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Quantitative Bounds Versus Existence of Weakly Coupled Bound States for Schrödinger Type Operators

Vu Hoang, Dirk Hundertmark, Johanna Richter and Semjon Vugalter

Abstract. It is well-known that for usual Schrödinger operators weakly coupled bound states exist in dimensions one and two, whereas in higher dimensions the famous Cwikel–Lieb–Rozenblum bound holds. We show for a large class of Schrödinger-type operators with general kinetic energies that these two phenomena are complementary. We explicitly get a natural semi-classical type bound on the number of bound states precisely in the situation when weakly coupled bound states exist not.

Contents

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1. Introduction: Quantitative Bounds Versus Weakly Coupled Bound States

In this paper, we study operators of the form

$$T(p) + V$$

where $p = -i\nabla$ is the quantum-mechanical momentum operator and the kinetic energy symbol $T : \mathbb{R}^d \to [0, \infty)$ is a measurable function. See Sect. 3 for the precise definition of $T(p) + V$ and the condition we need on the kinetic energy symbol T and the interaction potential V. There has been an enormous amount of interest in the study of bound states for such operators. Usually, in standard quantum mechanics the symbol is given by $T(\eta) = \eta^2$, that is, the kinetic energy is given by the the Laplacian $p^2 = -\Delta$. In this case, it is well-known that quantum mechanics in three and more dimensions is quite different from one and two dimensions. In three and more dimensions the perturbed operator $p^2 + V$ can be *unitarily equivalent* to the free operator p^2 for potentials V which are small in some sense [40, Theorem XIII.27], and for a suitable class of potentials the famous Cwikel–Lieb–Rozenblum bound (CLR) holds, which gives a bound on the number of bound states in terms of a semi-classical phase-space volume, see [11,30,41,42]. The works of Cwikel and Lieb had been motivated by Simon [48], the work of Rozenblum by the St. Petersburg school of mathematical physics around Birman and Solomyak, whose work had been virtually unnoticed in the west until the mid 1970s.[1] Rozenblum's paper [41] was an announcement of his result and, typically for the journal, did not contain any proofs. A version of his result also appeared in the summer school lecture notes by Birman and Solomyak [7]. The version with full proofs was published in [42]. Similarly, Lieb's paper [29] is announcement of his result and the details of his proof had been published later in [30,51]

In one and two dimensions, arbitrarily small attractive potentials produce a bound state, see Problem 2 on page 156 in [25], or [10,60] and also [38]. More precisely, it was rigorously shown[2] by Barry Simon many years ago, see [50], that it suffices that V is not identically zero and $V \in L^1$ with $\int V \,\mathrm{d}x \leq 0$,

[1] See the "note added in proof" in [48].
[2] For a textbook treatment of weakly coupled bound states for the Schrödinger operator in one and two dimensions, see page 679–682 in [53].

together with some mild moment condition on V, so that $-\Delta + \lambda V$ has a strictly negative bound state in one and two dimensions, for any $\lambda > 0$. This had been generalized in [37,59] to higher order Schrödinger-type operators.

Recently renewed interest in weakly coupled bound states arose due to the observation that such states can be found in many other physically interesting cases and they are responsible for different physical behavior of these systems compared to what one is used to from usual quantum mechanics in high dimensions. These systems include quantum wave guides, systems with homogenous or increasing magnetic fields, the Bardeen–Cooper–Schrieffer (BCS) model for superconductivity. These examples are not necessarily one or two-dimensional, but they are described by Schrödinger type operators with *strongly degenerate* kinetic energies [16,18,26,38], that is, the kinetic energy T can be degenerate, $T(\eta) = 0$, not only at a single point but on a "large" set in momentum space. For example, the kinetic energy could have a zero set which is an embedded hypersurface in $\mathbb{R}^d$. At this point it is important to emphasize that the results of [16] concern the special BCS Hamiltonian in three dimensions where, in particular, the zero set of the kinetic energy is a two-dimensional sphere in $\mathbb{R}^3$. The works [18,26,38] require that the kinetic energy T is locally bounded, satisfying some growth conditions at infinity, the zero set of the kinetic energy T is a smooth co-dimension one submanifold of $\mathbb{R}^d$, in [38] only locally, and that $\int V \, \mathrm{d}x < 0$ or, even stronger, $V \leq 0$ and $V \neq 0$. These last two conditions are stronger than the conditions on the potential in the original work of Simon. In particular, they leave open the question what happens if $V \neq 0$ and $\int V \, \mathrm{d}x = 0$ or if the co-dimension of the zero set of the kinetic energy T is larger than one.

Motivated by the above questions, we consider a *very general class* of kinetic energies and potentials. More importantly, we have a *sharp existence result* for weakly coupled bound states: We give conditions under which weakly coupled bound states exist for any non-trivial attractive potential, $\int V \, \mathrm{d}x \leq 0$, and if our conditions are not met, then for any strictly negative but sufficiently small potential weakly coupled bound states exist. Moreover, in the second case, we prove a quantitative bound on the number of negative bound states.

We are able to do this by *identifying the mechanism* which is responsible for the creation of weakly coupled bound states: Roughly speaking $\eta \mapsto T(\eta)^{-1}$ being integrable or not in a small neighborhood of the zero set of T distinguishes between having a quantitative bound on the number of negative bound states in the first case or having weakly coupled bound states for arbitrarily small attractive potentials in the second, see Theorems 1.1 and 1.3, for the precise conditions.[3]

In the following we will always assume, without further mentioning it, that the potential V is relatively form small with respect to the kinetic energy $T(p)$. That is, there exists $0 < a < 1$ and $b > 0$ such that

[3]And Theorem 1.5, which shows that under some slight additional assumptions these are complementary.

$$|\langle \varphi, V\varphi\rangle| \leq \|\sqrt{T(p)}\varphi\|^2 + b\|\varphi\|^2$$

for all $\varphi \in \mathcal{D}(\sqrt{T(p)}) = \mathcal{Q}(T(p))$, the form domain of $T(p)$. Here we identify, for simplicity, $\langle \varphi, V\varphi\rangle = \langle \sqrt{|V|}\varphi, \mathrm{sgn}(V)\sqrt{|V|}\varphi\rangle$ for the quadratic form of the potential V and also assume that its quadratic form domain $\mathcal{Q}(V)$ contains $\mathcal{Q}(T(p))$. In this case, the famous KLMN theorem guarantees that one can define the generalized Schrödinger operator as a sum of the quadratic forms corresponding to $T(p)$ and V, see [40,56]. By a slight abuse of notation, we denote this operator by $T(p) + V$.

Our first theorem concerns the existence of bound states. We write $V \neq 0$ if V is not the zero function, more precisely, if $V \neq 0$ on a set of positive Lebesgue measure.

Theorem 1.1. (Weakly coupled bound states) *Let $T : \mathbb{R}^d \to [0, \infty)$ be measurable. Assume that there exists a compact set $M \subset \mathbb{R}^d$ such that*

$$\int_{M_\delta} T(\eta)^{-1}\,\mathrm{d}\eta = \infty \textit{ for all } \delta > 0, \tag{1.1}$$

where $M_\delta := \{\eta \in \mathbb{R}^d : \mathrm{dist}(\eta, M) \leq \delta\}$. Then $\inf \sigma(T(p)) = 0$ and if the potential $V \neq 0$, obeys the basic assumptions from Sect. 2 and $V \leq 0$, then $T(p)+V$, defined in the quadratic form sense, has at least one strictly negative bound state.

Moreover, if T is locally bounded this also holds for sign indefinite *potentials in the sense that if $V \in L^1(\mathbb{R}^d)$, $V \neq 0$, obeys the basic assumptions from Sect. 2 and $\int V\,\mathrm{d}x \leq 0$, then the operator $T(p) + V$ has again at least one strictly negative eigenvalue.*

Remarks 1.2. (i) Our theorem poses rather weak conditions on the zero set of the kinetic energy symbol T. Moreover, T does not have to satisfy any growth conditions at infinity. Of course, if T does not satisfy a growth condition at infinity, then, even if $V \in L^1$, it does not have to be relatively form bounded with respect to $T(p)$. Assuming this and $\sigma_{\mathrm{ess}}(T(p) + V) = \sigma(T(p))$, one can formulate a version of Theorem 1.1 which still guarantees the existence of some negative spectrum, but not necessarily discrete, we leave the details to the interested reader.

(ii) If the potential is negative, $V \leq 0$, and the zero set of the kinetic energy is somewhat 'thick', there can be even be infinitely many bound states below the essential spectrum, see Theorem 3.4 and Corollaries 3.6 and 3.7 below. Moreover, if the kinetic energy symbol T is continuous, then the compact set M above can be chosen to be a subset of the zero set of T. In this case, the behaviour of T near its zero set determines whether (1.1) holds or not.

(iii) We would like to emphasize that our result also holds when $\int V\,\mathrm{d}x = 0$. This makes our criterion applicable in several cases, when previous results [16,18,26,38] fall short. One example is when the Fourier transform $\widehat{V}$ of the potential is zero on a large ball centered at the origin.

(iv) Our method in the proof of Theorem 1.1 relies on a very simple and natural variational calculation, which also works in the *critical case* where $\int V\,\mathrm{d}x = 0$. It does not require a detailed analysis of the generalized Schrödinger operator $T(p) + V$. Its main advantage is its simplicity and its wide range of applications.

The situation discussed in Theorem 1.1 changes drastically when $\eta \mapsto T(\eta)^{-1}$ is integrable near the zero set of T, more precisely, when $T^{-1}\mathbf{1}_{\{T<\delta\}} \in L^1(\mathbb{R}^d)$ for some $\delta > 0$. Not only do weakly coupled bound states cease to exist but we have even a quantitative bound on the number of negative eigenvalues in this case. More precisely, introduce the function $G : [0,\infty] \to [0,\infty]$ by

$$G(u) := u \int_{T(\eta)<u} T(\eta)^{-1} \frac{\mathrm{d}\eta}{(2\pi)^d} \tag{1.2}$$

for $u \geq 0$. It is clear from the definition that $G(u) < \infty$ if and only if $\int_{T<u} T(\eta)^{-1}\,\mathrm{d}\eta < \infty$ and if $G(u_0) < \infty$ for some $u_0 > 0$, then $G(u) < \infty$ for all $0 \leq u \leq u_0$ and in this case $\lim_{u\to 0} G(u) = 0$. The function G is the central object in our quantitative bound on the number of bound states for the Schrödinger-type operator $T(p) + V$.

Theorem 1.3. (Quantitative bound) *Let $T : \mathbb{R}^d \to [0,\infty)$ be measurable and assume that the potential V obeys the basic assumptions from Sect. 2. Then for $T(p)+V$, defined in the quadratic form sense, we have the bound*

$$N(T(p)+V) \leq \frac{\alpha^2}{(1-2\alpha)^2} \int G(V_-(x)/\alpha^2)\,\mathrm{d}x, \tag{1.3}$$

for all $0 < \alpha < \frac{1}{2}$. Here $V_- = \max(0,-V)$ is the negative part of V and $N(T(p)+V)$ is the number of eigenvalues of $T(p)+V$ which are strictly negative.

Moreover, if for a given potential V there exists a perturbation $W > 0$ with $G(V_- + W) < \infty$, then the above bound also includes zero-energy eigenvalues, that is,

$$\begin{aligned} N_0(T(p)+V) &:= \#\{\textit{eigenvalues of } T(P)+V \leq 0\} \\ &\leq \frac{\alpha^2}{(1-2\alpha)^2} \int G(V_-(x)/\alpha^2)\,\mathrm{d}x, \end{aligned} \tag{1.4}$$

Remarks 1.4. (i) As we will see in Section A, in many practical cases, even when the kinetic energy symbol T is not homogeneous, the function G from Theorem 1.3 can be straightforwardly evaluated and in most cases the result of this evaluation agrees with the precise semi-classical guess up to a small factor. In particular, when $T(\eta) = \eta^2$, we recover Cwikel's version of the CLR bound.

(ii) A straightforward argument shows that if for all $u > 0$ the sublevel sets $\{T < u\}$ have finite Lebesgue measure, then

$$\int_{T<u} T(\eta)^{-1}\,\mathrm{d}\eta < \infty \text{ for some } u > 0$$
$$\iff \int_{T<u} T(\eta)^{-1}\,\mathrm{d}\eta < \infty \text{ for all } u > 0.$$

In this case, $G(u) < \infty$ for all $u \geq 0$ is equivalent to $G(u_0) < \infty$ for some $u_0 > 0$. Moreover, in the case that the sublevel sets $\{T < u\}$ have finite Lebesgue measure, Lemma B.2 in the appendix yields a non-trivial relative form compactness criterium of a potential V which does not require that T diverges to infinity at infinity.

Of course, in all the applications we know of one usually has $\lim_{\eta\to\infty} T(\eta) = \infty$ or, in the case of discrete Schrödinger operators, the range of possible momenta η is a bounded subset of $\mathbb{R}^d$. Thus in these applications one always has $G(u) < \infty$ for all $u > 0$ once $G(u_0) < \infty$ for some $u_0 > 0$.

(iii) The function G above has a nice semi-classical interpretation. We note

$$\begin{aligned}
G(u) &= \int\limits_{T/u<1} (T(\eta)/u)^{-1}\,\frac{\mathrm{d}\eta}{(2\pi)^d} = \int\limits_{T/u<1} \frac{\mathrm{d}\eta}{(2\pi)^d} \int_0^\infty s^{-2}\mathbf{1}_{\{T/u<s\}}\,\mathrm{d}s \\
&= \int_0^\infty \int_{\mathbb{R}^d} \mathbf{1}_{\{T(\eta)-\min(1,s)u<0\}}\,\frac{\mathrm{d}\eta}{(2\pi)^d}\,\frac{\mathrm{d}s}{s^2} \\
&= \int_{\mathbb{R}^d} \mathbf{1}_{\{T(\eta)-u<0\}}\,\frac{\mathrm{d}\eta}{(2\pi)^d} + \int_0^1 \int_{\mathbb{R}^d} \mathbf{1}_{\{T(\eta)-\min(1,s)u<0\}}\,\frac{\mathrm{d}\eta}{(2\pi)^d}\,\frac{\mathrm{d}s}{s^2}.
\end{aligned}$$

Thus with the classical phase-space volume, given by

$$\begin{aligned}
N^{\mathrm{cl}}(T+V) &:= \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{\{T(\eta)+V(x)<0\}}\,\frac{\mathrm{d}\eta\mathrm{d}x}{(2\pi)^d} \\
&= \int_{\mathbb{R}^d} \left|\{\eta\in\mathbb{R}^d : T(\eta)+V(x)<0\}\right|\,\frac{\mathrm{d}x}{(2\pi)^d},
\end{aligned}$$

a reformulation of the bound in (1.3) is

$$\begin{aligned}
N(T(p)+V) &\leq \frac{\alpha^2}{(1-2\alpha)^2} N^{\mathrm{cl}}(T+V/\alpha^2) \\
&\quad + \frac{\alpha^2}{(1-2\alpha)^2} \int_0^1 N^{\mathrm{cl}}\left(T+sV(x)/\alpha^2\right)\,\frac{\mathrm{d}s}{s^2}
\end{aligned} \tag{1.5}$$

for all $0 < \alpha < 1/2$. The first part on the right hand side above is clearly related to the classical phase space volume guess suggested by the uncertainty principle and the second part can be considered as a *quantum correction.* We would like to emphasize that for the usual Schrödinger operator one has

$$N^{\mathrm{cl}}(|\eta|^2+V/\alpha^2) = \frac{|B_1^d|}{(2\pi)^d} \int_{\mathbb{R}^d} V_-(x)^{d/2}\,\mathrm{d}x \tag{1.6}$$

which for a large class of potentials, e.g., $V \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ is *finite* in all dimensions. On the other hand, it is well known that in dimensions one and two the Cwikel–Lieb–Rozenblum bound does not hold. Thus even in this well understood case, finiteness of the classical phase space volume *does not imply* the existence of a quantitative bound of CLR-type. Our quantum correction, the second term in (1.5) takes care of this. Theorem 1.3 shows that a simple general quantitative bound on the number of bound states holds, *exactly when* the contribution from the quantum correction, i.e., the second term in (1.5), is finite.

To show that under some slightly stronger conditions on T, Theorems 1.1 and 1.3 are complementary, we provide

Theorem 1.5. *Let $Z := \{T = 0\}$ be the zero set of the kinetic energy symbol $T : \mathbb{R}^d \to [0, \infty)$. Assume that Z is compact and that T is small only close to its zero set, more precisely, we assume that for all $\delta > 0$ there exists $u > 0$ with*

$$\int_{\{T<u\}\cap Z_\delta^c} \frac{1}{T(\eta)}\,\mathrm{d}\eta < \infty, \tag{1.7}$$

where Z_δ^c is the complement of $Z_\delta := \{\eta \in \mathbb{R}^d : \mathit{dist}(\eta, Z) \leq \delta\}$. Then

(a)

$$\int_{Z_\delta} \frac{1}{T(\eta)}\,\mathrm{d}\eta = \infty \quad \textit{for some } \delta > 0 \tag{1.8}$$

is equivalent to $T(p)+V$ having weakly coupled bound states for any non-trivial attractive potential V. That is, for any $V \leq 0$, $V \neq 0$, which obeys the basic assumptions from Sect. 2 the operator $T(p)+V$ has at least one strictly negative eigenvalue.
Moreover, if in addition the kinetic energy symbol T is locally bounded, then (1.8) is also equivalent to $T(p)+V$ having a strictly negative eigenvalue for non-trivial sign changing potentials V in the sense that if $V \in L^1(\mathbb{R}^d)$, $V \neq 0$, obeys the basic assumptions from Sect. 2 and $\int V\,\mathrm{d}x \leq 0$, then the operator $T(p)+V$ has again at least one strictly negative eigenvalue.

(b)

$$\int_{Z_\delta} \frac{1}{T(\eta)}\,\mathrm{d}\eta < \infty \quad \textit{for some } \delta > 0 \tag{1.9}$$

is equivalent to the existence of a quantitative bound on the number of bound states in the following sense: There exists a function $G_0 : [0, \infty] \to [u, \infty]$ with $G_0(u) < \infty$ for all small enough $u > 0$ and $\lim_{u\to 0+} G_0(u) = 0$ such that for any potential V which is relatively form compact with respect to $T(p)$ one has the bound

$$N(T(p)+V) \leq \int G_0(V_-(x))\,\mathrm{d}x,$$

where $V_- = \max(0, -V)$ *is the negative part of* V *and* $N(T(p)+V)$ *is the number eigenvalues of* $T(p)+V$ *which are strictly negative. Moreover, in this case one can take* $G_0(u) = \frac{\alpha^2}{(1-2\alpha)^2} G(\alpha^{-2}u)$ *with* G *defined in* (1.2) *and* $0 < \alpha < 1/2$.

Remarks 1.6. (i) To see that (1.7) is, indeed, a rather weak growth and regularity condition on T we note that (1.7) is fulfilled under the following two conditions on T:

$$1) \quad \text{There exists } 0 < \varepsilon, R < \infty \text{ with } T(\eta) \geq \varepsilon \text{ for all } |\eta| > R, \tag{1.10}$$

which is, for example, the case if $T(\eta) \to \infty$ for

$$2) \quad T \text{ is lower semi-continuous.} \tag{1.11}$$

Indeed, under the above two conditions one has

$$\text{for any } \delta > 0 \text{ there exists } u > 0 \text{ with } \{T < u\} \cap Z_\delta^c = \emptyset, \tag{1.12}$$

which clearly implies (1.7). To see (1.12), define $r(u) := \sup\{\operatorname{dist}(\eta, Z) : T(\eta) < u\}$. Then $\{T < u\} \subset Z_\delta$ is equivalent to $r(u) \leq \delta$, so it is enough to show $\lim_{u\to 0+} r(u) = 0$. Clearly, $0 < u \mapsto r(u)$ is increasing. Assume that there exists $\varepsilon_0 > 0$ with $r(u) \geq 2\varepsilon_0$ for all $u > 0$. Taking $u = 1/n$, this yields the existence of a sequence $\xi_n \in \mathbb{R}^d$ with $\operatorname{dist}(\xi_n, Z) > \varepsilon_0$ and $T(\xi_n) < 1/n$. Because of (1.10) we have $\xi_n \leq R$ for all large enough n. Thus we can take a subsequence ξ_{n_l} such that $\eta = \lim_{l\to\infty} \xi_{n_l}$ exists. Because of the lower semicontinuity of T one has $0 \leq T(\eta) \leq \liminf_{l\to\infty} T(\xi_{n_l}) = 0$, so $\eta \in Z$, which contradicts $\operatorname{dist}(\eta, Z) = \lim_{l\to\infty} \operatorname{dist}(\xi_{n_l}, Z) \geq \varepsilon_0 > 0$. So $r(u) \to 0$ as $u \to 0$, which proves (1.12).

(ii) The existence of a quantitative bound on the number of strictly negative eigenvalues of $T(p)+V$ clearly implies that weakly coupled bound states do not exist, see Remark 1.4.i. On the other hand, we do not know of any result in the literature which shows, without assuming strong additional conditions on the kinetic energy $T(P)$, that the absence of weakly coupled bound states implies the existence of a quantitative semi-classical type bound on the number of strictly negative bound states of Schrödinger-type operators $T(P) + V$. These strong additional assumptions refer to Markov properties [27], positivity-preservation of the associated semi-group [12,30,43] or the inverse of the kinetic energy being in a weak L^p-space, thus imposing a kind of homogeneity condition, [14,15], see also [46,47]. The main point of our Theorem 1.5 is that under *very weak regularity assumptions* on T *near its zero set*, which are fulfilled in all physically relevant cases, the two phenomena of weakly coupled bound states and the existence of quantitative semi-classical type bound on the number of strictly negative bound states are indeed complementary.

(iii) Condition (1.7) ensures that we have the equivalences

$$\int_{Z_\delta} \frac{1}{T(\eta)}\,\mathrm{d}\eta = \infty \text{ for all } \delta > 0$$
$$\iff \int_{Z_\delta} \frac{1}{T(\eta)}\,\mathrm{d}\eta = \infty \text{ for some } \delta > 0$$
$$\iff \int_{T<u} \frac{1}{T(\eta)}\,\mathrm{d}\eta = \infty \text{ for all } u > 0. \tag{1.13}$$

This clearly ensures that conditions (1.8) and (1.9) are complementary. The equivalence (1.13) follows from Lemma 6.4 in Sect. 6.2.

To put our work into perspective: Previously, the weakest condition on the potential which guaranteed existence of at least one strictly negative eigenvalue is due to Pankrashkin [38] who showed that a strictly negative eigenvalue exists if $V \in L^1$ and $\int V\,\mathrm{d}x < 0$. This condition is weaker than the condition of [16,18] where the authors have to assume that the Fourier transform $\widehat{V}$ is non-vanishing within a large enough ball centered at the origin.[4] In [16,18,26], they adapt the method of Simon, using the Birman–Schwinger principle [5,6,55], to identify a singular piece of the Birman–Schwinger operator. This approach needs *global assumptions* on the zero set of the kinetic energy, see also Remark 1.2.iii . The work [38] uses a construction of appropriate trial functions, as such the assumptions on the zero set of the kinetic energy in [38] are local. More precisely, there is an open set such that locally within this set the zero set of T is a smooth submanifold of $\mathbb{R}^d$. The work [26] establishes the precise asymptotic rate of the negative eigenvalues, but for this they need strong assumptions.

It is easy to construct examples of potentials $V \in L^1$ such that its Fourier transform $\widehat{V}$ is zero on a large centered ball. Simply take any spherically symmetric Schwartz function $\widehat{V}$ which is supported on a large enough centered annulus in Fourier space and let V be the inverse Fourier transform of $\widehat{V}$. In this case $\int V\,\mathrm{d}x = \widehat{V}(0) = 0$ and our Theorem 1.1 shows that there exists at least one strictly negative eigenvalue under suitable conditions on the kinetic energy T, whereas the previous results in [18,26,38] are not applicable. The only exception is the pioneering work of Simon, which for the Laplace operator $T(p) = p^2$ in one and two dimensions, shows that there are strictly negative eigenvalues if $\int V\,\mathrm{d}x \le 0$, V does not vanish identically, and, for some technical reasons, some high enough moments of V are finite.

Some additional remarks concerning Theorem 1.3 are

Remarks 1.7. (i) As already mentioned, from the definition of G together with a simple monotonicity argument it is clear that if for some u_0 one has $G(u_0) < \infty$ then $G(u) < \infty$ for all $0 \le u \le u_0$ and $\lim_{u\to 0+} G(u) = 0$. Thus, if $G(u_0) < \infty$, or equivalently, $T\mathbf{1}_{T<u_0} \in L^1(\mathbb{R}^d)$, for some $u_0 > 0$, a simple construction yields a potential $V < 0$ such that[5]

[4] More precisely, in [18] they require that $\widehat{V}$ is non-vanishing on the set $S - S = \{\eta_1 - \eta_2 : \eta_j \in S\}$, where S is the zero set of T.

[5] for your favorite choice of $0 < \alpha < 1/2$, e.g., $\alpha = 1/42$.

$\int_{\mathbb{R}^d} G(V_-(x)/\alpha^2)\,\mathrm{d}x < \infty$. But then, replacing V by λV for $0 < \lambda \le 1$, the monotone convergence theorem gives $\lim_{\lambda\to 0+} \int_{\mathbb{R}^d} G(\lambda V_-(x)/\alpha^2)\,\mathrm{d}x = 0$ and the bound provided by (1.3) shows

$$N(T(p) + \lambda V) \le \frac{\alpha^2}{(1-2\alpha)^2} \int_{\mathbb{R}^d} G(\lambda V_-(x)/\alpha^2)\,\mathrm{d}x < 1$$

for all small enough $\lambda > 0$. So in this case there exists a strictly negative potential for which $T(p) + V$ has no strictly negative eigenvalues. So Theorems 1.1 and 1.3 appear to be *complementary.* More precisely, as Theorem 1.5 below shows this is, indeed, the case under some slight additional global assumptions on the kinetic energy T, which seem to be fulfilled in all physically relevant applications.

(ii) Theorem 1.1 shows that strictly negative eigenvalues of $T(P) + V$ exist once the integral of $T(\eta)^{-1}$ diverges in a neighborhood of a compact set. On the other hand, Theorem 1.3 yields a quantitative bound on the number of negative eigenvalues under the condition that $T^{-1}\mathbf{1}_{\{T<\delta\}}$ is integrable for some $\delta > 0$. Naturally, one can ask the question what could happen if T^{-1} is integrable over every compact set, but diverges globally. As an example of such a situation, we consider the operator

$$H_\lambda = (p_1^2 + p_2^2)^{1/2} - \lambda U(x_1, x_2, x_3) \quad \text{on } L^2(\mathbb{R}^3),$$

for some $0 \le U \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, $\lambda \ge 0$. For sufficiently small λ this operator does not have negative spectrum (no weakly coupled bound states). On the other hand, after some critical λ_{cr} the infimum of the essential spectrum immediately goes down and discrete eigenvalues still do not exist. More precisely, by construction we have

$$\begin{aligned} \sigma(H_\lambda) \cap (-\infty, 0) &= \sigma_{\mathrm{ess}}(H) \cap (-\infty, 0) \\ &= \bigcup_{x_3 \in \mathbb{R}} \big(\sigma((p_1^2 + p_2^2)^{1/2} - \lambda U(\cdot, x_3)) \cap (-\infty, 0)\big). \end{aligned} \tag{1.14}$$

For fixed $x_3 \in \mathbb{R}$ consider the operator $(p_1^2 + p_2^2)^{1/2} - \lambda U(\cdot, x_3)$ as an operator on $L^2(\mathbb{R}^2)$. Its quadratic form is monotonically decreasing in $\lambda > 0$ and for fixed $x_3 \in \mathbb{R}$ we can apply Theorem 1.3 to show that for small enough λ the operator $(p_1^2 + p_2^2)^{1/2} - \lambda U(\cdot, x_3)$ is positive. This example shows that for the existence of weakly coupled bound states one needs that the kinetic energy goes to zero fast enough near its zero set.

We would like to stress that in all or our results, the assumptions on the kinetic energy symbol T are very weak and fulfilled in all physically interesting cases. Our theorems have several applications, discussed in Section A, including Schrödinger operators with fractional Laplacians, different types of Schrödinger type operators with degenerate kinetic energies such as pseudo-relativistic Schrödinger operators with positive mass and two-particle pseudo-relativistic Schrödinger operators with different masses, including *very* different masses, BCS-type operators, and discrete Schrödinger operators.

Our paper is organized as follows: We first address the question of existence of negative bound states. The main idea in the proof of Theorem 1.1 is first shown in a simple model case in Sect. 4. In Sect. 5 we give the proof of Theorem 1.1 and its refinement Theorem 3.4 and their corollaries. In Sect. 6 we give the proof of Theorem 1.3 and in Sect. 6.2 the proof of Theorem 1.5. The applications of Theorems 1.1 and 1.3 are discussed in Appendix A.

2. Basic Assumptions

We consider operators of the form

$$H = T(p) + V \tag{2.1}$$

where $p = -i\nabla$ is the quantum-mechanical momentum operator and the symbol of the kinetic energy T is a measurable nonnegative function on $\mathbb{R}^d$. We define $T(p)$ as the Fourier multiplier,

$$T(p)\varphi := \mathcal{F}^{-1}[T(\cdot)\widehat{\varphi}(\cdot)] \tag{2.2}$$

where we use the convention

$$\widehat{f}(\eta) := \mathcal{F}(f)(\eta) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\eta\cdot x} f(x)\,\mathrm{d}x$$

and

$$\check{g}(x) := \mathcal{F}^{-1}(g)(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\eta\cdot x} g(\eta)\,\mathrm{d}\eta$$

for the Fourier transform and its inverse. A-priori the above expressions are only defined when f, g are Schwartz class, but they extend to unitary operators to all of $L^2(\mathbb{R}^d)$ by density of Schwartz functions in $L^2(\mathbb{R})$, see [31,52].

For a positive self-adjoint operator A we denote by $\mathcal{Q}(A)$ its form domain and by $\mathcal{D}(A)$ its domain. Thus $\mathcal{Q}(A) = \mathcal{D}(\sqrt{A})$. In particular,

$$\mathcal{Q}(T(p)) = \left\{ f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} T(\eta)|\widehat{f}(\eta)|^2\,\mathrm{d}\eta < \infty \right\} \tag{2.3}$$

The potential $V : \mathbb{R}^d \to \mathbb{R}$ is a Borel-measurable function. In order to define the Schrödinger type operator $T(P) + V$ as a sum of quadratic forms, it is enough to assume, for simplicity, that its modulus $|V|$ is form small with respect to $T(p)$, that is, for some $0 < a < 1$ and $b > 0$ we have

$$\langle \varphi, |V|\varphi\rangle \leq \|\sqrt{T(p)}\varphi\|^2 + b\|\varphi\|^2 \tag{2.4}$$

for all $\varphi \in \mathcal{D}(\sqrt{T(p)}) = \mathcal{Q}(T(p))$, the form domain of $T(p)$. In this case the quadratic form domain $\mathcal{Q}(V)$ of V is given by the domain of the multiplication operator $|V|^{1/2}$, $\mathcal{Q}(V) = \mathcal{D}(|V|^{1/2})$, and we identify, for simplicity, $\langle \varphi, V\varphi\rangle = \langle \sqrt{|V|}\varphi, \mathrm{sgn}(V)\sqrt{|V|}\varphi\rangle$ for the quadratic form of the potential V. We also assume that $\mathcal{Q}(T(p)) \subset \mathcal{Q}(V)$.

More generally, let $V_+ = \max(V, 0)$, respectively $V_- = \max(-V, 0)$, be the positive, respectively negative, part of the potential V and assume that $\mathcal{Q}(T(p) \cap \mathcal{Q}(V_+)$ is dense in $L^2(\mathbb{R}^d)$ and that V_- is relatively form small with respect to $T(p) + V_+$. That is, $\mathcal{Q}(T(p) \cap \mathcal{Q}(V_+) \subset \mathcal{Q}(V_-)$ and there exists $0 \le \alpha < 1$ and $0 \le \beta < \infty$ such that

$$\|\sqrt{V_-} f\|^2 \le \alpha\Big(\|\sqrt{T(p)} f\|^2 + \|\sqrt{V_+} f\|^2\Big) + \beta\|f\|^2 \tag{2.5}$$

Under either conditions (2.4) or (2.5), the famous KLMN theorem, see, e.g., [56, Theorem 6.24] or [53, Theorem 7.5.7], shows that the natural quadratic form corresponding to $T(p) + V$ is closed on $\mathcal{Q}(T(p)) \cap \mathcal{Q}(V_+)$ and defines a lower bounded self-adjoint operator, which we will denote by $T(p) + V$, for simplicity.

Since we have the form smallness condition on $|V|$, or we split V into its positive and negative parts, we do not discuss highly singular oscillating potentials in this work.

Since $T(p)$ is a Fourier multiplier, i.e., multiplication by a function on the Fourier side, it has purely essential spectrum, $\sigma(T(p)) = \sigma_{\mathrm{ess}}(T(p)) = \mathrm{essrange}(T) \subset [0, \infty)$. Here the essential range of T is given by

$$\mathrm{essrange}(T) = \{E \in \mathbb{R} : \, |T^{-1}((E - \delta, E + \delta))| > 0 \text{ for all } \delta > 0\}.$$

We also need a condition on the essential spectrum of $T(p)+V$. We will always assume that

$$\sigma_{\mathrm{ess}}(T(p) + V) \subset \sigma_{\mathrm{ess}}(T(p)). \tag{2.6}$$

For sufficient conditions, which imply (2.6) and (2.5), respectively (2.4), see Remark 2.1.

For the quantitative bound on the number of bund states we need a slightly stronger assumption, where in (2.5) we replace V_+ by zero and in (2.6) we replace V by $-V_-$.

Remarks 2.1. (i) The conditions (2.5) and (2.6) are, in particular, fulfilled when $|V|$ is relatively form compact with respect to $T(p)$, i.,e, $(T(p) + 1)^{-1/2}|V|(T(p) + 1)^{-1/2}$ is a compact operator on $L^2(\mathbb{R}^d)$. This is well-known, see [56, Section 6.3] or [53, Section 7.5] for example. In this case, V is automatically relatively form small with respect to $T(p)$ with relative bound zero, [56, Lemma 6.26]. Moreover, by the relative form compactness $\sigma_{\mathrm{ess}}(T(p) + V) = \sigma_{\mathrm{ess}}(T(p)) = \sigma(T(p))$, [56, Lemma 6.26] or [53, Theorem 7.8.4],

(ii) *Necessary and sufficient* conditions are given in [35] for a potential V to be relatively form bounded, respectively relative form compact, with respect to the usual kinetic energy $T(p) = p^2$. These conditions cover, in particular, examples where the potential is highly oscillatory. Since a characterization of relatively form bound and form compact potentials V in the spirit of [35] is not known for the general class of kinetic energies we are interested in, we provide in Appendix B two sufficient conditions for the invariance of the essential spectrum of $T(p)$ under perturbation by a potential V which is relative form bounded with respect to $T(p)$. These condition are far from optimal but are easy to apply in a wide variety of cases.

For example, assume that V is relative form small with respect to $T(p)$. Then one has $\sigma_{\mathrm{ess}}(T(p)+V) = \sigma_{\mathrm{ess}}(T(p))$ as soon as $|V|^{1/2}(T(p)+1)^{-1}$ is a compact operator or if $V \in L^1(\mathbb{R}^d)$, assuming in addition that $\lim_{\eta\to\infty} T(\eta) = \infty$, see Lemma B.1 in the appendix. A different criterium, which still needs that T diverges at infinity, is discussed in Lemma B.2.

3. Existence of Bound States: The General Setup

Physical heuristic suggests that a weak attractive potential can create a bound state if T is small close to its zero set. To make this precise we introduce a local version of this.

Definition 3.1. Let $T : \mathbb{R}^d \to [0,\infty)$ be measurable. T has a thick zero set near $\omega \in Z$ if

$$\int_{B_\delta(\omega)} T(\eta)^{-1}\,\mathrm{d}\eta = \infty \text{ for all } \delta > 0, \tag{3.1}$$

where $B_\delta(\omega) = \{\eta \in \mathbb{R}^d : \mathrm{dist}(\eta,\omega) < \delta\}$ is the open ball of radius δ centered at ω.

In the following, we will assume, without mentioning all the time, that the assumptions of Theorem 1.1 hold and $T(p)+V$ is defined in the quadratic form sense. A local version of Theorem 1.1 is given by

Theorem 3.2. *Suppose that $V \le W \ne 0$ obey the basic assumptions from Sect. 2, $W \in L^1$, $\int W\,\mathrm{d}x \le 0$, and T has a thick zero set near some point $\omega \in Z$. Then $T(p)+V$ has at least one strictly negative eigenvalue.*

Remark 3.3. The sole role of the comparison potential W is to be able to easily include potentials V obey the basic assumptions from Sect. 2, but are not integrable. For example, if the positive part $V_+ \in L^1$ and $V_- \notin L^1$, and V obeys the basic assumptions from Sect. 2, we can choose

$$W_R^m = V_+ - \min(V_-, m)\mathbf{1}_{B_R}$$

where B_R is a centered ball of radius $R > 0$. Then $V \le W_R^m \in L^1$, W_R^m obeys the basic assumptions from Sect. 2, and $\int W_R^m\,\mathrm{d}x < 0$ for m, R large enough. Thus Theorem 3.2 yields the existence of a negative eigenvalue of $T(p)+V$ in this case.

In particular, the existence of a strictly negative eigenvalue of $T(p)+V$ for potentials $V \ne 0$, which obey the basic assumptions from Sect. 2 and which are *sign definite*, that is, $V \le 0$, follows at once from Theorem 3.2. We do not, however, consider sign changing potentials which are not in $L^1(\mathbb{R}^d)$.

A refinement of Theorem 3.2, when the zero set of the kinetic energy T has many disjoint thick parts is given by the next theorem, which also yields an easy criterion for the infinitude of weakly coupled bound states.

Theorem 3.4. *Assume that $V, W \ne 0$ obey the basic assumptions from Sect. 2 and that for some $k \in \mathbb{N}$ the kinetic energy T has a thick zero set near k pairwise distinct points $\omega_1, \ldots, \omega_k \in Z$.*

(a) *If $V \le 0$, then $T(p)+V$ has at least k strictly negative eigenvalues.*
(b) *If $V \le W$, $W \in L^1$, and the $k \times k$ matrix $M = (\widehat{W}(\omega_l - \omega_m))_{l,m=1,\ldots,k}$, where $\widehat{W}$ is the Fourier transform of W, is strictly negative definite, then $T(p)+V$ has at least k strictly negative eigenvalues.*
(c) *If $V \le W$, $W \in L^1$, the $k \times k$ matrix $M = (\widehat{W}(\omega_l - \omega_m))_{l,m=1,\ldots,k}$, is negative semi-bounded, the eigenvalue 0 of this matrix is non-degenerate, and the function T is locally bounded, then $T(p)+V$ has at least k strictly negative eigenvalues.*

Remarks 3.5. (i) In the spirit of Theorem 3.4, one can formulate a condition under which the operator $T(p)+V$ has at least k eigenvalues when $V \le W$, for a semi-bounded matrix $M = (\widehat{W}(\omega_l - \omega_m))_{l,m=1,\ldots,k}$ with a *degenerate* eigenvalue zero. We are not doing this for the sake of simplicity, but leave it to the interested reader.

(ii) Similar to part (a) of Theorem 3.4, in [9,38] the authors also had the condition that the matrix $M = (\widehat{V}(\omega_l - \omega_m))_{l,m=1,\ldots,k}$ is negative definite, but the conditions on the zero set of the kinetic energy are much stronger than ours. Moreover, we can also handle the case of a non-degenerate zero eigenvalue of M.

In the following we use for two real-valued functions f, g the notation $f \lesssim g$ if there exists a constant $C > 0$ such that $f \le Cg$. We also write $f \sim g$ if $f \lesssim g$ and $g \lesssim f$.

Useful corollaries of Theorems 3.2 and 3.4 are

Corollary 3.6. *Assume that $V \le W \ne 0$ obey the basic assumptions from Sect. 2, that there are k isolated points $\omega_1, \ldots, \omega_k$ and that near $F = \{\omega_1, \ldots, \omega_k\}$, i.e., in an open neighborhood $\mathcal{O}$ containing F, the kinetic energy symbol obeys the bound*

$$T(\eta) \lesssim \operatorname{dist}(\eta, F)^\gamma \quad \textit{for all } \eta \in \mathcal{O} \textit{ and some } \gamma \ge d. \tag{3.2}$$

(a) *If $V \le 0$ or if $W \in L^1(\mathbb{R}^d)$ and the matrix $M = (\widehat{W}(\omega_l - \omega_m))_{l,m=1,\ldots,k}$ is negative definite, then $T(p)+V$ has at least k strictly negative eigenvalues.*
(b) *If $W \in L^1(\mathbb{R}^d)$ and $\int W \,\mathrm{d}x \le 0$, then $T(p)+V$ has at least one strictly negative eigenvalue.*

Corollary 3.7. *Assume that $V \le W \ne 0$ obey the basic assumptions from Sect. 2and that there is a $\mathcal{C}^2$ submanifold Σ of codimension $1 \le m \le d-1$ such that near Σ, i.e., in an open neighborhood $\mathcal{O}$ containing Σ, the kinetic energy symbol obeys the bound*

$$T(\eta) \lesssim \operatorname{dist}(\eta, \Sigma)^\gamma \quad \textit{for all } \eta \in \mathcal{O} \textit{ and some } \gamma \ge m. \tag{3.3}$$

(a) *If $V \le 0$ then $T(p)+V$ has infinitely many strictly negative eigenvalues.*
(b) *If $V \le W$, $W \in L^1(\mathbb{R}^d)$, and $\int W \,\mathrm{d}x \le 0$, then $T(p)+V$ has at least one strictly negative eigenvalue.*

Remark 3.8. In most applications T is continuous and the zero set of T is either a point, a collection of points, or a smooth submanifold in $\mathbb{R}^d$. So Corollaries

3.6 and 3.7 would be enough to cover all applications we can think of. However, we find that the proof for the general case is so simple, that adding further structure to its assumptions only obscures the simplicity of the proof. So we prefer to state Theorem 1.1 and its local versions, Theorems 3.2 and 3.4, in their generality.

Some of the most interesting applications are considered in Appendix A.

4. Existence of Bound States: A Simple Model Case

We want to construct a test function φ such that $\langle \varphi, (T(p) + V)\varphi \rangle < 0$. Once one has such a state together with $\sigma_{\text{ess}}(T(p)+V) \subset [0,\infty)$, the Rayleigh–Ritz variational principle shows that strictly negative discrete spectrum exists. Of course, the catch is how to guess such a variational state φ in a systematic way.

To motivate our construction for our general set-up, we will discuss here the simple model case, where $T(\eta) = |\eta|^\gamma$, i.e., $T(p) = (-\Delta)^{\gamma/2}$ is a fractional Laplacian.

4.1. The Case $\int V \,\mathrm{d}x < 0$: Learning from Failure

We will work mainly in Fourier-space and, for simplicity, consider $\int V \,\mathrm{d}x < 0$ first. In order to make the kinetic energy small, a natural first guess for the Fourier transform of the test function would be

$$\widetilde{w}_\delta := \mathbf{1}_{A_\delta},$$

for a suitably chosen set A_δ of finite positive measure which concentrates around zero, since this makes the kinetic energy small. However, it turns out that this is not a good ansatz and it is instructive to see why. In order to use the assumption $\int V \,\mathrm{d}x < 0$, we want our test function to converge to a constant, so its Fourier transform should converge to a delta-function at zero. Thus we need to normalize $\widetilde{w}_\delta$ and are led to consider

$$\widetilde{\varphi}_\delta := \frac{\widetilde{w}_\delta}{\|\widetilde{w}_\delta\|_{L^1_\eta}}.$$

Note that this choice fulfills two crucial assumptions: Let

$$\kappa_\delta(x) := \mathcal{F}^{-1}(\widetilde{\varphi}_\delta)(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\eta\cdot x} \widetilde{\varphi}_\delta(\eta) \,\mathrm{d}\eta$$

be the inverse Fourier transform of $\widetilde{\varphi}_\delta$. We always have

$$|\kappa_\delta(x)| \le \frac{\|\widetilde{\varphi}_\delta\|_{L^1_\eta}}{(2\pi)^{d/2}} = \frac{1}{(2\pi)^{d/2}}$$

by our normalization of $\widetilde{\varphi}_\delta$. Moreover, as long as A_δ concentrates to the single point zero in a suitable way, we also have

$$\lim_{\delta\to 0} \kappa_\delta(x) = \frac{1}{(2\pi)^{d/2}} = \frac{1}{(2\pi)^{d/2}}$$

for all $x \in \mathbb{R}^d$. Since, by assumption the potential V is integrable, we conclude with Lebesgue's dominated convergence theorem

$$\lim_{\delta\to 0}\langle \kappa_\delta, V\kappa_\delta\rangle = \frac{1}{(2\pi)^d}\int V\,\mathrm{d}x. \tag{4.1}$$

So if $\int V\,\mathrm{d}x < 0$, the choice for φ_δ yields a test function which makes the potential contribution strictly negative.

It only remains to see whether the kinetic energy vanishes and, since the set A_δ concentrates near zero, this should be the case, but there is a catch: Note that

$$\langle \kappa_\delta, (-\Delta)^{\gamma/2}\kappa_\delta\rangle = \langle \widetilde{\varphi}_\delta, |\eta|^\gamma \widetilde{\varphi}_\delta\rangle = \frac{1}{\|\widetilde{w}_\delta\|^2_{L^1_\eta}}\int_{A_\delta}|\eta|^\gamma\,\mathrm{d}\eta.$$

Since $\|\widetilde{w}_\delta\|_{L^1_\eta} = |A_\delta|$, the Lebesgue measure of the set A_δ, we can use rearrangement inequalities, see, e.g., [31], to make the kinetic energy smallest by chosing A_δ to be centered ball of radius δ, say. In this case $|A_\delta| \sim \delta^d$ and thus

$$\langle \kappa_\delta, (-\Delta)^{\gamma/2}\kappa_\delta\rangle \sim \frac{1}{\delta^{2d}}\int_0^\delta r^{\gamma+d-1}\,\mathrm{d}r \sim \delta^{\gamma-d} \tag{4.2}$$

and this goes to zero as $\delta \to 0$ *only if* $\gamma > d$ and it misses the critical case where $\gamma = d$.

So we have to modify the test functions. A better choice, which also works for $\gamma = d$, turns out[6] to be given by

$$\widehat{w}_\delta(\eta) := |\eta|^{-\gamma}\,\mathbf{1}_{A_\delta}(\eta) \tag{4.3}$$

where now the set A_δ has to stay away from zero to make $\widehat{w}_\delta(\eta)$ normalizable. Note that $|\eta|^{-\gamma}$ is just the inverse of the symbol $T(\eta) = |\eta|^\gamma$.

We further set, as before,

$$\widehat{\varphi}_\delta := \frac{\widehat{w}_\delta}{\|\widehat{w}_\delta\|_{L^1_\eta}}.$$

With this choice

$$\langle w_\delta, (-\Delta)^{\gamma/2} w_\delta\rangle = \langle \widehat{w}_\delta, |\eta|^\gamma \widehat{w}_\delta\rangle = \int_{A_\delta}|\eta|^{-\gamma}\,\mathrm{d}\eta = \|\widehat{w}_\delta\|_{L^1_\eta},$$

hence

$$\langle \varphi_\delta, (-\Delta)^{\gamma/2}\varphi_\delta\rangle = \frac{1}{\|\widehat{w}_\delta\|_{L^1_\eta}}. \tag{4.4}$$

As before, we still have

$$\langle \varphi_\delta, V\varphi_\delta\rangle \to \frac{1}{(2\pi)^d}\int V\,\mathrm{d}x \quad \text{as } \delta \to 0$$

as soon as A_δ concentrates near zero in the limit $\delta \to 0$. Since the function $\mathbb{R}^d \ni \eta \mapsto |\eta|^{-\gamma}$ has a *non-integrable singularity* near zero for $\gamma \geq d$, we can make A_δ concentrate near zero, thus having $\|\widehat{w}_\delta\|_{L^1_\eta}$ blow up and, because of

[6] We will further generalize this ansatz in Sect. 5.

(4.4), we get $\lim_{\delta\to 0}\langle \varphi_\delta, (-\Delta)^{\gamma/2}\varphi_\delta\rangle = 0$, i.e., the kinetic energy vanishes in the limit $\delta \to 0$ as soon as $\gamma \geq d$.

Explicitly, choosing A_δ to be the annulus

$$A_\delta := \{r_{1,\delta} < |\eta| < r_{2,\delta}\}$$

we have

$$\|\widehat{w}_\delta\|_{L^1_\eta} \sim \int_{r_{1,\delta}}^{r_{2,\delta}} r^\gamma r^{d-1}\,\mathrm{d}r = \begin{cases} \ln(\frac{r_{2,\delta}}{r_{1,\delta}}) & \text{if } \gamma = d \\ \frac{1}{\gamma-d}\left[r_{1,d}^{-(\gamma-d)} - r_{2,\delta}^{-(\gamma-d)}\right] & \text{if } \gamma > d \end{cases}$$

and choosing $r_{1,\delta} = \delta^2$ and $r_{2,\delta} = \delta$ we see $\lim_{\delta\to 0}\|\widehat{w}_\delta\|_{L^1_\eta} = \infty$. With (4.4)

$$\lim_{\delta\to 0}\langle \varphi_\delta, ((-\Delta)^{\gamma/2} + V)\varphi_\delta\rangle = \frac{1}{(2\pi)^d}\int V\,\mathrm{d}x,$$

follows. So bound states with strictly negative energy exist once $\int V\,\mathrm{d}x < 0$.

4.2. The Case of $\int V\,\mathrm{d}x = 0$

To include the case where V does not vanish identically but $\int V\,\mathrm{d}x = 0$, we have to further modify the test function. Second order perturbation theory suggest that the test function should be modified by adding a suitable multiple of the potential V. This suggests the ansatz

$$\varphi_\delta + \alpha\phi \tag{4.5}$$

for some $\alpha \in \mathbb{R}$ and a suitably nice function ϕ, to be determined later, as a trial state for the computation of the energy. Using this we get, with $T(p) = |p|^\gamma = (-\Delta)^{\gamma/2}$,

$$\begin{aligned} E(\delta,\alpha) &:= \langle \varphi_\delta + \alpha\phi, (T(p) + V)(\varphi_\delta + \alpha\phi)\rangle \\ &= \langle \varphi_\delta, T(p)\varphi_\delta\rangle + \langle \varphi_\delta, V\varphi_\delta\rangle + 2\alpha\mathrm{Re}(\langle \varphi_\delta, T(p)\phi\rangle) + 2\alpha\mathrm{Re}(\langle \varphi_\delta, V\phi\rangle) \\ &\quad + \alpha^2\langle \phi, T(p)\phi\rangle + \alpha^2\langle \phi, V\phi\rangle. \end{aligned}$$

From the discussion above we know

$$\begin{aligned} \lim_{\delta\to 0}\langle \varphi_\delta, T(p)\varphi_\delta\rangle &= 0, \\ \lim_{\delta\to 0}\langle \varphi_\delta, V\varphi_\delta\rangle &= \frac{1}{(2\pi)^d}\int V\,\mathrm{d}x = 0, \\ \lim_{\delta\to 0}\langle \varphi_\delta, V\phi\rangle &= \frac{1}{(2\pi)^{d/2}}\int V\phi\,\mathrm{d}x, \end{aligned} \tag{4.6}$$

and, since $T(p)$ is a positive operator, we also have

$$|\langle \varphi_\delta, T(p)\phi\rangle| \leq \langle \varphi_\delta, T(p)\varphi_\delta\rangle^{1/2}\langle \phi, T(p)\phi\rangle^{1/2} \to 0 \quad \text{as } \delta \to 0. \tag{4.7}$$

Thus

$$E(\alpha) := \lim_{\delta\to 0} E(\delta,\alpha) = 2\alpha\frac{1}{(2\pi)^{d/2}}\mathrm{Re}\int V\phi\,\mathrm{d}x + \alpha^2\langle \phi, T(p)\phi\rangle + \alpha^2\langle \phi, V\phi\rangle$$

and

$$\lim_{\alpha\to 0}\frac{E(\alpha)}{\alpha} = \frac{2}{(2\pi)^{d/2}}\mathrm{Re}\int V\phi\,\mathrm{d}x.$$

This shows that we will have $E(\delta,\alpha) < 0$ for some finite $\delta > 0$ and $\alpha > 0$, if we can find a Schwartz function $\phi \in \mathcal{Q}(T(p))$ such that $\int V\phi \, \mathrm{d}x < 0$.

Split $V = V_+ - V_-$, the positive and negative parts of V. By assumption, $\int V_- \, \mathrm{d}x = \int V_+ \, \mathrm{d}x > 0$. Take a big centered ball B such that $\int_B V_- \, \mathrm{d}x > 0$ and consider the set

$$D := B \cap \{V_- > 0\}.$$

Let $\kappa_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ be an approximate delta-function and set

$$\phi_\varepsilon := \kappa_\varepsilon * \mathbf{1}_D.$$

This is a nice infinitely often differentiable function with compact support, in particular a Schwartz function, so its Fourier transform decays rapidly, hence hence ϕ_ε is in the form domains of $T(p)$ and $V_\pm$. By the properties of convolutions [31], we have $0 \le \phi_\varepsilon \le 1$ and $\phi_\varepsilon \to \mathbf{1}_D$ in L^1 for $\varepsilon \to 0$, hence, after taking a subsequence, also pointwise almost everywhere. With slight abuse of notation we denote this subsequence still by ϕ_ε. With the help of Lebesgue's dominated convergence theorem one sees

$$\lim_{\varepsilon\to 0} \int V\phi_\varepsilon \, \mathrm{d}x = -\int_B V_- \, \mathrm{d}x < 0,$$

so using ϕ_ε instead of ϕ for some small enough $\varepsilon > 0$ in the above argument shows that there are $\alpha, \delta, \varepsilon > 0$ such that

$$\langle \varphi_\delta + \alpha\phi_\varepsilon, (T(p) + V)(\varphi_\delta + \alpha\phi_\varepsilon)\rangle < 0.$$

Hence the variational principle shows that we have a strictly negative eigenvalue of $T(p) + V$ also in the case where V does not vanish identically but $\int V \, \mathrm{d}x = 0$.

5. Existence of Bound States: Proof of the General Case

In this section we give the proof of Theorems 1.1 and 3.4 and Colloraries 3.6 and 3.7. To prepare for this, we give a lemma first, which is a convenient replacement for (4.4). We have to be a little bit careful here, to ensure that the constructed function is normalizable. The construction in (4.3) worked, because there the kinetic energy was bounded away from zero in any open set not containing zero. In the general case, where T is just measurable, this is not so clear. As an easy way out we simply cut the kinetic energy close to zero.

Lemma 5.1. *Let $T, \chi : \mathbb{R}^d \to [0,\infty)$ be measurable and $0 \le \chi \le 1$. For $\tau > 0$ define the function $\widehat{w}_\tau$ by*

$$\widehat{w}_\tau(\eta) = \max(T(\eta), \tau)^{-1}\chi(\eta) \quad \textit{for } \eta \in \mathbb{R}^d.$$

Then $w_\tau := \mathcal{F}^{-1}(\widehat{w}_\tau) \in \mathcal{Q}(T(p))$, the quadratic form domain of $T(p)$, and we have the bound

$$\langle w_\tau, T(p)w_\tau\rangle \le \|\widehat{w}_\tau\|_{L^1_\eta}$$

for its kinetic energy.

Remark 5.2. At first sight the bound provided by Lemma 5.1 seems surprising, since the left hand side of the bound scales quadratically in w but the right hand side is linear in $\widehat{w}$. This is not a contradiction, though, since we assume that $\widehat{w}_\tau = \max(T,\tau)^{-1}\chi$ and $0 \le \chi \le 1$, which breaks the scaling.

Proof of Lemma 5.1. This is a simple calculation. Since T is positive and by Plancherel,

$$\begin{aligned}
\langle w_\tau, T(p) w_\tau\rangle &= \langle \sqrt{T(p)} w_\tau, \sqrt{T(p)} w_\tau\rangle = \langle \widehat{w}_\tau, T\widehat{w}_\tau\rangle \\
&= \int_{\mathbb{R}^d} T(\eta)\max(T(\eta),\tau)^{-2}\chi(\eta)^2\,\mathrm{d}\eta \\
&\le \int_{\mathbb{R}^d} \max(T(\eta),\tau)^{-1}\chi(\eta)^2\,\mathrm{d}\eta \\
&\le \int_{\mathbb{R}^d} \max(T(\eta),\tau)^{-1}\chi(\eta)\,\mathrm{d}\eta = \|\widehat{w}_\tau\|_{L^1_\eta},
\end{aligned}$$

since, by assumption $0 \le \chi \le 1$, thus also $0 \le \chi^2 \le \chi$. ☐

Now we come to the

Proof of Theorem 3.2. Since $V \le W$, we have $\langle \varphi, V_+\varphi\rangle \le \langle \varphi, W_+\varphi\rangle < \infty$ for all $\varphi \in \mathcal{Q}(W_+)$, i.e., $\mathcal{Q}(W_+) \subset \mathcal{Q}(V_+)$. Hence $\mathcal{Q}(T(p)+W) = \mathcal{Q}(T(p)) \cap \mathcal{Q}(W_+) \subset \mathcal{Q}(T(p)+V)$ and $\langle \varphi, (T(p)+V)\varphi\rangle \le \langle \varphi, (T(p)+W)\varphi\rangle$ for all $\varphi \in \mathcal{Q}(T(p)+W)$. The variational principle shows that $T(p)+V$ has at least as many negative eigenvalues as $T(p)+W$ has, [4,8,53]. So replacing V with W, if necessary, we can, without loss of generality, assume $V \in L^1(\mathbb{R}^d)$ and $\int V\,\mathrm{d}x \le 0$.

Let $\omega \in \mathbb{R}^d$ be such that

$$\int_{B_{1/n}(\omega)} T(\eta)^{-1}\,\mathrm{d}\eta = \infty$$

for every $n \in \mathbb{N}$. By monotone convergence, we have

$$\lim_{\tau\to 0}\int_{B_{1/n}(\omega)} \max(T(\eta),\tau)^{-1}\,\mathrm{d}\eta = \int_{B_{1/n}(\omega)} T(\eta)^{-1}\,\mathrm{d}\eta = \infty,$$

so there exists a sequence $\tau_{n+1} < \tau_n \to 0$, for $n \to \infty$, with

$$\lim_{n\to\infty}\int_{B_{1/n}(\omega)} \max(T(\eta),\tau_n)^{-1}\,\mathrm{d}\eta = \infty. \tag{5.1}$$

Define the functions $\widehat{w}_n$ and $\widehat{\varphi}_n$ by

$$\widehat{w}_n(\eta) := \max(T(\eta),\tau_n)^{-1}\mathbf{1}_{B_{1/n}(\eta)} \text{ and } \widehat{\varphi}_n(\eta) := \frac{\widehat{w}_n(\eta)}{\|\widehat{w}_n\|_{L^1_\eta}}$$

for every $\eta \in \mathbb{R}^d$. Note that $\widehat{w}_n \in L^1_\eta$, so φ_n is non-trivial. Because of Lemma 5.1, $w_n \in \mathcal{Q}(T(p)) \subset \mathcal{Q}(V)$. By construction,

$$\|\widehat{w}_n\|_{L^1_\eta} \to \infty \text{ as } n \to \infty.$$

In addition, since the sets $B_{1/n}(\omega)$ concentrate around ω and $\widehat{\varphi}_n$ is L^1 normalized, we also have that $\widehat{\varphi}_n$ is a sequence of approximate delta-functions which concentrates at ω. Thus we have the uniform bound

$$|\varphi_n(x)| \le \frac{1}{(2\pi)^{d/2}} \|\widehat{\varphi}\|_{L^1_\eta} = \frac{1}{(2\pi)^{d/2}}$$

and the pointwise limit

$$\lim_{n\to\infty} \varphi_n(x) = \frac{1}{(2\pi)^{d/2}} e^{i\omega\cdot x} \text{ for all } x \in \mathbb{R}^d.$$

Using Lebesgue's dominated convergence theorem this shows

$$\lim_{n\to\infty} \langle \varphi_n, V\varphi_n\rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} V \,\mathrm{d}x.$$

For the kinetic energy we simply note that Lemma 5.1 yields

$$\langle \varphi_n, T(p)\varphi_n\rangle = \frac{1}{\|\widehat{w}_n\|^2_{L^1_\eta}} \langle w_n, T(p)w_n\rangle \le \frac{1}{\|\widehat{w}_n\|_{L^1_\eta}} \to 0 \text{ as } n \to \infty.$$

So if $\int V \,\mathrm{d}x < 0$ we can immediately conclude

$$\lim_{n\to\infty} \langle \varphi_n, (T(p) + V)\varphi_n\rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} V \,\mathrm{d}x < 0$$

and the variational principle implies that there is a strictly negative eigenvalue of $T(p) + V$.

In the case $\int V_+ \,\mathrm{d}x = \int V_- \,\mathrm{d}x > 0$, so $\int V \,\mathrm{d}x = 0$, we use the construction of Sect. 4.2 to see that there exists a positive function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ with

$$\int_{\mathbb{R}^d} V\phi \,\mathrm{d}x < 0.$$

Similarly to the discussion in Sect. 4.2, we modify the trial state to the form

$$\varphi(x) = \varphi_n(x) + \alpha e^{i\omega\cdot x}\phi(x) \text{ for } x \in \mathbb{R}^d.$$

Setting $\widetilde{\phi}(x) = e^{i\omega\cdot x}\phi(x)$ we have, analogously to the calculation in Sect. 4.2,

$$\begin{aligned}
&\lim_{\alpha\to 0} \alpha^{-1} \lim_{n\to\infty} \langle \varphi_n + \alpha\widetilde{\phi}, (T(p) + V)(\varphi_n + \alpha\widetilde{\phi})\rangle \\
&\quad = \frac{2}{(2\pi)^{d/2}} \mathrm{Re} \int e^{-i\omega\cdot x} V(x)\widetilde{\phi}(x) \,\mathrm{d}x \\
&\quad = \frac{2}{(2\pi)^{d/2}} \int V(x)\phi(x) \,\mathrm{d}x < 0.
\end{aligned}$$

So for all large enough $n \in \mathbb{N}$ and small enough $\alpha > 0$

$$\langle \varphi_n + \alpha\widetilde{\phi}, (T(p) + V)(\varphi_n + \alpha\widetilde{\phi})\rangle < 0,$$

which, by the variational principle, implies the existence of at least one negative eigenvalue for $T(p) + V$. □

Proof of Theorem 3.4. We will first prove part b: Assume that there are k distinct points $\omega_1, \dots, \omega_k$ such that

$$\int_{B_\delta(\omega_l)} T(\eta)^{-1}\,\mathrm{d}\eta = \infty \tag{5.2}$$

for all $r = 1, \dots, k$ and all $\delta > 0$. Using the previous construction, we see that for each $r = 1, \dots, k$ there exist functions $\varphi_{r,n}$ where the support of $\widehat{\varphi}_{r,n}$ concentrates in Fourier space near ω_r. Then

$$\begin{aligned}
&\lim_{n\to\infty} \left\langle \sum_{r=1}^k c_r\varphi_{r,n}, (T(p)+W)\left(\sum_{r=1}^k c_r\varphi_{r,n}\right)\right\rangle \\
&\quad = \lim_{n\to\infty} \sum_{r,s=1}^k \overline{c_r}c_s\langle\varphi_{r,n}, (T(p)+W)\varphi_{s,n}\rangle \\
&\quad = \frac{1}{(2\pi)^d}\sum_{r,s=1}^k \overline{c_r}c_s\int_{\mathbb{R}^d} W(x)e^{-ix(\omega_r-\omega_s)}\,\mathrm{d}x = \frac{1}{(2\pi)^{d/2}}\sum_{r,s=1}^k \widehat{W}(\omega_r-\omega_s)\overline{c_r}c_s \\
&\quad = \frac{1}{(2\pi)^{d/2}}\langle c, Mc\rangle_{\mathbb{C}^k}
\end{aligned}$$

with the matrix $M = (\widehat{W}(\omega_r - \omega_s))_{r,s=1,\dots,k}$. If this matrix is negative definite, then $\langle c, Mc\rangle_{\mathbb{C}^k} < 0$ for all $c \neq 0$, thus $T(p) + W$, hence also $T(p) + V$, will be strictly negative on the subspace $N_{k,n} = \operatorname{span}(\varphi_{r,n}, r = 1, \dots, k)$. For large n the functions $\varphi_{r,n}$ do not overlap in Fourier-space, thus $\dim N_{k,n} = k$ for all large enough n. This gives the existence of at least k strictly negative eigenvalues of $T(p) + V$ by the usual variational arguments, see [4,8,53].

To prove part a, we simply note that if $V \le 0$ and $V \neq 0$, then for B_R a centered ball of radius R

$$W_R^m := -\min(V_-, m)\mathbf{1}_{B_R}$$

is integrable for all $m, R > 0$ and $V \le W_R^m \le 0$. Since $V \neq 0$, one has $\int W_R^m\,\mathrm{d}x < 0$ for large enough m, R. Using the variational principle again, we can assume $V = W \in L^1$, $W \le 0$, and $W \neq 0$, without loss of generality.

The matrix M above will be negative definite. For $|c|^2 = \sum_{r=1}^k |c_r|^2 = 1$, we have

$$\langle c, Mc\rangle_{\mathbb{C}^k} = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} W(x)\left|\sum_{r=1}^k c_r e^{-ix\omega_r}\right|^2\,\mathrm{d}x < 0$$

since $x \mapsto \sum_{r=1}^k c_r e^{-ix\omega_r}$ is real-analytic, thus not zero on any open set of positive Lebesgue measure and $W \le 0$, $W \neq 0$. Moreover, let S^{k-1} be the unit sphere in $\mathbb{C}^k$ and note that the map $S^{k-1} \ni c \mapsto \langle c, Mc\rangle_{\mathbb{C}^k}$ is continuous and M is a hermitian matrix. Thus by the above, we see that the largest eigenvalue of M is negative. So M is negative definite and by part b, we conclude that $T(p) + V$ has at least k strictly negative eigenvalues.

To prove part c, we note that it is enough to show that $T(p) + W$ has at least k negative eigenvalues. Let M be the $k \times k$ matrix as above and let

$a = (a_1, \dots, a_k)^t$ be the normalized eigenvector corresponding to the eigenvalue zero, which we assume to be non-degenerate. Let U_a be the $k-1$ dimensional orthogonal complement of the vector a in $\mathbb{C}^k$. Furthermore, we set

$$N_{a,n} := \left\{ \sum_{r=1}^{k} c_r \varphi_{r,n} : c = (c_1, \dots, c_k)^t \in U_a \right\},$$

where the functions $\varphi_{r,n}$ are defined just below Eq. (5.2). Note that $N_{a,n}$ is a $k-1$-dimensional subspace of $L^2(\mathbb{R}^d)$ for large enough n since then $\varphi_{r,m}$, $r = 1, \dots, k$, have disjoint support in Fourier space. Define, for some positive α,

$$\widetilde{\varphi} := \sum_{r=1}^{k} a_r \varphi_{r,n} + \alpha\phi$$

and $L_{k,\alpha,n} := \operatorname{span}\{N_{a,n}, \widetilde{\varphi}\}$. That is, any vector in $L_{k,\alpha,n}$ can be written as

$$\psi_{n,\alpha} = \psi_{n,\alpha}(\gamma, c) = \sum_{r=1}^{k} c_r \varphi_{r,n} + \gamma \left(\sum_{r=1}^{k} a_r \varphi_{r,n} + \alpha\phi \right) = \sum \tilde{c}_r \varphi_{r,n} + \gamma\alpha\phi \tag{5.3}$$

were we set $\tilde{c} = \tilde{c}(\gamma, c) = \gamma a + c \in \mathbb{C}^k$. Since U_a is the orthogonal complement of a in C^k, the map $\tilde{c} : \mathbb{C} \times U_a \to C^k$ is a bijection.

Our goal is to show that the dimension of $L_{k,n}$ is k and $T(p) + W$ is negative on $L_{k,\alpha,n} \setminus \{0\}$, for large enough n and a suitable choice of ϕ and $\alpha \in \mathbb{R}$. Writing $\psi_{n,\alpha} = (c + \gamma a)\varphi_{\cdot,n} + \gamma\alpha\phi$, we have

$$\begin{aligned} \langle \psi_{n,\alpha}, (T(p) + W)\psi_{n,\alpha} \rangle &= \langle (c + \gamma a)\varphi_{\cdot,n}, (T(p) + W)(c + \gamma a)\varphi_{\cdot,n} \rangle \\ &\quad + 2\operatorname{Re}\langle \gamma\alpha\phi, (T(p) + W)(c + \gamma a)\varphi_{\cdot,n} \rangle \\ &\quad + |\gamma\alpha|^2 \langle \phi, (T(p) + W)\phi \rangle. \end{aligned} \tag{5.4}$$

Since the $\varphi_{r,n}$, $r = 1, \dots, k$, have disjoint supports in Fourier space when $n \in \mathbb{N}$ is large and, as before, $\langle \varphi_{r,n}, T(p)\varphi_{r,n} \rangle = o_n(1) \to 0$ as $n \to \infty$, we have

$$\langle (c + \gamma a)\varphi_{\cdot,n}, T(p)(c + \gamma a)\varphi_{\cdot,n} \rangle = o_n(1)|c + \gamma a|^2 = o_n(1)(|c|^2 + |\gamma|^2)$$

where we also used that since a is normalized in C^k and $c \perp a$, we have $|c + \gamma a|^2 = |c|^2 + |\gamma|^2$. Also

$$\langle (c + \gamma a)\varphi_{\cdot,n}, W(c + \gamma a)\varphi_{\cdot,n} \rangle = \langle (c + \gamma a), (M + \Delta M_n)(c + \gamma a) \rangle_{C^k},$$

where $M = (\widehat{W}(\omega_r - \omega_s))_{r,s=1,\dots,k}$ is the $k \times k$ matrix as before, but now it has a single zero eigenvalue since $Ma = 0$ and $\langle c, Mc \rangle_{C^k} \le -\lambda_1 |c|^2$ for some $\lambda_1 > 0$ and all $c \perp a$. Moreover, ΔM_n is a $k \times k$ matrix which converges to zero as $n \to \infty$, that is, $\langle \tilde{c}, \Delta M_n \tilde{c} \rangle = o_n(1)|\tilde{c}|^2$ for all $\tilde{c} \in C^k$. Using $Ma = 0$ and $c \perp a$, we get

$$\begin{aligned} \langle (c + \gamma a), (M + \Delta M_n)(c + \gamma a) \rangle_{C^k} &= \langle c, Mc \rangle_{C^k} + \langle (c + \gamma a), \Delta M_n(c + \gamma a) \rangle_{C^k} \\ &\le -\lambda_1 |c|^2 + o_n(1)(|c|^2 + |\gamma|^2). \end{aligned}$$

For the part due to the kinetic energy in the cross term we use the Cauchy Schwarz inequality to bound it as

$$\begin{aligned}&|\langle\gamma\alpha\phi, T(p)(c+\gamma a)\varphi_{\cdot,n}\rangle|\\&\quad\le|\gamma\alpha|\langle\phi, T(p)\phi\rangle^{1/2}\left\langle(c+\gamma a)\varphi_{\cdot,n}, T(p)(c+\gamma a)\varphi_{\cdot,n}\right\rangle^{1/2}\\&\quad=o_n(1)\alpha|\gamma|(|c|^2+|\gamma|^2)^{1/2}\le o_n(1)\alpha(|c|^2+|\gamma|^2).\end{aligned}$$

The part due to the potential in the cross term is bounded as

$$\begin{aligned}\mathrm{Re}\langle\gamma\alpha\phi, W(c+\gamma a)\varphi_{\cdot,n}\rangle&=\alpha\mathrm{Re}(\overline{\gamma}\langle\phi, Wc\varphi_{\cdot,n}\rangle+|\gamma|^2\alpha\mathrm{Re}\langle\phi, Wa\varphi_{\cdot,n}\rangle\\&\le\alpha|\gamma|\||W|^{1/2}\phi\|\||W|^{1/2}c\varphi_{\cdot,n}\|+|\gamma|^2\alpha\mathrm{Re}\langle\phi, Wa\varphi_{\cdot,n}\rangle\\&\le C\alpha|\gamma||c|+|\gamma|^2\alpha\mathrm{Re}\langle\phi, Wa\varphi_{\cdot,n}\rangle,\end{aligned}$$

where we also used

$$\||W|^{1/2}\phi\|^2=\langle\phi,|W|\phi\rangle\le C\text{ and }\||W|^{1/2}c\varphi_{\cdot,n}\|^2=\langle c\varphi_{\cdot,n},|W|c\varphi_{\cdot,n}\rangle\le C|c|^2$$

for some large enough constant C.

Plugging all of this into (5.4) and collecting terms we get the upper bound

$$\begin{aligned}&\langle\psi_{n,\alpha},(T(p)+W)\psi_{n,\alpha}\rangle\\&\quad\le o_n(1)(|c|^2+|\gamma|^2)-\lambda_1|c|^2+\alpha\big(\mathrm{Re}\langle\phi, Wa\varphi_{\cdot,n}\rangle+C\alpha\big)|\gamma|^2+C\alpha|\gamma||c|.\end{aligned}\tag{5.5}$$

for some large enough constant C, any $\psi_{n,\alpha}\in L_{k,n,\alpha}$ in the form given by (5.3), and $\alpha\ge 0$.

Note that

$$\begin{aligned}\langle\phi, Wa\varphi_{\cdot,n}\rangle=(2\pi)^{-d/2}\int\overline{\phi(x)}W(x)\sum_{r=1}^{k}a_r\varphi_{r,n}\,\mathrm{d}x\to(2\pi)^{-d/2}\\\times\int\overline{\phi(x)}W(x)\sum_{r=1}^{k}a_re^{i\omega_r\cdot x}\,\mathrm{d}x\end{aligned}$$

so with the choice $\phi(x)=\zeta(x)\sum_{r=1}^k a_re^{i\omega_r\cdot x}$ for some real-valued function ζ

$$\langle\phi, Wa\varphi_{\cdot,n}\rangle\to(2\pi)^{-d/2}\int\zeta(x)W(x)\left|\sum_{r=1}^{k}a_re^{i\omega_r\cdot x}\right|^2\mathrm{d}x.$$

Of course,

$$(2\pi)^{-d/2}\int W(x)\left|\sum_{r=1}^{k}a_re^{i\omega_r\cdot x}\right|^2\mathrm{d}x=\langle a, Ma\rangle_{C^k}=0$$

but, since $\left|\sum_{r=1}^k a_re^{i\omega_r\cdot x}\right|$ can vanish only on a set of measure zero and W is not identically zero, there must exist a real-valued Schwartz function ζ_0 with compact support such that $\int\zeta_0(x)W(x)\left|\sum_{r=1}^k a_re^{i\omega_r\cdot x}\right|^2\mathrm{d}x<0$. We cut this Schwartz function in Fourier space with a spherically symmetric and smooth cut-off function, to get a real-valued Schwartz function ζ which has compact support in Fourier space and for which, by making the cut-off in Fourier space large enough, one has

$$\int \zeta(x)W(x)\left|\sum_{r=1}^{k} a_r e^{i\omega_r\cdot x}\right|^2 \,\mathrm{d}x < 0.$$

With this choice we then have

$$\mathrm{Re}\langle\phi, Wa\varphi_{\cdot,n}\rangle \le -\rho$$

for some $\rho > 0$ and all large enough n. Thus (5.5) implies

$$\begin{aligned}
&\langle\psi_{n,\alpha}, (T(p) + W)\psi_{n,\alpha}\rangle\\
&\quad\le o_n(1)(|c|^2 + |\gamma|^2) - \lambda_1|c|^2 - \alpha\big(\rho - C\alpha\big)|\gamma|^2 + C\alpha|\gamma||c|\\
&\quad\le o_n(1)(|c|^2 + |\gamma|^2) - \lambda_1|c|^2 - \alpha\big(\rho - C\alpha\big)|\gamma|^2 + C\alpha^{1/2}|c|^2 + C\alpha^{3/2}|\gamma|^2\\
&\quad= -\big(o_n(1) + C\alpha^{1/2} - \lambda_1\big)|c|^2 + \alpha\big(C(\alpha + \alpha^{1/2}) - \rho\big)|\gamma|^2\\
&\quad\le -\frac{\lambda_1}{2}|c|^2 - \frac{\alpha\rho}{2}|\gamma|^2 < 0,
\end{aligned} \tag{5.6}$$

for all large enough n and all small enough and positive α and all $c + \gamma a \neq 0$. Thus the quadratic form $T(p) + W$ is negative definite on the space $L_{k,n}$ and the usual min–max variational arguments show that $T(p)+V$ has no less than $\dim(L_{k,\alpha,n})$ negative eigenvalues.

It remains to show that $L_{k,n}$ has dimension k. Assume that

$$\psi_{\alpha,n}(c,\gamma) = (c + \gamma a)\varphi_{\cdot,n} + \gamma\alpha\phi = 0 \tag{5.7}$$

and $\alpha \neq 0$. Since in the construction above, we chose ζ_0 to have compact support, its Fourier transform $\widehat{\zeta_0}$ will not have compact support. The function $\widehat{\varphi}_{r,n}$ has support in a small ball, depending on how large n is, around each center ω_r, $r = 1,\dots,k$. So for large enough n the supports of $\widehat{\varphi}_{r,n}$ are pairwise disjoint and, moreover, since $\widehat{\zeta_0}$ does not have compact support, we can choose the cut-off in Fourier space so large that the support of $\widehat{\zeta}$ is not contained in the union of the supports of the $\widehat{\varphi}_{r,n}$. But then (5.7) immediately implies that $\gamma = 0$. Once this is the case, the linear independence of the $\varphi_{r,n}, r = 1,\dots,k$ for large n shows that also $c = 0$.

Thus for fixed $a \in C^k\backslash\{0\}$, the map $C^k \ni (c+\gamma a) \mapsto \psi_{\alpha,n}(c,\gamma) \in L_{k,n,\alpha}$, with $c \perp a$, is a bijection. Hence $L_{k,n,\alpha}$ is k-dimensional for all large enough n. This finishes the proof. □

Now we come to the

Proof of Corollary 3.6. A simple calculation shows that the assumption of Theorem 3.4 is fulfilled at k distinct points $\omega_1,\dots,\omega_k$. So Theorem 3.4 applies. □

For the proof of Corollary 3.7 the following Lemma is helpful, which gives the so-called nearest point projection parametrization of a suitable open neighborhood of Σ.

Lemma 5.3. *Let Σ be a $\mathcal{C}^2$ submanifold in $\mathbb{R}^d$ of codimension $1 \le n \le d-1$. Then for each point $\omega \in \Sigma$, there exists a neighborhood $\mathcal{O}$ of ω in $\mathbb{R}^d$ and neighborhoods $\mathcal{U}_1$ in $\mathbb{R}^{d-n}$ and $\mathcal{U}_2$ in $\mathbb{R}^n$ both containing zero and a $\mathcal{C}^1$ diffeomorphism $\Psi : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{O}$, such that*

$$\Psi(0,0) = \omega \text{ and } \Psi(y,0) \in \Sigma \text{ for all } y \in \mathcal{U}_1.$$

Moreover,

$$\operatorname{dist}(\Psi(y,t),\Sigma) = |t|.$$

This type of result seems to be well-known to geometers, at least in the analytic category, see for example [54]. However, we could not find a reference which assumes only that Σ is a $\mathcal{C}^2$ manifold. So for the convenience of the reader, and ours, we give the proof of this Lemma in Appendix C. We now come to the

Proof of Corollary 3.7. Assume that Σ has codimension $1 \le m \le d-1$. Pick a point $\omega \in \Sigma$ and let $\mathcal{O}$, $\mathcal{U}_1$, $\mathcal{U}_2$ be the neighborhoods and ψ the $\mathcal{C}^1$ diffeomorphism from Lemma 5.3. Since $\mathcal{O}$ is open there exists $\delta_0 > 0$ such that $B_\delta(\omega) \subset \mathcal{O}$ for all $0 < \delta \le \delta_0$. Fix such a δ and choose $A_1 \subset \mathcal{U}_1$ and $A_2 \subset \mathcal{U}_2$ both centered closed balls in $\mathbb{R}^{d-m}$, respectively, $\mathbb{R}^m$, with

$$\psi(A_1 \times A_2) \subset B_\delta(\omega).$$

We use ψ to change coordinates in the calculation of a lower bound for $\int_{B_\delta(\omega)} T(\eta)^{-1}\,\mathrm{d}\eta$. Parametrize η as $\eta = \psi(y,t)$, then the change of variables formula gives

$$\begin{aligned}
\int_{B_\delta(\omega)} T(\eta)^{-1}\,\mathrm{d}\eta &\ge \int_{\psi(A_1\times A_2)} T(\eta)^{-1}\,\mathrm{d}\eta \\
&= \iint_{A_1\times A_2} T(\psi(y,t))\,|\det(D(\psi(y,t)))|\,\mathrm{d}y\mathrm{d}t \\
&\gtrsim \iint_{A_1\times A_2} |t|^{-\gamma}\,\mathrm{d}y\mathrm{d}t \sim \int_{A_1}\Big(\int_0^{\operatorname{diam}(A_2)} r^{-\gamma+d-1}\,\mathrm{d}r\Big)\,\mathrm{d}y \\
&= \infty \int_{A_1} \mathrm{d}y = \infty,
\end{aligned}$$

where in the second inequality we used the assumption on the symbol T, and the fact that $D\psi$ is continuous, so $|\det(D\psi(y,t))| \gtrsim 1$ on the compact set $A_1 \times A_2$. In the last steps we simply used $\gamma \ge m$. Together with the $k=1$ case of part c of Theorem 3.4 this shows that $T(p)+V$ has at least one strictly negative eigenvalue if V is relatively form compact with respect to $T(p)$, $V \in L^1(\mathbb{R}^d)$, and $\int V\,\mathrm{d}x \le 0$.

Of course, we can pick arbitrarily many distinct points $\omega_l \in \Sigma$ and then the above shows that for arbitrarily many distinct points $\omega_l \in \Sigma$ one has

$$\int_{B_\delta(\omega_l)} T(\eta)^{-1}\,\mathrm{d}\eta = \infty$$

for all small enough δ, hence by monotonicity also for all $\delta > 0$. Thus if $V \ne 0$ and $V \le 0$, the assumption of part a of Theorem 3.4 is fulfilled for any $k \in \mathbb{N}$

and so $T(p) + V$ has infinitely many strictly negative bound states in this case. $\square$

Finally, we will prove Theorem 1.1, by reducing it to the $k = 1$ case of Theorem 3.4.c. For this, the following Lemma is useful.

Lemma 5.4. *Assume that $T : \mathbb{R}^d \to [0, \infty)$ is measurable and that there exists a compact set $M \subset \mathbb{R}^d$ such that (1.1) holds. Then there exists a point $\omega \in M$ such that T has a thick zero set near ω.*

Proof. By assumption we know that there exist a compact subset $M \subset \mathbb{R}^d$ with

$$\int_{M_\delta} T(\eta)^{-1}\,\mathrm{d}\eta = \infty$$

for all $\delta > 0$, where M_δ is the closed δ-neighborhood $M_\delta = \{\eta \in \mathbb{R}^d : \mathrm{dist}(\eta, M) \le \delta\}$.

Assume, by contradiction, that for every $\omega \in M$ there exists $\delta_\omega > 0$ with

$$\int_{B_{\delta_\omega}(\omega)} T(\eta)^{-1}\,\mathrm{d}\eta < \infty.$$

We clearly have

$$M \subset \bigcup_{\omega \in M} B_{\delta_\omega}(\omega)$$

and by compactness of M there exist a finite subcover, i.e., $N \in \mathbb{N}$ and points $\omega_l \in M$, $l = 1, \ldots, N$, such that

$$\mathcal{O} := \bigcup_{l=1}^{N} B_{\delta_{\omega_l}}(\omega_l) \supset M.$$

Clearly

$$\int_{\mathcal{O}} T(\eta)^{-1}\,\mathrm{d}\eta < \infty \tag{5.8}$$

by construction of $\mathcal{O}$. Since M is compact and contained in the open set $\mathcal{O}$, it has a strictly positive distance from the closed set $\mathcal{O}^c$. Thus there exists $\delta > 0$ such that $M_\delta \subset \mathcal{O}$, but then with (5.8) we arrive at the contradiction

$$\infty > \int_{\mathcal{O}} T(\eta)^{-1}\,\mathrm{d}\eta \ge \int_{M_\delta} T(\eta)^{-1}\,\mathrm{d}\eta = \infty.$$

Hence there exists $\omega \in M$ for which (3.1) holds. $\square$

Now we can give the short

Proof of Theorem 1.1. From the assumption of the Theorem and Lemma 5.4 we have that there exists a point $\omega \in \mathbb{R}^d$ such that T has a thick zero set near ω and hence the $k = 1$ case of Theorem 3.4.c applies. $\square$

6. Quantitative Bounds

6.1. Proof of Theorem 1.3: Quantitative Bound

Our approach is inspired by Cwikel's proof of the Cwikel–Lieb–Rozenblum inequality. We will give a slight modification of Cwikel's proof, which enables us to reduce his constant by a factor of two, see Lemma 6.1. For different modifications of Cwikel's proof see [57,59] and, in particular, [61]. Since $T(p)+V \geq T(p)-V_-$, in the sense of quadratic forms, the variational principle shows

$$N(T(p)+V) \leq N(T(p)-V_-)$$

where $N(A)$ denotes the number of negative eigenvalues of an operator A. Thus it is enough to bound the number of strictly negative eigenvalues of $T(p)-U$, where $U \geq 0$.

Since U is relatively form compact with respect to $T(p)$, the operator $\sqrt{U}(T(p)+E)^{-1/2}$ is compact for all $E > 0$, see [56, Lemma 6.28]. Let A be a compact operator with singular values $s_j(A)$, $j \in \mathbb{N}$, and let

$$n(A;1) := \#\{j \in \mathbb{N} :\ s_j(A) \geq 1\}$$

be the number of singular values of A greater or equal to one. Furthermore, for $E > 0$ let $N(T(p)-U, -E)$ be the number of eigenvalues of $T(p)-U$ which are less or equal to $-E$. The Birman–Schwinger principle, [53, Theorem 7.9.4], shows

$$N(T(p)-U, -E) = n(K_E; 1) \tag{6.1}$$

with the so-called Birman–Schwinger operator $K_E = \sqrt{U}(T(p)+E)^{-1}\sqrt{U}$, which is also a compact operator for any $E > 0$. Factorizing $K_E = A_E A_E^*$ with $A = f(x)g_E(p)$, where we introduced the multiplication operator $f = \sqrt{U}$ and the Fourier-multiplier $g_E(p) = (T(p)+E)^{-1/2}$, the Birman–Schwinger principle shows

$$N(T(p)-U, -E) = n(A_E; 1)$$

since the singular values of A_E are just the square roots of the positive eigenvalues of K_E. Since $N(T(p)-U) = \lim_{E\to 0+} N(T(p)-U, -E)$, we have to control $n(A_E;1)$ for small $E > 0$. For convenience, we will write g for g_E below. Following Cwikel, we decompose f and g as

$$f = \sum_{n\in\mathbb{Z}} f_n \quad \text{and } g = \sum_{n\in\mathbb{Z}} g_n,$$
$$\text{where } f_n := f\mathbf{1}_{\{\alpha r^{n-1} < f \leq \alpha r^n\}} \text{ and } g_n := g\mathbf{1}_{\{r^{n-1} < g \leq r^n\}}$$

for some $\alpha > 0$ and $r > 1$ and introduce the operators

$$B_{\alpha,r} := \sum_{k+l\leq 1} f_k(x)g_l(p), \quad H_{\alpha,r} := \sum_{k+l\geq 2} f_k(x)g_l(p).$$

We have the bounds ☐

Lemma 6.1. *For any $\alpha > 0$ and $r > 1$ and any functions $f, g \geq 0$ the operator B_α is bounded and its operator norm is bounded by*

$$\|B_{\alpha,r}\| \leq \alpha \left(\frac{r^4}{r^2 - 1} \right)^{1/2}.$$

Moreover, define $\widetilde{G}_\alpha(u)$ for $u, \alpha > 0$ by

$$\widetilde{G}_\alpha(u) := u^2 \int\limits_{ug(\eta)>\alpha} g(\eta)^2 \, \frac{\mathrm{d}\eta}{(2\pi)^d}.$$

If for $\alpha > 0$ we have $\int_{\mathbb{R}^d} G_\alpha(f(x))\,\mathrm{d}x < \infty$, then $H_{\alpha,r}$ is a Hilbert–Schmidt operator for all $r > 1$ and its Hilbert–Schmidt norm is bounded by

$$\|H_{\alpha,r}\|_{HS}^2 \leq \int_{\mathbb{R}^d} \widetilde{G}_\alpha(f(x))\,\mathrm{d}x.$$

Remarks 6.2. (i) If g is 'locally' L^2 in the sense that $g\mathbf{1}_{\{g>\alpha\}} \in L^2(\mathbb{R}^d)$ for any $\alpha > 0$, then $\widetilde{G}_\alpha(u) < \infty$ for all $u, \alpha > 0$ and $\lim_{u\to 0} \widetilde{G}_\alpha(u) = 0$ for any $\alpha > 0$.

(ii) Note that the right hand side of the bound on the operator norm of $B_{\alpha,r}$ is minimized by the choice $r = \sqrt{2}$ and the bound for the Hilbert–Schmidt norm of $H_{\alpha,r}$ is *independent* of $r > 1$. This improves the constant from Cwikel's original proof by a factor of two. We will use the choice $r = \sqrt{2}$ later.

Before we give the proof of the lemma, we state and prove an immediate consequence.

Corollary 6.3. *If $\int_{\mathbb{R}^d} \widetilde{G}_\alpha(f(x))\,\mathrm{d}x < \infty$ for all $\alpha > 0$, then the operator $f(x)g(p)$ is compact.*

Proof. By Lemma 6.1 we have $f(x)g(p) = B_\alpha + H_\alpha$, where B_α is bounded and H_α is a Hilbert–Schmidt, in particular, a compact operator. Since the operator norm of B_α is bounded by $\|B_\alpha\| \leq 2\alpha$, where we chose $r = 2$ for convenience, we see that $f(x)g(p)$ is the norm limit, as $\alpha \to 0$, of the compact operators H_α, so it must be compact. □

Proof of Lemma 6.1. The proof of the bound for the operator norm of $B_{\alpha,r}$ follows Cwikel's ideas closely, with the difference that we defined $B_{\alpha,r}$ slightly differently[7] than Cwikel in [11]. We give the short proof for the convenience of the reader and in order to implement a little trick, which allows to improve on Cwikel's constant by a factor of two:

[7]This slight change in definition is the reason why the bound on the Hilbert–Schmidt norm of $H_{\alpha,r}$ is *independent* of $r > 1$, so we can freely optimize the bound on $B_{\alpha,r}$ from Lemma 6.1 in $r > 1$. This was already noticed in [22].

Let $\Psi, \Phi \in L^2(\mathbb{R}^d)$, $\widetilde{f}_k := \alpha^{-1} r^{-k} f_k$ and $\widetilde{g}_l := r^{-l} g_l$. Then

$$\begin{aligned}\langle \Psi, B_{\alpha,r}\Phi\rangle &= \sum_{k+l\le 1} \langle f_k(x)\psi, g_l(p)\Phi\rangle = \alpha \sum_{k+l\le 1} r^{k+l} \langle \widetilde{f}_k\Psi, \widetilde{g}_l\widehat{\Phi}\rangle \\ &= \alpha \sum_{k\in\mathbb{Z}} \Big\langle \widetilde{f}_k\Psi, \sum_{n\le 1} r^n\, \widetilde{g}_{n-k}\widehat{\Phi}\Big\rangle\end{aligned}$$

Thus

$$\begin{aligned}|\langle \Psi, B_{\alpha,r}\Phi\rangle|\alpha &\le \sum_{k\in\mathbb{Z}} \Big|\Big\langle \widetilde{f}_k\Psi, \sum_{n\le 1} r^n\, \widetilde{g}_{n-k}\widehat{\Phi}\Big\rangle\Big| \le \alpha \sum_{k\in\mathbb{Z}} \big\|\widetilde{f}_k\Psi\big\| \Big\|\sum_{n\le 1} r^n\, \widetilde{g}_{n-k}\widehat{\Phi}\Big\| \\ &\le \alpha \Big(\sum_{k\in\mathbb{Z}} \|(\widetilde{f}_k\Psi\|^2\Big)^{1/2} \Big(\sum_{n\le 1}\Big\|\sum_{n\le 1} r^n\, \widetilde{g}_{n-k}\widehat{\Phi}\Big\|^2\Big)^{1/2} .\end{aligned}$$

Moreover, $0 \le \widetilde{f}_k \le 1$ and they have disjoint supports, so $\sum_{k\in\mathbb{Z}}(\widetilde{f}_k)^2 \le 1$ pointwise, hence

$$\sum_{k+l\in\mathbb{Z}} \|\widetilde{f}_k\psi\|^2 = \Big\langle \Psi, \sum_{k\in\mathbb{Z}} (\widetilde{f}_k)^2\Psi\Big\rangle \le \langle \Psi, \mathbf{1}_{\{f>0\}}\Psi\rangle \le \|\Psi\|^2.$$

For the second term, we note

$$\Big\|\sum_{n\le 1} r^n\, \widetilde{g}_{n-k}\widehat{\Phi}\Big\|^2 = \sum_{n_1,n_2\le 1} r^{n_1+n_2} \langle \widehat{\Phi}, \widetilde{g}_{n_1-k}\widetilde{g}_{n_2-k}\widehat{\Phi}\rangle = \sum_{n\le 1} r^{2n} \langle \widehat{\Phi}, (\widetilde{g}_{n-k})^2\widehat{\Phi}$$

since $\widetilde{g}_{n_1-k}$ and $\widetilde{g}_{n_2-k}$ have disjoint supports when $n_1 \neq n_2$. In addition, we also have $\sum_{k\in\mathbb{Z}}(\widetilde{g}_{n-k})^2 \le 1$ for any $n \in \mathbb{Z}$, hence

$$\sum_{k\in\mathbb{Z}} \Big\|\sum_{n\le 1} r^n\, \widetilde{g}_{n-k}\widehat{\Phi}\Big\|^2 = \sum_{n\le 1} r^{2n} \Big\langle \widehat{\Phi}, \sum_{k\in\mathbb{Z}} (\widetilde{g}_{n-k})^2\widehat{\Phi} \le \sum_{n\le 1} r^{2n}\|\widehat{\Phi}\|^2 = \frac{r^4}{r^2-1}\|\widehat{\Phi}\|^2$$

follows. So we get the bound

$$\|B_{\alpha,r}\| \le \alpha \left(\frac{r^4}{r^2-1}\right)^{1/2}$$

for the operator norm of $B_{\alpha,r}$.

The bound of the Hilbert–Schmidt norm of H_α is a simple calculation. It is convenient to consider the operator $\widetilde{H}_\alpha = f(x)\mathcal{F}^{-1}g(\eta)$, since $H_\alpha^* H_\alpha$ is unitarily equivalent to $\widetilde{H}_\alpha^*\widetilde{H}_\alpha$, so their Hilbert–Schmidt norms are the same. The advantage is that one can easily read off the kernel of $\widetilde{H}_\alpha$, for which we have the bound

$$\begin{aligned}|\widetilde{H}_t(x,\eta)| &\le (2\pi)^{-d/2} \sum_{k+l\ge 2} f_k(x) g_l(\eta) = (2\pi)^{-d/2} \sum_{k+l\ge 2} f_k(x) g_l(\eta) \mathbf{1}_{\{f(x)g(\eta)>\alpha\}} \\ &\le (2\pi)^{-d/2} f(x) g(\eta) \mathbf{1}_{\{f(x)g(\eta)>\alpha\}},\end{aligned}$$

since the supports of f_k, g_l, respectively, are pairwise disjoint and for (x, η) in the support of $f_k g_l$ we have $f(x)g(\eta) = f_k(x)g_l(\eta) > tr^{k+l-2} \ge t$ by construction of f_k and g_l and since $k + l \ge 2$ in the above sum. Thus with Tonelli's theorem one sees

$$\begin{aligned}\|H_\alpha\|_{HS}^2 = \|\widetilde{H}_\alpha\|_{HS}^2 &= \iint |\widetilde{H}_\alpha(x,\eta)|^2 \,\mathrm{d}x\mathrm{d}\eta \\ &\le (2\pi)^{-d} \iint_{f(x)g(\eta)>\alpha} f(x)^2 g(\eta)^2 \,\mathrm{d}\eta\mathrm{d}x \\ &= \int f(x)^2 \int_{f(x)g(\eta)>\alpha} g(\eta)^2 \,\frac{\mathrm{d}\eta\mathrm{d}x}{(2\pi)^d} = \int_{\mathbb{R}^d} \widetilde{G}_\alpha(f(x))\,\mathrm{d}x\end{aligned}$$

with

$$\widetilde{G}_\alpha(u) = u^2 \int_{ug(\eta)>\alpha} g(\eta)^2 \,\frac{\mathrm{d}\eta}{(2\pi)^d}.$$

as claimed. □

Now we come to the

Proof of Theorem 1.3. The usual arguments, see [56, Lemma 6.26] or [53, Theorem 7.8.3], show that the essential spectrum does not change, $\sigma_{\mathrm{ess}}(T(p)+V) = \sigma_{\mathrm{ess}}(T(p))$, when V is a relatively form compact perturbation of $T(p)$. That $\sigma_{\mathrm{ess}}(T(p)) = \sigma(T(p)) \subset [0,\infty)$ is clear, since $T(p)$ is a Fourier multiplier with a positive symbol T.

It remains to prove the bound (1.3): As already discussed in the beginning of this section, setting $f = V_-^{1/2}$ and $g = g_E = (T+E)^{-1/2}$, the Birman Schwinger principle and the variational theorem yield

$$\begin{aligned}N(T(p)+V,-E) &= n(\sqrt{V_-}(T(p)+E)^{-1/2};1) = \#\{n:\ s_n(f(x)g(p)) \ge 1\} \\ &\le \sum_{n\in\mathbb{N}} \frac{(s_n(f(x)g(p))-\mu)_+^2}{(1-\mu)^2} = \sum_{n\in\mathbb{N}} \frac{(s_n(B_\alpha+H_\alpha)-\mu)_+^2}{(1-\mu)^2}\end{aligned} \tag{6.2}$$

for any $0 \le \mu < 1$, where the inequality follows from the simple bound $(s-\mu)_+^2/(1-\mu)^2 \ge 1$ for all $s \ge 1$ and where we split $f(x)g(p) = B_\alpha + H_\alpha$, with the optimal choice of $r = 2$.

Ky–Fan's inequality for the singular values and the first part of Lemma 6.1 gives

$$s_n(B_\alpha + H_\alpha) \le s_1(B_\alpha) + s_n(H_\alpha) = \|B_\alpha\| + s_n(H_\alpha) \le 2\alpha + s_n(H_\alpha).$$

So choosing $\mu = 2\alpha$ in (6.2), we arrive at the bound

$$N(T(p)+V,-E) \le (1-2\alpha)^{-2}\|H_\alpha\|_{HS}^2 \le (1-2\alpha)^{-2}\int_{\mathbb{R}^d} \widetilde{G}_\alpha(f(x))\,\mathrm{d}x,$$

for all $0 < \alpha < 1/2$. Since $g = g_E = (T+E)^{-1/2}$ and $f = \sqrt{V_-}$ a straightforward calculation and a simple monotonicity argument shows

$$\begin{aligned}\widetilde{G}_\alpha(u) &= u^2 \int_{T+E<u^2/\alpha^2} \frac{1}{T(\eta)+E}\frac{\mathrm{d}\eta}{(2\pi)^d}\\ &\le u^2 \int_{T<u^2/\alpha^2} \frac{1}{T(\eta)}\frac{\mathrm{d}\eta}{(2\pi)^d} = \alpha^2 G\left(\frac{u^2}{\alpha^2}\right)\end{aligned}$$

with G from (1.2). So, since $f(x) = \sqrt{V_-(x)}$, we have

$$N(T(p)+V, -E) \le \frac{\alpha^2}{(1-2\alpha)^2}\int_{\mathbb{R}^d} G(V_-(x)/\alpha^2)\,\mathrm{d}x \tag{6.3}$$

and letting $E \to 0$ finishes the proof. $\square$

6.2. Proof of Theorem 1.5: Dichotomy

We start with

Lemma 6.4. *Under the conditions of Theorem 1.5 we have*

$$\begin{aligned}\int_{Z_\delta} \frac{1}{T}\,\mathrm{d}\eta = \infty \textit{ for some } \delta > 0 &\Longrightarrow \int_{T<u}\frac{1}{T}\,\mathrm{d}\eta = \infty \textit{ for all } u > 0\\ &\Longrightarrow \int_{Z_\delta}\frac{1}{T}\,\mathrm{d}\eta = \infty \textit{ for all } \delta > 0.\end{aligned}$$

Remark 6.5. Lemma 6.4 clearly shows

$$\int_{Z_\delta} \frac{1}{T}\,\mathrm{d}\eta = \infty \text{ for some } \delta > 0 \Longleftrightarrow \int_{Z_\delta}\frac{1}{T}\,\mathrm{d}\eta = \infty \text{ for all } \delta > 0$$

and

$$\int_{Z_\delta} \frac{1}{T}\,\mathrm{d}\eta < \infty \text{ for some } \delta > 0 \Longleftrightarrow \int_{T<u}\frac{1}{T}\,\mathrm{d}\eta = \infty \text{ for some } u > 0,$$

which explains Remark 1.6.iii.

Proof of Lemma 6.4. Note the simple identity

$$\begin{aligned}\int_{T<u}\frac{1}{T}\,\mathrm{d}\eta &= \int_{\{T<u\}\cap Z_\delta}\frac{1}{T}\,\mathrm{d}\eta + \int_{\{T<u\}\cap Z_\delta^c}\frac{1}{T}\,\mathrm{d}\eta\\ &= \int_{Z_\delta}\frac{1}{T}\,\mathrm{d}\eta - \int_{\{T\ge u\}\cap Z_\delta}\frac{1}{T}\,\mathrm{d}\eta + \int_{\{T<u\}\cap Z_\delta^c}\frac{1}{T}\,\mathrm{d}\eta\end{aligned} \tag{6.4}$$

where $\int_{\{T<u\}\cap Z_\delta^c}\frac{1}{T}\,\mathrm{d}\eta < \infty$ for all $\delta > 0$ and all small enough $u > 0$, depending on δ, because of (1.7). Also $\int_{\{T\ge u\}\cap Z_\delta^c}\frac{1}{T}\,\mathrm{d}\eta \le |Z_\delta|/u < \infty$ by assumption. So the left hand side of (6.4) is infinite for all small enough $u > 0$ if for some $\delta > 0$ we have $\int_{Z_\delta}\frac{1}{T}\,\mathrm{d}\eta = \infty$. But by monotonicity, then also $\int_{T<u}\frac{1}{T}\,\mathrm{d}\eta = \infty$ for all $u > 0$, which proves the first implication in Lemma 6.4.

On the other hand, once $\int_{T<u}\frac{1}{T}\,\mathrm{d}\eta = \infty$ for all $u > 0$, one sees from (6.4) that $\int_{Z_\delta}\frac{1}{T}\,\mathrm{d}\eta = \infty$ for any $\delta > 0$, since the last two terms in (6.4) are finite for all small enough $u > 0$. $\square$

Now we come to the

Proof of Theorem 1.5. For part (a) we note that by Lemma 6.4 one has

$$\int_{Z_\delta} \frac{1}{T}\,\mathrm{d}\eta = \infty \text{ for some } \delta > 0 \Longleftrightarrow \int_{Z_\delta} \frac{1}{T}\,\mathrm{d}\eta = \infty \text{ for all } \delta > 0$$

so one can use Theorem 1.1 to see that one weakly coupled bound states exist once (1.8) holds.

On the other hand, assume that (1.8) fails. Then, again by Lemma 6.4, we have $\int_{T<u} \frac{1}{T}\,\mathrm{d}\eta < \infty$ for all small enough $u > 0$. Thus $G(u)$ defined in (1.2) is finite for all small enough $u > 0$ and $\lim_{u\to 0+} G(u) = 0$. Then a simple argument, see Remark 1.4.i, yields a strictly negative potential V such that $T(p)+V$ has no negative spectrum. Thus condition (1.8) is equivalent to having weakly coupled bound states.

For part (b) we simply note that Lemma 6.4 shows that $\int_{Z_\delta} \frac{1}{T}\mathrm{d}\eta < \infty$ for some $\delta > 0$ implies $\int_{T<0} \frac{1}{T}\mathrm{d}\eta < \infty$ for all small enough $u > 0$. Then Theorem 1.3 shows that a quantitative bound on the number of strictly negative eigenvalues of $T(p) + V$ in the form (1.3) holds.

Conversely, assume that (1.9) fails. Then Theorem 1.1 applies. Thus weakly coupled bound states always exist for any non-trivial attractive potentials, hence no quantitative bound on the number of strictly negative bound states can exist. □

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Appendix A. Applications

In this section we discuss the following applications of Theorems 1.1 and 1.3 and their corollaries.

- A.1 Schrödinger operators with a fractional Laplacian.
- A.2 Relativistic one-particle operators with positive mass.
- A.3 Relativistic pair operators with positive mass.
- A.4 Ultra-relativistic pair operators.
- A.5 Relativistic pair operators: one heavy and one extremely light particle.
- A.6 Operators arising in the mathematical treatment of the Bardeen–Cooper–Schrieffer theory of superconductivity (BCS).
- A.7 Discrete Schrödinger operators on a lattice.

In all cases, except one, when there exists a finite bound for the number of bound states, these bounds agree, up to constants, exactly with what one would guess from semi-classics.

While not all of the applications we discuss in this appendix are completely new, they can be straightforwardly analyzed within our framework. We do not need to bound singular values of compact operator or show that a Birman–Schwinger type operator has a singular part. All we need is to analyze a real-valued integral, which, in most cases, is straightforward.

The one example, where we do not get a semi-classical type bound, is considered in Theorem A.12 in section A.3. In this *critical case* one *expects to have corrections* to the semi-classical picture and our quantum correction gives the right logarithmic correction term to the semi-classical picture. Moreover, the analysis of weakly coupled bound states in section A.4 solves a conjecture in [57].

We would also like to emphasize that Theorem 1.3 easily allows to get these semi-classical bounds even for kinetic energies which are *not homogenous*!

A.1 Schrödinger Operators with a Fractional Laplacian

We consider the operator $(-\Delta)^{\gamma/2} + V = |p|^\gamma + V$ in $\mathbb{R}^d$ assuming that V satisfies the conditions of Theorems 1.1 and 1.3. It follows immediately from Corollary 3.7 that

Theorem A.1. *Suppose $V \neq 0$ is an attractive potential in the sense that $\int V \mathrm{d}x \leq 0$. Then for $\gamma \geq d$ the operator $|p|^\gamma + V$ has at least one striclty negative eigenvalue.*

On the other hand, Theorem 1.3 implies for this operator that

Theorem A.2. *Assume that, $\gamma < d$ then the number of negative eigenvalues of the operator $|p|^\gamma + V$ satisfies*

$$N(|p|^\gamma + V) \le \left(\frac{4d}{d-\gamma}\right)^{2d/\gamma} \frac{d-\gamma}{d(4\gamma)^2} \frac{|B_1^d|}{(2\pi)^d} \int_{\mathbb{R}^d} V_-(x)^{d/\gamma}\, \mathrm{d}x, \tag{A.1}$$

where $|B_1^d|$ denotes the the volume of the unit ball in $\mathbb{R}^d$.

Proof. Inequality (A.1) follows from (1.3) for the optimal choice of $\alpha = \frac{d-\gamma}{4d}$. □

Of course, if $\gamma \ge d$, then there *cannot* be a quantitative bound on the number of negative eigenvalues, but for $\beta > 0$ the number of eigenvalues below $-\beta$, which we denote by $N(|p|^\gamma + V, -\beta)$, can be bounded.

Define, for $\gamma > d$ and $\beta > 0$,

$$G_{d,\gamma,\beta}(u) = \beta^{\frac{d-\gamma}{\gamma}} \frac{|B_1^d|}{(2\pi)^d}\, u \min\left(\left(\frac{u}{\beta} - 1\right)_+^{\frac{d}{\gamma}}, \frac{\pi d/\gamma}{\sin(\pi d/\gamma)}\right) \tag{A.2}$$

and, for $\gamma = d$,

$$G_{d,d,\beta}(u) = \frac{|B_1^d|}{(2\pi)^d}\, u \ln\left(1 + \left(\frac{u}{\beta} - 1\right)_+\right) \tag{A.3}$$

for $u \ge 0$. Then we have

Theorem A.3. *Assume that $\gamma \ge d$ and $\beta > 0$. Then for all $0 < \alpha < 1/2$*

$$N(|p|^\gamma + V, -\beta) \le \frac{\alpha^2}{(1-2\alpha)^2} \int_{\mathbb{R}^d} G_{d,\gamma,\beta}(\alpha^{-2} V_-(x))\, \mathrm{d}x.$$

Remarks A.4. (i) If $V_- \le \beta$, then $|p|^\gamma + V$ cannot have any spectrum below $-\beta$ and the bound from Theorem A.3 reflects this.

(ii) Note that $L^{\mathrm{cl}}_{0,d} = \frac{|B_1^d|}{(2\pi)^d}$ is one of the so-called classical Lieb–Thirring constants [23,33].

Proof. Of course, $N(|p|^\gamma + V, -\beta) = N(\beta + |p|^\gamma + V) = N(T(p) + V)$ with $T(\eta) = \beta + |\eta|^\gamma$ by a simple shifting argument. Thus Theorem 1.3 shows that a bound of the form of (1.3) holds for $N(|p|^\gamma + V, -\beta)$ with G given by

$$G(u) = u \int_{T<u} \frac{1}{T(\eta)} \frac{\mathrm{d}\eta}{(2\pi)^d} = u \int_{|\eta|^\gamma < u-\beta} \frac{1}{\beta + |\eta|^\gamma} \frac{\mathrm{d}\eta}{(2\pi)^d}. \tag{A.4}$$

We have

$$\int_{|\eta|^\gamma < \beta s} \frac{1}{\beta + |\eta|^\gamma}\, \mathrm{d}\eta = \beta^{\frac{d}{\gamma}-1} \int_{|\eta|^\gamma < s} \frac{1}{1 + |\eta|^\gamma}\, \mathrm{d}\eta = |S^{d-1}| \beta^{\frac{d}{\gamma}-1} \int_0^{s^{\frac{1}{\gamma}}} \frac{r^{d-1}}{1+r^\gamma}\, \mathrm{d}r$$

by scaling and going to spherical coordinates, $|S^{d-1}|$ is the surface area of the unit sphere in $\mathbb{R}^d$. Thus, if $\gamma = d$, then

$$G(u) = \frac{|S^{d-1}|}{d(2\pi)^d}\, u \ln\left(1 + \left(\frac{u}{\beta} - 1\right)_+\right)$$

and since $|B_1^d| = |S^{d-1}|/d$ this proves the claim for $\gamma = d$. If $\gamma > d$, then

$$\int_0^\infty \frac{r^{d-1}}{1+r^\gamma}\,\mathrm{d}r = \frac{1}{\gamma}\int_0^\infty t^{\frac{d}{\gamma}-1}(1+t)^{-1}\,\mathrm{d}t = \frac{\pi/\gamma}{\sin(\pi d/\gamma)}$$

by a contour integral over a keyhole contour encircling the positive real axis [2]. On the other hand, $\int_0^{s^{\frac{1}{\gamma}}} \frac{r^{d-1}}{1+r^\gamma}\,\mathrm{d}r \le \int_0^{s^{\frac{1}{\gamma}}} r^{d-1}\,\mathrm{d}r = \frac{s^{\frac{d}{\gamma}}}{d}$, so

$$\int_{|\eta|^\gamma < u-\beta} \frac{1}{\beta + |\eta|^\gamma}\frac{\mathrm{d}\eta}{(2\pi)^d} \le \frac{|S^{d-1}|}{d(2\pi)^d}\min\left(\left(\frac{u}{\beta}-1\right)_+^{\frac{d}{\gamma}}, \frac{\pi d/\gamma}{\sin(\pi d/\gamma)}\right)$$

which shows $G(u) \le G_{d,d,\beta}(u)$ if $\gamma > d$. This proves the theorem. □

A.2 Relativistic one-Particle Operators with Positive Mass

If one wants to include relativistic effects, one is often lead to pseudorelativistic operators where the kinetic energy is of the form $T_{c^2m}(p) = \sqrt{|p|^2 + c^4m^2} - c^2m$ [3,12,13,20,21,32,34]. Here m is the mass of the particle and c is the velocity of light. In the limit of $c \to 0$, i.e., the ultra-relativistic limit, and in the limit of vanishing mass, i.e., massless particles, this becomes simply the operator $|p|$, which was already discussed in Section A.1. For non-vanishing mass, one can set $c = 1$ by absorbing c^2 into m with a simple scaling argument. We have

Theorem A.5. *Let $d = 1$, or $d = 2$, $V \in L^1(\mathbb{R}^d)$ is relatively form compact with respect to $T_m(p)$ and an attractive potential in the sense that $V \ne 0$ and $\int V\,\mathrm{d}x \le 0$, then the operator $\sqrt{|p|^2+m^2} - m + V$ has at least one strictly negative eigenvalue.*

Proof. Since $\sqrt{|\eta|^2+m^2} - m = \frac{|\eta|^2}{2m} + O(\frac{|\eta|^4}{m^3})$, the claim follows from Corollary 3.6. □

For larger dimensions, we have a quantitative bound on the number of negative eigenvalues, counting multiplicity.

Theorem A.6. *For $d \ge 3$ and $m \ge 0$ let*

$$G_{d,m}(u) := \frac{d}{d-2}\frac{|B_1^d|}{(2\pi)^d}u^{d/2}(u+2m)^{d/2},$$

where $|B_1^d|$ denotes the volume of the ball of radius one in $\mathbb{R}^d$. Then the number of negative eigenvalues of $\sqrt{|p|^2+m^2} - m + V$ on $L^2(\mathbb{R}^d)$ satisfies, for any $0 < \alpha < \frac{1}{2}$,

$$N(\sqrt{|p|^2+m^2} - m + V) \le \frac{\alpha^2}{(1-2\alpha)^2}\int_{\mathbb{R}^d} G_{d,m}(\alpha^{-2}V_-(x))\,\mathrm{d}x. \qquad \text{(A.5)}$$

Remarks A.7. (i) We have

$$G_{d,m}(u) = \frac{d}{d-2}\frac{|B_1^d|}{(2\pi)^d}\left|\left\{\eta \in \mathbb{R}^d : T_m(\eta) < u\right\}\right|,$$

where $|A|$ denotes the volume of a Borel set $A \subset \mathbb{R}^d$. So up to a factor of $d/(d-2)$, the function $G_{d,m}$ is exactly what one would expect from a semi-classical guess.

(ii) In the limit $m \to 0$, one recovers, with a slightly worse constant, the bound from Theorem A.2 as long as $d \geq 3$.

(iii) Physical intuition suggests that for bound states large (negative) values of the potential correspond to a large momentum and small values to a small momentum. Since $\sqrt{|\eta|^2 + m^2} - m \simeq |\eta|$ for large and $\sqrt{|\eta|^2 + m^2} - m \simeq \frac{|\eta|^2}{2m}$ for small momentum, physical heuristics thus suggests that a pseudo-relativistic system has a *finite* number of bound states if the "large values" of V_- are in $L^d(\mathbb{R}^d)$ and the "small values" are in $L^{d/2}(\mathbb{R}^d)$. It is easy to see that

$$G_{d,m}(u) \sim \min(u,1)^{d/2} + \max(u,1)^d,$$

where the implicit constants depend only on m and d, so our bound (A.5) corroborates this physical heuristic argument quantitatively.

Proof of Theorem A.6. With $T(\eta) = T_m(\eta) = \sqrt{|\eta|^2 + m^2} - m$ we rewrite G from (1.2) as

$$\begin{aligned} G(u) &= u \int_{T<u} \frac{1}{T(\eta)} \frac{\mathrm{d}\eta}{(2\pi)^d} = \int_{T/u<1} \int_0^\infty s^{-2} \mathbf{1}_{\{T(\cdot)/u<s\}}(\eta)\, \mathrm{d}s \frac{\mathrm{d}\eta}{(2\pi)^d} \\ &= \frac{1}{(2\pi)^d} \int_0^\infty s^{-2} \big|\{T < \min(1,s)u\}\big|\, \mathrm{d}s \\ &= \frac{1}{(2\pi)^d} \left[\big|\{T<u\}\big| + u \int_0^u s^{-2} \big|\{T<s\}\big|\, \mathrm{d}s \right]. \end{aligned} \tag{A.6}$$

Since $\big|\{T < u\}\big| = \int_{|\eta|<(u(u+2m))^{1/2}} \mathrm{d}\eta = |B_1^d| u^{d/2}(u+2m)^{d/2}$, we get from (A.6)

$$G(u) = \frac{|B_1^d|}{(2\pi)^d} \left[u^{d/2}(u+2m)^{\frac{d}{2}} + u \int_0^u s^{\frac{d}{2}-2}(s+2m)^{d/2}\, \mathrm{d}s \right].$$

Using the simple bound

$$u \int_0^u s^{\frac{d}{2}-2}(s+2m)^{\frac{d}{2}}\, \mathrm{d}s \leq u(u+2m)^{\frac{d}{2}} \int_0^u s^{\frac{d}{2}-2}\, \mathrm{d}s = \frac{2}{d-2} u^{\frac{d}{2}}(u+2m)^{\frac{d}{2}},$$

we get the upper bound

$$G(u) \leq \frac{d}{d-2} \frac{|B_1^d|}{(2\pi)^d} u^{\frac{d}{2}}(u+2m)^{\frac{d}{2}} = G_{d,m}(u)$$

and Theorem 1.3 applies. □

A.3 Relativistic Pair Operators with Positive Masses

Considering two relativistic particles of masses $m_\pm$ interacting with each other one is lead to study the operator

$$\sqrt{p_+^2 + m_+^2} - m_+ + \sqrt{p_-^2 + m_-^2} - m_- + V(x_+ - x_-)$$

on $L^2(\mathbb{R}^{2d})$, with $p_\pm = -i\nabla_{x_\pm}$, the momenta of the first $(+)$ and second particle $(-)$, where $\mathbb{R}^{2d} \ni x = (x_+, x_-) \in \mathbb{R}^d \times \mathbb{R}^d$. See, for example, [28], where they study the essential spectrum, the extension of the famous HVZ

Theorem to semi-relativistic particles, for an arbitrary but fixed number of particles.

It turns out that, due to the fact that this operator does not transform in a simple way under Gallilei transformations, this system has some unusual features. Transforming this two-particle operator into center of mass coordinates, one gets a direct integral decomposition $\int_{\mathbb{R}^d}^{\oplus} H_{\mathrm{rel},m_\pm}(P)\,\mathrm{d}P$, see [28,57], where $H_{\mathrm{rel},m_\pm}(P)$ is the *pair-operator*

$$\begin{aligned} H_{\mathrm{rel},m_\pm}(P) &:= \sqrt{|\mu_+ P - q|^2 + \mu_+^2 M^2} + \sqrt{|\mu_- P + q|^2 + \mu_-^2 M^2} \\ &\quad - \sqrt{P^2 + M^2} + V(y) \\ &=: T_{P,M,\mu_\pm}(q) + V(y) \end{aligned} \tag{A.7}$$

on $L^2(\mathbb{R}^d)$. Here $y \in \mathbb{R}^d$ is the relative coordinate and $q = -i\nabla_y$ the relative momentum of the two particles, $M = m_- + m_+$ is the total mass, $P \in \mathbb{R}^3$ the total momentum, and we set $\mu_\pm := m_\pm / M$.

Note that the dependence on the total momentum is much more complicated than in the non-relativistic case, where the two particle operator

$$\frac{p_+^2}{2m_+} + \frac{p_-^2}{2m_-} + V(x_+ - x_-)$$

on $L^2(\mathbb{R}^d)$ is unitarily equivalent to a direct integral $\int_{\mathbb{R}^d}^{\oplus} H_{\text{non-rel}}(P)\,\mathrm{d}P$ with the non-relativistic pair-operator

$$H_{\text{non-rel}}(P) := \frac{M}{2m_+ m_-} p^2 + V(y) + \frac{P^2}{2M}.$$

Here the term $\frac{P^2}{2M}$ is simply the kinetic energy of the center of mass frame and the shift by $\frac{P^2}{2M}$ corresponds to the covariance of non-relativistic Schrödinger operators under Galilei transformations.

Bounds for the number of bound states for the relativistic pair-operator (A.7) were considered in [57]. Here we want to show how their results are an easy consequence of our approach. Moreover, in the following section, we will consider the ultra-relativistic pair-operator and prove a conjecture made in [57] concerning the limit of vanishing masses when both particles are ultra-relativistic in Section A.4. Moreover, we will also see how within our approach one can easily discuss a mixed relativistic-ultra relativistic case, where one particle has positive mass while the other one has zero mass, see Section A.5.

For positive masses and low dimensions we have

Theorem A.8. *Suppose that $V \neq 0$, $V \in L^1(\mathbb{R}^d)$ is relatively form-compact with respect to $T_{P,M,\mu_\pm}(p)$ and is an attractive potential in the sense that $\int V \mathrm{d}x \leq 0$. Then for $d = 1, 2$ and any $P \in \mathbb{R}^d$, $m_\pm > 0$ the operator $H_{rel,m_\pm}(P)$ has at least one negative eigenvalue.*

Proof. Using the Taylor expansion $\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+O(x^3)$ and $\mu_+ + \mu_- = 1$ one sees

$$\begin{aligned}
&T_{P,M,\mu_\pm}(\eta)\\
&\quad= \sqrt{P^2+M^2}\left(\mu_+\sqrt{1+\frac{\eta^2-2\mu_+P\cdot\eta}{\mu_+^2(P^2+M^2)}}+\mu_-\sqrt{1+\frac{\eta^2+2\mu_-P\cdot\eta}{\mu_-^2(P^2+M^2)}}-1\right)\\
&\quad= \frac{(P^2+M^2)\eta^2-(P\cdot\eta)^2}{2\mu_+\mu_-(P^2+M^2)^{3/2}}+O\left(\frac{|\eta|^4+\mu_\pm\eta^2|P\eta|}{\mu_\pm^3(P^2+M^2)^{3/2}}\right)\\
&\quad= \frac{M^2\eta^2+P^2\eta_\perp^2}{2\mu_+\mu_-(P^2+M^2)^{3/2}}+O\left(\frac{|\eta|^4+\mu_\pm\eta^2|P\eta|}{\mu_\pm^3(P^2+M^2)^{3/2}}\right)
\end{aligned}$$

where $M = m_+ + m_- > 0$, $\mu_\pm = m_\pm/M > 0$, and $\eta = sP + \eta_\perp$ with $s \in \mathbb{R}$ and $\eta_\perp$ orthogonal to P if $P \neq 0$. Thus the kinetic energy vanishes quadratically near $\eta = 0$ and Corollary 3.6 applies. □

To give a quantitative bound on the number of negative bound states we need a little bit more notation. Let $d \in \mathbb{N}$, $P \in \mathbb{R}^d$, $M \geq 0$, $g_{P,M} = \sqrt{P^2+M^2}$, $-1/2 \leq \widetilde{\mu} \leq 1/2$, and define

$$\begin{aligned}
G^d_{P,M,\widetilde{\mu}}(u) &:= \frac{3|B_1^d|}{(4\pi)^d}\frac{u^{d/2}(u+g_{P,M})\,(u+2g_{P,M})^{d/2}}{(u^2+2ug_{P,M}+M^2)^{1/2}}\\
&\quad\times\left(\frac{u^2+2ug_{P,M}+(1-4\widetilde{\mu}^2)M^2}{u^2+2ug_{P,M}+M^2}\right)^{d/2}. \qquad \text{(A.8)}
\end{aligned}$$

Remark A.9. The function $G^d_{P,M,\widetilde{\mu}}$ is quite natural. Up to a factor of three it is exactly what one would guess semi-classically, that is,

$$G^d_{P,M,\widetilde{\mu}}(u) = \frac{3}{(2\pi)^d}\left|\{\eta \in \mathbb{R}^d : T_{P,M,\mu_\pm}(\eta) < u\}\right|$$

with $\widetilde{\mu} = (\mu_- - \mu_+)/2$. For the convenience of the reader, we sketch the calculation of $\left|\{\eta \in \mathbb{R}^d : T_{P,M,\mu_\pm}(\eta) < u\}\right|$ from [57] in Appendix D.

In four and more dimensions we have a simple bound for the number of bound states of the relativistic pair-operator.

Theorem A.10. (Bound states in high dimension) *If $d \geq 4$, then the number of negative eigenvalues of the relativistic pair-operator $H_{rel,m_\pm}(P)$ on $L^2(\mathbb{R}^d)$ satisfies for $0 < \alpha < \frac{1}{2}$*

$$N(H_{rel,m_\pm}(P)) \leq \frac{\alpha^2}{(1-2\alpha)^2}\int_{\mathbb{R}^d} G^d_{P,M,\widetilde{\mu}}\left(\alpha^{-2}V_-(x)\right)\,\mathrm{d}x. \qquad \text{(A.9)}$$

Remark A.11. For positive total mass $M > 0$ we have

$$G^d_{P,M,\widetilde{\mu}}(u) \sim \min(u,1)^{d/2} + \max(u,1)^d$$

where the implicit constants depend only on the total momentum $P \in \mathbb{R}^d$ and the total mass $M > 0$. So the right hand side of (A.9) is finite if and only if the interaction potential is locally $L^d(\mathbb{R}^d)$ and globally $L^{d/2}(\mathbb{R}^d)$, as in the single particle case.

In the limit of zero total mass one has

$$G^d_{P,0}(u) = \lim_{M\to 0} G^d_{P,M,\widetilde{\mu}}(u) = \frac{3|B^d_1|}{(4\pi)^d} u^{(d-1)/2}(u+|P|)(u+2|P|)^{(d-1)/2}$$
$$\sim \min(u,1)^{(d-1)/2} + \max(u,1)^d$$

Compared to the massive case, where $V \in$ there is a *loss of one dimension* in the massless case: the negative part of the interaction potential V_- has to be locally in $L^d(\mathbb{R}^d)$ and globally in $L^{(d-1)/2}(\mathbb{R}^d)$ so that the right hand side of semi-classical bound (A.9) is finite.

Informally, setting $d = 3$, one sees that $G^3_{P,0}$ contains a term *linear* in u. This hints at the fact that in this case a quantitative bound on the number of bound states should not exist. As Theorem A.12 below shows, this is indeed the case. Moreover, any bound for positive masses $m_\pm > 0$ should diverge as $M = m_+ + m_- \to 0$. We will see in the next theorem, that this divergence is at most logarithmic in the limit of vanishing total mass M when $d = 3$. That such a divergent term is necessary is shown in Theorem A.13.

Proof of Theorem A.10. With

$$T(\eta) = T_{P,M,\mu_\pm}(\eta) = \sqrt{|\mu_+ P - \eta|^2 + \mu_+^2 M^2}$$
$$+ \sqrt{|\mu_- P + \eta|^2 + \mu_-^2 M^2} - \sqrt{P^2 + M^2} \tag{A.10}$$

we rewrite G from (1.2) as

$$G(u) = u \int_{T<u} \frac{1}{T(\eta)} \frac{\mathrm{d}\eta}{(2\pi)^d} = \frac{1}{(2\pi)^d} \int_0^\infty s^{-2} \big|\{T_{P,M,\mu_\pm} < \min(1,s)u\}\big| \,\mathrm{d}s$$
$$= \frac{1}{(2\pi)^d} \left\{ \big|\{T_{P,M,\mu_\pm} < u\}\big| + u \int_0^u s^{-2} \big|\{T_{P,M,\mu_\pm} < s\}\big| \,\mathrm{d}s \right\} \tag{A.11}$$

From Appendix D, see formula (D.2), we know

$$\big|\{T_{P,M,\mu_\pm} < u\}\big| = \frac{|B^d_1|}{2^d} \frac{u^{d/2}(u + g_{P,M})(u + 2g_{P,M})^{d/2}}{(u^2 + 2ug_{P,M} + M^2)^{1/2}}$$
$$\times \left(\frac{u^2 + 2ug_{P,M} + (1 - 4\widetilde{\mu}^2)M^2}{u^2 + 2ug_{P,M} + M^2} \right)^{d/2}$$

where $\widetilde{\mu} = (\mu_- - \mu_+)/2$ and $g_{P,M} = \sqrt{P^2 + M^2}$. So it is enough to show that

$$u \int_0^u s^{-2} \big|\{T_{P,M,\mu_\pm} < s\}\big| \,\mathrm{d}s \le 2\big|\{T_{P,M,\mu_\pm} < u\}\big|. \tag{A.12}$$

Since the map

$$0 \le u \mapsto \frac{u^2 + 2ug_{P,M} + (1 - 4\widetilde{\mu}^2)M^2}{u^2 + 2ug_{P,M} + M^2}$$

is increasing, we have for $d \geq 4$

$$\begin{aligned}
&2^d |B_1^d|^{-1} \int_0^u s^{-2} \big|\{T_{P,M,\mu_\pm} < s\}\big| \,\mathrm{d}s \\
&\quad \leq u^{\frac{d}{2}-2} (u + 2g_{P,M})^{d/2} \left(\frac{u^2 + 2ug_{P,M} + (1-4\widetilde{\mu}^2)M^2}{u^2 + 2ug_{P,M} + M^2} \right)^{d/2} \\
&\qquad \times \int_0^u \frac{s + g_{P,M}}{(s^2 + 2sg_{P,M} + M^2)^{1/2}} \,\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
\int_0^u \frac{s + g_{P,M}}{(s^2 + 2sg_{P,M} + M^2)^{1/2}} \,\mathrm{d}s &= (u^2 + 2ug_{P,M} + M^2)^{1/2} - M \\
&= \frac{u^2 + 2ug_{P,M}}{(u^2 + 2ug_{P,M} + M^2)^{1/2} + M} \\
&\leq \frac{2u(u + g_{P,M})}{(u^2 + 2ug_{P,M} + M^2)^{1/2}}.
\end{aligned}$$

Putting the above bounds together shows (A.12) and with Theorem 1.3 we conclude the proof. $\square$

In tandem with Theorem A.12, the following result shows that the relativistic pair-operator is "critical" in three dimensions.

Theorem A.12. (Bound states in dimension 3) *Let $P \in \mathbb{R}^3$, $m_\pm > 0$ and set $M = m_+ + m_-$, $\mu_\pm = m_\pm/M$, $\widetilde{\mu} = (\mu_- - \mu_+)/2$, $g_{P,M} = \sqrt{P^2 + M^2}$ and*

$$\begin{aligned}
&G^{\mathrm{mod}}_{P,M,\widetilde{\mu}}(u) \\
&\quad := G^3_{P,M,\widetilde{\mu}}(u) + u \frac{|B_1^3|}{2^{1/2}(2\pi)^3} (P^2 + M^2) \ln\left(\sqrt{1 + P^2/M^2} + \sqrt{2 + P^2/M^2} \right).
\end{aligned}$$

Then the number of negative eigenvalues of the relativistic pair-operator $H_{\mathrm{rel},m_\pm}(P)$ on $L^2(\mathbb{R}^3)$ satisfies for $0 < \alpha < \frac{1}{2}$

$$N(H_{\mathrm{rel},m_\pm}(P)) \leq \frac{\alpha^2}{(1-2\alpha)^2} \int_{\mathbb{R}^3} G^{\mathrm{mod}}_{P,M,\widetilde{\mu}} \left(\alpha^{-2} V_-(x) \right) \,\mathrm{d}x. \tag{A.13}$$

Proof. As in the proof of Theorem A.10, Theorem 1.3 yields a bound on the number of bound states with

$$G(u) = \frac{1}{(2\pi)^3} \left\{ \big|\{T_{P,M,\mu_\pm} < u\}\big| + u \int_0^u s^{-2} \big|\{T_{P,M,\mu_\pm} < s\}\big| \,\mathrm{d}s \right\}$$

where now $|A|$ denotes the volume of a Borel set $A \subset \mathbb{R}^3$. So it is enough to show

$$\begin{aligned}
&u \int_0^u s^{-2} \big|\{T_{P,M,\mu_\pm} < s\}\big| \,\mathrm{d}s \\
&\quad \leq 2 \big|\{T_{P,M,\mu_\pm} < u\}\big| + u 2^{5/2} g^2 |B_1^3| \ln(g/M + \sqrt{1 + (g/M)^2})
\end{aligned} \tag{A.14}$$

where we abbreviated $g = \sqrt{P^2+M^2}$ and $\tau = |P|/M$. Using (D.2) for $\big|\{T_{P,M,\mu_\pm} < s\}\big|$, we have, similarly as in the proof of Theorem A.10,

$$\begin{aligned}&\frac{2^3}{|B_1^3|}\int_0^u s^{-2}\big|\{T_{P,M,\mu_\pm} < s\}\big|\,\mathrm{d}s\\ &\qquad\qquad \le \left(\frac{u^2+2ug+(1-4\widetilde{\mu}^2)M^2}{u^2+2ug+M^2}\right)^{3/2}\int_0^u \frac{s^{-1/2}(s+2g)^{3/2}(s+g)}{(s^2+2sg+M^2)^{1/2}}\,\mathrm{d}s\end{aligned}$$

and an integration by parts shows

$$\begin{aligned}&\int_0^u \frac{s^{-1/2}(s+2g)^{3/2}(s+g)}{(s^2+2sg+M^2)^{1/2}}\,\mathrm{d}s\\ &\quad= \Big[s^{-1/2}(s+2g)^{3/2}\left((s^2+2sg+M^2)^{1/2}-M\right)\Big]_0^u\\ &\qquad -\int_0^u\left(-\frac{1}{2}s^{-3/2}(s+2g)^{3/2}+s^{-1/2}\frac{3}{2}(s+2g)^{1/2}\right)\left((s^2+2sg+M^2)^{1/2}-M\right)\mathrm{d}s\\ &\quad= \frac{u^{1/2}(u+2g)^{5/2}}{(u^2+2ug+M^2)^{1/2}+M}+\int_0^u s^{-3/2}(s+2g)^{1/2}\frac{(g-s)s(s+2g)}{(s^2+2sg+M^2)^{1/2}+M}\,\mathrm{d}s\\ &\quad\le \frac{2u^{1/2}(u+g)(u+2g)^{3/2}}{(u^2+2ug+M^2)^{1/2}}+\int_0^g \frac{s^{-1/2}(s+2g)^{3/2}(g-s)}{(s^2+2sg+M^2)^{1/2}}\,\mathrm{d}s.\end{aligned}$$

Since $(s+M^2/g)(s+g)\le s^2+2sg+M^2$, we have

$$\begin{aligned}&\int_0^g \frac{s^{-1/2}(s+2g)^{3/2}(g-s)}{(s^2+2sg+M^2)^{1/2}}\,\mathrm{d}s\\ &\quad\le \int_0^g \frac{s^{-1/2}(s+2g)^{3/2}(g-s)}{(s+g)^{1/2}(s+M^2/g)^{1/2}}\,\mathrm{d}s\\ &\quad\le 2^{3/2}g^2\int_0^g s^{-1/2}(s+M^2/g)^{-1/2}\,\mathrm{d}s = 2^{5/2}g^2\int_0^{\sqrt{g}}(s^2+M^2/g)^{-1/2}\,\mathrm{d}s\\ &\quad= 2^{5/2}g^2\int_0^{g/M}(s^2+1)^{-1/2}\,\mathrm{d}s = 2^{5/2}g^2\ln(g/M+\sqrt{1+(g/M)^2})\end{aligned}$$

where we also used that $s\mapsto \frac{(s+2g)^{3/2}(g-s)}{(s+g)^{1/2}}$ is decreasing on $[0,g]$. The last three bounds together with the trivial bound $\frac{u^2+2ug+(1-4\widetilde{\mu}^2)M^2}{u^2+2ug+M^2}\le 1$ show that (A.14) holds, which finishes the proof. □

A.4 Ultra-relativistic Pair Operators

In the ultra-relativistic limit one takes the velocity of light to zero. Equivalently, one takes the limit of vanishing masses. The kinetic energy symbol of the pair operator becomes

$$T_{P,0,\mu_\pm}(\eta) = \lim_{M\to 0} T_{P,M,\mu_\pm}(\eta) = |\mu_+P-\eta|+|\mu_-P+\eta|-|P|. \tag{A.15}$$

The triangle inequality shows $T_{P,0,\mu_\pm}(\eta) = |\mu_+P-\eta|+|\mu_-P+\eta|-|P| \ge |P|-|P| = 0$, so T is positive.[8] It was noted already in [57] that the kinetic energy $T_{P,0,\mu_\pm}$ is, for $P\neq 0$, zero on a 'large set': If η is parallel to P we can

[8]This also follows from the fact that $T_{P,0,\mu_\pm}$ is the limit of positive terms.

write it as $\eta = sP$ with $s \in \mathbb{R}$, and then, assuming also $-\mu_- \le s \le \mu_+$, one has

$$\begin{aligned} T_{P,0,\mu_\pm}(\eta) &= |\mu_+ P - sP| + |\mu_- P + sP| - |P| \\ &= (\mu_+ - s)|P| + (\mu_- + s)|P| - |P| = 0, \end{aligned}$$

since $\mu_- + \mu_+ = 1$, so the kinetic energy symbol has the whole segment $[-\mu_- P, \mu_+ P]$ as a zero set. This observation led to the conjecture in [57] that the ultra-relativistic pair operator $T_{P,0,\mu_\pm} + V$ should possess weakly coupled bound states in three dimensions whenever the total momentum does not vanish. Our next theorem confirms this.

Theorem A.13. *Assume that $d = 3$, the total momentum $P \in \mathbb{R}^3 \backslash \{0\}$, V is relatively form compact with respect to $T_{P,0,\mu_\pm}$ and an attractive potential in the sense that $V \ne 0$ and $\int V \,\mathrm{d}x \le 0$. Then $T_{P,0,\mu_\pm} + V$ has at least one strictly negative eigenvalue and if, in addition, $V \le 0$ then it has infinitely many strictly negative eigenvalues.*

Proof. We already convinced ourselves that $T_{P,0,\mu_\pm}$ is non-negative and zero on the line segment $[-\mu_- P, \mu_+ P]$.

Now let $\eta = sP + \eta_\perp$, where $-\mu_- < s < \mu_+$ and $\eta_\perp$ is perpendicular to P. Then

$$\begin{aligned} T_{P,0,\mu_\pm}(\eta) &= \sqrt{(\mu_+ - s)^2 P^2 + \eta_\perp^2} + \sqrt{(\mu_- + s)^2 P^2 + \eta_\perp^2} - |P| \\ &= \frac{\eta_\perp^2}{2(\mu_+ - s)(\mu_- + s)|P|} + O\left(\frac{|\eta_\perp|^4}{|\mu_\pm \mp s|^3 |P|^3} \right) \end{aligned}$$

using the Taylor expansion $\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2)$.

Since the line segment $[-\mu_- P, \mu_+ P]$ on which the kinetic energy vanishes has codimension 2 in $\mathbb{R}^3$, we can apply Corollary 3.7. $\square$

In dimension $d \ge 4$, there is a useful bound on the number of bound states even in the ultra-relativistic limit. It has the interesting feature that even though the kinetic energy of the massless pair still remembers, through $\mu_\pm$, the ratio of the two masses before taking the limit of vanishing total mass, the semiclassical bound is independent of this.

Theorem A.14. *Let $d \ge 4$ and define*

$$G^d_{P,0}(u) := \frac{(d-1)|B^d_1|}{(d-3)(4\pi)^d} u^{(d-1)/2} (u + |P|) \left(u + 2|P|\right)^{(d-1)/2}.$$

Then the number of negative bound states of the ultra-relativistic pair operator $T_{P,0,\mu_\pm} + V$ on $L^2(\mathbb{R}^d)$ is bounded by

$$N(T_{P,0,\mu_\pm}(q) + V) \le \frac{\alpha^2}{(1-2\alpha)^2} \int_{\mathbb{R}^d} G^d_{P,0}\left(\alpha^{-2} V_-(x)\right) \mathrm{d}x, \tag{A.16}$$

for all $0 < \alpha < 1/2$.

Remarks A.15. (i) Note $G^d_{P,0}(u) = \frac{d-3}{d-1} \lim_{M\to 0} G^d_{P,M,\widetilde{\mu}}(u) < \lim_{M\to 0} G^d_{P,M,\widetilde{\mu}}(u)$. So Theorem A.14 improves upon the $M \to 0$ limit in Theorem A.10.

(ii) Since $G^d_{P,0}(u) \lesssim \min(u,1)^{(d-1)/2} + \max(u,1)^d$, one needs $\min(V_-,1) \in L^{(d-1)/2}$ and $\max(V_-,1) \in L^d$ in order that the right hand side of (A.16) is finite.

A.5 Relativistic Pair Operators: One Heavy and One Massless Particle

Considering a pair of relativistic particles in the center of mass frame when the particles have *very* different masses, say the first one is much heavier than the other, it makes sense to consider the idealized limit where $m_+ = m$ is kept fixed, while $m_- \to 0$. In this case $\mu_+ = m_+/(m_+ + m_-) \to 1$, $\mu_- m_-/(m_+ m_-) \to 0$, and $\widetilde{\mu} = (\mu_- - \mu_+)/2 \to -1/2$, so the kinetic energy of the pair becomes

$$T_{P,m,1,0}(\eta) := \lim_{\mu_- \to 0} T_{P,M,\mu_\pm}(\eta) = \sqrt{|P-\eta|^2 + m^2} + |\eta| - \sqrt{P^2+m^2}. \tag{A.17}$$

In this case

$$|\{T_{P,m,1,0} < u\}| = \lim_{\mu_+ \to 1} \left|\{T_{P,M,\mu_\pm} < u\}\right| = \frac{|B^d_1|}{2^d} \frac{u^{d/2}(u+g_{P,m})\,(u+2g_{P,m})^d}{(u^2+2ug_{P,m}+m^2)^{(d+1)/2}} \tag{A.18}$$

where we recall $g_{P,m} = \sqrt{P^2+m^2}$. We define

$$\begin{aligned} G^d_{P,m,1}(u) := \frac{|B^d_1|}{(4\pi)^d} \Bigg\{ & \frac{u^{d/2}(u+g_{P,m})(u+2g_{P,m})^d}{(u^2+2ug_{P,m}+m^2)^{(d+1)/2}} \\ & + 2^d \left(\frac{g_{P,m}}{m}\right)^d \frac{u^{d/2}(u+2g_{P,m})}{(u^2+2ug_{P,m}+m^2)^{1/2}+m} \Bigg\}. \end{aligned} \tag{A.19}$$

With this function we have

Theorem A.16. *For all $d \geq 2$ the number of negative bound states of the relativistic pair operator $T_{P,m,1,0}(q) + V$ on $L^2(\mathbb{R}^d)$, describing one heavy and one massless particle, is bounded by*

$$N(T_{P,m,1,0}(q)+V) \leq \frac{\alpha^2}{(1-2\alpha)^2} \int_{\mathbb{R}^d} G^d_{P,m,1}\left(\alpha^{-2}V_-(x)\right) \mathrm{d}x, \tag{A.20}$$

for all $0 < \alpha < 1/2$.

Proof. Using a by now familiar argument, Theorem 1.3 yields a bound for $N(T_{P,m,1,0}(q)+V)$ with

$$G(u) = \frac{1}{(2\pi)^d} \left\{ \left|\{T_{P,m,1,0} < u\}\right| + u \int_0^u s^{-2} \left|\{T_{P,m,1,0} < s\}\right| \mathrm{d}s \right\}$$

and we have (A.18) for $\left|\{T_{P,m,1,0} < u\}\right|$. Since the map $0 \leq s \mapsto \frac{(s+2g_{P,m})^2}{(s^2+2sg_{P,m}+m^2)}$ is decreasing,

$$\begin{aligned}
&|B_1^d|^{-1} \int_0^u s^{-2} \left|\{T_{P,m,1,0} < s\}\right| \mathrm{d}s \\
&\quad = \frac{1}{2^d} \int_0^u \frac{s^{\frac{d}{2}-2}(s+g_{P,m})\,(s+2g_{P,m})^d}{(s^2+2sg_{P,m}+m^2)^{(d+1)/2}} \mathrm{d}s \\
&\quad \leq u^{\frac{d}{2}-2} \left(\frac{g_{P,m}}{m}\right)^d \int_0^u \frac{s+g_{P,m}}{(s^2+2sg_{P,m}+m^2)^{1/2}} \mathrm{d}s \\
&\quad = u^{\frac{d}{2}-2} \left(\frac{g_{P,m}}{m}\right)^d \left((u^2+2ug_{P,m}+m^2)^{1/2} - m\right) \\
&\quad = u^{\frac{d}{2}-1} \left(\frac{g_{P,m}}{m}\right)^d \frac{u+2g_{P,m}}{(u^2+2ug_{P,m}+m^2)^{1/2}+m}.
\end{aligned}$$

So $G(u) \leq G^d_{P,m,1}(u)$ for all $u \geq 0$. □

A.6 BCS Type Operators

In the Bardeen–Cooper–Schrieffer theory of super-conductivity the symbol of the kinetic energy is given by the function

$$K_{\beta,\mu}(\eta) = (\eta^2 - \mu)\frac{e^{\beta(\eta^2-\mu)}+1}{e^{\beta(\eta^2-\mu)}-1}$$

where $\beta = \frac{1}{T}$ is the inverse temperature and μ the chemical potential, i.e., the Fermi energy of the system, see [19] for a review. A-priori, the function $\mathbb{R}^d \ni \eta \mapsto K_{\beta,\mu}(\eta)$ is only defined for $\eta^2 \neq \mu$, but we can extend it to $\eta^2 = \mu$, by setting $K_{\beta,\mu}(\eta) = 2\beta^{-1}$ whenever $\eta^2 = \mu$. Extended in this way, $K_{\beta,\mu}$ is even $\mathcal{C}^\infty(\mathbb{R}^d)$, see the proof of Theorem A.17 below, and

$$\begin{aligned}
&\lim_{\eta\to\infty} K_{\beta,\mu}(\eta) = \infty,\ K_{\beta,\mu}(\eta) \geq 2\beta^{-1} \text{ for all } \eta \in \mathbb{R}^d \text{ and} \\
&K_{\beta,\mu}(\eta) = 2\beta^{-1} \Leftrightarrow |\eta| = \sqrt{\mu}.
\end{aligned}$$

Hence $\sigma(K_{\beta,\mu}(p)) = \sigma_{\mathrm{ess}}(K_{\beta,\mu}(p)) = [2\beta^{-1},\infty)$.

The function $K_{\beta,\mu}$ decreases pointwise in $\beta > 0$ and the limit of the kinetic energy as $\beta \to \infty$, i.e., the zero temperature limit, is given by

$$K_{\infty,\mu}(p) = |p^2 - \mu|.$$

In In BCS theory the operator

$$K_{\beta,\mu}(p) + V,$$

models the binding of Cooper pairs of electrons, where V describes the interaction of electrons. The free kinetic energy is modified to the above form to take into account the filled Fermi sea and finite temperature effects, μ is the Fermi energy.

The critical inverse temperature is given by

$$T_{\mathrm{cr}}(V)^{-1} := \beta_{\mathrm{cr}}(V) := \sup\{\beta > 0 : \inf \sigma(K_{\beta,\mu}(p) + V) \geq 0\}. \tag{A.21}$$

It was shown in [17] that the BCS gap equation has a non-trivial solution if $\beta > \beta_{\mathrm{cr}}(V)$ while for $0 \le \beta \le \beta_{\mathrm{cr}}(V)$ it does not. Thus the phase transition from a normal state to the superconducting state is determined by $\beta_{\mathrm{cr}}(V)$ and there is a phase transition at positive temperature if and only if $\beta_{\mathrm{cr}}(V) > 0$. Our method yields a painless simple criterion for it.

Theorem A.17. *Assume that $V \ne 0$, $\int V\,\mathrm{d}x \le 0$, and the Fermi energy $\mu > 0$. Then*

- (i) *The operator $|p^2 - \mu| + V$ in $\mathbb{R}^d$, $d \ge 2$ has at least one strictly negative eigenvalue and if, in addition, $V \le 0$, then it has infinitely many strictly negative eigenvalues.*
- (ii) *For all $\beta > 0$ the operator $K_{\beta,\mu}(p) + V$ has at least one eigenvalue strictly below $2\beta^{-1}$ and if, in addition, $V \le 0$, then it has infinitely many eigenvalues strictly below $2\beta^{-1}$. Its ground state eigenvalue is strictly decreasing and becomes strictly negative for large enough $\beta > 0$.*
- (iii) *$\beta_{\mathrm{cr}}(V)$ defined in (A.21) is finite, hence the critical temperature $T_{\mathrm{cr}}(V) > 0$. Moreover, if $0 < \beta < \beta_{\mathrm{cr}}$, then $\inf \sigma(K_{\beta,\mu}(p) + V) > 0$, and if $\beta > \beta_{\mathrm{cr}}$, then $\inf \sigma(K_{\beta,\mu}(p) + V) < 0$.*

Remarks A.18. (i) Our theorem shows that for arbitrary weak attractive interaction, even in the limiting case where $V \ne 0$ but $\int V\,\mathrm{d}x = 0$, Cooper pairs will always bind. This is a key ingredient for the BCS theory of superconductivity.

- (ii) In [16,18] a criterion was proven which implies positivity of the critical temperature for the BCS model for potentials λV for arbitrary small coupling λ once a suitable integral operator has a strictly negative eigenvalue. However, their approach, modeled after the one of Simon [50], identifies a singular part of the Birman–Schwinger operator, which forces weakly eigenvalues to exists. But, in order that this 'singular part' is not vanishing, this requires $\widehat{V}$ to be non-vanishing in a centered ball $B_{2\sqrt{\mu}}(0) = \{\eta \in \mathbb{R}^d : |\eta| < 2\sqrt{\mu}\}$. On the other hand, our Theorem shows that the full Birman–Schwinger operator is singular, even for potentials whose Fourier transform vanishes on $B_{2\sqrt{\mu}}(0)$: Just take for $\widehat{V}$ any spherically symmetric Schwartz function which is supported on a centered annulus disjoint from $B_{2\sqrt{\mu}}(0)$. Then our Theorem A.17 shows the existence of weakly coupled bound states, whereas the criteria in [16,18] are not applicable.
- (iii) According to Theorem 1.1 the operator $|p^2 - \mu| + V$ has weakly coupled bound states for arbitrarily small V also in the one-dimensional case. However, in this case the number of negative eigenvalues is finite.
- (iv) One can generalize the results of Theorem A.17 (i) and consider the operator $|p^2 - \mu|^\gamma + V$ for $\gamma > 0$ and $d \ge 2$. Theorem 1.1 implies in this case that the operator has an infinite number of weakly coupled bound states for all $\gamma \ge 1$, independently of $d \ge 2$. On the other hand, for $\gamma < 1$ Theorem 1.3 implies a quantitive bound on the number of negative eigenvalues.

Proof. Of course, part (iii) follows from part (ii), since the supremum on the right hand side of (A.21) is finite once the lowest eigenvalue of $K_{\beta,\mu}(p)+V$ is strictly negative for large enough $\beta > 0$.

For part (i) we simply note that the zero set of $\eta \mapsto |\eta^2 - \mu|$ is equal to the centered sphere $S^{d-1}_{\sqrt{\mu}}$ of radius $\sqrt{\mu} > 0$ and

$$|\eta^2 - \mu| = (|\eta| - \sqrt{\mu})(|\eta| + \sqrt{\mu}) \sim \operatorname{dist}\left(\eta, S^{d-1}_{\sqrt{\mu}}\right).$$

Since $S^{d-1}_{\sqrt{\mu}}$ has codimension 1 in $\mathbb{R}^d$ Corollary 3.7 applies.

Instead of using Corollary 3.7, we could use Theorem 3.4, since for any point $\omega \in S^{d-1}_{\sqrt{\mu}}$ one can easily see that $\int_{B_\delta(\omega)} |\eta^2 - \mu|^{-1}\, \mathrm{d}\eta = \infty$ for any $\delta > 0$.

For the proof of part (ii) consider the map $\mathbb{R} \ni a \mapsto a\frac{e^{\beta a}+1}{\mathrm{e}^{\beta a}-1}$, at first defined only for $a \neq 0$. Since

$$a\frac{e^{\beta a}+1}{\mathrm{e}^{\beta a}-1} = \beta^{-1}\frac{e^{\beta a}+1}{\frac{\mathrm{e}^{\beta a}-1}{\beta a}} \to 2\beta^{-1} \text{ as } a \to 0,$$

we can extend this to all $a \in \mathbb{R}$ by continuity. Moreover, this map is clearly infinitely often differentiable for $a \neq 0$, close to zero a short calculation reveals

$$a\frac{e^{\beta a}+1}{\mathrm{e}^{\beta a}-1} - 2\beta^{-1} = \frac{\beta a(e^{\beta a}+1) - 2\frac{e^{\beta a}-1}{\beta a}}{\frac{\mathrm{e}^{\beta a}-1}{\beta a}} = \frac{1}{6\beta}\big((\beta a)^2 + O((\beta a)^3)\big) \quad \text{(A.22)}$$

and one sees that it is even infinitely often differentiable for all $a \in \mathbb{R}$ and growing linearly in a for large a. Moreover,

$$\frac{\partial}{\partial a}\left(a\frac{e^{\beta a}+1}{e^{\beta a}-1}\right) = \frac{1}{2}e^{-\beta a}\frac{\sinh(\beta) - \beta a}{(\sinh(\beta a))^2} = \begin{cases} > 0 & \text{for } a > 0 \\ < 0 & \text{for } a < 0 \end{cases},$$

so

$$K_{\beta,\mu}(\eta) \geq 2\beta^{-1} \text{ for all } \eta \in \mathbb{R}^d \text{ and } K_{\beta,\mu}(\eta) = 2\beta^{-1} \Leftrightarrow |\eta| = \sqrt{\mu},$$

that is, $K_{\beta,\mu}$ attains its minimal value $2\beta^{-1}$ exactly at the sphere $S^{d-1}_{\sqrt{\mu}}$.

Furthermore,

$$\frac{\partial}{\partial \beta}a\frac{e^{\beta a}+1}{e^{\beta a}-1} = \begin{cases} -2\frac{a^2 e^{\beta a}}{(e^{\beta a}-1)^2} & \text{if } a \neq 0 \\ -2\beta^{-2} & \text{if } a = 0 \end{cases} < 0.$$

So the symbol $K_{\beta,\mu}(\eta)$ is strictly decreasing in $\beta > 0$. In particular, $K_{\beta,\mu}(\eta) > |\eta^2 - \mu|$ for all $\beta > 0$ and $\eta \in \mathbb{R}^d$.

The asymptotics (A.22) shows

$$K_{\beta,\mu}(\eta) - 2\beta^{-1} = \frac{\beta}{6}\operatorname{dist}\left(\eta, S^{d-1}_{\sqrt{\mu}}\right)^2 + O\left(\beta^2 \operatorname{dist}\left(\eta, S^{d-1}_{\sqrt{\mu}}\right)^3\right),$$

and again Corollary 3.7, or Theorem 3.4, show that $K_{\beta,\mu}(p)+V$ has infinitely many eigenvalues strictly below $2\beta^{-1}$.

Let $E_\beta := \inf \sigma(K_{\beta,\mu}(p)+V)$ be the ground state energy of $K_{\beta,\mu}(p)+V$. We claim that it is strictly decreasing in $\beta > 0$ and $\lim_{\beta\to\infty} E_\beta = E_\infty = \inf(|P^2 - \mu| + V) < 0$.

Let $\beta_2 > \beta_1 > 0$ and let ψ_1 be an eigenfunction of $K_{\beta_1,\mu}(p) + V$ corresponding to the ground state energy E_{β_1}. Then the variational principle and the strict monotonicity of the symbol $K_{\beta,\mu}(\eta)$ in $\beta > 0$ implies

$$\begin{aligned} E_{\beta_2} - E_{\beta_1} &\le \langle \psi_1, K_{\beta_2,\mu}(p) + V)\psi_1\rangle - \langle \psi_1, K_{\beta_1,\mu}(p) + V)\psi_1\rangle \\ &= \langle \widehat{\psi}_1, (K_{\beta_2,\mu}(\cdot) - K_{\beta_1,\mu}(\cdot))\widehat{\psi}_1\rangle < 0, \end{aligned}$$

so the ground state energy is strictly decreasing in $\beta > 0$. Moreover this implies $E_\beta > E_\infty := \inf \sigma(|p^2 - \mu| + V)$, the ground state energy of $|p^2 - \mu| + V$ and, again by the variational principle, letting ψ_∞ be a ground state of $|p^2 - \mu| + V$, we have

$$E_\beta \le \langle \psi_\infty, (K_{\beta,\mu} + V)\psi_\infty\rangle \to \langle \psi_\infty, (|p^2 - \mu| + V)\psi_\infty\rangle = E_\infty < 0 \text{ as } \beta \to \infty.$$

So E_β decreases strictly to $E_\infty < 0$. In particular, there is a unique $\beta_{\mathrm{cr}} > 0$ such that $\inf \sigma(K_{\beta,\mu}(p)+V) > 0$ for all $0 < \beta < \beta_{\mathrm{cr}}$ and $\inf \sigma(K_{\beta,\mu}(p)+V) < 0$ for all $\beta > \beta_{\mathrm{cr}}$. $\square$

A.7 Discrete Schrödinger Operators

We give the details of the method for discrete Schrödinger operators on $\mathbb{Z}^d$. In principle, one can consider a general d-dimensional lattice. One just has to use the dual lattice and adjust the notion of the discrete Fourier transform accordingly.

On $l^2(\mathbb{Z}^d)$ we consider operators $T(p)$ similar to (2.2) given by

$$T(p) := \mathcal{F}^{-1} T \mathcal{F}, \tag{A.23}$$

where for this section $\mathcal{F}$ now denotes the discrete Fourier transform given by

$$\mathcal{F}h(\eta) = \sum_{n\in\mathbb{Z}^d} e^{-i\eta n} h(n)$$

for $\eta \in \Gamma^*$, the d-dimensional Brillouin zone[9] $\Gamma^* = [-\pi, \pi)^d$. The inverse Fourier transform is given by

$$\mathcal{F}^{-1}g(n) = \int_{\Gamma^*} e^{i\eta n} g(\eta)\, \frac{\mathrm{d}\eta}{(2\pi)^d},$$

where $\mathrm{d}\eta/(2\pi)^d$ is the normalized Haar measure on the torus. A priori they are defined when $h \in l^1(\mathbb{Z}^d)$ and $g \in L^1(\Gamma^*)$, and it is well-known that $\mathcal{F}$ extends to a unitary operator $\mathcal{F} : l^2(\mathbb{Z}^d) \to L^2(\Gamma^*)$ with adjoint $\mathcal{F}^{-1}$. We call the function T admissible if $T(\eta) \in [0, T_{\max}]$ for some finite $T_{\max}$ and $\inf(\sigma(T(p))) = 0$. This is, for example, the case if T is continuous with $\min_{\eta\in\Gamma^*} T(\eta) = 0$ and $T_{\max} := \max_{\eta\in\Gamma^*} T(\eta)$. In this case one even has $\sigma(T(p)) = [0, T_{\max}]$.

Our Theorems 1.1 and 1.3 easily extend to this setting, yielding

Theorem A.19. (Weakly coupled bound states for discrete Schrödinger operators) *Let $T : \Gamma^* \to [0, T_{max}]$ be measurable. Assume that there exists a compact set $M \subset \Gamma^*$ such that*

$$\int_{M_\delta} T(\eta)^{-1}\, \mathrm{d}\eta = \infty \textit{ for all } \delta > 0, \tag{A.24}$$

[9] Here simply the torus

where $M_\delta := \{\eta \in \Gamma^* : \operatorname{dist}(\eta, M) \le \delta\}$. *Then* $\inf \sigma(T(p)) = 0$ *and, if the potential* $V \in l^1(\mathbb{Z}^d)$, *also* $\inf \sigma_{\mathrm{ess}}(T(p) + V) = 0$. *Moreover, if* $V \ne 0$ *with* $\sum_{n\in\mathbb{Z}^d} V(n) \le 0$, *then the operator* $T(p) + V$ *has at least one strictly negative eigenvalue.*

Assume that there exists a compact set $N \subset \Gamma^*$ *such that*

$$\int_{N_\delta} (T_{\max} - T(\eta))^{-1}\, \mathrm{d}\eta = \infty \text{ for all } \delta > 0, \tag{A.25}$$

where $N_\delta := \{\eta \in \Gamma^* : \operatorname{dist}(\eta, N) \le \delta\}$. *Then* $\sup \sigma(T(p)) = T_{\max}$, *and, if the potential* $V \in l^1(\mathbb{Z}^d)$, *also* $\sup \sigma_{\mathrm{ess}}(T(p) + V) = T_{\max}$. *Moreover, if* $V \ne 0$ *with* $\sum_{n\in\mathbb{Z}^d} V(n) \ge 0$, *then the operator* $T(p) + V$ *has at least one eigenvalue strictly greater than* $T_{\max}$.

Theorem 1.3 also has a counterpart in the discrete setting.

Theorem A.20. *Let* $T : \Gamma^* \to [0, T_{\max}]$ *be measurable,*

$$G^-_\alpha(u) := u \int_{T<u/\alpha^2} T(\eta)^{-1} \frac{\mathrm{d}\eta}{(2\pi)^d} \text{ for } u \ge 0, \tag{A.26}$$

and

$$G^+_\alpha(u) := u \int_{T_{\max}-T<u/\alpha^2} \big(T_{\max} - T(\eta)\big)^{-1} \frac{\mathrm{d}\eta}{(2\pi)^d} \text{ for } u \ge 0. \tag{A.27}$$

Then

(i) $G^-_\alpha(u) < \infty$ *for all* $\alpha, u > 0 \Leftrightarrow T^{-1}\mathbf{1}_{\{T<\delta\}} \in L^1(\Gamma^*)$ *for some* $\delta > 0$.

(ii) $G^+_\alpha(u) < \infty$ *for all* $\alpha, u > 0 \Leftrightarrow (T_{\max} - T)^{-1}\mathbf{1}_{\{T_{\max}-T<\delta\}} \in L^1(\Gamma^*)$ *for some* $\delta > 0$.

(iii) *Assume that the potential* V *is bounded and* $\inf \sigma_{\mathrm{ess}}(T(p)+V) \ge 0$. *Then one has, for all* $0 < \alpha < \frac{1}{2}$, *the bound*

$$N_-(T(p) + V) \le \frac{1}{(1-2\alpha)^2} \sum_{n\in\mathbb{Z}^d} G^-_\alpha(V_-(n)), \tag{A.28}$$

where $V_- = \max(0, -V)$ *is the negative part of* V *and* $N_-(T(p) + V)$ *is the number eigenvalues of* $T(p) + V$ *which are strictly negative.*
Similarly, if $\sup \sigma_{\mathrm{ess}}(T(p) + V) \le T_{\max}$, *then one has, for all* $0 < \alpha < \frac{1}{2}$, *the bound*

$$N_+(T(p) + V) \le \frac{1}{(1-2\alpha)^2} \sum_{n\in\mathbb{Z}^d} G^+_\alpha(V_+(n)), \tag{A.29}$$

where $V_+ = \max(0, V)$ *is the positive part of* V *and* $N_+(T(p)+V)$ *is the number of eigenvalues of* $T(p) + V$ *which are strictly above* $T_{\max}$.

Remarks A.21. (i) Theorem A.20 is a generalization of the results by Bach, de Siqueira Pedra, and Lakaev [1], who considered the case that T is a Morse function, in particular it is smooth and has no degenerate singular points, and T has exactly one zero. The work of Rozenblum and Solomyak

[44,45] considers only the standard discrete Schrödinger operator, which is given by a Morse function, but their approach also needs the fact that the standard discrete Schrödinger operator generates a positivity preserving semi-group on $l^2(\mathbb{Z}^d)$. In contrast, Theorems A.19 and A.20 hold and are complementary to each other when the kinetic energy symbol T is, for example, lower semi-continuous. In particular, T may have an *arbitrary sub-manifold*, or even a more general set, as its zero set.

(ii) The proofs of the two theorems above are a literal translation of the continuous case to the discrete setting. We leave the proofs as an exercise to the interested reader.

(iii) In the case of the usual discrete Schrödinger operator Δ_d on $l^2(\mathbb{Z}^d)$ one has $T(\eta) = \sum_{j=1}^d 2(1-\cos(\eta_j)) = \sum_{j=1}^d 4\sin^2(\eta_j/2)$. So in this case we can take $M = \{0\}$ and $N = \{(\pi,\dots,\pi)^t\}$. Since T behaves quadratically near M and N, one sees that (A.24) and (A.25) hold in dimension $d \le 2$.

(iv) Our Theorem 3.4 and Corollaries 3.6 and 3.7 have a natural counterpart in the discrete world with virtually the same proofs as in the continuous setting.

Remarks A.22. (i) If $G^{\pm}_{\alpha}(u)$ is finite for some α and u, then it is finite for all $\alpha, u > 0$. Moreover, since the integration in (A.24) and (A.25) is over a subset of the compact set Γ^*, the functions $G^{\pm}_{\alpha}(u)$ behave *linearly* in u for u large, once they are finite. We can improve this a little bit, see Corollary A.24.

(ii) As in the continuous case, one can reformulate the bounds on the discrete spectrum as

$$N_\pm(T(p)+V) \le (1-2\alpha)^{-2}\int_0^\infty N_\pm^{\mathrm{cl}}(\max(\alpha^2,s)T+V)\,\mathrm{d}s \tag{A.30}$$

with the semiclassical expressions

$$N_-^{\mathrm{cl}}(T+V) := \sum_{n\in\mathbb{Z}^d}\int_{\Gamma^*}\mathbf{1}_{\{T(\eta)+V<0\}}\frac{\mathrm{d}\eta}{(2\pi)^d}$$

and

$$N_+^{\mathrm{cl}}(T+V) := \sum_{n\in\mathbb{Z}^d}\int_{\Gamma^*}\mathbf{1}_{\{T(\eta)+V>T_{\max}\}}\frac{\mathrm{d}\eta}{(2\pi)^d}.$$

The proof of Theorem A.20 is a straightforward adaptation of the proof in the continuous case. We leave it to the interested reader.

In the case of the usual discrete Schrödinger operator Δ_d on $l^2(\mathbb{Z}^d)$ Theorem A.20 gives us the following explicit bounds.

Theorem A.23. *The numbers $N_-(-\Delta_d+V)$ and $N_+(-\Delta_d+V)$ of eigenvalues of $-\Delta_d+V$ below 0 and above $4d$, respectively, satisfy for any $\alpha\in(0,1/4)$*

$$N_-(-\Delta_d+V) \le (1+2\alpha)^{-2}\frac{|S^{d-1}|}{2^{2d}(d-2)\alpha^{d-2}} \times \sum_{n\in\mathbb{Z}^d} V_-(n)\min(V_-(n),4d\alpha^2)^{d/2-1}, \tag{A.31}$$

and

$$N_-(-\Delta_d + V) \le (1+2\alpha)^{-2}\frac{|S^{d-1}|}{2^{2d}(d-2)\alpha^{d-2}} \times \sum_{n\in\mathbb{Z}^d} V_+(n)\min(V_+(n), 4d\alpha^2)^{d/2-1}. \tag{A.32}$$

Proof. Using that

$$\sin(x/2) \ge \frac{x}{\pi}$$

for $x \in [-\pi,\pi]$ we estimate

$$T(\eta) \ge \frac{4}{\pi^2}|\eta|^2.$$

To get the first bound (A.31) we have to estimate $G_\alpha^-(u)$. We have

$$\begin{aligned} G_\alpha^-(u) &= u\int_{\{\eta\in[-\pi,\pi]^d \mid \sum_{j=1}^d 4\sin^2(\eta_j/2)<u/\alpha^2\}} \left(\sum_{j=1}^d 4\sin^2(\eta_j/2)\right)^{-1}\frac{\mathrm{d}\eta}{(2\pi)^d} \\ &\le u\int_{|\eta|\le\pi\min(\frac{u^{1/2}}{2\alpha}, d^{1/2})} \frac{\pi^2}{4}|\eta|^{-2}\frac{\mathrm{d}\eta}{(2\pi)^d} \\ &= \frac{|S^{d-1}|}{2^{2d}(d-2)\alpha^{d-2}} u\min(u, 4d\alpha^2)^{\frac{d}{2}-1}. \end{aligned} \tag{A.33}$$

Now, the claimed inequality follows immediately from (A.28).
For the proof of the second bound (A.32), we need to investigate

$$G_\alpha^+(u) = u\int_{\{\eta\in[0,2\pi]\mid 4d-\sum_{j=1}^d 4\sin^2(\eta_j/2)<u/\alpha^2\}} \times \left(4d - \sum_{j=1}^d 4\sin^2(\eta_j/2)\right)^{-1}\frac{\mathrm{d}\eta}{(2\pi)^d}. \tag{A.34}$$

But by the change of variables $\eta_j = \zeta_j + \pi, 1 \le j \le d$, we see that (A.34) and (A.34) are exactly the same. Now, the claim follows from (A.29). □

Copying a simple trick from [36], that exploits special properties of the discrete setting, we can improve our result a little bit.

Corollary A.24. *The numbers $N_-(-\Delta_d+V)$ and $N_+(-\Delta_d+V)$ of eigenvalues of $-\Delta_d + V$ below 0 and above* $4d$, *respectively, satisfy for any* $\alpha \in (0, 1/4)$

$$N_-(-\Delta_d + V) \le (1+2\alpha)^{-2}\frac{|S^{d-1}|}{2^{2d}(d-2)\alpha^{d-2}}\sum_{V_-\le 4d\alpha^2} V_-(n)^{d/2} + \#\{n\in\mathbb{Z}^d \mid V_-(n) > 4d\alpha^2\}, \tag{A.35}$$

and

$$N_+(-\Delta_d + V) \le (1+2\alpha)^{-2} \frac{|S^{d-1}|}{2^{2d}(d-2)\alpha^{d-2}} \sum_{V_+ \le 4d\alpha^2} V_+(n)^{d/2}$$
$$+ \#\{n \in \mathbb{Z}^d \mid V_+(n) > 4d\alpha^2\}. \tag{A.36}$$

Proof. We split the potential $V = V_+ - V_- = V_{+,1} + V_{+,2} - V_{-,1} - V_{-,2}$, where

$$V_{\pm,1}(n) \le h, \quad \text{and} \quad V_{\pm,2}(n) > h$$

for all $n \in \mathbb{Z}^d$ and some $h > 0$. Furthermore, let

$$n^{\pm}(h, V) = \#\{n \in \mathbb{Z}^d \mid V_{\pm}(n) > h\}$$

denote the number of values $n \in \mathbb{Z}^d$ for which the positive and negative part of the potential, respectively, are greater than h. Assuming that V is suitably summable, both $n^+(h,V)$ as well as $n^-(h,V)$ are finite and $V_{-,2}$ and $V_{+,2}$ can be considered finite rank perturbations.
Thus, for any $\varepsilon > 0$

$$\begin{aligned} N^-(-\Delta_d + V) &\le N^-(-(1-\varepsilon)\Delta_d - V_{-,1}) + N^-(-\varepsilon\Delta_d - V_{-,2}) \\ &\le N^-(-(1-\varepsilon)\Delta_d - V_{-,1}) + n^-(h,V), \end{aligned}$$

and

$$\begin{aligned} N^+(-\Delta_d + V) &\le N^+(-(1-\varepsilon)\Delta_d + V_{+,1}) + N^+(-\varepsilon\Delta_d + V_{+,2}) \\ &\le N^+(-(1-\varepsilon)\Delta_d + V_{+,1}) + n^+(h,V). \end{aligned}$$

Now, we choose $h = 4d\alpha^2$ and apply Theorem A.23 to $-(1-\varepsilon)\Delta_d - V_{-,1}$ and $-(1-\varepsilon)\Delta_d + V_{+,1}$. Since the resulting estimates are valid for any $\varepsilon > 0$, we pass to the limit $\varepsilon \to 0$ and end up with inequalities (A.35) and (A.36). □

Remark A.25. Of course, with virtually the same proof a version of Corollary A.24 holds also for more general kinetic energies T than just the discrete Laplacian.

Appendix B. Invariance of the Essential Spectrum

The following Lemma was used in Remark 2.1.ii.

Lemma B.1. *Assume that $|V|$, the modulus of the potential V, is relative form small with respect to $T(p)$, where $T : \mathbb{R}^d \to [0,\infty)$ is a measurable function with $\lim_{\eta\to\infty} T(\eta) = \infty$ and $p = -i\nabla$.*

(a) *If $V \in L^1(\mathbb{R}^d)$, then*

$$\sigma_{\text{ess}}(T(p) + V) = \sigma_{\text{ess}}(T(p)) = \text{essrange}(T). \tag{B.1}$$

(b) *More generally, if the operator $|V|^{1/2}(T(p)+1)^{-1}$ is compact, then again (B.1) holds.*

Recall that the essential range of T is given by

$$\mathrm{essrange}(T) = \{E \in \mathbb{R} : |T^{-1}((E-\delta, E+\delta))| > 0 \text{ for all } \delta > 0\}.$$

If T is continuous, then $\mathrm{essrange}(T) = \mathrm{range}(T)$.

For the second compactness criterium we need one more notation. Let

$$G_1(u) = u \int_{T+1<u} \frac{1}{T(\eta)+1} \frac{\mathrm{d}\eta}{(2\pi)^d} \tag{B.2}$$

for $u \geq 0$.

Lemma B.2. *Assume that*

$$\int_{\mathbb{R}^d} G_1(s|V(x)|)\mathrm{d}x < \infty \text{ for all } s > 0.$$

Then $|V|$, in particular V, is relatively form compact with respect to $T(p)$.

Proof of Lemma B.1. We will show that the resolvent difference $(T(p)+V+\lambda)^{-1} - (T(p)+\lambda)^{-1}$ is a compact operator for large enough $\lambda > 0$. Then Weyl's theorem, see, e.g., [56, Theorem 6.24] shows that $\sigma_{\mathrm{ess}}(T(p)+V) = \sigma_{\mathrm{ess}}(T(p)) = \mathrm{essrange}(T)$, where the last equality holds since $T(p)$ is a Fourier multiplier.

Let $\lambda > 0$ and $C_\lambda = (T(p)+\lambda)^{-1/2} V (T(p)+\lambda)^{-1/2}$, i.e., C_λ is the operator given by the quadratic form

$$\langle \psi, C_\lambda \psi\rangle = \langle |V|^{1/2}(T(p)+\lambda)^{-1/2}\psi, \mathrm{sgn}(V)|V|^{1/2}(T(p)+\lambda)^{-1/2}\psi\rangle.$$

Since $|V|$ is relatively form small with respect to $T(p)$ we know that C_λ is bounded with operator norm $\|C_\lambda\| < 1$ for large enough λ, see [56, Theorem 6.30].

Tiktopoulos' formula, see [49] or [56, Theorem 6.30] then shows

$$(T(p)+V+\lambda)^{-1} = (T(p)+\lambda)^{-1/2}(1+C_\lambda)^{-1}(T(p)+\lambda)^{-1/2} \tag{B.3}$$

for any $\lambda > 0$ such that $\|C_\lambda\| < 1$. Using (B.3) and $(1+\mathbb{C}_\lambda)^{-1} - C_\lambda = -(1+\mathbb{C}_\lambda)^{-1}C_\lambda$, the difference of the resolvents is given by

$$\begin{aligned}
&(T(p)+V+\lambda)^{-1} - (T(p)+\lambda)^{-1}\\
&\quad = -(T(p)+\lambda)^{-1/2}(1+C_\lambda)^{-1}C_\lambda(T(p)+\lambda)^{-1/2}\\
&\quad = -(T(p)+\lambda)^{-1/2}(1+C_\lambda)^{-1}(T(p)+\lambda)^{1/2}|V|^{1/2}\mathrm{sgn}(V)|V|^{1/2}(T(p)+\lambda)^{-1}
\end{aligned} \tag{B.4}$$

Since $|V|$ is relatively form small with respect to $T(p)$ the operators $|V|^{1/2}$ $(T(p)+\lambda)^{1/2}$ and $(T(p)+\lambda)^{1/2}|V|^{1/2}$ are bounded. Thus also $(T(p)+\lambda)^{1/2}|V|^{1/2}$ $\mathrm{sgn}(V)$ is bounded on $L^2(\mathbb{R}^d)$.

$(1+C_\lambda)^{-1}$ is bounded since $\|C_\lambda\| < 1$ for large λ. So (B.4) shows that the difference of the resolvents is a compact operator when λ is large as soon as $|V|^{1/2}(T(p)+\lambda)^{-1}$ is a compact operator and, in this case, the essential spectra of $T(p)+V$ and $T(p)$ will be the same. So we only need to show that $|V|^{1/2}(T(p)+\lambda)^{-1}$ is compact.

If $|V|^{1/2}(T(p)+1)^{-1}$, then for all $\lambda > 0$ also

$$|V|^{1/2}(T(p)+\lambda)^{-1} = |V|^{1/2}(T(p)+1)^{-1}(T(p)+1)(T(p)+\lambda)^{-1}$$

is compact. This proves the second claim of the lemma.

Fix $\lambda > 0$ so large that $\|C_\lambda\| < 1$. Let $\mathbf{1}_{\le L} = \mathbf{1}_{\{|\eta|\le L\}}$ be the characteristic function of the closed centered ball of radius L in momentum space, $\mathbf{1}_{\le L}(p)$ the corresponding Fourier multiplier, and $\mathbf{1}_{>L}(p) = \mathbf{1} - \mathbf{1}_{\le L}(p)$. If $V \in L^1(\mathbb{R}^d)$, then

$$A_L = |V|^{\frac{1}{2}}(T(p)+\lambda)^{-\frac{1}{2}}\mathbf{1}_{\le L}(p)$$

is compact, in fact, it is a Hilbert–Schmidt operator. To see this, it is enough to show that $B_L = |V|^{\frac{1}{2}}\mathcal{F}^{-1}(T(\eta)+1)^{-\frac{1}{2}}$ is a Hilbert–Schmidt operator, since $A_L^* A_L$ is unitarily equivalent to $B_L^* B_L$. The operator B_L has the kernel

$$B_L(x,\eta) = (2\pi)^{-\frac{d}{2}}|V(x)|^{\frac{1}{2}}e^{ix\cdot\eta}(T(\eta)+\lambda)^{-\frac{1}{2}}\mathbf{1}_{\le L}(\eta)$$

and from $V \in L^1(\mathbb{R}^d)$ it follows that the kernel of B_L is square-integrable with respect to $(x,\eta) \in \mathbb{R}^d \times \mathbb{R}^d$, which shows that B_L, hence also A_L is a Hilbert–Schmidt operator. Since $T(\eta) \to \infty$ as $\eta \to \infty$ one sees that

$$\begin{aligned} &\||V|^{\frac{1}{2}}(T(p)+\lambda)^{-1} - |V|^{\frac{1}{2}}(T(p)+\lambda)^{-1}\mathbf{1}_{\le L}(p)\| \\ &= \||V|^{\frac{1}{2}}(T(p)+\lambda)^{-\frac{1}{2}}(T(p)+\lambda)^{-\frac{1}{2}}\mathbf{1}_{\ge L}(p)\| \\ &\le \||V|^{\frac{1}{2}}(T(p)+\lambda)^{-\frac{1}{2}}\| \sup_{|\eta|\ge L}(T(\eta)+\lambda)^{-\frac{1}{2}} \to 0 \end{aligned}$$

as $L \to \infty$. Thus $|V|^{\frac{1}{2}}(T(p)+\lambda)^{-1}$ is the norm limit of compact operators, hence compact. This proves the first claim of the Lemma. $\square$

Proof of Lemma B.2. Note that Corollary 6.3 applies with the choice $f = |V|^{1/2}$ and $g = (T+1)^{-1/2}$ since then $G_1(su) = s^{-2}\widetilde{G}_\alpha(s^2\sqrt{u})$. $\square$

Appendix C. Nearest Point Projection

In this section we give a sketch of the proof of Lemma 5.3 for completeness and, more importantly, we could not find a reference in the $\mathcal{C}^2$ setting, which we need. We follow the presentation of [54] where it was done in the analytic setting.

Proof of Lemma 5.3. We assume that M is a C^2 submanifold of codimension n embedded in $\mathbb{R}^d$. At each point $\omega_0 \in \Sigma$, there exists an open set $\omega \in \mathcal{O} \subset \mathbb{R}^d$ and a chart, i.e., an open set $0 \in \mathcal{U}_1 \subset \mathbb{R}^{d-n}$ and a $\mathcal{C}^2$ map $\Phi : \mathcal{U}_1 \to \Sigma$ such that every point $\omega \in \Sigma \cap \mathcal{O}$ can be uniquely written as $\omega = \Phi(y)$ for $y \in \mathcal{U}_1$. Without loss of generality we can always assume $\omega_0 = \Phi(0)$.

The vectors $\{\partial_j\phi(y)\}_{j=1}^{d-n}$ form a basis of the tangent space of M at $\phi(y)$. Using the Gram–Schmidt orthogonalization, we can find n additional vectors $\nu_1(y), \dots, \nu_n(y) \in \mathbb{R}^d$ such that the vectors

$$\partial_1\phi(y), \dots, \partial_{d-n}\phi(y), \nu_1(y), \dots, \nu_n(y)$$

form a basis of $\mathbb{R}^d$ at the point $\phi(y)$. Additionally, since Φ is $\mathcal{C}^2$, we have that the above basis vectors of $\mathbb{R}^d$ depend continuously differentiable on $y \in \mathcal{U}_1$.

Now define a map on $\mathcal{U}_1 \times \mathbb{R}^n$ by

$$\psi(y,t) := \Phi(y) + \sum_{j=1}^{n} t_j \nu_j(y).$$

A computation shows $D\psi(0,0) = I_{d\times d}$, so by the inverse function theorem, ψ is a $\mathcal{C}^1$-diffeomorphism on a suitable neighborhood $\mathcal{U}_1 \times \mathcal{U}_2$ of $(0,0) \in \mathbb{R}^{d-n} \times \mathbb{R}^n$.

For all η in a small enough δ neighborhood of $\Sigma \cap \mathcal{O}$, the problem of minimizing

$$|\phi(y) - \eta|^2$$

over $y \in \mathcal{U}_1$ has a unique solution for which we must have that all partial derivatives of $|\phi(y) - \eta|^2$ vanish, i.e.,

$$0 = \partial_j \sum_{l=1}^{d} (\Phi_l(y) - \eta_l)^2 = 2 \sum_{l=1}^{d} \partial_j \Phi_l(y)(\Phi_l(y) - \eta_l) = 2\partial_j \Phi(y) \cdot (\Phi(y) - \eta).$$

Thus for y minimizing the distance $|\Phi(y) - \eta|$, the vector $\Phi(y) - \eta$ is perpendicular to the tangent plane of Σ at $\omega = \Phi(y)$. It follows that $\phi(y) - \eta$ can be written as a linear combination

$$\Phi(y) - \eta = \sum_{j=1}^{n} t_j \nu_j(y),$$

or

$$\eta = \Phi(y) + \sum_{j=1}^{n} t_j \nu_j(y) = \psi(y,t).$$

So locally the function ψ yields a parametrization of a neighborhood of Σ in $\mathbb{R}^d$ with the property that for $\eta = \psi(y,t)$ we have $\mathrm{dist}(\eta, \Sigma) = \mathrm{dist}(\psi(y,t), \Sigma) = |t|$. $\square$

Appendix D. The Classical Phase Volume of the Relativistic Pair Operator

Recall that the kinetic energy of the relativistic pair operator in (A.7) is given by

$$T_{P,M,\mu_\pm}(\eta) = \sqrt{|\mu_+ P - \eta|^2 + \mu_+^2 M^2} + \sqrt{|\mu_- P + \eta|^2 + \mu_-^2 M^2} - \sqrt{P^2 + M^2} \tag{D.1}$$

$\eta \in \mathbb{R}^d$ is the the relative momentum of the two particles, $M = m_- + m_+$ is the total mass, $P \in \mathbb{R}^3$ the total momentum, and we set $\mu_\pm := m_\pm / M$.

The the volume of the set $\{T_{P,M,\mu_\pm} < u\} = \{\eta \in \mathbb{R}^d : T_{P,M,\mu_\pm}(\eta) < u\}$ is given by

$$|\{T_{P,M,\mu_\pm} < u\}| = \frac{|B_d| u^{d/2}(u + g_{P,M})(u + 2g_{P,M})^{d/2}(u^2 + 2g_{P,M}u + (1 - 4\widetilde{\mu}^2)M^2)^{d/2}}{2^d(u^2 + 2g_{P,M}u + M^2)^{(d+1)/2}} \tag{D.2}$$

with $g_{P,M} = \sqrt{P^2 + M^2}$. The calculation of the volume $|\{T_{P,M,\mu_\pm} < u\}|$ was done in [57, Appendix I]. Since we need it in a slightly different form and notation, we sketch the calculation for the convenience of the reader.

Using a suitable rotation, we can assume $P = |P|e_1$, with $e_1 = (1, 0, \ldots, 0)^t$, the standard first unit vector in $\mathbb{R}^d$, and by scaling one has

$$|\{T_{P,M,\mu_\pm} < u\}| = |P||\{\zeta \in \mathbb{R}^d : T_{\tau,g,\mu_\pm}(\zeta) < u/|P|\}|$$

with $\tau = M/|P|$, $g = \sqrt{1 + \tau^2}$, and

$$T_{\tau,g,\mu_\pm}(\zeta) = \sqrt{|\mu_+ e_1 - \zeta|^2 + \mu_+^2\tau^2} + \sqrt{|\mu_- e_1 + \zeta|^2 + \mu_-^2\tau^2} - g.$$

Split $\zeta = (\vartheta, \xi) \in \mathbb{R} \times \mathbb{R}^{d-1}$ and put $T_\pm = \sqrt{|\xi|^2 + |\mu_\pm \mp \vartheta|^2 + \mu_\pm^2\tau^2}$. Then $T_{\tau,g,\mu_\pm}(\zeta) < s$ is equivalent to $T_+ + T_- < s + g$, that is,

$$\begin{aligned} 2T_+T_- &< (s+g)^2 - T_+^2 - T_-^2 \\ &= (s+g)^2 - (2|\xi|^2 + |\mu_+ - \vartheta|^2 + |\mu_- + \vartheta| + (\mu_+^2 + \mu_-^2)\tau^2). \end{aligned} \tag{D.3}$$

Define $\widetilde{\mu} = \frac{1}{2}(\mu_- - \mu_+)$. Clearly $4\widetilde{\mu}^2 = (\mu_- - \mu_+)^2$ and since $\mu_+ + \mu_- = 1$ we also have $1 = (\mu_+ + \mu_-)^2$. Thus

$$\mu_+^2 + \mu_-^2 = 2\left(\frac{1}{4} + \widetilde{\mu}^2\right).$$

Set $\vartheta = \varphi - \widetilde{\mu}$ and $A = s + g$. Then $\mu_\pm \mp \vartheta = \frac{1}{2} \mp \varphi$, so (D.3) is equivalent to

$$T_+T_- < \frac{A^2}{2} - \left(|\xi|^2 + \varphi^2 + \frac{1}{4} + \left(\frac{1}{4} + \widetilde{\mu}^2\right)\tau^2\right), \tag{D.4}$$

in particular, the right hand side of (D.4) is positive. Note that

$$\begin{aligned} \mu_+^2 &= \frac{\mu_+^2}{2} + \frac{\mu_-^2}{2} + \frac{\mu_+^2}{2} - \frac{\mu_-^2}{2} = \frac{\mu_+^2 + \mu_-^2}{2} + \frac{(\mu_+ + \mu_-)(\mu_+ - \mu_-)}{2} \\ &= \frac{1}{4} + \widetilde{\mu}^2 - \widetilde{\mu} \end{aligned}$$

and similarly $\mu_-^2 = \frac{1}{4} + \widetilde{\mu}^2 + \widetilde{\mu}$. Thus $T_+^2 T_-^2$ equals

$$\left(|\xi|^2 + \varphi^2 + \frac{1}{4} + \left(\frac{1}{4} + \widetilde{\mu}^2\right)\tau^2 - (\varphi + \widetilde{\mu}\tau^2)\right)$$
$$\left(|\xi|^2 + \varphi^2 + \frac{1}{4} + \left(\frac{1}{4} + \widetilde{\mu}^2\right)\tau^2 + (\varphi + \widetilde{\mu}\tau^2)\right)$$
$$= \left(|\xi|^2 + \varphi^2 + \frac{1}{4} + \left(\frac{1}{4} + \widetilde{\mu}^2\right)\tau^2\right)^2 - (\varphi + \widetilde{\mu}\tau^2)^2,$$

hence (D.4) is equivalent to

$$A^2\left(|\xi|^2 + \varphi^2 + \frac{1}{4} + \left(\frac{1}{4} + \widetilde{\mu}^2\right)\tau^2\right) - (\varphi + \widetilde{\mu}\tau^2)^2 < \frac{A^4}{4},$$

which in turn is equivalent to

$$\begin{aligned} |\xi|^2 &+ \frac{A^2-1}{A^2}\left(\varphi - \frac{\widetilde{\mu}\tau^2}{A^2-1}\right)^2 \\ &< \frac{A^2}{4} - \left(\frac{1}{4} + \left(\frac{1}{4} + \widetilde{\mu}^2\right)\tau^2\right) + \frac{1}{A^2-1}(\widetilde{\mu}\tau^2)^2 \\ &= \frac{(A^2-1)(A^2-1-(1+4\widetilde{\mu}^2\tau^2)) + 4\widetilde{\mu}^2\tau^4}{4(A^2-1)} \\ &= \frac{(A^2-1-\tau^2)(A^2-1-4\widetilde{\mu}^2\tau^2)}{4(A^2-1)}. \end{aligned}$$

So the set where $T_{\tau,g,\mu_\pm}(\zeta) < s$ is an ellipse with $d-1$ semiaxis of length

$$\frac{1}{2}\sqrt{\frac{(A^2-1-\tau^2)(A^2-1-4\widetilde{\mu}^2\tau^2)}{A^2-1}}$$

and one semiaxis of length

$$\frac{1}{2}\frac{A\sqrt{(A^2-1-\tau^2)(A^2-1-4\widetilde{\mu}^2\tau^2)}}{(A^2-1)}.$$

Thus its volume is given by

$$\begin{aligned} |\{T_{\tau,g,\mu_\pm} < s\}| &= \frac{|B_d|}{2^d}\frac{A(A^2-1-\tau^2)^{d/2}(A^2-1-4\widetilde{\mu}^2\tau^2)^{d/2}}{(A^2-1)^{(d+1)/2}} \\ &= \frac{|B_d|}{2^d}\frac{s^{d/2}(s+g)(s+2g)^{d/2}(s^2+2gs+(1-4\widetilde{\mu}^2)\tau^2)^{d/2}}{(s^2+2gs+\tau^2)^{(d+1)/2}}, \end{aligned}$$

since $A = s+g$ and $g = \sqrt{1+\tau^2}$. Rescaling this one arrives at (D.2).

References

[1] Bach, V., de Siqueira Pedra, W., Lakaev, S.N.: Bounds on the discrete spectrum of lattice Schrödinger operators. J. Math. Phys. **59**(2), 25 (2018)

[2] Bak, J., Newman, D.J.: Complex Analysis, 3rd edn. Springer, Berlin (2010)

[3] Balinsky, A., Evans, W.D.: Spectral Analysis of Relativistic Operators. Imperial College Press, London (2010)

[4] Berezin, F.A., Shubin, M.A.: The Schrödinger Equation. Mathematics and Its Applications (Soviet Series), vol. 66. Springer, Berlin (1991)

[5] Birman, M.S.: Perturbation of the spectrum of a singular elliptic operator under variation of the boundary and boundary conditions. Dokl. Akad. Nauk SSSR **137**(4), 761–763 (1961); English translation: Soviet Math. Dokl. **2**, 326–328 (1961)

[6] Birman, M.S.: On the spectrum of singular boundary-value problems. Mat. Sb. (N.S.) **55**(97), 125–174 (1961). (**Russian**); English translation: 11 Papers on Analysis, Am. Math. Soc. Transl. **53**(2), 23–60

[7] Birman, M.S., Solomjak, M.Z.: Quantitative analysis in Sobolev's imbedding theorems and applications to spectral theory. (Russian) Tenth Mathematical School (Summer School, Kaciveli/Nalchik, 1972), pp. 5–189. Izdanie Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev (1974)

[8] Birman, M.S., Solomyak, M.Z.: Spectral Theory of Self-adjoint Operators in Hilbert Space. Translated from Russian. Mathematics and its Applications (Soviet Series), p. xvi+301. D. Reidel Publishing Co., Dordrecht (1987)

[9] Brüning, J., Geyler, V., Pankrashkin, K.: On the discrete spectrum of spin-orbit Hamiltonians with singular interactions. Russ. J. Math. Phys. **14**(4), 423–429 (2007). Preprint arXiv:0709.0213v2

[10] Buell, W.F., Shadwick, B.A.: Potentials and bound states. Am. J. Phys. **63**, 256 (1995)

[11] Cwikel, M.: Weak type estimates for singular values and the number of bound states of Schrödinger operators. Ann. Math. **106**, 93–100 (1977)

[12] Daubechies, I.: An uncertainty principle for fermions with generalized kinetic energy. Commun. Math. Phys. **90**(4), 511–520 (1983)

[13] Fefferman, C., de la Llave, R.: Relativistic stability of matter. I. Rev. Mat. Iberoam. **2**(1–2), 119–213 (1986)

[14] Frank, R.: Cwikel's theorem and the CLR inequality. J. Spectr. Theory **4**, 1–21 (2014)

[15] Frank, R.: Eigenvalue bounds for the fractional Laplacian: a review. In: Recent Developments in Nonlocal Theory, pp. 210–235. De Gruyter Open, Warsaw (2017)

[16] Frank, R., Hainzl, C., Naboko, S., Seiringer, R.: The critical temperature for the BCS equation at weak coupling. J. Geom. Anal. **17**(4), 559–567 (2007)

[17] Hainzl, C., Hamza, E., Seiringer, R., Solovej, J.P.: The BCS functional for general pair interactions. Commun. Math. Phys. **281**, 349–367 (2008)

[18] Hainzl, C., Seiringer, R.: Asymptotic behavior of eigenvalues of Schrödinger type operators with degenerate kinetic energy. Math. Nachr. **283**(3), 489–499 (2010)

[19] Hainzl, C., Seiringer, R.: The Bardeen–Cooper–Schrieffer functional of superconductivity and its mathematical properties. J. Math. Phys. **57**, 021101 (2016). Preprint arXiv:1511.01995 [math-ph]

[20] Hardekopf, G., Sucher, J.: Critical coupling constant for relativistic equations and vacuum breakdown in quantum electrodynamics. Phys. Rev. A **31**, 2020–2029 (1985)

[21] Herbst, I.W.: Spectral theory of the operator $(p^2 + m^2)^{1/2} + Ze^2/r$. Commun. Math. Phys. **53**(3), 285–294 (1977)

[22] Hundertmark, D.: On the number of bound states for Schrödinger operators with operator-valued potentials. Arkiv Mat. **40**, 73–87 (2002)

[23] Hundertmark, D.: Bound state problems in quantum mechanics. In: Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th birthday, 463–496, Proceedings of the Symposium Pure Mathematics, vol. 76, Part 1. Amer. Math. Soc., Providence, RI (2007)

[24] Hundertmark, D., Kunstmann, P., Ried, T., Vugalter, S.: Cwikel's bound reloaded. Invent. Math. (2022). https://doi.org/10.1007/s00222-022-01144-7

[25] Landau, L.D., Lifshitz, E.M.: Quantum Mechanics, Non-relativistic Theory, vol. 3. Pergamon Press, London (1965)

[26] Laptev, A., Safranov, O., Weidl, T.: Bound state asymptotics for elliptic operators with strongly degenerated symbols. In: Nonlinear Problems in Mathematical Physics and Related Topics I, International Mathematical Series (New York) Vol. 1, pp. 233–246. Kluwer/Plenum New York (2002)

[27] Levin, D., Solomyak, M.: The Rozenblum–Lieb–Cwikel inequality for Markov generators. J. Anal. Math. **71**, 173–193 (1997)

[28] Lewis, R.T., Siedentop, H., Vugalter, S.: The essential spectrum of relativistic multi-particle operators. Ann. Inst. H. Poincaré Phys. Theor. **67**(1), 1–28 (1997)

[29] Lieb, E.: Bounds on the eigenvalues of the Laplace and Schrödinger operators. Bull. Am. Math. Soc. **82**(5), 751–753 (1976)

[30] Lieb, E.H.: The number of bound states of one-body Schrödinger operators and the Weyl problem. In: Proceedings of Symposia in Pure Mathematics, vol. 36, pp. 241–252. Springer, Berlin (1980)

[31] Lieb, E.H., Loss, M.: Analysis. AMS, Providence (2001)

[32] Lieb, E.H., Seiringer, R.: The Stability of Matter in Quantum Mechanics. Cambridge Univ. Press, Cambridge (2010)

[33] Lieb, E.H., Thirring, W.: Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities. In: Studies in Mathematical Physics, Essays in Honor of Valentine Bargmann, Princeton (1976)

[34] Lieb, E.H., Yau, H.-T.: The stability and instability of relativistic matter. Commun. Math. Phys. **118**, 177–213 (1988)

[35] Maz'ya, V.G., Verbitsky, I.E.: The Schrödinger operator on the energy space: boundedness and compactness criteria. Acta Math. **188**(2), 263–302 (2002)

[36] Molchanov, S., Vainberg, B.: On general Cwikel–Lieb–Rozenblum and Lieb–Thirring inequalities (2012). Preprint arXiv:0812.2968v4 [math-ph]

[37] Netrusov, Y., Weidl, T.: On Lieb–Thirring inequalities for higher order operators with critical and subcritical powers. Commun. Math. Phys. **182**, 355–370 (1996)

[38] Pankrashkin, K.: Variational principle for Hamiltonians with degenerate bottom. In: Beltita, I., Nenciu, G., Purice, R. (eds.) Mathematical Results in Quantum Mechanics, pp. 231–240. World Sci. Publ, Hackensack (2008). Preprint arXiv:0710.4790

[39] Parzygnat, A., Lee, K.K.Y., Avniel, Y., Johnson, S.G.: Sufficient conditions for two-dimensional localization by arbitrarily weak defects in periodic potentials with band gaps. Phys. Rev. B **81**, 155324 (2010)

[40] Reed, M., Simon, B.: Methods of Modern Mathematical Physics. IV. Operator Theory. Academic Press, San Diego (2005)

[41] Rozenblum, G.V.: Distribution of the discrete spectrum of singular differential operators. Dokl. Akad. Nauk SSSR **202**, 1012–1015 (1972) (**Russian**); English translation: Soviet Math. Dokl. **13**, 245–249 (1972)

[42] Rozenblum, G.V.: Distribution of the discrete spectrum of singular differential operators. Izv. Vyss. Uchebn. Zaved. Matematika **164**(1), 75–86 (1976) (*Russian*); English translation: Soviet Math. (Iz. VUZ) **20**(1), 63–71 (1976)

[43] Rozenblum, G.V., Solomyak, M.: The Cwikel–Lieb–Rozenblyum estimator for generators of positive semigroups and semigroups dominated by positive semigroups. (Russian. Russian summary) Algebra i Analiz **9**(6), 214–236 (1997); translation in St. Petersburg Math. J. **9**(6), 1195–1211 (1998)

[44] Rozenblum, G., Solomyak, M.: On the spectral estimates for the Schrödinger operator on,. J. Math. Sci. (N.Y.) **159**(2), 241–263 (2009)

[45] Rozenblum, G., Solomyak, M.: On spectral estimates for Schrödinger-type operators: the case of small local dimension. (Russian. Russian summary) Funktsional. Anal. i Prilozhen. **44**(4), 21–33 (2010); Translation in Funct. Anal. Appl. **44**(4), 259–269 (2010)

[46] Rumin, M.: Spectral density and Sobolev inequalities for pure and mixed states. Geom. Funct. Anal. **20**, 817–844 (2010)

[47] Rumin, M.: Balanced distribution-energy inequalities and related entropy bounds. Duke Math. J. **160**, 567–597 (2011)

[48] Simon, B.: Analysis with weak trace ideals and the number of bound states of Schrödinger operators. Trans. Am. Math. Soc. **224**(2), 367–380 (1976)

[49] Simon, B.: Quantum Mechanics for Hamiltonians defined as Quadratic Forms. Princeton Series in Physics, Princeton University Press, Princeton (1971)

[50] Simon, B.: The bound states of weakly coupled Schrödinger operators in one and two dimensions. Ann. Phys. **97**(2), 279–288 (1976)

[51] Simon, B.: Functional Integration and Quantum Physics. Pure and Applied Mathematics, vol. 86, p. ix+296. Academic Press Inc, New York-London (1979). ISBN: 0-12-644250-9

[52] Simon, B.: Real Analysis. A Comprehensive Course in Analysis, Part 1. American Mathematical Society AMS, Providence (2015)

[53] Simon, B.: Operator Theory. A Comprehensive Course in Analysis, Part 4. American Mathematical Society AMS, Providence (2015)

[54] Simon, L.: Theorems on Regularity and Singularity of Energy-Minimizing Maps. Birkhäuser, Basel (1996)

[55] Schwinger, J.: On the bound states of a given potential. Proc. Nat. Acad. Sci. U.S.A. **47**, 122–129 (1961)

[56] Teschl, G.: mathematical methods in Quantum Mechanics. With applications to Schrödinger Operators. Graduate Studies in Mathematics, vol. 157, 2nd edn. American Mathematical Society, Providence (2014)

[57] Vugalter, S., Weidl, T.: On the discrete spectrum of a pseudo-relativistic two-body pair operator. Ann. Henri Poincaré **4**(2), 301–341 (2003)

[58] Weidl, T.: Another look at Cwikel's inequality. In: Differential Operators and Spectral Theory. M.Sh. Birman's 70th Anniversary Collection. AMS Translations Series 2, vol. 189, pp. 247–254 (1999)

[59] Weidl, T.: Remarks on virtual bound states for semi-bounded operators. Commun. Part. Differ. Equ. **24**(1–2), 25–60 (1999)

[60] Weidl, T.: Nonstandard Cwikel Type Estimates. Contemporary Mathematics, vol. 445, pp. 337–357. American Mathematical Society, Providence (2007)

[61] Yang, K., de Llano, M.: Simple variational proof that any two-dimensional potential well supports at least one bound state. Am. J. Phys. **57**, 85 (1989)

Vu Hoang
Department of Mathematics
University of Texas at San Antonio
One UTSA Circle
San Antonio TX78249
USA
e-mail: duynguyenvu.hoang@utsa.edu

Dirk Hundertmark
Institute for Analysis
Karlsruhe Institute of Technology
76131 Karlsruhe
Germany

and

Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W. Green Street (MC-382)
Urbana IL61801
USA
e-mail: dirk.hundertmark@kit.edu

Johanna Richter
Hausdorff Center for Mathematics
Endenicher Allee 62
53115 Bonn
Germany
e-mail: johannar@math.uni-bonn.de

Semjon Vugalter
Department of Mathematics, Institute for Analysis
Karlsruhe Institute of Technology
76131 Karlsruhe
Germany
e-mail: semjon.wugalter@kit.edu

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