A remark on the non-uniqueness in L^∞ of the solutions to the two-dimensional Stokes problem in exterior domains

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To Matthias Hieber on his 60th birthday.

Abstract. The paper is concerned with the IBVP in exterior domains of the two-dimensional Stokes equations. The goal was to investigate the well-posedness in the set of solutions assuming an initial data $u_0 \in L^\infty(\Omega)$, divergence free, and enjoying the property $\|u(t)\|_\infty \leq c\|u_0\|_\infty$ for all $t > 0$ and c independent of u. For all $u_0 \in L^\infty$, divergence-free one shows examples of non-uniqueness in the above set of solutions.

1. Introduction

After the articles [25,27,28], in the last decades, the Stokes initial boundary value problem with an initial datum in L^∞, jointly with L^∞-estimates of the solutions, has been considered by several authors, both with homogeneous boundary data, see, e.g., [3–6,14,17], and with non-homogeneous; see, e.g., [7]. In the case of a two-dimensional exterior domain, apart from the contributions given in [27,28] related to the non-homogeneous and homogeneous boundary data, respectively, based on the theory of the hydrodynamic potentials, the quoted literature, based on methods of functional analysis, achieves some results in a sequence of different papers [3,4] and [1,2]. Actually, the result in [3,4] is partial, in the sense that the L^∞-estimate *a priori* holds locally in time:

$$\|u(t)\|_\infty \leq c\|u_0\|_\infty, \text{ for } t \in [0, T_0), \tag{1}$$

where the constant c and the size T_0, where T_0 *a priori* is finite, are independent of u. Subsequently in [1] estimate (1) is obtained for all $t > 0$, but the result holds paying in terms of generality. Indeed, in [1], the author considers the set of solutions for which the net force satisfies $\int_{\partial\Omega} \nu \cdot T(u, \pi_u) d\mathcal{H}^1 = 0$, where the symbol $T(u, \pi_u)$ denotes the stress tensor and ν is the normal to $\partial\Omega$. Finally, in [2], the author proves that the

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Stokes operator is a bounded analytic semigroup of angle $\frac{\pi}{2}$ on the subset of $L^{\infty}(\Omega)$ whose elements are divergence free, and without restriction estimate (1) holds for all $t > 0$.

In this note, we consider the case of initial data $u_0 \in L^{\infty}(\Omega)$, divergence free in weak sense. If from one side our result partially solves the question, from another side, it goes beyond the question, and it brings new facts on the well-posedness problem that can be of some interest. To better introduce the last sentence, we need a short digression. In two-dimensional exterior domains, the steady boundary Stokes problem presents a "pathologic property." One can solve the boundary value problem with a limit (say) a_{∞} at infinity ($|x| \to \infty$), but this limit can not be given as a datum. The solution in the set of the ones assuming value a_{∞} at infinity is unique (see Sect. 2 for the details). This set of solutions are else called "exceptional solutions," see specifically, Ch. 5 Sect. 4 of [11] or also [13]. In this note, the indetermination of the quoted "boundary condition at infinity" for the steady Stokes boundary value problem becomes a key tool to prove the non uniqueness of solutions to the Stokes initial boundary value problem in exterior domains.[1]

We limit ourselves to consider the two-dimensional problem, although some aspects of the construction can be considered in the n-dimensional case too.

The argument lines of the proof partially follow the ideas already employed in [17]; indeed, a new approach is considered.

In order to introduce our results, we need some notation. The symbol Ω denotes an exterior domain of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We set

$$C([0, \infty)):= \big\{ k \text{ continuous on } [0, \infty) \text{ with } \sup_{[0, \infty)} |k(t)| < \infty \big\},$$

$$C([0, \infty) \times \overline{\Omega}):= \big\{ u \text{ continuous on } [0, \infty) \times \overline{\Omega} \text{ and } \sup_{[0, \infty) \times \overline{\Omega}} |u(t, x)| < \infty \big\},$$

$$C^h((0, \infty) \times \overline{\Omega}) := \big\{ D^{\alpha} u, |\alpha| \leq h, \text{ continuous on } (0, \infty) \times \overline{\Omega} \text{ and } \sup_{\substack{[\varepsilon, T] \times \overline{\Omega} \\ 0 < \varepsilon < T < \infty}} |D^{\alpha} u(t, x)| < \infty \big\}.$$

A field $u \in L^{\infty}(\Omega)$ is said divergence free in weak sense if $\int_{\Omega} u \cdot \nabla\varphi \mathrm{d}x = 0$ for all φ such that $\nabla\varphi \in L^1(\Omega)$. As it is easy to understand, by the assumption of bounded boundary $\partial\Omega$ and its regularity, the property of divergence free implies in particular that there exists $u \cdot \nu$ on $\partial\Omega$ as distribution in $W^{-\frac{1}{p}, p}(\partial\Omega)$ for all $p \in (1, \infty)$.

We consider the initial boundary value problem

$$\begin{aligned} u_t - \Delta u = -\nabla\pi_u \,, \text{ in } (0, T) \times \Omega, \ \nabla \cdot u = 0 \,, \text{ in } (0, T) \times \Omega, \\ u = 0 \,, \text{ on } (0, T) \times \partial\Omega, \ u = u_0(x) \,, \text{ on } \{0\} \times \Omega. \end{aligned} \tag{2}$$

We set

$$\mathbb{L}^{\infty}(\Omega) := \{ u : u \in L^{\infty} \text{ and } \lim_{|x| \to \infty} |u(x)| = 0 \} \,.$$

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[1] The connection between *exceptional solutions* to the steady Stokes problem and some "unexpected" results for the solutions to the unsteady Stokes problem in exterior domains ($\Omega \subset \mathbb{R}^n$, $n \geq 2$) is present in other questions. Actually, the properties of the *exceptional solutions*, that we employ to distort an intuitive result of the solutions to problem (2), are also employed in the papers [22] and [20] to justify the sharpness of some time asymptotic behaviors of the solutions to problem (2): The behaviors are different from the analogous ones of the IBVP in half-space and of the Cauchy problem related to system (2).

For $R > 0$, we indicate by

$$\Omega_R := \{x \in \Omega : dist(x, \partial\Omega) > R\}.$$

For the functions of one real variable, we use the Newton symbol "·" to mean the derivate. We are going to prove.

THEOREM 1. *Let be $R > 0$. For all $u_0 \in \mathbb{L}^\infty(\Omega)$ divergence free in weak sense, there exists a unique vector function $u_R^\infty \in C^1([0, \infty))$ and a unique solution (u, π_u) to problem* (2) *such that*

$$for\ h \in \mathbb{N}_0,\ \nabla\pi_u,\ u \in C^h((0, \infty) \times \overline{\Omega}),\ \lim_{t \to 0}(u(t), \psi) = (u_0, \psi),\ for\ all\ \psi \in \mathscr{C}_0(\Omega),$$
$$u_R^\infty(0) = 0,\quad \lim_{|x| \to \infty} u(t, x) = u_R^\infty(t),\ for\ all\ t > 0,$$
$$|u_R^\infty(t)| \le c\|u_0\|_\infty,\quad \|u(t)\|_\infty \le c(R)\|u_0\|_\infty,\ for\ all\ t > 0, \tag{3}$$
$$t\|u_t(t)\|_\infty + \left(\frac{t}{t+1}\right)^{\frac{1}{2}} \|\nabla u(t)\|_\infty \le \overline{c}(R)\|u_0\|_\infty,\ for\ all\ t > 0,$$

with $c(R) := c\left[1 + c(r)R^{-\frac{2}{r}} + c(\alpha)R^{2\alpha}\right]$, $\alpha \in (0, 1)$, $r \in (2, \infty)$ and $c, c(\overline{R})$ independent of u_0. Moreover, for the pressure field holds:

$$\pi_u := x \cdot \dot{u}_R^\infty + \Pi_u,$$
$$with\quad \nabla\Pi_u \in L^\infty(\eta, T; L^r(\Omega))\ for\ all\ 0 < \eta < T \tag{4}$$
$$and,\ for\ \mu \in \left(0, \frac{1}{2}\right),\quad \|\pi_u(t)\|_{C(\overline{\Omega}')} \le \overline{c}(R)c(\Omega')\|u_0\|_\infty t^{\mu-1},\ t > 0,$$

for all $\Omega' \subset \Omega$ bounded with $\partial(\Omega - \Omega') \cap \partial\Omega = \emptyset$. Finally, we get $\lim_{t \to \infty} \|u(t)\|_\infty = \lim_{t \to \infty} u_R^\infty(t) = 0$.

The uniqueness holds in a wider set, actually

THEOREM 2. *Let (v, π_v) be a solution to problem* (2) *satisfying* (3) *and* (4)$_3$. *Then solutions (u, π_u) given in Theorem* 1 *and (v, π_v) coincide up to a function of t for the pressure field.*

Set $M := \min_{R \in (0, \infty)} c(R)$. Denoted by R_{min} the point of minimum of $c(R)$ and $u^\infty(t) := u_{R_{min}}^\infty$. We can state.

COROLLARY 1. *Let $u_0 \in \mathbb{L}^\infty(\Omega)$ divergence free. Then the solution (u, π_u) of Theorem* 1 *corresponding to $R = R_{min}$ solves problem* (2) *and enjoys the limit property* (3)$_2$ *with $u^\infty(t)$, estimate* (3)$_3$ *with $c(R) = M$ and* (3)$_4$ *and* (4)$_1$ *with $\overline{c}(R_{min})$.*

We stress that the statement of the theorems, like the one of the corollary, is furnished assuming $u_0 \in \mathbb{L}^\infty(\Omega)$, but it also holds under the more general assumption of $u_0 \in L^\infty(\Omega)$. In this case, the essential change is related to the limit properties (3)$_2$, that should be replaced by

$$\lim_{|x| \to \infty} u(t, x) - U(t, x) = u_R^\infty(t),$$

where U is the unique solution to the Cauchy problem (5) below. The behavior (3)$_4$ for $\|\nabla u(t)\|_2$ is sharp. This can be proved as in [17] (see also [14]).

i. *At the moment what is possible to state*

As far as we know, in the literature, the result related to the maximum modulus theorem of solutions to problem (2), which is not based on the theory of hydrodynamic potentials as made in [27,28], is given, in n-dimensional case, $n \geq 3$, by means of a suitable coupling of the results proved in [3] and in [17]; see also [6]. The first paper is concerned with local in time estimates and the second paper is concerned with the extension of the estimates to large time. In the two-dimensional case, the results of the first paper still work, while the results of the second paper do not work. The result in [17] is based on a technique of duality which does not work in two dimensions because, roughly speaking, the solution $\varphi(t, x)$ of the adjoint problem has the behavior $\|\varphi(t)\|_\infty \leq c\|\varphi\|_1 t^{-1}$ where the exponent -1 is sharp. Actually, following the approach of [17], the sharp behavior allows us to deduce an estimate of the kind $\|u(t)\|_\infty \leq c\|u_0\|_\infty \log(t + e)$. Another result, consequence of the above arguments used in [17] is that for all ball B centered in 0, $\|u(t)\|_{L^\infty(\Omega \cap B)} \leq c(B)\|u_0\|_\infty$, for all $t > 0$ and $u_0 \in L^\infty(\Omega)$. Both the results are not suitable for the maximum modulus theorem.

Moreover, in [1,2], the author proves that the Stokes operator is bounded and analytic on $L^\infty(\Omega)$ whose elements are divergence free. In paper [2], estimate (1) holds for all $t > 0$.

ii. *The developments given in this note*

In order to give a better comment to the special statement of the theorem, we firstly give an outline of the proof.

We consider a particular construction of the solution to the problem. Thank to the linear character of the equations, the solution of the theorem is seen as the sum of two solutions to problem (2): $u := u^1 + u^2$ and $\pi_u := \pi_{u^1} + \pi_{u^2}$. The solutions are derived by considering $u_0 := u_0^1 + u_0^2$. The datum u_0^1 has support in Ω_R and u_0^2 has compact support in $\overline{\Omega} - \Omega_{2R}$. We get $\|u_0^1\|_\infty + \|u_0^2\|_\infty \leq c\|u_0\|_\infty$ with a constant c independent of R and of u_0. The elements of the decomposition play a different role. As a consequence of the compact support, the solution u^2 verifies the estimate $\|u^2(t)\|_\infty \leq c|\Omega - \Omega_R|^{\frac{1}{r}}\|u_0^2\|_r t^{-\frac{1}{r}}$, $t > 0$, for all $r \in [1, \infty)$ (see [9]). We achieve the result of maximum modulus theorem for solution u^2 considering the above estimate and the results obtained (in [3,4]) by Abe and Giga on a local interval $(0, T_0)$ (see Theorem 7). Instead, solution u^1 comes by a special construction. We look for $u^1 := U + u_R^\infty + V + W$ and $\pi_{u^1} := -\dot{u}_R^\infty \cdot x + \pi_U + \pi_V + \pi_W$. The fields U, V, W are solutions to the following problems:

$$U_t - \Delta U = -\nabla \pi_U, \ \nabla \cdot U = 0, \ \text{in } (0, T) \times \mathbb{R}^2, \ U = u_0^1, \ \text{on } \{0\} \times \mathbb{R}^2, \tag{5}$$

$$\Delta V = \nabla \pi_V, \ \nabla \cdot V = 0 \ \text{in } \{t\} \times \Omega, \ \lim_{|x| \to \infty} V = 0, \ V = -U(t) - u_R^\infty(t) \ \text{on } \{t\} \times \Omega, \tag{6}$$

where we set $u_R^\infty(t) := -\dfrac{1}{|\partial\Omega|} \displaystyle\int_{\partial\Omega} U \mathrm{d}\mathcal{H}^1$, and

$$W_t - \Delta W = -\nabla\pi_W - V_t \text{ in } (0, T) \times \Omega, \quad \nabla \cdot W = 0 \text{ in } (0, T) \times \Omega,$$
$$W = 0 \text{ on } (0, T) \times \partial\Omega, \ W = 0 \text{ on } \{0\} \times \Omega. \tag{7}$$

As it is known the field U is unique and the pressure field π_U is a function $c(t)$. For problem (6), since the boundary data has zero integral media, we determine the existence of a unique (V, π_V) such that $|V(t, x)| \leq \|U(t) + u^\infty(t)\|_{L^\infty(\partial\Omega)}(1 + |x|)^{-1} \leq \|u_0\|_\infty(1 + |x|)^{-1}$. Moreover the regularity of U allows us to consider $V_t(t, x)$ that we employ to determine the solution to problem (7). The solution W is special, and we refer to Sect. 4 for the details. Here we limit ourselves to stress that the support of u_0^1 far from the boundary $\partial\Omega$ implies $t\|U_t(t)\|_{L^\infty(\Omega-\Omega_{\frac{R}{2}})} +$

$\|U(t)\|_{L^\infty(\Omega-\Omega_{\frac{R}{2}})} \leq ce^{-\frac{R^2}{8t}}\|u_0\|_\infty t^\mu$, $\mu \geq 0$. In turn it allows us to deduce, for all $q \in (2, \infty)$, $\|V_t(t)\|_q \leq c(R)\|u_0\|_\infty t^{-1+\mu}$ in a neighborhood of $t = 0$. This is one of the property that we need in order to deduce the estimates for W which are uniform in $t > 0$ with constant $c(R)$.

iii. *The intriguing "uniqueness" of the solution*

We begin saying that the uniqueness claimed by Theorem 2 has also another reading. Actually, if one is able to prove the existence of a solution (v, π_v) enjoying the properties (3) and (4)$_3$, then necessary this solution has a pressure field of the kind (4)$_1$. Hence if (v, π_v) is independent of our special construction of the solution, then (4)$_1$ should be an *a priori* property of a L^∞-solution which takes the "boundary datum" $u_R^\infty(t)$ at infinity.

The above remark also leads to claim that no solution stated in [3,4] can coincide with a solution of Theorem 1. In fact, denoted by $d_\Omega(x) := dist(x, \partial\Omega)$, the set of solution (u, π_u) considered in [3,4] enjoys the property $d_\Omega(x)|\nabla\pi_u(t, x)| \leq c\|\nabla u(t)\|_{L^\infty(\partial\Omega)}$, that is conflicting with (4)$_1$.

Although the solution (u, π_u) is unique in the class of existence detected in Theorem 1, that is, fixed $R > 0$, u is the unique solution corresponding to u_0 and assuming $u_R^\infty(t)$, the solution is not unique with respect to u_0 in the sense that varying $R > 0$, two different solutions could correspond to u_0. This is a consequence of our construction. Actually, for all $R > 0$, we can determine a unique field U solution to the Cauchy problem (5) with initial datum u_0^1. Via U we define univocally $u_R^\infty(t)$. One could think to fix the solution (u, π_u), with respect to $R > 0$, by requiring the best constant for the validity of estimate (3)$_3$. Although $M = \min_{R>0} c(R)$ exists, we are not able to obtain the proof related to the sharpness of M as constant in inequality (3)$_3$. We stress that the "anomalous" aspect of the result is independent of the fact that $u_0 \in \mathbb{L}^\infty(\Omega)$. Actually, we arrive at the same result also assuming $u_0 \in C(\overline{\Omega})$. As well we stress that if $u_0 \in L^\infty(\Omega) - \mathbb{L}^\infty(\Omega)$ the result of existence continues to hold. The difference between the claims is in the limit property (3)$_2$ which has to be

replaced with the following:

$$\lim_{|x|\to\infty} u(t,x) - U(t) = u_R^\infty(t).$$

In the light of the results, one understands that $u^\infty(t)$ cannot be fixed *a priori*. This is the analogous of the well-posedness for the 2D-steady Stokes problem (11) in exterior domains, which plays a crucial role in our construction. Actually, for the steady problem (11) (below), without the hypothesis $|\partial\Omega|^{-1}\int_{\partial\Omega} a\,d\mathcal{H}^1 = 0$, one proves that the solution (V, π_V) has the kinetic field which admits a limit a_∞ at infinity. But a_∞ cannot be a datum of the problem! Also the pair (V, π_V) is the unique solution enjoying the limit property with value a_∞. In dimension $n \geq 3$ one deduces the well-posedness in the L^∞-setting, avoiding the previous indetermination, by requiring $u^\infty(t) = 0$, for all $t \geq 0$. We stress that $u^\infty(t) = 0$ for all $t \geq 0$ does not means that $\lim_{|x|\to\infty} u(t,x) \to 0$. Actually, the right claim is $u(t,x) - U(t,x) \to 0$.

iv. *The comparison of non-uniqueness result with the examples of non-uniqueness related to problem* (2)

In the paper [26], for the initial boundary value problem in the half-space $(x_n > 0)$, the well-posedness of a solution (u, π_u) in the L^∞-setting is achieved coupling estimate (3) for u and the request that the $\nabla\pi_u \to 0$ for $x_n \to \infty$. This result is doubly remarkable. One side is in connection with the example of non-uniqueness exhibited in [4]. From another side, because in the L^∞-setting the representation formula, by means of the Green function, works for u but not for π_u. Actually, in the paper [26], assumed $u_0 \in L^\infty(\mathbb{R}_+^n)$, the kinetic field $u := \mathbb{G}[u_0]$, given by means of the Green function ($\mathbb{G}[\cdot]$), is solution because $\frac{\partial}{\partial t} - \Delta$ on $\mathbb{G}[u_0]$ matches the gradient of a function that *a priori* can not be represented by the formula of the pressure field because, as already said, it does not work for L^∞-datum. As it is known, this is in contrast with the L^p-theory where the representation formula is exhaustive. Coming back to the question of uniqueness, the example given in [4] is, for $n \geq 2$, a solution (u, π_u) where $u := \underbrace{(h(t, x_n), 0, \cdots, 0)}_{n \text{ components}} \in L^\infty((0, T) \times \mathbb{R}_+^n)$, with $h(t, x_n)$ solution of the heat equation on the domain $(0, \infty) \times (0, \infty)$ with $h(0, x_n) = 0$, $h(t, 0) = 0$ and $h_t - \Delta h = b(t)$, then the pressure field is $\pi_u := b(t)x_n$. The example of [24] (which is the first of this kind) is considered in [12] where it is extended to the case of exterior domains. The extension for the IBVP in exterior domains is given by means of the one related to the Cauchy problem, that is

$$v := g(t) \quad \text{and} \quad \pi_v := -\dot{g}(t) \cdot x, \quad \text{with } g(0) = 0.$$

All these examples of non-uniqueness, in different ways, exhibit pressure fields, for all say $P(t, x)$, which depend on the x-space variable in a linear way and with $\nabla P \in L^\infty(\Omega)$ for $t > 0$ (here Ω is meant as one of the domains considered, that is the whole space or a half-space or an exterior domain). This peculiarity is crucial for the non-uniqueness. By comparing $P(t, x)$ and the pressure field indicated in formula $(4)_1$, one concludes that both depend on x in a linear way, and ∇P as well $\nabla \pi_u \in L^\infty$ for $t > 0$. It is natural to inquire why for a fixed $R > 0$ the solution of the theorem is unique. Actually, uniqueness is performed in the set of solutions which satisfy among other the limit property $(3)_2$. This requirement is consistent as, for fixed R, the field $U(t, x)$ and the vector function $u_R^\infty(t)$ are uniquely determined by the initial data u_0, and the limit property $(3)_2$ for $u_0 \equiv 0$ means that $u \to 0$ for $|x| \to \infty$. This is not true in the case of the counterexamples. Actually, the initial data being null should give $u_R^\infty(t) = 0$ for all $t > 0$, instead, for $t > 0$, the limit is $g(t) \neq 0$.

v. *The special character of the maximum modulus theorem related to the Stokes problem*

In connection with estimate $(3)_3$, there is a discordance with the theory of *Maximum Principle* for parabolic and elliptic equations, or its variants for parabolic and elliptic systems. In this framework, denoted by $P(u)$ a problem and by a the datum of the problem, the result related to the estimate (3) reads as follows: all the solutions u of problem $P(u)$ belonging to the class $\mathfrak{C} \subset L^\infty$ enjoy the estimate

$$\|u\|_\infty \leq c\|a\|_\infty, \tag{8}$$

with the constant c which depends on the domain and on the kind of problem $P(u)$.

So that in this statement estimate (8) becomes an *a priori estimate* in the set $\mathfrak{C}$ of the solutions. In particular, for the linear character of $P(u)$, it allows us to deduce the uniqueness in $\mathfrak{C}$. In the light of Theorem 1, related to the case of unbounded domains, this is not possible. Of course, all this is a consequence of the further unknown of the problem, that is the pressure field, which being a dynamical variable, in particular it represents the dynamical response of the fluid, cannot be conditioned by data of the problem. From this point of view, the Stokes problem in the L^∞-setting is conflicting with the one in the L^p-setting, where the existence of u leads to an exhaustive result concerning the continuous dependence and uniqueness of the solutions. (Actually, these solutions tend to 0 at infinity.)

The plan of the paper is the following. In Sect. 2 we recall some estimates and results related to the L^p-theory of the steady and unsteady Stokes problem. In Sect. 3 we recall some implications that hold for the solution of Theorem 7. In Sect. 4, we furnish the first solution, that is (u^1, π_{u^1}). In Sect. 5, we furnish the second solution, that is (u^2, π_{u^2}). In Sect. 6, we conclude giving the proof of Theorems 1 and 2.

2. Some auxiliary results

In the following paper as it is usual, for $h \in \mathbb{N}$, we denote by $D^h w$ the partial derivatives of w related to a multi-index of length h. By $\mathscr{C}_0(\Omega)$ we mean the set of function $C_0^\infty(\Omega)$ divergence free. We denote by $J^r(\Omega)$ and $J^{1,r}(\Omega)$, $r \in (1, \infty)$, the completion of $\mathscr{C}_0(\Omega)$ in $L^r(\Omega)$ and in $W^{1,r}(\Omega)$, respectively.

We recall the following results.

LEMMA 1. *Let* $w(x) \in L^q(\Omega)$ *and* $D^m w \in L^p(\Omega)$, *$p, q \in [1, \infty)$. Then, for $k \in \{0, 1, \ldots, m-1\}$, the following inequality holds*

$$\|D^k w\|_r \leq c\|D^m w\|_p^b \|w\|_q^{1-b}, \tag{9}$$

where the exponents satisfy the dimensional balance:

$$\frac{1}{r} = \frac{k}{m} + a\left(\frac{1}{p} - \frac{m}{n}\right) + (1-b)\frac{1}{q},$$

with $b \in [\frac{k}{m}, 1]$ *either if* $p = 1$ *or* $p > 1$ *and* $m - k - \frac{n}{p} \notin \mathbb{N} \cup \{0\}$, *while* $a \in [\frac{k}{m}, 1)$ *if* $p > 1$ *and* $m - k - \frac{n}{p} \in \mathbb{N} \cup \{0\}$. *Moreover, if D is abounded domain, then inequality* (9) *holds in the following form:*

$$\|D^k w\|_r \leq c\left[\|D^m w\|_p^b \|w\|_q^{1-b} + \|w\|_q\right], \tag{10}$$

provided that the dimensional balance of the inequality is satisfied.

Proof. This lemma is part of an interpolation inequality proved in [8]. The difference with respect to the well known Gagliardo–Nirenberg interpolation inequality is in the fact that the value of $D^k w$ is not zero on the boundary. □

LEMMA 2 (Bogovski's lemma). *Let* $g \in L^r(\Omega)$ *with compact support and* $\int_\Omega g \, dx = 0$. *Then there exists a field* $G \in W_0^{1,r}(\Omega)$ *with compact support such that* $\nabla \cdot G = g$ *and*

$$\|G\|_r \leq c_P \cdot c\|g\|_r \text{ and } \|\nabla G\|_r \leq c\|g\|_r,$$

where c_P *is the Poincaré constant and c is a constant independent of g.*

Proof. See, e.g., [11]. □

We consider the boundary value problem for the Stokes system:

$$\Delta V - \nabla \pi_V = 0, \ \nabla \cdot V = 0, \text{ in } \Omega, \ V = a \text{ on } \partial\Omega. \tag{11}$$

THEOREM 3. *Let* $a \in C(\partial\Omega)$ *and* $|\partial\Omega|^{-1} \int_{\partial\Omega} a(y) d\sigma = 0$. *Then there exists a unique solution* (V, π_V) *to problem* (11) *such that* $V \in C(\overline{\Omega}) \cap C^2(\Omega)$ *and* $V \in C^1(\Omega)$ *with*

$$|V(x)| \leq c\|a\|_\infty (1 + |x|)^{-1}, \quad \text{for all} \quad x \in \Omega, \tag{12}$$

where c is a constant independent of a. Moreover, with further hypothesis of a $\in$ $W^{2-\frac{1}{q},q}(\partial\Omega)$, then

$$\|D^2 V\|_q + \|\nabla \pi_V\|_q \le c\|a\|_{W^{2-\frac{1}{q},q}(\partial\Omega)}, \tag{13}$$

where c is a constant independent of a.

Proof. Existence, uniqueness and estimate (12) can be found in [15] (Ch. 3, sect. 3) or in [21]. For similar results and regularity up to the boundary see also [11]. □

Let us consider the equation for the pressure:

$$\Delta \Pi = 0 \text{ in } \Omega, \quad \frac{d}{d\nu}\Pi = \nabla \times \nabla \times N \cdot \nu =: \nabla \times a \cdot \nu \text{ on } \partial\Omega, \quad \Pi \to c \text{ for } |x| \to \infty. \tag{14}$$

We are interested to the following result:

LEMMA 3. *Assume that $N \in W^{2-\frac{1}{q},q}(\partial\Omega)$. Then a solution of problem (14) is such that*

$$\|\Pi\|_{C(\overline{\Omega}')} \le c\Big[\|a\|_{L^q(\Omega')}^{(1-\frac{1}{d})(1-\alpha)}\|\nabla a\|_{L^q(\Omega')}^{\frac{1}{d}(1-\alpha)+\alpha} + \|a\|_{L^q(\Omega')}^{(1-\frac{1}{q})(1-\frac{1}{d})(1-\alpha)}\|\nabla a\|_{L^q(\Omega')}^{\left(\frac{1}{q}(1-\frac{1}{d})+\frac{1}{d}\right)(1-\alpha)+\alpha}\Big]$$

$$+ c\left(\|a\|_{L^q(\Omega')} + \|\nabla a\|_{L^q(\Omega')}^{\frac{1}{q}}\|a\|_{L^q(\Omega')}^{1-\frac{1}{q}}\right)^{1-\frac{1}{d}}\|\nabla a\|_{L^q(\Omega')}^{\frac{1}{d}}, \tag{15}$$

where c is a constant independent of N, $q > 2$, $\alpha := \frac{2}{q}$, $d := \frac{q}{1+\lambda q}$, $\lambda \in (0, 1-\frac{1}{q})$, and $\Omega' \subset \Omega$ bounded with $\partial(\Omega - \Omega') \cap \partial\Omega = \emptyset$.

Proof. It is well known that, for solutions problem to problem (14), the following estimate holds (the estimate is due to Solonnikov in [25], and recently, it is also reproduced in [19]):

$$\lambda \in (0,1), \quad \|\Pi\|_{L^q(\Omega')} \le c < a >_q^\lambda \text{ and } \quad \|\nabla\Pi\|_q \le c < a >_q^{1-\frac{1}{q}}, \tag{16}$$

where $\Omega' \subset \Omega$, bounded, with $\partial(\Omega - \Omega') \cap \partial\Omega = \emptyset$, and seminorm $< a >_q^\lambda :=$ $\left[\int_{\partial\Omega}\int_{\partial\Omega}\frac{|a(x)-a(y)|^q}{|x-y|^{1+\lambda q}}d\mathcal{H}^1 d\mathcal{H}^1\right]^{\frac{1}{q}}$. For $\lambda = 1 - \frac{1}{q}$, we get the Gagliardo seminorm. For $q > 2$ we obtain

$$\|\Pi\|_{C(\overline{\Omega}')} \le c(\|\nabla\Pi\|_{L^q(\Omega')}^\alpha\|\Pi\|_{L^q(\Omega')}^{1-\alpha} + \|\Pi\|_{L^q(\Omega')})$$
$$\le c\Big[\|\nabla a\|_{L^q(\Omega')}^\alpha\big(< a >_q^\lambda\big)^{1-\alpha} + < a >_q^\lambda\Big], \text{ with } \alpha := \frac{2}{q}.$$

We have

$$< a >_q^\lambda \le c\|a\|_{L^q(\partial\Omega)}^{1-\frac{1}{d}}\big(< a >_q^{1-\frac{1}{q}}\big)^{\frac{1}{d}} \le c\|a\|_{L^q(\partial\Omega)}^{1-\frac{1}{d}}\|\nabla a\|_{L^q(\Omega')}^{\frac{1}{d}}$$
$$\le c\big(\|a\|_{L^q(\Omega')} + \|\nabla a\|_{L^q(\Omega')}^{\frac{1}{q}}\|a\|_{L^q(\Omega')}^{1-\frac{1}{q}}\big)^{1-\frac{1}{d}}\|\nabla a\|_{L^q(\Omega')}^{\frac{1}{d}}, \text{ with } d = \frac{q}{1+\lambda q}.$$

Hence, we get

$$\|\Pi\|_{C(\overline{\Omega}')} \leq c\left[\|a\|_{L^q(\Omega')}^{(1-\frac{1}{d})(1-\alpha)}\|\nabla a\|_{L^q(\Omega')}^{\frac{1}{d}(1-\alpha)+\alpha} + \|a\|_{L^q(\Omega')}^{(1-\frac{1}{q})(1-\frac{1}{d})(1-\alpha)}\|\nabla a\|_{L^q(\Omega')}^{\left(\frac{1}{q}(1-\frac{1}{d})+\frac{1}{d}\right)(1-\alpha)+\alpha}\right]$$
$$+c\left(\|a\|_{L^q(\Omega')} + \|\nabla a\|_{L^q(\Omega')}^{\frac{1}{q}}\|a\|_{L^q(\Omega')}^{1-\frac{1}{q}}\right)^{1-\frac{1}{d}}\|\nabla a\|_{L^q(\Omega')}^{\frac{1}{d}}.$$

$\square$

LEMMA 4. *Assume that $N(t,x)$ in Lemma 3 is a smooth one-parameter family of function in $W^{2-\frac{1}{q},q}(\partial\Omega)$, the time $t > 0$ is the parameter. Assume that $t^{\frac{1}{2}}\|\nabla N(t)\|_\infty + t\|\nabla\nabla N(t)\|_\infty \leq A$. Then, in a right neighborhood of $t = 0$, we get*

$$\|\Pi(t)\|_{C(\overline{\Omega}')} \leq cAt^{\mu-1}, \tag{17}$$

with $c = c(T)$ independent of N and $t \in (0,T)$, exponent $\mu \in (0,\frac{1}{2})$, and $\Omega' \subset \Omega$ bounded with $\partial(\Omega - \Omega') \cap \partial\Omega = \emptyset$.

Proof. We have to estimate the right hand side of (15). Recalling that we are studying the behavior of $\|\Pi(t)\|_{C(\overline{\Omega}')}$ in a right neighborhood of $t = 0$, we can limit ourselves to consider the terms with the major singularity in $t = 0$. This is conditioned by the greater exponent for t^{-1}. Recalling that in estimate (15), we have $a = \nabla \times N(t,x)$ and the domain Ω' is bounded, employing the assumptions $\|\nabla N\|_\infty \leq c\|u_0\|_\infty t^{-\frac{1}{2}}$ and $\|\nabla\nabla N\|_\infty \leq c\|u_0\|_\infty t^{-1}$, then we get

$$\|\Pi(t)\|_{C(\overline{\Omega}')} \leq cAt^{-\beta},$$

with exponent $\beta := -\frac{1}{2}(1-\frac{1}{q})(1-\frac{1}{d})(1-\alpha) - \left(\frac{1}{q}(1-\frac{1}{d})+\frac{1}{d}\right)(1-\alpha) - \alpha$. By a computation we obtain

$$\beta = \frac{1}{2} + \frac{\alpha}{2} + (1-\alpha)\left[\frac{1}{2q}\left(1-\frac{1}{d}\right)+\frac{1}{2d}\right] =: \frac{1}{2} + \overline{\mu}$$

where we recall that $\alpha = \frac{2}{q}, q > 2, d = \frac{q}{1+\lambda q}$ and $\lambda \in (0, 1-\frac{1}{q})$. For large q and small λ, we arrive at $\overline{\mu} \in (0,\frac{1}{2})$. Thus we set $\mu - 1$ as exponent in (17). $\square$

LEMMA 5. *Let $q \in (1,\infty)$. Assume that $(v, \nabla\pi_v) \in W^{2,q}(\Omega) \cap J^{1,q}_{loc}(\overline{\Omega}) \times L^q(\Omega)$. Then there exists a constant c independent of (v, π_v) such that*

$$\|D^2 v\|_q + \|\nabla\pi_v\|_q \leq c(\|P\Delta v\|_q + \|v\|_{L^q(\Omega')}), \tag{18}$$

where $\Omega' \subset \Omega$ is bounded with $\partial(\Omega - \Omega') \cap \partial\Omega = \emptyset$.

Proof. See, e.g., [11] or [22]. $\square$

We use a special formulation of problem (2), that is given $\phi \in C_0(\Omega)$

$$\begin{cases} \varphi_t - \Delta\varphi = -\nabla\pi_\varphi, \text{ in } (0,T) \times \Omega, \ \nabla \cdot \varphi = 0, \text{ in } (0,T) \times \Omega, \\ \varphi = 0, \text{ on } (0,T) \times \partial\Omega, \\ \lim_{t \to 0}(\varphi(t), \psi) = (\phi_0, \psi), \text{ on for all } \psi \in \mathscr{C}_0(\Omega). \end{cases} \tag{19}$$

Due to condition $(19)_3$, problem (19) is a weak form of the usual initial boundary value problem for the Stokes equations. This weak formulation allows us to consider initial data in the Lebesgue spaces L^p and not in the space of the hydrodynamics J^p. It was introduced in [16]. Its interest is connected with the possibility of deducing estimates in L^r-Lebesgue spaces with $r \in (1, \infty]$ by means of duality arguments. Of course, for an initial data in J^p we come back to the classical Stokes solutions. For each $T > 0$, for q' conjugate exponent of q, we set $W_{q'} := \{\zeta(t, x) : \zeta \in C^1([0, T] \times \overline{\Omega}) \cap C(0, T; ; J^{1,q'}(\Omega))$ and $\zeta_t \in C([0, T]; L^{q'}(\Omega))\}$. For problem (2.1), the following result holds:

THEOREM 4. *Let $\phi \in L^1(\Omega)$. Then there exists a unique solution (φ, π_φ) to problem* (19) *such that*

i. $\eta > 0$, $q > 1$, $\varphi \in C(\eta, T; J^q(\Omega)) \cap L^\infty(\eta, T; J^{1,q}(\Omega))$, $D^2\varphi \in$, $\nabla\pi_\varphi \in L^\infty(\eta, T; L^q(\Omega))$;

ii. $q \in (1, \infty]$,

$$\|\varphi(t)\|_q \le c\|\phi\|_1 t^{-\mu}, \quad \mu = 1 - \tfrac{1}{q}, \; t > 0;$$

$$\|\nabla\varphi(t)\|_q \le c\|\phi\|_1 t^{-\mu_1}, \; \mu_1 = \begin{cases} \tfrac{1}{2} + \mu & \text{if } t \in (0, 1]; \\ \tfrac{1}{2} + \mu & \text{if } t > 0 \text{ and } q \in (1, 2], \\ 1 & \text{if } t > 1 \text{ and } q > 2; \end{cases}$$

$$\|\varphi_t(t)\|_q \le c\|\phi\|_1 t^{-\mu_2} \quad \mu_2 = 1 + \mu, \; t > 0;$$

where the constant c is independent of ϕ and the exponent μ_1 is sharp;

iii. $\int_0^t \left[(\varphi(\tau), \zeta_\tau(\tau)) - (\nabla\varphi(\tau), \nabla\zeta(\tau))\right]d\tau = (\varphi(t), \zeta(t)) - (\phi, \zeta(0))$, *for all $\zeta \in W_{r'}$ provided that $\tfrac{1}{2} + \tfrac{n}{2}(1 - \tfrac{1}{r}) < 1$; finally, $\lim_{t \to 0}(\varphi(t), \psi) = (\phi, \psi)$ for all $\psi \in \mathscr{C}_0(\Omega)$.*

REMARK 1. We emphasize that item iii. expresses a weak formulation just in a neighborhood of $t = 0$ that is employed for the weakness of the initial data ϕ. For further considerations we refer to the paper [16]. The result of Theorem 4 for the two-dimensional case is proved in [18]. It is a suitable coupling of the ones proved in [10] and those proved in [16].

In the following theorem is reproduced the classical result concerning (2), which in two-dimensional case in its complete form is proved in [9, 10]:

THEOREM 5. *The Stokes operator $-P\Delta$ generates an analytic semigroup on $J^p(\Omega)$, $p \in (1, \infty)$. Moreover, for all $\varphi_0 \in J^p(\Omega)$ and $t > s \ge 0$ the following estimates holds:*

$$\|\varphi(t)\|_q \le c\|\varphi(s)\|_p(t - s)^{-\mu}, \quad \mu = 1 - \tfrac{1}{q}, \; q \in [p, \infty], \; t > 0;$$

$$\|\nabla\varphi(t)\|_q \le c\|\varphi(s)\|_p(t - s)^{-\mu_1}, \; \mu_1 = \begin{cases} \tfrac{1}{2} + \mu & \text{if } t - s \in (0, 1]; \\ \tfrac{1}{2} + \mu & \text{if } t - s > 0 \text{ and } q \in (1, 2], \\ 1 & \text{if } t - s > 1 \text{ and } q > 2; \end{cases}$$

$$\|\varphi_t(t)\|_q \le c\|\varphi(s)\|_p(t - s)^{-\mu_2} \quad \mu_2 = 1 + \mu, \; t > 0;$$

where the constant c is independent of φ and the exponent μ_1 is sharp.

Proof. See [9] and [10] Theorem 1.1 and Theorem 1.2. □

We also need to consider

$$v_t - \Delta v = -\nabla \pi_u + f, \text{ in } (0, T) \times \Omega, \ \nabla \cdot u = 0, \text{ in } (0, T) \times \Omega,$$
$$u = 0, \text{ on } (0, T) \times \partial \Omega, \ u = 0, \text{ on } \{0\} \times \Omega. \tag{20}$$

THEOREM 6. *For all $f \in L^r(0, T; L^r(\Omega)), r \geq 2$ there exists a unique solution (v, π_v) to problem (20) such that $v \in C([0, T); J^r(\Omega)) \cap L^r(0, T; W^{2,r}(\Omega))$ and $\nabla \pi_v, v_t \in L^r(0, T; L^r(\Omega))$.*

Proof. See, e.g., [23]. □

We conclude our considerations on these auxiliary results by stressing that the solutions of Theorems 3 and 6 are smooth up to the boundary if data are smooth. As well solution of Theorems 5 is smooth up to the boundary for all $t > 0$. In the next sections these properties of regularity are tacitly employed. For example, they are considered in order to achieve the regularity claimed in Theorem 1.

3. Some results obtained in [3,4]

We recall that

THEOREM 7. *Let us consider the initial boundary value problem (2). Then there exists a $T_0 > 0$ such that, for all $u_0 \in L^\infty(\Omega)$ divergence free, there exists a unique solution $(u, \pi_u) \in C^2((0, T_0 \times \overline{\Omega}) \times C^1((0, T_0 \times \overline{\Omega})$ to the Stokes problem (2), with $u(t, x)$ $*$-weakly continuous in $t = 0$. Also the following estimates hold*

$$|u(t, x)| \leq c\|u_0\|_\infty, \quad \text{for all} \quad (t, x) \in C([0, T_0) \times \overline{\Omega}),$$
$$\sum_{|\alpha|=1} t^{\frac{|\alpha|}{2}} \|D^\alpha u(t)\|_\infty + t\|u_t(t)\|_\infty + t\|\nabla \pi_u(t)\|_\infty \leq c\|u_0\|_\infty, \quad \text{for all} \quad t \in [0, T_0] \tag{21}$$

where c is independent of u_0.

LEMMA 6. *Let $u_0 \in L^\infty(\Omega) \cap L^p(\Omega)$, $p \in (1, \infty)$. Denoted by (u, π_u) and (v, π_v) the solutions corresponding to u_0 by virtue of Theorems 5 and 7, respectively. Then the solutions coincide up to function of t for the pressure fields.*

Proof. This result is an immediate consequence of the approach employed in [3]–[4]. Hence we omit details. □

LEMMA 7. *Let (u, π_u) be the solution of Theorem 7. Set $\overline{\pi}_u(t) := |\Omega'|^{-1} \int_{\Omega'} \pi_u(t, x)dx$ where $\Omega' \subset \Omega$ bounded with $\partial(\Omega - \Omega') \cap \partial\Omega = \emptyset$, then, for some $\mu \in (0, \frac{1}{2})$, we get*

$$\|\pi_u(t) - \overline{\pi}_u(t)\|_{C(\Omega')} \leq c\|u_0\|_\infty t^{\mu-1}, \tag{22}$$

with constant c independent of u_0.

Proof. The pressure field π_u satisfied equation (14) with $N := \nabla \times u$. By virtue of (21), we satisfy the hypotheses of Lemma 4. Hence estimate (17) holds, which proves (22). □

4. Existence of a solution to problem (2) with a special initial datum

We are going to prove the following result:

THEOREM 8. *Let $u_0 \in \mathbb{L}^\infty(\Omega)$ divergence free in weak sense. Assume that for some $R > 0$, supp $u_0 \subset \Omega_R$. Then there exist $u_R^\infty(t) \in C([0, T))$ with $u_R^\infty(0) = 0$ and a solution (u, π_u) to problem (2) such that*

$$|u_R^\infty(t)| + \|u(t)\|_\infty \leq c_1(R)\|u_0\|_\infty \,, \text{ for all } t > 0 \,, \tag{23}$$

where we set $u_R^\infty := -|\partial\Omega|^{-1}\int_{\partial\Omega} U(t, \xi)\mathrm{d}\xi$ with U solution furnished by Lemma 8, and (u, π_u) enjoys the properties (3)–(4) and $c_1(R) := c + c(r)[1 + R^{-2}], r \in (2, \infty)$, where the constants c and $c(r)$ are independent of the datum of the problem. Finally, we get $\lim_{t\to\infty} \|u(t)\|_\infty = \lim_{t\to\infty} u_R^\infty(t) = 0$.

We need some lemmas.

LEMMA 8. *Let u_0 be as in Theorem 8. Then, for some $c(t) \in C(0, T)$, there exists a unique solution $(U(t, x), c(t))$ to problem (5) such that, for all $\eta > 0$, U is divergence free, $U \in C(\eta, T; C(\mathbb{R}^2) \cap C^2(\mathbb{R}^2))$ with $U_t \in C(\eta, T; C(\mathbb{R}^2))$ and*

$$\begin{aligned}
&\|U(t)\|_\infty \leq \|u_0\|_\infty \text{ for all } t > 0 \,, \\
&t\|U_{tt}(t)\|_\infty + \|D^2 U(t)\|_\infty + \|U_t(t)\|_\infty \leq ct^{-1}\|u_0\|_\infty \,, \text{ for all } t > 0 \,, \\
h \in \mathbb{N}, \ &t^h\|D_t^h U(t)\|_{C(\mathbb{R}^2 - \Omega_{\frac{R}{2}})} + t\|D^2 U(t)\|_{C(\mathbb{R}^2 - \Omega_{\frac{R}{2}})} \leq ce^{-\frac{R^2}{8t}}\|u_0\|_\infty \,, \text{ and for all } t > 0 \,, \\
&\text{for all } t \geq 0 \,, \ \lim_{|x|\to\infty} |U(t, x)| = 0 \,, \text{ for all } x \in \mathbb{R}^2 \,, \ \lim_{t\to\infty} |U(t, x)| = 0 \,,
\end{aligned} \tag{24}$$

where c is a constant independent of u_0. Finally, we also get $U \in C([0, T); C(\mathbb{R}^2 - \Omega_{\frac{R}{2}}))$ with $\lim_{t\to 0} \|U(t)\|_{C(\mathbb{R}^2 - \Omega_{\frac{R}{2}})} = 0$, and $\lim_{t\to 0} (U(t), \psi) = (u_0, \psi)$ for all $\psi \in \mathscr{C}_0(\mathbb{R}^2)$.

Proof. We consider the solution (U, π_U) with the kinetic field represented by means of the fundamental solution of the heat equation and the pressure field given by any continuous function $c(t) \in C((0, T))$. Estimates $(24)_{1,2,4}$ are well known, as well $\lim_{t\to 0}(U(t) - u_0, \psi) = 0$ for all $\psi \in \mathscr{C}_0(\mathbb{R}^2)$ and the asymptotic estimates of $U(t, x)$. For estimate $(24)_3$ we recall that $|x - y| \geq |y| - |x| > |y| - \frac{R}{2} > \frac{R}{2}$ for all $x \in \mathbb{R}^2 - \Omega_{\frac{R}{2}}$ and $y \in \Omega_R$. Hence, by virtue of our assumption for the initial data, we get

$$\begin{aligned}
|U(t, x)| &\leq \int_{\Omega_R} H(t, x - y)|u_0(y)|\mathrm{d}y \\
&\leq ce^{-\frac{R^2}{8t}}\|u_0\|_\infty \int_{\Omega_R} \frac{e^{-\frac{|x-y|^2}{8t}}}{t} \mathrm{d}y \,, \text{ for all } (t, x) \in (0, \infty) \times \mathbb{R}^2 - \Omega_{\frac{R}{2}} \,.
\end{aligned}$$

Hence the thesis holds. Analogously, we get the estimates for $t^h\|D^h U(t)\|_\infty$, which completes $(24)_3$. The above estimates also imply $U \in C([0, T); C(\mathbb{R}^2 - \Omega_{\frac{R}{2}}))$. $\quad\square$

LEMMA 9. *Let U be the solution of Lemma 8. For the vector function $u_R^\infty(t)$ introduced in Theorem 8, we get*

$$
\begin{aligned}
|u_R^\infty(t)| &\leq c e^{-\frac{R^2}{8t}} \|u_0\|_\infty, \ t > 0, \\
|\dot{u}_R^\infty(t)| &\leq c e^{-\frac{R^2}{8t}} \|u_0\|_\infty t^{-1}, \ t > 0.
\end{aligned}
\tag{25}
$$

Proof. Estimates (25) are immediate from estimates (24). □

LEMMA 10. *Assume in Theorem 3 $a := -U(t, x) - u_R^\infty(t)$ where $u_R^\infty(t)$ is the vector function introduced in Theorem 8. Then we get*

$$
\begin{aligned}
(1 + |x|)|V(t, x)| &\leq c g_0(t) \|u_0\|_\infty, \ \text{for all } x \in \Omega, \\
\|D^2 V(t)\|_q + \|\nabla \pi_V(t)\|_q &\leq c g(t) \|u_0\|_\infty,
\end{aligned}
\tag{26}
$$

where c is independent of a (i.e. u_0). In particular, set $\overline{\pi}_V := |\Omega'|^{-1} \int_{\Omega'} \pi_V dx$, we get

$$
\|\pi_V(t) - \overline{\pi}_V(t)\|_{C(\overline{\Omega}')} \leq \overline{c}(\Omega') g(t) \|u_0\|_\infty,
\tag{27}
$$

where $g_0(t) \in [0, 1]$ and $\lim_{t \to 0} g_0(t) = 0$, $g(t) = g_0(t) R^{-2}$ for $t \in [0, 1]$ and $g(t) = t^{-1}$ for $t \geq 1$.

Proof. Taking the right hand side of estimate $(24)_3$ into account, we get $\|U(t)\|_{C(\partial\Omega)}$ less than $c\|u_0\|_\infty g_0(t)$, and $\|U(t)\|_{W^{2-\frac{1}{q},2}(\partial\Omega)}$ less then $c\|u_0\|_\infty g(t)$. Estimates (26) are consequence of (12)–(13) and $(24)_3$. The Sobolev embedding, applied with $q > 2$, and the Poincaré inequality ensure that $\|\pi_V(t) - \overline{\pi}_V(t)\|_{C(\overline{\Omega}')} \leq c(\Omega')\|\nabla\pi_{V(t)}\|_q$. Then $\|\nabla\pi_V(t)\|_q$ is estimated by means of $(26)_2$, thus we arrive at (27). □

LEMMA 11. *In Theorem 3 assume $a := -U(t, x) - u_R^\infty(t)$, where $u_R^\infty(t)$ is the vector function introduced in Theorem 8. Then, $h \in \mathbb{N}$, we get*

$$
(1 + |x|)\|D_t^h V(t)\|_\infty \leq c\|D_t^h U(t)\|_{L^\infty(\partial\Omega)} \leq c g(t, h),
\tag{28}
$$

with c independent of a (i.e. u_0), and $g(t, h) = c(h, R) g_0(t)$ for $t \in [0, 1]$ and $g(t, h) \leq t^{-h}$ for $t \geq 1$, where function $g_0(t)$ is the same of the above lemma.

Proof. The first inequality is a consequence of (12) and the last inequality is deduced increasing the right hand side of $(24)_3$. □

REMARK 2. For the next arguments, we explicitly point out that $g(t, 1) = g(t)$.

Proof of Theorem 8

In the following we indicate by $\overline{c}(R)$ a constant $< \infty$, for all $R > 0$, and independent of u_0, whose value is not important in the computation. We look for a solution $u(t, x) := U(t, x) + u_R^\infty(t) + V(t, x) + W(t, x)$ and $\pi_u(t, x) := -\dot{u}_R^\infty(t) \cdot x + \pi_V(t, x) + \pi_W(t, x) + c(t)$, where $(U, c(t))$ is the solution furnished by Lemmas 10–11. For all $t > 0$, $u_R^\infty(t)$ is the integral media of $-U$ on $\partial\Omega$. By virtue of the regularity properties of U in a neighborhood of $\partial\Omega$, we have that $u_R^\infty(t)$ is continuous function

of t, uniformly bounded in t by $\|u_0\|_\infty$, and differentiable for $t > 0$. The pair (V, π_V) is the solution furnished by Lemma 3 and assuming boundary data $a := -U - u_\infty$ on $\{t\} \times \partial\Omega$. The pair (W, π_W) is the solution to the following problem:

$$W_t - \Delta W = -\nabla\pi_W - V_t, \ \nabla \cdot W = 0, \ \text{in } (0, T) \times \Omega, \tag{29}$$
$$W = 0 \text{ on } (0, T) \times \partial\Omega, \ W = 0 \text{ on } \{0\} \times \Omega,$$

where V_t is the derivate of V, hence solution to problem (11) and corresponding to the boundary data $-U_t - \dot{u}_\infty$. Taking into account of estimates (28), we find $r > 2$ such that $V_t \in L^r(0, T; L^r(\Omega))$. Hence, by virtue of Theorem 6, there exists a unique solution to problem (29), with $W \in C([0, T; J^{1,r}(\Omega)) \cap L^r(0, T; W^{2,r}(\Omega))$ and $W_t, \nabla\pi_W \in L^r(0, T; L^r(\Omega))$. By embedding, we deduce $W \in C([0, T); C(\overline{\Omega}))$ with $W(0, x) = 0$. Now, our aim is to find a bound of $\|W(t)\|_\infty$ by means of $\|u_0\|_\infty$. For $t > 0$, we set $\widehat{\varphi}(\tau, x) := \varphi(t - \tau, x)$ for all $(\tau, x) \in (0, t) \times \Omega$, where (φ, π_φ) is the solution to problem (19). Taking into account Theorem 4, multiplying by $\widehat{\varphi}$ the equation $(29)_1$, integrating by parts on $(0, t) \times \Omega$, we get

$$(W(t), \phi) = -\int_0^t (V_\tau, \varphi(t - \tau))d\tau. \tag{30}$$

Applying Hölder's inequality and recalling the estimates (28) for V_t, and recalling estimate ii. of Theorem 4 for φ, we get

$$|(V_\tau(\tau), \varphi(t - \tau))| \le \|V_\tau(\tau)\|_r \|\varphi(t - \tau)\|_{r'} \le c\|u_0\|_\infty\|\phi\|_1 (t - \tau)^{-\frac{1}{r}} g(\tau) \tag{31}$$

with $c = c(r)$ independent of u_0, ϕ and t. Hence from (30) by an easy computation we deduce

$$|(W(t), \phi)| \le c(1 + R^{-2})\|u_0\|_\infty\|\phi\|_1.$$

Since ϕ is arbitrary we arrive at

$$\|W(t)\|_\infty \le c_1(R)\|u_0\|_\infty, \ \text{for all } t > 0. \tag{32}$$

Collecting the L^∞-estimates related to U, that is $(24)_1$, estimate (25) for $u_R^\infty(t)$, and $(26)_1$ related to V, finally, estimate (32) for $W(t)$, we complete the proof of the existence of (u, π_u) with estimate (23). Concerning the regularity properties of (u, π_u) they are straightforward for any term of the sum which defines our solution. Finally, we give estimates $(3)_4$ and $(4)_1$. In the light of our construction of u, we consider separately any term of the sum. The behaviors of the time derivates of the terms U, u_R^∞ are immediate as well, by virtue of (28), the one for V. So we limit ourselves to consider W. In order to estimate $\|D_t W(t)\|_\infty$, we before achieve the estimates for $\|D_t W(t)\|_r$, then we achieve the estimates for L^∞-norm. Via estimates (28), for all $r_1 > 2$ and $\overline{\mu}_0 \in (0, 1]$, we have $\|V_t\|_{r_1} \le g_1(t, 1)\|u_0\|_\infty t^{\mu_0 - 1}$, $t > 0$. Hence, employing Theorems 5 for φ, we obtain

$$|(V_\tau(\tau), \varphi(t - \tau))| \le \|V_\tau(\tau)\|_{r_1}\|\varphi(t - \tau)\|_{r_1'} \le cg_1(\tau, 1)\|u_0\|_\infty\|\phi\|_{r'}(t - \tau)^{-\frac{1}{r}}, \ \tau \in (0, t),$$

with constants which are independent of u_0, ϕ and t. So that via (30), by an easy computation, we deduce

$$|(W(t), \phi)| \leq \overline{c}(R)\|u_0\|_\infty \|\phi\|_{r'} G_1(t),$$

where function $G_1(t) = t$ for $t \in [0, 1]$ and $G_1(t) = 1$ for $t \geq 1$. Since ϕ is arbitrary, we arrive at

$$\|W(t)\|_r \leq \overline{c}(R)\|u_0\|_\infty G_1(t), \quad \text{for all } t > 0. \tag{33}$$

Now, we consider W_t. By differentiating with respect to t the equation of W and considering the adjoint problem, integrating by parts on $(\frac{t}{2}, t) \times \Omega$, we get the following:

$$
\begin{aligned}
(W_t(t), \phi) &= \left(W_t\left(\frac{t}{2}\right), \varphi\left(\frac{t}{2}\right) \right) - \int_{\frac{t}{2}}^{t} (V_{\tau\tau}, \varphi(t - \tau)) \mathrm{d}\tau \\
&= \left(P\Delta W\left(\frac{t}{2}\right), \varphi\left(\frac{t}{2}\right) \right) - \left(PV_t\left(\frac{t}{2}\right), \varphi\left(\frac{t}{2}\right) \right) - \int_{\frac{t}{2}}^{t} (V_{\tau\tau}, \varphi(t - \tau)) \mathrm{d}\tau \\
&= - \left(W\left(\frac{t}{2}\right), \varphi_t\left(\frac{t}{2}\right) \right) - \left(PV_t\left(\frac{t}{2}\right), \varphi\left(\frac{t}{2}\right) \right) - \int_{\frac{t}{2}}^{t} (V_{\tau\tau}, \varphi(t - \tau)) \mathrm{d}\tau.
\end{aligned}
$$

Hence, applying Hölder's inequality, we get

$$
\begin{aligned}
|(W_t(t), \phi)| \leq & \left\| W\left(\tfrac{t}{2}\right) \right\|_r \left\| \varphi_t\left(\tfrac{t}{2}\right) \right\|_{r'} + \left\| V_t\left(\tfrac{t}{2}\right) \right\|_r \left\| \varphi\left(\tfrac{t}{2}\right) \right\|_{r'} \\
& + \int_{\frac{t}{2}}^{t} \|V_{\tau\tau}(\tau)\|_r \|\varphi(t - \tau)\|_{r'} \mathrm{d}\tau
\end{aligned} \tag{34}
$$

From (28) we have $\|V_t(t)\|_r \leq cg(t, 1)\|u_0\|_\infty$ and $\|V_{\tau\tau}(\tau)\|_r \leq c\overline{c}(R)\|u_0\|_\infty g(t, 2)$, recalling ii. of Theorem 4, by virtue of (33), we obtain

$$
\begin{aligned}
|(W_t(t), \phi)| \leq & \,\overline{c}(R)\|u_0\|_\infty \|\phi\|_1 \Big[G_1(t) + t^{-1-\frac{1}{r}} g(t, 1) \\
& + \int_{\frac{t}{2}}^{t} \left[\tau \chi_{[0,1]}(\tau) + (1 - \chi_{[0,1]}(\tau)\tau^{-2})\tau^{-1-\frac{1}{r}} \right], \quad t > 0.
\end{aligned}
$$

Since ϕ is arbitrary we easily deduce

$$\|W_t(t)\|_\infty \leq \overline{c}(R)t^{-1}\|u_0\|_\infty, \quad t > 0.$$

By the same arguments employed to obtain estimate (33), we get

$$\|W_t(t)\|_r \leq \overline{c}(R)G_1(t)\|u_0\|_\infty, \quad t > 0. \tag{35}$$

Via inequality (9) and estimate (18), we easily deduce that

$$
\begin{aligned}
\|\nabla W(t)\|_\infty &\leq c\|D^2 W(t)\|_r^{\frac{2+r}{2r}} \|W(t)\|_r^{\frac{r-2}{2r}} \leq c(\|W_t + PV_t\|_r + \|W\|_r)^{\frac{2+r}{2r}} \|W\|^{\frac{r-2}{r}} \\
&\leq c(\|W_t + V_t\|_r^{\frac{2+r}{2r}} \|W\|_r^{\frac{r-2}{2r}} + \|W\|_r.
\end{aligned}
$$

By virtue of (28), (33), and (35), we easily arrive at

$$\|\nabla W(t)\|_\infty \le \overline{c}(R)\|u_0\|_\infty\,,\ \ t > 0\,.$$

Our construction, by straightforward considerations related to U and V, furnishes also for the term $\|\nabla u(t)\|_\infty$ the wanted estimate, so we arrive at the proof of (3)$_4$. Now we are in a position to prove (4)$_3$. Since for all $t > 0$ we get $(W, \nabla\pi_W) \in W^{2,r}\cap J^{1,r}(\Omega)\times L^r(\Omega)$, by virtue of Lemma 5, we deduce estimate (18) with $P\Delta W := -W_t - PV_t$. In particular, set $\overline{\pi}_W := |\Omega'|^{-1}\int_{\Omega'}\pi_W\mathrm{d}x$, we get

$$\begin{aligned}
&\|\pi_W(t) - \overline{\pi}_W(t)\|_{C(\Omega')}\\
&\le c\big[\|\nabla\pi_W(t)\|_{L^r(\Omega')}^b\|\pi_W(t) - \overline{\pi}_W(t)\|_{L^r(\Omega')}^{1-b} + \|\pi_W(t) - \overline{\pi}_W(t)\|_{L^r(\Omega')}\big]\\
&\le c\|\nabla\pi_W(t)\|_r \le c\big[\|W_t(t)\|_r + \|V_t\|_r + \|W(t)\|_r\big]\,,\ \ t > 0\,.
\end{aligned}$$

Employing estimates (27) for π_V, (28), (33) and (35) for V_t, W and W_t, respectively, we arrive at

$$\|\pi_V(t) + \pi_W(t) - |\Omega'|^{-1}\int_{\Omega'}\big[\pi_V(t,x) - \pi_W(t,x)\big]\mathrm{d}x\|_{C(\Omega')} \le \overline{c}(R)c(\Omega')\|u_0\|_\infty\,,\ \ t > 0\,. \tag{36}$$

Thanks to estimate (25)$_2$ we also have $\|\dot{u}_R^\infty(t)\cdot x - |\Omega'|^{-1}\int_{\Omega'}\dot{u}_R^\infty(t)\cdot x\mathrm{d}x\|_{C(\Omega')} \le \overline{c}(R)\|u_0\|_\infty$. Hence we can consider achieved (4)$_3$. The last claims of the theorem related to the asymptotic behaviors are not difficult to prove. For the sake of the brevity, we omit the details. This completes the proof of the theorem. □

5. Existence of a solution to problem (2) for data with compact support

For the results of the following theorem, we employ on some neighborhood of $t = 0$ the results by Abe and Giga, and then the ones typical of the L^q-theory. We are going to prove

THEOREM 9. *Let u_0 be in (2) with compact support enclosed in $\Omega - \Omega_{\overline{R}}$, and assume $u_0 \in L^\infty(\Omega)$ divergence free. Then there exists a unique solution (u, π_u) to problem (2) such that, for all $\eta,\ T > 0$, $(u, \pi_u) \in C(\eta, T; C^2(\overline{\Omega})) \times C(\eta, T; C^1(\overline{\Omega}))$ and*

$$\begin{aligned}
&\lim_{t\to 0}(u(t), \psi) = (u_0, \psi)\,,\ \textit{for all } \psi \in \mathscr{C}_0(\Omega)\,,\\
&\|u(t)\|_\infty \le c|\Omega - \Omega_{\overline{R}}|^\alpha\|u_0\|_\infty\,,\ \textit{for all } t > 0\,.
\end{aligned} \tag{37}$$

Solution (u, π_u), with $\pi_u \equiv \Pi_u + c(t)$, enjoys the regularity properties (3)$_4$ and (4)$_{2,3}$. Finally, we get $\lim_{t\to\infty}\|u(t)\|_\infty = 0$.

Proof. By virtue of Theorem 7 we know that there exists a solution (u, π_u) such that, for all $\eta,\ T > 0$, $(u, \pi_u) \in C(\eta, T; C^2(\overline{\Omega})) \times C(\eta, T; C^1(\overline{\Omega}))$, and there exist an interval $(0, T_0)$ and constant c, both independent of u_0, such that

$$\|u(t)\|_\infty \le c\|u_0\|_\infty\,,\ \text{ for all } t \in (0, T_0)\,. \tag{38}$$

Finally, the limit property $(37)_1$ holds. So that we have to complete the proof of $(37)_2$ for all $t \geq T_0$. Since, for all $r > 1$, $u_0 \in J^r(\Omega)$, by virtue of Theorem 5, there exists a constant $c(r)$ such that

$$\|u(t)\|_\infty \leq ct^{-\frac{1}{r}} \|u_0\|_r \leq ct^{-\frac{1}{r}} |\Omega - \Omega_{\overline{R}}|^{\frac{1}{q}} \|u_0\|_\infty \,, \text{ for all } t \geq T_0 \,.$$

Since Lemma 6 holds, coupling the last estimate and (38) we complete the proof of $(37)_2$. Estimates $(3)_4$ for u, by virtue of Theorem 7, are true on $(0, T_0)$. In order to extend them for $t \geq T_0$, as already made for (37), it is enough to employ the properties of the solution u in L^r-setting. Concerning π_u we get estimate (22) on $(0, T_0)$. Employing again the results in L^r, we complete the proof of $(4)_{2,3}$. Finally, the asymptotic property is immediate.   □

REMARK 3. Without losing the generality, we can assume $0 \in \mathbb{R}^2 - \overline{\Omega}$. Hence a possible estimate of $|\Omega - \Omega_{\overline{R}}|^{\frac{1}{q}}$ is given by $c(\overline{R} + \delta)^{\frac{2}{q}}$, where $\delta := diam(\mathbb{R}^2 - \overline{\Omega})$.

6. Proof of the Theorems 1 and 2

6.1. Existence

Proof. We introduce a nonnegative smooth cutoff function h_R with $h_R = 1$ for $x \in \overline{\Omega} - \Omega_R$ and $h_R = 0$ for $x \in \overline{\Omega}_{2R}$. We denote by $C_{R,2R}$ the compact support of ∇h_R. The following decomposition holds:

$$u_0 = u_0(1 - h_R) + u_0 h_R = \left[u_0(1 - h_R) + b^1\right] + \left[u_0 h_R + b^2\right] = u_0^1 + u_0^2 \,,$$

where the fields b^i, $i = 1, 2$, are the Bogovski solutions to the problems

$$\nabla \cdot b^1 = u_0 \cdot \nabla h_R \text{ and } \nabla \cdot b^2 = -u_0 \cdot \nabla h_R \,, \text{ in } C_{R,2R} \,.$$

It is known that by the Bogowski representation formula we get

$$b^1 + b^2 = 0 \text{ in } C_{R,2R} \equiv \text{supp} \, \nabla h_R \,,$$

and we get $\|u_0^i\|_\infty \leq c\|u_0\|_\infty$, $i = 1, 2$, with c independent of R and u_0. Hence $u_0^i \in L^\infty(\Omega)$, $i = 1, 2$, is divergence free, function u_0^1 has support far from $\partial\Omega$, that is $R = dist(\text{supp} \, u_0^1, \partial\Omega) > 0$, and trivially $u_0^1 \in L^\infty(\Omega) \cap C(\overline{\Omega} - \Omega_R)$. Instead, u_0^2 has compact support. Corresponding to these data, we obtain (u^1, π_{u^1}) and (u^2, π_{u^2}) solutions to problem (2), by virtue of Theorem 8 and of Theorem 9, respectively. Setting $(u, \pi_u) \equiv (u^1 + u^2, \pi_{u^1} + \pi_{u^2})$, the pair (u, π_u) is a solution to problem (2) with an initial datum u_0. The field u satisfies estimates $(3)_3$ with constant $c(R) = c\left[1 + c(r)R^{-\frac{2}{r}} + R^{2\alpha}\right]$ as a consequence of estimate (23) for u^1 and of estimate (37) for u^2, provided that one considers Remark 3. The regularity as well properties (3) and (4) are consequence of the properties of the component solutions.   □

6.2. Uniqueness

We premise the following

LEMMA 12. *Assume that $u \in L^r_{loc}(\overline{\Omega})$, $r \in (1, \infty)$, $u \to 0$ for large $|x|$ and weakly divergence free. Assume also*

$$|(u, \varphi)| \leq A\|\varphi\|_{L^{r'}(\Omega_R)}, \text{ for all } \varphi \in \mathscr{C}_0(\Omega_R), \tag{39}$$

where, we recall, $\Omega_R := \{x \in \Omega : dist(x, \partial\Omega) > R\}$. Then we get

$$\|u\|_r \leq A + c(R)\|u\|_{L^r(\Omega \cap B(R+1))}. \tag{40}$$

Proof. Here $c(R) > 0$ is a constant which is independent of u and whose value is not interesting for us. We introduce a smooth non negative cutoff function h_R such that $h_R(x) = 1$ for $|x| \leq R + \varepsilon$ and $h_R = 0$ for $|x| \geq R + 1 - \varepsilon$. We consider the decomposition

$$u = uh_R + b_1 + (1 - h_R)u + b_2,$$

where we indicated by b_i, $i = 1, 2$, the Bogovski solutions to the problems

$$\nabla \cdot b_1 = -u \cdot \nabla h_R \text{ and } \nabla \cdot b_2 = u \cdot \nabla h_R \text{ in } R < |x| < R + 1,$$
$$\text{with } b_i = 0 \text{ on } |x| = R \cup |x| = R + 1.$$

We mean that the solutions b_i are extended to 0 on Ω. Moreover, we recall that $\|b_i\|_r \leq C(R)\|\nabla b_i\|_r \leq c(R)\|u\|_{L^r(\Omega \cap B(R+1))}$, and by construction, the functions $uh_R + b_1$ and $(1 - h_R)u + b_2$ have weak divergence free. From the assumption (39), we get

$$|(uh_R + b_1 + (1 - h_R)u + b_2, \varphi)| = |(u, \varphi)| \leq A\|\varphi\|_{r'}, \text{ for all } \varphi \in \mathscr{C}_0(\Omega),$$

which furnishes

$$|((1 - h_R)u + b_2, \varphi)| \leq A\|\varphi\|_{r'} + c(R)\|u\|_{L^r(\Omega \cap B(R+1))}\|\varphi\|_{r'},$$

here we toke the compact support of b_1 into account. Since by hypothesis $(1 - h_R)u + b_2 \to 0$ for $|x| \to \infty$ and by construction $(1 - h_R)u + b_2 = 0$ on $\partial\Omega_R$, from the last estimate we deduce

$$\|(1 - h_R)u + b_2\|_{L^r(\Omega_R)} \leq A + (1 + c(R))\|u\|_\infty.$$

Recalling that $h_R = 1$ for $|x| > R + 1$ and b_2 has compact support, the last estimate, in particular, furnishes

$$\|u\|_{L^r(\Omega_{R+1})} \leq A + (1 + c(R))\|u\|_\infty.$$

On the other hand we have $u \in L^r_{loc}(\overline{\Omega})$, hence in the end we obtain $u \in J^r(\Omega)$. $\square$

Proof of the uniqueness. The first step is to prove that $u - v$ belongs to $L^r(\Omega)$ for some $r \in (1, \infty)$. For this task we argue by duality. Let $\varphi_0 \in \mathscr{C}_0(\Omega_R)$ (that is $dist(\partial supp, \partial\Omega) > R$) and let $\varphi(t, x)$ be the solution to the Cauchy problem (5) corresponding to φ_0. It is well known that φ is smooth, belongs to $L^1(\mathbb{R}^2)$, for all $t \geq 0$, and $|\varphi(t, x)| \leq c(\varphi_0)t^{-1}\exp[-|x|^2/8t]$ for all $t > 0$ and x such that $dist(x, \partial \operatorname{supp}\varphi_0) > \frac{|x|}{2}$. Moreover, by the assumption on support of $\varphi_0 \subset \Omega_R$, we have

$$|\varphi(t, x)| \leq ce^{-\frac{R^2}{8t}}\frac{1}{t}\int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{8t}}|\varphi_0(y)|dy$$

$$\leq ct^{-\frac{1}{r'}}e^{-\frac{R^2}{8t}}\|\varphi_0\|_{r'}, \text{ for all } t > 0 \text{ and } x \in \mathbb{R}^2 - \Omega_{\frac{R}{2}}. \quad (41)$$

We set $(w, \pi_w) := (u - v, \pi_u - \pi_v)$. Since both the kinetic fields satisfy the limit property at infinity given in (3)$_2$, the difference $w \to 0$ for large $|x|$. By virtue of (3)$_1$ for both the pressure fields, then we have $|\pi_w(t, x)| \leq c(t)(1 + |x|)$, as well for π_w (4)$_3$ holds. Multiplying by $\varphi(t - \tau, x)$, $\tau \in [0, t]$, the equation of (w, π_w) and integrating by parts on $(s, t) \times \Omega$, we get

$$(w(t), \varphi_0) = (w(s), \varphi(t - s)) - \int_s^t \int_{\partial\Omega} \nu \cdot T(w(\tau), \pi_w(\tau)) \cdot \varphi(t - \tau)d\mathcal{H}^1 d\tau. \quad (42)$$

In the previous integration by parts we take into account that, on $(0, t) \times \Omega, \varphi(t - \tau, x)$ is a solution to the Stokes adjoint Cauchy problem. Employing estimate (41)$_1$ related to φ, recalling estimate (3)$_4$ for $\|\nabla w(t)\|_\infty$ and estimate (4)$_3$ for $\|\pi_w(t)\|_\infty$, we have that the right hand side of (42) is finite for any $\varphi \in \mathscr{C}_0(\Omega_R)$ and $s \geq 0$. Hence, letting $s \to 0$, the $*$-weak continuity of w and properties of φ ensure a limit for the right-hand side with a bound, for all $\mu \in (0, \frac{1}{2})$, of the kind

$$|(w(t), \varphi_0)| \leq c\|u_0\|_\infty(1 + t^\mu)\|\varphi_0\|_{r'}, \text{ for all } \varphi_0 \in \mathscr{C}_0(\Omega_R).$$

Since $w \to 0$ for $|x| \to \infty$, via the last estimate, by virtue of Lemma 12, we deduce

$$\|w(t)\|_r \leq c(R)(1 + t^\mu)\|u_0\|_\infty, \text{ for all } t > 0. \quad (43)$$

Now we consider the solution to problem (2) with an initial data $\varphi_0 \in \mathscr{C}_0(\Omega)$. Employing Theorem 5, we again denote the solution (no confusion occurs) by (φ, π_φ). Introduced a sequence of smooth and nonnegative cutoff functions, say $\{\zeta^m(x)\}$, with $\zeta^m = 1$ for $|x| \leq m$ and $\zeta^m = 0$ for $|x| \geq 2m$, by virtue of Bogovski's lemma, we can construct a sequence $\{\varphi^m\}$ divergence free and with compact support in $\overline{\Omega}$ that converges to φ with respect to the metric $C(0, T; J^{1,r'}(\Omega)) \cap L^r(0, T; W^{2,r'}(\Omega))$ and with $\{\varphi_t^m\}$ that converges to φ_t with respect to the metric $L^{r'}(0, T; L^{r'}(\Omega))$. Multiplying equation (2) of (w, π_w) by $\varphi^m(t - \tau, x)$ and integrating by parts on $(s, t) \times \Omega$, we obtain

$$(w(t), \varphi_0^m) = (w(s), \varphi^m(t - s)) - \int_s^t \int_\Omega \left[(w(\tau), \varphi_\tau^m(t - \tau)) + (w(\tau), \Delta\varphi^m(t - \tau))\right]d\tau.$$

Letting $m \to \infty$, we have

$$(w(t), \varphi_0) = (w(s), \varphi(t-s))$$
$$- \int_s^t \int_\Omega \Big[(w(\tau), \varphi_\tau(t-\tau)) + (w(\tau), P\Delta\varphi(t-\tau)) \Big] \mathrm{d}\tau = (w(s), \varphi(t-s)),$$

thus we write the formula

$$(w(t), \varphi_0) = (w(s), \varphi^m(t)) + (w(s), \varphi^m(t-s) - \varphi^m(t)) + (w(s), \varphi(t-s) - \varphi^m(t-s)).$$

Recalling the $*$-weak continuity in t of w and estimate (43), letting $s \to 0$ and subsequently $m \to \infty$, we get $(w(t), \varphi_0) = 0$ for all $\varphi_0 \in \mathscr{C}_0(\Omega)$. Since $w \in J^r(\Omega)$ for all $t > 0$, we arrive at $w \equiv 0$, and from the equation $\nabla\pi_w = 0$. $\qquad\square$

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