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A comparison among various notions of viscosity solution for Hamilton–Jacobi equations on networks

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ABSTRACT

Three definitions of viscosity solutions for Hamilton–Jacobi equations on networks recently appeared in literature (Achdou et al. (2013) [1], Imbert et al. (2013) [4], Schieborn and Camilli (2013) [6]). Being motivated by various applications, they appear to be considerably different. The aim of this note is to establish their equivalence.

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1. Introduction

The theory of viscosity solutions (see [2] for an overview) has been extensively studied and refined by many authors, and, among the numerous contributions in the literature, one can find several adaptations to very different settings. In recent times there has been increasing interest in the study of nonlinear PDEs on networks and, concerning Hamilton–Jacobi equations, three different notions of viscosity solution have been introduced [1,4,6].

A major task in the theory of PDEs on network is to establish the correct transition conditions the solutions are subjected to at vertices. It is easy to see that classical transition conditions such the Kirchhoff condition, based on the divergence structure of the problem, are not adequate to characterize the expected viscosity solution of the equation. Hence in all the three approaches the equation is considered also at the vertices. On the other side, since the three papers are motivated by different applications (respectively, a control problem constrained to a network in [1], the study of traffic flow at a junction in [4] and Eikonal equations and distance functions on networks in [6]), they differ for the assumptions made on the Hamiltonians and mainly for the definitions of viscosity solution at the vertices (while inside the edges all the definitions coincide with the classical one).

Nevertheless, since all the definitions give existence and uniqueness of the solution, it is worth to compare them. In this paper, we show that, at least when restated in a common framework, the three definitions are equivalent. For comparing them, we shall obviously impose common assumptions but, since each definition requires adequate (and somehow limiting) conditions, it turns out that our assumptions are quite restrictive: for example, the comparison of the definitions in [4,6] requires that the Hamiltonian depends only on the gradient (i.e., it is independent of the state variable and of the edge) and it is strictly convex. However, for the sake of generality, in this paper we shall keep assumptions as weak as possible.

This paper is organized as follows: in Section 2 we introduce the network and the Hamilton–Jacobi equation we consider on it. Section 3 contains the three definitions of solution. Section 4 (respectively, Section 5) is devoted to establishing the equivalence between the definition in [1] (resp., in [6]) and the one in [4].

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2. Setting of the problem

We consider a planar star-shaped network Γ composed of a transition vertex v and of a finite number of straight edges $e_j, j \in J := \{1, \ldots, N\}$, i.e.

$$\Gamma = \{v\} \cup \bigcup_{j \in J} e_j \subset \mathbb{R}^2, \quad v = (0, 0), \ e_j = (0, 1)\eta_j \tag{2.1}$$

where $(\eta_j)_{j \in J}$ is a set of unit vectors in $\mathbb{R}^2$ with $\eta_j \neq \eta_k$ if $j \neq k$. For each edge e_j, we fix a parametrization $\pi_j : [0, l_j] \to \mathbb{R}^2$ such that $e_j = \pi_j((0, l_j))$ and $\bar{e}_j = \pi_j([0, l_j])$. Moreover we assume: $v = \pi_j(0)$ for any $j \in J$; in this way, we fix an orientation of e_j.

Consider a function $u : \Gamma \to \mathbb{R}$; for $j \in J$, we denote by $u^j : [0, l_j] \to \mathbb{R}$ the restriction of u to $\bar{e}_j$, i.e. $u^j(y) = u(x)$ for $y = \pi_j^{-1}(x) \in [0, l_j]$. We say that u is continuous (resp., upper or lower semi-continuous) when it is so with respect to the topology induced on Γ from $\mathbb{R}^2$ and we shall write $u \in C(\Gamma)$ (resp., $u \in USC(\Gamma)$ or $u \in LSC(\Gamma)$). As in [4,6], we consider the derivate with respect to the parametrization

$$D_j u(x) := \frac{du^j}{dy}(y) \quad \text{for } y = \pi_j^{-1}(x). \tag{2.2}$$

We denote by $\partial_j^+ u(v)$ the super-differential of u at v along the edge e_j, i.e.

$$\partial_j^+ u(v) := \{p \in \mathbb{R} : u^j(y) \leq u^j(0) + py + o(y) \text{ for } x \in e_j \to v, y = \pi_j^{-1}(x)\}$$

and we set $\partial_j^- u(v) = -\partial_j^+(-u)(v)$.

We consider the Hamilton–Jacobi equation

$$u + H(x, Du) = 0, \quad x \in \Gamma \tag{2.3}$$

where $u : \Gamma \to \mathbb{R}$ and $H : \Gamma \times \mathbb{R} \to \mathbb{R}$. In the following $H^j : [0, l_j] \times \mathbb{R} \to \mathbb{R}$ denotes the restriction of H to $\bar{e}_j$; throughout this paper, we shall assume

$$H^j \in C^0([0, l_j] \times \mathbb{R}), \quad \lim_{|p| \to \infty} H^j(x, p) = +\infty \text{ unif. in } x. \tag{2.4}$$

3. Three definitions of viscosity solution

In this section, we recall the three definitions of viscosity solutions of problem (2.3) introduced in [1,4,6]. Even though in [4] it is considered an evolutive equation, in order to compare the different notions of solution we restate the definition in terms of the stationary equation (2.3). Moreover we will only consider the definitions at the vertex v, since in the other points of the network they coincide with the standard one of viscosity solution.

We first define the admissible test functions in the sense of [1,4]

Definition 3.1. A function $\phi \in C(\Gamma)$ is an {ACCT, IMZ}-admissible test function at v if, for any $j \in J$, ϕ^j belongs to $C^1([0, l_j])$. We denote by $C_*^1(\Gamma)$ the set of {ACCT, IMZ}-admissible test functions.

We now give the definition of admissible test function in the sense of [6].

Definition 3.2. Let $\phi \in C(\Gamma), j, k \in J, j \neq k$. A function $\phi : \Gamma \to \mathbb{R}$ is said to be a CS-admissible (j, k)-test function at v if both $D_j \phi(v)$ and $D_k \phi(v)$ exist with

$$D_j \phi(v) + D_k \phi(v) = 0. \tag{3.1}$$

Remark 3.1. In Definition 3.1 admissible test functions can have different derivates at v along different edges, while in Definition 3.2 an admissible (j, k)-test function is differentiable if restricted to couple e_j, e_k taking into account the orientation. On the other hand, in Definition 3.1 admissible test functions have derivatives along each incident arc, while in Definition 3.2 an admissible test function needs to be differentiable only along two arcs.

3.1. The ACCT solution

In [1], the Hamiltonian is the control-theoretic one

$$H(x, p) = \sup_{a \in A} \{-f(x, a) \cdot p - \ell(x, a)\} \tag{3.2}$$

where f, ℓ are continuous functions while $A := \cup_{j \in J} A^j$ is a compact set with $A_j \neq \emptyset$ for each $j \in J$. It is assumed that, for every $x \in e_j, f(x, a) \in \mathbb{R}\eta_j$ if and only if $a \in A^j$ while, in the vertex v, the set of admissible controls reduces to the set $A_v := \cup_{j \in J} \{a \in A^j : f(v, a) \in \mathbb{R}_+ \eta_j\} \neq \emptyset$ (here, $\mathbb{R}_+ := [0, +\infty)$). In particular, we can write $H^j = \sup_{a \in A^j} \{-f \cdot p - \ell\}$. In [1], the authors introduced a *relaxed gradient* for a function u defined on Γ. Consider $\zeta \in \mathbb{R}^2$ for which there exists a

continuous function $z : [0, 1] \to \Gamma$ and a sequence $(t_n)_{n \in \mathbb{N}}$, $0 < t_n \leq 1$ with $t_n \to 0$ such that $\lim_{n \to \infty} \frac{z(t_n) - x}{t_n} = \zeta$, and such that $\lim_{n \to \infty} \frac{u(z(t_n)) - u(x)}{t_n}$ exists and does not depend on z and $(t_n)_{n \in \mathbb{N}}$. In this case, they define the relaxed gradient by

$$Du(x, \zeta) := \lim_{n \to \infty} \frac{u(z(t_n)) - u(x)}{t_n}.$$

Set

$$FL_j(v) := \overline{co} \left((f(v, a), \ell(v, a)) : a \in A^j \right) \cap (\mathbb{R}_+ \eta_j \times \mathbb{R}), \quad FL(v) := \cup_{j \in J} FL_j(v).$$

Definition 3.3. – A function $u \in USC(\Gamma)$ is a subsolution of (2.3) at v if for any $\phi \in C^1_*(\Gamma)$ s.t. $u - \phi$ has a local maximum point at v, there holds

$$u(v) + \sup_{(\zeta, \xi) \in FL(v)} \{-D\phi(v, \zeta) - \xi\} \leq 0.$$

– A function $u \in LSC(\Gamma)$ is a supersolution of (2.3) at v if for any $\phi \in C^1_*(\Gamma)$ s.t. $u - \phi$ has a local minimum point at v, there holds

$$u(v) + \sup_{(\zeta, \xi) \in FL(v)} \{-D\phi(v, \zeta) - \xi\} \geq 0.$$

Remark 3.2. Even if the coefficients in (3.2) are continuous functions, the Hamiltonian is in general discontinuous as a function of the state variable. In fact, (3.2) is the Hamiltonian of a control problem in $\mathbb{R}^2$ constrained to the network Γ. Hence the set of the admissible controls, i.e. the controls corresponding to a tangential direction to the network, is in general discontinuous passing from a point inside an edge to the vertex v.

Taking advantage to the straightforward identity

$$D\phi(v, \zeta) = D_j \phi(v) \eta_j \cdot \zeta \quad \text{for } \zeta \in \mathbb{R} \eta_j,$$

let us rewrite the previous definition in terms of the derivatives with respect to the parametrization:

Lemma 3.1. *A function* $u \in USC(\Gamma)$ *(respectively,* $u \in LSC(\Gamma)$*) is a subsolution (resp., supersolution) of* (2.3) *at* v *if for any* $\phi \in C^1_*(\Gamma)$ *s.t.* $u - \phi$ *attains a local maximum (resp., minimum) at* v*, there holds*

$$u(v) + \max_{j \in J} \sup_{(\zeta, \xi) \in FL_j(v)} \{-D_j \phi(v) \eta_j \cdot \zeta - \xi\} \leq 0 \quad (resp., \geq 0).$$

3.2. The IMZ solution

In [4], the Hamiltonian H is assumed to satisfy

$$\exists p_0^j \in \mathbb{R} \text{ s.t. } H^j \text{ is non-increasing on } (-\infty, p_0^j), \text{ non-decreasing on } (p_0^j, +\infty) \tag{3.3}$$

for each $j \in J$. Define the function H_j^- by

$$H_j^-(p) := \inf_{q \leq 0} H^j(v, p + q) = \begin{cases} H^j(v, p), & \text{if } p < p_0^j \\ H^j(v, p_0^j), & \text{if } p \geq p_0^j. \end{cases} \tag{3.4}$$

Remark 3.3. Inside each edge, the Hamiltonian depends only on p and on the edge itself while its definition in the vertex involves every H^j. It is important to observe that no continuity condition on the Hamiltonian in p at v is assumed.

Definition 3.4. – A function $u \in USC(\Gamma)$ is a subsolution of (2.3) at v if for any $\phi \in C^1_*(\Gamma)$ s.t. $u - \phi$ has a local maximum point at v, then

$$u(v) + \max_{j \in J} H_j^-(D_j \phi(v)) \leq 0.$$

– A function $u \in LSC(\Gamma)$ is a supersolution of (2.3) at v if for any $\phi \in C^1_*(\Gamma)$ s.t. $u - \phi$ has a local minimum point at v, then

$$u(v) + \max_{j \in J} H_j^-(D_j \phi(v)) \geq 0.$$

3.3. The CS solution

In [6], the Hamiltonian is assumed to satisfy

$$H^j(v, p) = H^k(v, p) \quad \text{for any } p \in \mathbb{R}, j, k \in J \tag{3.5}$$

$$H^j(v, p) = H^j(v, -p) \quad \text{for any } p \in \mathbb{R}, j \in J. \tag{3.6}$$

Assumptions (3.5) and (3.6) are the continuity of H in p and its independence on the orientation of the incident arc, respectively.

Definition 3.5. – A function $u \in USC(\Gamma)$ is a subsolution of (2.3) at v if for any $j, k \in J$ and any (j, k)-test function ϕ for which $u - \phi$ attains a local maximum at v relatively to $\bar{e}_j \cup \bar{e}_k$, then

$$u(v) + H^k(v, D_k\phi(v)) = u(v) + H^j(v, D_j\phi(v)) \leq 0. \tag{3.7}$$

– A function $u \in LSC(\Gamma)$ is a supersolution of (2.3) at v if for any $j \in J$, there exists $k \in J$, $k \neq j$, (said feasible for j at v) such that for any (j, k)-test function ϕ for which $u - \phi$ attains a local minimum at v relatively to $\bar{e}_j \cup \bar{e}_k$, then

$$u(v) + H^k(v, D_k\phi(v)) = u(v) + H^j(v, D_j\phi(v)) \geq 0. \tag{3.8}$$

Remark 3.4. Note that the definitions of subsolution and supersolution in Definition 3.5 are not symmetric, unlike the ones in Definitions 3.3 and 3.4.

4. Comparison between ACCT and IMZ

To fix a common framework for the two settings, we assume that H verifies the assumptions of both Sections 3.1 and 3.2. Recall that the admissible test functions coincide for the two definitions.

Theorem 4.1. *The definitions of (ACCT)-solution and (IMZ)-solution are equivalent.*

Proof. We assume wlog $u(v) = 0$ and that, for $p_j^- \leq \underline{p}_j \leq p_0^j \leq \bar{p}_j \leq p_j^+$, we have $H^j(v, p) < 0$ if, and only if, $p \in (\underline{p}_j, \bar{p}_j)$ (which may be empty) and $H^j(v, p) > 0$ if, and only if, $p \in (-\infty, p_j^-) \cup (p_j^+, +\infty)$.

1. (ACCT)-subsolution implies (IMZ)-subsolution. Let u be an (ACCT)-subsolution and ϕ be an admissible upper test function for u at v. To prove $\max_j H_j^-(D_j\phi(v)) \leq 0$ we assume by contradiction that $H_j^-(D_j\phi) > 0$ for some $j \in J$. By the definition of H_j^- this is equivalent to $D_j\phi(v) < p_j^-$. Hence, we have

$$\sup_{(\zeta, \xi) \in FL(v)} \{-D\phi(v, \zeta) - \xi\} \geq \sup_{(\zeta, \xi) \in FL_j(v)} \{-D_j\phi(v)\eta_j \cdot \zeta - \xi\}$$

$$\geq \sup_{a \in A^j \cap A_v} \{-D_j\phi(v)\eta_j \cdot f(v, a) - \ell(v, a)\} = H^j(v, D_j\phi(v)) > 0$$

which contradicts the fact that u is an (ACCT)-subsolution.

2. (IMZ)-subsolution implies (ACCT)-subsolution. Let u be a (IMZ)-subsolution and ϕ be an admissible test function for u at v. Assume by contradiction

$$\sup_{(\zeta, \xi) \in FL(v)} \{-D\phi(v, \zeta) - \xi\} > 0. \tag{4.1}$$

Being a (IMZ)-subsolution, u verifies $\max_j H_j^-(D\phi) \leq 0$ namely

$$D_j\phi(v) \geq p_j^- \quad \forall j \in J.$$

We recall that $FL_j(v) \subset \mathbb{R}_+ \eta_j \times \mathbb{R}$; therefore, the previous inequality implies

$$-D_j\phi(v)\eta_j \cdot \zeta - \xi \leq -p_j^- \eta_j \cdot \zeta - \xi \quad \forall(\zeta, \xi) \in FL_j(v).$$

We deduce

$$\sup_{(\zeta, \xi) \in FL(v)} \{-D\phi(v, \zeta) - \xi\} \leq \max_j \sup_{(\zeta, \xi) \in FL_j(v)} \{-p_j^- \eta_j \cdot \zeta - \xi\}. \tag{4.2}$$

On the other hand, since $H_j(v, p_j^-) = 0$, by (3.2) we obtain that $-p_j^- \eta_j \cdot f(v, a) - \ell(v, a) \leq 0$ for every $a \in A^j$. By linearity, we infer $-p_j^- \eta_j \cdot \zeta - \xi \leq 0$ for every $(\zeta, \xi) \in FL_j(v)$ and consequently

$$\max_j \sup_{(\zeta, \xi) \in FL_j(v)} \{-p_j^- \eta_j \cdot \zeta - \xi\} \leq 0.$$

Replacing this inequality in (4.2), we get a contradiction to (4.1).

3. (ACCT)-supersolution implies (IMZ)-supersolution. Let u be a (ACCT)-supersolution. We want to prove that, for each admissible lower test function ϕ for u at v, we have $\max_j H_j^-(D_j\phi(v)) \geq 0$, i.e. that there exists $j \in J$ such that $D_j\phi \leq \underline{p}_j$.

We observe that $\partial_j^- u(v) = (-\infty, d_j]$ (possibly, $d_j = \pm\infty$) for each $j \in J$. If $d_j = -\infty$ for some $j \in J$, then $\partial_j^- u(v)$ is empty and, in particular, there is no lower test function for u at v; thus, there is nothing to prove. If $d_j \neq -\infty$ for any $j \in J$, we claim that there exists $j \in J$ such that

$$d_j \leq \underline{p}_j. \tag{4.3}$$

By contradiction, assume that $d_j > \underline{p}_j$ for every $j \in J$; hence, there exists an admissible lower test function ϕ such that $D_j\phi(v) \in (\underline{p}_j, \bar{p}_j)$ for each $j \in J$. By Lemma 3.1, for some $j \in J$, we have

$$\sup_{(\zeta,\xi)\in FL_j(v)} \{-D_j\phi(v)\eta_j \cdot \zeta - \xi\} \geq 0. \tag{4.4}$$

On the other hand, for each $j \in J$, there holds $H^j(v, D_j\phi(v)) < 0$ and, in particular,

$$-D_j\phi(v)\eta_j \cdot f(v, a) - \ell(v, a) < 0 \quad \forall a \in A^j.$$

By linearity, we infer

$$-D_j\phi(v)\eta_j \cdot \zeta - \xi < 0 \quad \forall(\zeta, \xi) \in FL_j(v)$$

which contradicts (4.4). This achieves the proof of (4.3).

Finally, let ϕ be an admissible lower test function; in particular, we have $D_j\phi(v) \in (-\infty, d_j]$ for every $j \in J$. By relation (4.3), we get that there exists $j \in J$ such that $H_j^-(D_j\phi(v)) \geq 0$. By the arbitrariness of ϕ, we deduce that u is an (IMZ)-supersolution.

4. (IMZ)-supersolution implies (ACCT)-supersolution. Let u be a (IMZ)-supersolution and ϕ an admissible lower test function for u at v. The definition of (IMZ)-supersolution reads $\max_j H_j^-(D_j\phi(v)) \geq 0$, namely, for some $j \in J$, there holds $H_j^-(D_j\phi(v)) \geq 0$. This fact is equivalent to $D_j\phi(v) \leq \underline{p}_j$. Whence, we get $H^j(v, D_j\phi(v)) = H_j^-(D_j\phi) \geq 0$ and we accomplish the proof observing that $\sup_{(\zeta,\xi)\in FL(v)}\{-D\phi(v, \zeta) - \xi\} \geq \max_j H^j(v, D_j\phi(v))$. $\square$

Remark 4.1. In the previous proof, we actually established that the definition of (ACCT)-supersolution (resp., subsolution) is equivalent to the one of (IMZ)-supersolution (resp., subsolution).

5. Comparison between IMZ and CS

We assume that H satisfies the assumptions of both Section 3.2 and of Section 3.3 with $p_0^i = 0$ (this is due to (3.6)). In particular, by (3.5), in v the H^j's are independent of j; for the sake of simplicity, we drop the index "j".

Theorem 5.1. *The definitions of (CS)-solution and (IMZ)-solution are equivalent.*

The proof of the theorem is postponed at the end of the section. Let us first establish the following result.

Proposition 5.1. *The definitions of (IMZ)-subsolution and (CS)-subsolution are equivalent, while (IMZ)-supersolution implies (CS)-supersolution.*

Proof. Wlog we assume: $u(v) = 0$ and, for some nonnegative values $\bar{p}$ and p^*, $H(v, p) > 0$ if, and only if, $p \in (-\infty, -\bar{p}) \cup (\bar{p}, +\infty)$ and $H(v, p) < 0$ if, and only if, $p \in (-p^*, p^*)$.

1. (IMZ)-subsolution implies (CS)-subsolution. By (3.4), (3.6) the function H^- is defined as follows: $H^-(p) = H(v, p)$ if $p \leq 0$, $H^-(v, p) = H(v, 0) < 0$ if $p \geq 0$. Let u be a (IMZ)-subsolution. Following the same arguments of [4, Lemma 5.5], one can easily prove that u is Lipschitz continuous; in particular, $\partial_j^+ u(v) \neq \emptyset$ for every $j \in J$. Set $\partial_j^+ u(v) =: [p_j, +\infty)$ for each $j \in J$. We deduce that the function $\psi \in C_*^1(\Gamma)$ with $\partial_j\psi = p_j$ is an (IMZ)-admissible upper test function. By the definition of (IMZ)-subsolution, we infer $H^-(p_j) \leq 0$ for each $j \in J$. By the definition of $\bar{p}$ and of H^-, this relation can be rewritten as:

$$p_j \geq -\bar{p} \quad \forall j \in J. \tag{5.1}$$

Consider now a (CS)-admissible (j, k)-upper test function ϕ for u at v. We want to prove that $H(v, D_j\phi) = H(v, D_k\phi) \leq 0$, namely $D_j\phi(v), D_k\phi(v) \in [-\bar{p}, \bar{p}]$. To this end, we assume by contradiction that $D_j\phi(v) > \bar{p}$. Hence, by relation (3.1), we have

$$-\bar{p} > -D_j\phi(v) = D_k\phi(v) \geq p_k$$

that contradicts relation (5.1). Therefore, we have $D_j\phi(v) \leq \bar{p}$ and, similarly, $D_k\phi(v) \leq \bar{p}$. Owing to the relation $D_k\phi(v) = -D_j\phi(v)$, it holds that $D_j\phi(v), D_k\phi(v) \in [-\bar{p}, \bar{p}]$. Taking into account the arbitrariness of the function ϕ and of (j, k), we accomplish the proof of point (1).

2. (CS)-subsolution implies (IMZ)-subsolution. We will use the following lemma (see [3, Lemma 5.4]).

Lemma 5.1. *Let $y_m \in [0, l_j]$ ($m \in \mathbb{N}$) with $\lim_m y_m = 0$. Then there holds*

$$u^j(0) + \lim_{m \to +\infty} H\left(v, \frac{u^j(y_m) - u^j(0)}{y_m}\right) \leq 0. \tag{5.2}$$

Assume by contradiction that there exists $\phi \in C^1_*(\Gamma)$ and $j \in J$ such that $H^-(D_j\phi(v)) \geq \delta > 0$. Set $p = D_j\phi(v)$; by the continuity of H^-, fix η sufficiently small in such a way that $H^-(p + \eta) \geq \delta/2$. Since $p \in \partial_j^+ u(v)$, if $x_m \in e_j$, $x_m \to v$ and $y_m = \pi_j^{-1}(x_m)$ we have that

$$p_m := \frac{u^j(y_m) - u^j(0)}{y_m} \leq p + \eta$$

for m sufficiently large. Moreover $u(x_m) \to u(v) = 0$ and therefore for m sufficiently large $u(x_m) + H^-(p_m) \geq \frac{\delta}{4}$. On the other side by Lemma 5.1 and $u(v) = 0$, it follows that $u(x_m) + H(v, p_m) < \delta/4$ for m sufficiently large and therefore a contradiction.

   3. (IMZ)-supersolution implies (CS)-supersolution. Let us observe that it is possible that, for some $j \in J$, there holds $\partial_j^- u(v) = \emptyset$. In this case there is no (IMZ)-admissible lower test function. In order to prove that u is a (CS)-supersolution, it suffices to choose the edge e_j as the feasible one; with this choice, there is no (CS)-admissible test function neither, i.e. there is nothing to prove.

   Let us now assume $\partial_j^- u(v) \neq \emptyset$ for every $j \in J$. Set $\partial_j^- u(v) =: (-\infty, p_j]$ (possibly $p_j = +\infty$). We deduce that the function $\psi \in C^1_*(\Gamma)$ with $D_j\psi(v) = p_j$ ($D_j\psi = 2p^*$ if $p_j = +\infty$) is an (IMZ)-admissible lower test function. By the definition of (IMZ)-supersolution, there exists $j \in J$ such that $H^-(D_j\psi(v)) \geq 0$. By the definition of H^- and of ψ when $p_j = +\infty$, we infer that there exists $j \in J$ such that

$$p_j \leq -p^*. \tag{5.3}$$

In order to prove that u is a (CS)-supersolution, let us choose e_j as the feasible edge; in other words, we claim that, for every $k \in J \setminus \{j\}$, for every (CS)-admissible (j, k)-lower test function ϕ, there holds $H(v, D_j\phi(v)) = H(v, D_k\phi(v)) \geq 0$. To this end, we observe that the definition of lower test function and relation (5.3) entail: $D_j\phi(v) \leq p_j \leq -p^*$. By our assumptions, we get $H(v, D_j\phi(v)) \geq 0$. $\square$

Remark 5.1. Thanks to Theorem 4.1, we also have that the definitions of (ACCT)-subsolution and (CS)-subsolution are equivalent, while (ACCT)-supersolution implies (CS)-supersolution.

Proof of Theorem 5.1. By Proposition 5.1, we have only to prove that a (CS)-solution is a (IMZ)-supersolution. Let u be a (CS)-solution; by the same arguments of [6, Proposition 3.3], it follows that u is Lipschitz continuous. We observe that, for each $j \in J$, there holds: $\partial_j^- u(v) = (-\infty, p_j]$ for some $p_j \in \mathbb{R}$. A function $\psi \in C^1_*(\Gamma)$ is a (IMZ)-admissible lower test function if, and only if, $D_j\psi(v) \leq p_j$ for each $j \in J$. For such a ψ, our aim is to prove:

$$u(v) + H^-(D_j\psi) \geq 0 \quad \text{for some } j \in J. \tag{5.4}$$

Let us split our arguments according to the existence or not existence of (CS)-admissible lower test functions. Fix an edge, say e_1; wlog assume that e_2 is the feasible edge in the definition of (CS)-supersolution. We observe that there exists an admissible $(1, 2)$-lower test function if, and only if, there holds $p_1 \geq -p_2$.
   Case (i): Non existence of a (CS)-lower test function. We have

$$p_1 < -p_2. \tag{5.5}$$

By (5.5) either p_1 or p_2 is negative and wlog we assume $p_1 < 0$. We observe that, for the network $\bar{e}_1 \cup \bar{e}_2$, the definition of (CS)-solution coincides with the classical definition of solution in viscosity sense for a segment. We can consider such a segment as the one given by the orientation of e_1 inverting the orientation of e_2. By [5, Theorem 1] it follows that either u is differentiable in v (this cannot be our case because $p_1 \neq p_2$) or there holds

$$H(v, p_1) = H(v, -p_2) = -u(v).$$

Since $p_1 < 0$, we deduce

$$u(v) + H^-(p_1) = 0,$$

and also, by the monotonicity of H^-,

$$u(v) + H^-(p) \geq 0, \quad \forall p \leq p_1$$

which entails inequality (5.4).

`Case (ii): Existence of some (CS)-lower test function.` We assume that there exists some (CS)-admissible $(1, 2)$-lower test function ϕ for the function u at v. In this case, we have $p_1 \geq D_1\phi(v) = -D_2\phi(v) \geq -p_2$. Moreover, observe that any $(1, 2)$-lower test function $\bar{\phi}$ with $D_1\bar{\phi}(v) \in [-p_2, p_1]$ is admissible. Therefore, the definition of (CS)-supersolution yields

$$u(v) + H(v, p) \geq 0, \quad \forall p \in [-p_2, p_1], \tag{5.6}$$
$$u(v) + H(v, p) \geq 0, \quad \forall p \in [-p_1, p_2]. \tag{5.7}$$

Introduce $h(p) := u(v) + H(v, p)$. If $h(p) \geq 0 \ \forall p \in \mathbb{R}$, then there is nothing to prove; otherwise, for some $p^* > 0$, $h(p)$ is negative if, and only if, $p \in (-p^*, p^*)$. Relation (5.6) ensures that the interval $(-p_2, p_1) \cap (-p^*, p^*)$ must be empty. Whence, we have either $p_1 \leq -p^*$ or $p_2 \leq -p^*$. For $p_1 \leq -p^*$, the definition of H^- and relation (5.6) ensure

$$u(v) + H^-(p) = u(v) + H(v, p) \geq 0, \quad \forall p \leq p_1.$$

For $p_2 \leq -p^*$, the definition of H^- and relation (5.7) ensure

$$u(v) + H^-(p) = u(v) + H(v, p) \geq 0, \quad \forall p \leq p_2.$$

Both the last two relations amount to inequality (5.4). $\square$

Remark 5.2. In the previous proof the hypothesis that u is a (CS)-solution is needed only when there is no admissible (CS)-lower test function. In this case, it seems that the information available from the (CS)-definition of supersolution are not sufficient to obtain an equivalent property for (IMZ)-definition.

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