work by van den Berg (1996). Strong mixing results for hard core Gibbs processes in $\mathbb{R}^d$ are given in the unpublished work of Heinrich (1992). For a very recent contribution, we also mention the preprint by Michelen and Perkins (2022) who prove (with analytic methods) that repulsive finite-range pair interaction processes have strong spatial mixing properties for low activities.

3.4. Gibbsian particle processes

So far we have justified the fact that we study Gibbs processes in general spaces by a somewhat puristic argument, namely that such a study allows to identify truly structural properties and separate them from topological and other assumptions. Still, if there were no applications beyond $\mathbb{R}^d$, the utility of the general theory would be limited. To demonstrate that the latter concern is, at least to a large extent, unfounded, we discuss in this section particle processes. Indeed, to study random systems of general particles or advanced germ-grain models in stochastic geometry, one benefits greatly from having a theory for point processes in the space of compact subsets of an underlying space, cf. Schneider and Weil (2008). Most results focus on Poisson particle processes, but the consideration of Gibbsian models is an interesting extension, as witnessed by Stucki (2013) and Jansen (2016) in the context of percolation in Gibbsian ball models as well as by Flimmel and Beneš (2018) and Beneš et al. (2020) for more general particles. To study such processes, however, the theory of Gibbs processes has to be available at least on metric spaces.

Let $(\mathbb{B}, d)$ be a complete separable metric space. Let $\mathbb{X} = \mathcal{C}^*(\mathbb{B})$ be the space of non-empty compact subsets (particles) of $\mathbb{B}$ equipped with the Hausdorff metric d_H. For a definition of d_H and the fact that $\mathcal{C}^*(\mathbb{B})$ equipped with d_H is a complete separable metric space we refer to Appendix A.9 and to Molchanov (2017). Denote the corresponding Borel σ-field by $\mathcal{X} = \mathcal{B}\big(\mathcal{C}^*(\mathbb{B})\big)$ and write $\mathcal{X}_b = \mathcal{B}_b\big(\mathcal{C}^*(\mathbb{B})\big)$ for the collection of all Borel sets which are bounded with respect to d_H. Let λ be a locally finite measure on $\mathcal{C}^*(\mathbb{B})$. In line with our setting in Section 2.2 we consider the space of locally finite counting measures

$$\mathbf{N}\big(\mathcal{C}^*(\mathbb{B})\big) = \Big\{\mu \text{ is a measure on } \mathcal{C}^*(\mathbb{B}) : \mu\big(B_H(K, r)\big) \in \mathbb{N}_0 \text{ for all } K \in \mathcal{C}^*(\mathbb{B}) \text{ and } r \geq 0\Big\},$$

where $B_H(K, r)$ denotes a ball of radius r in $\mathcal{C}^*(\mathbb{B})$ around K with respect to d_H. Our general setup from previous sections covers this setting and yields a definition and properties of Gibbs processes in $\mathcal{C}^*(\mathbb{B})$, which are called Gibbsian particle processes. As regards their existence, we mention three settings that are of principal interest. The first concerns pair interactions for which we refer to the general results in Section 4.4. The other two settings correspond to Sections 3.2.2 and 3.3.2, namely finite range and cluster-dependent interactions. We discuss these two results as corollaries of Corollary 3.20 and Theorem 3.28.

Corollary 3.39. *Let $\kappa : \mathcal{C}^*(\mathbb{B}) \times \mathbf{N}\big(\mathcal{C}^*(\mathbb{B})\big) \to [0, \infty)$ be a measurable map which satisfies the cocycle relation (2.8). Assume there exists a λ-locally integrable map $\vartheta : \mathcal{C}^*(\mathbb{B}) \to [0, \infty)$ such that*

$$\sup_{\mu \in \mathbf{N}(\mathcal{C}^*(\mathbb{B}))} \kappa(K, \mu) \leq \vartheta(K)$$

for λ-a.e. $K \in \mathcal{C}^*(\mathbb{B})$. Also suppose that there exists some $r > 0$ such that, for $K \in \mathcal{C}^*(\mathbb{B})$ and $\mu \in \mathbf{N}\big(\mathcal{C}^*(\mathbb{B})\big)$,

$$\kappa(K, \mu) = \kappa\big(K, \mu_{B_H(K,r)}\big).$$

Then there exists a Gibbs particle process with PI κ and reference measure λ.

Before we turn to cluster-dependent interactions, we apply the previous corollary to prove existence of the particle processes considered by Beneš et al. (2020), a question left unanswered in their work. In the following example, we show the existence in a slightly more general setting which fully includes theirs, thus proving their conjecture. Any issues of measurability in the context of the space $\mathcal{C}^*(\mathbb{B})$ are omitted at this point to make reading more convenient, but they are settled completely in Appendix A.9.

Example 3.40. Let $\mathbb{B} = \mathbb{R}^d$ endowed with the Euclidean distance and the Lebesgue measure $\mathcal{L}^d$. Consistent with the general setting above, we consider the space $\mathcal{C}^{(d)}$ of non-empty compact subsets of $\mathbb{R}^d$ equipped with the Borel σ-field induced by the Hausdorff metric. We let $c : \mathcal{C}^{(d)} \to \mathbb{R}^d$ be a center function, that is, c is measurable and satisfies $c(K+x) = c(K) + x$ for all $K \in \mathcal{C}^{(d)}$ and $x \in \mathbb{R}^d$, where

$$K + x = \{z + x : z \in K\}.$$

Let $\mathbb{Q}$ be a probability measure on $\mathcal{C}^{(d)}$ with $\mathbb{Q}(\mathcal{C}_0^{(d)}) = 1$, where

$$\mathcal{C}_0^{(d)} = \big\{K \in \mathcal{C}^{(d)} : c(K) = 0\big\}.$$

Moreover, assume that, for each $C \in \mathcal{C}^{(d)}$,

$$\int_{\mathcal{C}^{(d)}} \mathcal{L}^d(L + C) \,\mathrm{d}\mathbb{Q}(L) < \infty,$$

which is certainly fulfilled if the particle size is bounded, that is, if $\mathbb{Q}\big(\{K \in \mathcal{C}^{(d)} : K \subset B(0,R)\}\big) = 1$ for a fixed radius $R > 0$. For some fixed intensity parameter $\gamma > 0$, consider on $\mathcal{C}^{(d)}$ the measure

$$\lambda(\cdot) = \gamma \int_{\mathbb{R}^d} \int_{\mathcal{C}^{(d)}} \mathbb{1}\{L + x \in \cdot\,\} \,\mathrm{d}\mathbb{Q}(L) \,\mathrm{d}\mathcal{L}^d(x).$$

Then λ is locally finite. Indeed, for any $K \in \mathcal{C}^{(d)}$ and $r \geq 0$ the definition of the Hausdorff metric (and the fact that we work in $\mathbb{R}^d$) yields the existence of a set $C \in \mathcal{C}^{(d)}$ such that $L \subset C$ for all $L \in B_H(K,r)$, so

$$\begin{aligned}
\lambda\big(B_H(K,r)\big) \leq \lambda\big(\{L \in \mathcal{C}^{(d)} : L \cap C \neq \varnothing\}\big) &= \gamma \int_{\mathbb{R}^d} \int_{\mathcal{C}^{(d)}} \mathbb{1}\big\{(L + x) \cap C \neq \varnothing\big\} \,\mathrm{d}\mathbb{Q}(L) \,\mathrm{d}\mathcal{L}^d(x) \\
&= \gamma \int_{\mathcal{C}^{(d)}} \mathcal{L}^d\big(L + (-C)\big) \,\mathrm{d}\mathbb{Q}(L) \\
&< \infty.
\end{aligned}$$

Let $V = \{V_n : n \in \mathbb{N}, n \geq 2\}$ be a collection of higher-order interaction potentials in the sense that $V_n : (\mathcal{C}^{(d)})^n \to (-\infty, \infty]$ is a measurable and symmetric function for each $n \geq 2$. Assume that there exists some $R_V > 0$ such that $V_n(K_1, \dots, K_n) = 0$ for all $K_1, \dots, K_n \in \mathcal{C}^{(d)}$ with

$$\max\big\{d_H(K_i, K_j) : 1 \leq i < j \leq n\big\} > R_V$$

and all $n \geq 2$. Moreover, assume that, for every $K \in \mathcal{C}^{(d)}$ and $\mu \in \mathbf{N}(\mathcal{C}^{(d)})$,

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)!} \max\bigg\{ -\int_{(\mathcal{C}^{(d)})^{n-1}} V_n(K, L_1, \dots, L_{n-1}) \,\mathrm{d}\mu^{(n-1)}(L_1, \dots, L_{n-1}),\ 0\bigg\} < \infty.$$

Define $\kappa : \mathcal{C}^{(d)} \times \mathbf{N}(\mathcal{C}^{(d)}) \to [0, \infty)$ as

$$\kappa(K, \mu) = \exp\Bigg(-\sum_{n=2}^{\infty} \frac{1}{(n-1)!} \int_{(\mathcal{C}^{(d)})^{n-1}} V_n(K, L_1, \dots, L_{n-1}) \,\mathrm{d}\mu^{(n-1)}(L_1, \dots, L_{n-1})\Bigg).$$

Notice that κ is well-defined by the summability assumption on the V_n. Beneš et al. (2020) multiply κ with an indicator function $\mathbb{1}\big\{\mu(\{K\}) = 0\big\}$, a modification that changes nothing about the following observations. The map κ is measurable by Lemma A.4 and it satisfies the cocycle relation, since Exercise 4.3 of Last and Penrose (2017), as recalled in (A.4), and the symmetry of the V_n imply

$$\kappa(K, \mu) \cdot \kappa(C, \mu + \delta_K) = \exp\Bigg(-\sum_{n=2}^{\infty} \frac{1}{(n-1)!} \bigg(\int_{(\mathcal{C}^{(d)})^{n-1}} V_n(K, L_1, \dots, L_{n-1}) \,\mathrm{d}\mu^{(n-1)}(L_1, \dots, L_{n-1})$$

$$
\begin{aligned}
&\qquad + \int_{(\mathcal{C}^{(d)})^{n-1}} V_n(C, L_1, \dots, L_{n-1}) \,\mathrm{d}(\mu + \delta_K)^{(n-1)}(L_1, \dots, L_{n-1}) \Big) \Big) \\
&= \exp\bigg(- \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \Big(\int_{(\mathcal{C}^{(d)})^{n-1}} V_n(K, L_1, \dots, L_{n-1}) \,\mathrm{d}\mu^{(n-1)}(L_1, \dots, L_{n-1}) \\
&\qquad + \int_{(\mathcal{C}^{(d)})^{n-1}} V_n(C, L_1, \dots, L_{n-1}) \,\mathrm{d}\mu^{(n-1)}(L_1, \dots, L_{n-1}) \\
&\qquad + (n-1) \int_{(\mathcal{C}^{(d)})^{n-2}} V_n(C, K, L_1, \dots, L_{n-2}) \,\mathrm{d}\mu^{(n-2)}(L_1, \dots, L_{n-2}) \Big) \bigg) \\
&= \exp\bigg(- \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \Big(\int_{(\mathcal{C}^{(d)})^{n-1}} V_n(C, L_1, \dots, L_{n-1}) \,\mathrm{d}\mu^{(n-1)}(L_1, \dots, L_{n-1}) \\
&\qquad + \int_{(\mathcal{C}^{(d)})^{n-1}} V_n(K, L_1, \dots, L_{n-1}) \,\mathrm{d}(\mu + \delta_C)^{(n-1)}(L_1, \dots, L_{n-1}) \Big) \bigg) \\
&= \kappa(C, \mu) \cdot \kappa(K, \mu + \delta_C).
\end{aligned}
$$

Furthermore, κ has finite interaction range, as

$$
\begin{aligned}
\kappa(K, \mu) &= \exp\bigg(- \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \int_{(\mathcal{C}^{(d)})^{n-1}} V_n(K, L_1, \dots, L_{n-1}) \,\mathrm{d}\mu^{(n-1)}(L_1, \dots, L_{n-1}) \bigg) \\
&= \exp\bigg(- \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \int_{(\mathcal{C}^{(d)})^{n-1}} V_n(K, L_1, \dots, L_{n-1}) \,\mathrm{d}\mu_{B_H(K, R_V)}^{(n-1)}(L_1, \dots, L_{n-1}) \bigg) \\
&= \kappa\big(K, \mu_{B_H(K, R_V)}\big)
\end{aligned}
$$

by assumption on V. Finally, if we assume that $\sup_{\mu \in \mathbf{N}(\mathcal{C}^{(d)})} \kappa(K, \mu) \leq \vartheta(K)$ for λ-a.e. $K \in \mathcal{C}^{(d)}$ and a locally integrable function $\vartheta : \mathcal{C}^{(d)} \to [0, \infty)$, then Corollary 3.39 guarantees the existence of a Gibbsian particle process with PI κ. Beneš et al. (2020) even assume that $\kappa \leq 1$ uniformly. ○

In the context of particle processes there is a natural notion of clusters. In the language of Section 3.3.2, consider on $\mathcal{C}^*(\mathbb{B})$ the binary relation given by

$$
K \sim L \quad \text{if, and only if,} \quad K \cap L \neq \varnothing. \tag{3.15}
$$

This relation is clearly symmetric and the set $\big\{(K, L) \in \mathcal{C}^*(\mathbb{B}) \times \mathcal{C}^*(\mathbb{B}) : K \sim L\big\}$ is measurable by Lemma A.24. The relation also leads to a very intuitive interpretation of the clusters in (3.7) which, in this case, describe the connected components of a germ-grain model. Notice that the assumption in Theorem 3.32 then simply describes absence of percolation in the Poisson-Boolean model. In the remainder of this section we use results on percolation in the Boolean model due to Gouéré (2008) to establish existence and uniqueness of Gibbs particle processes in the subcritical phase. As such we recover the uniqueness results of Hofer-Temmel (2019) and Hofer-Temmel and Houdebert (2019), which is a benefit as these works contain technical gaps that are now filled. We also extend those results to particle processes with more general grains, similar to Beneš et al. (2020) but allowing for unbounded grains. Moreover, we newly provide the existence results for those same processes in the region of their uniqueness.

We consider the setting from Example 3.40, thus restricting to $\mathbb{B} = \mathbb{R}^d$. For a center function let $c : \mathcal{C}^{(d)} \to \mathbb{R}^d$ denote the map that assign to each compact set in $\mathbb{R}^d$ the center of its circumball, and let $\mathrm{rad} : \mathcal{C}^{(d)} \to [0, \infty)$ denote the map that assigns to a compact set the radius of its circumball. Both of these maps are continuous with respect to the Hausdorff metric by Lemma A.28. With the relation in (3.15) define clusters as in (3.7).

Corollary 3.41. *Let $\mathbb{Q}$ be a probability measure on $\mathcal{C}^{(d)}$ with $\mathbb{Q}(\mathcal{C}_0^{(d)}) = 1$ and*

$$\int_{\mathcal{C}^{(d)}} \mathrm{rad}(L)^d \, \mathrm{d}\mathbb{Q}(L) < \infty,$$

and, for $\gamma > 0$, put

$$\lambda_\gamma(\cdot) = \gamma \int_{\mathbb{R}^d} \int_{\mathcal{C}^{(d)}} \mathbb{1}\{L + x \in \cdot\,\} \, \mathrm{d}\mathbb{Q}(L) \, \mathrm{d}\mathcal{L}^d(x).$$

Let $\kappa : \mathcal{C}^{(d)} \times \mathbf{N}(\mathcal{C}^{(d)}) \to [0, \infty)$ be a measurable map that satisfies the cocycle property (2.8) and which is such that, for any $K \in \mathcal{C}^{(d)}$ and $\mu \in \mathbf{N}(\mathcal{C}^{(d)})$,

$$\kappa(K, \mu) \leq 1 \quad \text{as well as} \quad \kappa(K, \mu) = \kappa\big(K, C(K, \mu)\big).$$

Then there is a constant $\gamma_c = \gamma_c(\mathbb{Q}, d) > 0$ such that for any $\gamma < \gamma_c$ there exists, up to equality in distribution, exactly one Gibbs particle process with PI κ, activity γ, and grain distribution $\mathbb{Q}$, that is, exactly one Gibbs process in $\mathcal{C}^{(d)}$ with PI κ and reference measure λ_γ.

Proof. Let Z be a random element of $\mathcal{C}^{(d)}$ with distribution $\mathbb{Q}$ (the typical grain). We first note that the measure λ_γ is locally finite. Indeed, it follows as in Example 3.40 that, for any $K \in \mathcal{C}^{(d)}$ and $r > 0$,

$$\lambda_\gamma\big(B_H(K, r)\big) \leq \gamma \int_{\mathcal{C}^{(d)}} \mathcal{L}^d\big(L + (-C)\big) \, \mathrm{d}\mathbb{Q}(L) = \gamma \cdot \mathbb{E}\big[\mathcal{L}^d\big(Z + (-C)\big)\big]$$

for some suitable set $C \in \mathcal{C}^{(d)}$. As C is compact, we have $C \subset B(0, s)$ for $s > 0$ large enough, and we also have $Z \subset B\big(0, \mathrm{rad}(Z)\big)$ almost surely. Thus, by assumption on $\mathbb{Q}$,

$$\begin{aligned}
\lambda_\gamma\big(B_H(K, r)\big) \leq \gamma \cdot \mathbb{E}\big[\mathcal{L}^d\big(B(0, \mathrm{rad}(Z) + s)\big)\big] &= \gamma \cdot \mathcal{L}^d\big(B(0,1)\big) \cdot \mathbb{E}\big[\big(\mathrm{rad}(Z) + s\big)^d\big] \\
&\leq \gamma \cdot \mathcal{L}^d\big(B(0,1)\big) \cdot (1+s)^d \cdot \max\big\{1, \, \mathbb{E}\big[\mathrm{rad}(Z)^d\big]\big\} \\
&< \infty.
\end{aligned}$$

Let $\Phi = \sum_{j=1}^\infty \delta_{K_j}$ be a Poisson particle process with intensity measure λ_γ. Theorem 3.32 yields the claim if we can show that the Boolean model based on Φ (in other words, the Boolean model with intensity γ and grain distribution $\mathbb{Q}$) is subcritical. To this end, let us define on $[0, \infty)$ the probability measure

$$m(\cdot) = \int_{\mathcal{C}^{(d)}} \mathbb{1}\big\{\mathrm{rad}(L) \in \cdot\,\big\} \, \mathrm{d}\mathbb{Q}(L) = \mathbb{P}\big(\mathrm{rad}(Z) \in \cdot\,\big),$$

which satisfies

$$\int_0^\infty r^d \, \mathrm{d}m(r) = \int_{\mathcal{C}^{(d)}} \mathrm{rad}(L)^d \, \mathrm{d}\mathbb{Q}(L) < \infty.$$

The Boolean model $\bigcup_{j=1}^\infty K_j$ is almost surely contained in $\bigcup_{j=1}^\infty B\big(c(K_j), \mathrm{rad}(K_j)\big)$, which is itself a Boolean model, since

$$\Psi = \int_{\mathcal{C}^{(d)}} \mathbb{1}\big\{B\big(c(L), \mathrm{rad}(L)\big) \in \cdot\,\big\} \, \mathrm{d}\Phi(L)$$

is a (stationary) Poisson particle process with intensity measure

$$\begin{aligned}
\int_{\mathcal{C}^{(d)}} \mathbb{1}\big\{B\big(c(L), \mathrm{rad}(L)\big) \in \cdot\,\big\} \, \mathrm{d}\lambda_z(L) &= \gamma \int_{\mathbb{R}^d} \int_{\mathcal{C}^{(d)}} \mathbb{1}\big\{B\big(x, \mathrm{rad}(L)\big) \in \cdot\,\big\} \, \mathrm{d}\mathbb{Q}(L) \, \mathrm{d}\mathcal{L}^d(x) \\
&= \gamma \int_{\mathbb{R}^d} \int_0^\infty \mathbb{1}\big\{B(x, r) \in \cdot\,\big\} \, \mathrm{d}m(r) \, \mathrm{d}\mathcal{L}^d(x)
\end{aligned}$$

by the mapping theorem for Poisson processes (Theorem 5.1 of Last and Penrose, 2017). Theorem 2.1 of Gouéré

(2008) (and the stationarity of Ψ) provides a constant $\gamma_c(\mathbb{Q}, d) > 0$ such that

$$\mathbb{P}\big(C(K, \Phi)(\mathbb{X}) < \infty\big) \geq \mathbb{P}\big(C(K, \Psi)(\mathbb{X}) < \infty\big) = 1$$

for all $K \in \mathcal{C}^{(d)}$ and every $\gamma < \gamma_c(\mathbb{Q}, d)$. □

To extend an example from the literature, we consider segment processes in $\mathbb{R}^d$, cf. Example 2.2 of Flimmel and Beneš (2018). Proceeding as in Example 3.40 gives an existence result which covers the processes discussed by Flimmel and Beneš (2018) (even in arbitrary dimension), where a global and deterministic bound on the length of the segments is assumed. We use Corollary 3.41 to provide an existence and uniqueness result for segment processes which also allow for unbounded length distributions.

Example 3.42 (**Segment processes**)**.** Let m be a probability measure on $[0, \infty)$ with existing d-th moment, which yields (half) the length of the segments, and let Q be a probability measure on $\mathbb{R}^d$ concentrated on the unit sphere S^{d-1}, which yields the orientation of the segments. Consider on $\mathcal{C}_0^{(d)}$ the probability measure

$$\mathbb{Q}(\cdot) = \int_0^\infty \int_{S^{d-1}} \mathbb{1}\Big\{\big\{s \cdot v : s \in [-r, r]\big\} \in \cdot\Big\} \, \mathrm{d}Q(v) \, \mathrm{d}m(r).$$

The particle processes corresponding to the reference measure λ_γ based on $\mathbb{Q}$ are random segments in $\mathbb{R}^d$. Let $V : \mathcal{C}^{(d)} \cup \{\varnothing\} \to [0, \infty]$ be measurable with $V(\varnothing) = 0$ (where the σ-field on $\mathcal{C}^{(d)} \cup \{\varnothing\}$ is constructed from the one on $\mathcal{C}^{(d)}$ by adding the singleton $\{\varnothing\}$ as a measurable set, as recalled in Appendix A.9). With the PI

$$\kappa(K, \mu) = \exp\bigg(-\beta \int_{\mathcal{C}^{(d)}} V(K \cap L) \, \mathrm{d}\mu(L)\bigg),$$

where $\beta > 0$, this fits into the setting of Corollary 3.41. Indeed, κ is measurable by Corollary A.25, is easily seen to satisfy the cocycle property (as in Lemma 4.1), and is bounded by 1. The fact that $\kappa(K, \cdot\,)$ depends only on the respective clusters of K follows from $V(\varnothing) = 0$. A specific example for V is $V(K) = c \cdot \mathbb{1}\{K \neq \varnothing\}$ for $c \in [0, \infty]$, but V could also be the restriction of a locally finite measure on $\mathbb{R}^d$ onto $\mathcal{C}^{(d)} \cup \{\varnothing\}$ (which gives a measurable map by Proposition E.13 of Molchanov, 2017).

The existence result for this particular κ can be improved by noting that it corresponds to a pair interaction and using Corollary 4.14 or Theorem B.1 of Jansen (2019). But of course more complicated κ can be considered as well. Moreover, we argue in Example 5.19 that the uniqueness result in Corollary 3.41 can be improved in the pair potential setting (by replacing the Boolean model with a more refined random connection model as in Theorem 5.14). ○

Remark 3.43. If, in the setting of Corollary 3.41, the grain distribution $\mathbb{Q}$ satisfies

$$\int_{\mathcal{C}^{(d)}} \mathbb{1}\big\{\mathrm{rad}(L) \geq r\big\} \, \mathrm{d}\mathbb{Q}(L) \leq \mathrm{e}^{-c \cdot r}$$

for all $r \geq 1$ and some constant $c > 0$, then it might be possible to establish that the Gibbs process in $\mathcal{C}^{(d)}$ with PI κ and reference measure λ_γ decorrelates exponentially for all $\gamma < \gamma_c$ and that the process is both α- and β-mixing for each such γ. With the help of Lemma 3 of Beneš et al. (2020) and Remark 3.36 this follows immediately from Theorems 3.34 and 3.38 if the grains are bounded deterministically (as Beneš et al., 2020, assume them to be). The more general result for unbounded grains might be provable if Lemma 3 of Beneš et al. (2020) is replaced by a similar but stronger result using Theorem 1.4 of Duminil-Copin et al. (2020) instead of Equation (3.7) of Ziesche (2018). This is not pursued further at this point. ○

CHAPTER 4

The essentials of pair interaction processes

While the results in this chapter, similar to the previous chapters, are derived on more general state spaces than considered in the literature, the results are not new in their essence. Still, the purpose of this chapter is manifold. First of all, the mentioned generalization in the allowed state spaces gives some additional structural insights and provides a framework for known results. As this generalization requires the improvement of technicalities underlying the theory of pair interaction processes, these technical results might be of interest in their own right. Moreover, the setting of pair potentials gives another opportunity to apply the existence result of the previous chapter. We will see that Theorem 3.12 includes some of the strongest existence results from the literature. As such, the study in this chapter shows that the abstraction of Theorem 3.12 does not lead to a loss in more specific and structured settings. A final objective of this chapter is to provide the basics of pair interaction processes in order to prepare for the upcoming chapter on corresponding uniqueness results.

The structure of this chapter is as follows. In Section 4.1 we make precise the concept of a pair potential and pair interaction processes. Section 4.2 is devoted to the discussion of technical preliminaries which play an important role later on in the chapter but also in Chapter 5. In Section 4.3 we provide a general and rather generic proof of the Kirkwood–Salsburg equations. The sole purpose of Section 4.4 is to see what Theorem 3.12 yields in the pair potential setting, thus proving a general existence result which includes pair potentials with negative parts. The chapter is completed by Section 4.5 which is devoted exclusively to examples.

We consider the general setting from Section 2.2, namely $(\mathbb{X}, \mathcal{X})$ is a measurable space with localizing structure $B_1 \subset B_2 \subset \ldots$ and λ is a locally finite reference measure on $\mathbb{X}$. Whenever it is not stated otherwise, the results in this chapter refer to this setting.

4.1. Pair potentials

Suppose that $v : \mathbb{X} \times \mathbb{X} \to (-\infty, \infty]$ is a measurable and symmetric function, which we call a pair potential. The Papangelou intensity $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ corresponding to v is given by

$$\kappa(x, \mu) = \exp\bigg(- \int_{\mathbb{X}} v(x, y)\, \mathrm{d}\mu(y)\bigg) \cdot \mathbb{1}\bigg\{ \int_{\mathbb{X}} v^-(x, y)\, \mathrm{d}\mu(y) < \infty \bigg\} \tag{4.1}$$

for $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$, with v^- the negative part of v. We consider the given PI throughout this chapter. The basic properties of κ are collected in the following lemma.

Lemma 4.1. *The PI κ given in (4.1) is measurable and satisfies the cocycle relation. Moreover, for $x \in \mathbb{X}$ and $\mu, \psi \in \mathbf{N}$, one has*

$$\kappa(x, \mu + \psi) = \kappa(x, \mu) \cdot \kappa(x, \psi),$$

and $\kappa(x, \delta_y) = \mathrm{e}^{-v(x,y)} = \kappa(y, \delta_x)$ for any $x, y \in \mathbb{X}$.

Proof. Measurability of κ follows from Lemma A.4 since the integral of v can be split into

$$\int_{\mathbb{X}} v(x, y)\, \mathrm{d}\mu(y) = \int_{\mathbb{X}} v^+(x, y)\, \mathrm{d}\mu(y) - \int_{\mathbb{X}} v^-(x, y)\, \mathrm{d}\mu(y),$$

using that the indicator function in (4.1) ensures that the integral is well-defined. For $x, z \in \mathbb{X}$ and $\mu \in \mathbf{N}$ we have, by symmetry of v,

$$
\begin{aligned}
&\kappa(x,\mu)\cdot\kappa(z,\mu+\delta_x)\\
&\quad= \exp\Big(-\int_{\mathbb{X}} v(x,y)\,\mathrm{d}\mu(y) - \int_{\mathbb{X}} v(z,y)\,\mathrm{d}(\mu+\delta_x)(y)\Big)\cdot \mathbb{1}\Big\{\int_{\mathbb{X}} \big(v^-(x,y)+v^-(z,y)\big)\,\mathrm{d}\mu(y)<\infty\Big\}\\
&\quad= \exp\Big(-\int_{\mathbb{X}} v(z,y)\,\mathrm{d}\mu(y) - \int_{\mathbb{X}} v(x,y)\,\mathrm{d}(\mu+\delta_z)(y)\Big)\cdot \mathbb{1}\Big\{\int_{\mathbb{X}} \big(v^-(x,y)+v^-(z,y)\big)\,\mathrm{d}\mu(y)<\infty\Big\}\\
&\quad= \kappa(z,\mu)\cdot\kappa(x,\mu+\delta_z),
\end{aligned}
$$

so κ satisfies the cocycle assumption. Moreover, we have, for $x\in\mathbb{X}$ and $\mu,\psi\in\mathbf{N}$,

$$
\begin{aligned}
\kappa(x,\mu+\psi) &= \exp\Big(-\int_{\mathbb{X}} v(x,y)\,\mathrm{d}\mu(y)\Big)\cdot\exp\Big(-\int_{\mathbb{X}} v(x,y)\,\mathrm{d}\psi(y)\Big)\cdot\mathbb{1}\Big\{\int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}(\mu+\psi)(y)<\infty\Big\}\\
&= \kappa(x,\mu)\cdot\kappa(x,\psi).
\end{aligned}
$$

The remaining claim of the lemma follows immediately from (4.1) and the symmetry of v. □

Lemma 4.1 establishes that κ is indeed a valid PI for the study of corresponding Gibbs processes. A Gibbs process with PI κ, that is, with pair potential v, is called pair interaction process. Existence of such processes is discussed in Section 4.4.

Remark 4.2. Note that the indicator function which appears in the definition of κ in (4.1) can be omitted whenever $\mu\in\mathbf{N}_f$. Indeed, Lemma A.6 yields that $\int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}\mu(y)<\infty$ for all $\mu\in\mathbf{N}_f$. ○

We conclude this introductory section by calculating the Hamiltonian corresponding to a pair potential.

Remark 4.3. For $x_1,\dots,x_m\in\mathbb{X}$ and $\psi\in\mathbf{N}$ with

$$
\sum_{j=1}^m \int_{\mathbb{X}} v^-(x_j,y)\,\mathrm{d}\psi(y)<\infty, \tag{4.2}
$$

Lemma 4.1 and the definition of κ imply

$$
\begin{aligned}
\kappa_m(x_1,\dots,x_m,\psi) &= \kappa(x_1,\psi)\cdot\kappa(x_2,\psi+\delta_{x_1})\cdot\kappa(x_3,\psi+\delta_{x_1}+\delta_{x_2})\cdot\ldots\cdot\kappa(x_m,\psi+\delta_{x_1}+\ldots+\delta_{x_{m-1}})\\
&= \kappa(x_1,\psi)\cdot\ldots\cdot\kappa(x_m,\psi)\prod_{1\le i<j\le m}\mathrm{e}^{-v(x_i,x_j)}\\
&= \exp\Big(-\sum_{j=1}^m\int_{\mathbb{X}} v(x_j,y)\,\mathrm{d}\psi(y)\Big)\prod_{1\le i<j\le m}\mathrm{e}^{-v(x_i,x_j)}.
\end{aligned}
$$

Given (4.2), we thus have

$$
-\log\big(\kappa_m(x_1,\dots,x_m,\psi)\big) = \sum_{j=1}^m\int_{\mathbb{X}} v(x_j,y)\,\mathrm{d}\psi(y) + \sum_{1\le i<j\le m} v(x_i,x_j).
$$

If (4.2) is violated, then Lemma 4.1 and the definition of κ amount to $\kappa_m(x_1,\dots,x_m,\psi)=0$ and therefore $-\log\big(\kappa_m(x_1,\dots,x_m,\psi)\big)=\infty$. Proposition A.5 implies that, for $\mu\in\mathbf{N}$ with $\mu(\mathbb{X})=m\in\mathbb{N}$ and $\psi\in\mathbf{N}$,

$$
\begin{aligned}
&\frac{1}{m!}\int_{\mathbb{X}^m} -\log\big(\kappa_m(x_1,\dots,x_m,\psi)\big)\,\mathrm{d}\mu^{(m)}(x_1,\dots,x_m)\\
&\quad= \Big(\int_{\mathbb{X}}\int_{\mathbb{X}} v(x,y)\,\mathrm{d}\psi(y)\,\mathrm{d}\mu(x) + \frac{1}{2}\int_{\mathbb{X}^2} v(x,y)\,\mathrm{d}\mu^{(2)}(x,y)\Big)\cdot\mathbb{1}\Big\{\int_{\mathbb{X}}\int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}\psi(y)\,\mathrm{d}\mu(x)<\infty\Big\}\\
&\qquad+\infty\cdot\mathbb{1}\Big\{\int_{\mathbb{X}}\int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}\psi(y)\,\mathrm{d}\mu(x)=\infty\Big\}.
\end{aligned}
$$

By definition of the Hamiltonian and Remark 2.32, we have, for $\mu, \psi \in \mathbf{N}$,

$$\begin{aligned} H(\mu,\psi) &= \infty \cdot \mathbb{1}\big\{\mu(\mathbb{X}) = \infty\big\} + \infty \cdot \mathbb{1}\bigg\{\int_{\mathbb{X}}\int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}\psi(y)\,\mathrm{d}\mu(x) = \infty\bigg\} \\ &\quad + \int_{\mathbb{X}}\int_{\mathbb{X}} v(x,y)\,\mathrm{d}\psi(y)\,\mathrm{d}\mu(x)\cdot \mathbb{1}\bigg\{\int_{\mathbb{X}}\int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}\psi(y)\,\mathrm{d}\mu(x) < \infty\bigg\} \\ &\quad + \frac{1}{2}\int_{\mathbb{X}^2} v(x,y)\,\mathrm{d}\mu^{(2)}(x,y)\cdot \mathbb{1}\big\{\mu(\mathbb{X}) < \infty\big\}, \end{aligned}$$

using that $\int_{\mathbb{X}^2} v^-(x,y)\,\mathrm{d}\mu^{(2)}(x,y) < \infty$ if $\mu \in \mathbf{N}_f$ (by Lemma A.6) to reorganize the indicator functions. We conclude that the energy function corresponding to v is

$$H(\mu) = H(\mu,\mathbf{0}) = \infty \cdot \mathbb{1}\big\{\mu(\mathbb{X}) = \infty\big\} + \frac{1}{2}\int_{\mathbb{X}^2} v(x,y)\,\mathrm{d}\mu^{(2)}(x,y)\cdot \mathbb{1}\big\{\mu(\mathbb{X}) < \infty\big\},$$

which also includes

$$H\Big(\sum_{j=1}^{m}\delta_{x_j}\Big) = \sum_{1\le i<j\le m} v(x_i,x_j),$$

for $x_1,\dots,x_m \in \mathbb{X}$ and $m \in \mathbb{N}$. The formula in the last display (which might be somewhat more familiar to readers of Ruelle, 1969) emphasizes that in the context of pair interaction processes only interactions between pairs of points are relevant and no interactions of higher order, involving multiple points, appear. The definition of κ itself already allows for a similar interpretation. Note that the lecture notes by Jansen (2017) as well as the text book chapter by Dereudre (2019) also start with energies functions and arrive at Papangelou intensities when discussing the GNZ equation. ○

4.2. Technical preliminaries

We now collect some technical lemmata in the context of pair potentials. In principle, the results we present are rather standard, but Lemmata 4.6 and 4.8 are newly established in arbitrary measurable spaces.

Recall that the limit relations in Theorem 3.12 need to hold only on a suitable sub-collection of the bounded sets in $\mathbb{X}$. In the pair potential setting such choices are most useful and discussed in Corollaries 4.5 and 5.8. To cover these results, we provide a more general construction, which is implicitly mentioned in Appendix B of Jansen (2019).

Lemma 4.4. *Let $g : \mathbb{X} \to [0,\infty]$ be a measurable map such that $g(x) < \infty$ for λ-a.e. $x \in \mathbb{X}$. There exist sets $C_1 \subset C_2 \subset \dots$ from $\mathcal{X}_b$ such that $\bigcup_{\ell=1}^{\infty} C_\ell = \mathbb{X}$ and*

$$\int_{C_\ell} g(x)\,\mathrm{d}\lambda(x) < \infty, \quad \ell \in \mathbb{N}.$$

Moreover, if $\mathcal{X}_b^{(g)}$ is the collection of all sets from $\mathcal{X}$ which are contained in one of the C_ℓ, then $\mathcal{X}_b^{(g)}$ constitutes a ring over $\mathbb{X}$ which generates $\mathcal{X}$.

Proof. By assumption, $\lambda\big(\{x \in \mathbb{X} : g(x) = \infty\}\big) = 0$. Put, for $i, j \in \mathbb{N}$,

$$\tilde{C}_{i,j} = B_i \cap \{x \in \mathbb{X} : g(x) \le j\}$$

and $\tilde{C}_0 = \{x \in \mathbb{X} : g(x) = \infty\}$. Then g is λ-integrable over each of the sets $\tilde{C}_0, \tilde{C}_{i,j}$, for $i, j \in \mathbb{N}$, and $\tilde{C}_0 \cup \bigcup_{i,j=1}^{\infty} \tilde{C}_{i,j} = \mathbb{X}$. Successively uniting these sets gives the required sets $C_1 \subset C_2 \subset \dots$. The fact that $\mathcal{X}_b^{(g)}$ is a ring over $\mathbb{X}$ is obvious and clearly $\sigma(\mathcal{X}_b^{(g)}) \subset \sigma(\mathcal{X}_b) = \mathcal{X}$. For any $B \in \mathcal{X}$, we have $B \cap C_\ell \in \mathcal{X}_b^{(g)}$ for all

$\ell \in \mathbb{N}$, so

$$B = \bigcup_{\ell=1}^{\infty} (B \cap C_\ell) \in \sigma(\mathcal{X}_b^{(g)}). \qquad \square$$

The first of two relevant special cases for us is the following corollary to Lemma 4.4. This construction is essentially due to Jansen (2019). Recall that $\mathcal{Z}$ is the algebra of local events from Definition A.9.

Corollary 4.5. *Let $\vartheta : \mathbb{X} \to [0, \infty]$ be measurable and assume that, for λ-a.e. $x \in \mathbb{X}$,*

$$\int_{\mathbb{X}} \left| \mathrm{e}^{-v(x,y)} - 1 \right| \vartheta(y) \, \mathrm{d}\lambda(y) < \infty.$$

There exist sets $C_1 \subset C_2 \subset \dots$ from $\mathcal{X}_b$ such that $\bigcup_{\ell=1}^{\infty} C_\ell = \mathbb{X}$ and

$$\int_{C_\ell} \exp\left(\int_{\mathbb{X}} \left| \mathrm{e}^{-v(x,y)} - 1 \right| \vartheta(y) \, \mathrm{d}\lambda(y) \right) \mathrm{d}\lambda(x) < \infty, \quad \ell \in \mathbb{N}.$$

If $\mathcal{X}_b^*$ is the collection of all sets from $\mathcal{X}$ which are contained in one the C_ℓ, then $\mathcal{X}_b^*$ is a ring over $\mathbb{X}$ with $\sigma(\mathcal{X}_b^*) = \mathcal{X}$. Moreover, the collection $\{C \times A : C \in \mathcal{X}_b^*, A \in \mathcal{Z}\}$ is a π-system that generates $\mathcal{X} \otimes \mathcal{N}$.

The first two parts of the corollary follow immediately from Lemma 4.4 upon choosing

$$g(x) = \exp\left(\int_{\mathbb{X}} \left| \mathrm{e}^{-v(x,y)} - 1 \right| \vartheta(y) \, \mathrm{d}\lambda(y) \right).$$

The final part of Corollary 4.5 is an almost literal copy of part (ix) in Lemma A.10. Notice that if

$$\operatorname*{ess\,sup}_{x \in \mathbb{X}} \int_{\mathbb{X}} \left| \mathrm{e}^{-v(x,y)} - 1 \right| \vartheta(y) \, \mathrm{d}\lambda(y) < \infty,$$

we can stick to our original localizing structure and take $\mathcal{X}_b^* = \mathcal{X}_b$. The essential supremum in the previous display is with respect to λ. If ϑ is constant, the integrability assumption in the lemma reduces to the classical assumption

$$\int_{\mathbb{X}} \left| \mathrm{e}^{-v(x,y)} - 1 \right| \mathrm{d}\lambda(y) < \infty$$

already considered by Ruelle (1963b) in $\mathbb{R}^d$. A pair potential with this property is usually referred to as *regular* or *tempered*.

We now prove a very useful expansion for κ in (4.1). The result is known in complete separable metric spaces and generalizes immediately to Borel spaces, but the following is the first proof of the result in an arbitrary measurable space.

Lemma 4.6. *Let $x \in \mathbb{X}$ and $\psi \in \mathbf{N}_f$. Then*

$$\kappa(x, \psi) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \left(\mathrm{e}^{-v(x,y_j)} - 1 \right) \mathrm{d}\psi^{(k)}(y_1, \dots, y_k).$$

Proof. For $\psi = \mathbf{0}$ the claim follows immediately from the definition of κ in (4.1). Next, consider $\psi \in \mathbf{N}_f$ with $\psi(\mathbb{X}) = m \in \mathbb{N}$. By part (iv) of Proposition A.2 we have $\psi^{(k)} = \mathbf{0} \in \mathbf{N}(\mathbb{X}^k)$ for any $k > m$. Moreover,

$$\int_{\mathbb{X}^k} \prod_{j=1}^{k} \left| \mathrm{e}^{-v(x,y_j)} - 1 \right| \mathrm{d}\psi^{(k)}(y_1, \dots, y_k) < \infty,$$

for all $k \leq m$ by Lemma A.6. We conclude that the infinite sum in the statement of the lemma is well-defined and finite. For $x_1, \dots, x_m \in \mathbb{X}$ we have

$$\begin{aligned}\kappa\Big(x, \sum_{j=1}^m \delta_{x_j}\Big) = \prod_{j=1}^m \mathrm{e}^{-v(x,x_j)} &= \sum_{J\subset[m]} \prod_{j\in J} \big(\mathrm{e}^{-v(x,x_j)} - 1\big) \\ &= 1 + \sum_{k=1}^m \frac{1}{k!} \sum_{j_1,\dots,j_k\in[m]}^{\neq} \prod_{i=1}^k \big(\mathrm{e}^{-v(x,x_{j_i})} - 1\big) \\ &= 1 + \sum_{k=1}^\infty \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{i=1}^k \big(\mathrm{e}^{-v(x,y_i)} - 1\big)\, \mathrm{d}\Big(\sum_{j=1}^m \delta_{x_j}\Big)^{(k)}(y_1,\dots,y_k),\end{aligned}$$

where the first two equalities follow from Lemma 4.1 and the subsequent Remark 4.7, respectively. Applying Proposition A.5 twice gives

$$\begin{aligned}\kappa(x,\psi) &= \frac{1}{m!} \int_{\mathbb{X}^m} \kappa\Big(x, \sum_{j=1}^m \delta_{x_j}\Big)\, \mathrm{d}\psi^{(m)}(x_1,\dots,x_m) \\ &= \frac{1}{m!} \int_{\mathbb{X}^m} \left[1 + \sum_{k=1}^\infty \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{i=1}^k \big(\mathrm{e}^{-v(x,y_i)} - 1\big)\, \mathrm{d}\Big(\sum_{j=1}^m \delta_{x_j}\Big)^{(k)}(y_1,\dots,y_k)\right] \mathrm{d}\psi^{(m)}(x_1,\dots,x_m) \\ &= 1 + \sum_{k=1}^\infty \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^k \big(\mathrm{e}^{-v(x,y_j)} - 1\big)\, \mathrm{d}\psi^{(k)}(y_1,\dots,y_k).\end{aligned}$$ □

Remark 4.7. In the proof of Lemma 4.6 we have used the following simple observation which is included for the readers convenience. Let $a_1, a_2, \dots \in \mathbb{R}$. For each $n \in \mathbb{N}$ we have

$$\prod_{j=1}^n a_j = \sum_{J\subset[n]} \prod_{j\in J} (a_j - 1),$$

recalling that $[n] = \{1,\dots,n\}$, with the standard convention of empty products being set to 1. This follows by induction as

$$\prod_{j=1}^1 a_j = a_1 = 1 + (a_1 - 1) = \sum_{J\subset[1]} \prod_{j\in J} (a_j - 1)$$

and, if the formula holds for any fixed $n \in \mathbb{N}$, then

$$\sum_{J\subset[n+1]} \prod_{j\in J} (a_j - 1) = \sum_{J\subset[n]} \prod_{j\in J} (a_j - 1) + \sum_{J\subset[n]} \prod_{j\in J\cup\{n+1\}} (a_j - 1) = \prod_{j=1}^n a_j + (a_{n+1} - 1) \prod_{j=1}^n a_j = \prod_{j=1}^{n+1} a_j.$$

Note that the formula can be extended to infinite products in the lines of Exercise 2.10 of Jansen (2017). ○

The expansion of the previous lemma cannot be expected to hold for every $\psi \in \mathbf{N}$, but under suitable assumptions it holds almost surely with respect to the distribution of some point process. In the case of non-negative v, this observation appears in (A.1) and (A.2) of Jansen (2019), to mention just one recent appearance of the expansion.

Lemma 4.8. *Let η be a point process in $\mathbb{X}$ whose factorial moment measures satisfy $\alpha_{\eta,m} \le (\vartheta\lambda)^m$ for each $m \in \mathbb{N}$ and a measurable map $\vartheta : \mathbb{X} \to [0,\infty)$ which is subject to*

$$\int_{\mathbb{X}} \left|\mathrm{e}^{-v(x,y)} - 1\right| \vartheta(y)\, \mathrm{d}\lambda(y) < \infty$$

for λ-a.e. $x \in \mathbb{X}$. Also assume that, for λ-a.e. $x \in \mathbb{X}$,

$$\mathbb{P}\bigg(\int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}\eta(y) < \infty\bigg) = 1.$$

Then

$$\kappa(x,\psi) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big(\mathrm{e}^{-v(x,y_j)} - 1\big)\,\mathrm{d}\psi^{(k)}(y_1,\dots,y_k)$$

for $(\lambda \otimes \mathbb{P}^\eta)$-a.e. $(x,\psi) \in \mathbb{X} \times \mathbf{N}$.

Proof. Denote by $\mathbb{X}_1 \in \mathcal{X}$ the set of all $x \in \mathbb{X}$ which satisfy the two assumptions in the lemma, and notice that $\lambda(\mathbb{X} \setminus \mathbb{X}_1) = 0$. For each $x \in \mathbb{X}_1$ we have

$$\begin{aligned}
\mathbb{E}\Bigg[\sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big|\mathrm{e}^{-v(x,y_j)} - 1\big|\,\mathrm{d}\eta^{(k)}(y_1,\dots,y_k)\Bigg] &= \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big|\mathrm{e}^{-v(x,y_j)} - 1\big|\,\mathrm{d}\alpha_{\eta,k}(y_1,\dots,y_k) \\
&\le \sum_{k=1}^{\infty} \frac{1}{k!} \bigg(\int_{\mathbb{X}} \big|\mathrm{e}^{-v(x,y)} - 1\big|\,\vartheta(y)\,\mathrm{d}\lambda(y)\bigg)^k \\
&\le \exp\bigg(\int_{\mathbb{X}} \big|\mathrm{e}^{-v(x,y)} - 1\big|\,\vartheta(y)\,\mathrm{d}\lambda(y)\bigg).
\end{aligned}$$

Fix $x \in \mathbb{X}_1$. By the previous observation and the assumptions, there exists a set $\mathbf{N}_{1,x} \in \mathcal{N}$ with $\mathbb{P}^\eta(\mathbf{N} \setminus \mathbf{N}_{1,x}) = 0$ such that, for each $\psi \in \mathbf{N}_{1,x}$,

$$\sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big|\mathrm{e}^{-v(x,y_j)} - 1\big|\,\mathrm{d}\psi^{(k)}(y_1,\dots,y_k) < \infty$$

as well as $\int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}\psi(y) < \infty$. Fix $\psi \in \mathbf{N}_{1,x}$. By Lemma 4.6 and dominated convergence we obtain

$$\begin{aligned}
\lim_{\ell\to\infty} \kappa(x,\psi_{B_\ell}) &= 1 + \lim_{\ell\to\infty} \sum_{k=1}^{\infty} \frac{1}{k!} \int_{B_\ell^k} \prod_{j=1}^{k} \big(\mathrm{e}^{-v(x,y_j)} - 1\big)\,\mathrm{d}\psi^{(k)}(y_1,\dots,y_k) \\
&= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big(\mathrm{e}^{-v(x,y_j)} - 1\big)\,\mathrm{d}\psi^{(k)}(y_1,\dots,y_k).
\end{aligned}$$

By monotone convergence we have

$$\lim_{\ell\to\infty} \int_{B_\ell} v^+(x,y)\,\mathrm{d}\psi(y) = \int_{\mathbb{X}} v^+(x,y)\,\mathrm{d}\psi(y) \in [0,\infty]$$

as well as

$$\lim_{\ell\to\infty} \int_{B_\ell} v^-(x,y)\,\mathrm{d}\psi(y) = \int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}\psi(y) \in [0,\infty).$$

We obtain that

$$\lim_{\ell\to\infty} \kappa(x,\psi_{B_\ell}) = \lim_{\ell\to\infty} \exp\bigg(-\int_{B_\ell} v(x,y)\,\mathrm{d}\psi(y)\bigg) = \exp\bigg(-\int_{\mathbb{X}} v(x,y)\,\mathrm{d}\psi(y)\bigg) = \kappa(x,\psi),$$

which concludes the proof. □

The final part of the proof of Lemma 4.8 yields the following corollary.

Corollary 4.9. *For $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$ with $\int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}\mu(y) < \infty$, one has $\lim_{\ell\to\infty} \kappa(x,\mu_{B_\ell}) = \kappa(x,\mu)$.*

In the context of existence results in Section 4.4 we need the following corollary of Lemma 4.8.

Corollary 4.10. *Let the point process η and the function ϑ be as in Lemma 4.8. Let ξ be a point process such that $\xi \le \eta$ almost surely. Then*

$$\kappa(x,\psi) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big(\mathrm{e}^{-v(x,y_j)} - 1\big)\, \mathrm{d}\psi^{(k)}(y_1,\dots,y_k)$$

for $(\lambda \otimes \mathbb{P}^{\xi})$-a.e. $(x,\psi) \in \mathbb{X} \times \mathbf{N}$.

Proof. Note that, for each $m \in \mathbb{N}$, part (v) of Proposition A.2 implies

$$\alpha_{\xi,m}(\cdot) = \mathbb{E}\big[\xi^{(m)}(\cdot)\big] \le \mathbb{E}\big[\eta^{(m)}(\cdot)\big] = \alpha_{\eta,m}(\cdot) \le (\vartheta\lambda)^m(\cdot).$$

Moreover, we have

$$\mathbb{P}\bigg(\int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}\xi(y) = \infty\bigg) \le \mathbb{P}\bigg(\int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}\eta(y) = \infty\bigg) = 0$$

for λ-a.e. $x \in \mathbb{X}$. Applying Lemma 4.8 to ξ completes the proof. □

4.3. The Kirkwood–Salsburg equations

We now prove the Kirkwood–Salsburg equations, which every pair interaction Gibbs process satisfies. These equations are well-known and go back to Mayer (1947), see also Hill (1956) and Ruelle (1969), but again the known result is extended to arbitrary measurable spaces. The arguments below adapt the proof of the equations given for non-negative pair potentials by Jansen (2019).

Theorem 4.11. *Let η be a Gibbs process in $\mathbb{X}$ with PI κ from (4.1). Assume that the factorial moment measures satisfy $\alpha_{\eta,m} \le (\vartheta\lambda)^m$ for all $m \in \mathbb{N}$ and a measurable map $\vartheta : \mathbb{X} \to [0,\infty)$ with*

$$\int_{\mathbb{X}} \big|\mathrm{e}^{-v(x,y)} - 1\big|\, \vartheta(y)\,\mathrm{d}\lambda(y) < \infty$$

for λ-a.e. $x \in \mathbb{X}$. Also assume that, for λ-a.e. $x \in \mathbb{X}$,

$$\mathbb{P}\bigg(\int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}\eta(y) < \infty\bigg) = 1.$$

Then the correlation functions $\rho_m = \rho_{\eta,m}$ ($m \in \mathbb{N}$) of η satisfy the Kirkwood–Salsburg equations

$$\begin{aligned}\rho_{m+1}(x,x_1,\dots,x_m) = \Bigg(&\sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big(\mathrm{e}^{-v(x,y_j)} - 1\big)\, \rho_{m+k}(x_1,\dots,x_m,y_1,\dots,y_k)\,\mathrm{d}\lambda^k(y_1,\dots,y_k)\\ &+ \rho_m(x_1,\dots,x_m)\Bigg) \cdot \prod_{j=1}^{m} \mathrm{e}^{-v(x,x_j)}\end{aligned}$$

for λ^{m+1}-a.e. $(x,x_1,\dots,x_m) \in \mathbb{X}^{m+1}$ and each $m \in \mathbb{N}_0$, where we put $\rho_0 = 1$.

Notice that the Gibbs processes from the existence result, Theorem 4.13, which we prove in Section 4.4, satisfy all assumptions of the theorem at hand.

Proof. For $m = 0$ we have, by Lemmata 2.20 and 4.8, as well as Fubini's theorem,

$$\rho_1(x) = \mathbb{E}\big[\kappa(x,\eta)\big] = 1 + \mathbb{E}\Bigg[\sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big(\mathrm{e}^{-v(x,y_j)} - 1\big)\,\mathrm{d}\eta^{(k)}(y_1,\dots,y_k)\Bigg]$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big(\mathrm{e}^{-v(x,y_j)} - 1\big)\, \rho_k(y_1, \dots, y_k)\, \mathrm{d}\lambda^k(y_1, \dots, y_k)$$

for λ-a.e. $x \in \mathbb{X}$. Now, fix $m \in \mathbb{N}$. By Lemmata 2.20 and 4.1 we have, for λ^{m+1}-a.e. $(x, x_1, \dots, x_m) \in \mathbb{X}^{m+1}$,

$$\rho_{m+1}(x, x_1, \dots, x_m) = \mathbb{E}\Big[\kappa_{m+1}(x, x_1, \dots, x_m, \eta)\Big] = \kappa\Big(x, \sum_{j=1}^{m} \delta_{x_j}\Big) \cdot \mathbb{E}\big[\kappa_m(x_1, \dots, x_m, \eta) \cdot \kappa(x, \eta)\big],$$

which, due to Lemma 4.8 (as well as Lemmata 4.1 and 2.20), equals

$$\prod_{j=1}^{m} \mathrm{e}^{-v(x,x_j)} \Bigg(\rho_m(x_1, \dots, x_m) + \mathbb{E}\Bigg[\sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big(\mathrm{e}^{-v(x,y_j)} - 1\big)\, \kappa_m(x_1, \dots, x_m, \eta)\, \mathrm{d}\eta^{(k)}(y_1, \dots, y_k)\Bigg]\Bigg).$$

Applying Fubini's theorem and the multivariate GNZ equation (Lemma 2.14), the expectation in the previous display is equal to

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big(\mathrm{e}^{-v(x,y_j)} - 1\big) \cdot \mathbb{E}\Bigg[\kappa_m\Big(x_1, \dots, x_m, \eta + \sum_{i=1}^{k} \delta_{y_i}\Big) \cdot \kappa_k(y_1, \dots, y_k, \eta)\Bigg]\, \mathrm{d}\lambda^k(y_1, \dots, y_k) \\
&\quad = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big(\mathrm{e}^{-v(x,y_j)} - 1\big)\, \rho_{m+k}(x_1, \dots, x_m, y_1, \dots, y_k)\, \mathrm{d}\lambda^k(y_1, \dots, y_k),
\end{aligned}$$

which establishes the Kirkwood–Salsburg equation. Notice that the use of Fubini's theorem and the extension of Lemma 2.14 to sufficiently integrable functions is justified by observing that

$$\begin{aligned}
&\mathbb{E}\Bigg[\sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big|\mathrm{e}^{-v(x,y_j)} - 1\big|\, \kappa_m(x_1, \dots, x_m, \eta)\, \mathrm{d}\eta^{(k)}(y_1, \dots, y_k)\Bigg] \\
&\quad = \sum_{k=1}^{\infty} \frac{1}{k!}\, \mathbb{E}\Bigg[\int_{\mathbb{X}^k} \prod_{j=1}^{k} \big|\mathrm{e}^{-v(x,y_j)} - 1\big|\, \kappa_m(x_1, \dots, x_m, \eta)\, \mathrm{d}\eta^{(k)}(y_1, \dots, y_k)\Bigg] \\
&\quad = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big|\mathrm{e}^{-v(x,y_j)} - 1\big| \cdot \mathbb{E}\Big[\kappa_{m+k}(x_1, \dots, x_m, y_1, \dots, y_k, \eta)\Big]\, \mathrm{d}\lambda^k(y_1, \dots, y_k) \\
&\quad \le \vartheta(x_1) \cdot \ldots \cdot \vartheta(x_m) \cdot \exp\Bigg(\int_{\mathbb{X}} \big|\mathrm{e}^{-v(x,y)} - 1\big|\, \vartheta(y)\, \mathrm{d}\lambda(y)\Bigg) \\
&\quad < \infty. \qquad \square
\end{aligned}$$

4.4. Existence results for pair interaction processes

The major obstacle in applying Theorem 3.12 to prove existence of pair interaction Gibbs processes with infinite range is to guarantee the validity of the two limit assumptions in that theorem. Some technical calculations show, however, that these limit relations hold with only minor adaptations of the assumptions made in Theorem 3.12 anyhow. The following lemma is essential. In this whole section, let $\mathcal{X}_b^*$ be as in Corollary 4.5.

Lemma 4.12. *Let I be some index set and, for each $i \in I$, let η_i be a point processes in $\mathbb{X}$ such that $\alpha_{\eta_i,m} \le (\vartheta\lambda)^m$ for each $m \in \mathbb{N}$, where $\vartheta : \mathbb{X} \to [0, \infty)$ is a fixed measurable map with*

$$\int_{\mathbb{X}} \big|\mathrm{e}^{-v(x,y)} - 1\big|\, \vartheta(y)\, \mathrm{d}\lambda(y) < \infty$$

for λ-a.e. $x \in \mathbb{X}$. Also assume that, for λ-a.e. $x \in \mathbb{X}$ and each $i \in I$,

$$\mathbb{P}\Big(\int_{\mathbb{X}} v^-(x,y)\,\mathrm{d}\eta_i(y) < \infty \Big) = 1.$$

Then, for every $B \in \mathcal{X}_b^*$,

$$\lim_{\ell \to \infty} \sup_{i \in I} \int_B \mathbb{E}\big|\kappa(x,\eta_i) - \kappa\big(x,(\eta_i)_{B_\ell}\big)\big|\,\mathrm{d}\lambda(x) = 0.$$

Proof. By Lemma 4.8 and Corollary 4.10 we have, for λ-a.e. $x \in \mathbb{X}$, all $i \in I$, and each $\ell \in \mathbb{N}$,

$$\kappa(x,\eta_i) - \kappa\big(x,(\eta_i)_{B_\ell}\big) = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k \setminus B_\ell^k} \prod_{j=1}^{k} \big(\mathrm{e}^{-v(x,y_j)} - 1\big)\,\mathrm{d}\eta_i^{(k)}(y_1,\dots,y_k)$$

almost surely, so

$$\begin{aligned}\mathbb{E}\big|\kappa(x,\eta_i) - \kappa\big(x,(\eta_i)_{B_\ell}\big)\big| &\le \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k \setminus B_\ell^k} \prod_{j=1}^{k} \big|\mathrm{e}^{-v(x,y_j)} - 1\big|\,\mathrm{d}\alpha_{\eta_i,k}(y_1,\dots,y_k) \\ &\le \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k \setminus B_\ell^k} \prod_{j=1}^{k} \big|\mathrm{e}^{-v(x,y_j)} - 1\big|\,\vartheta(y_1)\cdot\ldots\cdot\vartheta(y_k)\,\mathrm{d}\lambda^k(y_1,\dots,y_k).\end{aligned}$$

Fix $B \in \mathcal{X}_b^*$. Since

$$\begin{aligned}\int_B \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} \big(\big|\mathrm{e}^{-v(x,y_j)} - 1\big|\,\vartheta(y_j)\big)\,\mathrm{d}\lambda^k(y_1,\dots,y_k)\,\mathrm{d}\lambda(x) &\le \int_B \exp\Big(\int_{\mathbb{X}} \big|\mathrm{e}^{-v(x,y)} - 1\big|\,\vartheta(y)\,\mathrm{d}\lambda(y)\Big)\,\mathrm{d}\lambda(x) \\ &< \infty,\end{aligned}$$

dominated convergence yields

$$\begin{aligned}&\limsup_{\ell \to \infty} \sup_{i \in I} \int_B \mathbb{E}\big|\kappa(x,\eta_i) - \kappa\big(x,(\eta_i)_{B_\ell}\big)\big|\,\mathrm{d}\lambda(x) \\ &\quad\le \limsup_{\ell \to \infty} \int_B \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k \setminus B_\ell^k} \prod_{j=1}^{k} \big(\big|\mathrm{e}^{-v(x,y_j)} - 1\big|\,\vartheta(y_j)\big)\,\mathrm{d}\lambda^k(y_1,\dots,y_k)\,\mathrm{d}\lambda(x) \\ &\quad= 0.\end{aligned}$$

□

We are now able to provide a first general existence result in the pair potential setting, which serves as a foundation for all other results in this context. The proof consists of checking that the stated assumptions suffice to invoke Theorem 3.12.

Theorem 4.13. *Let $(\mathbb{X},\mathcal{X})$ be a substandard Borel space with localizing structure $B_1 \subset B_2 \subset \ldots$ and let λ be a locally finite measure on $\mathbb{X}$. Let $v : \mathbb{X} \times \mathbb{X} \to (-\infty,\infty]$ be measurable and symmetric such that $\inf_{x,y \in \mathbb{X}} v(x,y) \ge -l$ for a constant $l \ge 0$. Let κ be given by (4.1) and assume that $Z_{B_n}(\mathbf{0}) < \infty$ for each $n \in \mathbb{N}$. Suppose that*

$$\sup_{n \in \mathbb{N}} \frac{1}{Z_{B_n}(\mathbf{0})} \sum_{k=m}^{\infty} \frac{1}{(k-m)!} \int_{B_n^{k-m}} \exp\Big(- \sum_{1 \le i < j \le k} v(x_i,x_j)\Big)\,\mathrm{d}\lambda^{k-m}(x_{m+1},\dots,x_k) \le \vartheta(x_1)\cdot\ldots\cdot\vartheta(x_m)$$

for λ^m-a.e. $(x_1,\dots,x_m) \in \mathbb{X}^m$, each $m \in \mathbb{N}$, and some measurable, locally λ-integrable map $\vartheta : \mathbb{X} \to [0,\infty)$ which is such that, for λ-a.e. $x \in \mathbb{X}$,

$$\int_{\mathbb{X}} \big|\mathrm{e}^{-v(x,y)} - 1\big|\,\vartheta(y)\,\mathrm{d}\lambda(y) < \infty \quad \textit{as well as} \quad \int_{\mathbb{X}} v^-(x,y)\,\vartheta(y)\,\mathrm{d}\lambda(y) < \infty.$$

Then there exists a Gibbs process in $\mathbb{X}$ with pair potential v and reference measure λ.

Proof. First notice that, since v is bounded from below by $-l$, we have

$$\kappa(x,\mu) \leq \exp\big(l \cdot \mu(\mathbb{X})\big)$$

for all $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$. For each $n \in \mathbb{N}$, the assumption $Z_{B_n}(\mathbf{0}) < \infty$ ensures that a finite Gibbs process ξ_n with PI $\kappa^{(B_n,\mathbf{0})}$ exists. Recall from Remark 3.15 that their correlation functions are given by

$$\begin{aligned}\rho_{\xi_n,m}(x_1,\dots,x_m) &= \frac{1}{Z_{B_n}(\mathbf{0})}\sum_{k=m}^{\infty}\frac{1}{(k-m)!}\int_{B_n^{k-m}}\kappa_k(x_1,\dots,x_k,\mathbf{0})\,\mathrm{d}\lambda^{k-m}(x_{m+1},\dots,x_k)\\ &= \frac{1}{Z_{B_n}(\mathbf{0})}\sum_{k=m}^{\infty}\frac{1}{(k-m)!}\int_{B_n^{k-m}}\exp\Big(-\sum_{1\leq i<j\leq k}v(x_i,x_j)\Big)\,\mathrm{d}\lambda^{k-m}(x_{m+1},\dots,x_k)\end{aligned}$$

for λ^m-a.e. $(x_1,\dots,x_m) \in B_n^m$ and all $m,n \in \mathbb{N}$. Since $\rho_{\xi_n,m} = 0$ on $\mathbb{X}^m \setminus B_n^m$, we have

$$\sup_{n\in\mathbb{N}}\rho_{\xi_n,m}(x_1,\dots,x_m) \leq \vartheta(x_1)\cdot\ldots\cdot\vartheta(x_m)$$

for λ^m-a.e. $(x_1,\dots,x_m) \in \mathbb{X}^m$ and each $m \in \mathbb{N}$. We obtain from (2.3) that

$$\sup_{n\in\mathbb{N}} j_{\xi_n,B,m}(x_1,\dots,x_m) \leq \frac{1}{m!}\sup_{n\in\mathbb{N}}\rho_{\xi_n,m}(x_1,\dots,x_m) \leq \frac{\vartheta(x_1)\cdot\ldots\cdot\vartheta(x_m)}{m!}$$

for λ^m-a.e. $(x_1,\dots,x_m) \in \mathbb{X}^m$, each $m \in \mathbb{N}$, and every $B \in \mathcal{X}_b$. The bound on the correlation functions immediately gives $\alpha_{\xi_n,m} \leq (\vartheta\lambda)^m$, for each $n \in \mathbb{N}$, and by assumption we have

$$\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\int_{\mathbb{X}}v^-(x,y)\,\mathrm{d}\xi_n(y)\Big] = \sup_{n\in\mathbb{N}}\int_{\mathbb{X}}v^-(x,y)\,\mathrm{d}\alpha_{\xi_n,1}(y) \leq \int_{\mathbb{X}}v^-(x,y)\,\vartheta(y)\,\mathrm{d}\lambda(y) < \infty$$

for λ-a.e. $x \in \mathbb{X}$, which yields, for each such x and every $n \in \mathbb{N}$,

$$\mathbb{P}\Big(\int_{\mathbb{X}}v^-(x,y)\,\mathrm{d}\xi_n(y) < \infty\Big) = 1.$$

Invoking Lemma 4.12 gives

$$\lim_{\ell\to\infty}\sup_{n\in\mathbb{N}}\int_B\mathbb{E}\big|\kappa(x,\xi_n) - \kappa\big(x,(\xi_n)_{B_\ell}\big)\big|\,\mathrm{d}\lambda(x) = 0$$

for each $B \in \mathcal{X}_b^*$. It remains to prove that any of the limit processes η we obtain by applying Theorem 3.10 to $(\xi_n)_{n\in\mathbb{N}}$ also satisfies such a limit relation. To this end, notice first that by proceeding exactly as in the first part of the proof of Lemma 3.8, we get $\alpha_{\eta,m} \leq (\vartheta\lambda)^m$ for each $m \in \mathbb{N}$. Observe that, for λ-a.e. $x \in \mathbb{X}$,

$$\mathbb{E}\Big[\int_{\mathbb{X}}v^-(x,y)\,\mathrm{d}\eta(y)\Big] = \int_{\mathbb{X}}v^-(x,y)\,\mathrm{d}\alpha_{\eta,1}(y) \leq \int_{\mathbb{X}}v^-(x,y)\,\vartheta(y)\,\mathrm{d}\lambda(y) < \infty,$$

so, for each such x, we have $\mathbb{P}\big(\int_{\mathbb{X}}v^-(x,y)\,\mathrm{d}\eta(y) < \infty\big) = 1$. Lemma 4.12 gives

$$\lim_{\ell\to\infty}\int_B\mathbb{E}\big|\kappa(x,\eta) - \kappa(x,\eta_{B_\ell})\big|\,\mathrm{d}\lambda(x) = 0$$

for each $B \in \mathcal{X}_b^*$. The claim follows from Theorem 3.12. □

Before we start to consider explicit examples, we discuss a special case of Theorem 4.13, namely the setting of non-negative pair potentials. Such potentials are often called *repulsive*. The following corollary is a very slight generalization of Theorem B.1 of Jansen (2019) in terms of the admissible state space.

Corollary 4.14. *Let $(\mathbb{X}, \mathcal{X})$ be a substandard Borel space with localizing structure $B_1 \subset B_2 \subset \dots$ and let λ be a locally finite measure on $\mathbb{X}$. Let $v : \mathbb{X} \times \mathbb{X} \to [0, \infty]$ be measurable and symmetric. Assume that, for λ-a.e. $x \in \mathbb{X}$,*

$$\int_{\mathbb{X}} \left(1 - \mathrm{e}^{-v(x,y)}\right) \mathrm{d}\lambda(y) < \infty.$$

Then there exists a Gibbs process in $\mathbb{X}$ with pair potential v and reference measure λ.

Proof. Since v is non-negative, we have $v^- \equiv 0$ and, for all $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$,

$$\kappa(x, \mu) = \exp\left(- \int_{\mathbb{X}} v(x, y) \, \mathrm{d}\mu(y) \right) \leq 1.$$

With this bound we obtain $Z_{B_n}(\mathbf{0}) \leq \mathrm{e}^{\lambda(B_n)} < \infty$, for each $n \in \mathbb{N}$. Moreover, we have

$$\begin{aligned}
&\frac{1}{Z_{B_n}(\mathbf{0})} \sum_{k=m}^{\infty} \frac{1}{(k-m)!} \int_{B_n^{k-m}} \exp\left(- \sum_{1 \leq i < j \leq k} v(x_i, x_j) \right) \mathrm{d}\lambda^{k-m}(x_{m+1}, \dots, x_k) \\
&\quad \leq \frac{1}{Z_{B_n}(\mathbf{0})} \sum_{k=m}^{\infty} \frac{1}{(k-m)!} \int_{B_n^{k-m}} \exp\left(- \sum_{m+1 \leq i < j \leq k} v(x_i, x_j) \right) \mathrm{d}\lambda^{k-m}(x_{m+1}, \dots, x_k) \\
&\quad = \frac{1}{Z_{B_n}(\mathbf{0})} \sum_{k=m}^{\infty} \frac{1}{(k-m)!} \int_{B_n^{k-m}} \kappa_{k-m}(x_{m+1}, \dots, x_k, \mathbf{0}) \, \mathrm{d}\lambda^{k-m}(x_{m+1}, \dots, x_k) \\
&\quad = \frac{1}{Z_{B_n}(\mathbf{0})} \cdot \left(1 + \sum_{k=1}^{\infty} \int_{B_n^k} \kappa_k(x_1, \dots, x_k, \mathbf{0}) \, \mathrm{d}\lambda^k(x_1, \dots, x_k) \right) \\
&\quad = 1
\end{aligned}$$

uniformly in $n \in \mathbb{N}$ for all $x_1, \dots, x_m \in \mathbb{X}$ and $m \in \mathbb{N}$. Choosing $\vartheta \equiv 1$, Theorem 4.13 yields the claim. □

4.5. EXAMPLES

We conclude this chapter on pair interaction processes by discussing a number of examples. As we mentioned at the beginning of this chapter, the existence result for pair interaction processes, Theorem 4.13, despite allowing for more general state spaces, does not lend itself immediately to proving existence of new classes of processes, but rather serves as an overlying structure for previous results. For instance, it includes the general result on non-negative pair interactions by Jansen (2019) as Corollary 4.14 shows. The generality in the state space allows to include marked Gibbs processes as well, for instance Example 2 of Rœlly and Zass (2020) and the Gibbsian diffusion processes of Zass (2022). Another setting, which we introduce below, allows for negative parts in the potential, but requires lower regularity and superstability, cf. Ruelle (1970). This classical setting is also included in our abstraction.

The first example we consider are hard core processes in metric spaces. In fact we have already introduced them as hard spheres in Example 3.21 and their existence is guaranteed by Corollary 3.20 due to the finite range. Still, reconsidering this example under the guise of a pair interaction is illuminating.

Example 4.15 (Hard core process). Let $\mathbb{X}$ be a (Borel subset of a) complete separable metric space, whose metric is denoted by d, and let λ be a locally finite measure on $\mathbb{X}$. Define $v : \mathbb{X} \times \mathbb{X} \to [0, \infty]$ as

$$v(x, y) = \infty \cdot \mathbb{1}\{d(x, y) \leq R\}$$

for some fixed $R > 0$. Then

$$\int_{\mathbb{X}} \big(1 - \mathrm{e}^{-v(x,y)}\big) \,\mathrm{d}\lambda(y) = \int_{\mathbb{X}} \mathbb{1}\big\{d(x,y) \leq R\big\} \,\mathrm{d}\lambda(y) = \lambda\big(B(x,R)\big) < \infty$$

for all $x \in \mathbb{X}$, so Corollary 4.14 provides the existence of a corresponding Gibbs process. ○

For the remainder of this section, we consider the state space $\mathbb{X} = \mathbb{R}^d$, for $d \in \mathbb{N}$, endowed with the Lebesgue measure $\mathcal{L}^d$. We denote by $|\cdot|$ the Euclidean norm.

As opposed to the hard core processes in the previous example, the following soft core process has an infinite interaction range.

Example 4.16 (Very soft core process). Consider the pair potential $v : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ given by

$$v(x,y) = -\log\left(1 - \exp\left(-\frac{|x-y|^2}{c}\right)\right), \quad x, y \in \mathbb{R}^d,$$

where $c > 0$. In this case, we have

$$\int_{\mathbb{R}^d} \big(1 - \mathrm{e}^{-v(x,y)}\big) \,\mathrm{d}\mathcal{L}^d(y) = \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{c}\right) \mathrm{d}\mathcal{L}^d(y) = \int_{\mathbb{R}^d} \exp\left(-\frac{|y|^2}{c}\right) \mathrm{d}\mathcal{L}^d(y) = \sqrt{(c\cdot\pi)^d} < \infty$$

for all $x \in \mathbb{R}^d$, so Corollary 4.14 provides the existence of a corresponding Gibbs process whose reference measure can be any multiple of $\mathcal{L}^d$. ○

We consider one further example of a non-negative pair interaction, namely the Gaussian core potential due to Stillinger (1976).

Example 4.17 (Gaussian core model). The Gaussian core pair potential is given as

$$v(x,y) = a \cdot \exp\left(-\frac{|x-y|^2}{c}\right), \quad x, y \in \mathbb{R}^d,$$

where $a, c > 0$. Using that $1 - \mathrm{e}^{-r} \leq r$ for all $r \in \mathbb{R}$, we get, for any $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \big(1 - \mathrm{e}^{-v(x,y)}\big) \,\mathrm{d}\mathcal{L}^d(y) \leq \int_{\mathbb{R}^d} v(x,y) \,\mathrm{d}\mathcal{L}^d(y) = a \int_{\mathbb{R}^d} \exp\left(-\frac{|y|^2}{c}\right) \mathrm{d}\mathcal{L}^d(y) = a \cdot \sqrt{(c\cdot\pi)^d} < \infty,$$

so Corollary 4.14 yields the existence of a Gibbs process with pair potential v and any multiple of $\mathcal{L}^d$ as a reference measure. ○

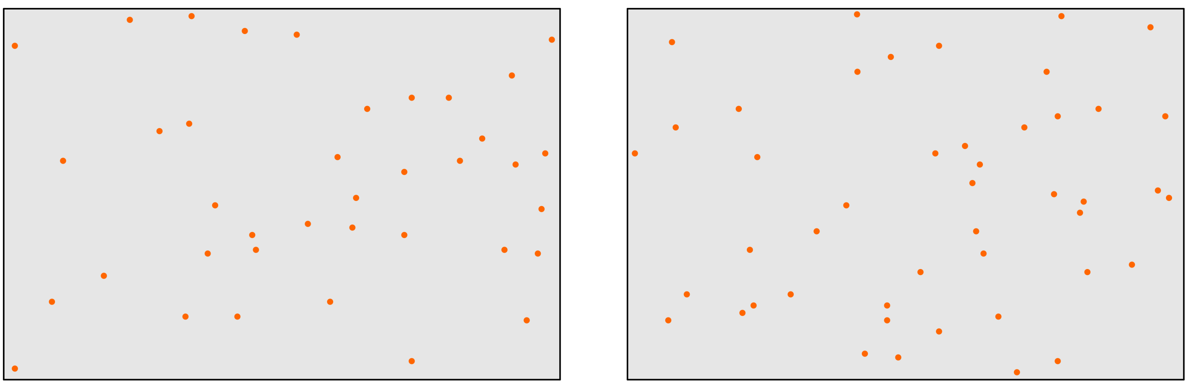


Figure 4.1.: Realizations of a very soft core process with parameter $c = 0.01$ (left) and a Gaussian core process with parameters $a = 1$ and $c = 0.01$ (right). Both realizations correspond to processes with reference measure $100 \cdot \mathcal{L}^2$ restricted to $[0, 1.5] \times [0, 1]$.

While Corollary 4.14, in providing a handle for non-negative pair potentials, is a good starting point for dealing with pair interactions, it does not include important cases like the Lennard-Jones or Morse potential. However, we have not yet used the full power of Theorem 4.13. In the following we show that it also includes the diffusion processes of Zass (2022) as well as the existence result discussed by Ruelle (1970).

Example 4.18 (Gibbsian diffusion processes). Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a Banach space endowed with a finite measure Q. Consider a measurable and symmetric function $v : (\mathbb{R}^d \times \mathbb{B})^2 \to (-\infty, \infty]$ such that

$$\sum_{1 \leq i < j \leq m} v(x_i, b_i, x_j, b_j) \geq -s \cdot m \tag{4.3}$$

for all $x_1, \ldots, x_m \in \mathbb{R}^d$ and $b_1, \ldots, b_m \in \mathbb{B}$, each $m \in \mathbb{N}$, and some constant $s \geq 0$. This corresponds to the stability assumption which is discussed below. Moreover, assume that, for each $x, x' \in \mathbb{R}^d$ and $b, b' \in \mathbb{B}$,

$$v(x, b, x', b') = 0$$

whenever $|x - x'| > R + \|b\|_{\mathbb{B}} + \|b'\|_{\mathbb{B}}$, where $R \geq 0$ is fixed. Finally, assume a local stability bound in that

$$\sup_{\mu \in \mathbf{N}_f(\mathbb{R}^d \times \mathbb{B})} \kappa(x, b, \mu) \leq \vartheta(b)$$

for $(\mathcal{L}^d \otimes Q)$-a.e. $(x, b) \in \mathbb{R}^d \times \mathbb{B}$, with κ as in (4.1), and where $\vartheta : \mathbb{B} \to [0, \infty)$ is measurable and satisfies

$$\int_{\mathbb{B}} \big(1 + \|b\|_{\mathbb{B}}^d\big)\, \vartheta(b) \,\mathrm{d}Q(b) < \infty. \tag{4.4}$$

Note that these assumptions are weaker than Assumptions 1 and 2 of Zass (2022) if his self interaction term is put to 0 (including a non-zero self interaction is possible in our setting as well but requires a few adaptations). In the given setting, a Gibbs process in $\mathbb{R}^d \times \mathbb{B}$ with pair interaction v exists by Theorem 4.13. Indeed, the stability assumption (4.3) implies that v is bounded from below by $-2 \cdot s$ and it also implies that the required partition functions are finite, a calculation carried out in the proof of Corollary 4.22 below. The local stability property of κ yields the bound on the correlation function according to Remark 3.15. Moreover, the range assumption, the fact that v is bounded from below, and (4.4) imply

$$\begin{aligned}
&\int_{\mathbb{R}^d \times \mathbb{B}} \big|\mathrm{e}^{-v(x,b,x',b')} - 1\big|\, \vartheta(b') \,\mathrm{d}(\mathcal{L}^d \otimes Q)(x', b') \\
&\quad \leq (1 + \mathrm{e}^{2s}) \int_{\mathbb{R}^d \times \mathbb{B}} \mathbb{1}\big\{|x - x'| \leq R + \|b\|_{\mathbb{B}} + \|b'\|_{\mathbb{B}}\big\}\, \vartheta(b') \,\mathrm{d}(\mathcal{L}^d \otimes Q)(x', b') \\
&\quad = (1 + \mathrm{e}^{2s}) \cdot \mathcal{L}^d\big(B(0,1)\big) \int_{\mathbb{B}} \big(R + \|b\|_{\mathbb{B}} + \|b'\|_{\mathbb{B}}\big)^d\, \vartheta(b') \,\mathrm{d}Q(b') \\
&\quad < \infty
\end{aligned}$$

as well as

$$\int_{\mathbb{R}^d \times \mathbb{B}} v^-(x, b, x', b')\, \vartheta(b') \,\mathrm{d}(\mathcal{L}^d \otimes Q)(x', b') \leq 2s \cdot \mathcal{L}^d\big(B(0,1)\big) \int_{\mathbb{B}} \big(R + \|b\|_{\mathbb{B}} + \|b'\|_{\mathbb{B}}\big)^d\, \vartheta(b') \,\mathrm{d}Q(b') < \infty$$

for all $x \in \mathbb{R}^d$ and $b \in \mathbb{B}$. This shows that all assumptions of Theorem 4.13 are satisfied.

In one special case of the given setting, $\mathbb{B}$ is the space of all continuous functions $f : [0, 1] \to \mathbb{R}^d$ with $f(0) = 0$. Each point (x, f) of the resulting Gibbs processes in $\mathbb{R}^d \times \mathbb{B}$ is then interpreted as a diffusion starting in x with trajectory $x + f(\cdot)$. The examples given in Section 2 of Zass (2022) are included in this setting. ○

Turning to the classical setting of Ruelle (1970) we assume, with a slight ambiguity in notation, that

$v : \mathbb{R}^d \to (-\infty, \infty]$ is a measurable function with $v(x) = v(-x)$ for all $x \in \mathbb{R}^d$, and we define

$$v : \mathbb{R}^d \times \mathbb{R}^d \to (-\infty, \infty], \quad v(x, y) = v(y - x).$$

Definition 4.19 (Superstability and lower regularity).

- The map v is called *superstable* if $v = v^* + v^{**}$, where $v^* : \mathbb{R}^d \to (-\infty, \infty]$ is a measurable function with $v^*(x) = v^*(-x)$ (for all $x \in \mathbb{R}^d$) which is *stable*, that is,

$$\sum_{1 \leq i < j \leq m} v^*(x_j - x_i) \geq -s \cdot m$$

  for all $x_1, \ldots, x_m \in \mathbb{R}^d$, each $m \in \mathbb{N}$, and a constant $s \geq 0$, and where $v^{**} : \mathbb{R}^d \to [0, \infty)$ is continuous with $v^{**}(0) > 0$.

- The map v is called *lower regular* if there exists a decreasing function $\varphi : [0, \infty) \to [0, \infty)$ with

$$\int_0^\infty r^{d-1} \varphi(r) \, \mathrm{d}r < \infty$$

  such that $v(x) \geq -\varphi(|x|)$ for all $x \in \mathbb{R}^d$.

Notice that a bit more general definitions of these properties are given in Section 1 of Ruelle (1970).

Remark 4.20. The stability assumption on pair potentials is one of the most basic assumptions in the context of Gibbs processes. In fact, the assumption can be regarded as necessary to discuss such processes. Indeed, in Section 2.5 we established that finite Gibbs processes only exist if the partition functions are finite. In Proposition 3.2.2 of Ruelle (1969) it is proven that for real-valued and upper semicontinuous v, stability is equivalent to the partition functions being finite and hence equivalent to the existence of finite Gibbs processes.

The lower stability condition yields

$$\inf_{x \in \mathbb{R}^d} v(x) \geq \inf_{x \in \mathbb{R}^d} \big(-\varphi(|x|) \big) = -\sup_{x \in \mathbb{R}^d} \varphi(|x|) \geq -\varphi(0)$$

with $\varphi(0) \geq 0$. Hence, v is bounded from below. This also follows if v is stable, as $v(x) = v(x - 0) \geq -2s$.

In his Proposition 2.6, Ruelle (1970) shows that superstability and lower regularity of v ensure the classical Ruelle bound (Definition 3.7) on the correlation function. This fact is used in the upcoming Corollary 4.22 to cover the bound in Theorem 4.13. ○

The following Proposition gives straightforward sufficient conditions for a pair potential to be both superstable and lower regular. These assumptions are the result of several refinements throughout the 1960s and prove quite handy in working with the given properties. The result is stated as Proposition 1.4 by Ruelle (1970), referring to Proposition 3.2.8 of Ruelle (1969) for a proof. Ruelle (1969), in turn, gives credit to Dobrushin (1964) and Fisher and Ruelle (1966). Several other references should also be mentioned. Related assumptions on pair potentials were already considered by Ruelle (1963b) and Fisher (1964), paving the way for further considerations. Dobrushin (1969) also works with similar assumptions, for instance in his Equation (4.1), but focusing on a hard core (instead of a bound by a function φ_1 as below) due to the applications that are considered. The papers by Dobrushin (1964) and Dobrushin and Minlos (1967) collect various reasonable assumptions on pair potentials, virtually including superstability and lower regularity (though it appears these designations were only coined by Ruelle a bit later), particularly in relation to the existence of infinite volume Gibbs distributions.

Proposition 4.21. *Let $v : \mathbb{R}^d \to (-\infty, \infty]$ be measurable with $v(x) = v(-x)$ for all $x \in \mathbb{R}^d$. Assume that $\inf_{x \in \mathbb{R}^d} v(x) \geq -l$ for a constant $l \geq 0$ and that there exist $0 < r_1 < r_2 < \infty$ as well as decreasing functions*

$\varphi_1 : [0, r_1] \to [0, \infty]$ and $\varphi_2 : [r_2, \infty) \to [0, \infty)$ with

$$\int_0^{r_1} r^{d-1} \, \varphi_1(r) \, \mathrm{d}r = \infty \quad \text{and} \quad \int_{r_2}^{\infty} r^{d-1} \, \varphi_2(r) \, \mathrm{d}r < \infty$$

such that $v(x) \geq \varphi_1(|x|)$ for $|x| \leq r_1$ as well as $|v(x)| \leq \varphi_2(|x|)$ for $|x| \geq r_2$. Then v is superstable and lower regular.

The following result is a modern point process theoretic version of Theorem 5.5 of Ruelle (1970).

Corollary 4.22. *Let $v : \mathbb{R}^d \to (-\infty, \infty]$ be a measurable function with $v(x) = v(-x)$ for all $x \in \mathbb{R}^d$. Moreover, suppose that v is superstable and lower regular, and satisfies*

$$\int_{\mathbb{R}^d} \left| \mathrm{e}^{-v(y)} - 1 \right| \mathrm{d}\mathcal{L}^d(y) < \infty.$$

Then there exists a Gibbs process with pair interaction v and reference measure $\lambda = \gamma \cdot \mathcal{L}^d$ for any $\gamma > 0$.

Proof. As detailed in Remark 4.20, the lower regularity of v implies that v is bounded from below. Since v is superstable, hence stable, there exists a constant $s \geq 0$ such that

$$\kappa_m(x_1, \dots, x_m, \mathbf{0}) = \exp\Bigg(- \sum_{1 \leq i < j \leq m} v(x_j - x_i) \Bigg) \leq \exp\big(s \cdot m\big)$$

for all $x_1, \dots, x_m \in \mathbb{R}^d$ and each $m \in \mathbb{N}$. Therefore, we have, for each $n \in \mathbb{N}$,

$$Z_{B_n}(\mathbf{0}) \leq 1 + \sum_{m=1}^{\infty} \frac{\gamma^m}{m!} \cdot \mathrm{e}^{s \cdot m} \cdot \mathcal{L}^d(B_n)^m = \exp\Big(\gamma \cdot \mathrm{e}^s \cdot \mathcal{L}^d(B_n)\Big) < \infty,$$

where we choose the localizing structure $B_n = B(0, n)$. By Proposition 2.6 of Ruelle (1970), superstability and lower regularity of v imply

$$\sup_{n \in \mathbb{N}} \frac{1}{Z_{B_n}(\mathbf{0})} \sum_{k=m}^{\infty} \frac{\gamma^{k-m}}{(k-m)!} \int_{B_n^{k-m}} \exp\Bigg(- \sum_{1 \leq i < j \leq k} v(x_j - x_i) \Bigg) \mathrm{d}(\mathcal{L}^d)^{k-m}(x_{m+1}, \dots, x_k) \leq \vartheta^m$$

for $(\mathcal{L}^d)^m$-a.e. $(x_1, \dots, x_m) \in (\mathbb{R}^d)^m$, each $m \in \mathbb{N}$, and a constant $\vartheta \geq 1$. We also have, for any $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \left| \mathrm{e}^{-v(y-x)} - 1 \right| \mathrm{d}\mathcal{L}^d(y) = \int_{\mathbb{R}^d} \left| \mathrm{e}^{-v(y)} - 1 \right| \mathrm{d}\mathcal{L}^d(y) < \infty$$

as well as (due to the lower regularity and using hyperspherical coordinates)

$$\int_{\mathbb{R}^d} v^-(y - x) \, \mathrm{d}\mathcal{L}^d(y) = \int_{\mathbb{R}^d} v^-(y) \, \mathrm{d}\mathcal{L}^d(y) \leq \int_{\mathbb{R}^d} \varphi(|y|) \, \mathrm{d}\mathcal{L}^d(y) = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_0^{\infty} r^{d-1} \, \varphi(r) \, \mathrm{d}r < \infty,$$

where Γ is the gamma function. Theorem 4.13 yields the claim. □

We conclude our list of examples by discussing the Lennard-Jones potential (named after Jones, 1924a,b) as well as the Morse potential (due to Morse, 1929). We discuss them in full detail with regard to the properties of pair potentials introduces above. In the literature, the author could only find rigorous considerations on the Morse potential in dimension three. Below, arbitrary dimensions are included.

Example 4.23 (Lennard-Jones interaction). Fix $d \in \mathbb{N}$ with $d \leq 5$. Define

$$v(x) = \frac{a}{|x|^{12}} - \frac{b}{|x|^6}, \quad x \in \mathbb{R}^d \setminus \{0\},$$

where $a > 0$ and $b \in \mathbb{R}$, with the interesting case being $b > 0$. We put $v(0) = \infty$. A somewhat tedious calculation, which is completely detailed at the end of this example, gives (for any value of $b \in \mathbb{R}$)

$$\int_{\mathbb{R}^d} \left| \mathrm{e}^{-v(x)} - 1 \right| \mathrm{d}\mathcal{L}^d(x) < \infty.$$

For $b \leq 0$ we have $v(\cdot) \geq 0$ and the existence of a Gibbs process with pair interaction v and reference measure $\gamma \cdot \mathcal{L}^d$, for $\gamma > 0$, is guaranteed by Corollary 4.14. If $b > 0$, simple calculus gives $\inf_{x \in \mathbb{R}^d} v(x) = -\frac{b^2}{4a}$. Moreover, the two following choices can be made.

- Let $r_1 = \sqrt[6]{\frac{a}{2b}}$ and $\varphi_1 : [0, r_1] \to [0, \infty]$ with $\varphi_1(r) = \frac{a}{2r^{12}}$, putting $\varphi_1(0) = \infty$. For $x \in \mathbb{R}^d$ with $0 < |x| \leq r_1$ we have

$$v(x) = \frac{1}{|x|^{12}} \left(a - b \cdot |x|^6 \right) \geq \frac{1}{|x|^{12}} \left(a - b \cdot \frac{a}{2b} \right) = \frac{a}{2 \cdot |x|^{12}} = \varphi_1(|x|),$$

  and the function φ_1 is clearly decreasing and satisfies

$$\int_0^{r_1} r^{d-1} \varphi_1(r) \, \mathrm{d}r = \frac{a}{2} \int_0^{r_1} r^{d-13} \, \mathrm{d}r = \infty,$$

  where we use that $d \leq 12$.

- Fix any $r_2 > r_1$ and let $\varphi_2 : [r_2, \infty) \to [0, \infty)$ be given as $\varphi_2(r) = \frac{a}{r^{12}} + \frac{b}{r^6}$. For $x \in \mathbb{R}^d$ we have $|v(x)| \leq \varphi_2(|x|)$ and the function φ_2 is decreasing and satisfies

$$\int_{r_2}^{\infty} r^{d-1} \varphi_2(r) \, \mathrm{d}r = \int_{r_2}^{\infty} \left(a r^{d-13} + b r^{d-7} \right) \mathrm{d}r < \infty,$$

  using that $d \leq 5$.

Proposition 4.21 implies that v is superstable and lower regular, so Corollary 4.22 guarantees the existence of a Gibbs process with pair interaction v and reference measure $\gamma \cdot \mathcal{L}^d$.

For completeness, we give a full account of the integrability assumption on v (with arbitrary b). A simple transformation with hyperspherical coordinates in $\mathbb{R}^d$ gives

$$\begin{aligned}
\int_{\mathbb{R}^d} \left| \mathrm{e}^{-v(x)} - 1 \right| \mathrm{d}\mathcal{L}^d(x) &= \int_{\mathbb{R}^d} \left| \exp\left(- \frac{a}{|x|^{12}} + \frac{b}{|x|^6} \right) - 1 \right| \mathrm{d}\mathcal{L}^d(x) \\
&= \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int_0^{\infty} r^{d-1} \left| \exp\left(- \frac{a}{r^{12}} + \frac{b}{r^6} \right) - 1 \right| \mathrm{d}r \\
&= \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int_0^{\infty} s^{-1-d} \left| \exp\left(- a s^{12} + b s^6 \right) - 1 \right| \mathrm{d}s.
\end{aligned}$$

For $s \geq \sqrt[6]{|b| \,/\, a}$ we have $\frac{b}{a} \leq s^6$ and hence $-as^{12} + bs^6 \leq 0$. Therefore,

$$\int_{\sqrt[6]{|b|/a}}^{\infty} s^{-1-d} \cdot \left| \exp\left(- a s^{12} + b s^6 \right) - 1 \right| \mathrm{d}s \leq \int_{\sqrt[6]{|b|/a}}^{\infty} s^{-1-d} \, \mathrm{d}s = \frac{\left(\sqrt[6]{a \,/\, |b|} \right)^d}{d} < \infty.$$

We also have

$$\begin{aligned}
\int_0^1 s^{-1-d} \left| \exp\left(- a s^{12} + b s^6 \right) - 1 \right| \mathrm{d}s &\leq \int_0^1 s^{-1-d} \sum_{k=1}^{\infty} \frac{\left(a s^{12} + |b| \cdot s^6 \right)^k}{k!} \, \mathrm{d}s \\
&\leq \int_0^1 s^{-1-d} \sum_{k=1}^{\infty} \frac{(a + |b|)^k \cdot s^{6k}}{k!} \, \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{(a+|b|)^k}{k!} \cdot \frac{1}{6k-d} \\
&\leq \exp\big(a+|b|\big) < \infty,
\end{aligned}
$$

using once more that $d \leq 5$. If $\frac{|b|}{a} \leq 1$ this already gives the finiteness of the integral in consideration. If $\frac{|b|}{a} > 1$, we also confirm this claim, since

$$
\int_1^{\sqrt[6]{|b|/a}} s^{-1-d} \big| \exp\big(-as^{12} + bs^6\big) - 1 \big| \, \mathrm{d}s < \infty
$$

is trivially true. ○

Example 4.24 (**Morse interaction**). Let $d \in \mathbb{N}$ be arbitrary. We consider the pair potential

$$
v(x) = b \cdot \big(\mathrm{e}^{-2a(|x|-R)} - 2\mathrm{e}^{-a(|x|-R)}\big), \quad x \in \mathbb{R}^d,
$$

where $a, b, R > 0$ are parameters with the additional assumption that

$$
\mathrm{e}^{aR} > 4^{\frac{d+1}{2}}. \tag{4.5}
$$

This condition already appeared in Ruelle (1963a), where only the case $d = 3$ is considered. Simple calculus yields $\inf_{x \in \mathbb{R}^d} v(x) = -b$ and, using hyperspherical coordinates, we obtain

$$
\int_{\mathbb{R}^d} \big|\mathrm{e}^{-v(x)} - 1\big| \, \mathrm{d}\mathcal{L}^d(x) = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_0^{\infty} r^{d-1} \big| \exp\big(2b \cdot \mathrm{e}^{-a(r-R)} - b \cdot \mathrm{e}^{-2a(r-R)}\big) - 1 \big| \, \mathrm{d}r. \tag{4.6}
$$

Note that $2\mathrm{e}^{-a(r-R)} - \mathrm{e}^{-2a(r-R)} \geq 0$ if, and only if, $r \geq R - \frac{\log(2)}{a}$. We thus put $r^* = \max\big\{0, R - \frac{\log(2)}{a}\big\}$ and split the integral on the right hand side of (4.6) into an integral over $[0, r^*]$, which is trivially finite, and into

$$
\begin{aligned}
\int_{r^*}^{\infty} r^{d-1} \Big(\exp\big(2b \cdot \mathrm{e}^{-a(r-R)} - b \cdot \mathrm{e}^{-2a(r-R)}\big) - 1 \Big) \, \mathrm{d}r &= \int_{r^*}^{\infty} r^{d-1} \sum_{k=1}^{\infty} \frac{b^k}{k!} \big(2\mathrm{e}^{-a(r-R)} - \mathrm{e}^{-2a(r-R)}\big)^k \, \mathrm{d}r \\
&\leq \sum_{k=1}^{\infty} \frac{b^k}{k!} \int_{r^*}^{\infty} r^{d-1} \big(2\mathrm{e}^{-a(r-R)}\big)^k \, \mathrm{d}r \\
&\leq \sum_{k=1}^{\infty} \frac{(2b \cdot \mathrm{e}^{aR})^k}{k!} \int_0^{\infty} r^{d-1} \, \mathrm{e}^{-akr} \, \mathrm{d}r.
\end{aligned}
$$

By an iterative application of integration by parts, we further calculate the right hand side as

$$
\sum_{k=1}^{\infty} \frac{(2b \cdot \mathrm{e}^{aR})^k}{k!} \cdot \frac{(d-1)!}{(ak)^d} \leq \frac{(d-1)!}{a^d} \sum_{k=0}^{\infty} \frac{(2b \cdot \mathrm{e}^{aR})^k}{k!} = \frac{(d-1)!}{a^d} \cdot \exp\big(2b \cdot \mathrm{e}^{aR}\big),
$$

hence the integral on the left hand side of (4.6) is finite. Next, observe that, for all $x \in \mathbb{R}^d$,

$$
v(x) \geq -2b \cdot \mathrm{e}^{-a(|x|-R)}.
$$

Upon defining the decreasing function $\varphi(r) = 2b \cdot \mathrm{e}^{-a(r-R)}$ for $r \geq 0$, we have $v(x) \geq -\varphi(|x|)$ for all $x \in \mathbb{R}^d$, and φ satisfies

$$
\int_0^{\infty} r^{d-1} \, \varphi(r) \, \mathrm{d}r = 2b \cdot \mathrm{e}^{aR} \int_0^{\infty} r^{d-1} \, \mathrm{e}^{-ar} \, \mathrm{d}r = 2b \cdot \mathrm{e}^{aR} \cdot \frac{(d-1)!}{a^d} < \infty.
$$

Therefore, v is lower regular. To show that Corollary 4.22 can be applied with the Morse potential, it remains to prove that v is superstable. Observe that if there existed a function φ_1 as in Proposition 4.21, then

$$\infty = \int_0^{r_1} r^{d-1}\,\varphi_1(r)\,\mathrm{d}r = \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}}\int_{B(0,r_1)} \varphi_1(|x|)\,\mathrm{d}\mathcal{L}^d(x) \le \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}}\int_{B(0,r_1)} v(x)\,\mathrm{d}\mathcal{L}^d(x)$$

which is clearly a contradiction, as $v \in L^1(\mathbb{R}^d, \mathcal{L}^d)$. Consequently, we cannot rely on Proposition 4.21 to prove superstability, but will resort to the definition of the term. Assumption (4.5) on the parameters of the model ensures that

$$\varepsilon = \frac{\mathrm{e}^{aR}}{4^{\frac{d+1}{2}}} - 1 > 0.$$

For $x \in \mathbb{R}^d$ we define $v^{**}(x) = 2b\varepsilon \cdot \mathrm{e}^{-a(|x|-R)}$ as well as

$$v^*(x) = v(x) - v^{**}(x) = b \cdot \mathrm{e}^{2aR} \cdot \mathrm{e}^{-2a|x|} - 2b(1+\varepsilon)\cdot \mathrm{e}^{aR}\cdot \mathrm{e}^{-a|x|}.$$

Then clearly $v = v^* + v^{**}$, where v^{**} maps $\mathbb{R}^d$ continuously to $[0,\infty)$ with $v^{**}(0) = 2b\varepsilon \cdot \mathrm{e}^{aR} > 0$ and where $v^* : \mathbb{R}^d \to \mathbb{R}$ is measurable and satisfies $v^*(x) = v^*(-x)$ for all $x \in \mathbb{R}^d$. It remains to establish that v^* is stable. As $v^* \in L^1(\mathbb{R}^d, \mathcal{L}^d)$, the Fourier transform of v^* is well-defined, so we consider (sticking to Definition 3.8.1 of Bogachev, 2007a)

$$\widehat{v^*}(y) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle y,x\rangle}\, v^*(x)\,\mathrm{d}\mathcal{L}^d(x),$$

where i denotes the imaginary unit and $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{R}^d$. With the transformation formula we get

$$\begin{aligned}\widehat{v^*}(y) &= \frac{(2\pi)^d}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} \mathrm{e}^{-2\pi\mathrm{i}\langle y,x\rangle}\, v^*(2\pi x)\,\mathrm{d}\mathcal{L}^d(x)\\ &= (2\pi)^{d/2}\cdot b\cdot \mathrm{e}^{aR}\bigg(\mathrm{e}^{aR}\int_{\mathbb{R}^d}\mathrm{e}^{-2\pi\mathrm{i}\langle y,x\rangle}\,\mathrm{e}^{-2\pi\cdot 2a|x|}\,\mathrm{d}\mathcal{L}^d(x) - 2(1+\varepsilon)\int_{\mathbb{R}^d}\mathrm{e}^{-2\pi\mathrm{i}\langle y,x\rangle}\,\mathrm{e}^{-2\pi\cdot a|x|}\,\mathrm{d}\mathcal{L}^d(x)\bigg).\end{aligned}$$

The integrals on the right hand side are (roughly) Fourier transforms of Abel kernels and thus can be written explicitly in terms of Poisson kernels. To be precise, Theorem 1.14 of Stein and Weiss (1971) yields

$$\begin{aligned}\widehat{v^*}(y) &= (2\pi)^{d/2}\cdot b\cdot\mathrm{e}^{aR}\left(\mathrm{e}^{aR}\cdot\frac{\Gamma\big(\frac{d+1}{2}\big)}{\pi^{\frac{d+1}{2}}}\cdot\frac{2a}{(4a^2+|y|^2)^{\frac{d+1}{2}}} - 2(1+\varepsilon)\cdot\frac{\Gamma\big(\frac{d+1}{2}\big)}{\pi^{\frac{d+1}{2}}}\cdot\frac{a}{(a^2+|y|^2)^{\frac{d+1}{2}}}\right)\\ &= 2ab\cdot 2^{d/2}\mathrm{e}^{aR}\cdot\frac{\Gamma\big(\frac{d+1}{2}\big)}{\sqrt{\pi}}\left(\frac{\mathrm{e}^{aR}}{(4a^2+|y|^2)^{\frac{d+1}{2}}} - \frac{1+\varepsilon}{(a^2+|y|^2)^{\frac{d+1}{2}}}\right).\end{aligned}$$

Hence, we have $\widehat{v^*} \in L^1(\mathbb{R}^d, \mathcal{L}^d)$. Since v^* is continuous, Corollary 3.8.12 of Bogachev (2007a) allows us to invert the Fourier transform, in the sense that, for $x \in \mathbb{R}^d$,

$$v^*(x) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}\langle x,y\rangle}\,\widehat{v^*}(y)\,\mathrm{d}\mathcal{L}^d(y).$$

Moreover, a simple calculation shows that $\max_{r\ge 0}\frac{4a^2+r^2}{a^2+r^2} = 4$, which gives, for any $y \in \mathbb{R}^d$,

$$(1+\varepsilon)\cdot\left(\frac{4a^2+|y|^2}{a^2+|y|^2}\right)^{\frac{d+1}{2}} \le (1+\varepsilon)\cdot 4^{\frac{d+1}{2}} = \mathrm{e}^{aR}$$

or, equivalently,

$$\frac{1+\varepsilon}{(a^2+|y|^2)^{\frac{d+1}{2}}} \le \frac{\mathrm{e}^{aR}}{(4a^2+|y|^2)^{\frac{d+1}{2}}}.$$

We conclude that $\widehat{v^*}(y) \ge 0$ for all $y \in \mathbb{R}^d$. Combining our observations, it follows that, for $x_1,\ldots,x_m \in \mathbb{R}^d$

and $m \in \mathbb{N}$,

$$\sum_{j,k=1}^{m} v^*(x_k - x_j) = \sum_{j,k=1}^{m} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle x_k - x_j, y\rangle}\, \widehat{v^*}(y)\, \mathrm{d}\mathcal{L}^d(y) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left| \sum_{j=1}^{m} \mathrm{e}^{\mathrm{i}\langle x_j, y\rangle} \right|^2 \widehat{v^*}(y)\, \mathrm{d}\mathcal{L}^d(y),$$

where $|\cdot|$ is the complex modulus. By symmetry of v^* we end up with

$$\sum_{1 \le j < k \le m} v^*(x_k - x_j) = \frac{1}{2} \left(\sum_{j,k=1}^{m} v^*(x_k - x_j) - v^*(0) \cdot m \right) \ge -\frac{v^*(0)}{2} \cdot m$$

and, since $v^*(0) = b \cdot \mathrm{e}^{aR} \cdot \big(\mathrm{e}^{aR} - 2(1+\varepsilon)\big) = b \cdot \mathrm{e}^{2aR} \cdot \big(1 - 2 \cdot 4^{-\frac{d+1}{2}}\big) \ge \frac{b \cdot \mathrm{e}^{2aR}}{2} > 0$, the function v^* is stable and hence v is superstable. ○

CHAPTER 5

Uniqueness results for repulsive and subcritical pair interaction processes

In this final chapter of the thesis we come back to the question of uniqueness of Gibbs measures. In contrast to the general setting in Section 3.3, we now focus on repulsive pair interaction processes as a special case of the discussion in Chapter 4. The main question is if this specific setting allows for precise and interpretable conditions as to when the distribution of the pair interaction process is unique.

Recall from Section 3.3 that the uniqueness results we obtained with the disagreement coupling are linked to some notion of connectedness in terms of a binary relation on the state space. If we had a pair potential with finite range $R > 0$, we might associate with each point in the pair interaction process a ball of radius $R/2$ and observe that two points can only interact if the corresponding balls intersect. Similar to the discussion of particle processes in Corollary 3.41, this interpretation brings into play the Boolean model. More precisely, we might infer that as soon as the Boolean model whose grains are balls of radius $R/2$ does not percolate, the pair interaction process is unique in distribution. This is the result derived by Hofer-Temmel and Houdebert (2019). However, it is apparent that this construction wastes a lot of information. In fact, the only information which is used is whether two points interact or not, without any regard to the extent of the interaction. It is reasonable to ask if the Boolean model can be replaced by another random graph which takes into account the full information contained in the pair potential. The natural candidate for such a model is the random connection model, a random graph whose vertices are the points of a point process, with two points x, y being connected with probability $\varphi(x, y)$ independently for all pairs of points. The merit of this model is that the connection function φ, which determines the connection probabilities, allows for almost arbitrary choices. So the question is whether we can find a suitable connection function, in dependence of the given pair potential, such that absence of percolation in the corresponding random connection model implies uniqueness of the Gibbs measure.

Indeed, this question is answered positively and the accompanying result reads as follows. If the random connection model with connection function $\varphi = 1 - \mathrm{e}^{-v}$ based on a Poisson process does not percolate, then the pair interaction process with pair potential v is unique in distribution. The proof of this result, however, is not as straightforward as the above discussion might indicate. The first major difficulty arises from the observation that the random connection model cannot simply be written as a Poisson process in some state space which is endowed with a deterministic binary relation. Even though the random connection model can be interpreted as a Poisson process in a rich product space, the precise construction of the edges depends on the configuration (meaning the locations of the Poisson points) which is not allowed in the setting of Section 3.3. As a remedy, we consider an approximation of the random connection model in a suitable extension of our initial state space and apply the disagreement coupling in the enlarged space. Proceeding in such a way, however, comes with another difficulty. Namely, once we work in a larger space, all results with regard to uniqueness of the Gibbs measures refer to Gibbs processes in the larger space and one has to find a way to go back to pair interaction processes in the initial state space. In order to achieve this goal, we introduce a new projection property which allows to recover the pair interaction process in the original space as a projection of another Gibbs process in the larger space. One technical difficulty we carry with us throughout all proofs is a common problem, namely that most arguments are comparably intuitive for pair potentials with finite range, but require a bit of technical sophistication to extend to infinite interaction ranges.

The structure of this chapter is as follows. We first introduce, in Section 5.1, the random connection model and recall basic facts about its behavior in the context of percolation. We devote the short Section 5.2 to the extension of the state space, thus fixing notation which is required throughout the chapter. In Section 5.3 we discuss the approximation of the random connection model as mentioned above. We continue, in Section 5.4, by a discussion of the projection property to go from the extended space back to the initial state space. In Section 5.5 we give a formal statement of the main result and provide a rigorous proof based on the technical preliminaries discussed in the preceding sections. Once the result is established, the question arises if, and by how much, it improves previous results. We therefore simulate, in Section 5.6, so-called critical intensities of the random connection model, derive thresholds for the uniqueness of Gibbs measures, and compare those results to earlier studies.

While almost everything we do in this chapter works in any Borel space, it is essential to use the fact that a Gibbs process with non-negative pair potential is dominated by a suitable Poisson process, a feature established in Section 3.3.1. To use this property, we have to restrict to complete separable metric spaces and, in order to be consistent with the setting and avoid confusion, we fix such a state space throughout this chapter. More specifically, let $(\mathbb{X}, d)$ be a complete separable metric space, denote its Borel σ-field by $\mathcal{X}$ and the collection of d-bounded Borel sets by $\mathcal{X}_b$. Furthermore, let λ be a locally finite measure on $\mathbb{X}$. As a localizing structure we choose $B_n = B(x_0, n)$ for $n \in \mathbb{N}$ and some fixed origin $x_0 \in \mathbb{X}$. Denote by $v : \mathbb{X} \times \mathbb{X} \to [0, \infty]$ a non-negative pair potential, that is, a measurable and symmetric function. Recall from Equation (4.1) in Section 4.1 that the Papangelou intensity corresponding to v is

$$\kappa(x, \mu) = \exp\Big(- \int_{\mathbb{X}} v(x, y) \, \mathrm{d}\mu(y) \Big)$$

for $x \in \mathbb{X}$ and $\mu \in \mathbf{N}(\mathbb{X})$. We use the given state space, pair potential, and κ throughout the chapter without further reference to their definition. Also recall, from Corollary 4.14, that a Gibbs process with pair interaction v is guaranteed to exist if, for λ-a.e. $x \in \mathbb{X}$,

$$\int_{\mathbb{X}} \big(1 - \mathrm{e}^{-v(x,y)}\big) \, \mathrm{d}\lambda(y) < \infty.$$

Notice that, as we encounter point processes in different spaces, we always make precise the underlying space in our notation $\mathbf{N}(\mathbb{X})$, as opposed to previous chapters, where there was virtually no possibility of confusion.

5.1. The random connection model and continuum percolation

In this section we collect basic definitions and facts on the random connection model. With regard to notation, we roughly follow Nestmann (2019) and Last et al. (2021).

Suppose that $\varphi : \mathbb{X} \times \mathbb{X} \to [0, 1]$ is a measurable and symmetric function and let $\Phi = \sum_{k=1}^{\Phi(\mathbb{X})} \delta_{X_k}$ be a point process in $\mathbb{X}$. Let $U_{i,j}$, $i, j \in \mathbb{N}$, be independent random variables, uniformly distributed on the unit interval $[0, 1]$, such that the double sequence $(U_{i,j})_{i,j \in \mathbb{N}}$ is independent of Φ. Let $\prec$ be a strict total order on $\mathbb{X}$ with $\big\{(x, y) \in \mathbb{X}^2 : x \prec y\big\} \in \mathcal{X}^{\otimes 2}$. Define

$$\mathbb{X}^{[2]} = \big\{\mathbf{e} \in \mathbf{N}(\mathbb{X}) : \mathbf{e}(\mathbb{X}) = 2\big\},$$

which is a Borel space when endowed with the restriction of $\mathcal{N}(\mathbb{X})$ onto $\mathbb{X}^{[2]}$ (by Theorem 1.5 of Kallenberg, 2017, and the fact that measurable subsets of a Borel space are Borel spaces themselves). We interpret $\mathbb{X}^{[2]}$ as the space of all sets $e \subset \mathbb{X}$ containing exactly two elements. In a formal understanding, the *random connection* *model* (RCM) based on Φ with *connection function* φ is a point process Γ in $\mathbb{X}^{[2]}$ defined by

$$\Gamma = \sum_{i,j=1}^{\Phi(\mathbb{X})} \mathbb{1}\big\{X_i \prec X_j,\, U_{i,j} \le \varphi(X_i, X_j)\big\} \cdot \delta_{\{X_i, X_j\}}. \tag{5.1}$$

We interpret Γ as a random graph with vertex set Φ, where two vertices X_i and X_j of Φ are connected with probability $\varphi(X_i, X_j)$, independently for all pairs of vertices. The order on the space $\mathbb{X}$ is used to determine which of the uniform random variables is formally used to simulate a connection between a given pair of points. While the definition of Γ depends on the specific ordering, its distribution does not.

For $\mu \in \mathbf{N}(\mathbb{X})$ we write $x \in \mu$ if $x \in \mathrm{supp}(\mu)$. We say that $x, y \in \Phi$ are connected via Γ, writing $x \overset{\Gamma}{\longleftrightarrow} y$, if either $x = y$ or there exist $n \in \mathbb{N}$ and $e_1, \ldots, e_n \in \Gamma$ such that $x \in e_1$, $y \in e_n$ and $e_i \cap e_{i+1} \neq \varnothing$ for $i \in \{1, \ldots, n-1\}$.

We frequently augment the vertex set of the RCM with additional points. Formally, if $x_1, \ldots, x_m \in \mathbb{X}$, for $m \in \mathbb{N}$, we denote by $\Gamma^{x_1,\ldots,x_m}$ the RCM with connection function φ based on the point process

$$\Phi^{x_1,\ldots,x_m} = \Phi + \delta_{x_1} + \ldots + \delta_{x_m}.$$

The cluster of a point $z \in \mathbb{X}$ in the RCM Γ^z is the point process C_z in $\mathbb{X}$ given by

$$C_z = \int_{\mathbb{X}} \mathbb{1}\{x \in \cdot\,\}\, \mathbb{1}\big\{z \overset{\Gamma^z}{\longleftrightarrow} x\big\}\, \mathrm{d}\Phi^z(x). \tag{5.2}$$

It charges z as well as all points from Φ which are connected to z via Γ^z. The proof of the fact that C_z is a point process is similar to the proof of Lemma 3.26 using the definition of Γ in (5.1) and the notion of connectedness via Γ.

Once we specify to Φ being a Poisson process with intensity measure λ, we say that the RCM is *subcritical* if, for λ-a.e. $x \in \mathbb{X}$,

$$\mathbb{P}\big(C_x(\mathbb{X}) < \infty\big) = 1.$$

Intuitively, this simply means that the connected components of the random graph are finite. As λ completely determines the Poisson process, the RCM is determined by λ and φ, hence we also say that the pair (φ, λ) is subcritical. If $\varphi = 1 - \mathrm{e}^{-v}$ for a pair potential v, then we say that the pair (v, λ) is subcritical.

Remark 5.1. Observe that if the RCM is subcritical, then each point can only be directly connected to finitely many points. Given a point $x \in \mathbb{X}$, the points of Φ which are directly connected to x in Γ^x are an independent thinning of Φ with thinning probability

$$y \mapsto p(y) = \varphi(x, y).$$

For the formal definition of such a thinning, see Definition 5.7 of Last and Penrose (2017). We denote this thinning by Φ_p. By Corollary 5.9 and Proposition 5.5 of Last and Penrose (2017), Φ_p is a Poisson process with intensity measure

$$\int_{\mathbb{X}} \mathbb{1}\{y \in \cdot\,\}\, p(y)\, \mathrm{d}\lambda(y) = \int_{\mathbb{X}} \mathbb{1}\{y \in \cdot\,\}\, \varphi(x, y)\, \mathrm{d}\lambda(y).$$

In particular, $\Phi_p(\mathbb{X})$ is a Poisson random variable with parameter

$$\int_{\mathbb{X}} \varphi(x, y)\, \mathrm{d}\lambda(y).$$

The fact that the RCM is subcritical implies $\Phi_p(\mathbb{X}) < \infty$ almost surely, so the integral in the previous display, being the parameter of the Poisson random variable $\Phi_p(\mathbb{X})$, has the be finite as well.

To summarize this observation in the case where $\varphi = 1 - \mathrm{e}^{-v}$, for a pair potential v, note that if (v, λ) is subcritical then, for λ-a.e. $x \in \mathbb{X}$,

$$\int_{\mathbb{X}} \big(1 - \mathrm{e}^{-v(x,y)}\big) \,\mathrm{d}\lambda(y) < \infty.$$

This is the condition that ensures existence of a corresponding pair interaction process in Corollary 4.14. ○

From what we have done in this thesis so far, not to mention the innumerable applications in the literature, the value of Mecke's formula and, more generally, the GNZ equations became obvious. In the context of the random connection model it is most useful that a version of Mecke's formula for the RCM exists. We recall the following special case from Equation (4.1) of Last et al. (2021) (cf. Theorem 3.3.3 of Nestmann, 2019, for a proof).

Proposition 5.2 (Mecke's formula for the RCM). *Let Φ be a Poisson process with intensity measure λ and consider the RCM based on Φ with connection function φ. Then, for any $x \in \mathbb{X}$ and all measurable functions $f : \mathbb{X} \times \mathbf{N}(\mathbb{X}^{[2]}) \to [0, \infty]$,*

$$\mathbb{E}\bigg[\int_{\mathbb{X}} f(y, \Gamma^x) \,\mathrm{d}\Phi(y)\bigg] = \mathbb{E}\bigg[\int_{\mathbb{X}} f(y, \Gamma^{x,y}) \,\mathrm{d}\lambda(y)\bigg].$$

In the following subsection we briefly summarize the relevant existing knowledge with regard to subcriticality of the random connection model.

5.1.1. Absence of percolation in the RCM

From a percolation perspective it is helpful to consider for each $\gamma \geq 0$ a RCM with connection function φ based on a Poisson process with intensity measure $\gamma\lambda$. Then γ can be interpreted as an *intensity* or *activity*. Define a *critical intensity* by

$$\gamma_c = \sup\big\{\gamma \geq 0 : (\varphi, \gamma\lambda) \text{ is subcritical}\big\}. \tag{5.3}$$

The following result ensures that $(\varphi, \gamma\lambda)$ is subcritical for each $\gamma < \gamma_c$.

Proposition 5.3. *Fix $x \in \mathbb{X}$. In dependence of $\gamma \geq 0$, let $C_{x,\gamma}$ be the cluster of x in the RCM with connection function φ based on a Poisson process Φ with intensity measure $\gamma\lambda$. Then the probability $\mathbb{P}\big(C_{x,\gamma}(\mathbb{X}) < \infty\big)$ is decreasing in γ.*

The proof of Proposition 5.3 is a simple coupling, where a Poisson process with smaller intensity is embedded into one with larger intensity while the uniform random variables used to determine the connections are managed such that any edge in the smaller RCM appears in the larger RCM as well. This coupling is identical to the one which is used in the proof of Proposition 2 of Penrose (1991).

Almost all results in the literature concerning percolation in the RCM consider the stationary case. Hence, for the remainder of this subsection, let $\mathbb{X} = \mathbb{R}^d$ with $d \geq 2$, $\lambda = \mathcal{L}^d$, and consider

$$\varphi(x, y) = \varphi(y - x)$$

for $x, y \in \mathbb{R}^d$, where $\varphi : \mathbb{R}^d \to [0, 1]$ is a measurable function with $\varphi(-x) = \varphi(x)$ for all $x \in \mathbb{R}^d$. The following result follows from Theorem 1 of Penrose (1991) (see also Theorem 6.1 of Meester and Roy, 1996) and shows that, in the given setting, the critical intensity is non-trivial.

Proposition 5.4. *If $0 < \int_{\mathbb{R}^d} \varphi(x) \,\mathrm{d}\mathcal{L}^d(x) < \infty$, then $0 < \gamma_c < \infty$.*

While we demonstrate in Section 5.6 that critical intensities of the RCM can be simulated with reasonable precision, it is rather difficult to formally obtain explicit numerical bounds on this quantity. One crude lower bound is the so-called branching lower bound already considered by Penrose (1991), which is implicit in the proof of the above proposition. Indeed, Equation (6.2) of Penrose (1991) shows that, under the assumption of Proposition 5.4,

$$\gamma_c \geq \left(\int_{\mathbb{R}^d} \varphi(x) \, \mathrm{d}\mathcal{L}^d(x) \right)^{-1}.$$

In the following subsection this branching bound is generalized and discussed. The simulations in Section 5.6 show that the branching bound strongly underestimates the critical intensity in low dimensions, but the bound improves as the dimension grows. This is in line with Theorem 1 of Meester et al. (1997) who show that if, for all $d \geq 2$,

$$I(d) = \int_{\mathbb{R}^d} \varphi(x) \, \mathrm{d}\mathcal{L}^d(x) < \infty$$

(and a further technical condition is satisfied), then, as $d \to \infty$,

$$\gamma_c(d) \cdot I(d) \longrightarrow 1,$$

where $\gamma_c(d)$ denotes the critical intensity from (5.3) in dependence of the dimension.

As the above results refer to dimensions $d \geq 2$, we conclude this digression with a short discussion of the one-dimensional case, at least for connection functions with finite range.

Example 5.5 (One-dimensional RCMs). Let $\varphi : [0, \infty) \times [0, \infty) \to [0, 1]$ be measurable and symmetric. Assume there exists some $R > 0$ such that $\varphi(s, t) = 0$ for all $s, t \geq 0$ with $|t - s| > R$. Fix an arbitrary $\gamma > 0$ and let Φ be a Poisson process in $[0, \infty)$ with intensity measure $\gamma \mathcal{L}^1_{[0,\infty)}$. Following Chapter 7 of Last and Penrose (2017), we denote the points of Φ by $T_1, T_2, \ldots$ and set $T_0 = 0$. By Theorem 7.2 of Last and Penrose (2017), $T_1, T_2 - T_1, T_3 - T_2, \ldots$ are independent random variables, each being exponentially distributed with parameter γ. Let $x \geq 0$. Note that if there exist consecutive Poisson points larger than x which have distance greater than R then, by the finite range of φ, the cluster of x must be finite. Therefore,

$$\begin{aligned}
\mathbb{P}\Big(C_x([0,\infty)) < \infty\Big) &\geq \mathbb{P}\left(\bigcup_{j=1}^{\infty} \left(\{T_j > x\} \cap \bigcup_{k=j+1}^{\infty} \{T_k - T_{k-1} > R\} \right) \right) \\
&\geq \liminf_{n\to\infty} \mathbb{P}\left(\{T_n > x\} \cap \bigcup_{k=n+1}^{\infty} \{T_k - T_{k-1} > R\} \right) \\
&= \liminf_{n\to\infty} \mathbb{P}(T_n > x) \cdot \mathbb{P}\left(\bigcup_{k=n+1}^{\infty} \{T_k - T_{k-1} > R\} \right), \qquad (5.4)
\end{aligned}$$

where the last equality follows from the independence properties of the Poisson points. Now, observe that for any $n \in \mathbb{N}$

$$\mathbb{P}\left(\bigcup_{k=n+1}^{\infty} \{T_k - T_{k-1} > R\} \right) = 1 - \lim_{m\to\infty} \prod_{k=n+1}^{m} \mathbb{P}\big(T_k - T_{k-1} \leq R\big) = 1 - \lim_{m\to\infty} \big(1 - \mathrm{e}^{-\gamma R}\big)^{m-n} = 1.$$

Furthermore, using the fact that $\Phi([0,x])$ is Poisson-distributed with parameter $\gamma \cdot x$, we have, as $n \to \infty$,

$$\mathbb{P}(T_n > x) = \mathbb{P}\big(\Phi([0,x]) \leq n-1\big) = \mathrm{e}^{-\gamma x} \sum_{i=0}^{n-1} \frac{(\gamma x)^i}{i!} \longrightarrow \mathrm{e}^{-\gamma x} \cdot \mathrm{e}^{\gamma x} = 1.$$

We conclude from (5.4) that the RCM based on Φ with connection function φ is subcritical. As $\gamma > 0$ was arbitrary, we have $\gamma_c = \infty$. This result easily extends to a RCM on the whole real line. ○

5.1.2. General branching bounds of first order

As indicated in the previous subsection, we now provide a rigorous result as to when the RCM is subcritical. The proof is established via a bound on a suitable branching construction. We work in the overall setting of this chapter, but the following arguments work in any Borel space without changes. As before, let $\varphi : \mathbb{X} \times \mathbb{X} \to [0, 1]$ be a measurable and symmetric function and denote by Φ a Poisson process in $\mathbb{X}$ with intensity measure λ. All random connection models which appear in this subsection are based on φ and Φ. We define the *pair connectedness* function $\tau : \mathbb{X} \times \mathbb{X} \to [0, 1]$,

$$\tau(x, y) = \mathbb{P}\big(x \xleftrightarrow{\Gamma^{x,y}} y\big).$$

We can immediately state the announced result, without introducing further notation.

Theorem 5.6. *Assume there exists a measurable function $g : \mathbb{X} \to [0, \infty)$ such that, for λ-a.e. $x \in \mathbb{X}$,*

$$\int_{\mathbb{X}} \varphi(x, y)\, \mathrm{d}\lambda(y) + \int_{\mathbb{X}} \varphi(x, y)\, g(y)\, \mathrm{d}\lambda(y) \leq g(x). \tag{5.5}$$

Then (φ, λ) is subcritical.

Proof. First of all, notice that, by the RCM version of Mecke's formula in Proposition 5.2, we have

$$\mathbb{E}\big[C_x(\mathbb{X})\big] = 1 + \mathbb{E}\bigg[\int_{\mathbb{X}} \mathbb{1}\big\{x \xleftrightarrow{\Gamma^{x}} y\big\}\, \mathrm{d}\Phi(y)\bigg] = 1 + \int_{\mathbb{X}} \mathbb{P}\big(x \xleftrightarrow{\Gamma^{x,y}} y\big)\, \mathrm{d}\lambda(y) = 1 + \int_{\mathbb{X}} \tau(x, y)\, \mathrm{d}\lambda(y).$$

Observe that if two points $x, y \in \mathbb{X}$ are connected via $\Gamma^{x,y}$, then either x and y are directly connected or there lies at least one Poisson point in between them. Therefore,

$$\tau(x, y) \leq \varphi(x, y) + \mathbb{E}\bigg[\int_{\mathbb{X}} \varphi(x, z)\, \mathbb{1}\big\{z \xleftrightarrow{\Gamma^{y}} y\big\}\, \mathrm{d}\Phi(z)\bigg].$$

Another application of Mecke's formula for the RCM (Proposition 5.2) shows that the right hand side of the previous display is equal to

$$\varphi(x, y) + \int_{\mathbb{X}} \varphi(x, z) \cdot \mathbb{P}\big(z \xleftrightarrow{\Gamma^{z,y}} y\big)\, \mathrm{d}\lambda(z) = \varphi(x, y) + \int_{\mathbb{X}} \varphi(x, z)\, \tau(z, y)\, \mathrm{d}\lambda(z).$$

For two functions $h_1, h_2 : \mathbb{X} \times \mathbb{X} \to [0, \infty]$ we define a convolution type operator via

$$(h_1 * h_2)(x, y) = \int_{\mathbb{X}} h_1(x, z)\, h_2(z, y)\, \mathrm{d}\lambda(z).$$

With this definition, the inequality derived above reads as $\tau \leq \varphi + \varphi * \tau$. Iteration of this inequality yields that, for every $n \in \mathbb{N}$,

$$\tau \leq \varphi + \varphi^{*2} + \ldots + \varphi^{*n} + \varphi^{*n} * \tau, \tag{5.6}$$

where φ^{*k} denotes the k-fold convolution of φ with itself. By a similar iteration of (5.5), we see that, for λ-a.e. $x \in \mathbb{X}$ and every $n \in \mathbb{N}$,

$$g(x) \geq \sum_{k=1}^{n} \int_{\mathbb{X}} \varphi^{*k}(x, y)\, \mathrm{d}\lambda(y) + \int_{\mathbb{X}} \varphi^{*n}(x, y)\, g(y)\, \mathrm{d}\lambda(y) \geq \sum_{k=1}^{n} \int_{\mathbb{X}} \varphi^{*k}(x, y)\, \mathrm{d}\lambda(y).$$

Taking $n \to \infty$ gives, for λ-a.e. $x \in \mathbb{X}$,

$$g(x) \geq \sum_{k=1}^{\infty} \int_{\mathbb{X}} \varphi^{*k}(x, y)\, \mathrm{d}\lambda(y). \tag{5.7}$$

For each such x we thus have

$$\lim_{k\to\infty}\int_{\mathbb{X}}\varphi^{*k}(x,y)\,\mathrm{d}\lambda(y)=0. \tag{5.8}$$

It follows from (5.6) that, for $x\in\mathbb{X}$ and all $n,\ell\in\mathbb{N}$,

$$\begin{aligned}\int_{B_\ell}\tau(x,y)\,\mathrm{d}\lambda(y)&\le\sum_{k=1}^{n}\int_{B_\ell}\varphi^{*k}(x,y)\,\mathrm{d}\lambda(y)+\int_{B_\ell}\int_{\mathbb{X}}\varphi^{*n}(x,z)\,\tau(z,y)\,\mathrm{d}\lambda(z)\,\mathrm{d}\lambda(y)\\&\le\sum_{k=1}^{n}\int_{B_\ell}\varphi^{*k}(x,y)\,\mathrm{d}\lambda(y)+\lambda(B_\ell)\int_{\mathbb{X}}\varphi^{*n}(x,z)\,\mathrm{d}\lambda(z).\end{aligned}$$

Letting $n\to\infty$, we obtain from (5.8) that, for λ-a.e. $x\in\mathbb{X}$ and all $\ell\in\mathbb{N}$,

$$\int_{B_\ell}\tau(x,y)\,\mathrm{d}\lambda(y)\le\sum_{k=1}^{\infty}\int_{B_\ell}\varphi^{*k}(x,y)\,\mathrm{d}\lambda(y).$$

With monotone convergence, letting $\ell\to\infty$, together with the formula for the expected cluster size at the beginning of this proof and (5.7), we arrive at

$$\mathbb{E}\big[C_x(\mathbb{X})\big]=1+\int_{\mathbb{X}}\tau(x,y)\,\mathrm{d}\lambda(y)\le1+\sum_{k=1}^{\infty}\int_{\mathbb{X}}\varphi^{*k}(x,y)\,\mathrm{d}\lambda(y)\le1+g(x)<\infty$$

for λ-a.e. $x\in\mathbb{X}$. Hence, the cluster of such x in Γ^x is finite almost surely, that is, (φ,λ) is subcritical. □

We make some remarks on Theorem 5.6.

Remark 5.7. In Remark 5.1 we have seen that a necessary condition for subcriticality is that

$$\int_{\mathbb{X}}\varphi(x,y)\,\mathrm{d}\lambda(y)<\infty$$

for λ-a.e. $x\in\mathbb{X}$. A substantial tightening of this assumption gives a sufficient condition for subcriticality. More precisely, assume that

$$q=\operatorname*{ess\,sup}_{x\in\mathbb{X}}\int_{\mathbb{X}}\varphi(x,y)\,\mathrm{d}\lambda(y)<1.$$

Then, choosing $g\equiv\frac{q}{1-q}$, we obtain, for λ-a.e. $x\in\mathbb{X}$,

$$\int_{\mathbb{X}}\varphi(x,y)\,\mathrm{d}\lambda(y)+\int_{\mathbb{X}}\varphi(x,y)\,g(y)\,\mathrm{d}\lambda(y)\le q+\frac{q}{1-q}\cdot q=\frac{q}{1-q}=g(x).$$

If $\varphi=1-\mathrm{e}^{-v}$ for a pair potential v then this is essentially the requirement for uniqueness of the pair interaction process by Houdebert and Zass (2022). Apart from the fact that Houdebert and Zass (2022) require additional technical assumptions and work in $\mathbb{R}^d$ only, it follows from Theorem 5.6 that our uniqueness result via the RCM provides substantial improvement, a fact that the simulations in Section 5.6 further illustrate.

Assumption (5.5) in Theorem 5.6 is also weaker than condition (KPU$_t$) of Jansen (2019). Indeed, if there exists a measurable function $g:\mathbb{X}\to[0,\infty)$ and some $t\ge0$ such that, for λ-a.e. $x\in\mathbb{X}$,

$$\mathrm{e}^{t}\int_{\mathbb{X}}\varphi(x,y)\,\mathrm{e}^{g(y)}\,\mathrm{d}\lambda(y)\le g(x),$$

then (5.5) follows from $\mathrm{e}^{g(y)}\ge1+g(y)$ and $\mathrm{e}^{t}\ge1$. In particular, Theorem 5.6 improves Corollary C.1 of Jansen (2019). ○

5.2. An extension of the state space

In the introduction of this chapter we described how working exclusively in the original state space $\mathbb{X}$ appears to be insufficient when trying to combine the RCM with the disagreement coupling. We now introduce the setting for our technical workaround. The notation we establish in this section is fixed throughout the following three sections as well. Right now, and also in the following two sections, the restriction to a diffuse reference measure is necessary. In the proof of our uniqueness result we argue, as in Theorem 3.32, why this is no loss of generality.

Let the space $(\mathbb{X}, d)$ and the pair (v, λ) be as in the introduction of this chapter and assume, in addition, that λ is diffuse. Suppose that, for λ-a.e. $x \in \mathbb{X}$,

$$\int_{\mathbb{X}} \big(1 - \mathrm{e}^{-v(x,y)}\big) \, \mathrm{d}\lambda(y) < \infty. \tag{5.9}$$

Recall from Remark 5.1 that this integrability assumption is immediately satisfied if (v, λ) is subcritical. However, for the approximation of the RCM in Section 5.3 and the projection property for Gibbs processes in Section 5.4 the weaker assumption suffices.

Similar to Corollary 4.5 we consider the following second relevant specialization of Lemma 4.4. The first two parts of the corollary follow immediately from Lemma 4.4 and the last part is a literal copy of the proof of part (viii) of Lemma A.10, replacing $\mathcal{X}_b$ by $\mathcal{X}_b^{**}$.

Corollary 5.8. *There exist sets $C_1 \subset C_2 \subset \ldots$ from $\mathcal{X}_b$ such that $\bigcup_{\ell=1}^{\infty} C_\ell = \mathbb{X}$ and*

$$\int_{C_\ell} \int_{\mathbb{X}} \big(1 - \mathrm{e}^{-v(x,y)}\big) \, \mathrm{d}\lambda(y) \, \mathrm{d}\lambda(x) < \infty, \quad \ell \in \mathbb{N}. \tag{5.10}$$

Let $\mathcal{X}_b^{**}$ be the collection of all sets from $\mathcal{X}$ which are contained in one of the C_ℓ. Then $\mathcal{X}_b^{**}$ is a ring over $\mathbb{X}$ with $\sigma(\mathcal{X}_b^{**}) = \mathcal{X}$. Moreover,

$$\mathcal{Z}^{**} = \bigcup_{C \in \mathcal{X}_b^{**}} \mathcal{N}_C(\mathbb{X})$$

is an algebra over $\mathbf{N}(\mathbb{X})$ which generates $\mathcal{N}(\mathbb{X})$.

The collection $\mathcal{X}_b^{**}$ as well as the sets $C_1, C_2, \ldots$ are used throughout the remainder of this chapter. The construction will prove useful later on in order to avoid additional integrability assumptions.

Put $C_0 = \varnothing$. Having in mind that Borel subsets of $\mathbb{X}$ are Borel spaces, we consider, for each $\ell \in \mathbb{N}$, a bi-measurable bijection ι_ℓ from the Borel space $\big(C_\ell \setminus C_{\ell-1}, \mathcal{X} \cap (C_\ell \setminus C_{\ell-1})\big)$ onto a Borel subset of $(\ell - 1, \ell]$. We define the injective and measurable map $\iota : \mathbb{X} \to (0, \infty)$ as

$$\iota(x) = \sum_{k=1}^{\infty} \iota_k(x) \, \mathbb{1}_{C_k \setminus C_{k-1}}(x).$$

On the space $\mathbb{X}$ we let a strict total order be given by $x \prec y$ whenever $\iota(x) < \iota(y)$. Observe that for $x \in C_\ell$ and $y \in \mathbb{X} \setminus C_\ell$ the construction always yields $x \prec y$. Moreover, the measurability of ι ensures that

$$\big\{(x, y) \in \mathbb{X}^2 : x \prec y\big\} = \big\{(x, y) \in \mathbb{X}^2 : \iota(x) < \iota(y)\big\} \in \mathcal{X}^{\otimes 2}.$$

From now on, we fix $\varphi = 1 - \mathrm{e}^{-v}$ as a function on $\mathbb{X} \times \mathbb{X}$. In order to apply the disagreement coupling from Last and Otto (2021), we want to approximate the RCM with connection function φ by suitably interpreting a Poisson process on a rich product space. To this end, we consider as a mark space

$$\mathbb{M} = [0, 1]^{\mathbb{N} \times \mathbb{N}},$$

the space of doubly indexed sequences in $[0,1]$, endowed with the product σ-field $\mathcal{M}$, and we denote by $\mathbb{Q}$ the probability measure on $\mathbb{M}$ given as an infinite product of uniform distributions on $[0,1]$.

Later on, we need to be able to separate the points in our space with the help of countable partitions. For each $\delta > 0$, let $D_{1,\delta}, D_{2,\delta}, \ldots \in \mathcal{X}$ be a partition of $\mathbb{X}$ such that for any two points $x, y \in \mathbb{X}$ $(x \neq y)$ there exists some $\delta_0 > 0$ so that x and y are separated by the δ-partition for every $\delta < \delta_0$, where the points being separated means that they lie in different sets of the partition. Such directed partitions can always be constructed in separable metric spaces, comparable to Lemma 1.3 of Kallenberg (2017) and Appendix A.8.

Connections in the approximating model are determined by a deterministic binary relation on the product space $\mathbb{X} \times \mathbb{M}$. For the definition of the relation we need the measurable map $R_\delta : (\mathbb{X} \times \mathbb{M})^2 \to [0,1]$,

$$R_\delta(x,r,y,s) = \sum_{i,j=1}^{\infty} \mathbb{1}_{D_{i,\delta}}(x)\, \mathbb{1}_{D_{j,\delta}}(y) \left(\mathbb{1}\{x \prec y\} \cdot r_{i,j} + \mathbb{1}\{y \prec x\} \cdot s_{j,i} \right)$$

for $\delta > 0$. Based on R_δ we define a reflexive and symmetric relation $\sim_\delta$ on $\mathbb{X} \times \mathbb{M}$ via

$$(x,r) \sim_\delta (y,s) \quad \Longleftrightarrow \quad R_\delta(x,r,y,s) \leq \varphi(x,y).$$

Observe that, while the formal construction is a bit tedious, it is rather easy to make sense of. The intuition is as follows. To see if two points $x, y \in \mathbb{X}$ with $x \neq y$ are connected, we look for the partition sets in which these points lie and endow each (ordered) pair of partition sets with a mark (being the realization of a standard uniform random variable). Given the two partition sets in which the points lie, the order relation on the state space determines which of the two possible marks is chosen to determine whether there is a connection. Just like for the RCM, the probability that the two points are connected is given by $\varphi(x,y)$, which is formalized by the points being connected if the corresponding mark is smaller than $\varphi(x,y)$. By construction, the marks of different partition sets are realized by independent random variables, so if all considered points lie in different partition sets, their connections are drawn independently of one another. Indeed, we show in the following section that, as $\delta \searrow 0$, the given relation does recover the RCM.

Before we turn to the merits of the given construction, we introduce and recall some notation. We endow the space $\mathbb{X} \times \mathbb{M}$ with the localizing structure $B_1 \times \mathbb{M}, B_2 \times \mathbb{M}, \ldots$ and, in line with our general definition, let $\mathbf{N}(\mathbb{X} \times \mathbb{M})$ be the set of all measures ψ on $\mathbb{X} \times \mathbb{M}$ such that $\psi(B \times \mathbb{M}) \in \mathbb{N}_0$ for each $B \in \mathcal{X}_b$.

Whenever ψ is a measure on $\mathbb{X} \times \mathbb{M}$ we write

$$\bar{\psi} = \psi(\,\cdot \times \mathbb{M})$$

for its projection onto $\mathbb{X}$. Conversely, if μ is a counting measure on $\mathbb{X}$, we construct a measure $\hat{\mu}$ on $\mathbb{X} \times \mathbb{M}$ by endowing each point of μ with a fixed (but arbitrary) mark $s_* \in \mathbb{M}$.

Given some set $B \in \mathcal{X}$ and a measure ψ on $\mathbb{X} \times \mathbb{M}$, we write

$$\psi_B = \psi\big(\,\cdot \cap (B \times \mathbb{M})\big)$$

for the restriction of ψ onto $B \times \mathbb{M}$, and we denote in a generic way by $\mathbf{N}_{fs}$ the set of finite simple counting measures.

To complement the notation from Section 3.3.2 we write, for a point $(x,r) \in \mathbb{X} \times \mathbb{M}$ and a set $S \subset \mathbb{X} \times \mathbb{M}$,

$$(x,r) \sim_\delta S$$

if $(x,r) \sim_\delta (y,s)$ for some $(y,s) \in S$, and likewise $(x,r) \not\sim_\delta S$ if (x,r) is not $\sim_\delta$-connected to any point in S. We also use this notation for $\psi \in \mathbf{N}(\mathbb{X} \times \mathbb{M})$, formally meaning that $S = \mathrm{supp}(\psi)$. Recall that (x,r) and (y,s) are $\sim_\delta$-connected via ψ, written as $(x,r) \overset{\psi}{\sim}_\delta (y,s)$, if there exist $n \in \mathbb{N}_0$ and $(y_1,s_1), \ldots, (y_n,s_n) \in \psi$ such that

$(y_j, s_j) \sim_\delta (y_{j+1}, s_{j+1})$ for $j = 0, \ldots, n$, with $(y_0, s_0) = (x, r)$ as well as $(y_{n+1}, s_{n+1}) = (y, s)$. Any questions of measurability of quantities involving $\sim_\delta$ are settled by Lemma 3.26 and Corollary 3.27.

5.3. APPROXIMATING THE RANDOM CONNECTION MODEL

The intuition behind the relation $\sim_\delta$ introduced in the previous section has already been discussed. In this section we formalize the resulting approximation property. We prove, in two steps, that connections via $\sim_\delta$ with respect to a Poisson process in $\mathbb{X} \times \mathbb{M}$ translate to connections via the RCM in $\mathbb{X}$. In the first step, we restrict our attention to connections between finitely many points and, in a second step, we extend the result to infinitely many points. Afterwards we provide another preliminary technicality concerning the behavior of the RCM in the subcritical regime. All of these results are tailored to their later use.

Notice that, due to λ being diffuse, Remark A.14 and Lemma 2.23 imply that all Poisson and Gibbs processes we consider in the following are simple.

Lemma 5.9. *Let Ψ be a Poisson process in $\mathbb{X} \times \mathbb{M}$ with intensity measure $\lambda \otimes \mathbb{Q}$. Let $\ell \in \mathbb{N}$ and fix $\mu \in \mathbf{N}_{fs}(\mathbb{X}) \cap \mathbf{N}_{\mathbb{X} \setminus C_\ell}(\mathbb{X})$ as well as $x \in C_\ell$. Denote by $\Gamma^{x,\mu}_{C_\ell}$ the RCM with connection function $\varphi = 1 - \mathrm{e}^{-v}$ based on $\bar{\Psi}_{C_\ell} + \mu + \delta_x$. Then*

$$\lim_{\delta \searrow 0} \int_{\mathbb{M}} \mathbb{P}\Big((x, r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\mu}\Big) \, \mathrm{d}\mathbb{Q}(r) = \mathbb{P}\Big(x \overset{\Gamma^{x,\mu}_{C_\ell}}{\longleftrightarrow} \mu\Big),$$

where the notation $x \overset{\Gamma^{x,\mu}_{C_\ell}}{\longleftrightarrow} \mu$ means that x is connected via $\Gamma^{x,\mu}_{C_\ell}$ to some point of μ.

Proof. To lighten the notation, set $C = C_\ell$. For $\mathbb{P}^{\bar{\Psi}_C}$-a.e. $\nu \in \mathbf{N}_{fs}(\mathbb{X}) \cap \mathbf{N}_C(\mathbb{X})$ (write $\nu = \sum_{i=1}^k \delta_{x_i}$) the conditional distribution of Ψ_C given $\bar{\Psi}_C = \nu$ is

$$\mathbb{P}\big(\Psi_C \in \cdot \mid \bar{\Psi}_C = \nu\big) = \int_{\mathbb{M}^k} \mathbb{1}\bigg\{ \sum_{i=1}^k \delta_{(x_i, r_i)} \in \cdot \bigg\} \, \mathrm{d}\mathbb{Q}^k(r_1, \ldots, r_k).$$

Choose $\delta > 0$ so small that the finitely many points in $\mu + \nu + \delta_x$ are separated by the δ-partition. By definition of $\sim_\delta$, if we try to determine whether (x, r) is connected via Ψ_C and $\sim_\delta$ to some point in $\hat{\mu}$, then the marks of the points of $\hat{\mu}$ (which, in fact, all carry the same mark s_*) are irrelevant as μ only has points in $\mathbb{X} \setminus C$ which are larger than x and the points of ν with respect to $\prec$. As the relevant points lie in different sets of the partition, the connections are realized by independent standard uniform random variables (by choice of $\sim_\delta$ and $\mathbb{Q}$), so we have, by definition of the RCM,

$$\int_{\mathbb{M}} \mathbb{P}\Big((x, r) \overset{\Psi_C}{\sim_\delta} \hat{\mu} \mid \bar{\Psi}_C = \nu\Big) \, \mathrm{d}\mathbb{Q}(r) = \mathbb{P}\Big(x \overset{\Gamma^{x,\mu}_C}{\longleftrightarrow} \mu \mid \bar{\Psi}_C = \nu\Big).$$

Dominated convergence gives

$$\begin{aligned}
\lim_{\delta \searrow 0} \int_{\mathbb{M}} \mathbb{P}\Big((x, r) \overset{\Psi_C}{\sim_\delta} \hat{\mu}\Big) \, \mathrm{d}\mathbb{Q}(r) &= \lim_{\delta \searrow 0} \int_{\mathbf{N}(\mathbb{X})} \int_{\mathbb{M}} \mathbb{P}\Big((x, r) \overset{\Psi_C}{\sim_\delta} \hat{\mu} \mid \bar{\Psi}_C = \nu\Big) \, \mathrm{d}\mathbb{Q}(r) \, \mathrm{d}\mathbb{P}^{\bar{\Psi}_C}(\nu) \\
&= \int_{\mathbf{N}(\mathbb{X})} \mathbb{P}\Big(x \overset{\Gamma^{x,\mu}_C}{\longleftrightarrow} \mu \mid \bar{\Psi}_C = \nu\Big) \, \mathrm{d}\mathbb{P}^{\bar{\Psi}_C}(\nu) \\
&= \mathbb{P}\Big(x \overset{\Gamma^{x,\mu}_C}{\longleftrightarrow} \mu\Big).
\end{aligned}$$

□

The limiting factor of Lemma 5.9 is that μ has to be finite. It might be somewhat intuitive that it is possible to extend this to infinite configuration if the location of the points and the decline of the connection function is controlled suitably. What we require later on is that μ can be replaced by the realization of an infinite volume pair interaction process. The fact that the pair potential in consideration is repulsive, and the integrability assumption (5.10), suffice to extend Lemma 5.9 accordingly.

Lemma 5.10. *Let Ψ be a Poisson process in $\mathbb{X} \times \mathbb{M}$ with intensity measure $\lambda \otimes \mathbb{Q}$. Let $\ell \in \mathbb{N}$ and $D \in \mathcal{X}$ with $D \subset C_\ell$. Suppose that η is a Gibbs process in $\mathbb{X}$ with pair potential v and reference measure λ such that η is independent of Ψ and the double sequence of uniform random variables used to construct the RCMs. For each $x \in \mathbb{X}$ let $\Gamma_\ell^{x,\eta}$ be a RCM with connection function $\varphi = 1 - \mathrm{e}^{-v}$ based on $\bar{\Psi}_{C_\ell} + \eta_{\mathbb{X}\setminus C_\ell} + \delta_x$. Then*

$$\lim_{\delta \searrow 0} \int_{D \times \mathbb{M}} \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}_{\mathbb{X}\setminus C_\ell}\Big) \, \mathrm{d}(\lambda \otimes \mathbb{Q})(x,r) = \int_D \mathbb{P}\Big(x \xleftrightarrow{\Gamma_\ell^{x,\eta}} \eta_{\mathbb{X}\setminus C_\ell}\Big) \, \mathrm{d}\lambda(x).$$

Proof. For $n \in \mathbb{N}$ with $n > \ell$ we write $\Gamma_{\ell,n}^{x,\eta}$ for the restriction of $\Gamma_\ell^{x,\eta}$ onto C_n, meaning that only vertices inside C_n and their connections among each other remain. For such n, we have

$$\begin{aligned} &\left| \int_{D \times \mathbb{M}} \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}_{\mathbb{X}\setminus C_\ell}\Big) \, \mathrm{d}(\lambda \otimes \mathbb{Q})(x,r) - \int_D \mathbb{P}\Big(x \xleftrightarrow{\Gamma_\ell^{x,\eta}} \eta_{\mathbb{X}\setminus C_\ell}\Big) \, \mathrm{d}\lambda(x) \right| \\ &\quad \le \int_{D \times \mathbb{M}} \left| \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}_{\mathbb{X}\setminus C_\ell}\Big) - \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}_{C_n\setminus C_\ell}\Big) \right| \mathrm{d}(\lambda \otimes \mathbb{Q})(x,r) \\ &\qquad + \left| \int_{D \times \mathbb{M}} \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}_{C_n\setminus C_\ell}\Big) \, \mathrm{d}(\lambda \otimes \mathbb{Q})(x,r) - \int_D \mathbb{P}\Big(x \xleftrightarrow{\Gamma_{\ell,n}^{x,\eta}} \eta_{C_n\setminus C_\ell}\Big) \, \mathrm{d}\lambda(x) \right| \\ &\qquad + \int_D \left| \mathbb{P}\Big(x \xleftrightarrow{\Gamma_{\ell,n}^{x,\eta}} \eta_{C_n\setminus C_\ell}\Big) - \mathbb{P}\Big(x \xleftrightarrow{\Gamma_\ell^{x,\eta}} \eta_{\mathbb{X}\setminus C_\ell}\Big) \right| \mathrm{d}\lambda(x). \end{aligned} \tag{5.11}$$

We denote the three terms that appear on the right hand side of (5.11) by $I_{\delta,n}^{(1)}$, $I_{\delta,n}^{(2)}$, and $I_n^{(3)}$, respectively, and consider them separately. As for the first term, note that, for $\delta > 0$ and $n > \ell$,

$$\begin{aligned} I_{\delta,n}^{(1)} &= \int_{D \times \mathbb{M}} \left| \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}_{\mathbb{X}\setminus C_\ell}\Big) - \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}_{C_n\setminus C_\ell}\Big) \right| \mathrm{d}(\lambda \otimes \mathbb{Q})(x,r) \\ &\le \int_{D \times \mathbb{M}} \mathbb{E}\left| \mathbb{1}\Big\{(x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}_{\mathbb{X}\setminus C_\ell}\Big\} - \mathbb{1}\Big\{(x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}_{C_n\setminus C_\ell}\Big\} \right| \mathrm{d}(\lambda \otimes \mathbb{Q})(x,r). \end{aligned}$$

The only case where the difference of the indicator functions yields a value different from 0 is if (x,r) is $\sim_\delta$-connected via Ψ_{C_ℓ} to $\hat{\eta}_{\mathbb{X}\setminus C_n}$ but not to $\hat{\eta}_{C_n\setminus C_\ell}$. In particular, the right hand side of the previous display is bounded by

$$\int_{D \times \mathbb{M}} \mathbb{E}\Big[\mathbb{1}\Big\{(x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}_{\mathbb{X}\setminus C_n}\Big\}\Big] \, \mathrm{d}(\lambda \otimes \mathbb{Q})(x,r).$$

If $(x,r) \in D \times \mathbb{M}$ is $\sim_\delta$-connected via Ψ_{C_ℓ} to $\hat{\eta}_{\mathbb{X}\setminus C_n}$, then either $(x,r) \sim_\delta (y,s_*)$ for some point $y \in \eta_{\mathbb{X}\setminus C_n}$ or one of the Poisson points is connected to $\hat{\eta}_{\mathbb{X}\setminus C_n}$. Thus, the previous term is bounded by

$$\begin{aligned} &\int_{D \times \mathbb{M}} \mathbb{E}\left[\int_{\mathbb{X}\setminus C_n} \mathbb{1}\{(x,r) \sim_\delta (y,s_*)\} \, \mathrm{d}\eta(y) \right] \mathrm{d}(\lambda \otimes \mathbb{Q})(x,r) \\ &\quad + \int_{D \times \mathbb{M}} \mathbb{E}\left[\int_{C_\ell \times \mathbb{M}} \int_{\mathbb{X}\setminus C_n} \mathbb{1}\{(z,t) \sim_\delta (y,s_*)\} \, \mathrm{d}\eta(y) \, \mathrm{d}\Psi(z,t) \right] \mathrm{d}(\lambda \otimes \mathbb{Q})(x,r). \end{aligned}$$

Using that η is independent of Ψ and that the Papangelou intensity κ of η is bounded by 1, the GNZ equation (for η) and Mecke's formula (for Ψ) yield that the term is further bounded by

$$\begin{aligned} &\int_{D \times \mathbb{M}} \int_{\mathbb{X}\setminus C_n} \mathbb{1}\{(x,r) \sim_\delta (y,s_*)\} \, \mathrm{d}\lambda(y) \, \mathrm{d}(\lambda \otimes \mathbb{Q})(x,r) \\ &\quad + \lambda(D) \int_{C_\ell \times \mathbb{M}} \int_{\mathbb{X}\setminus C_n} \mathbb{1}\{(z,t) \sim_\delta (y,s_*)\} \, \mathrm{d}\lambda(y) \, \mathrm{d}(\lambda \otimes \mathbb{Q})(z,t). \end{aligned}$$

As $D \subset C_\ell \subset C_n$, the points in D and C_ℓ are always smaller (with respect to $\prec$) than the points in $\mathbb{X} \setminus C_n$ so, by construction of $\sim_\delta$ and $\mathbb{Q}$, the latter sum equals

$$\int_D \int_{\mathbb{X}\setminus C_n} \varphi(x,y)\,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda(x) + \lambda(D) \int_{C_\ell} \int_{\mathbb{X}\setminus C_n} \varphi(z,y)\,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda(z),$$

which converges to 0 as $n \to \infty$ by dominated convergence, using (5.10). Hence, the first term on the right hand side of (5.11) converges to 0 as $n \to \infty$ uniformly in δ, that is,

$$\lim_{n\to\infty} \sup_{\delta>0} I^{(1)}_{\delta,n} = 0.$$

As for the second term on the right hand side of (5.11), observe that by the independence of η and Ψ, dominated convergence, and Lemma 5.9 we have, for any $n > \ell$,

$$\begin{aligned}
\limsup_{\delta\searrow 0} I^{(2)}_{\delta,n} &= \limsup_{\delta\searrow 0} \left| \int_{D\times\mathbb{M}} \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}_{C_n\setminus C_\ell}\Big)\,\mathrm{d}(\lambda\otimes\mathbb{Q})(x,r) - \int_D \mathbb{P}\Big(x \xleftrightarrow{\Gamma^{x,\eta}_{\ell,n}} \eta_{C_n\setminus C_\ell}\Big)\,\mathrm{d}\lambda(x) \right| \\
&\le \limsup_{\delta\searrow 0} \int_{\mathbf{N}(\mathbb{X})} \int_D \left| \int_{\mathbb{M}} \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\mu}_{C_n\setminus C_\ell}\Big)\,\mathrm{d}\mathbb{Q}(r) - \mathbb{P}\Big(x \xleftrightarrow{\Gamma^{x,\mu}_{\ell,n}} \mu_{C_n\setminus C_\ell}\Big) \right| \mathrm{d}\lambda(x)\,\mathrm{d}\mathbb{P}^\eta(\mu) \\
&= 0.
\end{aligned}$$

To handle the third term on the right hand side of (5.11), note that, for $n > \ell$,

$$\begin{aligned}
I^{(3)}_n &= \int_D \left| \mathbb{P}\Big(x \xleftrightarrow{\Gamma^{x,\eta}_{\ell,n}} \eta_{C_n\setminus C_\ell}\Big) - \mathbb{P}\Big(x \xleftrightarrow{\Gamma^{x,\eta}_{\ell}} \eta_{\mathbb{X}\setminus C_\ell}\Big) \right| \mathrm{d}\lambda(x) \\
&\le \int_D \mathbb{E}\left| \mathbb{1}\Big\{x \xleftrightarrow{\Gamma^{x,\eta}_{\ell,n}} \eta_{C_n\setminus C_\ell}\Big\} - \mathbb{1}\Big\{x \xleftrightarrow{\Gamma^{x,\eta}_{\ell}} \eta_{\mathbb{X}\setminus C_\ell}\Big\} \right| \mathrm{d}\lambda(x).
\end{aligned}$$

The difference of the indicator functions appearing in the expectation can only be distinct from 0 if there exists a connection from x to $\eta_{\mathbb{X}\setminus C_n}$ via $\Gamma^{x,\eta}_\ell$ which uses no point in $\eta_{C_n\setminus C_\ell}$. Thus, either x or one of the Poisson points in C_ℓ has to be connected directly to one of the points in $\eta_{\mathbb{X}\setminus C_n}$. Together with the given independence properties, the quantity is therefore bounded by

$$\int_D \mathbb{E}\left[\int_{\mathbb{X}\setminus C_n} \varphi(x,y)\,\mathrm{d}\eta(y) \right] \mathrm{d}\lambda(x) + \int_D \mathbb{E}\left[\int_{C_\ell} \int_{\mathbb{X}\setminus C_n} \varphi(z,y)\,\mathrm{d}\eta(y)\,\mathrm{d}\bar{\Psi}(z) \right] \mathrm{d}\lambda(x).$$

Another application of the GNZ and Mecke equation, together with the fact that the PI of η is smaller than or equal to 1, the previous term is bounded by

$$\int_D \int_{\mathbb{X}\setminus C_n} \varphi(x,y)\,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda(x) + \lambda(D) \int_{C_\ell} \int_{\mathbb{X}\setminus C_n} \varphi(z,y)\,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda(z).$$

By choice of D and C_ℓ, referring to (5.10), dominated convergence implies that

$$\lim_{n\to\infty} I^{(3)}_n = 0.$$

For any $\varepsilon > 0$ we may now choose $n_0 \in \mathbb{N}$ with $n_0 > \ell$ such that

$$\sup_{\delta>0} I^{(1)}_{\delta,n_0} < \frac{\varepsilon}{3} \quad \text{and} \quad I^{(3)}_{n_0} < \frac{\varepsilon}{3}.$$

Moreover, we find $\delta_0 > 0$ such that, for any $\delta < \delta_0$,

$$I^{(2)}_{\delta,n_0} < \frac{\varepsilon}{3}.$$

Then (5.11), applied to n_0, gives

$$\left| \int_{D\times\mathbb{M}} \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}_{\mathbb{X}\setminus C_\ell}\Big)\,\mathrm{d}(\lambda\otimes\mathbb{Q})(x,r) - \int_D \mathbb{P}\Big(x \overset{\Gamma_\ell^{x,\eta}}{\longleftrightarrow} \eta_{\mathbb{X}\setminus C_\ell}\Big)\,\mathrm{d}\lambda(x)\right| \le I^{(1)}_{\delta,n_0} + I^{(2)}_{\delta,n_0} + I^{(3)}_{n_0} < \varepsilon$$

for all $\delta < \delta_0$. This concludes the proof. □

As announced at the beginning of this section, we now investigate the behavior of the RCM in the subcritical regime. More specifically, we establish that the probability of a point x being connected to a Gibbs point in $\mathbb{X}\setminus C_\ell$ via the RCM based on a Poisson process in C_ℓ goes to 0 as $\ell\to\infty$.

Lemma 5.11. *Assume that (v,λ) is subcritical. Let Φ be a Poisson process in $\mathbb{X}$ with intensity measure λ. Let η be a Gibbs process with pair potential v and reference measure λ such that η is independent of Φ and the double sequence of uniform random variables used to construct the RCMs. For each $x\in\mathbb{X}$ let $\Gamma_\ell^{x,\eta}$ be a RCM with connection function $\varphi = 1-\mathrm{e}^{-v}$ and vertex set $\Phi_{C_\ell} + \eta_{\mathbb{X}\setminus C_\ell} + \delta_x$. Then, for λ-a.e. $x\in\mathbb{X}$,*

$$\lim_{\ell\to\infty} \mathbb{P}\Big(x \overset{\Gamma_\ell^{x,\eta}}{\longleftrightarrow} \eta_{\mathbb{X}\setminus C_\ell}\Big) = 0.$$

Proof. First of all, observe that, since the PI κ of η is bounded by 1, Corollary 3.24 lets us assume, without loss of generality, that $\eta \le \Phi'$ almost surely, where Φ' is a Poisson process with intensity measure λ independent of Φ and the double sequence of uniform random variables used for the RCM. Thus, for each $x\in\mathbb{X}$ and any $\ell\in\mathbb{N}$, we have

$$\mathbb{P}\Big(x \overset{\Gamma_\ell^{x,\eta}}{\longleftrightarrow} \eta_{\mathbb{X}\setminus C_\ell}\Big) \le \mathbb{P}\Big(x \overset{\Gamma_\ell^{x,\Phi'}}{\longleftrightarrow} \Phi'_{\mathbb{X}\setminus C_\ell}\Big),$$

where $\Gamma_\ell^{x,\Phi'}$ denotes the RCM with connection function φ based on $\Phi_{C_\ell} + \Phi'_{\mathbb{X}\setminus C_\ell} + \delta_x$. Since the two Poisson processes are independent, $\Phi_{C_\ell} + \Phi'_{\mathbb{X}\setminus C_\ell}$ is (for every $\ell\in\mathbb{N}$) a Poisson process in $\mathbb{X}$ with intensity measure $\lambda_{C_\ell} + \lambda_{\mathbb{X}\setminus C_\ell} = \lambda$ by the superposition principle (Theorem 3.3 of Last and Penrose, 2017). Therefore, again using the independence, we can replace $\Phi'_{\mathbb{X}\setminus C_\ell}$ by $\Phi_{\mathbb{X}\setminus C_\ell}$ and $\Gamma_\ell^{x,\Phi'}$ by Γ^x (the RCM based on $\Phi+\delta_x$) in the above probability, which yields

$$\mathbb{P}\Big(x \overset{\Gamma_\ell^{x,\eta}}{\longleftrightarrow} \eta_{\mathbb{X}\setminus C_\ell}\Big) \le \mathbb{P}\big(x \overset{\Gamma^x}{\longleftrightarrow} \Phi_{\mathbb{X}\setminus C_\ell}\big).$$

However, if x is connected via Γ^x to $\Phi_{\mathbb{X}\setminus C_\ell}$, then the cluster of x in Γ^x has at least one point in $\mathbb{X}\setminus C_\ell$. Thus, the probability in question is bounded by

$$\mathbb{P}\big(C_x(\mathbb{X}\setminus C_\ell) > 0\big).$$

Similar to (3.9), we use the subcriticality assumption on (v,λ) to conclude that, for λ-a.e. $x\in\mathbb{X}$,

$$\limsup_{\ell\to\infty} \mathbb{P}\Big(x \overset{\Gamma_\ell^{x,\eta}}{\longleftrightarrow} \eta_{\mathbb{X}\setminus C_\ell}\Big) \le \limsup_{\ell\to\infty} \mathbb{P}\big(C_x(\mathbb{X}\setminus C_\ell) > 0\big) = \mathbb{P}\big(C_x(\mathbb{X}) = \infty\big) = 0.$$ □

5.4. A projection property of pair interaction processes

Notice that everything in the previous two sections works for arbitrary φ. The reason for the choice $\varphi = 1-\mathrm{e}^{-v}$ becomes apparent in the following. We show that the projection of a hard core (with respect to $\sim_\delta$) Gibbs process in the extended space $\mathbb{X}\times\mathbb{M}$ onto the original state space $\mathbb{X}$ is a pair interaction process in $\mathbb{X}$. To obtain this result it is essential that the relation $\sim_\delta$ be defined with the specific choice of φ.

For $\delta>0$, let $\kappa_\delta : \mathbb{X}\times\mathbb{M}\times\mathbf{N}(\mathbb{X}\times\mathbb{M}) \to [0,\infty)$ be given by

$$\kappa_\delta(x,r,\psi) = \mathbb{1}\big\{(x,r)\not\sim_\delta \psi\big\} = \exp\bigg(-\int_{\mathbb{X}\times\mathbb{M}} -\log\big(\mathbb{1}\big\{(x,r)\not\sim_\delta (y,s)\big\}\big)\,\mathrm{d}\psi(y,s)\bigg).$$

The map κ_δ corresponds to the Papangelou intensity of a pair interaction process with hard core type pair potential $\big((x,r),(y,s)\big) \mapsto \infty \cdot \mathbb{1}\big\{(x,r) \sim_\delta (y,s)\big\}$, hence it is measurable and satisfies the cocycle property by Lemma 4.1. Recall from (3.7) that, for $(x,r) \in \mathbb{X} \times \mathbb{M}$ and $\psi \in \mathbf{N}(\mathbb{X} \times \mathbb{M})$, we call

$$C_\delta(x,r,\psi) = \int_{\mathbb{X}\times\mathbb{M}} \mathbb{1}\big\{(y,s) \in \cdot\,\big\}\, \mathbb{1}\big\{(x,r) \overset{\psi}{\sim}_\delta (y,s)\big\}\, \mathrm{d}\psi(y,s)$$

the ψ-cluster of (x,r) with respect to $\sim_\delta$. By Lemma 3.26, $(x,r,\psi) \mapsto C_\delta(x,r,\psi) \in \mathbf{N}(\mathbb{X} \times \mathbb{M})$ is a measurable mapping.

Note that if $(x,r) \not\sim_\delta \psi$, then $(x,r) \not\sim_\delta C_\delta(x,r,\psi)$ since $C_\delta(x,r,\psi) \leq \psi$ as measures on $\mathbb{X}$. Conversely, if $(x,r) \not\sim_\delta C_\delta(x,r,\psi)$, then $(x,r) \not\sim_\delta \psi$, for if (x,r) was connected to a point of ψ this point would also be part of the ψ-cluster of (x,r). Therefore,

$$\kappa_\delta(x,r,\psi) = \kappa_\delta\big(x,r,C_\delta(x,r,\psi)\big),$$

which is the main assumption on the PI in Section 3.3. Observe that κ_δ now satisfies all properties required for the disagreement coupling (Proposition 3.30) and Theorem 3.31.

As in Section 2.8, we denote by $\mathrm{P}_{B,\nu}$ the distribution of a finite Gibbs process with PI $\kappa^{(B,\nu)}$ and reference measure λ, where $B \in \mathcal{X}_b$ and $\nu \in \mathbf{N}(\mathbb{X})$. Since κ is the PI of a Gibbs process with pair potential v, the distribution $\mathrm{P}_{B,\nu}$ is that of a pair interaction process in B with pair potential v and boundary condition ν.

We now prove, in two steps, the projection property for Gibbs processes. More specifically, we show that (in the limit $\delta \searrow 0$) the projection of a Gibbs process in $\mathbb{X} \times \mathbb{M}$ with PI κ_δ onto $\mathbb{X}$ gives a Gibbs process with PI κ. The projection property that is established in Proposition 2.1 of Georgii and Häggström (1996), though dealing specifically with the Potts model in $\mathbb{R}^d$, is conceptually related.

Lemma 5.12. *Let $\ell \in \mathbb{N}$ and $\psi \in \mathbf{N}_{fs}(\mathbb{X} \times \mathbb{M})$ with $\psi(C_\ell \times \mathbb{M}) = 0$. For each $\delta > 0$, let ξ_δ be a Gibbs process in $\mathbb{X} \times \mathbb{M}$ with PI $\kappa_\delta^{(C_\ell \times \mathbb{M},\psi)}$ and reference measure $\lambda \otimes \mathbb{Q}$. Then, for every set $E \in \mathcal{N}(\mathbb{X})$,*

$$\lim_{\delta \searrow 0} \mathbb{P}\big(\bar{\xi}_\delta \in E\big) = \mathrm{P}_{C_\ell,\bar{\psi}}(E).$$

Proof. For notational convenience we abbreviate $C = C_\ell$. For $\delta > 0$ and $E \in \mathcal{N}(\mathbb{X})$, Lemma 2.27 and (2.10) imply that the probability $\mathbb{P}\big(\bar{\xi}_\delta \in E\big)$ is given by

$$\frac{1}{Z_{\delta,C\times\mathbb{M}}(\psi)} \Bigg[\mathbb{1}_E(\mathbf{0}) + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{C^m} \mathbb{1}_E\Big(\sum_{j=1}^{m} \delta_{x_j}\Big) \cdot \Big(\int_{\mathbb{M}^m} \kappa_{\delta,m}(x_1,r_1,\dots,x_m,r_m,\psi)\, \mathrm{d}\mathbb{Q}^m(r_1,\dots,r_m)\Big)\, \mathrm{d}\lambda^m(x_1,\dots,x_m) \Bigg],$$

where $Z_{\delta,C\times\mathbb{M}}$ is the partition function corresponding to κ_δ and $\lambda \otimes \mathbb{Q}$. Recall from Section 2.3 that

$$\begin{aligned} &\kappa_{\delta,m}(x_1,r_1,\dots,x_m,r_m,\psi) \\ &\quad = \kappa_\delta(x_1,r_1,\psi) \cdot \kappa_\delta(x_2,r_2,\psi + \delta_{(x_1,r_1)}) \cdot \ldots \cdot \kappa_\delta(x_m,r_m,\psi + \delta_{(x_1,r_1)} + \ldots + \delta_{(x_{m-1},r_{m-1})}). \end{aligned}$$

Hence, by definition of κ_δ, the term $\kappa_{\delta,m}(x_1,r_1,\dots,x_m,r_m,\psi)$ is the indicator function that there exist no $\sim_\delta$-connections among the points $(x_1,r_1),\dots,(x_m,r_m)$ and none of these points is $\sim_\delta$-connected to ψ.

Denote by $y_1,\dots,y_k \in \mathbb{X} \setminus C$ the points of $\bar{\psi}$. For $x_1,\dots,x_m \in C$ with $x_1 \prec \ldots \prec x_m$ we have $x_m \prec y_j$ for each $j \in \{1,\dots,k\}$ and we find $\delta_0 > 0$ such that the points $x_1,\dots,x_m,y_1,\dots,y_k$ lie in different sets of the δ-partition for each $\delta < \delta_0$. Observe that the marks of the points $y_1,\dots,y_k$ do not matter in the determination of $\sim_\delta$-connections of these points with any of the points $x_1,\dots,x_m$. The definition of $\sim_\delta$ and $\mathbb{Q}$, together with the specific choice of κ_δ, yield

$$\int_{\mathbb{M}^m} \kappa_{\delta,m}(x_1, r_1, \ldots, x_m, r_m, \psi) \, \mathrm{d}\mathbb{Q}^m(r_1, \ldots, r_m) = \prod_{1 \le i < j \le m} \bigl(1 - \varphi(x_i, x_j)\bigr) \prod_{i=1}^m \prod_{j=1}^k \bigl(1 - \varphi(x_i, y_j)\bigr).$$

Now, since $\varphi = 1 - \mathrm{e}^{-v}$, Lemma 4.1 and the definition of κ_m show that this term equals

$$\prod_{1 \le i < j \le m} \mathrm{e}^{-v(x_i, x_j)} \prod_{i=1}^m \prod_{j=1}^k \mathrm{e}^{-v(x_i, y_j)} = \kappa_m(x_1, \ldots, x_m, \bar{\psi}).$$

With the symmetry properties of $\kappa_{\delta,m}$ and the fact that λ is diffuse (which allows to introduce an indicator function with $x_1 \prec \ldots \prec x_m$), dominated convergence (using $\kappa_{\delta,m} \le 1$) implies for each $F \in \mathcal{N}(\mathbb{X})$ that

$$\begin{aligned}
&\lim_{\delta \searrow 0} \sum_{m=1}^\infty \frac{1}{m!} \int_{C^m} \mathbb{1}_F\Bigl(\sum_{j=1}^m \delta_{x_j}\Bigr) \cdot \Bigl(\int_{\mathbb{M}^m} \kappa_{\delta,m}(x_1, r_1, \ldots, x_m, r_m, \psi) \, \mathrm{d}\mathbb{Q}^m(r_1, \ldots, r_m)\Bigr) \mathrm{d}\lambda^m(x_1, \ldots, x_m) \\
&\quad = \sum_{m=1}^\infty \frac{1}{m!} \int_{C^m} \mathbb{1}_F\Bigl(\sum_{j=1}^m \delta_{x_j}\Bigr) \kappa_m(x_1, \ldots, x_m, \bar{\psi}) \, \mathrm{d}\lambda^m(x_1, \ldots, x_m).
\end{aligned}$$

Applied to $F = \mathbf{N}(\mathbb{X})$ this yields $\lim_{\delta \searrow 0} Z_{\delta, C \times \mathbb{M}}(\psi) = Z_C(\bar{\psi})$. Another application of the limit relation (to $F = E$), the observation from the beginning of this proof, and Lemma 2.27 imply

$$\begin{aligned}
\lim_{\delta \searrow 0} \mathbb{P}\bigl(\bar{\xi}_\delta \in E\bigr) &= \frac{1}{Z_C(\bar{\psi})} \Bigl[\mathbb{1}_E(\mathbf{0}) + \sum_{m=1}^\infty \frac{1}{m!} \int_{C^m} \mathbb{1}_E\Bigl(\sum_{j=1}^m \delta_{x_j}\Bigr) \kappa_m(x_1, \ldots, x_m, \bar{\psi}) \, \mathrm{d}\lambda^m(x_1, \ldots, x_m)\Bigr] \\
&= \mathrm{P}_{C, \bar{\psi}}(E).
\end{aligned}$$

□

Let $\mathrm{P}^\delta_{C_\ell \times \mathbb{M}, \psi}$ denote the distribution of a finite Gibbs process with PI $\kappa_\delta^{(C_\ell \times \mathbb{M}, \psi)}$ and reference measure $\lambda \otimes \mathbb{Q}$. Then the previous lemma reads as

$$\lim_{\delta \searrow 0} \mathrm{P}^\delta_{C_\ell \times \mathbb{M}, \psi}\bigl(\{\nu \in \mathbf{N}(\mathbb{X} \times \mathbb{M}) : \bar{\nu} \in E\}\bigr) = \mathrm{P}_{C_\ell, \bar{\psi}}(E), \quad E \in \mathcal{N}(\mathbb{X}).$$

However, this relation being true only for finite boundary conditions ψ is not enough to consider infinite range interactions. Fortunately, just like in Section 5.3, the integrability assumption (5.10) allows us to extract more information.

Lemma 5.13. *Let $\ell \in \mathbb{N}$. Assume that η is a Gibbs process in $\mathbb{X}$ with pair potential v and reference measure λ. Then, for each $E \in \mathcal{N}(\mathbb{X})$,*

$$\lim_{\delta \searrow 0} \mathbb{E}\Bigl|\mathrm{P}^\delta_{C_\ell \times \mathbb{M}, \hat{\eta}_{\mathbb{X} \setminus C_\ell}}\bigl(\{\nu \in \mathbf{N}(\mathbb{X} \times \mathbb{M}) : \bar{\nu} \in E\}\bigr) - \mathrm{P}_{C_\ell, \eta_{\mathbb{X} \setminus C_\ell}}(E)\Bigr| = 0.$$

Proof. First of all, note that, for any $n > \ell$ and $x_1, \ldots, x_m \in C_\ell$,

$$\begin{aligned}
&\mathbb{E}\Bigl|\kappa_m(x_1, \ldots, x_m, \eta_{\mathbb{X} \setminus C_\ell}) - \int_{\mathbb{M}^m} \kappa_{\delta,m}(x_1, r_1, \ldots, x_m, r_m, \hat{\eta}_{\mathbb{X} \setminus C_\ell}) \, \mathrm{d}\mathbb{Q}^m(r_1, \ldots, r_m)\Bigr| \\
&\quad \le \mathbb{E}\bigl|\kappa_m(x_1, \ldots, x_m, \eta_{\mathbb{X} \setminus C_\ell}) - \kappa_m(x_1, \ldots, x_m, \eta_{C_n \setminus C_\ell})\bigr| \\
&\qquad + \mathbb{E}\Bigl|\kappa_m(x_1, \ldots, x_m, \eta_{C_n \setminus C_\ell}) - \int_{\mathbb{M}^m} \kappa_{\delta,m}(x_1, r_1, \ldots, x_m, r_m, \hat{\eta}_{C_n \setminus C_\ell}) \, \mathrm{d}\mathbb{Q}^m(r_1, \ldots, r_m)\Bigr| \\
&\qquad + \int_{\mathbb{M}^m} \mathbb{E}\bigl|\kappa_{\delta,m}(x_1, r_1, \ldots, x_m, r_m, \hat{\eta}_{C_n \setminus C_\ell}) - \kappa_{\delta,m}(x_1, r_1, \ldots, x_m, r_m, \hat{\eta}_{\mathbb{X} \setminus C_\ell})\bigr| \, \mathrm{d}\mathbb{Q}^m(r_1, \ldots, r_m).
\end{aligned} \tag{5.12}$$

By Corollary 4.9 we have $\lim_{n \to \infty} \kappa(x, \mu_{C_n}) = \kappa(x, \mu)$ for all $x \in \mathbb{X}$ and $\mu \in \mathbf{N}(\mathbb{X})$. Thus, by definition of κ_m

and dominated convergence (using that $\kappa_m \leq 1$),

$$\lim_{n\to\infty} \mathbb{E}\big|\kappa_m(x_1,\dots,x_m,\eta_{\mathbb{X}\setminus C_\ell}) - \kappa_m(x_1,\dots,x_m,\eta_{C_n\setminus C_\ell})\big| = 0.$$

For each fixed $n > \ell$, and if $x_1 \prec \dots \prec x_m$, the proof of Lemma 5.12 and dominated convergence yield

$$\lim_{\delta\searrow 0} \mathbb{E}\left|\kappa_m(x_1,\dots,x_m,\eta_{C_n\setminus C_\ell}) - \int_{\mathbb{M}^m} \kappa_{\delta,m}(x_1,r_1,\dots,x_m,r_m,\hat{\eta}_{C_n\setminus C_\ell})\,\mathrm{d}\mathbb{Q}^m(r_1,\dots,r_m)\right| = 0.$$

As for the last term in (5.12), recall that $\kappa_{\delta,m}(x_1,r_1,\dots,x_m,r_m,\hat{\eta}_{\mathbb{X}\setminus C_\ell})$ is equal to 1 if there is no $\sim_\delta$-connection between any of the points $(x_1,r_1),\dots,(x_m,r_m)$ and none of these points is connected to $\hat{\eta}_{\mathbb{X}\setminus C_\ell}$, and it is equal to 0 otherwise. Hence, if $\kappa_{\delta,m}(x_1,r_1,\dots,x_m,r_m,\hat{\eta}_{\mathbb{X}\setminus C_\ell}) = 1$ then $\kappa_{\delta,m}(x_1,r_1,\dots,x_m,r_m,\hat{\eta}_{C_n\setminus C_\ell}) = 1$, so in the only case where the difference appearing in the last term of (5.12) can give a value different from 0, one of the points $(x_1,r_1),\dots,(x_m,r_m)$ has to be connected to $\hat{\eta}_{\mathbb{X}\setminus C_n}$. Thus, for each $\delta > 0$ and $n > \ell$,

$$\begin{aligned}
I_{\delta,n} &= \int_{\mathbb{M}^m} \mathbb{E}\big|\kappa_{\delta,m}(x_1,r_1,\dots,x_m,r_m,\hat{\eta}_{C_n\setminus C_\ell}) - \kappa_{\delta,m}(x_1,r_1,\dots,x_m,r_m,\hat{\eta}_{\mathbb{X}\setminus C_\ell})\big|\,\mathrm{d}\mathbb{Q}^m(r_1,\dots,r_m)\\
&\leq \int_{\mathbb{M}^m} \mathbb{E}\Big[\mathbb{1}\big\{(x_j,r_j)\sim_\delta \hat{\eta}_{\mathbb{X}\setminus C_n} \text{ for at least one } j\in\{1,\dots,m\}\big\}\Big]\,\mathrm{d}\mathbb{Q}^m(r_1,\dots,r_m)\\
&\leq \sum_{j=1}^m \mathbb{E}\bigg[\int_{(\mathbb{X}\setminus C_n)\times\mathbb{M}} \int_{\mathbb{M}^m} \mathbb{1}\big\{(x_j,r_j)\sim_\delta (x,r)\big\}\,\mathrm{d}\mathbb{Q}^m(r_1,\dots,r_m)\,\mathrm{d}\hat{\eta}(x,r)\bigg].
\end{aligned}$$

By construction, the marks of η are not used in any decision about connections in the above term as $x_1,\dots,x_m$ are points in C_ℓ and thus always smaller than points of $\eta_{\mathbb{X}\setminus C_n}$ with respect to the order $\prec$ on $\mathbb{X}$, so only the marks $r_1,\dots,r_m$ matter. We obtain

$$\begin{aligned}
I_{\delta,n} &\leq \sum_{j=1}^m \mathbb{E}\bigg[\int_{\mathbb{X}\setminus C_n} \int_{\mathbb{M}^m} \mathbb{1}\big\{(x_j,r_j)\sim_\delta (x,s_*)\big\}\,\mathrm{d}\mathbb{Q}^m(r_1,\dots,r_m)\,\mathrm{d}\eta(x)\bigg]\\
&= \sum_{j=1}^m \mathbb{E}\bigg[\int_{\mathbb{X}\setminus C_n} \varphi(x_j,x)\,\mathrm{d}\eta(x)\bigg]\\
&\leq \sum_{j=1}^m \int_{\mathbb{X}\setminus C_n} \varphi(x_j,x)\,\mathrm{d}\lambda(x),
\end{aligned}$$

using the GNZ equation for η (and the fact that the corresponding PI κ is bounded by 1). The last term in the previous display, however, goes to 0 as $n\to\infty$ by (5.9) for λ^m-a.e. $(x_1,\dots,x_m)\in\mathbb{X}^m$. We conclude that $\lim_{n\to\infty}\sup_{\delta>0} I_{\delta,n} = 0$ and the same $\varepsilon/3$-argument used in the proof of Lemma 5.10 shows that the left hand side of (5.12) converges to 0, as $\delta\searrow 0$, for λ^m-a.e. $(x_1,\dots,x_m)\in C_\ell^m$.

Dominated convergence implies for each $E\in\mathcal{N}(\mathbb{X})$ that, as $\delta\searrow 0$,

$$\begin{aligned}
&\sum_{m=1}^\infty \frac{1}{m!}\int_{C_\ell^m} \mathbb{1}_E\Big(\sum_{j=1}^m \delta_{x_j}\Big)\cdot\bigg(\int_{\mathbb{M}^m} \kappa_{\delta,m}(x_1,r_1,\dots,x_m,r_m,\hat{\eta}_{\mathbb{X}\setminus C_\ell})\,\mathrm{d}\mathbb{Q}^m(r_1,\dots,r_m)\bigg)\mathrm{d}\lambda^m(x_1,\dots,x_m)\\
&\xrightarrow{L^1(\mathbb{P})} \sum_{m=1}^\infty \frac{1}{m!}\int_{C_\ell^m} \mathbb{1}_E\Big(\sum_{j=1}^m \delta_{x_j}\Big)\,\kappa_m(x_1,\dots,x_m,\eta_{\mathbb{X}\setminus C_\ell})\,\mathrm{d}\lambda^m(x_1,\dots,x_m).
\end{aligned}$$

By (2.10) (having Lemma 2.27 in mind) the previous line can be rewritten as

$$Z_{\delta,C_\ell\times\mathbb{M}}(\hat{\eta}_{\mathbb{X}\setminus C_\ell})\cdot \mathrm{P}^\delta_{C_\ell\times\mathbb{M},\hat{\eta}_{\mathbb{X}\setminus C_\ell}}\big(\{\nu\in\mathbf{N}(\mathbb{X}\times\mathbb{M}):\bar{\nu}\in E\}\big) \xrightarrow{L^1(\mathbb{P})} Z_{C_\ell}(\eta_{\mathbb{X}\setminus C_\ell})\cdot \mathrm{P}_{C_\ell,\eta_{\mathbb{X}\setminus C_\ell}}(E).$$

Moreover, the choice $E = \mathbf{N}(\mathbb{X})$ gives

$$Z_{\delta, C_\ell \times \mathbb{M}}(\hat{\eta}_{\mathbb{X} \setminus C_\ell}) \xrightarrow{L^1(\mathbb{P})} Z_{C_\ell}(\eta_{\mathbb{X} \setminus C_\ell}).$$

Due to the path-wise inequalities

$$1 \leq Z_{\delta, C_\ell \times \mathbb{M}}(\hat{\eta}_{\mathbb{X} \setminus C_\ell}) \leq \mathrm{e}^{\lambda(C_\ell)} \quad \text{and} \quad 1 \leq Z_{C_\ell}(\eta_{\mathbb{X} \setminus C_\ell}) \leq \mathrm{e}^{\lambda(C_\ell)}$$

we also have

$$\frac{1}{Z_{\delta, C_\ell \times \mathbb{M}}(\hat{\eta}_{\mathbb{X} \setminus C_\ell})} \xrightarrow{L^1(\mathbb{P})} \frac{1}{Z_{C_\ell}(\eta_{\mathbb{X} \setminus C_\ell})}$$

(using that the map $t \mapsto t^{-1}$, $t > 0$, is Lipschitz continuous when restricted to a compact set). Using the shorthand notation $\{\bar{\nu} \in E\} = \{\nu \in \mathbf{N}(\mathbb{X} \times \mathbb{M}) : \bar{\nu} \in E\}$ we obtain, from the previous observations,

$$\begin{aligned}
&\mathbb{E}\Big|\mathrm{P}^{\delta}_{C_\ell \times \mathbb{M}, \hat{\eta}_{\mathbb{X} \setminus C_\ell}}\big(\{\bar{\nu} \in E\}\big) - \mathrm{P}_{C_\ell, \eta_{\mathbb{X} \setminus C_\ell}}(E)\Big| \\
&\quad \leq \mathbb{E}\bigg|\frac{1}{Z_{\delta, C_\ell \times \mathbb{M}}(\hat{\eta}_{\mathbb{X} \setminus C_\ell})} \cdot \Big(Z_{\delta, C_\ell \times \mathbb{M}}(\hat{\eta}_{\mathbb{X} \setminus C_\ell}) \cdot \mathrm{P}^{\delta}_{C_\ell \times \mathbb{M}, \hat{\eta}_{\mathbb{X} \setminus C_\ell}}\big(\{\bar{\nu} \in E\}\big) - Z_{C_\ell}(\eta_{\mathbb{X} \setminus C_\ell}) \cdot \mathrm{P}_{C_\ell, \eta_{\mathbb{X} \setminus C_\ell}}(E)\Big)\bigg| \\
&\qquad + \mathbb{E}\bigg|\bigg(\frac{1}{Z_{\delta, C_\ell \times \mathbb{M}}(\hat{\eta}_{\mathbb{X} \setminus C_\ell})} - \frac{1}{Z_{C_\ell}(\eta_{\mathbb{X} \setminus C_\ell})}\bigg) \cdot Z_{C_\ell}(\eta_{\mathbb{X} \setminus C_\ell}) \cdot \mathrm{P}_{C_\ell, \eta_{\mathbb{X} \setminus C_\ell}}(E)\bigg| \\
&\quad \leq \mathbb{E}\Big|Z_{\delta, C_\ell \times \mathbb{M}}(\hat{\eta}_{\mathbb{X} \setminus C_\ell}) \cdot \mathrm{P}^{\delta}_{C_\ell \times \mathbb{M}, \hat{\eta}_{\mathbb{X} \setminus C_\ell}}\big(\{\bar{\nu} \in E\}\big) - Z_{C_\ell}(\eta_{\mathbb{X} \setminus C_\ell}) \cdot \mathrm{P}_{C_\ell, \eta_{\mathbb{X} \setminus C_\ell}}(E)\Big| \\
&\qquad + \mathrm{e}^{\lambda(C_\ell)} \cdot \mathbb{E}\bigg|\frac{1}{Z_{\delta, C_\ell \times \mathbb{M}}(\hat{\eta}_{\mathbb{X} \setminus C_\ell})} - \frac{1}{Z_{C_\ell}(\eta_{\mathbb{X} \setminus C_\ell})}\bigg|,
\end{aligned}$$

and the right hand side goes to 0 as $\delta \searrow 0$. □

5.5. THE MAIN RESULT

With the technical work in the previous three sections, we are now able to prove the main result of this chapter, and arguably of the whole thesis. Namely, we show that if (v, λ) is subcritical then the pair interaction Gibbs process with pair potential v and reference measure λ is unique in distribution. This result gives an affirmative answer to the question asked in Section 2.6 of Jansen (2019), whether absence of percolation in the random connection model implies uniqueness of a corresponding Gibbs measure.

For a more comfortable reading, we recall the given setting. We consider an arbitrary complete separable metric space $(\mathbb{X}, d)$ endowed with its Borel σ-field $\mathcal{X}$ and we let λ be a locally finite measure on $(\mathbb{X}, \mathcal{X})$. A non-negative (repulsive) pair potential is a measurable and symmetric map

$$v : \mathbb{X} \times \mathbb{X} \to [0, \infty].$$

Moreover, we consider a random connection model with connection function $\varphi = 1 - \mathrm{e}^{-v}$ based on a Poisson process with intensity measure λ, which is said to be subcritical if its clusters are finite almost surely.

Theorem 5.14. *If the random connection model based on $\varphi = 1 - \mathrm{e}^{-v}$ and λ is subcritical, then, up to equality in distribution, there exists exactly one Gibbs process with pair potential v and reference measure λ.*

Our strategy for the proof is as follows. We first show, similar to the proof of Theorem 3.32, that we can restrict to diffuse measures λ. We then apply Theorem 3.31 in the extended state space $\mathbb{X} \times \mathbb{M}$ from Section 5.2 and proceed as in the proof of Theorem 3.32. The observations thus made in $\mathbb{X} \times \mathbb{M}$ carry over to $\mathbb{X}$ with the approximation and projection properties established in Sections 5.3 and 5.4.

Proof. Recall from Remark 5.1 that the subcriticality assumption implies that, for λ-a.e. $x \in \mathbb{X}$,

$$\int_{\mathbb{X}} \left(1 - \mathrm{e}^{-v(x,y)}\right) \mathrm{d}\lambda(y) < \infty.$$

By Corollary 4.14, or Theorem B.1 of Jansen (2019), there exists a Gibbs process with pair potential v and reference measure λ. It remains to prove that this process is unique in distribution.

We first argue that we can assume, without loss of generality, that λ is diffuse. To this end, we consider the space $\tilde{\mathbb{X}} = \mathbb{X} \times [0,1]$, equipped with some complete metric that induces the product topology, and let $\tilde{\lambda} = \lambda \otimes \mathcal{L}^1_{[0,1]}$. Define

$$\tilde{v} : \tilde{\mathbb{X}} \times \tilde{\mathbb{X}} \to [0,\infty], \quad \tilde{v}\big((x,r),(y,s)\big) = v(x,y).$$

Let $\tilde{C}_{(z,r)}$ denote the cluster of $(z,r) \in \tilde{\mathbb{X}}$ in a RCM with connection function $\tilde{\varphi} = 1 - \exp(-\tilde{v})$ based on $\tilde{\Phi}^{(z,r)}$, where $\tilde{\Phi} = \sum_{k=1}^{\Phi(\mathbb{X})} \delta_{(X_k,U_k)}$ is a uniform randomization of a Poisson process $\Phi = \sum_{k=1}^{\Phi(\mathbb{X})} \delta_{X_k}$ in $\mathbb{X}$ with intensity measure λ. This randomization is defined with the help of independent random variables $U_1, U_2, \ldots$ that are uniformly distributed on $[0,1]$, with the whole sequence $(U_k)_{k\in\mathbb{N}}$ independent of Φ. By the marking theorem (Theorem 5.6 of Last and Penrose, 2017), $\tilde{\Phi}$ is a Poisson process in $\tilde{\mathbb{X}}$ with intensity measure $\tilde{\lambda}$. Choosing on $\tilde{\mathbb{X}}$ the lexicographical order (with the order $\prec$ on $\mathbb{X}$ in the first component), the definition of the RCM in (5.1) ensures that, for any decision to determine connections $(x,r) \leftrightarrow (y,s)$ and $x \leftrightarrow y$, the same uniform random variables are used. Thus, we have

$$\tilde{C}_{(z,r)}(\tilde{\mathbb{X}}) \le C_z(\mathbb{X}) \quad \mathbb{P}\text{-a.s.},$$

where C_z is the cluster of z in the RCM with connection function $\varphi = 1 - \mathrm{e}^{-v}$ based on Φ^z. (Notice that inequality enters since points $(x,r),(x,s)$ might not be connected in the randomized RCM.) Hence, if (v,λ) is subcritical then so is $(\tilde{v},\tilde{\lambda})$. Moreover, if κ is the PI corresponding to v then $\tilde{\kappa}\big((x,r),\psi\big) = \kappa\big(x,\psi(\,\cdot \times [0,1])\big)$ is the PI corresponding to $\tilde{v}$. Lemma 3.25 implies that, if η is a Gibbs process in $\mathbb{X}$ with pair potential v and reference measure λ, then any uniform randomization $\tilde{\eta}$ of η is a Gibbs process with pair potential $\tilde{v}$ and reference measure $\tilde{\lambda}$. Now, if (v,λ) is subcritical and Theorem 5.14 holds for diffuse reference measures, then the Gibbs process with pair interaction $\tilde{v}$ is unique in distribution. Consequently, uniform randomizations $\tilde{\eta}, \tilde{\eta}'$ of two Gibbs processes η, η' with pair potential v satisfy $\tilde{\eta} \overset{d}{=} \tilde{\eta}'$, and we obtain

$$\eta = \tilde{\eta}(\,\cdot \times [0,1]) \overset{d}{=} \tilde{\eta}'(\,\cdot \times [0,1]) = \eta'.$$

Hence, we may assume in the following that λ is diffuse.

For the remainder of the proof we rely heavily on the notation introduced in Sections 5.2 – 5.4. Fix $\ell \in \mathbb{N}$ and let Ψ be a Poisson process in $\mathbb{X} \times \mathbb{M}$ with intensity measure $\lambda \otimes \mathbb{Q}$. For $\psi, \psi' \in \mathbf{N}_{(\mathbb{X}\setminus C_\ell)\times\mathbb{M}}(\mathbb{X} \times \mathbb{M})$, every $\delta > 0$, and any $E \in \mathcal{N}_D(\mathbb{X})$ with $D \in \mathcal{X}$ and $D \subset C_\ell$, Theorem 3.31 immediately gives

$$\left|\mathrm{P}^\delta_{C_\ell\times\mathbb{M},\psi}\big(\{\bar{\nu} \in E\}\big) - \mathrm{P}^\delta_{C_\ell\times\mathbb{M},\psi'}\big(\{\bar{\nu} \in E\}\big)\right| \le \int_{D\times\mathbb{M}} \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} (\psi + \psi')\Big)\, \mathrm{d}(\lambda \otimes \mathbb{Q})(x,r),$$

with the abbreviation $\{\bar{\nu} \in E\} = \{\nu \in \mathbf{N}(\mathbb{X} \times \mathbb{M}) : \bar{\nu} \in E\}$, noting that $\{\bar{\nu} \in E\} \in \mathcal{N}_{D\times\mathbb{M}}(\mathbb{X} \times \mathbb{M})$.

Now, let η and η' be two Gibbs processes in $\mathbb{X}$ with pair potential v and reference measure λ. Assume, without loss of generality, that (η,η') is independent of Ψ and the double sequence of uniform random variables used to define the RCMs. Let $D \in \mathcal{X}_b^{**}$ be arbitrary and choose ℓ large enough so that $D \subset C_\ell$. Take $E \in \mathcal{N}_D(\mathbb{X})$. By the DLR equation from Lemma 2.47 we have

$$\left|\mathbb{P}^\eta(E) - \mathbb{P}^{\eta'}(E)\right| = \left|\mathbb{E}\Big[\mathrm{P}_{C_\ell,\eta_{\mathbb{X}\setminus C_\ell}}(E)\Big] - \mathbb{E}\Big[\mathrm{P}_{C_\ell,\eta'_{\mathbb{X}\setminus C_\ell}}(E)\Big]\right| \le \mathbb{E}\left|\mathrm{P}_{C_\ell,\eta_{\mathbb{X}\setminus C_\ell}}(E) - \mathrm{P}_{C_\ell,\eta'_{\mathbb{X}\setminus C_\ell}}(E)\right|$$

and, by the projection property from Lemma 5.13, the right hand side equals

$$\limsup_{\delta \searrow 0} \mathbb{E}\Big|\mathrm{P}^{\delta}_{C_\ell \times \mathbb{M}, \hat{\eta}_{\mathbb{X}\setminus C_\ell}}\big(\{\bar{\nu} \in E\}\big) - \mathrm{P}^{\delta}_{C_\ell \times \mathbb{M}, \hat{\eta}'_{\mathbb{X}\setminus C_\ell}}\big(\{\bar{\nu} \in E\}\big)\Big|.$$

Applying the bound from the previous paragraph, the above limit superior is bounded by

$$\begin{aligned}
&\limsup_{\delta \searrow 0} \int_{D\times\mathbb{M}} \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} (\hat{\eta}_{\mathbb{X}\setminus C_\ell} + \hat{\eta}'_{\mathbb{X}\setminus C_\ell})\Big)\,\mathrm{d}(\lambda \otimes \mathbb{Q})(x,r) \\
&\quad \le \limsup_{\delta \searrow 0} \int_{D\times\mathbb{M}} \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}_{\mathbb{X}\setminus C_\ell}\Big)\,\mathrm{d}(\lambda \otimes \mathbb{Q})(x,r) + \limsup_{\delta \searrow 0} \int_{D\times\mathbb{M}} \mathbb{P}\Big((x,r) \overset{\Psi_{C_\ell}}{\sim_\delta} \hat{\eta}'_{\mathbb{X}\setminus C_\ell}\Big)\,\mathrm{d}(\lambda \otimes \mathbb{Q})(x,r).
\end{aligned}$$

Applying Lemma 5.10 to each of the two terms, we arrive at

$$\big|\mathbb{P}^{\eta}(E) - \mathbb{P}^{\eta'}(E)\big| \le \int_D \mathbb{P}\Big(x \xleftrightarrow{\Gamma_\ell^{x,\eta}} \eta_{\mathbb{X}\setminus C_\ell}\Big)\,\mathrm{d}\lambda(x) + \int_D \mathbb{P}\Big(x \xleftrightarrow{\Gamma_\ell^{x,\eta'}} \eta'_{\mathbb{X}\setminus C_\ell}\Big)\,\mathrm{d}\lambda(x).$$

Using Lemma 5.11 and dominated convergences (for $\ell \to \infty$) twice, the right hand side is seen to converge to 0. As $D \in \mathcal{X}_b^{**}$ and $E \in \mathcal{N}_D(\mathbb{X})$ were arbitrary, the measures $\mathbb{P}^{\eta}$ and $\mathbb{P}^{\eta'}$ agree on the algebra $\mathcal{Z}^{**}$ from Corollary 5.8, which generates $\mathcal{N}(\mathbb{X})$. Hence, $\mathbb{P}^{\eta} = \mathbb{P}^{\eta'}$ and the proof is complete. □

In the remainder of this Section we discuss several corollaries of Theorem 5.14 and classify the result with regard to the existing literature. We also allude to possible further research.

Introducing an intensity parameter $\gamma \ge 0$ and recalling the notion of a critical intensity γ_c from (5.3) with $\varphi = 1 - \mathrm{e}^{-v}$, Theorem 5.14 and Proposition 5.3 immediately give the following result.

Corollary 5.15. *Assume that $\gamma < \gamma_c$. Then, up to equality in distribution, there exists exactly one Gibbs process with pair interaction v and reference measure $\gamma\lambda$.*

We think of Corollary 5.15 as of an explicit lower bound on the range of uniqueness. Even though γ_c is not explicitly known, it admits a clear probabilistic meaning and the Poisson-based RCM can be simulated easily. To get an impression of the critical intensities, we have simulated three examples, including two interaction functions from the physics literature. Corresponding results are presented in Section 5.6. Example 5.5 shows that, in one dimension, repulsive pair interaction processes with finite range are always unique in distribution.

In the stationary setting discussed in Section 5.1.1 we known that $0 < \gamma_c < \infty$ in dimensions $d \ge 2$ (cf. Penrose, 1991). But even in this case Theorem 5.14 raises the question for sufficient conditions implying that (v, λ) is subcritical. The rather crude criterion from Theorem 5.6 and Remark 5.7, based on a branching construction, allows for the following derivation from Theorem 5.14.

Corollary 5.16. *Assume that*

$$\operatorname*{ess\,sup}_{x\in\mathbb{X}} \int_{\mathbb{X}} \big(1 - \mathrm{e}^{-v(x,y)}\big)\,\mathrm{d}\lambda(y) < 1.$$

Then, up to equality in distribution, there exists exactly one Gibbs process with pair interaction v and reference measure λ.

Rewritten in terms of $\gamma\lambda$ and interpreting the branching bound as a lower bound on the critical intensity, the following result is also immediate.

Corollary 5.17. *Assume that*

$$\gamma < \bigg(\operatorname*{ess\,sup}_{x\in\mathbb{X}} \int_{\mathbb{X}} \big(1 - \mathrm{e}^{-v(x,y)}\big)\,\mathrm{d}\lambda(y)\bigg)^{-1}.$$

Then, up to equality in distribution, there exists exactly one Gibbs process with pair interaction v and reference measure $\gamma\lambda$. (If $v = 0$ almost everywhere, the result holds for any $\gamma \ge 0$.)

Corollary 5.17 is the main result of Houdebert and Zass (2022), proven there in $\mathbb{R}^d$ under two additional assumptions by applying a classical method due to Dobrushin (1968a) to a suitable discretization. Our Corollary 5.17 generalizes this in several ways. First of all we work on a general space $\mathbb{X}$ and not just on $\mathbb{R}^d$. Second, we do not need any hard core assumption on v. Neither do we need another technical assumption made by Houdebert and Zass (2022). Finally, and perhaps most importantly, the simulations in Section 5.6 show that Theorem 5.14 extends the bounds on the uniqueness region far beyond the branching bound.

Theorem 5.14 also considerably generalizes the percolation criterion derived by Hofer-Temmel and Houdebert (2019) for translation invariant pair potentials v in $\mathbb{R}^d$ with a finite range $R > 0$. Indeed, the uniqueness result in Section 3.2.1 of Hofer-Temmel and Houdebert (2019) requires the RCM with connection function $\varphi_R(x) = \mathbb{1}\big\{|x| \le R\big\}$ to be subcritical. Since $\varphi = 1 - \mathrm{e}^{-v} \le \varphi_R$, the critical intensity associated with φ is larger than the one associated with φ_R. As our bound takes into account the full information contained in the pair potential and not just its range, the difference can be very significant as illustrated in Section 5.6. The RCM associated with φ_R is referred to as the *Gilbert graph*, see e.g. Last and Penrose (2017). The corresponding pair potential is the hard core potential $v_R(x) = \infty \cdot \mathbb{1}\big\{|x| \le R\big\}$. One further setting which illustrates the shortcomings of the uniqueness result of Hofer-Temmel and Houdebert (2019) is the following observation.

Corollary 5.18. *Assume that $\gamma \ge 0$ and $\beta \ge 0$ are such that*

$$\gamma \cdot \operatorname*{ess\,sup}_{x \in \mathbb{X}} \int_{\mathbb{X}} \big(1 - \mathrm{e}^{-\beta \cdot v(x,y)}\big) \, \mathrm{d}\lambda(y) < 1.$$

Then, up to equality in distribution, there exists exactly one Gibbs process with pair interaction $\beta \cdot v$ and reference measure $\gamma\lambda$.

The constant β is called *inverse temperature*. The bounds by Hofer-Temmel and Houdebert (2019) do not change under multiplication of v with β, no matter how small β is, since only the range is taken into account. As pointed out by Jansen (2019) and demonstrated by Houdebert and Zass (2022), the branching bound provides an improvement over Hofer-Temmel and Houdebert (2019) for small β. This is backed by Corollary 5.18 as small values of β intuitively make the essential supremum smaller and hence allow for larger values of γ. Theorem 5.14 improves upon the branching bound even further, for all values of β. An example in Section 5.6 (see Table 5.2) shows that this improvement can be quite significant.

The Dobrushin criterion mentioned above was also used in other papers to establish uniqueness of Gibbs measures, for instance by Georgii and Häggström (1996) (without giving the details) and by Conache et al. (2018). The results of these papers allow the pair potential to take negative values but do not provide explicit information on the domain of uniqueness. A further drawback of this approach is the assumption of a finite range, which poses a severe restriction.

Another method for proving uniqueness of Gibbs distributions are fixed point methods based on the Kirkwood–Salsburg equations from Section 4.3 and the fact that the correlation functions determine the distribution of a point process under suitable assumptions. This method can be traced back to Ruelle (1969). For some recent contributions we refer to Jansen (2019), Zass (2022), and Jansen and Kolesnikov (2021). The uniqueness region identified by this method of cluster expansion is characterized by the contractivity of certain integral operators and does not seem to have an explicit probabilistic interpretation. In view of the method and the special cases discussed by Jansen (2019), we expect this region to be comparable with the branching bound or its generalization in Theorem 5.6. While the paper by Betsch and Last (2022), of which the chapter at hand essentially consists, has been under review, Michelen and Perkins (2021a) have extended their earlier preprint and completed the formal details of a new analytic approach to uniqueness, which applies to non-negative, translation invariant pair potentials on $\mathbb{R}^d$ and which significantly improves the branching bounds. In their follow-up work, Michelen and Perkins (2021b) extend the result to more general state spaces, dropping the assumption of translation invariance.

Yet another method for proving uniqueness is to identify a Gibbs distribution as a stationary and reversible measure with respect to a suitable Markovian dynamics, cf. Ferrari et al. (2002) and Schreiber and Yukich (2013). Uniqueness follows if the so-called ancestor clans, coming from an embedding into a space-time Poisson process, are finite. The resulting bounds on the domain of uniqueness are not explicit but might be comparable with the branching bounds (see also the discussion by Beneš et al., 2020).

As one of our main tools for proving Theorem 5.14 was the disagreement coupling of Hofer-Temmel and Houdebert (2019) and Last and Otto (2021), which was inspired by the results of van den Berg and Maes (1994) in the discrete setting, we should also mention at this point that a non-Poisson version of disagreement percolation was applied by Dereudre and Houdebert (2021) to prove uniqueness of the Widom-Rowlinson process for a certain range of parameters. This Gibbs process is not governed by a pair potential but enjoys nice and rather specific monotonicity and symmetry properties.

The connection function $\varphi = 1 - \mathrm{e}^{-v}$, which appears prominently in this chapter, was already used by Given and Stell (1990) and Georgii and Häggström (1996) for a Fortuin-Kasteleyn type coupling of a Potts model and the continuum random cluster model. While their coupling proceeds by conditioning, we use a marked Poisson process for embedding two Gibbs processes with different boundary conditions and, at the same time, for constructing a RCM. This way we are able to directly refer to the percolation properties of a Poisson-driven RCM. The function $-\varphi = \mathrm{e}^{-v} - 1$ is called *Mayer's f-function* and appears in the context of cluster expansions, referring once more to Jansen (2019) for an overview. Closely related to this last point is the fact the Mayer's f-function appears in the Kirkwood–Salsburg equation in Theorem 4.11.

Before we turn to a simulation study in explicit examples, we return to the large and slightly more abstract class of particle processes from Section 3.4, and we make three comments alluding to possible further examples in the discrete setting as well as future research.

Example 5.19. As in Examples 3.40 and 3.42, suppose that $\mathbb{X}$ is the space $\mathcal{C}^{(d)}$ of non-empty compact subsets of $\mathbb{R}^d$. Let $V : \mathcal{C}^{(d)} \cup \{\varnothing\} \to [0, \infty]$ be measurable with $V(\varnothing) = 0$. For instance, V could be the volume or, if the reference measure λ is concentrated on the convex bodies, a linear combination of the intrinsic volumes. Assume that the pair potential is given by

$$v(K, L) = V(K \cap L), \quad K, L \in \mathcal{C}^{(d)}.$$

As a percolation model, the associated RCM with connection function $\varphi = 1 - \mathrm{e}^{-v}$ is considerably more general than the Boolean model. The latter arises in the special case $V(K) = \infty \cdot \mathbb{1}\{K \neq \varnothing\}$, where the connection function is given by $\varphi_\infty(K, L) = \mathbb{1}\{K \cap L \neq \varnothing\}$ and the connections do not involve any additional randomness.

Theorem 5.14 requires (φ, λ) to be subcritical, while the previous results of Hofer-Temmel and Houdebert (2019) and Beneš et al. (2020), when specialized to non-negative pair potentials, require the Boolean model $(\varphi_\infty, \lambda)$ to be subcritical, as in Corollary 3.41. Since $\varphi(K, L) \leq \varphi_\infty(K, L)$ our result gives better bounds on the uniqueness region. In particular, $\varphi(K, L) < \varphi_\infty(K, L)$ whenever $V(K \cap L) > 0$. If, for instance, V is continuous at $\varnothing$, then $\varphi(K, L)$ can be arbitrarily small but still $\varphi_\infty(K, L) = 1$. ○

Remark 5.20. Since we allow for a non-diffuse intensity measure, our results cover the case of a discrete graph $G = (V, E)$. We may then take $\mathbb{X} = V$ and $\lambda = \gamma\lambda_0$, where λ_0 is the counting measure on V and $\gamma \geq 0$. A possible choice of a connection function is $\varphi(x, y) = p$ if $\{x, y\} \in E$ and $\varphi(x, y) = 0$ otherwise, where $p \in (0, 1)$ is a given probability. The resulting RCM is a Poisson version of a *mixed percolation* model, see Chayes and Schonmann (2000). However, φ could also be *long-ranged* as considered by Deijfen et al. (2013). In fact, it is easy to come up with a version of the model by Deijfen et al. (2013) driven by a Poisson process in $\mathbb{Z}^d$. In principle it might be possible to apply our uniqueness results to discrete models of statistical physics. ○

Remark 5.21. It is believed (see e.g. Dereudre, 2019, and the references given there) that in many Gibbs models there is some threshold $\gamma^* > 0$ such that more than one Gibbs measure exists once the activity is larger than γ^*. This phase transition was rigorously proven by Georgii and Häggström (1996) for a continuum Potts model and in particular for the Widom-Rowlinson model. Recently it was shown by Dereudre and Houdebert (2021) that the latter even undergoes a certain sharp phase transition. We expect γ_c, as defined by (5.3) with $\varphi = 1 - e^{-v}$, to be much smaller than γ^*. A careful analysis of our proofs suggests that a possible, but rather implicit, approximation of γ^* is a critical intensity which is defined in terms of random connection models based on suitable finite volume versions of a Gibbs process with pair potential v, comparable to Proposition 3.1 of Georgii and Häggström (1996). This is also supported by the discussion of Georgii et al. (2005). Improving the lower bound on the region of uniqueness from Corollary 5.15, or even understanding the phase transition, remains an active field of research. We believe that our methods can be further improved to contribute to this endeavor. ○

Remark 5.22. Another interesting question is whether repulsive pair interaction processes in the subcritical regime have mixing and decorrelation properties. In order to apply Theorems 3.34 and 3.38 to obtain such results, a bound similar to (3.13) is required for the random connection model. To the author's knowledge such bounds are not yet available. Even if they where, some technical result in the lines of Section 5.3 would be required to obtain the corresponding bounds for the approximating model of the RCM as well. At least for interactions with finite range, results of this kind should be feasible. ○

5.6. Simulation results for the critical threshold

Upon comparing the random connection model with a branching process, Theorem 5.14 implies Corollaries 5.17 and 5.18 which correspond to the uniqueness result of Houdebert and Zass (2022). However, it is known from simulations for the Gilbert graph, which corresponds to the hard sphere model, that the branching bounds are widely off the actual critical intensity in low dimensions. For an overview we refer to Ziesche (2018). Thus, we expect Theorem 5.14 to yield substantial improvements over Corollary 5.17 in low dimensions. To illustrate this point, we provide simulation results that give a rough overview on this difference in various models.

To approximate the true critical intensity of the random connection model we proceed as follows. For a given connection function in $\mathbb{R}^d$ with finite range $R > 0$, for instance one coming from a pair potential, in a given dimension d and for a given intensity $\gamma > 0$, we fix a large system size $S > R$. In the ball of radius S around the origin we now construct the cluster of the origin in the RCM based on a stationary Poisson process with intensity γ augmented by the origin. We start by simulating Poisson points (according to the given intensity) in the ball of radius R around the origin, corresponding to all points which could possibly be connected to the origin and we check each of those points for such a connection (only to the origin). In the following, we keep track of three types of points, namely *saturated* points which are part of the cluster and whose perspective has already been taken (which after the first step includes only the origin), those points which are part of the cluster but around which we might still have to simulate new Poisson points, and those Poisson points which are not yet connected to the cluster. Then we proceed algorithmically as follows. Of those cluster points from whose perspective we have not yet simulated we choose that point x which is furthest away from the origin to take its perspective, meaning that we check if that particular point connects to any of the Poisson points which already exist but are not yet part of the cluster, and then proceed to simulate new Poisson points (according to the given intensity) in that part of $B(x, R)$ which was not yet covered in previous steps and we check if any of the new points connects to x. Note that we do not look for connections between two cluster points as we already know that both are part of the cluster and an additional connection between them does not change the size of the cluster. Also notice that when we first generate new points, we do not check for connections among them immediately (and only for connections to the center of the given step), but as soon as we take the perspective of any of the cluster points we check for connections to the points in its neighborhood and thus miss no relevant connection. The algorithm terminates as soon as all cluster points are saturated, and hence the construction of the cluster was completed within $B(0, S)$, or if the cluster connects to the complement of

$B(0, S)$, meaning that some point in the cluster has a norm larger than S.

To make a decision whether the RCM percolates for a given intensity, we construct the cluster $5,000$ times. If the cluster connects to the complement of $B(0, S)$ a single time, we count that as percolation, even though, of course, a larger initial choice of the system size might have revealed that the cluster is actually finite. If the cluster lies within $B(0, S)$ in each of the $5,000$ runs, we count this as no percolation. To find a rough approximation of the critical intensity, we start at the branching lower bound, which is easily calculated for a given model, and increase the intensity by 10% as long as our algorithm decides that no percolation occurs. As soon as we first encounter percolation, we accept that intensity as an upper bound and the last intensity at which no percolation occurred as a lower bound. To refine the approximation further, we then slice the resulting interval by half two times, investigating the middle between the two bounds for percolation and adjusting the upper and lower bound accordingly. Note that the choices for S and the number of runs where made according to our computational resources, larger values for both quantities will surely lead to better estimates, but one has to observe that there is no theoretical guarantee that the simulation will provide lower bounds for the critical intensities (see the discussion below). Note that a highly related algorithmic way for exploring a cluster is explained in Chapter 5.2 of Janson et al. (2000) for the discrete setting with finitely many vertices. In particular, we followed Janson et al. (2000) in calling certain cluster points saturated.

To establish that the simulations yield plausible and useful results, we consider as a benchmark the hard sphere model which corresponds to the classical Gilbert graph, cf. Chapter 16 of Last and Penrose (2017).

Example 5.23 (**Hard spheres - Gilbert graph**)**.** The pair potential corresponding to the hard sphere model is $v_R(x, y) = \infty \cdot \mathbb{1}\big\{|x - y| \leq R\big\}$, for $x, y \in \mathbb{R}^d$, referring to the more general Example 4.15. This interaction corresponds to the Gilbert graph with range R, whose connection function is given by

$$\varphi_R(x, y) = 1 - \exp\big(-v_R(x, y)\big) = \mathbb{1}\big\{|x - y| \leq R\big\}.$$

The percolation properties of the Gilbert graph, in turn, are exactly those of the Boolean model with grains being balls with fixed radius $R/2$.

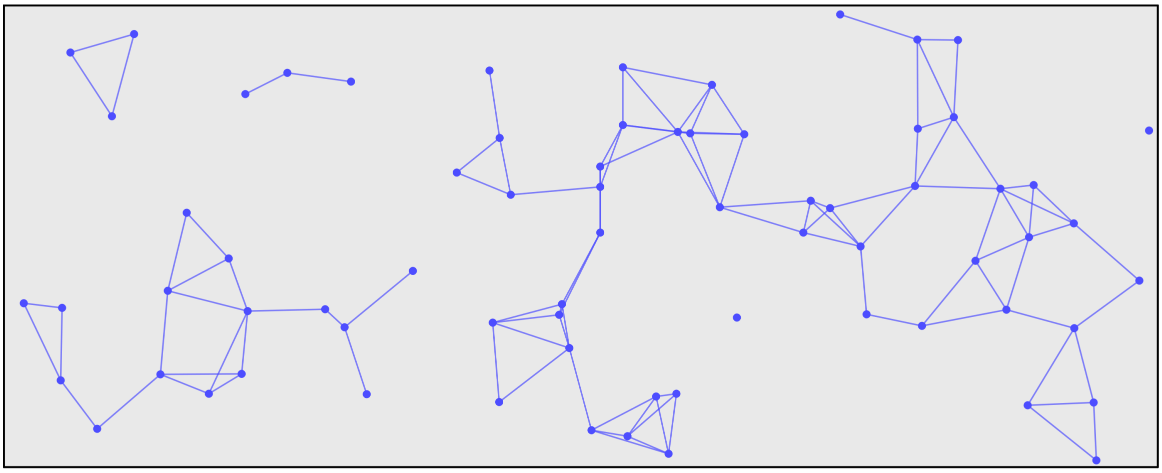


Figure 5.1.: Excerpt of a realization of a Gilbert graph with intensity $\gamma = 0.35$ and range $R = 2$. Depicted are only the Poisson points within $[0, 25] \times [0, 10]$ and their connections among each other.

The branching lower bound on the critical intensity is

$$\left(\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \varphi_R(x, y) \, \mathrm{d}\mathcal{L}^d(y)\right)^{-1} = \frac{1}{\mathcal{L}^d\big(B(0, R)\big)} = \frac{\Gamma\big(\frac{d}{2} + 1\big)}{R^d \cdot \pi^{d/2}}.$$

In the simulations in Table 5.1, we fix $R = 2$ and compare the approximation of our method of simulation with the branching bounds and some of the best approximations for the critical intensity from the literature, namely the results from Torquato and Jiao (2012). Note that it is common in physics to investigate the percolation behavior in terms of the reduced number density, so we had to convert the values from Torquato and Jiao (2012) into corresponding critical intensities by dividing with the volume of the unit ball in the given dimension. Throughout we round all values to five significant digits.

d	S	Approx. of γ_c	Branching bound	Values due to Torquato and Jiao
2	1000	0.34072	0.079577	0.35909
3	500	0.079338	0.029842	0.081621
4	300	0.025915	0.012665	0.026435
5	200	0.010039	0.0059368	0.010342

Table 5.1.: Gilbert graph

The comparison between our approximation and the lower bounds of Torquato and Jiao (2012) (with the correction reported by Torquato and Jiao, 2014), which are known to be very precise, show that our simulations, even with the very manageable choice of the simulation parameters, are reasonably well calibrated in that they provide conservative lower bounds for the critical intensity which are not wide off the mark and thus provide a solid reference for the order of magnitude of the critical intensity. The table shows that the critical intensities of the Gilbert graph are substantially larger than the branching lower bounds, which implies that our bounds on the region of uniqueness in the hard sphere model is substantially larger than the bounds by Houdebert and Zass (2022). For the hard spheres model this is not a new observation since our results agree, in this specific model, with the earlier disagreement percolation results of Hofer-Temmel and Houdebert (2019). This is due to the fact that, as discussed in Section 5.5, the range of the potential is the only information taken into account by Hofer-Temmel and Houdebert (2019) and for the hard spheres model it happens to be the only relevant parameter. In the upcoming examples a further improvement can be observed. ○

Next we consider a modification of the hard sphere model, where an arbitrary overlap of spheres is possible, namely the *penetrable spheres* model considered by Likos et al. (1998).

Example 5.24 (Penetrable spheres - adjusted Gilbert graph). Let $0 < c < \infty$ and consider the pair interaction

$$v(x, y) = c \cdot \mathbb{1}\big\{|x - y| \leq R\big\}, \quad x, y \in \mathbb{R}^d.$$

The parameter c (which in the hard sphere model is ∞) gives a measure of how valiantly spheres resist an overlap, but as c is a fixed constant, the manner of the overlap plays no role in the spheres resistance of it. The RCM corresponding to this interaction function has connection function $\varphi(x, y) = (1 - \mathrm{e}^{-c}) \cdot \mathbb{1}\big\{|x - y| \leq R\big\}$. As we simulate from the RCM perspective, we parameterize $p = 1 - \mathrm{e}^{-c} \in (0, 1)$ (in our case $p = 0.5$ and $p = 0.75$) which is then the probability that any two points with distance less than R connect. Hence, the model can be interpreted as a modified Gilbert graph with an adjusted connection probability.

The branching lower bounds for the adjusted Gilbert graph are simply those of the Gilbert graph divided by p. In order to be able to compare the simulation results for the different models, we again fix $R = 2$.

d	S	Approx. of γ_c	Branching bound
2	1000	0.48813	0.15915
3	500	0.12503	0.059683
4	300	0.041814	0.02533
5	200	0.01699	0.011874

d	S	Approx. of γ_c	Branching bound
2	1000	0.39376	0.1061
3	500	0.096166	0.039789
4	300	0.03216	0.016887
5	200	0.012459	0.0079157

Table 5.2.: Adjusted Gilbert graph with $p = 0.5$ (left) and $p = 0.75$ (right)

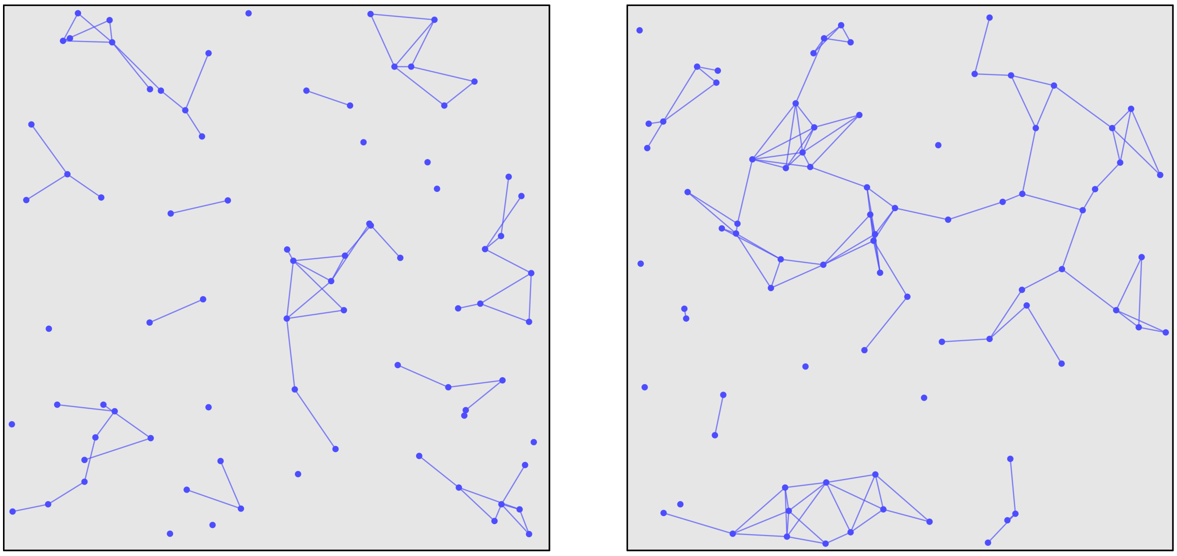


Figure 5.2.: Excerpts of realizations of adjusted Gilbert graphs, both with intensity $\gamma = 0.35$ and range $R = 2$, for $p = 0.5$ (left) and $p = 0.75$ (right). Depicted are only the Poisson points within $[0, 15] \times [0, 15]$ and their connections among each other.

The observations in the penetrable spheres model (Table 5.2) are similar to those in the case of hard spheres. Even our conservative approximations of the critical intensity, and hence the region of uniqueness, improve the branching bounds (or Dobrushin method, cf. Houdebert and Zass, 2022) by factors larger than 3 in two dimensions. In five dimensions the improvement is still by factors of more than 1.5. Also the approximated values for γ_c are substantially larger than in the Gilbert graph, which is an improvement over the classical disagreement percolation approach by Hofer-Temmel and Houdebert (2019), where only the range of v is taken into account. ○

As a last model we consider a pair interaction discussed in the physics literature, namely the *soft-sphere* or *inverse-power potential*, which can be traced back at least to Rowlinson (1964).

Example 5.25 (**Inverse-power potential).** The pair interaction function is given by

$$v(x, y) = b \cdot \frac{R^n}{|x-y|^n} \cdot \mathbb{1}\{|x-y| \leq R\}, \quad x, y \in \mathbb{R}^d,$$

where $b > 0$ is the characteristic energy and $n \in \mathbb{N}$ the hardness parameters. We fix $b = 1$ and consider $n \in \{6, 12\}$. The connection function of the corresponding RCM is

$$\varphi(x, y) = \left[1 - \exp\left(-\frac{R^n}{|x-y|^n}\right)\right] \cdot \mathbb{1}\{|x-y| \leq R\}.$$

d	S	Approx. of γ_c	Branching bound
2	1000	0.35494	0.084969
3	500	0.087095	0.03276
4	300	0.028471	0.014254
5	200	0.01128	0.0068329

d	S	Approx. of γ_c	Branching bound
2	1000	0.35272	0.082379
3	500	0.083445	0.031387
4	300	0.027011	0.013523
5	200	0.010873	0.00643

Table 5.3.: Soft-sphere model with $n = 6$ (left) and $n = 12$ (right)

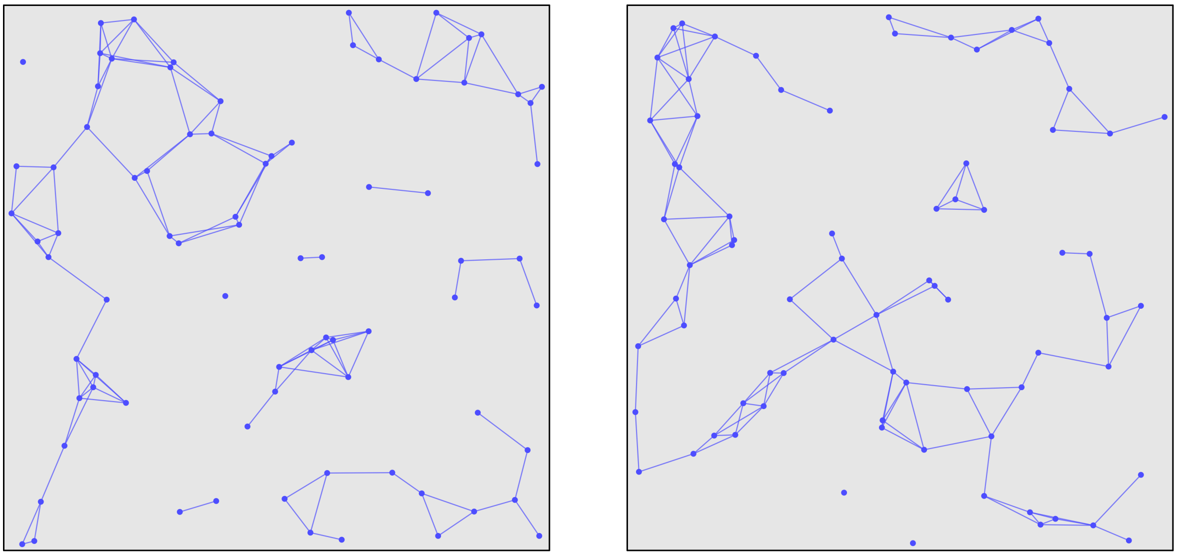


Figure 5.3.: Excerpts of realizations of the RCM corresponding to the inverse-power potential, both with intensity $\gamma = 0.35$ and range $R = 2$, for $n = 6$ (left) and $n = 12$ (right). Depicted are only the Poisson points within $[0, 15] \times [0, 15]$ and their connections among each other.

The branching lower bounds for the critical intensity calculate as

$$\left(\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int_0^R \left[1 - \exp\left(- \frac{R^n}{r^n} \right) \right] \cdot r^{d-1} \, \mathrm{d}r \right)^{-1}.$$

To ensure comparability, we again consider $R = 2$. The simulations in Table 5.3 indicate that in the soft-sphere model Theorem 5.14 yields improvements on the region of uniqueness qualitatively similar to the penetrable spheres model. ○

The results of our simulations lead to the following observations. In the dimensions we consider, the true critical intensities are larger than the branching bounds by factors between 1.4 and 4.5, depending on the model and (mostly) on the dimension. Thus, Theorem 5.14 improves the regions of uniqueness of the corresponding Gibbs process by those very same factors as compared to Houdebert and Zass (2022). Moreover, for the penetrable and soft spheres, the critical values are larger than those of the hard spheres which stands for an according improvement of the results by Hofer-Temmel and Houdebert (2019). While it is known from Meester et al. (1997) that the branching bound improves as the dimension grows, it also seems to improve (if much less so) if the overall connection probability in the RCM decreases and the critical intensity thus rises.

Note that our approximation approach is really just an approximation and not founded on a solid theoretical basis. The benchmark model (hard spheres) and the corresponding existing simulations indicate that a further improvement of our approximations (by somewhere around 2-5%) is possible, but to the author's knowledge the presented simulations are the first for general random connection models. As such, our hands-on approach provides fairly conservative approximations of the critical intensity of the RCM and gives a good idea of its overall magnitude, in particular compared to the branching bound. One possible approach to theoretically founded simulations, which would lead to approximations that come with confidence intervals, is to prove a mean-field lower bound for the RCM in lines with Equation (5.1) of Ziesche (2018) for the Gilbert graph. This is an open problem for further research. The rigorous lower bounds on the critical intensity of the Gilbert graph derived by Hall (1985) and Ziesche (2018) indicate that theoretical improvements over the branching bound should also be feasible.

APPENDIX A

Some details on measure theory and analysis

The appendix chapter contains some complements from measure theory, functional analysis, and topology. By nature, the choice of topics that belong not into the main document, but into the appendix, is subjective. The goal here is to collect technical results which are too specific to simply assume that the reader is familiar with them, but whose details are not important enough to appear in the main body of the thesis.

Such results as are well-documented in the literature are embedded in a way that makes each of the appendix sections reasonably self-contained, but these results are not proven anew. As before, we call any such result a *Proposition*, as we *propose* that the reader might accept them without proof. Claims which are significantly adapted or even generalized, as compared to the existing literature, or results which are completely new, as well as claims for which proofs are hard to find, are proven in full detail.

A.1. Factorial measures

We recall the definition of factorial measures on an arbitrary measurable space. The generality of the construction is due to Last and Penrose (2017). We mostly collect their results but add some new (and mostly technical) insights. Before we start, let us mention that a very slight technical error is made by Last and Penrose (2017): to guarantee the measurability of the term in their Equation (A.15), the localizing structure on the measurable space is explicitly needed, as laid out in Lemma A.4 below.

Fix a measurable space $(\mathbb{X}, \mathcal{X})$ with localizing structure $B_1 \subset B_2 \subset \dots$ and let the space $\mathbf{N}$ be defined as in Section 2.2. We write $[k] = \{1, \dots, k\}$ for $k \in \mathbb{N}$, $[0] = \varnothing$, and $[k] = \mathbb{N}$ if $k = \infty$. For a measure $\mu = \sum_{j=1}^{k} \delta_{x_j} \in \mathbf{N}$ with $k \in \mathbb{N}_0 \cup \{\infty\}$ and $x_j \in \mathbb{X}$, define the m-th *factorial measure* of μ on $(\mathbb{X}^m, \mathcal{X}^{\otimes m})$ as

$$\mu^{(m)} = \sum_{j_1, \dots, j_m \in [k]}^{\neq} \delta_{(x_{j_1}, \dots, x_{j_m})}, \tag{A.1}$$

where the superscript $\neq$ is used to indicate that the indices in the summation are pairwise distinct and where the term is defined as $\mathbf{0} \in \mathbf{N}(\mathbb{X}^m)$ if the sum is empty. Clearly, $\mu^{(m)} \in \mathbf{N}(\mathbb{X}^m)$ and $\mu^{(1)} = \mu$. It is also easy to verify that

$$\mu^{(m+1)} = \int_{\mathbb{X}^m} \bigg(\int_{\mathbb{X}} \mathbb{1}\big\{(x_1, \dots, x_{m+1}) \in \cdot\,\big\} \, \mathrm{d}\mu(x_{m+1}) - \sum_{j=1}^{m} \mathbb{1}\big\{(x_1, \dots, x_m, x_j) \in \cdot\,\big\} \bigg) \mathrm{d}\mu^{(m)}(x_1, \dots, x_m) \tag{A.2}$$

for each $m \in \mathbb{N}$. It is well-known that in general measurable spaces not every measure in $\mathbf{N}$ can be written as a sum of Dirac measures (cf. Exercise 2.5 of Last and Penrose, 2017). Still, for each $\mu \in \mathbf{N}$ there exists a unique sequence of symmetric measures $\mu^{(m)} \in \mathbf{N}(\mathbb{X}^m)$ with $\mu^{(1)} = \mu$ and such that (A.2) is valid for each $m \in \mathbb{N}$. This is guaranteed by the following result, stated as Proposition 4.3 by Last and Penrose (2017).

Proposition A.1 (Last and Penrose). *For each $\mu \in \mathbf{N}$ there exists a unique sequence of symmetric measures $\mu^{(m)} \in \mathbf{N}(\mathbb{X}^m)$ ($m \in \mathbb{N}$) such that $\mu^{(1)} = \mu$ and the recursion (A.2) is valid for all $m \in \mathbb{N}$. Moreover, the maps $\mu \mapsto \mu^{(m)} \in \mathbf{N}(\mathbb{X}^m)$ are measurable.*

A proof of this proposition is given in Appendix A of Last and Penrose (2017). As we have mentioned before, though the authors do not state this explicitly, it is essential to have the localizing structure on $\mathbb{X}$ to obtain the measurability of $\mu \mapsto \mu^{(m)}$. Proposition A.1, and thus, by extension, recursion (A.2), is the definition of the factorial measures that we adopt. The following proposition collects those properties of factorial measures that can be found in Chapter 4 and Appendix A of Last and Penrose (2017).

Proposition A.2. *Let $\mu, \nu \in \mathbf{N}$ and fix any $m \in \mathbb{N}$. The following properties are satisfied.*

(i) *If $D_1, \ldots, D_m \in \mathcal{X}$ are pairwise disjoint, then $\mu^{(m)}(D_1 \times \ldots \times D_m) = \prod_{j=1}^m \mu(D_j)$.*

(ii) *For $B \in \mathcal{X}$ it holds true that $\mu^{(m)}(B^m) = \mu(B) \cdot \big(\mu(B) - 1\big) \cdot \ldots \cdot \big(\mu(B) - m + 1\big)$.*

(iii) *For $B \in \mathcal{X}$ it holds true that $(\mu^{(m)})_{B^m} = \mu_B^{(m)}$.*

(iv) *For $B \in \mathcal{X}$ the relation $\mu^{(m)}(B^m) = 0$ holds whenever $\mu(B) < m$.*

(v) *If $\mu \leq \nu$, then $\mu^{(m)} \leq \nu^{(m)}$.*

We complement the previous proposition by the following additional property.

Lemma A.3. *Fix $m, k \in \mathbb{N}$ with $k \geq m$ and let $\mu \in \mathbf{N}$. Let $B \in \mathcal{X}$ and $D \in \mathcal{X}^{\otimes m}$. If $\mu(B) = k$, then*

$$\mu_B^{(k)}(D \times B^{k-m}) = (k-m)! \cdot \mu_B^{(m)}(D).$$

Proof. For $\mu \in \mathbf{N}$ with $\mu(B) = k$ and $k \geq m$, the recursion (A.2) gives

$$\mu_B^{(m+1)}(D \times B) = \big(\mu(B) - m\big) \int_{\mathbb{X}^m} \mathbb{1}\big\{(x_1, \ldots, x_m) \in D\big\} \, \mathrm{d}\mu_B^{(m)}(x_1, \ldots, x_m) = (k-m) \cdot \mu_B^{(m)}(D)$$

and, by applying this relation iteratively $k - m$ times, we obtain $\mu_B^{(k)}(D \times B^{k-m}) = (k-m)! \cdot \mu_B^{(m)}(D)$. □

Already in writing down the (multivariate) GNZ equations, the following measurability property is essential. We mostly use this result for $\mathbb{Y} = \mathbf{N}$ or $\mathbb{Y} = \mathbf{N} \times \mathbf{N}$, but different choices also appear occasionally.

Lemma A.4. *Let $(\mathbb{Y}, \mathcal{Y})$ be an arbitrary measurable space and fix $m \in \mathbb{N}$. The mapping*

$$\mathbf{N} \times \mathbb{Y} \ni (\mu, y) \mapsto \int_{\mathbb{X}^m} f(x_1, \ldots, x_m, y) \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m) \in [0, \infty]$$

is $\mathcal{N} \otimes \mathcal{Y}$-measurable for every measurable function $f : \mathbb{X}^m \times \mathbb{Y} \to [0, \infty]$.

Proof. By Proposition A.1, the mapping $\mu \mapsto \mu^{(m)}$ is measurable, so for $D \in \mathcal{X}^{\otimes m}$ and $A \in \mathcal{Y}$ the map

$$(\mu, y) \mapsto \int_{\mathbb{X}^m} \mathbb{1}_{D \times A}(x_1, \ldots, x_m, y) \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m) = \mathbb{1}_A(y) \cdot \mu^{(m)}(D)$$

is measurable. Denote by $\mathcal{D}$ the collection of all sets $E \in \mathcal{X}^{\otimes m} \otimes \mathcal{Y}$ for which

$$(\mu, y) \mapsto \int_{\mathbb{X}^m} \mathbb{1}_E(x_1, \ldots, x_m, y) \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m)$$

is measurable. The π-system $\{D \times A : D \in \mathcal{X}^{\otimes m},\ A \in \mathcal{Y}\}$ is contained in $\mathcal{D}$, we thus have $\mathbb{X}^m \times \mathbb{Y} \in \mathcal{D}$, and $\mathcal{D}$ is clearly closed with respect to countable disjoint unions. Let $E, F \in \mathcal{D}$ with $E \subset F$ and observe that

$$\int_{\mathbb{X}^m} \mathbb{1}_{F \setminus E}(x_1, \ldots, x_m, y) \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m) = \lim_{\ell \to \infty} \int_{\mathbb{X}^m} \mathbb{1}_{F \setminus E}(x_1, \ldots, x_m, y) \, \mathrm{d}\mu_{B_\ell}^{(m)}(x_1, \ldots, x_m)$$

by monotone convergence, using part (iii) of Proposition A.2. Since $\mu_{B_\ell}^{(m)}(\mathbb{X}^m) \leq \mu(B_\ell)! < \infty$, the following difference of integrals is well defined (for each $\ell \in \mathbb{N}$)

$$\int_{\mathbb{X}^m} \mathbb{1}_{F \setminus E}(x_1, \ldots, x_m, y) \, \mathrm{d}\mu_{B_\ell}^{(m)}(x_1, \ldots, x_m) = \int_{\mathbb{X}^m} \mathbb{1}_F(x_1, \ldots, x_m, y) \, \mathrm{d}\mu_{B_\ell}^{(m)}(x_1, \ldots, x_m)$$

$$- \int_{\mathbb{X}^m} \mathbb{1}_E(x_1, \ldots, x_m, y) \, \mathrm{d}\mu_{B_\ell}^{(m)}(x_1, \ldots, x_m).$$

The right hand side is a measurable function of (μ, y) since $E, F \in \mathcal{D}$ and $\mu \mapsto \mu_{B_\ell}$ is measurable. As limits of measurable functions are measurable, we conclude that

$$(\mu, y) \mapsto \int_{\mathbb{X}^m} \mathbb{1}_{F \setminus E}(x_1, \ldots, x_m, y) \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m)$$

is measurable, so $F \setminus E \in \mathcal{D}$. Therefore, $\mathcal{D}$ is a Dynkin system and Dynkin's π-λ-theorem implies $\mathcal{D} = \mathcal{X}^{\otimes m} \otimes \mathcal{Y}$. Standard monotone approximation completes the proof. □

Equation (4.19) of Last and Penrose (2017) indicates how factorial measures can be used to represent any functional on $\mathbf{N}$ when evaluated on $\mathbf{N}_f$. Indeed, refining that particular equation and extending it by monotone approximation immediately yields the following result.

Proposition A.5. *Let $F : \mathbf{N} \to [0, \infty]$ be a measurable map. For any $\mu \in \mathbf{N}_f$,*

$$F(\mu) = \mathbb{1}\{\mu(\mathbb{X}) = 0\} \cdot F(\mathbf{0}) + \sum_{m=1}^{\infty} \frac{1}{m!} \mathbb{1}\{\mu(\mathbb{X}) = m\} \int_{\mathbb{X}^m} F\Big(\sum_{j=1}^{m} \delta_{x_j}\Big) \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m).$$

We frequently use Proposition A.5 to argue that a function which is finite for sums of finitely many Dirac measures is finite on the whole of $\mathbf{N}_f$.

Lemma A.6. *Let $G : \mathbf{N} \to [0, \infty]$ be a measurable map such that $G(\mathbf{0}) < \infty$ and $G(\delta_{x_1} + \ldots + \delta_{x_m}) < \infty$ for all $x_1, \ldots, x_m \in \mathbb{X}$ and any $m \in \mathbb{N}$. Then $G(\mu) < \infty$ for each $\mu \in \mathbf{N}_f$.*

Proof. Applying Proposition A.5 to the measurable map $F(\mu) = \mathbb{1}\{G(\mu) < \infty\}$ and using the assumption, we obtain, for any $\mu \in \mathbf{N}_f$,

$$\begin{aligned}
\mathbb{1}\{G(\mu) < \infty\} &= \mathbb{1}\{\mu(\mathbb{X}) = 0\} \cdot \mathbb{1}\{G(\mathbf{0}) < \infty\} \\
&\qquad + \sum_{m=1}^{\infty} \frac{1}{m!} \mathbb{1}\{\mu(\mathbb{X}) = m\} \int_{\mathbb{X}^m} \mathbb{1}\Big\{G\Big(\sum_{j=1}^{m} \delta_{x_j}\Big) < \infty\Big\} \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m) \\
&= \mathbb{1}\{\mu(\mathbb{X}) = 0\} + \sum_{m=1}^{\infty} \frac{1}{m!} \mathbb{1}\{\mu(\mathbb{X}) = m\} \, \mu^{(m)}(\mathbb{X}^m) \\
&= \mathbb{1}\{\mu(\mathbb{X}) < \infty\} \\
&= 1,
\end{aligned}$$

where part (ii) of Proposition A.2 yielded the third equality. □

Note that the statement of Proposition A.5 also holds for any measurable map $F : \mathbf{N} \to \mathbb{R}$ (splitting $F = F^+ - F^-$) without additional integrability assumptions on F. This lack of integrability assumptions is justified by applying Lemma A.6 (with (A.1) in mind) to the maps

$$G(\mu) = \mathbb{1}\{\mu(\mathbb{X}) = m\} \int_{\mathbb{X}^m} \Big| F\Big(\sum_{j=1}^{m} \delta_{x_j}\Big) \Big| \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m)$$

for $m \in \mathbb{N}$. The following result related to Proposition A.5 is implicit in Lemma 2.44 of Jansen (2017).

Lemma A.7. *Let $F : \mathbf{N} \to \mathbb{R}$ be a measurable map. For any $\mu \in \mathbf{N}_f$,*

$$F(\mu) = F(\mathbf{0}) + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbb{X}^m} \sum_{J \subset [m]} (-1)^{m-|J|} F\Big(\sum_{j \in J} \delta_{x_j}\Big) \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m),$$

where $|J|$ denotes the cardinality of the set J.

Proof. Note that the claim is trivial for $\mu = \mathbf{0}$, so assume that $1 \le \mu(\mathbb{X}) < \infty$. The functions integrated on the right hand side of the equation in the statement of the lemma are integrable by Lemma A.6 and (A.1). As $\mu^{(m)} = \mathbf{0}$ for any $m > \mu(\mathbb{X})$ by Proposition A.2, the sum in the lemma is, in effect, always finite, so all terms are well-defined.

Now, let $y_1, y_2, \ldots \in \mathbb{X}$ and fix $k \in \mathbb{N}$. Then, for any measurable map $F : \mathbf{N} \to \mathbb{R}$, a change in the order of summation gives

$$\begin{aligned}\sum_{I \subset [k]} \sum_{J \subset I} (-1)^{|I|-|J|} F\Big(\sum_{j \in J} \delta_{y_j}\Big) &= \sum_{J \subset [k]} \Bigg(\sum_{J \subset I \subset [k]} (-1)^{|I|-|J|}\Bigg) F\Big(\sum_{j \in J} \delta_{y_j}\Big) \\ &= \sum_{J \subset [k]} \Bigg(\sum_{i=0}^{k-|J|} (-1)^i \cdot \binom{k-|J|}{i}\Bigg) F\Big(\sum_{j \in J} \delta_{y_j}\Big) \\ &= \sum_{J \subset [k]} \mathbb{1}\big\{|J| = k\big\} \cdot F\Big(\sum_{j \in J} \delta_{y_j}\Big) \\ &= F\Big(\sum_{i=1}^{k} \delta_{y_j}\Big).\end{aligned}$$

Rewriting the first sum on the right hand side gives

$$\begin{aligned}F\Big(\sum_{i=1}^{k} \delta_{y_j}\Big) &= F(\mathbf{0}) + \sum_{m=1}^{k} \frac{1}{m!} \sum_{j_1, \ldots, j_m \in [k]}^{\neq} \Bigg(\sum_{J \subset [m]} (-1)^{m-|J|} F\Big(\sum_{i \in J} \delta_{y_{j_i}}\Big)\Bigg) \\ &= F(\mathbf{0}) + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbb{X}^m} \sum_{J \subset [m]} (-1)^{m-|J|} F\Big(\sum_{j \in J} \delta_{x_j}\Big) \, \mathrm{d}\Big(\sum_{i=1}^{k} \delta_{y_i}\Big)^{(m)}(x_1, \ldots, x_m). \qquad \text{(A.3)}\end{aligned}$$

Proposition A.5 extends (A.3) to general $\mu \in \mathbf{N}_f$ and completes the proof. $\square$

Exercise 4.3 of Last and Penrose (2017) asks the reader to prove that, for any $\mu \in \mathbf{N}$, $x \in \mathbb{X}$, and $m \in \mathbb{N}$,

$$(\mu + \delta_x)^{(m+1)} = \mu^{(m+1)} + \int_{\mathbb{X}^m} \Big[\mathbb{1}\big\{(x, x_1, \ldots, x_m) \in \cdot\big\} + \ldots + \mathbb{1}\big\{(x_1, \ldots, x_m, x) \in \cdot\big\}\Big] \, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m). \qquad \text{(A.4)}$$

We formulate and prove the following lemma as an extension of this exercise. Therein, we denote by $S(n)$ the symmetric group of degree n containing all permutations of $[n] = \{1, \ldots, n\}$. For $z_1, \ldots, z_n \in \mathbb{X}$ and $\tau \in S(n)$ we write $\tau(z_1, \ldots, z_n) = \big(z_{\tau(1)}, \ldots, z_{\tau(n)}\big)$.

Lemma A.8. *Let $k, m \in \mathbb{N}$ and $\mu, \nu \in \mathbf{N}$ with $\mu(\mathbb{X}) = k$ and $\nu(\mathbb{X}) = m$. Then*

$$(\mu + \nu)^{(k+m)} = \frac{1}{k! \cdot m!} \int_{\mathbb{X}^k} \int_{\mathbb{X}^m} \sum_{\tau \in S(k+m)} \mathbb{1}\big\{\tau(z_1, \ldots, z_{k+m}) \in \cdot\big\} \, \mathrm{d}\nu^{(m)}(z_1, \ldots, z_m) \, \mathrm{d}\mu^{(k)}(z_{m+1}, \ldots, z_{m+k})$$

and, for any measurable and symmetric function $f : \mathbb{X}^{k+m} \to [0, \infty]$,

$$\begin{aligned}&\int_{\mathbb{X}^{k+m}} f(z_1, \ldots, z_{k+m}) \, \mathrm{d}(\mu + \nu)^{(k+m)}(z_1, \ldots, z_{k+m}) \\ &\quad = \frac{(k+m)!}{k! \cdot m!} \int_{\mathbb{X}^k} \int_{\mathbb{X}^m} f(x_1, \ldots, x_k, y_1, \ldots, y_m) \, \mathrm{d}\nu^{(m)}(y_1, \ldots, y_m) \, \mathrm{d}\mu^{(k)}(x_1, \ldots, x_k).\end{aligned}$$

Proof. We only show the first claim, with the second claim following by monotone approximation. Clearly, both $(\mu + \nu)^{(k+m)}$ and the term in the lemma constitute finite measures on $\mathbb{X}^{k+m}$. To prove that these measures are equal it suffices to show that they agree on sets of the form $C_1 \times \ldots \times C_{k+m} \in \mathcal{X}^{\otimes(k+m)}$. Thus, let $C_1, \ldots, C_{k+m} \in \mathcal{X}$. Denote by $\mathcal{A}$ the field generated by these sets. Lemma A.15 of Last and Penrose (2017) yields the existence of $x_1, \ldots, x_k, y_1, \ldots, y_m \in \mathbb{X}$ such that $\mu' = \sum_{i=1}^{k} \delta_{x_i}$ and $\nu' = \sum_{j=1}^{m} \delta_{y_j}$ satisfy

$$\mu^{(n)}(D) = (\mu')^{(n)}(D) \quad \text{and} \quad \nu^{(n)}(D) = (\nu')^{(n)}(D)$$

for all $n \in \mathbb{N}$ and $D \in \mathcal{A}^{\otimes n}$, where $\mathcal{A}^{\otimes n}$ is the field generated by the system $\{D_1 \times \ldots \times D_n : D_1, \ldots, D_n \in \mathcal{A}\}$. Similarly, there exist $z_1, \ldots, z_{k+m} \in \mathbb{X}$ such that $(\mu + \nu)' = \sum_{i=1}^{k+m} \delta_{z_i}$ satisfies

$$(\mu + \nu)^{(n)}(D) = \big((\mu + \nu)'\big)^{(n)}(D)$$

for all $n \in \mathbb{N}$ and $D \in \mathcal{A}^{\otimes n}$. Hence, (A.1) implies

$$\begin{aligned}(\mu + \nu)^{(k+m)}(C_1 \times \ldots \times C_{k+m}) &= \big((\mu + \nu)'\big)^{(k+m)}(C_1 \times \ldots \times C_{k+m}) \\ &= \sum_{i_1, \ldots, i_{k+m} \in [k+m]}^{\neq} \mathbb{1}\Big\{(z_{i_1}, \ldots, z_{i_{k+m}}) \in C_1 \times \ldots \times C_{k+m}\Big\}.\end{aligned}$$

From the construction in Lemma A.15 of Last and Penrose (2017) it is obvious that $(\mu + \nu)' = \mu' + \nu'$, so the previous term equals

$$\sum_{i_1, \ldots, i_k \in [k]}^{\neq} \sum_{j_1, \ldots, j_m \in [m]}^{\neq} \frac{1}{k! \cdot m!} \sum_{\tau \in S(k+m)} \mathbb{1}\Big\{\tau(x_{i_1}, \ldots, x_{i_k}, y_{j_1}, \ldots, y_{j_m}) \in C_1 \times \ldots \times C_{k+m}\Big\},$$

which, by (A.1) and the construction of μ' and ν', equals the right hand side of the claim. □

A.2. Local events and local functions

In this part of the appendix we deal with local events and functions. The results we present are mostly scattered in the literature and usually stated without proof, which is understandable as they involve only very basic measure theory. For one rather comprehensive reference we mention the lecture notes by Preston (2005b). We still provide full detail on these concepts and adapt them to our notation.

We consider the setting from Section 2.2, where $(\mathbb{X}, \mathcal{X})$ is a measurable space, which is localized by the sets $B_1 \subset B_2 \subset \ldots$, and $\mathbf{N}$ is the set of all locally finite counting measures on $\mathbb{X}$. Recall that $\pi_B : \mathbf{N} \to \mathbb{N}_0 \cup \{\infty\}$ is the evaluation map $\mu \mapsto \mu(B)$, for $B \in \mathcal{X}$, and that $\mathcal{N}$ is the σ-field on $\mathbf{N}$ generated by $\{\pi_B : B \in \mathcal{X}\}$.

For a set $B \in \mathcal{X}$ we define the map $p_B : \mathbf{N} \to \mathbf{N}$, $p_B(\mu) = \mu_B$, which assigns counting measures their restriction onto B. Put

$$\mathcal{N}_B = \sigma(p_B) = p_B^{-1}(\mathcal{N}),$$

the σ-field on $\mathbf{N}$ generated by p_B. Moreover, define $\mathcal{Z} = \bigcup_{B \in \mathcal{X}_b} \mathcal{N}_B$.

Definition A.9 (Local events and local functions).

- A set $A \in \mathcal{N}$ of counting measures is called a *B-local event* (for some $B \in \mathcal{X}$) if $A \in \mathcal{N}_B$. We say that $A \in \mathcal{N}$ is a *local event* if there exists a set $B \in \mathcal{X}_b$ such that A is B-local, that is, if $A \in \mathcal{Z}$.
- A measurable map $F : \mathbf{N} \to [-\infty, \infty]$ is called *B-local* (for some $B \in \mathcal{X}$) if $F(\mu) = F(\mu_B)$ for all $\mu \in \mathbf{N}$. The map F is called a *local function* if there exists a set $B \in \mathcal{X}_b$ such that F is B-local.

Notice that we define B-locality for arbitrary sets $B \in \mathcal{X}$, but whenever we call an event or a function local, we specify to bounded sets $B \in \mathcal{X}_b$. In the following lemma we provide detailed information about the properties of local events and functions.

Lemma A.10. *In parts* (i) – (v)*, fix a set* $B \in \mathcal{X}$*.*

(i) *The* σ*-field* $\mathcal{N}_B$ *is generated by the evaluation maps* π_D*, for* $D \in \mathcal{X}$ *with* $D \subset B$*, that is,*

$$\mathcal{N}_B = \sigma(\pi_D : D \in \mathcal{X},\, D \subset B)$$

In particular, $\mathcal{N}_B \subset \mathcal{N}$*.*

(ii) *For* $A \in \mathcal{N}_B$ *and* $\mu \in \mathbf{N}$ *it holds true that* $\mu \in A$ *if, and only if,* $\mu_B \in A$*.*

(iii) *The collection* $\mathcal{G}_B = \{A \cap A' : A \in \mathcal{N}_B,\, A' \in \mathcal{N}_{B^c}\}$ *satisfies* $\sigma(\mathcal{G}_B) = \mathcal{N}$*.*

(iv) *A measurable map* $F : \mathbf{N} \to [-\infty, \infty]$ *is* B*-local if, and only if,* F *is* $\mathcal{N}_B$*-measurable.*

(v) *A set* $A \in \mathcal{N}$ *is* B*-local if, and only if,* $\mathbb{1}_A$ *is a* B*-local function.*

Furthermore,

(vi) $D_1, D_2 \in \mathcal{X}$ *satisfy* $D_1 \subset D_2$ *if, and only if,* $\mathcal{N}_{D_1} \subset \mathcal{N}_{D_2}$*.*

(vii) *For each* $D_1, D_2 \in \mathcal{X}$ *it holds that* $\mathcal{N}_{D_1 \cup D_2} = \sigma(\mathcal{N}_{D_1} \cup \mathcal{N}_{D_2})$*.*

(viii) *The collection* $\mathcal{Z} = \bigcup_{B \in \mathcal{X}_b} \mathcal{N}_B$ *is an algebra of subsets of* $\mathbf{N}$ *with* $\sigma(\mathcal{Z}) = \mathcal{N}$*.*

(ix) *Define* $\mathcal{E} = \{B \times A : B \in \mathcal{X}_b,\, A \in \mathcal{Z}\}$*. Then* $\mathcal{E}$ *is a* π*-system and* $\sigma(\mathcal{E}) = \mathcal{X} \otimes \mathcal{N}$*.*

Proof. (i) For $A = \big\{\mu \in \mathbf{N} : \mu(C) = k\big\}$, with $C \in \mathcal{X}$ and $k \in \mathbb{N}_0$, we have

$$p_B^{-1}(A) = \{\mu \in \mathbf{N} : \mu_B \in A\} = \big\{\mu \in \mathbf{N} : \mu_B(C) = k\big\} = \big\{\mu \in \mathbf{N} : \pi_{C \cap B}(\mu) = k\big\},$$

and therefore $\mathcal{N}_B \subset \sigma(\pi_D : D \in \mathcal{X},\, D \subset B)$. To prove the converse inclusion, let $D \in \mathcal{X}$ with $D \subset B$ and $k \in \mathbb{N}_0$. Then

$$\pi_D^{-1}(\{k\}) = \big\{\mu \in \mathbf{N} : \pi_D(\mu) = k\big\} = \big\{\mu \in \mathbf{N} : \pi_D\big(p_B(\mu)\big) = k\big\} = p_B^{-1}\big(\pi_D^{-1}(\{k\})\big) \in \mathcal{N}_B,$$

so each π_D is $\mathcal{N}_B$-measurable. Since $\sigma(\pi_D : D \in \mathcal{X},\, D \subset B)$ is the smallest σ-field for which all of these maps are measurable, we have $\sigma(\pi_D : D \in \mathcal{X},\, D \subset B) \subset \mathcal{N}_B$.

(ii) Fix $\mu \in \mathbf{N}$. Define the collection of sets

$$\mathcal{H} = \big\{A \subset \mathbf{N} : \mu \in A \text{ if, and only if, } \mu_B \in A\big\}.$$

It is trivial to verify that $\mathcal{H}$ is a σ-field over $\mathbf{N}$. For each $D \in \mathcal{X}$ with $D \subset B$ and any $k \in \mathbb{N}_0$, we have $\mu \in \pi_D^{-1}(\{k\})$ if, and only if, $\mu_B \in \pi_D^{-1}(\{k\})$. Consequently, each such π_D is $\mathcal{H}$-measurable. Part (i) gives $\mathcal{N}_B \subset \mathcal{H}$, which implies the claim.

(iii) By part (i) we have $\mathcal{N}_B, \mathcal{N}_{B^c} \subset \mathcal{N}$. Hence $\mathcal{G}_B \subset \mathcal{N}$ and $\sigma(\mathcal{G}_B) \subset \mathcal{N}$. Now, let $D \in \mathcal{X}$ and $k \in \mathbb{N}_0$. Observe that

$$\begin{aligned}\pi_D^{-1}(\{k\}) = \big\{\mu \in \mathbf{N} : \mu(D) = k\big\} &= \bigcup_{j=0}^{k} \Big(\big\{\mu \in \mathbf{N} : \mu_B(D) = j\big\} \cap \big\{\mu \in \mathbf{N} : \mu_{B^c}(D) = k - j\big\}\Big) \\ &= \bigcup_{j=0}^{k} \Big(p_B^{-1}\big(\pi_D^{-1}(\{j\})\big) \cap p_{B^c}^{-1}\big(\pi_D^{-1}(\{k-j\})\big)\Big) \\ &\in \sigma(\mathcal{G}_B).\end{aligned}$$

We conclude that π_D is $\sigma(\mathcal{G}_B)$-measurable for each $D \in \mathcal{X}$, so $\mathcal{N} \subset \sigma(\mathcal{G}_B)$.

(iv) If F is B-local, then by definition $F = F \circ p_B$, so F is $\mathcal{N}_B$-measurable. Conversely, if F is $\mathcal{N}_B$-measurable, then the factorization lemma (cf. Lemma 1.13 of Kallenberg, 2002) provides an $\mathcal{N}$-measurable map $G : \mathbf{N} \to [-\infty, \infty]$ such that $F = G \circ p_B$, which shows that F is B-local.

(v) This follows immediately from (iv).

(vi) If $D_1 \subset D_2$, then (i) implies $\mathcal{N}_{D_1} \subset \mathcal{N}_{D_2}$. For the converse implication, assume that $\mathcal{N}_{D_1} \subset \mathcal{N}_{D_2}$. Let $A = \big\{\mu \in \mathbf{N} : \mu(D_1) = 1\big\}$. As before, we find that

$$A = \big\{\mu \in \mathbf{N} : \mu_{D_1}(D_1) = 1\big\} = \big\{\mu \in \mathbf{N} : \pi_{D_1}\big(p_{D_1}(\mu)\big) = 1\big\} = p_{D_1}^{-1}\big(\pi_{D_1}^{-1}(\{1\})\big) \in \mathcal{N}_{D_1} \subset \mathcal{N}_{D_2}$$

and part (v) yields that $\mathbb{1}_A$ is both D_1-local and D_2-local. Now, if there exists a point $x \in D_1$ such that $x \notin D_2$, then

$$1 = \mathbb{1}_A(\delta_x) = \mathbb{1}_A\big((\delta_x)_{D_2}\big) = \mathbb{1}_A(\mathbf{0}) = 0,$$

a contradiction. We conclude that $D_1 \subset D_2$.

(vii) From part (vi) it follows that $\mathcal{N}_{D_1}, \mathcal{N}_{D_2} \subset \mathcal{N}_{D_1 \cup D_2}$, so $\sigma(\mathcal{N}_{D_1} \cup \mathcal{N}_{D_2}) \subset \mathcal{N}_{D_1 \cup D_2}$. Moreover, for $D \in \mathcal{X}$ with $D \subset D_1 \cup D_2$ and $k \in \mathbb{N}_0$, we have

$$\pi_D^{-1}(\{k\}) = \bigcup_{j=0}^{k} \Big(\big\{\mu \in \mathbf{N} : \mu_{D_1}(D) = j\big\} \cap \big\{\mu \in \mathbf{N} : \mu_{D_2}(D \setminus D_1) = k - j\big\}\Big) \in \sigma(\mathcal{N}_{D_1} \cup \mathcal{N}_{D_2}),$$

so π_D is $\sigma(\mathcal{N}_{D_1} \cup \mathcal{N}_{D_2})$-measurable. Hence, $\mathcal{N}_{D_1 \cup D_2} \subset \sigma(\mathcal{N}_{D_1} \cup \mathcal{N}_{D_2})$.

(viii) Since $\varnothing, \mathbf{N} \in \mathcal{N}_D$ for each $D \in \mathcal{X}_b$, we clearly have $\varnothing, \mathbf{N} \in \mathcal{Z}$. For $A \in \mathcal{Z}$ we find $D \in \mathcal{X}_b$ with $A \in \mathcal{N}_D$ and, as $\mathcal{N}_D$ is a σ-field, we have $\mathbf{N} \setminus A \in \mathcal{N}_D \subset \mathcal{Z}$. For $A_1, A_2 \in \mathcal{Z}$ we find $D_1, D_2 \in \mathcal{X}_b$ with $A_1 \in \mathcal{N}_{D_1}$ and $A_2 \in \mathcal{N}_{D_2}$. By part (vii) we then have $A_1, A_2 \in \mathcal{N}_{D_1 \cup D_2}$ and therefore $A_1 \cup A_2 \in \mathcal{N}_{D_1 \cup D_2} \subset \mathcal{Z}$ (noting that $D_1 \cup D_2 \in \mathcal{X}_b$). Hence $\mathcal{Z}$ is an algebra. The inclusion $\sigma(\mathcal{Z}) \subset \mathcal{N}$ is clear from part (i). However, it also follows from (i) that π_B is $\mathcal{N}_B$-measurable for each $B \in \mathcal{X}_b$, so we can conclude that $\mathcal{N} \subset \sigma(\mathcal{Z})$ as soon as we establish that

$$\mathcal{N} = \sigma(\pi_B : B \in \mathcal{X}_b).$$

By definition of $\mathcal{N}$, we have $\sigma(\pi_B : B \in \mathcal{X}_b) \subset \sigma(\pi_B : B \in \mathcal{X}) = \mathcal{N}$. For the other inclusion, note that, for any $B \in \mathcal{X}$ and upon putting $B_0 = \varnothing$,

$$\pi_B(\mu) = \mu(B) = \sum_{\ell=1}^{\infty} \mu\big(B \cap (B_\ell \setminus B_{\ell-1})\big) = \sum_{\ell=1}^{\infty} \pi_{B \cap (B_\ell \setminus B_{\ell-1})}(\mu),$$

for each $\mu \in \mathbf{N}$, so π_B is measurable with respect to $\sigma(\pi_B : B \in \mathcal{X}_b)$.

(ix) It follows from part (viii) that $\mathcal{E}$ is a π-system. We clearly have

$$\sigma(\mathcal{E}) \subset \sigma\big(\{B \times A : B \in \mathcal{X},\ A \in \mathcal{N}\}\big) = \mathcal{X} \otimes \mathcal{N}.$$

Consider the projection maps $\mathrm{pr}_{\mathbb{X}} : \mathbb{X} \times \mathbf{N} \to \mathbb{X}$ and $\mathrm{pr}_{\mathbf{N}} : \mathbb{X} \times \mathbf{N} \to \mathbf{N}$. The σ-field $\mathcal{X} \otimes \mathcal{N}$ is the smallest σ-field for which both projections are measurable. Thus, if we show that both projections are measurable with respect to $\sigma(\mathcal{E})$, the proof is complete. For $A \in \mathcal{Z}$ we have

$$\mathrm{pr}_{\mathbf{N}}^{-1}(A) = \mathbb{X} \times A = \bigcup_{\ell=1}^{\infty} (B_\ell \times A) \in \sigma(\mathcal{E}),$$

so (viii) gives $\mathrm{pr}_{\mathbf{N}}^{-1}(\mathcal{N}) = \mathrm{pr}_{\mathbf{N}}^{-1}\big(\sigma(\mathcal{Z})\big) = \sigma\big(\mathrm{pr}_{\mathbf{N}}^{-1}(\mathcal{Z})\big) \subset \sigma(\mathcal{E})$. Similarly we obtain, for each $B \in \mathcal{X}_b$,

$$\mathrm{pr}_{\mathbb{X}}^{-1}(B) = B \times \mathbf{N} = \bigcup_{n=1}^{\infty} \big(B \times \{\mu \in \mathbf{N} : \mu(B) \leq n\}\big) \in \sigma(\mathcal{E}),$$

where we use that measures from $\mathbf{N}$ assign B finite value and that $\{\mu \in \mathbf{N} : \mu(B) \leq n\} \in \mathcal{N}_B \subset \mathcal{Z}$. We conclude that $\mathrm{pr}_{\mathbb{X}}^{-1}(\mathcal{X}) = \sigma\big(\mathrm{pr}_{\mathbb{X}}^{-1}(\mathcal{X}_b)\big) \subset \sigma(\mathcal{E})$. □

As in the proof of part (viii), it follows that $\bigcup_{\ell=1}^{\infty} \mathcal{N}_{B_\ell}$ is an algebra which generates $\mathcal{N}$.

A.3. Measurable diagonals

It is common in point process theory to assume that the state space is (at least) a Borel space in order to handle measurability issues. However, many properties can be established under a weaker assumption. For instance, the measurable removal of points from a counting measure as well as the study of simple counting measures requires only that the space has a measurable diagonal. Even though Section 6 of Last and Penrose (2017) contains some hints that this assumption might be sufficient, the rigorous details are laid out for the first time (to the authors knowledge).

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space with localizing structure $B_1 \subset B_2 \subset \ldots$ and let $\mathbf{N}$ be defined as in Section 2.2. We say that $\mathbb{X}$ has a *measurable diagonal* if

$$D_{\mathbb{X}} = \big\{(x, y) \in \mathbb{X}^2 : x = y\big\} \in \mathcal{X}^{\otimes 2},$$

borrowing the notation from Last and Penrose (2017).

Lemma A.11. *Assume that $\mathbb{X}$ has a measurable diagonal. Then $\mathbb{X}$ is separable, that is, $\{x\} \in \mathcal{X}$ for each $x \in \mathbb{X}$. Moreover, the map $\mathbb{X} \times \mathbf{N} \ni (x, \mu) \mapsto \mu(\{x\}) \in \mathbb{N}_0$ is well-defined and measurable.*

Proof. For each $x \in \mathbb{X}$ the map $h_x : \mathbb{X} \to \mathbb{X}^2$, $h_x(y) = (x, y)$ is measurable, so $\{x\} = h_x^{-1}(D_{\mathbb{X}}) \in \mathcal{X}$. As there exists some $\ell \in \mathbb{N}$ such that $x \in B_\ell$, we actually have $\{x\} \in \mathcal{X}_b$, so the definition of $\mathbf{N}$ ensures that $\mu(\{x\}) < \infty$. In particular, the map in consideration is well-defined. Since $\mathbb{X}$ has a measurable diagonal, the mapping $(x, y) \mapsto \mathbb{1}_{D_{\mathbb{X}}}(x, y)$ is measurable and Lemma A.4 implies the measurability of

$$(x, \mu) \mapsto \int_{\mathbb{X}} \mathbb{1}_{D_{\mathbb{X}}}(x, y)\, \mathrm{d}\mu(y) = \int_{\mathbb{X}} \mathbb{1}\{x = y\}\, \mathrm{d}\mu(y) = \int_{\mathbb{X}} \mathbb{1}_{\{x\}}(y)\, \mathrm{d}\mu(y) = \mu(\{x\}). \qquad □$$

Lemma A.12. *If $(\mathbb{X}, \mathcal{X})$ is countably generated and separable, then $\mathbb{X}$ has a measurable diagonal.*

Proof. Let $C_1, C_2, \ldots \in \mathcal{X}$ be the sets that generate $\mathcal{X}$. Define, for $n \in \mathbb{N}$,

$$\mathcal{X}_n = \left\{ \bigcap_{j=1}^{n} C_j^{\varepsilon_j} : \varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\} \right\},$$

where we set $C_j^0 = (C_j)^{\mathsf{c}}$ and $C_j^1 = C_j$. We claim that $D_{\mathbb{X}} = \bigcap_{n=1}^{\infty} \bigcup_{C \in \mathcal{X}_n} (C \times C)$ which is clearly a measurable subset of $\mathbb{X}^2$. Let $x \in \mathbb{X}$ and $n \in \mathbb{N}$ be arbitrary. For each $j \in \{1, \ldots, n\}$ either $x \in C_j = C_j^1$ or $x \in (C_j)^{\mathsf{c}} = C_j^0$. Hence, we find $C \in \mathcal{X}_n$ such that $(x, x) \in C \times C$. Now, let $(x, y) \in \bigcap_{n=1}^{\infty} \bigcup_{C \in \mathcal{X}_n} (C \times C)$. Assume that $x \neq y$. It is easy to verify that

$$\mathcal{H} = \big\{A \in \mathcal{X} : x \in A \text{ if, and only if, } y \in A\big\}$$

is a σ-field over $\mathbb{X}$. If we had $x \in C_j$ if, and only if, $y \in C_j$ for each $j \in \mathbb{N}$, then $\{C_1, C_2, \ldots\} \subset \mathcal{H}$ and therefore $\mathcal{H} = \mathcal{X}$. However, $\{x\} \in \mathcal{X}$ by separability, but $\{x\} \notin \mathcal{H}$ as $x \neq y$. Consequently, we find an index $k \in \mathbb{N}$ such that (possibly after relabeling) $x \in C_k$ and $y \in (C_k)^{\mathsf{c}}$. This implies that, for every $C \in \mathcal{X}_k$, we have $(x,y) \in (C \times C)^c$ which poses a contradiction to the fact that $(x,y) \in \bigcap_{n=1}^{\infty} \bigcup_{C \in \mathcal{X}_n} (C \times C)$. Thus the assumption $x \neq y$ is false and we must have $x = y$. □

Note that in Lemma A.12 a slightly weaker assumption than the measurable space being countably generated actually gives equivalence, see Musiał (1980). For our purposes, however, the above statement is enough since the substandard Borel spaces we consider (in Appendix A.6 and Sections 3.1, 3.2) are countably generated. Notice that Lemma A.12 includes every second countable Hausdorff space and thus in particular every separable metric space. It also includes every Borel space.

For a measure $\mu \in \mathbf{N}$ and a point $x \in \mathbb{X}$ we define

$$\mu \setminus \delta_x = \mu - \delta_x \mathbb{1}\big\{\mu(\{x\}) > 0\big\},$$

which is itself a counting measure on $(\mathbb{X}, \mathcal{X})$. The measure $\mu \setminus \delta_{x_1} \setminus \ldots \setminus \delta_{x_m}$, for $x_1, \ldots, x_m \in \mathbb{X}$ and $m \in \mathbb{N}$, is defined iteratively.

Lemma A.13. *Assume that $\mathbb{X}$ has a measurable diagonal. Then the map $d_m : \mathbb{X}^m \times \mathbf{N} \to \mathbf{N}$,*

$$d_m(x_1, \ldots, x_m, \mu) = \mu \setminus \delta_{x_1} \setminus \ldots \setminus \delta_{x_m}$$

is measurable for each $m \in \mathbb{N}$.

Proof. We prove the claim by induction, starting with the initial case $m = 1$. Note that, for any $B \in \mathcal{X}$,

$$(x, \mu) \mapsto d_1(x, \mu)(B) = (\mu \setminus \delta_x)(B) = \mu(B) - \mathbb{1}\{x \in B\} \cdot \mathbb{1}\big\{\mu(\{x\}) > 0\big\} \in \mathbb{N}_0 \cup \{\infty\}$$

is measurable by Lemma A.11. If the map d_m is measurable for any fixed $m \in \mathbb{N}$, then d_{m+1} is also measurable since $d_{m+1}(x_1, \ldots, x_{m+1}, \mu) = d_1\big(x_{m+1}, d_m(x_1, \ldots, x_m, \mu)\big)$. □

Remark A.14. Let $\mathbb{X}$ have a measurable diagonal. By Lemma A.11 the space is also separable, and we call a counting measure $\mu \in \mathbf{N}$ simple if

$$\mu(\{x\}) \leq 1, \quad x \in \mathbb{X}.$$

We denote by $\mathbf{N}_s = \mathbf{N}_s(\mathbb{X})$ the set of all such measures. We now generalize Proposition 6.7 of Last and Penrose (2017), showing that $\mu \in \mathbf{N}_s$ if, and only if, $\mu^{(2)}(D_{\mathbb{X}}) = 0$.

Indeed, if $\mu \notin \mathbf{N}_s$ we find $x \in \mathbb{X}$ such that $\mu(\{x\}) = m$ with $m \geq 2$ and Proposition A.5, applied to the map $F(\psi) = \psi^{(2)}(D_{\mathbb{X}})$ and the measure $\mu_{\{x\}}$, gives

$$\mu^{(2)}(D_{\mathbb{X}}) \geq \mu_{\{x\}}^{(2)}(D_{\mathbb{X}}) = \frac{1}{m!} \int_{\{x\}^m} \Big(\sum_{j=1}^{m} \delta_{x_j}\Big)^{(2)}(D_{\mathbb{X}})\, \mathrm{d}\mu^{(m)}(x_1, \ldots, x_m) = \binom{m}{2} > 0.$$

Conversely, if $\mu \in \mathbf{N}_s$ then, by the definition of the factorial measure in (A.2),

$$0 \leq \mu^{(2)}(D_{\mathbb{X}}) = \int_{\mathbb{X}} \bigg(\int_{\mathbb{X}} \mathbb{1}_{D_{\mathbb{X}}}(x,y)\, \mathrm{d}\mu(y) - \mathbb{1}_{D_{\mathbb{X}}}(x,x) \bigg) \mathrm{d}\mu(x) = \int_{\mathbb{X}} \big(\mu(\{x\}) - 1\big)\, \mathrm{d}\mu(x) \leq 0,$$

which shows that $\mu^{(2)}(D_{\mathbb{X}}) = 0$.

By the above equivalence and Proposition A.1, we get $\mathbf{N}_s = \big\{\mu \in \mathbf{N} : \mu^{(2)}(D_{\mathbb{X}}) = 0\big\} \in \mathcal{N}$. With the given generalization at hand, a literal copy of the proof of Proposition 6.9 of Last and Penrose (2017) shows that a Poisson process is simple if, and only if, its intensity measure is diffuse. Here a point process Φ is called simple if $\mathbb{P}(\Phi \in \mathbf{N}_s) = 1$ and a measure λ on $\mathbb{X}$ is diffuse if $\lambda(\{x\}) = 0$ for all $x \in \mathbb{X}$. Hence, the observations from Last and Penrose (2017) do not require the Borel structure but only a measurable diagonal. ○

A.4. The Banach–Alaoglu Theorem

This part of the appendix is a collection of well-known results for L^p-spaces, particularly in connection with the Banach–Alaoglu theorem. We adapt the presentation to our notation, but, for the mathematical content, rely on Chapter 4 of Bogachev (2007a) (who himself relies on Chapter V of Dunford and Schwartz, 1958) and Chapter 3 of Cohn (2013), where all of these results and considerations are detailed.

Let $(\mathbb{B}, \mathcal{O}(\mathbb{B}))$ be a topological vector space over $\mathbb{R}$. We denote by $\mathbb{B}'$ the (topological) dual space of $\mathbb{B}$ consisting of all continuous linear functionals $T : \mathbb{B} \to \mathbb{R}$. A sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{B}$ is said to converge weakly to $b \in \mathbb{B}$ (denoted by $b_n \rightharpoonup b$) if $T(b_n) \to T(b)$, as $n \to \infty$, for all $T \in \mathbb{B}'$. A sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathbb{B}'$ converges weakly* to $T \in \mathbb{B}'$ (denoted by $T_n \overset{*}{\rightharpoonup} T$) if $T_n(b) \to T(b)$, as $n \to \infty$, for all $b \in \mathbb{B}$. Hence, we can endow $\mathbb{B}$ and $\mathbb{B}'$ with the *weak* and *weak* topology*, respectively.

If $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is a normed vector space, then so is $\mathbb{B}'$ with a norm given by

$$\|T\|_{\mathbb{B}'} = \sup\big\{|T(b)| : b \in \mathbb{B},\ \|b\|_{\mathbb{B}} \le 1\big\}.$$

In fact, $(\mathbb{B}', \|\cdot\|_{\mathbb{B}'})$ is a Banach space. The two norms constitute the *strong topologies* on $\mathbb{B}$ and $\mathbb{B}'$. Note that in the context of metric spaces we call a space separable if it contains a countable dense subset.

Proposition A.15 (Banach–Alaoglu). *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a normed vector space. Then the closed unit ball in $(\mathbb{B}', \|\cdot\|_{\mathbb{B}'})$ is compact with respect to the weak* topology. If the space $\mathbb{B}$ is separable, then the weak* topology on the unit ball in $\mathbb{B}'$ is metrizable and hence the unit ball in $\mathbb{B}'$ is also sequentially compact. In particular, if $\mathbb{B}$ is separable, then any bounded sequence in $\mathbb{B}'$ admits a weakly* convergent subsequence.*

We now specialize to L^p-spaces. Let $(\mathbb{Y}, \mathcal{Y}, \nu)$ be a measure space and denote by $L^p(\mathbb{Y}, \nu) = L^p(\mathbb{Y}, \mathcal{Y}, \nu)$ the (equivalence classes of) measurable functions $f : \mathbb{Y} \to \mathbb{R}$ which satisfy

$$\|f\|_{L^p(\mathbb{Y},\nu)} = \left(\int_{\mathbb{Y}} |f|^p \,\mathrm{d}\nu\right)^{1/p} < \infty,$$

where $1 \le p < \infty$. Denote by $L^\infty(\mathbb{Y}, \nu) = L^\infty(\mathbb{Y}, \mathcal{Y}, \nu)$ the (equivalence classes of) measurable functions $f : \mathbb{Y} \to \mathbb{R}$ which satisfy

$$\|f\|_{L^\infty(\mathbb{Y},\nu)} = \inf\big\{c > 0 : |f(y)| \le c \text{ for } \nu\text{-a.e. } y \in \mathbb{Y}\big\} < \infty.$$

For each $1 \le p \le \infty$ the space $L^p(\mathbb{Y}, \nu)$ is a Banach space. For $1 \le p \le \infty$ we let $1 \le q \le \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ (exceptionally putting $\frac{1}{\infty} = 0$) and we define the map $T_p : L^q(\mathbb{Y}, \nu) \to \big(L^p(\mathbb{Y}, \nu)\big)'$ by

$$T_p(g)(f) = \int_{\mathbb{Y}} f\, g \,\mathrm{d}\nu, \quad f \in L^p(\mathbb{Y}, \nu).$$

For each $1 < p < \infty$ the map T_p is an isometric isomorphism between the Banach spaces $L^q(\mathbb{Y}, \nu)$ and $\big(L^p(\mathbb{Y}, \nu)\big)'$. Moreover, if ν is σ-finite, the same holds true for $p = 1$, that is, the map T_1 is an isometric isomorphism $L^\infty(\mathbb{Y}, \nu) \to \big(L^1(\mathbb{Y}, \nu)\big)'$. If the measurable space $(\mathbb{Y}, \mathcal{Y})$ is countably generated and ν is σ-finite, then $L^p(\mathbb{Y}, \nu)$ is separable for each $1 \le p < \infty$.

To apply the Banach–Alaoglu theorem to the space $L^1(\mathbb{Y}, \nu)$ we thus assume that $(\mathbb{Y}, \mathcal{Y})$ is a countably generated measurable space and that ν is a σ-finite measure on $(\mathbb{Y}, \mathcal{Y})$. Then $L^1(\mathbb{Y}, \nu)$ is a separable Banach space whose topological dual is isometrically isomorphic to $L^\infty(\mathbb{Y}, \nu)$. Consider a sequence of functions $(j_n)_{n\in\mathbb{N}}$ in $L^\infty(\mathbb{Y}, \nu)$ which is bounded, that is,

$$\sup_{n\in\mathbb{N}} \|j_n\|_{L^\infty(\mathbb{Y},\nu)} \le c$$

for some constant $c \geq 0$. Then the sequence $\big(T_1(j_n)\big)_{n\in\mathbb{N}}$ in $\big(L^1(\mathbb{Y},\nu)\big)'$ is bounded, since

$$\sup_{n\in\mathbb{N}} \big\|T_1(j_n)\big\|_{(L^1(\mathbb{Y},\nu))'} = \sup_{n\in\mathbb{N}} \|j_n\|_{L^\infty(\mathbb{Y},\nu)} \leq c,$$

using that T_1 is an isometry. By Proposition A.15 we find some $T \in \big(L^1(\mathbb{Y},\nu)\big)'$ and a suitable subsequence of $(j_n)_{n\in\mathbb{N}}$ such that $T_1(j_{n_k}) \xrightarrow{*} T$ as $k \to \infty$. As T_1 is isomorphic, we find a function $j \in L^\infty(\mathbb{Y},\nu)$ with $T_1(j) = T$ and we conclude that

$$\int_{\mathbb{Y}} f\, j_{n_k}\,\mathrm{d}\nu = T_1(j_{n_k})(f) \longrightarrow T_1(j)(f) = \int_{\mathbb{Y}} f\, j\,\mathrm{d}\nu,$$

as $k \to \infty$, for every $f \in L^1(\mathbb{Y},\nu)$.

A.5. The Dunford–Pettis Lemma

In the previous appendix section on the Banach–Alaoglu theorem we have discussed the extraction of weak* convergent subsequences of bounded sequences in L^q-spaces for $1 < q \leq \infty$. For $q = 1$ this approach does not give feasible results as the operator T_∞ in Appendix A.4 is not isomorphic. However, different techniques allow for a comparable result in L^1.

Let $(\mathbb{Y}, \mathcal{Y}, \nu)$ be a measure space. Following Definition 4.5.1 of Bogachev (2007a), we call a collection of functions $\mathcal{G} \subset L^1(\mathbb{Y},\nu)$ uniformly integrable if

$$\lim_{c\to\infty} \sup_{f\in\mathcal{G}} \int_{\mathbb{Y}} |f(y)| \cdot \mathbb{1}\big\{|f(y)| > c\big\}\,\mathrm{d}\nu(y) = 0.$$

With Appendix A.4 in mind, note that uniform integrability is a weaker assumption than the class $\mathcal{G}$ being bounded in $L^\infty(\mathbb{Y},\nu)$, but it is a stronger requirement than $\mathcal{G}$ being merely bounded in $L^1(\mathbb{Y},\nu)$.

The following proposition is provided as Theorem 4.7.18 by Bogachev (2007a).

Proposition A.16 (**Dunford–Pettis**). *Assume that the measure ν is finite and let $\mathcal{G} \subset L^1(\mathbb{Y},\nu)$. Then $\mathcal{G}$ is uniformly integrable if, and only if, its closure is compact in the weak topology of $L^1(\mathbb{Y},\nu)$.*

The Eberlein–Šmulian theorem, given as Theorem 4.7.10 by Bogachev (2007a) (cf. Chapter V of Dunford and Schwartz, 1958, for a proof), tells us that $\mathcal{G} \subset L^1(\mathbb{Y},\nu)$ (for a general measure ν) has a compact closure in the weak topology of $L^1(\mathbb{Y},\nu)$ if, and only if, every sequence in $\mathcal{G}$ has a weakly convergent subsequence in $L^1(\mathbb{Y},\nu)$. Moreover, as recalled in Appendix A.4, if the measure ν is σ-finite, the weak convergence in $L^1(\mathbb{Y},\nu)$ can be expressed via the isometry T_1. These observations give the following result.

Corollary A.17. *Assume that the measure ν is finite. Let $(j_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(\mathbb{Y},\nu)$. The collection $\{j_n : n \in \mathbb{N}\}$ is uniformly integrable if, and only, if every subsequence of $(j_n)_{n\in\mathbb{N}}$ contains a further subsequence, denoted by $(j_{n_k})_{k\in\mathbb{N}}$, such that*

$$\lim_{k\to\infty} \int_{\mathbb{Y}} g\, j_{n_k}\,\mathrm{d}\nu = \int_{\mathbb{Y}} g\, j\,\mathrm{d}\nu$$

for all $g \in L^\infty(\mathbb{Y},\nu)$ and some function $j \in L^1(\mathbb{Y},\nu)$.

The limit function arising in Corollary A.17 is ν-a.e. non-negative if each function j_n is non-negative (ν-a.e.). Indeed, if this were not the case, we would find a set $D \in \mathcal{Y}$ with $\nu(D) > 0$ such that $j < 0$ on D, which implies

$$0 > \int_{\mathbb{Y}} \mathbb{1}_D\, j\,\mathrm{d}\nu = \lim_{k\to\infty} \int_{\mathbb{Y}} \mathbb{1}_D\, j_{n_k}\,\mathrm{d}\nu \geq 0,$$

a contradiction.

A.6. A Kolmogorov extension result on the space of counting measures

The results in this part of the appendix are presented in full detail in the lecture notes by Preston (2005a,b). Though these are not peer reviewed publications in a scientific journal, the math is sound and the author has checked all results presented in this thesis as well as the preliminaries leading up to them, and, aside from a few typos, they are perfectly correct.

First, introduce the space $\mathbb{S} = \{0,1\}^{\mathbb{N}}$ of all sequences of 0's and 1's. Endowed with the metric

$$d\big((s_n)_{n\in\mathbb{N}},\,(s'_n)_{n\in\mathbb{N}}\big) = \sum_{n=1}^{\infty} \frac{|s_n - s'_n|}{2^n}$$

the space $\mathbb{S}$ is a compact metric space and we denote by $\mathcal{B}(\mathbb{S})$ its Borel σ-field. It is easy to verify that the σ-field of a measurable space $(\mathbb{X}, \mathcal{X})$ is countably generated if, and only if, there exists a map $f : \mathbb{X} \to \mathbb{S}$ such that $f^{-1}\big(\mathcal{B}(\mathbb{S})\big) = \mathcal{X}$.

A measurable space $(\mathbb{X}, \mathcal{X})$ is called substandard Borel space if there exists a map $f : \mathbb{X} \to \mathbb{S}$ such that $f^{-1}\big(\mathcal{B}(\mathbb{S})\big) = \mathcal{X}$ and $f(\mathbb{X}) \in \mathcal{B}(\mathbb{S})$. Thus, substandard Borel spaces are countably generated measurable spaces that satisfy some additional property. Any Borel space, and thus also any complete separable metric space, is substandard Borel, which can be shown with standard constructions in the lines of Theorem 6.1 of Preston (2007). Substandard Borel spaces provide enough structure to supply existence results for probability kernels and extension results such as the one we state below. Note that Preston (2007) discusses these spaces in much detail and calls them type-$\mathcal{B}$ spaces.

The results culminating in Proposition 18.2 (4) of Preston (2005a) and Lemma 2.2 of Preston (2005b) show that if $(\mathbb{X}, \mathcal{X})$ is a localized substandard Borel space, then $(\mathbf{N}, \mathcal{N}_B)$ is also substandard Borel for each $B \in \mathcal{X}_b$. Proposition 2.3 of Preston (2005b), which is proven via the general Kolmogorov extension result presented in Theorem 19.1 of Preston (2005a), shows that $(\mathbf{N}, \mathcal{N})$ is substandard Borel and an extension theorem holds for probability laws on the space of counting measures. The precise result reads as follows.

Proposition A.18. *Let $(\mathbb{X}, \mathcal{X})$ be a substandard Borel space with localizing structure $B_1 \subset B_2 \subset \ldots$. For each $\ell \in \mathbb{N}$ let P_ℓ be a probability measure on $(\mathbf{N}, \mathcal{N}_{B_\ell})$ such that $\mathrm{P}_\ell(A) = \mathrm{P}_i(A)$ for all $A \in \mathcal{N}_{B_i}$ whenever $i < \ell$. Then there exists a unique probability measure P on $(\mathbf{N}, \mathcal{N})$ such that, for all $\ell \in \mathbb{N}$ and $A \in \mathcal{N}_{B_\ell}$,*

$$\mathrm{P}(A) = \mathrm{P}_\ell(A).$$

The assumption on the measures $\{\mathrm{P}_\ell : \ell \in \mathbb{N}\}$ is called the Kolmogorov consistency property. Another reference which essentially provides a version of Proposition A.18 for Borel spaces is Theorem 4.1 in Chapter V of Parthasarathy (1967).

A.7. A metric structure on the space of counting measures

For the coupling constructions in the context of cluster-dependent Gibbs processes we need to consider weak convergence for probability measures on the space $\mathbf{N} \times \mathbf{N}$ and apply the Portmanteau theorem. In order to do so, we require a metric structure on this space and we have to ensure that a specific subset of the space is closed therein. To this end, we recall that a metric can be constructed on the space of counting measures. We also summarize the relevant properties of this metric, following Daley and Vere-Jones (2005, 2008).

Let $(\mathbb{X}, d)$ be a complete separable metric space. Denote by $\mathcal{X}$ the corresponding Borel σ-field and by $\mathcal{X}_b$ the collection of all Borel sets which are bounded with respect to d. Let the space of counting measures $(\mathbf{N}, \mathcal{N})$, as well as the set $\mathbf{N}_f$, be defined as in Section 2.2.

For a set $D \in \mathcal{X} \setminus \{\varnothing\}$ and a point $x \in \mathbb{X}$ we denote by $\operatorname{dist}(x, D) = \inf_{y \in D} d(x,y)$ the distance from x to D. For $\varepsilon > 0$ write D_ε for the halo set of D, that is,

$$D_\varepsilon = \big\{x \in \mathbb{X} : \operatorname{dist}(x, D) < \varepsilon\big\}.$$

Note that $D_\varepsilon \in \mathcal{X} \setminus \{\varnothing\}$ as $x \mapsto \operatorname{dist}(x, D)$ is Lipschitz-continuous, hence measurable. Following Appendices A2.5 and A2.6 of Daley and Vere-Jones (2005) we define the Prohorov distance of $\mu, \psi \in \mathbf{N}_f$ as

$$d_f(\mu, \psi) = \inf\Big\{\varepsilon > 0 : \mu(D) \leq \psi(D_\varepsilon) + \varepsilon \text{ and } \psi(D) \leq \mu(D_\varepsilon) + \varepsilon \text{ for all closed sets } D \in \mathcal{X} \setminus \{\varnothing\}\Big\}$$

and we define the map $d_{\mathbf{N}} : \mathbf{N} \times \mathbf{N} \to [0, \infty)$,

$$d_{\mathbf{N}}(\mu, \psi) = \int_0^\infty \frac{d_f(\mu_{O_r}, \psi_{O_r})}{1 + d_f(\mu_{O_r}, \psi_{O_r})} \cdot \mathrm{e}^{-r}\,\mathrm{d}r, \quad \mu, \psi \in \mathbf{N},$$

where $O_r = O_r(x_0) = \big\{x \in \mathbb{X} : d(x, x_0) < r\big\}$ is the open ball of radius $r > 0$ around a fixed origin $x_0 \in \mathbb{X}$. By Theorem A2.6.III of Daley and Vere-Jones (2005) as well as Lemma 9.1.V and Proposition 9.1.IV of Daley and Vere-Jones (2008), the map $d_{\mathbf{N}}$ is a metric on $\mathbf{N}$ which turns $(\mathbf{N}, d_{\mathbf{N}})$ into a complete separable metric space whose Borel σ-field is precisely $\mathcal{N}$. Note that Daley and Vere-Jones (2005) also argue why the topology generated by $d_{\mathbf{N}}$ does not depend on the choice of origin x_0.

Assume in the following that the product space $\mathbf{N} \times \mathbf{N}$ is equipped with any metric that generates the product topology on $\mathbf{N} \times \mathbf{N}$ induced by $(\mathbf{N}, d_{\mathbf{N}})$, for instance

$$\big((\mu, \psi), (\mu', \psi')\big) \mapsto \max\big\{d_{\mathbf{N}}(\mu, \mu'),\, d_{\mathbf{N}}(\psi, \psi')\big\}.$$

The product topology on $\mathbf{N} \times \mathbf{N}$ induces $\mathcal{N} \otimes \mathcal{N}$ as its Borel σ-field (by Lemma 1.2 of Kallenberg, 2002). Recall that two measures μ, ψ on $\mathbb{X}$ satisfy $\mu \leq \psi$, by definition, if $\mu(B) \leq \psi(B)$ for all $B \in \mathcal{X}$.

Lemma A.19. *The set $\big\{(\mu, \psi) \in \mathbf{N} \times \mathbf{N} : \mu \leq \psi\big\}$ is closed in $\mathbf{N} \times \mathbf{N}$.*

Proof. As $\mathbf{N} \times \mathbf{N}$ is endowed with a metric, it suffices to prove that the given set is sequentially closed. Thus, let $(\mu_n, \psi_n) \in \mathbf{N} \times \mathbf{N}$ with $\mu_n \leq \psi_n$, for each $n \in \mathbb{N}$, such that $(\mu_n, \psi_n) \to (\mu, \psi)$ in $\mathbf{N} \times \mathbf{N}$ as $n \to \infty$. As the coordinate projections are continuous with respect to the product topology, we get $\mu_n \to \mu$ and $\psi_n \to \psi$ each with respect to $d_{\mathbf{N}}$ as $n \to \infty$.

Assume that $\mu \not\leq \psi$. As $(\mathbb{X}, d)$ is a complete separable metric space, and hence μ and ψ can be written as sums of Dirac measures by Lemma 1.6 of Kallenberg (2017), there exists a point $z \in \mathbb{X}$ such that $\psi(\{z\}) < \mu(\{z\})$. Moreover, μ and ψ are locally finite, so we find some $\varepsilon > 0$ such that

$$\mu\big(O_{2\varepsilon}(z)\big) = \mu(\{z\}) \quad \text{and} \quad \psi\big(O_{2\varepsilon}(z)\big) = \psi(\{z\}),$$

meaning that no point of $\mu + \psi$ other than z lies in $O_{2\varepsilon}(z)$. In particular, we have

$$\mu\big(\partial O_\varepsilon(z)\big) = 0 \quad \text{and} \quad \psi\big(\partial O_\varepsilon(z)\big) = 0,$$

where $\partial O_\varepsilon(z)$ denotes the boundary of $O_\varepsilon(z)$. Proposition A2.6.II of Daley and Vere-Jones (2005) yields

$$\mu_n\big(O_\varepsilon(z)\big) \longrightarrow \mu\big(O_\varepsilon(z)\big) \quad \text{and} \quad \psi_n\big(O_\varepsilon(z)\big) \longrightarrow \psi\big(O_\varepsilon(z)\big)$$

as $n \to \infty$. We conclude that

$$\mu(\{z\}) = \mu\big(O_\varepsilon(z)\big) = \limsup_{n\to\infty} \mu_n\big(O_\varepsilon(z)\big) \leq \limsup_{n\to\infty} \psi_n\big(O_\varepsilon(z)\big) = \psi\big(O_\varepsilon(z)\big) = \psi(\{z\}) < \mu(\{z\}),$$

a contradiction. Therefore, we have $\mu \leq \psi$ and the proof is complete. □

A.8. Dissection systems and differentiation of measures

In Section 3.3.5 or, more specifically, in the proof of Theorem 3.34 we rely heavily on a differentiation property for measures. A detailed account of such differentiation properties is given in the book by Hayes and Pauc (1970) but we rely on rather specific results from Section 1.5 of Kallenberg (2017) involving dissection systems. While such differentiation properties were already considered by Lebesgue (1904), the more general results originate from de Possel (1936). We also use this appendix section to collect additional technical results which fit into the same formalism.

Let $(\mathbb{X}, d)$ be a complete separable metric space endowed with its Borel σ-field and denote by $\mathcal{X}_b$ the collection of Borel sets which are bounded with respect to the metric d. Following Kallenberg (2017) we call a family of sets

$$\{D_{j,n} : j, n \in \mathbb{N}\} \subset \mathcal{X}_b \setminus \{\varnothing\}$$

a *dissection system* in $\mathbb{X}$ if the following properties are satisfied. Namely,

- for each $n \in \mathbb{N}$ the sets $D_{1,n}, D_{2,n}, \ldots$ are pairwise disjoint with $\bigcup_{j=1}^{\infty} D_{j,n} = \mathbb{X}$,
- for all $n \in \mathbb{N}$ the partition $\{D_{j,n+1} : j \in \mathbb{N}\}$ is a refinement of $\{D_{j,n} : j \in \mathbb{N}\}$ in the sense that each set $D_{j,n}$ is a union of sets from $\{D_{j,n+1} : j \in \mathbb{N}\}$,
- for each $n \in \mathbb{N}$ any bounded set $B \in \mathcal{X}_b$ is covered by finitely many sets from $\{D_{j,n} : j \in \mathbb{N}\}$, and
- we have $\sigma(D_{j,n} : j, n \in \mathbb{N}) = \mathcal{X}$.

Such a dissection system exists in any localized Borel space, hence also in the setting at hand, by Lemma 1.3 of Kallenberg (2017). Moreover, since $\{x\} \in \mathcal{X}$ for each $x \in \mathbb{X}$, the last property implies that the dissection system separates points in $\mathbb{X}$ in the sense that for any $x, y \in \mathbb{X}$ with $x \neq y$ there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the points x, y lie in different sets from $\{D_{j,n} : j \in \mathbb{N}\}$.

For $m \in \mathbb{N}$ we endow the product space $\mathbb{X}^m$ with the metric

$$\big((x_1, \ldots, x_m), (y_1, \ldots, y_m)\big) \mapsto \max\big\{d(x_1, y_1), \ldots, d(x_m, y_m)\big\}$$

which metrizes the product topology and makes $\mathbb{X}^m$ itself a complete separable metric space.

Lemma A.20. *Let $\{D_{j,n} : j, n \in \mathbb{N}\}$ be a dissection system in $\mathbb{X}$ and fix any $m \in \mathbb{N}$. Then the collection*

$$\big\{D_{j_1,n} \times \ldots \times D_{j_m,n} : (j_1, \ldots, j_m) \in \mathbb{N}^m,\ n \in \mathbb{N}\big\}$$

is a dissection system in $\mathbb{X}^m$.

Proof. The first two properties of a dissection system are clearly satisfied. As a set $D_1 \times \ldots \times D_m$ is bounded in $\mathbb{X}^m$ if, and only if, $D_1, \ldots, D_m$ are bounded in $\mathbb{X}$, the third property also holds. By definition of the product σ-field, we have

$$\sigma\Big(D_{j_1,n} \times \ldots \times D_{j_m,n} : (j_1, \ldots, j_m) \in \mathbb{N}^m,\ n \in \mathbb{N}\Big) \subset \mathcal{X}^{\otimes m}. \tag{A.5}$$

As $\mathcal{X}^{\otimes m}$ is the smallest σ-field on $\mathbb{X}^m$ for which all projection maps $\mathrm{pr}_j(x_1, \ldots, x_m) = x_j$, for $j = 1, \ldots, m$, are measurable, it suffices to prove that the projection maps are measurable with respect to the σ-field on the left hand side of (A.5). Let $j \in \{1, \ldots, m\}$. For any $i, n \in \mathbb{N}$,

$$\mathrm{pr}_j^{-1}(D_{i,n}) = \underbrace{\mathbb{X} \times \ldots \times \mathbb{X}}_{j-1 \text{ times}} \times D_{i,n} \times \underbrace{\mathbb{X} \times \ldots \times \mathbb{X}}_{m-j \text{ times}}$$

$$\begin{aligned}
&= \bigcup_{\ell_1,\dots,\ell_{j-1},\ell_{j+1},\dots,\ell_m \in \mathbb{N}} D_{\ell_1,n} \times \ldots \times D_{\ell_{j-1},n} \times D_{i,n} \times D_{\ell_{j+1},n} \times \ldots \times D_{\ell_m,n} \\
&\in \sigma\Big(D_{j_1,n} \times \ldots \times D_{j_m,n} : (j_1,\dots,j_m) \in \mathbb{N}^m,\, n \in \mathbb{N}\Big),
\end{aligned}$$

which gives

$$\mathrm{pr}_j^{-1}(\mathcal{X}) = \sigma\big(\mathrm{pr}_j^{-1}(D_{i,n}) : i, n \in \mathbb{N}\big) \subset \sigma\Big(D_{j_1,n} \times \ldots \times D_{j_m,n} : (j_1,\dots,j_m) \in \mathbb{N}^m,\, n \in \mathbb{N}\Big).$$

We conclude that the σ-fields in (A.5) are identical and the last property of a dissection system is satisfied. □

The following result essentially follows from the proof of Theorem 1.28 of Kallenberg (2017) and the discussion thereafter. A little more detail concerning the proof can be obtained by combining Example A and Proposition 2 of Rao (1987) with the convergence theorem for uniformly integrable martingales (e.g. Theorem 10.3.13 of Bogachev, 2007b).

Proposition A.21. *Let $\{D_{j,n} : j, n \in \mathbb{N}\}$ be a dissection system in $\mathbb{X}$. Moreover, let α and λ be locally finite measures on $\mathbb{X}$ such that $\alpha \ll \lambda$ with density function $\rho : \mathbb{X} \to [0, \infty)$. Then, for λ-a.e. $x \in \mathbb{X}$,*

$$\rho(x) = \lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{\alpha(D_{j,n})}{\lambda(D_{j,n})} \cdot \mathbb{1}\{x \in D_{j,n}\},$$

where we set $0/0 = 0$ in this one instance.

Remark A.22. As $\{D_{j,n} : j \in \mathbb{N}\}$ is a partition of $\mathbb{X}$ for each $n \in \mathbb{N}$, we find for any $x \in \mathbb{X}$ and every $n \in \mathbb{N}$ precisely one set $D_{x,n} \in \{D_{j,n} : j \in \mathbb{N}\}$ such that $x \in D_{x,n}$. With this slight (but intuitive) abuse of notation, the statement of the previous proposition reads as

$$\rho(x) = \lim_{n\to\infty} \frac{\alpha(D_{x,n})}{\lambda(D_{x,n})}$$

for λ-a.e. $x \in \mathbb{X}$. With the new notation, the properties of the dissection system imply $D_{x,n} \supset D_{x,n+1}$ for each $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} D_{x,n} = \{x\}$, for every $x \in \mathbb{X}$. ○

For technical reasons (in the form of the upcoming lemma) we need to consider in Section 3.3.5 a *topological dissection system* which is a dissection system $\{D_{j,n} : j, n \in \mathbb{N}\}$ with the additional property that any open subset of $\mathbb{X}$ is a countable union of sets from $\{D_{j,n} : j, n \in \mathbb{N}\}$. For any locally finite measure λ on $\mathbb{X}$ the collection

$$\big\{B \in \mathcal{X}_b : \lambda(\partial B) = 0\big\},$$

where ∂B denotes the topological boundary of B, contains a topological dissection system by Lemma 1.9 of Kallenberg (2017). In particular, such a system exists. For two non-empty sets $D, D' \subset \mathbb{X}$ we denote their distance by

$$\mathrm{dist}(D, D') = \inf\big\{d(x, y) : x \in D,\, y \in D'\big\} = \inf_{x\in D} \mathrm{dist}(x, D'),$$

with $\mathrm{dist}(x, D') = \inf_{y\in D'} d(x, y)$ as in Section A.7.

Lemma A.23. *Let $\{D_{j,n} : j, n \in \mathbb{N}\}$ be a topological dissection system in $\mathbb{X}$. Fix $k, m \in \mathbb{N}$ and let $x_1, \dots, x_k, y_1, \dots, y_m \in \mathbb{X}$ be pairwise distinct. Then*

$$\lim_{n\to\infty} \mathrm{dist}\Bigg(\bigcup_{i=1}^{k} D_{x_i,n},\, \bigcup_{j=1}^{m} D_{y_j,n}\Bigg) = \mathrm{dist}\big(\{x_1,\dots,x_k\}, \{y_1,\dots,y_m\}\big).$$

Proof. First of all, let $z \in \mathbb{X}$ be arbitrary. By the additional property of a topological dissection system, any open ball around z contains all but finitely many of the sets $D_{z,n}$, $n \in \mathbb{N}$. Therefore, we have

$$\lim_{n\to\infty} \operatorname{diam}(D_{z,n}) = 0,$$

where $\operatorname{diam}(D) = \sup_{x,y\in D} d(x,y)$ denotes the diameter of a non-empty subset D of $\mathbb{X}$. Now, fix any $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,m\}$. For each $n \in \mathbb{N}$ we trivially have

$$\operatorname{dist}(D_{x_i,n}, D_{y_j,n}) \le \operatorname{dist}\big(\{x_i\},\{y_j\}\big) = d(x_i,y_j).$$

Moreover, the triangle inequality implies

$$\operatorname{dist}(D_{x_i,n}, D_{y_j,n}) \ge d(x_i,y_j) - \operatorname{diam}(D_{x_i,n}) - \operatorname{diam}(D_{y_j,n}),$$

so we obtain $\lim_{n\to\infty} \operatorname{dist}(D_{x_i,n}, D_{y_j,n}) = d(x_i,y_j)$. We conclude that, as $n \to \infty$,

$$\begin{aligned}
\operatorname{dist}\Bigg(\bigcup_{i=1}^{k} D_{x_i,n}, \bigcup_{j=1}^{m} D_{y_j,n}\Bigg) &= \min\Big\{\operatorname{dist}(D_{x_i,n}, D_{y_j,n}) : i \in \{1,\dots,k\},\, j \in \{1,\dots,m\}\Big\} \\
&\to \min\big\{d(x_i,y_j) : i \in \{1,\dots,k\},\, j \in \{1,\dots,m\}\big\} \\
&= \operatorname{dist}\big(\{x_1,\dots,x_k\},\{y_1,\dots,y_m\}\big),
\end{aligned}$$

where the equalities follow by iterating the following elementary observation. Let B, D, D' be non-empty subsets of $\mathbb{X}$. Then we trivially get

$$\operatorname{dist}(B, D\cup D') \le \min\big\{\operatorname{dist}(B,D),\, \operatorname{dist}(B,D')\big\}.$$

On the other hand we have, for any $x \in B$,

$$\min\big\{\operatorname{dist}(B,D),\, \operatorname{dist}(B,D')\big\} \le \min\big\{\operatorname{dist}(x,D),\, \operatorname{dist}(x,D')\big\} \le \operatorname{dist}(x, D\cup D')$$

by first bounding the terms in the minimum separately and then arguing by a simply contradiction. Taking $\inf_{x\in B}$ yields $\min\big\{\operatorname{dist}(B,D),\, \operatorname{dist}(B,D')\big\} \le \operatorname{dist}(B, D\cup D')$ and therefore

$$\operatorname{dist}(B, D\cup D') = \min\big\{\operatorname{dist}(B,D),\, \operatorname{dist}(B,D')\big\}. \qquad \square$$

A.9. Measurability properties of mappings on the space of compact sets

In Section 3.4 we discuss particles processes, that is, point processes in the space of compact subsets of a metric space. In Example 3.40, Equation (3.15), Corollary 3.41, and Example 3.42 there arise questions concerning measurability. In the appendix section at hand, we settle all relevant issues. We also give a short reminder of the topology on the space of compact particles. Note that we state and prove the results precisely in the generality we need them in and with almost no additional preliminary thoughts and notation.

As in Section 3.4, let $(\mathbb{B}, d)$ be a complete separable metric space with Borel sets $\mathcal{B}(\mathbb{B})$ and $\mathcal{C}^*(\mathbb{B})$ the collection of non-empty compact subsets of $\mathbb{B}$, equipped with the Hausdorff metric d_H. Recall from Appendix D of Molchanov (2017) that, for $K, L \in \mathcal{C}^*(\mathbb{B})$,

$$d_H(K,L) = \inf\big\{\varepsilon > 0 : K \subset L^\varepsilon \text{ and } L \subset K^\varepsilon\big\} = \sup_{x\in\mathbb{B}}\big|\operatorname{dist}(x,K) - \operatorname{dist}(x,L)\big|,$$

where $C^\varepsilon = \{x \in \mathbb{B} : \operatorname{dist}(x, C) \le \varepsilon\}$ is the ε-envelope of $C \in \mathcal{B}(\mathbb{B}) \setminus \{\varnothing\}$ and $\operatorname{dist}(x, C) = \inf_{y \in C} d(x, y)$. The metric space $\big(\mathcal{C}^*(\mathbb{B}), d_H\big)$ is complete by Theorem D.9 of Molchanov (2017) and, combining the remark before Proposition D.4 of Molchanov (2017) with his Corollary D.7, the space is also seen to be separable.

Denote the Borel σ-field of $\big(\mathcal{C}^*(\mathbb{B}), d_H\big)$ by $\mathcal{B}\big(\mathcal{C}^*(\mathbb{B})\big)$. By Theorem D.6 of Molchanov (2017), we have

$$\mathcal{B}\big(\mathcal{C}^*(\mathbb{B})\big) = \sigma\Big(\big\{K \in \mathcal{C}^*(\mathbb{B}) : K \cap O \ne \varnothing\big\} : O \in \mathcal{O}(\mathbb{B})\Big) \tag{A.6}$$

where $\mathcal{O}(\mathbb{B})$ are the open subsets of $(\mathbb{B}, d)$. It follows immediately that, for any $x \in \mathbb{B}$ and $r \ge 0$,

$$\big\{K \in \mathcal{C}^*(\mathbb{B}) : K \subset B(x, r)\big\} = \big\{K \in \mathcal{C}^*(\mathbb{B}) : K \cap B(x, r)^{\mathsf{c}} = \varnothing\big\} \in \mathcal{B}\big(\mathcal{C}^*(\mathbb{B})\big),$$

where $B(x, r)$ denotes the closed ball in $\mathbb{B}$ of radius r around x with respect to d. Note that $B(x, 0) = \{x\}$.

In Section 3.4 we frequently use an indicator function of two compact sets having non-empty intersection. The following lemma guarantees that the map

$$(K, L) \mapsto \mathbb{1}\{K \cap L \ne \varnothing\}$$

is measurable. Endow the space $\mathcal{C}^*(\mathbb{B}) \times \mathcal{C}^*(\mathbb{B})$ with the product topology (similar to Appendix A.7).

Lemma A.24. *Fix any closed set $F \subset \mathbb{B}$. Then the set $\big\{(K, L) \in \mathcal{C}^*(\mathbb{B}) \times \mathcal{C}^*(\mathbb{B}) : K \cap L \cap F \ne \varnothing\big\}$ is closed in $\mathcal{C}^*(\mathbb{B}) \times \mathcal{C}^*(\mathbb{B})$.*

Proof. It suffices to prove that the set is sequentially closed. Assume this were not the case. Then there exist $K_1, L_1, K_2, L_2, \ldots \in \mathcal{C}^*(\mathbb{B})$ such that, for each $n \in \mathbb{N}$,

$$K_n \cap L_n \cap F \ne \varnothing$$

and $(K_n, L_n) \to (K, L)$, as $n \to \infty$, for some $K, L \in \mathcal{C}^*(\mathbb{B})$ such that $K \cap L \cap F = \varnothing$. By the assumed convergence, the collections

$$\{K, K_n : n \in \mathbb{N}\} \quad \text{and} \quad \{L, L_n : n \in \mathbb{N}\}$$

are compact subsets of $\mathcal{C}^*(\mathbb{B})$, hence so is their union. By Theorem 3.1 of Christensen (1974) the set

$$C = K \cup L \cup \bigcup_{n=1}^{\infty} (K_n \cup L_n)$$

is compact in $\mathbb{B}$. By assumption on the sets K_n, L_n, we find some point $x_n \in K_n \cap L_n \cap F$ for each $n \in \mathbb{N}$. As the sequence $(x_n)_{n \in \mathbb{N}}$ is contained in the compact set C, there exists a subsequence which converges to a point $x \in \mathbb{B}$. If $x \notin K$ then $r = \operatorname{dist}(x, K)/2 > 0$ as K is closed, and $x \notin K^r$ (the r-envelope of K). However, for any large enough $n \in \mathbb{N}$ we have $K_n \subset K^r$ (by definition of the Hausdorff metric) which gives $\operatorname{dist}(x_n, x) \ge r$ in contradiction to the convergence $x_n \to x$ as $n \to \infty$. Therefore, $x \in K$. The identical argument with K replace by L shows that $x \in L$ and, since F is closed, we also have $x \in F$. We conclude that $x \in K \cap L \cap F$ which is a contradiction to the fact that $K \cap L \cap F = \varnothing$. $\square$

To ensure measurability of the Papangelou intensity in Example 3.42, we need a refinement of Lemma A.24. To this end, let $\mathcal{C}(\mathbb{B})$ be the space of all compact subsets of $\mathbb{B}$, in other words $\mathcal{C}(\mathbb{B}) = \mathcal{C}^*(\mathbb{B}) \cup \{\varnothing\}$. As a σ-field on $\mathcal{C}(\mathbb{B})$ we choose

$$\mathcal{B}\big(\mathcal{C}(\mathbb{B})\big) = \sigma\Big(\mathcal{B}\big(\mathcal{C}^*(\mathbb{B})\big) \cup \big\{\{\varnothing\}\big\}\Big) = \mathcal{B}\big(\mathcal{C}^*(\mathbb{B})\big) \cup \Big\{\mathcal{H} \cup \{\varnothing\} : \mathcal{H} \in \mathcal{B}\big(\mathcal{C}^*(\mathbb{B})\big)\Big\},$$

where the equality follows from the simple observation that the right hand side is a σ-field.

Corollary A.25. *The mapping $\mathcal{C}^*(\mathbb{B}) \times \mathcal{C}^*(\mathbb{B}) \ni (K, L) \mapsto K \cap L \in \mathcal{C}(\mathbb{B})$ is measurable.*

Proof. Denote the given intersection mapping by $\mathrm{Int}(K, L)$, for $K, L \in \mathcal{C}^*(\mathbb{B})$. Note that the set

$$\mathrm{Int}^{-1}(\{\varnothing\}) = \big\{(K, L) \in \mathcal{C}^*(\mathbb{B}) \times \mathcal{C}^*(\mathbb{B}) : K \cap L = \varnothing\big\}$$

is open in $\mathcal{C}^*(\mathbb{B}) \times \mathcal{C}^*(\mathbb{B})$ (hence measurable) due to Lemma A.24 (upon choosing $F = \mathbb{B}$). By definition of the σ-field $\mathcal{B}\big(\mathcal{C}(\mathbb{B})\big)$ and (A.6), it suffices to show that, for any $O \in \mathcal{O}(\mathbb{B})$, the set

$$\mathrm{Int}^{-1}\big(\{C \in \mathcal{C}^*(\mathbb{B}) : C \cap O \neq \varnothing\}\big)$$

is a measurable subset of $\mathcal{C}^*(\mathbb{B}) \times \mathcal{C}^*(\mathbb{B})$. Thus, let $O \in \mathcal{O}(\mathbb{B})$. As in the proof of Theorem D.6 of Molchanov (2017), consider, for each $m \in \mathbb{N}$,

$$F_m = \Big\{x \in \mathbb{B} : \mathrm{dist}(x, O^c) \geq \frac{1}{m}\Big\}.$$

Each of these sets is closed in $\mathbb{B}$ and we have $\bigcup_{m=1}^\infty F_m = O$. We conclude that

$$\begin{aligned}\mathrm{Int}^{-1}\big(\{C \in \mathcal{C}^*(\mathbb{B}) : C \cap O \neq \varnothing\}\big) &= \big\{(K, L) \in \mathcal{C}^*(\mathbb{B}) \times \mathcal{C}^*(\mathbb{B}) : K \cap L \cap O \neq \varnothing\big\} \\ &= \bigcup_{m=1}^\infty \big\{(K, L) \in \mathcal{C}^*(\mathbb{B}) \times \mathcal{C}^*(\mathbb{B}) : K \cap L \cap F_m \neq \varnothing\big\},\end{aligned}$$

which is a measurable subset of $\mathcal{C}^*(\mathbb{B}) \times \mathcal{C}^*(\mathbb{B})$ by Lemma A.24. □

We now specify to the setting of Example 3.40 and Corollary 3.41, where $\mathbb{B} = \mathbb{R}^d$ and the space of non-empty compact subsets of $\mathbb{R}^d$ is denoted by $\mathcal{C}^{(d)} = \mathcal{C}^*(\mathbb{R}^d)$ with corresponding Borel σ-field $\mathcal{B}(\mathcal{C}^{(d)})$. In $\mathbb{R}^d$ all balls are compact and they depend continuously on their center and radius.

Lemma A.26. *The mapping $\mathbb{R}^d \times [0, \infty) \to \mathcal{C}^{(d)}$, $(x, r) \mapsto B(x, r)$ is continuous.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ which converges to some $x \in \mathbb{R}^d$ and let $(r_n)_{n\in\mathbb{N}}$ be a sequence of non-negative numbers which converges to $r \geq 0$. Then, as $n \to \infty$,

$$d_H\big(B(x_n, r_n), B(x, r)\big) \leq |x_n - x| + |r_n - r| \longrightarrow 0.$$ □

For $x \in \mathbb{R}^d$ and $K \in \mathcal{C}^{(d)}$ we write $K + x = \{z + x : z \in K\} \in \mathcal{C}^{(d)}$ and $-K = \{-z : z \in K\} \in \mathcal{C}^{(d)}$. For $K, L \in \mathcal{C}^{(d)}$ we put $K + L = \{x + z : x \in K,\, z \in L\}$. Note that $+ : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $(x, z) \mapsto x + z$ is a continuous map and $K + L$ is the image of the compact set $K \times L$ under $+$, hence $K + L \in \mathcal{C}^{(d)}$. The following lemma collects our required measurability properties concerning these operations.

Lemma A.27.

(i) *The map $\mathbb{R}^d \times \mathcal{C}^{(d)} \to \mathcal{C}^{(d)}$, $(x, K) \mapsto K + x$ is continuous.*

(ii) *The map $\mathbb{R}^d \times \mathcal{C}^{(d)} \times \mathcal{C}^{(d)} \to \{0, 1\}$, $(x, L, C) \mapsto \mathbb{1}\big\{(L + x) \cap C \neq \varnothing\big\}$ is measurable.*

(iii) *The map $\mathcal{C}^{(d)} \times \mathcal{C}^{(d)} \to [0, \infty)$, $(L, C) \mapsto \mathcal{L}^d\big(L + (-C)\big)$ is measurable.*

Proof. We need only prove part (i). Part (ii) follows from (i) and Lemma A.24, while part (iii) follows from (ii) by observing that

$$\mathcal{L}^d\big(L + (-C)\big) = \mathcal{L}^d\big(C + (-L)\big) = \int_{\mathbb{R}^d} \mathbb{1}\big\{x \in C + (-L)\big\}\, \mathrm{d}\mathcal{L}^d(x) = \int_{\mathbb{R}^d} \mathbb{1}\big\{(L + x) \cap C \neq \varnothing\big\}\, \mathrm{d}\mathcal{L}^d(x).$$

For the proof of (i), let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ which converges to $x \in \mathbb{R}^d$ and let $K_1, K_2, \ldots \in \mathcal{C}^{(d)}$ converge to $K \in \mathcal{C}^{(d)}$ with respect to the Hausdorff metric. By the triangle inequality we get, for $n \in \mathbb{N}$,

$$d_H(K_n + x_n, K + x) \le d_H(K_n + x_n, K + x_n) + d_H(K + x_n, K + x). \tag{A.7}$$

The first term on the right hand side of (A.7) is given by

$$\begin{aligned}\sup_{z\in\mathbb{R}^d} \big|\mathrm{dist}(z, K_n + x_n) - \mathrm{dist}(z, K + x_n)\big| &= \sup_{z\in\mathbb{R}^d} \big|\mathrm{dist}(z - x_n, K_n) - \mathrm{dist}(z - x_n, K)\big| \\ &= \sup_{z\in\mathbb{R}^d} \big|\mathrm{dist}(z, K_n) - \mathrm{dist}(z, K)\big| \\ &= d_H(K_n, K).\end{aligned}$$

As the distance function $x \mapsto \mathrm{dist}(x, K)$ is Lipschitz-continuous, the second term on the right hand side of (A.7) equals

$$\sup_{z\in\mathbb{R}^d} \big|\mathrm{dist}(z, K + x_n) - \mathrm{dist}(z, K + x)\big| = \sup_{z\in\mathbb{R}^d} \big|\mathrm{dist}(z - x_n, K) - \mathrm{dist}(z - x, K)\big| \le |x_n - x|.$$

We conclude from (A.7) that, as $n \to \infty$,

$$d_H(K_n + x_n, K + x) \le d_H(K_n, K) + |x_n - x| \longrightarrow 0. \qquad \square$$

Note that, while Lemma A.24 is proven in a very general setting (which the author could not find elsewhere), the results in Lemma A.27 can be strengthened. For instance, more general linear spaces can be considered and the map $K \mapsto \mathcal{L}^d(K)$ is actually upper semicontinuous. Such stronger results are not necessary for us and as they require additional notation, we do not make them more precise and refer to Beer (1993), Schneider and Weil (2008), and Molchanov (2017) for further reading.

In the remainder of this appendix section, we provide basic properties of the circumball of a compact set which appears in Corollary 3.41. Let $K \in \mathcal{C}^{(d)}$. It is a very easy (though a bit tedious) exercise to show that among those balls in $\mathbb{R}^d$ which contain K there exists a unique ball with the smallest radius. This ball is called the circumball of K and we denote it by B_K. As in Corollary 3.41, let $c : \mathcal{C}^{(d)} \to \mathbb{R}^d$ be the map which assigns to each compact set the center of its circumball and denote by $\mathrm{rad} : \mathcal{C}^{(d)} \to [0, \infty)$ the map that assigns to a compact set the radius of its circumball. As the circumball of a compact set is unique, these maps are well-defined.

Lemma A.28. *Both* $c : \mathcal{C}^{(d)} \to \mathbb{R}^d$ *and* $\mathrm{rad} : \mathcal{C}^{(d)} \to [0, \infty)$ *are continuous.*

Proof. Let $K_1, K_2, \ldots \in \mathcal{C}^{(d)}$ converge to $K \in \mathcal{C}^{(d)}$ with respect to d_H. Let $\varepsilon > 0$. By definition of d_H there exists an index $n_0 \in \mathbb{N}$ such that, for any $n \ge n_0$,

$$K \subset (K_n)^{\varepsilon/2} \subset (B_{K_n})^{\varepsilon/2} \quad \text{as well as} \quad K_n \subset K^{\varepsilon/2} \subset (B_K)^{\varepsilon/2}.$$

As the function $\mathrm{rad} : \mathcal{C}^{(d)} \to [0, \infty)$ determines the smallest radius that a ball containing a given compact set can have, it follows that

$$\mathrm{rad}(K) < \mathrm{rad}(K_n) + \varepsilon \quad \text{and} \quad \mathrm{rad}(K_n) < \mathrm{rad}(K) + \varepsilon,$$

which gives $|\mathrm{rad}(K_n) - \mathrm{rad}(K)| < \varepsilon$. Hence the radius map is continuous. Put $r = 2 \cdot \sup_{n\in\mathbb{N}} \mathrm{rad}(K_n)$. Since

$\mathrm{rad}(K_n) \to \mathrm{rad}(K)$ as $n \to \infty$, we have $r < \infty$. As in the proof of Lemma A.24, the set

$$C = K \cup \bigcup_{n=1}^{\infty} K_n$$

is compact by the assumed convergence and Theorem 3.1 of Christensen (1974). With balls in $\mathbb{R}^d$ being compact, the set $C + B(0, r)$ is also compact. Since $c(K_n) \in K_n + B\big(0, 2 \cdot \mathrm{rad}(K_n)\big)$ for each $n \in \mathbb{N}$, we have

$$\big\{c(K_n) : n \in \mathbb{N}\big\} \subset C + B(0, r),$$

so $\big(c(K_n)\big)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}^d$. In order to finish the proof it suffices to show that any convergent subsequence of $\big(c(K_n)\big)_{n\in\mathbb{N}}$ converges to $c(K)$. Therefore, assume that $c(K_{n_i}) \to c^* \in [0, \infty)$ as $i \to \infty$. It follows from Lemma A.26 that $B\big(c(K_{n_i}), \mathrm{rad}(K_{n_i})\big) \to B\big(c^*, \mathrm{rad}(K)\big)$ in d_H. Now, if for $x \in K$ we had $x \notin B\big(c^*, \mathrm{rad}(K)\big)$, then

$$s = \frac{1}{2} \cdot \mathrm{dist}\Big(x, B\big(c^*, \mathrm{rad}(K)\big)\Big) > 0$$

and hence $\mathrm{dist}(x, K_{n_i}) \geq s$ for all but finitely many $i \in \mathbb{N}$ (similar to the argument toward the end of the proof of Lemma A.24). This is a contradiction to the fact that

$$\mathrm{dist}(x, K_{n_i}) = \big|\mathrm{dist}(x, K_{n_i}) - \mathrm{dist}(x, K)\big| \leq d_H(K_{n_i}, K) \longrightarrow 0$$

as $i \to \infty$. Hence, we have $K \subset B\big(c^*, \mathrm{rad}(K)\big)$ and, by the uniqueness of the circumball, $c^* = c(K)$. □

Note that if $K_1, K_2, \ldots \in \mathcal{C}^{(d)}$ converge to $K \in \mathcal{C}^{(d)}$ with respect to the Hausdorff metric, then Lemma A.28 implies $\mathrm{rad}(K_n) \to \mathrm{rad}(K)$ and $c(K_n) \to c(K)$, as $n \to \infty$, so Lemma A.26 shows that the full map

$$\mathcal{C}^{(d)} \to \mathcal{C}^{(d)}, \quad K \mapsto B_K$$

is continuous. The overall idea of the proof of Lemma A.28 is inspired by the course material for the exercise class of the stochastic geometry course taught at Karlsruhe Institute of Technology in the summer term 2020. For a different proof we refer to Lemma 4.1.1 of Schneider and Weil (2008).

Bibliography

Beer, G. (1993). *Topologies on Closed and Closed Convex Sets*, Mathematics and Its Application, Kluwer Academic Publishers, Dordrecht.

Beneš, V., Hofer-Temmel, C., Last, G. and Večeřa, J. (2020). Decorrelation of a class of Gibbs particle processes and asymptotic properties of U-statistics, *Journal of Applied Probability* **57**(3): 928–955.

Betsch, S. (2022). Structural properties of Gibbsian point processes in abstract spaces, *arXiv 2201.11415.*

Betsch, S. and Last, G. (2022). On the uniqueness of Gibbs distributions with a non-negative and subcritical pair potential, *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* (to appear).

Błaszczyszyn, B., Yogeshwaran, D. and Yukich, J. E. (2019). Limit theory for geometric statistics of point processes having fast decay of correlations, *The Annals of Probability* **47**(2): 835–895.

Bogachev, V. I. (2007a). *Measure Theory - Volume I*, Springer, Berlin.

Bogachev, V. I. (2007b). *Measure theory - Volume II*, Springer, Berlin.

Bray, A. and Schoenberg, F. P. (2013). Assessment of point process models for earthquake forecasting, *Statistical Science* **28**(4): 510–520.

Chayes, L. and Schonmann, R. H. (2000). Mixed percolation as a bridge between site and bond percolation, *The Annals of Applied Probability* **10**(4): 1182–1196.

Christensen, J. P. R. (1974). *Topology and Borel structure : Descriptive topology and set theory with applications to functional analysis and measure theory*, North-Holland Mathematics Studies Vol. 10; Notas de Matemática Vol. 51, North-Holland Publishing Co., Amsterdam.

Cohn, D. L. (2013). *Measure Theory (Second Edition)*, Birkhäuser Advanced Texts, Birkhäuser, New York.

Conache, D., Daletskii, A., Kondratiev, Y. G. and Pasurek, T. (2018). Gibbs states of continuum particle systems with unbounded spins: Existence and uniqueness, *Journal of Mathematical Physics* **59**(1): 013507.

Daley, D. J. and Vere-Jones, D. (2005). *An Introduction to the Theory of Point Processes Volume I: Elementary Theory and Methods (Second Edition)*, Probability and Its Applications, Springer, New York.

Daley, D. J. and Vere-Jones, D. (2008). *An Introduction to the Theory of Point Processes Volume II: General Theory and Structure (Second Edition)*, Probability and Its Applications, Springer, New York.

de Possel, R. (1936). Sur la dérivation abstraite des fonctions d'ensembles, *Journal de Mathématiques Pures et Appliquées* **15**: 391–409.

Deijfen, M., van der Hofstad, R. and Hooghiemstra, G. (2013). Scale-free percolation, *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* **49**(3): 817–838.

Dereudre, D. (2009). The existence of quermass-interaction processes for nonlocally stable interaction and nonbounded convex grains, *Advances in Applied Probability* **41**(3): 664–681.

Dereudre, D. (2019). Introduction to the theory of Gibbs point processes, *in* D. Coupier (ed.), *Stochastic geometry : Modern research frontiers*, Lecture Notes in Mathematics 2237, Springer, Cham, pp. 181–226.

Dereudre, D., Drouilhet, R. and Georgii, H.-O. (2012). Existence of Gibbsian point processes with geometry-dependent interactions, *Probability Theory and Related Fields* **153**: 643–670.

Dereudre, D. and Houdebert, P. (2021). Sharp phase transition for the continuum Widom–Rowlinson model, *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* **57**(1): 387–407.

Dereudre, D. and Vasseur, T. (2020). Existence of Gibbs point processes with stable infinite range interaction, *Journal of Applied Probability* **57**(3): 775–791.

Dobrushin, R. L. (1964). Investigation of conditions for the asymptotic existence of the configuration integral of Gibbs' distribution, *Theory of Probability & Its Applications* **9**(4): 566–581.

Dobrushin, R. L. (1968a). The description of a random field by means of conditional probabilities and conditions of its regularity, *Theory of Probability & Its Applications* **13**(2): 197–224.

Dobrushin, R. L. (1968b). Gibbsian random fields for lattice systems with pairwise interactions, *Functional Analysis and Its Applications* **2**(4): 292–301.

Dobrushin, R. L. (1968c). The problem of uniqueness of a Gibbsian random field and the problem of phase transitions, *Functional Analysis and Its Applications* **2**(4): 302–312.

Dobrushin, R. L. (1969). Gibbsian random fields. The general case, *Functional Analysis and Its Applications* **3**(1): 22–28.

Dobrushin, R. L. and Minlos, R. A. (1967). Existence and continuity of pressure in classical statistical physics, *Theory of Probability & Its Applications* **12**(4): 535–559.

Duminil-Copin, H., Raoufi, A. and Tassion, V. (2020). Subcritical phase of d-dimensional Poisson–Boolean percolation and its vacant set, *Annales Henri Lebesgue* **3**: 677–700.

Dunford, N. and Schwartz, J. T. (1958). *Linear operators, I. General Theory*, Pure and applied mathematics Vol. VII, Interscience Publishers, New York.

Ferrari, P. A., Fernández, R. and Garcia, N. L. (2002). Perfect simulation for interacting point processes, loss networks and Ising models, *Stochastic Processes and their Applications* **102**(1): 63–88.

Fisher, M. E. (1964). The free energy of a macroscopic system, *Archive for Rational Mechanics and Analysis* **17**: 377–410.

Fisher, M. E. and Ruelle, D. (1966). The stability of many-particle systems, *Journal of Mathematical Physics* **7**(2): 260–270.

Flimmel, D. and Beneš, V. (2018). Gaussian approximation for functionals of Gibbs particle processes, *Kybernetika* **54**(4): 765–777.

Föllmer, H. (1982). A covariance estimate for Gibbs measures, *Journal of Functional Analysis* **46**(3): 387–395.

Gabrielli, A., Labini, F. S., Joyce, M. and Pietronero, L. (2005). *Statistical physics for cosmic structures*, Springer, Berlin.

Georgii, H.-O. (1976). Canonical and grand canonical Gibbs states for continuum systems, *Communications in Mathematical Physics* **48**(1): 31–51.

Georgii, H.-O. (2011). *Gibbs Measures and Phase Transitions (Second Edition)*, De Gruyter Studies in Mathematics 9, De Gruyter, Berlin.

Georgii, H.-O. and Håggström, O. (1996). Phase transition in continuum Potts models, *Communications in Mathematical Physics* **181**(2): 507–528.

Georgii, H.-O. and Küneth, T. (1997). Stochastic comparison of point random fields, *Journal of Applied Probability* **34**(4): 868–881.

Georgii, H.-O., Lőrinczi, J. and Lukkarinen, J. (2005). The continuum Potts model at the disorder-order transition – a study by cluster dynamics, *Journal of Statistical Mechanics: Theory and Experiment* p. P06011.

Georgii, H.-O. and Zessin, H. (1993). Large deviations and the maximum entropy principle for marked point random fields, *Probability Theory and Related Fields* **96**: 177–204.

Gibbs, J. W. (1884). On the fundamental formula of statistical mechanics, with applications to astronomy and thermodynamics, *Proceedings of the American Association for the Advancement of Science* **33**: 57–58.

Gibbs, J. W. (1902). *Elementary principles in statistical mechanics, developed with especial reference to the rational foundation of thermodynamics*, Charles Scribner's Sons, New York.

Given, J. A. and Stell, G. (1990). The Kirkwood-Salsburg equations for random continuum percolation, *Journal of Statistical Physics* **59**(3-4): 981–1018.

Gouéré, J.-B. (2008). Subcritical regimes in the Poisson Boolean model of continuum percolation, *The Annals of Probability* **36**(4): 1209–1220.

Hall, P. (1985). On continuum percolation, *The Annals of Probability* **13**(4): 1250–1266.

Hayes, C. A. and Pauc, C. Y. (1970). *Derivation and martingales*, Ergebnisse der Mathematik und ihrer Grenzgebiete Vol. 49, Springer, Berlin.

Heinrich, L. (1992). Mixing properties of Gibbsian point processes and asymptotic normality of Takacs-Fiksel estimates, *Preprint 92 - 051*.

Hill, T. L. (1956). *Statistical mechanics: Principles and selected applications*, McGraw-Hill series in advanced chemistry, McGraw-Hill, New York.

Hofer-Temmel, C. (2019). Disagreement percolation for the hard-sphere model, *Electronic Journal of Probability* **24**(91): 1–22.

Hofer-Temmel, C. and Houdebert, P. (2019). Disagreement percolation for Gibbs ball models, *Stochastic Processes and Their Applications* **129**(10): 3922–3940.

Hohenwarter, M. (2002). *GeoGebra: Ein Softwaresystem für dynamische Geometrie und Algebra der Ebene*, Master's thesis, Paris Lodron University, Salzburg, Austria.

Houdebert, P. and Zass, A. (2022). An explicit Dobrushin uniqueness region for Gibbs point processes with repulsive interactions, *Journal of Applied Probability* **59**(2): 541–555.

Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes (Second Edition)*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen Vol. 288, Springer, Berlin.

Jansen, S. (2016). Continuum percolation for Gibbsian point processes with attractive interactions, *Electronic Journal of Probability* **21**(47): 1–22.

Jansen, S. (2017). Gibbs point processes, *Online lecture notes* (accessed May 26, 2022).
URL: *www.mathematik.uni-muenchen.de/~jansen/gibbspp.pdf*

Jansen, S. (2019). Cluster expansions for Gibbs point processes, *Advances in Applied Probability* **51**(4): 1129–1178.

Jansen, S. and Kolesnikov, L. (2021). Cluster expansions: Necessary and sufficient convergence conditions, *arXiv 2112.13134*.

Janson, S., Łuczak, T. and Ruciński, A. (2000). *Random Graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., New York.

Jones, J. E. (1924a). On the determination of molecular fields. –I. From the variation of the viscosity of a gas with temperature, *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **106**(738): 441–462.

Jones, J. E. (1924b). On the determination of molecular fields. –II. From the equation of state of a gas, *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **106**(738): 463–477.

Kallenberg, O. (2002). *Foundations of Modern Probability (Second Edition)*, Probability and Its Applications, Springer-Verlag, New York.

Kallenberg, O. (2017). *Random Measures, Theory and Applications*, Probability Theory and Stochastic Modelling Vol. 77, Springer International Publishing Switzerland, Cham.

Kelly, F. P. and Ripley, B. D. (1976). A note on Strauss's model for clustering, *Biometrika* **63**(2): 357–360.

Kondratiev, Y. G. and Kuna, T. (2002). Harmonic analysis on configuration space I: General theory, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **5**(2): 201–233.

Kondratiev, Y. G., Pasurek, T. and Röckner, M. (2012). Gibbs measures of continuous systems: An analytic approach, *Reviews in Mathematical Physics* **24**(10): 1250026.

Lanford, O. E. and Ruelle, D. (1969). Observables at infinity and states with short range correlations in statistical mechanics, *Communications in Mathematical Physics* **13**(3): 194–215.

Last, G., Nestmann, F. and Schulte, M. (2021). The random connection model and functions of edge-marked Poisson processes: Second order properties and normal approximation, *The Annals of Applied Probability* **31**(1): 128–168.

Last, G. and Otto, M. (2021). Disagreement coupling of Gibbs processes with an application to Poisson approximation, *arXiv 2104.00737*.

Last, G. and Penrose, M. D. (2017). *Lectures on the Poisson Process*, Institute of Mathematical Statistics Textbooks, Cambridge University Press, Cambridge.

Lebesgue, H. L. (1904). Leçons sur l'intégration et la recherche des fonctions primitives, *Oeuvres Scientifiques* **2**: 111–154.

Lenard, A. (1973). Correlation functions and the uniqueness of the state in classical statistical mechanics, *Communications in Mathematical Physics* **30**(1): 35–44.

Likos, C. N., Watzlawek, M. and Löwen, H. (1998). Freezing and clustering transitions for penetrable spheres, *Physical Review E* **58**(3): 3135–3144.

Lindvall, T. (1992). *Lectures on the coupling method*, Wiley series in probability and mathematical statistics 294, John Wiley & Sons, Inc., New York.

Lindvall, T. (1999). On Strassen's theorem on stochastic domination, *Electronic Communications in Probability* **4**: 51–59.

Malyshev, V. A. (1975). The central limit theorem for Gibbsian random fields, *Doklady Akademii Nauk SSSR* **224**(1): 35–38. English translation in *Soviet Mathematics Doklady* **16**(5): 1141–1145.

Matthes, K., Warmuth, W. and Mecke, J. (1979). Bemerkungen zu einer Arbeit von Nguyen Xuan Xanh und Hans Zessin, *Mathematische Nachrichten* **88**(1): 117–127.

Mayer, J. E. (1947). Integral equations between distribution functions of molecules, *The Journal of Chemical Physics* **15**(4): 187–201.

Mecke, J. (1967). Stationäre zufällige Maße auf lokalkompakten Abelschen Gruppen, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **9**(1): 36–58.

Meester, R., Penrose, M. D. and Sarkar, A. (1997). The random connection model in high dimensions, *Statistics & Probability Letters* **35**(2): 145–153.

Meester, R. and Roy, R. (1996). *Continuum Percolation*, Cambridge University Press, Cambridge.

Michelen, M. and Perkins, W. (2021a). Analyticity for classical gasses via recursion, *arXiv 2008.00972*.

Michelen, M. and Perkins, W. (2021b). Potential-weighted connective constants and uniqueness of Gibbs measures, *arXiv 2109.01094*.

Michelen, M. and Perkins, W. (2022). Strong spatial mixing for repulsive point processes, *arXiv 2202.08753*.

Molchanov, I. (2017). *Theory of Random Sets (Second Edition)*, Probability Theory and Stochastic Modelling 87, Springer, London.

Møller, J. and Waagepetersen, R. P. (2004). *Statistical inference and simulation for spatial point processes*, Monographs on Statistics and Applied Probability 100, Chapman & Hall/CRC, Boca Raton.

Morse, P. M. (1929). Diatomic molecules according to the wave mechanics. II. Vibrational levels, *Physical Review* **34**: 57–64.

Moyal, J. E. (1962). The general theory of stochastic population processes, *Acta Mathematica* **108**: 1–31.

Musiał, K. (1980). Projective limits of perfect measure spaces, *Fundamenta Mathematicae* **110**(3): 163–189.

Nestmann, F. (2019). Zentrale Grenzwertsätze im Random Connection Model, *Doctoral dissertation*, Karlsruhe Institute of Technology (KIT).
URL: *http://dx.doi.org/10.5445/IR/1000093879*

Nguyen, X. X. and Zessin, H. (1979). Integral and differential characterizations of the Gibbs process, *Mathematische Nachrichten* **88**(1): 105–115.

Papangelou, F. (1974). The conditional intensity of general point processes and an application to line processes, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **28**(3): 207–226.

Parthasarathy, K. R. (1967). *Probability measures on metric spaces*, Probability and mathematical statistics Vol. 3, Academic Press, New York.

Penrose, M. D. (1991). On a continuum percolation model, *Advances in Applied Probability* **23**(3): 536–556.

Poinas, A. (2019). A bound of the β-mixing coefficient for point processes in terms of their intensity functions, *Statistics & Probability Letters* **148**: 88–93.

Poinas, A., Delyon, B. and Lavancier, F. (2019). Mixing properties and central limit theorem for associated point processes, *Bernoulli* **25**(3): 1724–1754.

Preston, C. J. (1976). *Random Fields*, Lecture Notes in Mathematics 534, Springer-Verlag, Berlin.

Preston, C. J. (2005a). Some notes on measure theory, *Online lecture notes* (accessed May 26, 2022).
URL: *www.math.uni-bielefeld.de/~preston/rest/measures/files/measures.pdf*

Preston, C. J. (2005b). Specifications and their Gibbs states, *Online lecture notes* (accessed May 26, 2022).
URL: *www.math.uni-bielefeld.de/~preston/rest/gibbs/files/specifications.pdf*

Preston, C. J. (2007). Some notes on standard Borel and related spaces, *Online lecture notes* (accessed May 26, 2022).
URL: *www.math.uni-bielefeld.de/~preston/rest/measures/files/sborel.pdf*

Quesada, J. A., Melchor, I. and Nolasco, A. (2017). Point process methods in epidemiology: Application to the analysis of human immunodeficiency virus/acquired immunodeficiency syndrome mortality in urban areas, *Geospatial Health* **12**(1): 151–156.

Rao, M. M. (1987). *Measure theory and integration*, Pure and Applied Mathematics - A Wiley Interscience publication, John Wiley & Sons, Inc., New York.

Rio, E. (2017). *Asymptotic Theory of Weakly Dependent Random Processes*, Probability Theory and Stochastic Modelling Vol. 80, Springer-Verlag, Berlin.

Rœlly, S. and Zass, A. (2020). Marked Gibbs point processes with unbounded interaction: An existence result, *Journal of Statistical Physics* **179**(4): 972–996.

Rowlinson, J. S. (1964). The statistical mechanics of systems with steep intermolecular potentials, *Molecular Physics* **8**(2): 107–115.

Ruelle, D. (1963a). Classical statistical mechanics of a system of particles, *Helvetica Physica Acta* **36**: 183–197.

Ruelle, D. (1963b). Correlation functions of classical gases, *Annals of Physics* **25**(1): 109–120.

Ruelle, D. (1969). *Statistical Mechanics - Rigorous Results*, The Mathematical Physics Monograph Series, W. A. Benjamin, Inc., New York.

Ruelle, D. (1970). Superstable interactions in classical statistical mechanics, *Communications in Mathematical Physics* **18**(2): 127–159.

Schneider, R. and Weil, W. (2008). *Stochastic and Integral Geometry*, Probability and Its Applications, Springer-Verlag, Berlin.

Schreiber, T. and Yukich, J. E. (2013). Limit theorems for geometric functionals of Gibbs point processes, *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* **49**(4): 1158–1182.

Schuhmacher, D. and Stucki, K. (2014). Gibbs point process approximation: Total variation bounds using Stein's method, *The Annals of Probability* **42**(5): 1911–1951.

Stein, E. M. and Weiss, G. (1971). *Introduction to Fourier analysis on Euclidean spaces*, Monographs in harmonic analysis; [1]. Princeton mathematical series Vol. 32, Princeton University Press, Princeton.

Stillinger, F. H. (1976). Phase transitions in the Gaussian core system, *The Journal of Chemical Physics* **65**(10): 3968–3974.

Strauss, D. J. (1975). A model for clustering, *Biometrika* **62**(2): 467–475.

Stucki, K. (2013). Continuum percolation for Gibbs point processes, *Electronic Communications in Probability* **18**(67): 1–10.

Thompson, H. R. (1955). Spatial point processes, with applications to ecology, *Biometrika* **42**(1/2): 102–115.

Torquato, S. and Jiao, Y. (2012). Effect of dimensionality on the continuum percolation of overlapping hyperspheres and hypercubes. II. Simulation results and analyses, *The Journal of Chemical Physics* **137**(7): 074106.

Torquato, S. and Jiao, Y. (2014). Erratum: "Effect of dimensionality on the continuum percolation of overlapping hyperspheres and hypercubes. II. Simulation results and analyses" [J. Chem. Phys. 137, 074106 (2012)], *The Journal of Chemical Physics* **141**(15): 159901.

van den Berg, J. (1996). Uniqueness and mixing properties of Gibbs measures, *CWI Quarterly* **9**(1&2): 19–24.

van den Berg, J. and Maes, C. (1994). Disagreement percolation in the study of Markov fields, *The Annals of Probability* **22**(2): 749–763.

van Rossum, G. and Drake, F. L. (2009). *Python 3 Reference Manual*, CreateSpace, Scotts Valley, CA.

Wiener, N. (1938). The homogeneous chaos, *American Journal of Mathematics* **60**(4): 897–936.

Wiener, N. and Wintner, A. (1943). The discrete chaos, *American Journal of Mathematics* **65**(2): 279–298.

Zass, A. (2022). Gibbs point processes on path space: existence, cluster expansion and uniqueness, *Markov Processes and Related Fields* **28**(3): 329–364.

Ziesche, S. (2018). Sharpness of the phase transition and lower bounds for the critical intensity in continuum percolation on $\mathbb{R}^d$, *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* **54**(2): 866–878.