A Sampling Strategy of the Radiation Operator in Near-Zone based on an Asymptotic Kernel

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Abstract

In this paper, we address the problem of discretizing the singular system of the radiation operator concerning the case of a magnetic strip current whose radiated field is observed in near zone on a bounded line parallel to the source. This question has been already addressed in previous articles with the limitation that the extension of the observation domain does not overcome the source size. In this article, we remove such limitation, hence, we provide a discrete model that well approximates the singular values of the radiation operator in the case where the observation domain is larger than the source.

Index Terms – inverse source, radiation operator, asymptotic approach, discretization.

I. Introduction

The question of finding an efficient discretization of the field radiated by a source is a relevant task in the framework of inverse source problem. The latter consists in the inversion of the integral equation

$$E = A\,J \tag{1}$$

where A stands for the radiation operator, J represents the density current of the source, and E represents the radiated field. In the case of planar scanning, the most common sampling strategy is based on a uniform sampling with a step length of half-wavelength [1]. Despite its simplicity, such a strategy is not efficient since it requires to acquire a number of samples that may be significantly higher than the dimension of the unknown space. For this reason, the aims of the paper are those of 1) establishing the minimum number of sampling points, 2) finding their optimal position.

As concerns the first point, the minimum number of sampling points is equal to the number of degrees of freedom (NDF) of the radiated field. The latter can be evaluated by counting the number of the most relevant singular values of the radiation operator.

As regards the optimal positions of the sampling points, this issue can be recast in how to collect the radiated field in such a way to obtain a discrete model whose singular values well approximate the most relevant singular values of the radiation operator. For several far-field configurations, the kernel of the related eigenvalue problem involves a bandlimited kernel of difference type. In these cases, the sampling theory approach developed in [2] can be exploited to discretize the continuous model.

In this paper, we consider the case of a strip current whose radiated field is observed in near zone over a truncated line parallel to the source. Unfortunately, for such a configuration, the integral equation for the computation of the singular values involves a space-variant kernel; hence, the literature mentioned above cannot be directly applied to discretize the model. Despite this, as shown in [3] and [4], a change of the integration variable allows recasting the operator involved in the related eigenvalue problem, as a convolution operator with a bandlimited kernel, at least when the observation domain is smaller than the observation. In this paper, thanks to the use of a weighted adjoint, we remove such limitation; hence, we provide a discretization strategy of the radiation operator that works when the observation domain is larger than the source domain.

II. Geometry of the problem

A 1D magnetic current $\bar{J}(x) = J(x)\,\hat{\imath}_y$ directed along the y-axis and supported on the set $SD = [-a, a]$ of the x-axis radiates within a homogeneous medium with wavenumber β. The x component of the radiated electric field is observed in near non-reactive zone over a bounded observation domain $OD = [-X_o, X_o]$ that is parallel to the source and located along the axis $z = z_o$ (see Fig. 1).

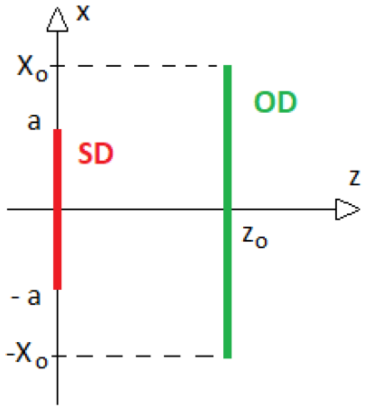

FIG. 1 – Geometry of the problem.

For the geometry at hand, the radiation operator is given by

$$A\,J(x') = \int_{-a}^{+a} \frac{z_o}{R^{3/2}(x',x_o)} e^{-j\,\beta\,R\,(x',\;x_o)}\,J(x')\,dx' \qquad (2)$$

with $R\,(x', x_o) = \sqrt{(x - x')^2 + z_o^2}$. Consequently, a weighted adjoint operator can be expressed in the form

$$A_w^\dagger\,E = \int_{-X_o}^{+X_o} w(x_o)\,\frac{z_o}{R^{3/2}(x',x_o)} e^{j\,\beta\,R\,(x',\;x_o)}\,E(x_o)dx_o \qquad (3)$$

where $w(x_o)$ is the weight function that prefilters the data. Further considerations on the effects of the weight function will be made in the next sections.

III. STUDY OF THE RADIATION OPERATOR

In this section, we first evaluate the kernel of $AA_w^\dagger$ by exploiting an asymptotic approach. Later, by introducing a suitable change of variables, we show that it is possible to approximate such kernel with a bandlimited function of difference type.

The operator $AA_w^\dagger$ is defined as below

$$AA_w^\dagger E = \int_{-X_o}^{+X_o}\left(z_o^2\,w(x_o) \int_{-a}^{+a} \frac{e^{-j\,\beta\left(R\,(x',x)\,-R\,(x',x_o)\right)}}{R^{\frac{3}{2}}(x',\;x)\,R^{\frac{3}{2}}(x',\;x_o)}\,dx' \right) E(x_o)\,dx_o \qquad (4)$$

By setting $f(x', x, x_o) = R^{-3/2}(x', x)\,R^{-3/2}(x', x_o)$ and $\phi(x', x, x_o) = \frac{R\,(x',x)\,-\,R\,(x',x_o)}{a}$, the kernel can be expressed by the following integral

$$H(x, x_o) = z_o^2\,w(x_o) \int_{-a}^{+a} f(x', x, x_o)\,e^{-j\,\beta a\,\phi(x',x,x_o)}\,dx' \qquad (5)$$

For $\beta a \gg 1$, and $\forall x \neq x_o$ the kernel $H(x, x_o)$ can be approximated through the asymptotic form

$$H(x,x_o) \approx -\frac{z_o^2}{j\beta a} w(x_o)\,e^{-j\frac{\beta a}{2}(\phi_{-a}(x,x_o)+\phi_a(x,x_o))} \cdot \left(\frac{f_a(x,x_o)}{\phi_a'(x,x_o)} e^{j\frac{\beta a}{2}(\phi_{-a}(x,x_o)-\phi_a(x,x_o))} - \frac{f_{-a}(x,x_o)}{\phi_{-a}'(x,x_o)} e^{-j\frac{\beta a}{2}(\phi_{-a}(x,x_o)-\,\phi_a(x,x_o))} \right) \qquad (6)$$

which corresponds to the first term of the integration by parts method [5]. The subscripts $-a$ or a in Eq. (6) denote that the correspondent function has been particularized in the point $x' = -a$ or $x' = a$.

As can be seen from Eq. (6), the operator $AA_w^\dagger$ is space-variant in the variables (x, x_o). With the aim to recast it in a form more similar to a convolution operator, the last expression of $H(x, x_o)$ suggests introducing the following variables

$$\eta(x_o) = \frac{\sqrt{(x_o+a)^2+z_o^2} - \sqrt{(x_o-a)^2+z_o^2}}{2a}, \qquad \gamma(x_o) = \frac{\sqrt{(x_o+a)^2+z_o^2} + \sqrt{(x_o-a)^2+z_o^2}}{2a} \tag{7}$$

The latter allows expressing the operator $AA_w^\dagger$ in the following form

$$AA_w^\dagger E = \int_{\eta(-X_o)}^{\eta(X_o)} \frac{z_o^2}{j\beta a} w(\eta_o) \frac{dx}{d\eta_o} e^{j\beta a(\gamma(\eta_o)-\gamma(\eta))} \left(\frac{f_{-a}(\eta,\eta_o)}{\phi'_{-a}(\eta,\eta_o)} e^{-j\beta a(\eta-\eta_o)} - \frac{f_a(\eta,\eta_o)}{\phi'_a(\eta,\eta_o)} e^{j\beta a(\eta-\eta_o)} \right) E(\eta_o) d\eta_o \tag{8}$$

where $\eta_o = \eta(x_o)$, $\eta = \eta(x)$, and $\frac{dx}{d\eta_o} = \frac{1}{\sqrt{a^2(1-\eta_o^2)+z_o^2}} \frac{a^2(1-\eta_o^2)^2+z_o^2}{(1-\eta_o^2)^{3/2}}$.

Note that the change of the integration variable introduces in the kernel the Jacobian term $dx/d\eta_o$ which is singular for $\eta_o = \pm 1$. Such singularities affect also on the kernel $H(\eta,\eta_o)$ if they are not compensated by $w(\eta_o)$. Since

$$\frac{dx}{d\eta_o} = O\left(\frac{1}{(\eta_o \mp 1)^{3/2}}\right), \quad \frac{f_a(\eta,\eta_o)}{\phi'_a(\eta,\eta_o)} = O\left((\eta_o \mp 1)^{3/4}\right), \quad \frac{f_{-a}(\eta,\eta_o)}{\phi'_{-a}(\eta,\eta_o)} = O\left((\eta_o \mp 1)^{3/4}\right) \tag{9}$$

as $\eta_o \to \pm 1$, it results that for $\eta_o \to \pm 1$ the terms $\frac{dx}{d\eta_o} \frac{f_a(\eta,\eta_o)}{\phi'_a(\eta,\eta_o)}$ and $\frac{dx}{d\eta_o} \frac{f_{-a}(\eta,\eta_o)}{\phi'_{-a}(\eta,\eta_o)}$ go to infinity as $(\eta_o \mp 1)^{-3/4}$. Hence, in order to compensate the singularities, we choose

$$w(\eta_o) = \left(\frac{dx}{d\eta_o}\right)^{-1/2} \tag{10}$$

In Fig. 2 the diagram of $w(\eta_o)$ for different values of z_o is shown.

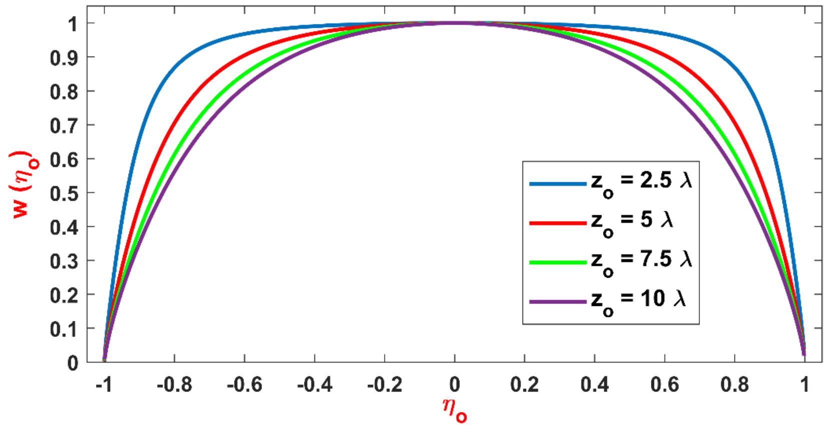

Fig. 2 – Diagram of $w(\eta_o)$ normalized with respect to its maximum for different values of z_o when the size source $a = 10\lambda$.

Such choice of the weight function makes the kernel free from singularities, and it allows rewriting $H(\eta,\eta_o)$ in the form below

$$H(\eta,\eta_o) \approx \frac{z_o^2}{j\beta a} \left(\frac{dx}{d\eta_o}\right)^{1/2} e^{j\beta a(\gamma(\eta_o)-\gamma(\eta))} \left(\frac{f_{-a}(\eta,\eta_o)}{\phi'_{-a}(\eta,\eta_o)} e^{-j\beta a(\eta-\eta_o)} - \frac{f_a(\eta,\eta_o)}{\phi'_a(\eta,\eta_o)} e^{j\beta a(\eta-\eta_o)} \right) \tag{11}$$

At this stage, the $H(\eta,\eta_o)$ is free from singularities but it does not appear space-invariant since the amplitude terms do not seem to depend on the difference $(\eta - \eta_o)$. However, as shown in [4], $\forall(\eta,\eta_o) \in D_1 = [\eta(-a),\eta(a)] \times [\eta(-a),\eta(a)]$ the terms f_a/ϕ'_a and f_{-a}/ϕ'_{-a} can be approximated as below

$$\frac{f_a(\eta,\eta_o)}{\phi'_a(\eta,\eta_o)} \approx \frac{f_{-a}(\eta,\eta_o)}{\phi'_{-a}(\eta,\eta_o)} \approx -\frac{a}{z_o^2 \frac{dx}{d\eta_o}(\eta-\eta_o)} \tag{12}$$

Hence, it results that $\forall(\eta,\eta_o) \in D_1$ the kernel can be rewritten apart for an unessential factor in the simple form

$$H(\eta,\eta_o) \approx e^{j\beta a(\gamma(\eta_o)-\gamma(\eta))} \frac{\sin(\beta a(\eta-\eta_o))}{\pi(\eta-\eta_o)} \tag{13}$$

where we have taken into account that for $\eta_o \in [\eta(-a),\eta(a)]$ the term $(dx/d\eta_o)^{-1/2}$ is almost constant and well approximated by the value in $\eta_o = 0$.

According to approximation in Eq. (12), for $\eta_o \to \pm 1$ the terms f_a/ϕ'_a and f_{-a}/ϕ'_{-a} are not an $O\left((\eta_o \mp 1)^{3/4}\right)$ as shown in Eq. (9). For this reason, such approximation does not work for $(\eta,\eta_o)\in D-D_1$ where $D=[\eta(-X_o),\eta(X_o)]\times[\eta(-X_o),\eta(X_o)]$. In such region, a reasonable approximation of the terms f_a/ϕ'_a and f_{-a}/ϕ'_{-a} is given by

$$\frac{f_a(\eta,\eta_o)}{\phi'_a(\eta,\eta_o)}\approx\frac{f_{-a}(\eta,\eta_o)}{\phi'_{-a}(\eta,\eta_o)}\approx-\frac{const}{\left(\frac{dx}{d\eta_o}\right)^{1/2}(\eta-\eta_o)} \tag{14}$$

By substituting Eq. (14) in Eq. (11), it follows that also for $(\eta,\eta_o)\in D-D_1$ the kernel $H(\eta,\eta_o)$ can be expressed as in Eq. (13). Hence, we can state that $\forall\,(\eta,\eta_o)\in D$ the kernel of $AA_w^{\dagger}$ can be approximated as in Eq. (13). In Fig. 3 the actual kernel of $AA_w^{\dagger}$ is compared with the sinc kernel given by Eq. (13). As it can be seen from the figure, the sinc kernel represents a good approximation of the actual kernel.

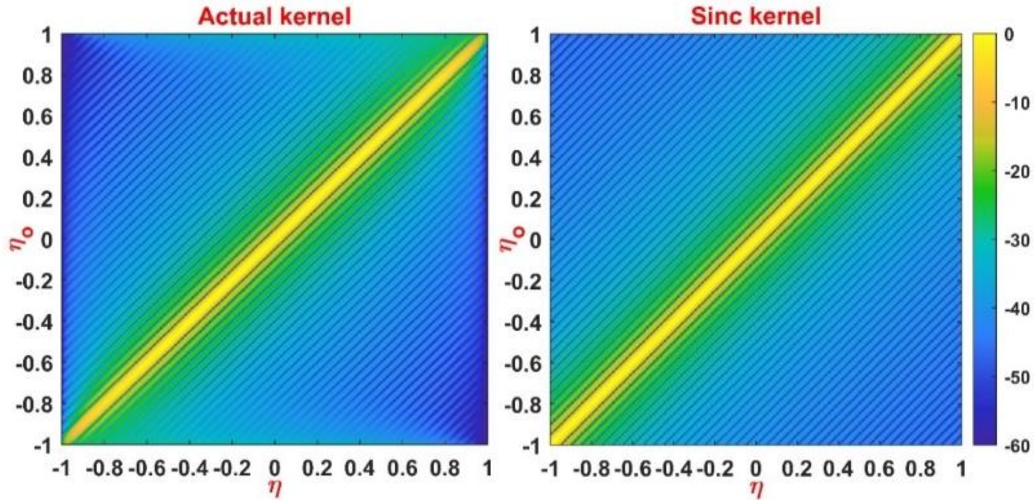

FIG. 3 – Amplitude of $K(\eta,\eta_o)$ (numerically computed), and amplitude of the sinc kernel given by Eq. (13). The diagrams are in dB, and they refer to the configuration $a=10\lambda$, $z_o=5\lambda$, $X_o=10a$.

IV. NDF AND DISCRETIZATION

At this stage let us discuss about the NDF, and the discretization of the radiation operator in the case where $X_o>a$. Naturally, the best thing would be that of solving such issues when the adjoint operator is not modified by the weight function $w(\eta_o)$. In the remaining part of the manuscript, we denote the usual adjoint operator by $A^{\dagger}$. As regards the NDF, let us note that the eigenvalues of the operators $AA^{\dagger}$ and $AA_w^{\dagger}$ exhibit the same behavior. To proof this, as first thing note that by definition the operators $A^{\dagger}A$ and $AA^{\dagger}$ have the same eigenvalues. Later, observe that the operator $AA_w^{\dagger}$ involves a convolution kernel of sinc type with a spatial bandwidth product $c=\beta a\,\eta(X_o)$ exactly like the kernel of $A^{\dagger}A$ [6]. This implies that, unless of a constant factor, $AA_w^{\dagger}$ and $A^{\dagger}A$ have the same eigenvalues. Consequently, we can conclude that the weight function does not change the behavior of the eigenvalues, hence, the eigenvalues of $AA^{\dagger}$ and $AA_w^{\dagger}$ have the same shape. Thanks to this, it is possible to compute the eigenvalues of $AA^{\dagger}$ by referring to the operator $AA_w^{\dagger}$. By exploiting the results in [7], we can state that the eigenvalues of $AA^{\dagger}$ exhibit a step-like behavior with the knee occurring at the index

$$N=\frac{2\beta a}{\pi}\eta(X_o) \tag{15}$$

As regards the discretization, let us start by considering the eigenvalue problem $AA_w^{\dagger}v_n^w=\lambda_n v_n^w$. The latter can be easily recast in the form

$$\int_{\eta(-X_o)}^{\eta(X_o)} sinc\big(\beta a(\eta-\eta_o)\big)\bar{v}_n^{w}(\eta_o)\,d\eta_o=\lambda_n\,\bar{v}_n^{w}(\eta) \tag{16}$$

where $\bar{v}_n(\eta_o)=v_n(\eta_o)e^{j\beta a\,\gamma(\eta_o)}$. The integral equation expressed by Eq. (16) involves a convolution operator with a bandlimited kernel whose maximum frequency with respect to η is equal to $\beta a/(2\pi)$. Consequently, it can be discretized by applying the sampling theory approach shown in [2] with a sampling step $\Delta\eta=\pi/(\beta a)$. The application of such sampling method provides an eigenvalue problem for a matrix which apart for the truncation error approximates very well the eigenvalues of the operator $AA_w^{\dagger}$.

However, as specified before, our aim is that of finding a discretization of $AA^{\dagger}$. Since if $X_o > a$ the sampling theory approach cannot be used to discretize the operator $AA^{\dagger}$, we check if the discretization introduced for $AA_w^{\dagger}$ works also for $AA^{\dagger}$. Figure (4) shows the eigenvalues of $AA^{\dagger}$, and those of the discrete model for different values of the sampling step $\Delta\eta = \pi/(\chi\beta a)$ (at Nyquist rate $\chi = 1$, with an oversampling factor $\chi = 1.1$ and $\chi = 1.1$) when $X_o = 10a$. As can be seen from the figure, if the observation domain is significantly larger that the source domain, a sampling step exactly equal to the Nyquist step $\Delta\eta = \pi/(\beta a)$ is not sufficient to approximate the eigenvalues of the continuous model. However, a sampling step $\Delta\eta$ a little bit smaller than $\pi/(\beta a)$ already suffices to approximate the eigenvalues of the continuous operator $AA^{\dagger}$.

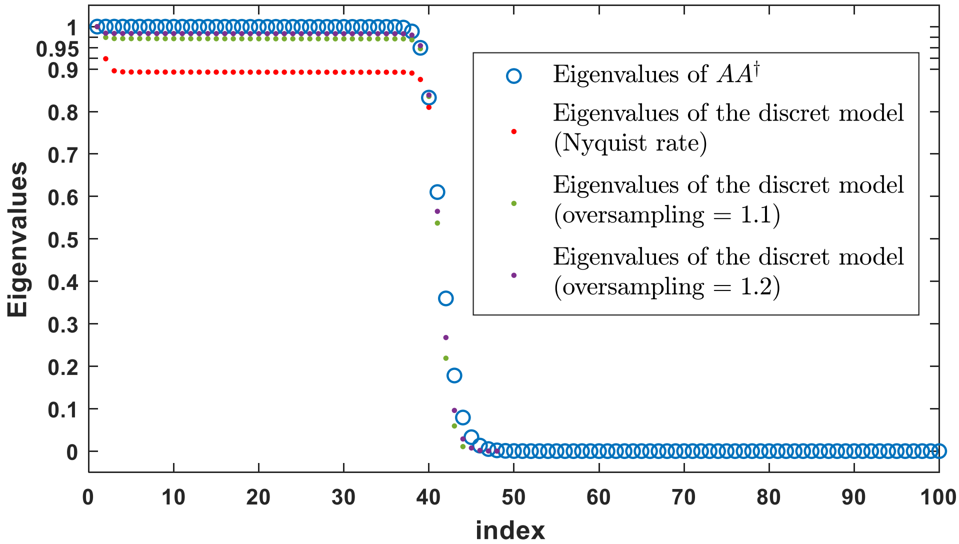

FIG. 4 – Eigenvalues of $AA^{\dagger}$ and those of the discretized model for different values of the sampling step $\Delta\eta$. The figure refers to the case $a = 10\,\lambda$, $z_o = 5\,\lambda$, $X_o = 10\,a$.

Note that the non-linear relation between η and x implies that the uniform sampling step $\Delta\eta$ maps into a spatially varying sampling step Δx. The position of the sampling points in x domain can be found by exploiting the equation

$$x_m = \eta_m \sqrt{a^2 + z_0^2/(1-\eta_m^2)} \quad \text{where } \eta_m = \eta(-X_o) + (m-1)\Delta\eta \qquad \forall m = 1, \ldots, N_{samples} \tag{17}$$

V. CONCLUSION

In this paper, with reference to a strip current observed in near zone on a finite line parallel to the source, a strategy to collect the radiated field in the case $X_o > a$ has been proposed. In particular, it has been shown that to obtain a good approximation of the singular values of the radiation operator the field must be collect according to Eq. (17).

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