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Theories and analyses of functionally graded circular plates

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ABSTRACT

This paper presents the governing equations and analytical solutions of the classical and shear deformation theories of functionally graded axisymmetric circular plates. The classical, first-order, and third-order shear deformation theories are presented, accounting for through-thickness variation of two-constituent functionally graded material, modified couple stress effect, and the von Kármán nonlinearity. Analytical solutions for bending of the linear theories, some of which are not readily available in the literature, are included to show the influence of the material variation, boundary conditions, and loads.

1. Background

Recently, the authors have published a comprehensive paper on theories and analytical solutions of the bending of functionally graded material (FGM) beams [1]. The present paper considers functionally graded axisymmetric circular plates. Like beam theories, plate theories are derived from the three-dimensional elasticity theory by making certain simplifying assumptions concerning the kinematics of deformation and stress states. The present study has the dual purpose of presenting the classical, first-order, and third-order theories of circular plates and analytical solutions of the associated linear theories for the case of bending. The theories to be described account for the through-thickness variation of two-constituent material, modified couple stress effect, and geometric nonlinearity in the form of the von Kármán nonlinear strains.

For the convenience of analysis, we use the cylindrical coordinate system (r, θ, z) to describe the deformation and stress state in circular plates. The word "axisymmetry" refers to the case in which the solution (i.e., displacements as well as stresses) is independent of the angular coordinate θ (see Fig. 1). This is possible if and only if the geometry, material properties, loads, and boundary conditions are also independent of θ. We assume such is the case in this paper.

The variation of properties through the thickness is considered to be of the power law type. A typical material property P is varied as a function of the thickness coordinate z as

$$P(z) = \left(P_1 - P_2\right) f(z) + P_2, \quad f(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n \tag{1}$$

where P_1 and P_2 are the material properties of material 1 (at the top) and 2 (at the bottom), respectively, and n is the volume fraction exponent (power-law index). Note that when $n = 0$, we obtain the single-material beam (with property P_1).

A large number of papers on modified couple stress theories for beams and plates, including circular plates, can be found in the literature (see, e.g., [2]). The modified couple stress theory brings a single length scale through a phenomenological constitutive model relating the couple stress to the curvature relation (see, e.g., [3–9]). The contribution due to the couple stress is included into a plate theory by modifying the strain energy expression of the plate. To this end, suppose that $\mathbf{u}$ denotes the displacement vector of an arbitrary point in the plate. Then the rotation vector $\boldsymbol{\omega}$, which represents the macro-rotation, is defined as

$$\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\nabla} \times \mathbf{u} \tag{2}$$

The curvature tensor χ represents the rate of change of rotations, which are assumed to be small:

$$\chi = \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{\omega} + (\boldsymbol{\nabla}\boldsymbol{\omega})^{\mathrm{T}}\right] \tag{3}$$

The modified couple stress theory is based on the hypothesis that the rate of change of macro-rotations cause additional stresses, called *couple stresses*, in the continuum. The modified couple stress tensor $\mathbf{m}$ is related to the curvature tensor χ through the constitutive relations [10]:

$$\mathbf{m} = 2G\ell^2\,\chi \tag{4}$$

where ℓ is the length scale parameter and G is the shear modulus.

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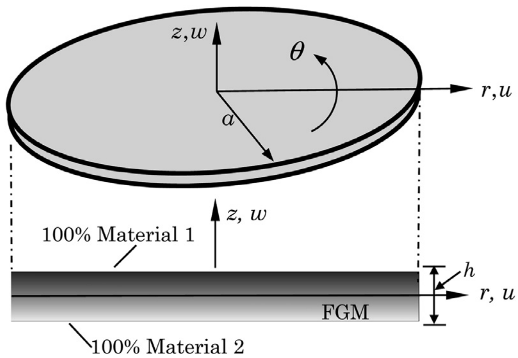

Fig. 1. Geometry and coordinate system used for circular plates.

The strain energy potential of a circular plate is modified to account for the energy due to modified couple stress as

$$U = \frac{1}{2}\int_a^b \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} (\boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \mathbf{m} : \boldsymbol{\chi}) dz\right] dx \tag{5}$$

where h is the thickness of the plate, b is the inner radius of an annular plate, a is the outer radius, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}$ is the simplified Green–Lagrange strain tensor, $\mathbf{m}$ is the deviatoric part of the symmetric couple stress tensor, and $\boldsymbol{\chi}$ is the symmetric curvature tensor. In the coming sections, these relations will be specialized to various plate theories.

2. Mechanics preliminaries

2.1. Modified Green–Lagrange strains

Let $\mathbf{u}$ denote the displacement vector with components (u_r, u_θ, u_z) in the (r, θ, z) coordinate directions, respectively. Due to the assumed axisymmetry (i.e., the material properties, loads, and boundary conditions are independent of the coordinate θ), we have $u_\theta = 0$ and u_r and u_z are independent of θ. In addition, if we *assume* the inextensibility of the transverse normal lines, then u_z is only a function of the radial coordinate r.

The modified Green–Lagrange strain tensor that accounts for moderate rotations of normal lines perpendicular to the plane of the plate is given by (see Reddy [11])

$$\mathbf{E} \approx \frac{1}{2}\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}} + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r\right] \equiv \boldsymbol{\varepsilon} \tag{6}$$

where $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z)$ are the basis vectors in the cylindrical coordinate system. Thus, the nonzero strain components in the cylindrical coordinate system for the axisymmetric case are:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial u_z}{\partial r}\right)^2, \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}. \tag{7}$$

2.2. Curvature tensor

The only nonzero component of the rotation vector $\boldsymbol{\omega}$ for axisymmetric deformation is

$$\omega_\theta = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) \tag{8}$$

Then the nonzero components of χ for the axisymmetric case are

$$\chi_{r\theta} = \frac{1}{2}\left(\frac{\partial \omega_\theta}{\partial r} - \frac{\omega_\theta}{r}\right) = \frac{1}{4}\left(\frac{\partial^2 u_r}{\partial z \partial r} - \frac{\partial^2 u_z}{\partial r^2}\right) - \frac{1}{r}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right), \tag{9a}$$

$$\chi_{z\theta} = \frac{1}{2}\frac{\partial \omega_\theta}{\partial z} = \frac{1}{4}\left(\frac{\partial^2 u_r}{\partial z^2} - \frac{\partial^2 u_z}{\partial z \partial r}\right). \tag{9b}$$

2.3. Stress–strain relations

For a two-constituent functionally graded linear elastic material, the plane stress–strain equations relating the nonzero stresses $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{rz})$ to the nonzero strains $(\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{rz})$ of the axisymmetric case are

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{Bmatrix} = \frac{E(z)}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ 2\varepsilon_{rz} \end{Bmatrix} \tag{10}$$

where Young's modulus E varies with z according to

$$E(z) = (E_1 - E_2)v_1(z) + E_2, \quad v_1(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n \tag{11}$$

and Poisson's ratio ν is assumed to be a constant. The modified couple stress constitutive relation becomes

$$m_{r\theta} = 2G\ell^2 \chi_{r\theta}, \quad G = \frac{E}{2(1+\nu)}, \tag{12}$$

where $m_{r\theta}$ is the nonzero component of the symmetric couple stress tensor $\mathbf{m}$.

2.4. Strain energy functional

According to the modified couple stress theory, the strain energy potential for linear elastic case can be expressed as (the common factor 2π is omitted throughout the development)

$$U = \frac{1}{2}\int_0^a \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} (\boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \mathbf{m} : \boldsymbol{\chi}) dz\right] r dr \tag{13a}$$

$$= \frac{1}{2}\int_0^a \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + 2\sigma_{rz}\varepsilon_{rz} + 2m_{r\theta}\chi_{r\theta}\right) dz\right] r dr, \tag{13b}$$

where a is radius of the plate, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}$ is the simplified Green–Lagrange strain tensor defined in Eq. (6), $\mathbf{m}$ is the deviatoric part of the symmetric couple stress tensor, and $\boldsymbol{\chi}$ is the symmetric curvature tensor defined in Eq. (3).

3. Governing equations of the CPT

3.1. Displacements and strains

The total displacements (u_r, u_z) along the coordinate directions (r, z), as implied by the Love–Kirchhoff hypothesis for axisymmetric bending of circular plates, are assumed in the form

$$\begin{aligned} \mathbf{u} &= u_r\,\hat{\mathbf{e}}_r + u_z\,\hat{\mathbf{e}}_z \\ u_r(r,z) &= u(r) - z\frac{dw}{dr}, \quad u_\theta = 0, \quad u_z(r,z) = w(r), \end{aligned} \tag{14}$$

where u is the radial displacement and w is the transverse deflection of a point on the midplane of the plate. The Love–Kirchhoff hypothesis amounts to neglecting both transverse shear and transverse normal effects.

The von Kármán strains in (7) for the classical plate theory take the form

$$\varepsilon_{rr} = \varepsilon_{rr}^{(0)} + z\varepsilon_{rr}^{(1)}, \quad \varepsilon_{\theta\theta} = \varepsilon_{\theta\theta}^{(0)} + z\varepsilon_{\theta\theta}^{(1)} \tag{15}$$

where

$$\begin{aligned} \varepsilon_{rr}^{(0)} &= \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2, & \varepsilon_{rr}^{(1)} &= -\frac{d^2w}{dr^2} \\ \varepsilon_{\theta\theta}^{(0)} &= \frac{u}{r}, & \varepsilon_{\theta\theta}^{(1)} &= -\frac{1}{r}\frac{dw}{dr} \end{aligned} \tag{16}$$

The rotation and curvature components are

$$\begin{aligned} \omega_\theta &= \frac{1}{2}\left(\frac{du_r}{dz} - \frac{du_z}{dr}\right) = -\frac{dw}{dr}, \quad \chi_{z\theta} = 0, \\ \chi_{r\theta} &= \frac{1}{2}\left(\frac{d\omega_\theta}{dr} - \frac{\omega_\theta}{r}\right) = \frac{1}{2}\left(-\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) \end{aligned} \tag{17}$$

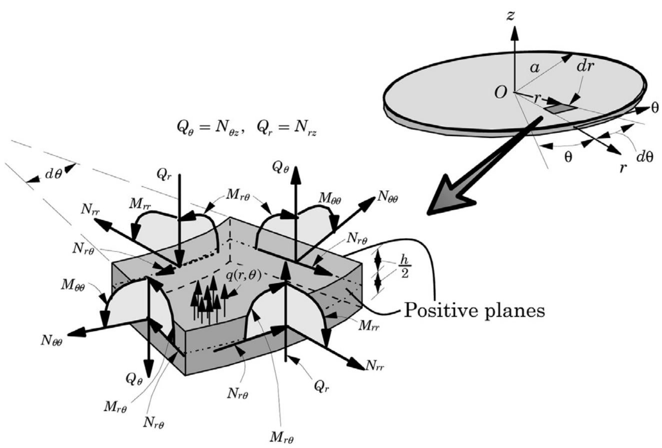

Fig. 2. An element of a circular plate with stress resultants at a "point".

3.2. Equations of equilibrium

The principle minimum total potential energy (or the principle of virtual displacements) can be used to derive the equations of equilibrium:

$$\frac{1}{r}\left[\frac{d}{dr}(rN_{rr}) - N_{\theta\theta}\right] = 0 \tag{18}$$

$$\frac{1}{r}\frac{d}{dr}(rN_{rz}) + \frac{1}{r}\frac{d}{dr}\left[\frac{d}{dr}(rP_{r\theta}) - P_{r\theta}\right] + \frac{1}{r}\frac{d}{dr}\left(rN_{rr}\frac{dw}{dr}\right) + q = 0 \tag{19}$$

$$-\frac{1}{r}\left[\frac{d}{dr}(rM_{rr}) - M_{\theta\theta}\right] + N_{rz} = 0, \tag{20}$$

where $q = q(r,t)$ is the distributed transverse load, and N_{rr}, $N_{\theta\theta}$, M_{rr}, $M_{\theta\theta}$, and $P_{r\theta}$ are the stress resultants (see Fig. 2 for the notation),

$$\begin{aligned}
N_{rr}(r) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{rr}\,dz, \qquad N_{\theta\theta}(r) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\theta\theta}\,dz\\
M_{rr}(r) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{rr} z\,dz, \qquad M_{\theta\theta}(r) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\theta\theta} z\,dz\\
P_{r\theta}(r) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} m_{r\theta}\,dz,
\end{aligned} \tag{21}$$

The boundary conditions involve specifying one element of each of the following duality pairs:

$$\begin{aligned}
&u \quad \text{or} \quad rN_{rr},\\
&w \quad \text{or} \quad r\left[V_r + \frac{d}{dr}(rP_{r\theta}) + P_{r\theta}\right] \equiv r\hat{V}_r,\\
&-\frac{\partial w}{\partial r} \quad \text{or} \quad rM_{rr} + rP_{r\theta} \equiv r\hat{M}_{rr}.
\end{aligned} \tag{22}$$

where V_r is the effective transverse shear force acting in the rz-plane

$$V_r = N_{rz} + N_{rr}\frac{dw}{dr} = \frac{1}{r}\left[\frac{d}{dr}(rM_{rr}) - M_{\theta\theta} + rN_{rr}\frac{dw}{dr}\right] \tag{23}$$

The equations of equilibrium and the duality pairs for the case in which the couple stress effect is not considered are obtained by setting $P_{r\theta} = 0$.

3.3. Plate constitutive relations

The stress resultants N_{rr}, $N_{\theta\theta}$, M_{rr}, $M_{\theta\theta}$, $P_{r\theta}$ of the classical plate theory are related to the displacements (u, w) according to the following equations:

$$N_{rr} = A_{rr}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right] - B_{rr}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right), \tag{24a}$$

$$N_{\theta\theta} = A_{rr}\left[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right] - B_{rr}\left(\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right), \tag{24b}$$

$$M_{rr} = B_{rr}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right] - D_{rr}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right), \tag{24c}$$

$$M_{\theta\theta} = B_{rr}\left[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right] - D_{rr}\left(\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right), \tag{24d}$$

$$P_{r\theta} = S_{r\theta}\left(-\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right), \tag{24e}$$

$$N_{rz} = \frac{1}{r}\left[\frac{d}{dr}(rM_{rr}) - M_{\theta\theta}\right], \tag{24f}$$

where A_{rr}, B_{rr}, D_{rr}, and $S_{r\theta}$ are the extensional, extensional-bending, bending, and shear stiffnesses, respectively:

$$\begin{aligned}
A_{rr} &= \frac{1}{(1-\nu^2)}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\,dz, \qquad B_{rr} = \frac{1}{(1-\nu^2)}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)z\,dz\\
D_{rr} &= \frac{1}{(1-\nu^2)}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)z^2\,dz, \quad S_{r\theta} = \frac{\ell^2}{2(1+\nu)}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\,dz
\end{aligned} \tag{25}$$

For the material distribution through the thickness according to Eq. (11), the following integrals are useful in computing A_{rr}, B_{rr}, and so on:

$$\begin{aligned}
&\int_{-\frac{h}{2}}^{\frac{h}{2}} v_1(z)\,dz = \frac{h}{n+1},\\
&\int_{-\frac{h}{2}}^{\frac{h}{2}} v_1(z)\,z\,dz = \frac{nh^2}{2(n+1)(n+2)},\\
&\int_{-\frac{h}{2}}^{\frac{h}{2}} v_1(z)\,z^2\,dz = \frac{(2+n+n^2)h^3(r)}{4(n+1)(n+2)(n+3)}.
\end{aligned} \tag{26}$$

3.4. Displacement formulation of the CPT

We now can write the governing equations of the CPT solely in terms of u and w with the help of the plate constitutive equations. The resulting differential equations would be second order in u and fourth order in w, the total order being six. Here we present such equations for the case in which the couple stress effect is omitted.

The equilibrium equations of the CPT without the couple stress effect are obtained by setting $P_{r\theta}$ to zero:

$$-\frac{1}{r}\left[\frac{d}{dr}(rN_{rr}) - N_{\theta\theta}\right] = 0, \tag{27}$$

$$-\frac{1}{r}\frac{d}{dr}\left[\frac{d}{dr}(rM_{rr}) - M_{\theta\theta}\right] - \frac{1}{r}\frac{d}{dr}\left[r\left(N_{rr}\frac{dw}{dr}\right)\right] - q = 0. \tag{28}$$

Substituting for N_{rr}, $N_{\theta\theta}$, M_{rr}, and $M_{\theta\theta}$ from Eqs. (24a)–(24f) into the equations of equilibrium, Eqs. (27) and (28), we obtain

$$\begin{aligned}
&-\frac{1}{r}\frac{d}{dr}\left\{rA_{rr}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right] - rB_{rr}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)\right\}\\
&\qquad + A_{rr}\frac{1}{r}\left[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right] - B_{rr}\frac{1}{r}\left(\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = 0,
\end{aligned} \tag{29}$$

$$\begin{aligned}
&-\frac{1}{r}\frac{d}{dr}\left[\frac{d}{dr}\left\{rB_{rr}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right] - rD_{rr}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)\right\}\right.\\
&\qquad \left. - B_{rr}\left[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right] + D_{rr}\left(\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right)\right]\\
&\qquad -\frac{1}{r}\frac{d}{dr}\left\{A_{rr}r\frac{dw}{dr}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right]\right.\\
&\qquad \left. - B_{rr}r\frac{dw}{dr}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)\right\} - q = 0.
\end{aligned} \tag{30}$$

3.5. Analytical solutions for bending

3.5.1. Governing equations

In this section, we develop the exact solutions of the linearized equilibrium equations governing functionally graded material (FGM) circular plates. Of course, the isotropic plate solutions can be deduced from the results to be derived here for the FGM plates.

First, we summarize the relevant equations for the purpose of this section. The equations of equilibrium in terms of the stress resultants are (without the couple stress effect and nonlinear terms):

$$-\frac{1}{r}\left[\frac{d}{dr}(rN_{rr})-N_{\theta\theta}\right]=0, \tag{31}$$

$$-\frac{1}{r}\frac{d}{dr}\left[\frac{d}{dr}(rM_{rr})-M_{\theta\theta}\right]-q=0. \tag{32}$$

The bending Eq. (32) can be cast as a pair of equations:

$$-\frac{1}{r}\frac{d}{dr}(rN_{rz})-q=0, \tag{33}$$

$$-\frac{d}{dr}(rM_{rr})+M_{\theta\theta}+rN_{rz}=0. \tag{34}$$

Equation (34) defines the bending moment–shear force relationship:

$$rN_{rz}=\frac{d}{dr}(rM_{rr})-M_{\theta\theta}.$$

In the linearized theory, the stress resultants N_{rr}, $N_{\theta\theta}$, M_{rr}, and $M_{\theta\theta}$ are related to the displacements by

$$N_{rr}=A_{rr}\left(\frac{du}{dr}+\nu\frac{u}{r}\right)-B_{rr}\left(\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right), \tag{35}$$

$$N_{\theta\theta}=A_{rr}\left(\frac{u}{r}+\nu\frac{du}{dr}\right)-B_{rr}\left(\nu\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right), \tag{36}$$

$$M_{rr}=B_{rr}\left(\frac{du}{dr}+\nu\frac{u}{r}\right)-D_{rr}\left(\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right), \tag{37}$$

$$M_{\theta\theta}=B_{rr}\left(\frac{u}{r}+\nu\frac{du}{dr}\right)-D_{rr}\left(\nu\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right), \tag{38}$$

$$\sigma_{rr}=\frac{E}{(1-\nu^2)}\left[\left(\frac{du}{dr}+\nu\frac{u}{r}\right)-z\left(\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right)\right], \tag{39}$$

$$\sigma_{\theta\theta}=\frac{E}{(1-\nu^2)}\left[\left(\nu\frac{du}{dr}+\frac{u}{r}\right)-z\left(\nu\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)\right], \tag{40}$$

where the expressions for σ_{rr} and $\sigma_{\theta\theta}$ are obtained using Eqs. (10) and (29) (omitting the nonlinear contributions).

3.5.2. Exact solutions

From Eq. (33), we obtain

$$rN_{rz}=-\int^r rq(\xi)\,d\xi+c_1. \tag{41}$$

Substituting Eqs. (37), (38), and (41) in Eq. (34), and integrating twice with respect to r, we obtain (after some algebraic manipulations)

$$B_{rr}ru-D_{rr}r\frac{dw}{dr}=-\int^r\left\{\xi\int^\xi\left[\frac{1}{\eta}\int^\eta\mu q(\mu)\,d\mu\right]d\eta\right\}d\xi+c_1\frac{r^2}{4}(2\log r-1)+c_2\frac{r^2}{2}+c_3. \tag{42}$$

Following the same procedure with Eq. (31) as we did with Eq. (34), we obtain

$$A_{rr}ru-B_{rr}r\frac{dw}{dr}=c_4\frac{r^2}{2}+c_5. \tag{43}$$

From Eqs. (42) and (43), we can solve for ru and $r(dw/dr)$ as

$$ru(r)=\bar{D}^*_{rr}\left(c_4\frac{r^2}{2}+c_5\right)-\bar{B}^*_{rr}\left(F(r)+c_2\frac{r^2}{2}+c_3\right), \tag{44}$$

$$r\frac{dw}{dr}=\bar{B}^*_{rr}\left(c_4\frac{r^2}{2}+c_5\right)-\bar{A}^*_{rr}\left(F(r)+c_2\frac{r^2}{2}+c_3\right), \tag{45}$$

where

$$\bar{A}^*_{rr}=\frac{A_{rr}}{D^*_{rr}},\quad \bar{B}^*_{rr}=\frac{B_{rr}}{D^*_{rr}},\quad \bar{D}^*_{rr}=\frac{D_{rr}}{D^*_{rr}},\quad D^*_{xx}=A_{rr}D_{rr}-B_{rr}B_{rr}, \tag{46}$$

$$F(r)=-\int^r\left\{\xi\int^\xi\left[\frac{1}{\eta}\int^\eta\mu q(\mu)\,d\mu\right]d\eta\right\}d\xi+c_1\frac{r^2}{4}(2\log r-1). \tag{47}$$

Integrating Eq. (45) once, we arrive at the expression for $w(r)$

$$w(r)=\bar{B}^*_{rr}\left(c_4\frac{r^2}{4}+c_5\log r\right)-\bar{A}_{rr}\left(\int^r\frac{1}{\xi}F(\xi)\,d\xi+c_2\frac{r^2}{4}+c_3\log r+c_6\right), \tag{48}$$

The six constants of integration can be determined using six boundary conditions, three at $r=b$ (b is the inner radius; $b=0$ for solid plate) and three at $r=a$ (the outer radius) from the duality pairs:

$$(u,N_{rr}),\quad (w,N_{rz}),\quad \left(\frac{dw}{dr},M_{rr}\right) \tag{49}$$

To facilitate the determination of the constants of integration using the boundary conditions, we write N_{rr} and M_{rr} in terms of the displacements u and w. First, we compute du/dr and d^2w/dr^2:

$$\frac{du}{dr}=\bar{D}^*_{rr}\left(\frac{1}{2}c_4-\frac{c_5}{r^2}\right)-\bar{B}^*_{rr}\left[\frac{d}{dr}\left(\frac{F}{r}\right)+\frac{1}{2}c_2-\frac{c_3}{r^2}\right], \tag{50}$$

$$\frac{d^2w}{dr^2}=\bar{B}^*_{rr}\left(\frac{1}{2}c_4-\frac{c_5}{r^2}\right)-\bar{A}^*_{rr}\left[\frac{d}{dr}\left(\frac{F}{r}\right)+\frac{1}{2}c_2-\frac{c_3}{r^2}\right], \tag{51}$$

Next, we write N_{rr} and M_{rr} in terms of the constants of integration (after some simplifications) as:

$$N_{rr}=\frac{1+\nu}{2}c_4-\frac{1-\nu}{r^2}c_5, \tag{52}$$

$$M_{rr}=(1+\nu)\left(\frac{c_2}{2}-\frac{c_3}{r^2}\right)+\frac{d}{dr}\left(\frac{F}{r}\right)+\nu\frac{F}{r^2}. \tag{53}$$

The function $F(r)$ depends on $q(r)$ and the constant of integration c_1. Two cases that are of interest are when $q(r)=0$ and $q(r)=q_0$, a constant. Then we have (for $q_0=0$ or q_0 is a constant)

$$F(r)=-\frac{q_0r^4}{16}+c_1\frac{r^2}{4}(2\log r-1). \tag{54}$$

The exact solutions for deflection, moments, and stresses in an FGM circular plate with clamped edge, $r=a$ and subjected to uniformly distributed load of intensity q_0 [the coefficients $\bar{A}^*_{rr}$ and $\bar{B}^*_{rr}$ are defined in Eq. (46)] are

$$u(r)=-\bar{B}^*_{rr}\frac{q_0a^3}{16}\frac{r}{a}\left(1-\frac{r^2}{a^2}\right), \tag{55}$$

$$w(r)=\bar{A}^*_{rr}\frac{q_0a^4}{64}\left[1-\left(\frac{r}{a}\right)^2\right]^2. \tag{56}$$

Expressions for the stress resultants from Eqs. (35)–(38) become ($N_{rr}=N_{\theta\theta}=0$)

$$M_{rr}(r)=\frac{q_0a^2}{16}\left[(1+\nu)-(3+\nu)\left(\frac{r}{a}\right)^2\right], \tag{57}$$

$$M_{\theta\theta}(r)=\frac{q_0a^2}{16}\left[(1+\nu)-(1+3\nu)\left(\frac{r}{a}\right)^2\right], \tag{58}$$

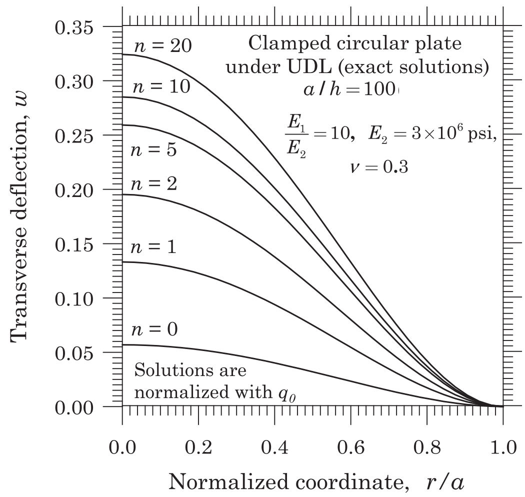

Fig. 3. Variation of the transverse $w(r)$ with r/a for various values of the volume fraction index n.

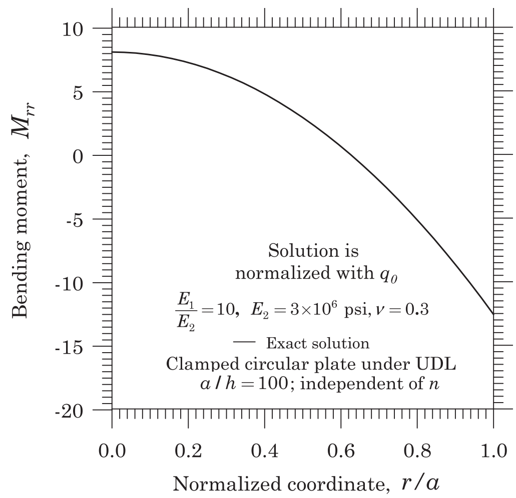

Fig. 4. Variation of the bending moment $M_{rr}(r)$ with r/a for various values of the n.

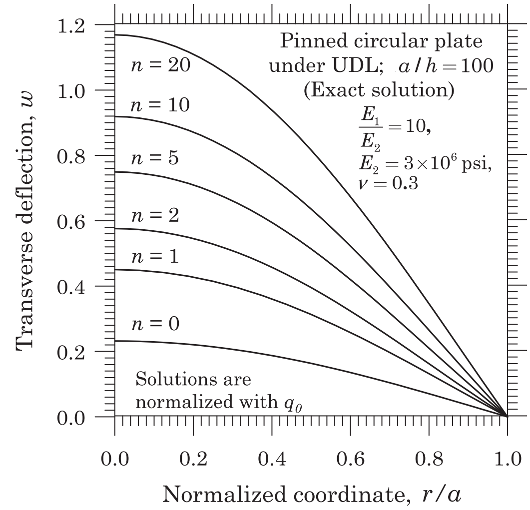

Fig. 5. Plots of the center deflection of pinned circular plates as functions of the normalized radial coordinate, r/a, for various value of n.

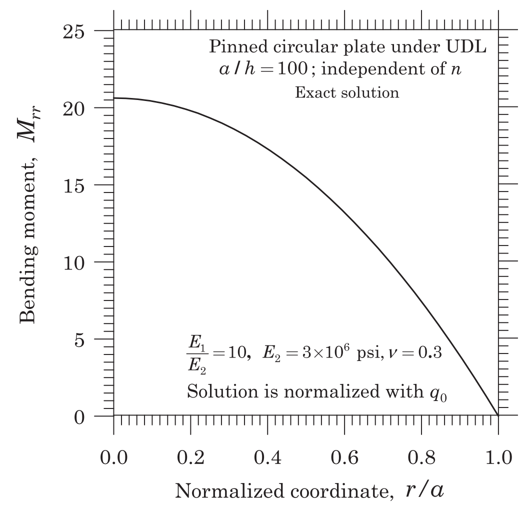

Fig. 6. Plots of the bending moment M_{rr} of pinned circular plates as a function of the normalized radial coordinate, r/a. The results are independent of the n.

$$\sigma_{rr}(r,z) = \frac{q_0 a^2}{16}\frac{E(z)}{1-\nu^2}(\bar{B}^*_{rr} + z\bar{A}^*_{rr})\left[(1+\nu) - (3+\nu)\left(\frac{r}{a}\right)^2\right], \tag{59}$$

$$\sigma_{\theta\theta}(r,z) = \frac{q_0 a^2}{16}\frac{E(z)}{1-\nu^2}(\bar{B}^*_{rr} + z\bar{A}^*_{rr})\left[(1+\nu) - (1+3\nu)\left(\frac{r}{a}\right)^2\right]. \tag{60}$$

The results presented can be simplified to an isotropic plate by setting $B_{xx} = \bar{B}^*_{xx} = 0$, $\bar{A}^*_{xx} = 1/D_{rr}$, and $\bar{D}^*_{rr} = 1/A_{rr}$.

To generate numerical results, we consider circular plates of radius $a = 10$ in., thickness $h = 0.1$ in., and modulus ratio $E_1/E_2 = 10$ with $E_2 = 30\times10^6$ psi and $\nu = 0.3$. Figure 3 shows plots of the transverse deflection $w(r)$ as a function of the normalized radial distance r/a for various values of the volume fraction index n ($n = 0$ corresponds to the isotropic plate). The deflections are normalized with respect to the load q_0. Figure 4 shows plots of the bending moment $M_{rr}(r)$ as a function of r/a for various values of n.

The expressions for the transverse deflection, bending moments, and stresses in a FGM circular plate with pinned edge at $r = a$ and subjected to uniformly distributed load of intensity q_0 and an applied bending moment M_a at $r = a$ are

$$u(r) = -\bar{B}^*_{rr}\frac{q_0 a^3}{16}\frac{r}{a}\left(1 - \frac{r^2}{a^2}\right), \tag{61}$$

$$w(r) = \bar{A}^*_{rr}\frac{q_0 a^4}{64}\left[\left(\frac{5+\nu}{1+\nu}\right) - 2\left(\frac{3+\nu}{1+\nu}\right)\frac{r^2}{a^2} + \frac{r^4}{a^4}\right] - \frac{\bar{B}^*_{rr}\bar{B}^*_{rr}}{(1+\nu)\bar{D}^*_{rr}}\frac{q_0 a^4}{16}\left(1 - \frac{r^2}{a^2}\right) + \frac{M_a a^2}{2(1+\nu)D_{rr}}\left(1 - \frac{r^2}{a^2}\right), \tag{62}$$

$$M_{rr}(r) = (3+\nu)\frac{q_0 a^2}{16}\left(1 - \frac{r^2}{a^2}\right) + M_a, \tag{63}$$

where the coefficients $\bar{A}^*_{rr}$, $\bar{B}^*_{rr}$, and $\bar{D}^*_{rr}$ are defined in Eq. (46). We nte that M_a does not contribute to $u(r)$.

Numerical results are presented for the case in which $M_a = 0$ and the following data:

$$a = 10 \text{ in.}, \quad h = 0.1 \text{ in.}, \quad \frac{E_1}{E_2} = 10, \quad E_2 = 30\times10^6 \text{ psi}, \quad \nu = 0.3. \tag{64}$$

All results are normalized by the load q_0. Figure 5 contains plots of the deflections $w(0)$ predicted for the pinned FGM plates as a function of the normalized radial coordinate, r/a, for various values of the power-law index n; Fig. 6 contains plots of the variation of the bending moment M_{rr} as function of the normalized radial coordinate, r/a; the results are independent of n.

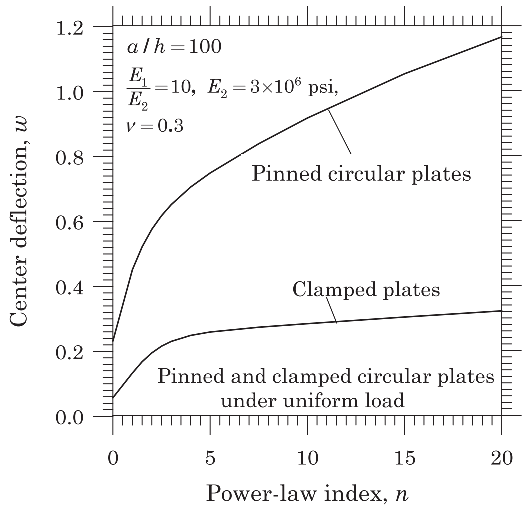

Fig. 7. Plots of the center deflection $w(0)$ of pinned and clamped circular plates as a function of the power-law index, n.

Figure 7 shows the center deflection $w(0)$ as a function of the power-law index n for the pinned and clamped circular plates. We note that the rate of increase of the deflection has two different regions; the first region has a rapid increase of the deflection while the second region is marked with a relatively slow increase. This is primarily because of the fact that the coupling coefficient B_{xx} varies with n rapidly for the smaller values of n followed by a slow decay after $n > 3$. The rate of increase in the deflection or slope in the second part is less for clamped plates than for the pinned plates. The reason is the fact that the clamped plate is relatively stiffer than the pinned plate.

4. The first-order shear deformation theory

4.1. Displacements and strains

The displacement field of the first-order shear deformation plate theory (FST) is

$$\mathbf{u} = u_r\,\hat{\mathbf{e}}_r + u_z\,\hat{\mathbf{e}}_z,\quad u_r(r,z,t) = u(r,t) + z\phi_r(r,t),\quad u_z(r,z,t) = w(r,t), \tag{65}$$

where ϕ_r denotes the rotation of a transverse normal in the plane θ =constant. The FST includes a constant state of transverse shear strain with respect to the thickness coordinate, and hence, requires the use of a shear correction factor.

The nonzero von Kármán strains of the theory are

$$\varepsilon_{rr} = \varepsilon_{rr}^{(0)} + z\varepsilon_{rr}^{(1)},\quad \varepsilon_{\theta\theta} = \varepsilon_{\theta\theta}^{(0)} + z\varepsilon_{\theta\theta}^{(1)},\quad \varepsilon_{rz} = \varepsilon_{rz}^{(0)}, \tag{66}$$

where

$$\begin{aligned} &\varepsilon_{rr}^{(0)} = \tfrac{du}{dr} + \tfrac{1}{2}\left(\tfrac{dw}{dr}\right)^2, && \varepsilon_{rr}^{(1)} = \tfrac{d\phi_r}{dr},\\ &\varepsilon_{\theta\theta}^{(0)} = \tfrac{u}{r}, \quad \varepsilon_{\theta\theta}^{(1)} = \tfrac{\phi_r}{r}, && 2\varepsilon_{rz}^{(0)} = \phi_r + \tfrac{dw}{dr}. \end{aligned} \tag{67}$$

The rotation and curvature components are

$$\omega_\theta = \frac{1}{2}\left(\phi_r - \frac{dw}{dr}\right),\quad \chi_{r\theta} = \frac{1}{4}\left[\frac{d\phi_r}{dr} - \frac{1}{r}\phi_r - \left(\frac{d^2w}{dr^2} - \frac{1}{r}\frac{dw}{dr}\right)\right]. \tag{68}$$

4.2. Equations of equilibrium

The governing equations of equilibrium of the FST are:

$$\frac{1}{r}\left[\frac{d}{dr}\left(rN_{rr}\right) - N_{\theta\theta}\right] = 0, \tag{69}$$

$$\frac{1}{r}\frac{d}{dr}\left(rV_r\right) + \frac{1}{2r}\frac{d}{dr}\left[\frac{d}{dr}\left(rP_{r\theta}\right) + P_{r\theta}\right] + q = 0, \tag{70}$$

$$\frac{1}{r}\left[\frac{d}{dr}\left(rM_{rr}\right) - M_{\theta\theta} + \frac{1}{2}\frac{d}{dr}\left(rP_{r\theta}\right) + \frac{1}{2}P_{r\theta}\right] - N_{rz} = 0, \tag{71}$$

where

$$\begin{aligned} (N_{rr}, M_{rr}) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} (1,z)\sigma_{rr}\,dz,\quad N_{rz} = K_s\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{rz}\,dz\\ (N_{\theta\theta}, M_{\theta\theta}) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} (1,z)\sigma_{\theta\theta}\,dz,\quad P_{r\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} m_{r\theta}\,dz\\ V_r &= N_{rz} + N_{rr}\frac{dw}{dr} \end{aligned} \tag{72}$$

and K_s denotes the shear correction coefficient.

The boundary conditions involve specifying one element of each of the following pairs:

$$\begin{aligned} &u \text{ or } rN_{rr};\quad w \text{ or } rV_r + \frac{1}{2}\left[\frac{d}{dr}\left(rP_{r\theta}\right) + P_{r\theta}\right] \equiv r\bar{V}_r\\ &-\frac{dw}{dr} \text{ or } -\tfrac{1}{2}rP_{r\theta};\quad \phi_r \text{ or } rM_{rr} + \tfrac{1}{2}rP_{r\theta} \equiv r\bar{M}_{rr} \end{aligned} \tag{73}$$

4.3. Plate constitutive relations

The stress resultants appearing in Eqs. (69)–(71) can be expressed in terms of the generalized displacements (u, w, ϕ_r) as (thermal effects are not included)

$$\begin{aligned} N_{rr} &= A_{rr}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right] + B_{rr}\left(\frac{d\phi_r}{dr} + \frac{\nu}{r}\phi_r\right),\\ N_{\theta\theta} &= A_{rr}\left[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right] + B_{rr}\left(\nu\frac{d\phi_r}{dr} + \frac{1}{r}\phi_r\right),\\ M_{rr} &= B_{rr}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right] + D_{rr}\left(\frac{d\phi_r}{dr} + \frac{\nu}{r}\phi_r\right),\\ M_{\theta\theta} &= B_{rr}\left[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right] + D_{rr}\left(\nu\frac{d\phi_r}{dr} + \frac{1}{r}\phi_r\right),\\ N_{rz} &= S_{rz}\left(\phi_r + \frac{dw}{dr}\right),\quad P_{r\theta} = \frac{1}{2}S_{r\theta}\left[\frac{d\phi_r}{dr} - \frac{d^2w}{dr^2} - \frac{1}{r}\left(\phi_r - \frac{dw}{dr}\right)\right], \end{aligned} \tag{74}$$

where A_{rr}, B_{rr}, D_{rr}, $S_{rz} = K_s\,A_{rz}$, and $S_{r\theta}$ are the extensional, extensional-bending, bending, shear, and couple stress stiffness coefficients, respectively:

$$\begin{aligned} A_{rr} &= \frac{1}{(1-\nu^2)}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\,dz, && B_{rr} = \frac{1}{(1-\nu^2)}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)z\,dz,\\ D_{rr} &= \frac{1}{(1-\nu^2)}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)z^2\,dz, && A_{rz} = \frac{1}{2(1+\nu)}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\,dz,\\ S_{r\theta} &= \frac{\ell^2}{2(1+\nu)}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\,dz, && m_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z)(z)^i\,dz. \end{aligned} \tag{75}$$

4.4. Equations of equilibrium in terms of the displacements

The equations of equilibrium in Eqs. (69)–(71) can be expressed in terms of the generalized displacements (u, w, ϕ_r) by invoking Eq. (74):

$$\begin{aligned} &-\frac{1}{r}\frac{d}{dr}\left\{rA_{rr}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right] + rB_{rr}\left(\frac{d\phi_r}{dr} + \frac{\nu}{r}\phi_r\right)\right\}\\ &\quad + \frac{1}{r}\left\{A_{rr}\left[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right] + B_{rr}\left(\nu\frac{d\phi_r}{dr} + \frac{1}{r}\phi_r\right)\right\} = 0, \end{aligned} \tag{76}$$

$$\begin{aligned} &-\frac{1}{r}\frac{d}{dr}\left\{rS_{rz}\left(\phi_r + \frac{dw}{dr}\right) + rA_{rr}\frac{dw}{dr}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right]\right.\\ &\qquad\qquad \left. + rB_{rr}\frac{\partial w}{\partial r}\left(\frac{\partial\phi_r}{\partial r} + \frac{\nu}{r}\phi_r\right)\right\}\\ &\quad -\frac{1}{r}\frac{d^2}{dr^2}\left\{rS_{r\theta}\left[\frac{d\phi_r}{dr} - \frac{d^2w}{dr^2} - \frac{1}{r}\left(\phi_r - \frac{dw}{dr}\right)\right]\right\}\\ &\quad -\frac{1}{r}\frac{d}{dr}\left\{S_{r\theta}\left[\frac{d\phi_r}{dr} - \frac{d^2w}{dr^2} - \frac{1}{r}\left(\phi_r - \frac{dw}{dr}\right)\right]\right\} - q = 0, \end{aligned} \tag{77}$$

$$-\frac{1}{r}\frac{d}{dr}\left\{rB_{rr}\left[\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2+\nu\frac{u}{r}\right]+rD_{rr}\left(\frac{d\phi_r}{dr}+\frac{\nu}{r}\phi_r\right)\right\}$$
$$+\frac{1}{r}\left\{B_{rr}\left[\frac{u}{r}+\nu\frac{du}{dr}+\frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right]+D_{rr}\left(\nu\frac{d\phi_r}{dr}+\frac{1}{r}\phi_r\right)\right\}$$
$$-\frac{1}{r}\frac{d}{dr}\left\{rS_{r\theta}\left[\frac{d\phi_r}{dr}-\frac{d^2w}{dr^2}-\frac{1}{r}\left(\phi_r-\frac{dw}{dr}\right)\right]\right\}$$
$$-\frac{1}{r}S_{r\theta}\left[\frac{d\phi_r}{dr}-\frac{d^2w}{dr^2}-\frac{1}{r}\left(\phi_r-\frac{dw}{dr}\right)\right]+S_{rz}\left(\phi_r+\frac{dw}{dr}\right)=0. \quad (78)$$

4.5. Exact solutions

4.5.1. Governing equations

In this section, we develop the exact solutions of functionally graded material (FGM) plates using the FST. The couples stress effect is not included here, although it is possible to include but algebraically a bit more complicated. The developments to be presented are similar to those presented in Section 3.5.

The equations of equilibrium of the FST in terms of the stress resultants are (without the couple stress effect and setting the nonlinear terms to zero) are:

$$-\frac{1}{r}\left[\frac{d}{dr}(rN_{rr})-N_{\theta\theta}\right]=0, \quad (79)$$

$$-\frac{1}{r}\frac{d}{dr}(rN_{rz})-q=0, \quad (80)$$

$$-\frac{d}{dr}(rM_{rr})+M_{\theta\theta}+rN_{rz}=0. \quad (81)$$

The stress resultants (N_{rr}, $N_{\theta\theta}$, M_{rr}, $M_{\theta\theta}$, N_{rz}) and stresses (σ_{rr}, $\sigma_{\theta\theta}$, σ_{rz}) of the linearized FST are related to the displacements by

$$N_{rr}=A_{rr}\left(\frac{du}{dr}+\nu\frac{u}{r}\right)+B_{rr}\left(\frac{d\phi_r}{dr}+\frac{\nu}{r}\phi_r\right), \quad (82)$$

$$N_{\theta\theta}=A_{rr}\left(\frac{u}{r}+\nu\frac{du}{dr}\right)+B_{rr}\left(\nu\frac{d\phi_r}{dr}+\frac{1}{r}\phi_r\right), \quad (83)$$

$$M_{rr}=B_{rr}\left(\frac{du}{dr}+\nu\frac{u}{r}\right)+D_{rr}\left(\frac{d\phi_r}{dr}+\frac{\nu}{r}\phi_r\right), \quad (84)$$

$$M_{\theta\theta}=B_{rr}\left(\frac{u}{r}+\nu\frac{du}{dr}\right)+D_{rr}\left(\nu\frac{d\phi_r}{dr}+\frac{1}{r}\phi_r\right), \quad (85)$$

$$\sigma_{rr}=\frac{E}{(1-\nu^2)}\left[\left(\frac{du}{dr}+\nu\frac{u}{r}\right)+z\left(\frac{d\phi_r}{dr}+\frac{\nu}{r}\phi_r\right)\right], \quad (86)$$

$$\sigma_{\theta\theta}=\frac{E}{(1-\nu^2)}\left[\left(\nu\frac{du}{dr}+\frac{u}{r}\right)+z\left(\nu\frac{d\phi_r}{dr}+\frac{1}{r}\phi_r\right)\right], \quad (87)$$

$$N_{rz}=S_{rz}\left(\phi_r+\frac{dw}{dr}\right),\quad \sigma_{rz}=\frac{E}{2(1+\nu)}\left(\phi_r+\frac{dw}{dr}\right). \quad (88)$$

From Eq. (80), we obtain

$$rN_{rz}=-\int^r rq(\xi)\,d\xi+c_1. \quad (89)$$

Substituting Eqs. (84), (85), and (89) in Eq. (81), and integrating twice with respect to r (and with several algebraic simplifications), we obtain

$$B_{rr}\,ru+D_{rr}\,r\phi_r=-\int^r\left\{\xi\int^\xi\left[\frac{1}{\eta}\int^\eta\mu q(\mu)\,d\mu\right]d\eta\right\}d\xi$$
$$+c_1\frac{r^2}{4}(2\log r-1)+c_2\frac{r^2}{2}+c_3. \quad (90)$$

Following the same procedure with Eq. (79) as we did with Eq. (81), we obtain

$$A_{rr}\left[\frac{d}{dr}\left(r\frac{du}{dr}\right)-\frac{u}{r}\right]+B_{rr}\left[\frac{d}{dr}\left(r\frac{d\phi_r}{dr}\right)-\frac{1}{r}\phi_r\right]=0, \quad (91)$$

and

$$A_{rr}\,ru+B_{rr}\,r\phi_r=c_4\frac{r^2}{2}+c_5. \quad (92)$$

Solving Eqs. (90) and (92) for $u(r)$ and $\phi_r(r)$, we obtain

$$u(r)=\bar{D}^*_{rr}\left(c_4\frac{r}{2}+\frac{c_5}{r}\right)-\bar{B}^*_{rr}\left(\frac{1}{r}F(r)+c_2\frac{r}{2}+\frac{c_3}{r}\right), \quad (93)$$

$$\phi_r(r)=-\bar{B}^*_{rr}\left(c_4\frac{r}{2}+\frac{c_5}{r}\right)+\bar{A}^*_{rr}\left(\frac{1}{r}F(r)+c_2\frac{r}{2}+\frac{c_3}{r}\right). \quad (94)$$

where

$$\bar{A}^*_{rr}=\frac{A_{rr}}{D^*_{rr}},\quad \bar{B}^*_{rr}=\frac{B_{rr}}{D^*_{rr}},\quad \bar{D}^*_{rr}=\frac{D_{rr}}{D^*_{rr}},\quad D^*_{xx}=A_{rr}D_{rr}-B_{rr}B_{rr}, \quad (95)$$

$$F(r)=-\int^r\left\{\xi\int^\xi\left[\frac{1}{\eta}\int^\eta\mu q(\mu)\,d\mu\right]d\eta\right\}d\xi+c_1\frac{r^2}{4}(2\log r-1). \quad (96)$$

Substituting for N_{rz} from Eq. (88) into Eq. (89) and solving for dw/dr, we obtain

$$\frac{dw}{dr}=\frac{1}{r}\left[\bar{B}^*_{rr}\left(c_4\frac{r^2}{2}+c_5\right)-\bar{A}^*_{rr}\left(F(r)+c_2\frac{r^2}{2}+c_3\right)\right]$$
$$-\frac{1}{rS_{rz}}\left(\int^r rq(\xi)\,d\xi+c_1\right), \quad (97)$$

and integrating once with respect to r

$$w(r)=\bar{B}^*_{rr}\left(c_4\frac{r^2}{4}+c_5\log r\right)-\bar{A}^*_{rr}\left(\int^r\frac{1}{\xi}F(\xi)\,d\xi+c_2\frac{r^2}{4}+c_3\log r+c_6\right)$$
$$-\frac{1}{S_{rz}}\left(\int^r\frac{1}{\xi}\int^\xi\eta q(\eta)\,d\eta\,d\xi+c_1\log r\right). \quad (98)$$

The constants of integration are determined using boundary conditions arising from the specification of one element of each of the following three duality pairs:

$$(u,N_{rr}),\quad (w,N_{rz}),\quad (\phi_r,M_{rr}) \quad (99)$$

Here, we consider couple of examples to illustrate the use of the boundary conditions to determine the exact solutions. Expressions for the function $F(r)$ [see Eq. (96)] and the integrals involving it are needed in the examples to be discussed. Two cases that are of interest are when $q(r)=0$ and $q=q_0$, a constant. In these two case we have

$$F(r)=c_1\frac{r^2}{4}(2\log r-1),\ \text{for } q=0, \quad (100)$$

$$F(r)=c_1\frac{r^2}{4}(2\log r-1)-\frac{q_0r^4}{16},\ \text{for } q=q_0. \quad (101)$$

$$\int\frac{1}{r}F(r)\,dr=c_1\frac{r^2}{4}(2\log r-1)-\frac{q_0r^4}{64},\ \text{for } q=q_0. \quad (102)$$

We also need the following integral when $q(r)=q_0$:

$$\int^r\frac{1}{\xi}\int^\xi\eta q(\eta)\,d\eta\,d\xi=\frac{q_0r^2}{4}. \quad (103)$$

The exact solutions for deflection, moments, and stresses in an FGM circular plate with clamped edge, $r=a$, are [the coefficients $\bar{A}^*_{rr}$ and $\bar{B}^*_{rr}$ are defined in Eq. (96)]

$$u(r)=-\bar{B}^*_{rr}\frac{q_0a^3}{16}\frac{r}{a}\left(1-\frac{r^2}{a^2}\right), \quad (104)$$

$$\phi_r(r)=\bar{A}^*_{rr}\frac{q_0a^3}{16}\frac{r}{a}\left(1-\frac{r^2}{a^2}\right), \quad (105)$$

$$w(r)=\bar{A}^*_{rr}\frac{q_0a^4}{64}\left[1-\left(\frac{r}{a}\right)^2\right]^2+\frac{1}{S_{rz}}\frac{q_0a^2}{4}\left(1-\frac{r^2}{a^2}\right). \quad (106)$$

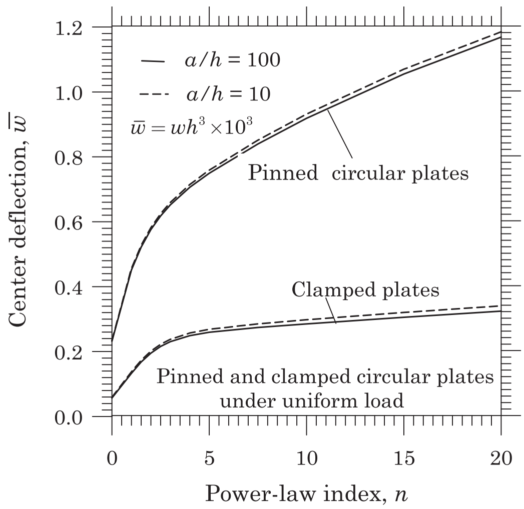

Fig. 8. Variation of the transverse maximum deflection $\bar{w}$ versus the power-law index n for clamped and pinned circular plates for two different radius-to-thickness ratios, $a/h = 10, 100$.

The displacements and bending moments of an FGM circular plate with pinned edge at $r = a$ and subjected to uniformly distributed load of intensity q_0 as well as an applied bending moment M_a at $r = a$ are:

$$u(r) = -\bar{B}^*_{rr}\frac{q_0 a^3}{16}\frac{r}{a}\left(1 - \frac{r^2}{a^2}\right), \tag{107}$$

$$\phi_r(r) = \frac{q_0 a^3}{16}\frac{r}{a}\left[\left(\frac{3+\nu}{1+\nu}\right)\bar{A}^*_{rr} - 2\frac{\bar{B}^*_{rr}\bar{B}^*_{rr}}{(1+\nu)\bar{D}^*_{rr}} - \bar{A}^*_{rr}\frac{r^2}{a^2}\right], \tag{108}$$

$$\begin{aligned} w(r) = {} & \bar{A}^*_{rr}\frac{q_0 a^4}{64}\left[\left(\frac{5+\nu}{1+\nu}\right) - 2\left(\frac{3+\nu}{1+\nu}\right)\frac{r^2}{a^2} + \frac{r^4}{a^4}\right] - \frac{\bar{B}^*_{rr}\bar{B}^*_{rr}}{(1+\nu)\bar{D}^*_{rr}}\frac{q_0 a^4}{16}\left(1 - \frac{r^2}{a^2}\right) \\ & + \frac{M_a a^2}{2(1+\nu)D_{rr}}\left(1 - \frac{r^2}{a^2}\right) + \frac{q_0 a^2}{4S_{rz}}\left(1 - \frac{r^2}{a^2}\right), \end{aligned} \tag{109}$$

$$M_{rr}(r) = (3+\nu)\frac{q_0 a^2}{16}\left(1 - \frac{r^2}{a^2}\right) + M_a, \tag{110}$$

where the coefficients $\bar{A}^*_{rr}$, $\bar{B}^*_{rr}$, and $\bar{D}^*_{rr}$ are defined in Eq. (46). We note that M_a does not contribute to $u(r)$.

The numerical results generated with the data

$$a = 10 \text{ in.}, \quad h = 0.1 \text{ in.}, \quad \frac{E_1}{E_2} = 10, \quad E_2 = 30 \times 10^6 \text{ psi}, \quad \nu = 0.3$$

coincide with the plots presented in Figs. 3 and 4, indicating that the effect of shear deformation is negligible for this thin plate ($a/h = 100$). Figure 8 shows $\bar{w} = w(0)h^3 \times 10^3$ versus the power-law index n for two different ratios $a/h = 10$ (thick) and $a/h = 100$ (thin), showing the effect of shear deformation on the transverse displacement w. Figure 8 also contains results for pinned circular plates.

5. Third-order shear deformation theory

5.1. Displacements and strains

In this section we develop the Reddy third-order shear deformation plate theory (TST) of the axisymmetric circular plates. We use an higher-order expansion of the radial displacement u_r through the thickness of the plate and thus further relax the Love–Kirchhoff hypothesis by removing the assumption of straightness of a transverse normal (in all theories the inextensibility of a transverse normal can be removed by assuming that the transverse deflection also varies through the thickness).

The third-order plate theory of Reddy [12–15] is based on the displacement field

$$\mathbf{u}(r,z,t) = u_r(r,z,t)\,\hat{\mathbf{e}}_r + u_z(r,z,t)\,\hat{\mathbf{e}}_z, \tag{111a}$$

$$\begin{aligned} u_r(r,z,t) &= u(r,t) + z\phi_r(r,t) - \alpha\, z^3\left(\phi_r + \frac{dw}{dr}\right), \\ u_z(r,z,t) &= w(r,t), \quad \alpha = \frac{4}{3h^2}\,, \end{aligned} \tag{111b}$$

where (u_r, u_z) are the total displacement components along the r and z coordinates, respectively, (u, w, ϕ_r) are the generalized displacements, and h is the total thickness of the plate. The displacement field accommodates quadratic variation of transverse shear strains and shear stresses and vanishing of transverse shear stress on the top $z = h/2$ and bottom $z = -h/2$ planes of a plate, and there is no need to use shear correction coefficient in the third-order theory.

The nonzero von Kármán nonlinear strains can be written as

$$\varepsilon_{rr} = \varepsilon^{(0)}_{rr} + z\varepsilon^{(1)}_{rr} + z^3\varepsilon^{(3)}_{rr}, \quad \varepsilon_{\theta\theta} = \varepsilon^{(0)}_{\theta\theta} + z\varepsilon^{(1)}_{\theta\theta} + z^3\varepsilon^{(3)}_{\theta\theta}, \quad \varepsilon_{rz} = \varepsilon^{(0)}_{rz} + z^2\varepsilon^{(2)}_{rz} \tag{112}$$

where

$$\begin{aligned} \varepsilon^{(0)}_{rr} &= \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2, \quad \varepsilon^{(1)}_{rr} = \frac{d\phi_r}{dr}, \quad \varepsilon^{(3)}_{rr} = -\alpha\left(\frac{d\phi_r}{dr} + \frac{d^2w}{dr^2}\right) \\ \varepsilon^{(0)}_{\theta\theta} &= \frac{u}{r}, \quad \varepsilon^{(1)}_{\theta\theta} = \frac{\phi_r}{r}, \quad \varepsilon^{(3)}_{\theta\theta} = -\alpha\frac{1}{r}\left(\phi_r + \frac{dw}{dr}\right) \\ 2\varepsilon^{(0)}_{rz} &= \phi_r + \frac{dw}{dr}, \quad 2\varepsilon^{(2)}_{rz} = -\beta\left(\phi_r + \frac{dw}{dr}\right), \quad \beta = \frac{4}{h^2}\,. \end{aligned} \tag{113}$$

The rotation and curvature components are

$$\begin{aligned} \omega_\theta &= \frac{1}{2}\left(\frac{du_r}{dz} - \frac{du_z}{dr}\right) = \frac{1}{2}\left[\phi_r - \frac{dw}{dr} - \beta z^2\left(\phi_r + \frac{dw}{dr}\right)\right], \\ \chi_{r\theta} &= \frac{1}{2}\left(\frac{d\omega_\theta}{dr} - \frac{\omega_\theta}{r}\right) = \frac{1}{4}\left[(1 - \beta\, z^2)\left(\frac{d\phi_r}{dr} - \frac{1}{r}\phi_r\right)\right. \\ &\qquad \left. - (1 + \beta\, z^2)\left(\frac{d^2w}{dr^2} - \frac{1}{r}\frac{dw}{dr}\right)\right]. \end{aligned} \tag{114}$$

5.2. Equations of equilibrium

Using the principle of virtual displacements for the third-order theory we obtain the following equations of equilibrium:

$$-\frac{1}{r}\left[\frac{d}{dr}(rN_{rr}) - N_{\theta\theta}\right] = 0, \tag{115}$$

$$-\frac{1}{r}\frac{d}{dr}\left(r\hat{V}_r\right) - \alpha\frac{1}{r}\frac{d}{dr}\left[\frac{d}{dr}(rP_{rr}) - P_{\theta\theta}\right] - \frac{1}{2}\frac{1}{r}\frac{d}{dr}\left[\hat{P}_{r\theta} + \frac{d}{dr}(r\hat{P}_{r\theta})\right] - q = 0, \tag{116}$$

$$-\frac{1}{r}\left[\frac{d}{dr}(r\bar{M}_{rr}) - \bar{M}_{\theta\theta}\right] + \bar{N}_{rz} - \frac{1}{2}\frac{1}{r}\left[\bar{P}_{r\theta} + \frac{d}{dr}(r\bar{P}_{r\theta})\right] = 0. \tag{117}$$

where the stress resultants $(N_{rr}, N_{\theta\theta}, N_{rz}, M_{rr}, M_{\theta\theta})$, higher-order stress resultants $(P_{rr}, P_{\theta\theta}, P_{rz})$, $(P_{r\theta}, Q_{r\theta})$ are defined by

$$(N_{rr}, N_{\theta\theta}) = \int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_{rr}, \sigma_{\theta\theta})dz, \quad (M_{rr}, M_{\theta\theta}) = \int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_{rr}, \sigma_{\theta\theta})z\,dz \tag{118a}$$

$$(N_{rz}, P_{rz}) = \int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{rz}\,(1, z^2)\,dz, \quad (P_{rr}, P_{\theta\theta}) = \int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_{rr}, \sigma_{\theta\theta})z^3\,dz, \tag{118b}$$

$$(P_{r\theta}, Q_{r\theta}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} m_{r\theta}\,(1, z^2)\,dz, \tag{118c}$$

$$\hat{V}_r = \hat{N}_{rz} + N_{rr}\frac{dw}{dr}\,. \tag{118d}$$

and

$$\bar{M}_{rr} = M_{rr} - \alpha\, P_{rr}, \quad \bar{M}_{\theta\theta} = M_{\theta\theta} - \alpha\, P_{\theta\theta}, \quad \bar{N}_{rz} = N_{rz} - \beta\, P_{rz}, \tag{119a}$$

$$\bar{P}_{r\theta} = P_{r\theta} - \beta Q_{r\theta}, \quad \hat{P}_{r\theta} = P_{r\theta} + \beta Q_{r\theta} \tag{119b}$$

The boundary conditions involve specifying one element of each of the following four duality pairs:

$$(u, rN_{rr}); \quad (w, r\bar{V}_r); \quad (\phi_r, r\bar{M}_{rr} + \tfrac{1}{2}r\bar{P}_{r\theta}); \quad \left(\theta_r, \alpha r\, P_{rr} + \tfrac{1}{2}r\hat{P}_{r\theta}\right), \tag{120}$$

where the slope θ_r and the effective shear force $\hat{V}_r$ are defined as

$$\theta_r = -\frac{dw}{dr}, \quad \hat{V}_r = \bar{N}_{rz} + N_{rr}\frac{dw}{dr} + \alpha\left[\frac{d}{dr}(rP_{rr}) - P_{\theta\theta}\right] + \tfrac{1}{2}\left[\hat{P}_{r\theta} + \frac{d}{dr}(r\hat{P}_{r\theta})\right]. \tag{121}$$

5.3. Plate constitutive equations

The Young's modulus E varies with z according to

$$E(z) = \left(E_1 - E_2\right)v_1(z) + E_2, \quad v_1(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n \tag{122}$$

and ν is a constant. The modified couple stress constitutive relation is

$$m_{r\theta} = 2G\ell^2\,\chi_{r\theta}, \quad G = \frac{E}{2(1+\nu)}, \tag{123}$$

where $m_{r\theta}$ is the nonzero component of the symmetric couple stress tensor $\mathbf{m}$, ℓ is the length scale parameter, and G is the shear modulus.

The stress resultants appearing in Eqs. (118a)–(118c) can be expressed in terms of the generalized displacements (u, w, ϕ_r) as

$$N_{rr} = A_{rr}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right] + \bar{B}_{rr}\left(\frac{d\phi_r}{dr} + \frac{\nu}{r}\phi_r\right) - \alpha E_{rr}\left(\frac{\nu}{r}\frac{dw}{dr} + \frac{d^2w}{dr^2}\right), \tag{124a}$$

$$N_{\theta\theta} = A_{rr}\left[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right] + \bar{B}_{rr}\left(\nu\frac{d\phi_r}{dr} + \frac{1}{r}\phi_r\right) - \alpha E_{rr}\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right), \tag{124b}$$

$$M_{rr} = B_{rr}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right] + \bar{D}_{rr}\left(\frac{d\phi_r}{dr} + \frac{\nu}{r}\phi_r\right) - \alpha F_{rr}\left(\frac{\nu}{r}\frac{dw}{dr} + \frac{d^2w}{dr^2}\right), \tag{124c}$$

$$M_{\theta\theta} = B_{rr}\left[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right] + \bar{D}_{rr}\left(\nu\frac{d\phi_r}{dr} + \frac{1}{r}\phi_r\right) - \alpha F_{rr}\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right), \tag{124d}$$

$$P_{rr} = E_{rr}\left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\frac{u}{r}\right] + \bar{F}_{rr}\left(\frac{d\phi_r}{dr} + \frac{\nu}{r}\phi_r\right) - \alpha H_{rr}\left(\frac{\nu}{r}\frac{dw}{dr} + \frac{d^2w}{dr^2}\right), \tag{124e}$$

$$P_{\theta\theta} = E_{rr}\left[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}\left(\frac{dw}{dr}\right)^2\right] + \bar{F}_{rr}\left(\nu\frac{d\phi_r}{dr} + \frac{1}{r}\phi_r\right) - \alpha H_{rr}\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right), \tag{124f}$$

$$N_{rz} = \bar{A}_{rz}\left(\phi_r + \frac{dw}{dr}\right), \quad R_{rz} = \bar{D}_{rz}\left(\phi_r + \frac{dw}{dr}\right), \tag{124g}$$

$$P_{r\theta} = A_{r\theta}\left(\frac{d\phi_r}{dr} - \frac{1}{r}\phi_r - \frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) + \beta D_{r\theta}\left(-\frac{d\phi_r}{dr} + \frac{1}{r}\phi_r + \frac{d^2w}{dr^2} - \frac{1}{r}\frac{dw}{dr}\right), \tag{124h}$$

$$Q_{r\theta} = D_{r\theta}\left(\frac{d\phi_r}{dr} - \frac{1}{r}\phi_r - \frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) + \beta F_{r\theta}\left(-\frac{d\phi_r}{dr} + \frac{1}{r}\phi_r + \frac{d^2w}{dr^2} - \frac{1}{r}\frac{dw}{dr}\right), \tag{124i}$$

where A_{rr}, B_{rr}, D_{rr}, E_{rr}, F_{rr}, H_{rr}, $A_{r\theta}$, $D_{r\theta}$, $F_{r\theta}$ are the extensional, extensional-bending, bending, and higher-order stiffness coefficients:

$$\begin{aligned}
(A_{rr}, B_{rr}, D_{rr}, E_{rr}, F_{rr}, H_{rr}) &= \frac{1}{(1-\nu^2)}\int_{-\frac{h}{2}}^{\frac{h}{2}}(1, z, z^2, z^3, z^4, z^6)E(z)\,dz,\\
(A_{rz}, D_{rz}, F_{rz}) &= \frac{1}{2(1+\nu)}\int_{-\frac{h}{2}}^{\frac{h}{2}}(1, z^2, z^4)E(z)\,dz,\\
(A_{r\theta}, D_{r\theta}, F_{r\theta}) &= \frac{\ell^2}{4(1+\nu)}\int_{-\frac{h}{2}}^{\frac{h}{2}}(1, z^2, z^4)E(z)\,dz
\end{aligned} \tag{125}$$

and

$$\begin{aligned}
\bar{B}_{rr} = B_{rr} \quad -\alpha E_{rr}, \quad \bar{D}_{rr} = D_{rr} - \alpha F_{rr}, \quad \bar{F}_{rr} = F_{rr} - \alpha H_{rr},\\
\bar{A}_{rz} = A_{rz} - \beta D_{rz}, \quad \bar{D}_{rz} = D_{rz} - \beta F_{rz}
\end{aligned} \tag{126}$$

The equations of equilibrium in Eqs. (115)–(117) can be expressed in terms of the generalized displacements using Eqs. (124a)–(124g).

5.4. Exact solution

As shown in the following pages, it is not possible to determine the exact solution of the TST equations due to the presence of higher-order stress resultants P_{rr} and $P_{\theta\theta}$. We outline the steps similar to those followed for the FST in Section 4.5 to find the exact solutions of the linearized equations without the foundation modulus ($k = 0$) and the couple stress terms.

We begin with some mathematical identities:

$$\begin{aligned}
\frac{d}{dr}(rN_{rr}) - N_{\theta\theta} &= A_{rr}\left[\frac{d}{dr}\left(r\frac{du}{dr}\right) - \frac{u}{r}\right] + \bar{B}_{rr}\left[\frac{d}{dr}\left(r\frac{d\phi_r}{dr}\right) - \frac{\phi_r}{r}\right] - \alpha E_{rr}\left[\frac{d}{dr}\left(r\frac{d^2w}{dr^2}\right) - \frac{1}{r}\frac{dw}{dr}\right]\\
&= A_{rr}r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] + \bar{B}_{rr}r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\phi_r)\right] - \alpha E_{rr}r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right],
\end{aligned} \tag{127}$$

$$\frac{d}{dr}(rM_{rr}) - M_{\theta\theta} = B_{rr}r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] + \bar{D}_{rr}r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\phi_r)\right] - \alpha F_{rr}r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right], \tag{128}$$

$$\frac{d}{dr}(rP_{rr}) - P_{\theta\theta} = E_{rr}r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] + \bar{F}_{rr}r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\phi_r)\right] - \alpha H_{rr}r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right], \tag{129}$$

Then from Eq. (115) we have

$$A_{rr}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] + \bar{B}_{rr}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\phi_r)\right] - \alpha E_{rr}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = 0. \tag{130}$$

Integration with respect to r twice yields

$$A_{rr}\,u + \bar{B}_{rr}\,\phi_r - \alpha E_{rr}\frac{dw}{dr} = \frac{c_1 r}{2} + \frac{c_2}{r}, \tag{131}$$

where c_1 and c_2 are constants of integration.

Integrating Eq. (116) (note that $\bar{V}_r = \bar{N}_{rz}$ for the linear case and $k = 0$) with respect to r results in

$$r\bar{N}_{rz} + \alpha\left[\frac{d}{dr}(rP_{rr}) - P_{\theta\theta}\right] = -\int r\,q(r)\,dr + c_3. \tag{132}$$

Substituting for $\bar{N}_{rz}$ from Eq. (117) into Eq. (132), we obtain

$$\left[\frac{d}{dr}(rM_{rr}) - M_{\theta\theta}\right] = -\int r\,q(r)\,dr + c_3. \tag{133}$$

Then, in view of the identity in Eq. (128), we obtain

$$B_{rr} r \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] + \bar{D}_{rr} r \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\phi_r)\right] - \alpha F_{rr} r \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = -\int r\, q(r)\, dr + c_3. \tag{134}$$

Integrating the above equation twice with respect to r, we obtain:

$$\begin{aligned} B_{rr}\, u + \bar{D}_{rr}\, \phi_r - \alpha F_{rr}\frac{dw}{dr} &= -\frac{1}{r}\int r\left[\int \frac{1}{r}\left(\int r\, q(r)\, dr\right) dr\right] dr \\ &\quad + c_3 \frac{r}{4}(2\log r - 1) + \frac{c_4\, r}{2} + \frac{c_5}{r} \\ &= -F(r) + c_3\frac{r}{4}(2\log r - 1) + \frac{c_4\, r}{2} + \frac{c_5}{r}, \end{aligned} \tag{135}$$

where

$$F(r) = \frac{1}{r}\int r\left[\int \frac{1}{r}\left(\int r\, q(r)\, dr\right) dr\right] dr. \tag{136}$$

Solving Eqs. (131) and (135) for u and ϕ_r in terms of dw/dr, we obtain

$$u(r) = \frac{\bar{D}_{rr}\, p(r) - \bar{B}_{rr}\, g(r)}{\bar{D}^*}, \quad \phi_r(r) = \frac{A_{rr}\, g(r) - B_{rr}\, p(r)}{\bar{D}^*}, \tag{137}$$

where

$$p(r) = \alpha E_{rr}\frac{dw}{dr} + \frac{c_1 r}{2} + \frac{c_2}{r}, \tag{138a}$$

$$g(r) = \alpha F_{rr}\frac{dw}{dr} - F(r) + c_3\frac{r}{4}(2\log r - 1) + \frac{c_4\, r}{2} + \frac{c_5}{r}, \tag{138b}$$

$$\bar{D}^* = A_{rr}\bar{D}_{rr} - B_{rr}\bar{B}_{rr}. \tag{138c}$$

We see that the solution for u and ϕ_r includes the unknown dw/dr. In the first-order shear deformation plate theory (FST), we have used Eq. (132) without the higher-order stress resultants. However, the presence of these higher-order terms makes the task of solving for dw/dr difficult. To see this, use Eqs. (129) and (132) and obtain

$$\begin{aligned} \hat{A}_{rz}\left(\phi_r + \frac{dw}{dr}\right) + \alpha \left\{ E_{rr} r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] + \bar{F}_{rr} r \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\phi_r)\right] \right. \\ \left. - \alpha H_{rr} r \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = -\int r\, q(r)\, dr + c_3 \end{aligned} \tag{139}$$

The form of the above equation makes it very difficult (if not impossible) to obtain the exact solution. The exact solutions of the simplified third-order beam theory (SBT) were discussed in [1]. One may follow similar approach here to determine the exact solutions to a *simplified TST*.

6. Summary

Three different plate theories, namely, the classical, first-order, and third-order plate theories are presented for axisymmetric bending of circular plates, accounting for the through-thickness variation of the material, modified couple stress effect, and the von Kármán nonlinearity. Exact solutions for bending of the first two theories are presented for several boundary conditions. The approach to develop exact solution for the third-order theory is presented but short of obtaining a solution as it involves additional unknown. It may be possible to obtain a simplified theory and obtain a solution as was done in the case of beams. Numerical examples are also presented to illustrate the accuracy of various models and bring out certain salient features of functionally graded circular plates. Finite element models of the nonlinear theories of circular plates presented herein can be found in the monograph by Reddy [2], which contains detailed discussions of obtaining analytical and numerical solutions. Extensions of the theories presented herein to buckling and vibration [2,16], especially accounting for nonlocal effects [17], are also awaiting.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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