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On the Cauchy Problem of Vectorial Thermostatted Kinetic Frameworks

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Abstract: This paper is devoted to the derivation and mathematical analysis of new thermostatted kinetic theory frameworks for the modeling of nonequilibrium complex systems composed by particles whose microscopic state includes a vectorial state variable. The mathematical analysis refers to the global existence and uniqueness of the solution of the related Cauchy problem. Specifically, the paper is divided in two parts. In the first part the thermostatted framework with a continuous vectorial variable is proposed and analyzed. The framework consists of a system of partial integro-differential equations with quadratic type nonlinearities. In the second part the thermostatted framework with a discrete vectorial variable is investigated. Real world applications, such as social systems and crowd dynamics, and future research directions are outlined in the paper.

Keywords: complexity; kinetic theory; integro-differential equation; Cauchy problem; nonlinearity

1. Introduction

The mathematical frameworks inferred from nonlinear analysis methods have gained much attention, in particular for real world applications. The main interest has been focused on the modeling of a complex living system, which is composed of a large number of entities, called particles, whose interactions require to be considered in a somehow "holistic" perspective, as they cannot be envisaged as a simple superposition of the interactions between couples of particles, and the functional dependence of multiple interactions on binary ones cannot be assumed *linear*. On the contrary, it cannot be even conjectured, so that the perturbations produced by surrounding particles make the results of binary interactions not deterministic. The reader interested to a more deeper understanding of the complex system is referred to the books [1,2] and paper [3].

In the inert matter case, Maxwell and Boltzmann have introduced and developed the statistical picture of systems made of a large number of particles, and the kinetic theory of gases [4]. However, it has become ever more apparent that their viewpoint could be exported to almost all the problems, from both hard and soft sciences, including the active matter which involves systems of objects/particles/individuals able to express a strategy, see [5] and the references cited therein. Recently, the mathematical framework of the thermostatted kinetic theory has been proposed in [6,7] for the modeling of nonequilibrium complex living systems, namely systems subjected to external force fields. According to this theory, the complex living system is divided into different particle subsystems consisting of particles expressing the same function or strategy. The strategy is modeled by introducing a scalar real variable, called activity variable, and consequently the microscopic state of the particles includes, but is not limited to, the activity variable. The evolution of each functional subsystem is depicted by a distribution function defined on the microscopic state of the particles. The particle evolution is driven by interactions which are particle-conservative (changing into the magnitude of the activity variable) and nonconservative (proliferation and mutation). A thermostat term is introduced in order to control the activity energy of the system and to allow the reaching of nonequilibrium stationary states [8–12]. Depending on the structure of the activity variable, continuous and discrete activity thermostatted frameworks have been derived. In particular, the continuous thermostatted framework consists into a system of partial integro-differential equations with quadratic type nonlinearities. On the other hand, the discrete thermostatted framework is a system of nonlinear ordinary differential equations. It is worth stressing that changing in the activity variable are modelled by employing the stochastic game theory [13]. The reader interested in recent applications to biology and pedestrian dynamics is referred to papers [14–16].

The above described thermostatted kinetic theory method is based on the main assumption that the evolution of each functional subsystem composing the complex system depends on a strategy only. On one hand this assumption simplifies the theory, at least from the viewpoint of its formal development; on the other hand the assumption seems to restrict its application to a rather small number of particular cases, which do not capture a sufficient number of relevant aspects of real phenomena. Indeed for a complex living system, the particles composing a functional subsystem can be able to express simultaneously different functions, for instance this is the case of social-economical systems where the behavior of agents depends not only on their own wealth, but also on their predisposition to be "criminals" in terms of evading taxes. Accordingly, the introduction in the particle microscopic state of several variables modeling different functions is required (vectorial structure). A first attempt has been proposed in [17,18] for the modeling of human feelings.

The present paper is devoted to the derivation of thermostatted kinetic frameworks with a vectorial activity structure. Specifically, the paper is divided in two parts: In the first part the thermostatted framework with a continuous vectorial variable is proposed and analyzed. The framework consists of a system of partial integro-differential equations with quadratic nonlinearities. In the second part, the thermostatted framework with a discrete vectorial variable is investigated. The mathematical analysis refers to the global existence and uniqueness of the solution of the related Cauchy problem and is gained by employing methods of nonlinear analysis and fixed-point arguments. To the best of our knowledge, this is the first time that a vectorial activity variable is proposed for a thermostatted framework. It is worth stressing that the introduction of a vector activity variable can complicate, especially, the numerical analysis as it increases the already great number of parameters but allows to obtain a faithful description of reality.

The contents of the present paper are organized into five more sections which follow this introduction. In detail, Section 2 is devoted to the fundamentals of the scalar thermostatted kinetic theory framework where the activity is assumed to be a scalar real variable. Section 3 deals with the generalization of the kinetic equation to the case of a vectorial activity variable and for a complex system at equilibrium. The related Cauchy problem is analyzed and the existence and uniqueness of the solution is proven by employing fixed-point arguments. In Section 4, the vectorial thermostatted framework is proposed for the modeling of nonequilibrium complex systems and the related Cauchy problem is investigated. The vectorial thermostatted framework in the case of a vectorial discrete activity variable is proposed and analyzed in Section 5. Finally, Section 6 concludes the paper with a references to application and future research directions.

2. The Scalar Thermostatted Kinetic Theory Framework

This section deals with the main elements of the thermostatted kinetic theory methods. Specifically, let $\mathcal{C}$ be a complex system composed of n functional subsystems, each of them characterized by particles which share the same strategy (active particles). The system $\mathcal{C}$ is assumed to be homogeneous with respect to the mechanical variables, i.e., space and velocity. Accordingly, the microscopic state of the particles is described by a *scalar variable* $u \in D_u \subseteq \mathbb{R}$, called *activity variable*, which models the particle strategy. The overall state of the ith functional subsystem, for $i \in \{1, 2, \dots, n\}$, at the time t is described by the *distribution function* $f_i = f_i(t, u) : [0, +\infty[\times D_u \to \mathbb{R}^+$. Accordingly $f_i(t, u)\, du$ represents the number of active particles whose microscopic state at the time t belongs to the elementary volume $[u, u + du]$. The overall state of the complex system $\mathcal{C}$ is described by the *distribution function vector* $\mathbf{f}(t, u) = (f_1(t, u), f_2(t, u), \dots, f_n(t, u))$. The computation of moments of $f_i(t, u)$ allows the definition of the local and global macroscopic quantities. Specifically, the *global pth-order moment* reads:

$$\mathbb{E}_p[\mathbf{f}](t) = \sum_{i=1}^{n} \int_{D_u} u^p f_i(t, u)\, du, \quad p \in \mathbb{N}.$$

The microscopic state of a particle of C evolves due to the conservative interactions among the particles. Specifically: A particle of the ith functional subsystem with microscopic state u_* interacts with the particle u^* of the jth functional subsystem and acquires, in probability, the state u of the particle of the ith functional subsystem. The *evolution equation* of the ith functional subsystem is obtained by balancing the inlet/outlet flux into the elementary volume of the microscopic states. Accordingly:

$$\partial_t f_i(t, u) = J_i[\mathbf{f}](t, u) = G_i[\mathbf{f}](t, u) - L_i[\mathbf{f}](t, u), \tag{1}$$

where $J_i[\mathbf{f}]$ is the operator which models the *conservative interactions*. In particular, $G_i[\mathbf{f}]$ denotes the following *gain-term operator*, while $L_i[\mathbf{f}]$ denotes the following *loss-term operator*:

$$G_i[\mathbf{f}] = \sum_{j=1}^{n} \int_{D_u} \eta_{ij}(u_*, u^*) \mathcal{A}_{ij}(u_*, u^*, u)\, f_i(t, u_*) f_j(t, u^*)\, du_*\, du^*,$$

$$L_i[\mathbf{f}] = f_i(t, u) \sum_{j=1}^{n} \int_{D_u} \eta_{ij}(u, u^*) f_j(t, u^*)\, du^*,$$

where:

- $\eta_{ij}(u_*, u^*) : D_u \times D_u \to \mathbb{R}^+$ denotes the *interaction rate* between the active particle u_* of the ith functional subsystem and the active particle u^* of the jth functional subsystem;
- $\mathcal{A}_{ij}(u_*, u^*, u) : D_u \times D_u \times D_u \to \mathbb{R}^+$ is the *probability density* that the particle of the ith functional subsystem with state u_* falls into the state u after an interaction with the particle u^* of the jth functional subsystem.

The framework (1) thus consists of a system of n partial-integro-differential equations.

Remark 1. *The probability density $\mathcal{A}_{ij}$ satisfies the following relation which ensures the conservation of the total number of particles:*

$$\int_{D_u} \mathcal{A}_{ij}(u_*, u^*, u)\, du = 1 \qquad \forall u_*, u^* \in D_u. \tag{2}$$

Assume that an *external force field* $\mathbf{F}(u) : D_u \to \mathbb{R}^n$ acts on the system $\mathcal{C}$. In order to reach a *non-equilibrium stationary state*, the overall system is constrained to keep constant the *global activation energy*:

$$\mathbb{E}_2[\mathbf{f}](t) = \int_{D_u} u^2 \tilde{f}(t, u)\, du, \tag{3}$$

where

$$\tilde{f}(t, u) = \sum_{i=1}^{n} f_i(t, u). \tag{4}$$

A gaussian-like thermostat has been proposed in order to ensure the conservation of the energy, see [6,11,16] and the references therein. If $F_i(u) = F$, for $i \in \{1, 2, \ldots, n\}$, the *thermostatted kinetic equation* for the ith functional subsystem reads:

$$\partial_t f_i(t, u) + \partial_u \left(F \left(1 - u \int_{D_u} u\tilde{f}(t.u)\, du \right) f_i \right) = J_i[\mathbf{f}](t, u). \tag{5}$$

In a compact way, Equation (5) can be rewritten as follows:

$$\partial_t \mathbf{f}(t, u) + \mathbf{F}\, \partial_u \left((1 - u\mathbb{E}_1[\mathbf{f}](t))\, \mathbf{f}(t, u) \right) = \mathbf{J}[\mathbf{f}](t, u), \tag{6}$$

where $\mathbf{J}[\mathbf{f}](t, u) = \mathbf{G}[\mathbf{f}](t, u) - \mathbf{L}[\mathbf{f}](t, u)$.

3. A Vector Activity Kinetic Framework

This section is devoted to an important generalization of the mathematical framework (5). Specifically, the microscopic state of the active particle is now composed by a *vector activity variable* $\mathbf{u} = (u_1, u_2, \ldots, u_m)$, where $u_j \in D_{u_j} \subseteq \mathbb{R}$, for $j \in \{1, 2, \ldots, m\}$. Let $f_i(t, \mathbf{u})$ be the *distribution function* of the ith functional subsystem and $\mathbf{f}(t, \mathbf{u}) = (f_1(t, \mathbf{u}), f_2(t, \mathbf{u}), \ldots, f_n(t, \mathbf{u})))$ the *distribution function vector*. The *local density* is defined as follows:

$$\mathbb{E}_0[f_i](t) = \int_{D_{\mathbf{u}}} f_i(t, \mathbf{u})\, d\mathbf{u}, \tag{7}$$

and the *global density* is:

$$\mathbb{E}_0[\mathbf{f}](t) = \sum_{i=1}^{n} \int_{D_{\mathbf{u}}} f_i(t, \mathbf{u})\, d\mathbf{u}. \tag{8}$$

The evolution equation of the ith functional subsystem now reads:

$$\partial_t f_i(t, \mathbf{u}) = J_i[\mathbf{f}](t, \mathbf{u}) = G_i[\mathbf{f}](t, \mathbf{u}) - L_i[\mathbf{f}](t, \mathbf{u}) \tag{9}$$

where $G_i[\mathbf{f}]$ (*gain-term operator*) and $L_i[\mathbf{f}]$ (*loss-term operator*) now write:

$$G_i[\mathbf{f}] = \sum_{h=1}^{m} \sum_{j=1}^{n} \int_{D_{\mathbf{u}} \times D_{\mathbf{u}}} \eta_{ij}((\mathbf{u}_*)_h, (\mathbf{u}^*)_h)\, \mathcal{A}_{ij}\left((\mathbf{u}_*)_h, (\mathbf{u}^*)_h, \mathbf{u}\right) f_i(t, (\mathbf{u}_*)_h) f_j(t, (\mathbf{u}^*)_h)\, d(\mathbf{u}_*)_h\, d(\mathbf{u}^*)_h, \tag{10}$$

$$L_i[\mathbf{f}] = f_i(t, \mathbf{u}) \sum_{h=1}^{m} \sum_{j=1}^{n} \int_{D_{\mathbf{u}}} \eta_{ij}(\mathbf{u}, (\mathbf{u}^*)_h) f_j(t, (\mathbf{u}^*)_h)\, d(\mathbf{u}^*)_h, \tag{11}$$

where $\mathcal{A}_{ij}\left((\mathbf{u}_*)_h, (\mathbf{u}^*)_h, \mathbf{u}\right)$, for $i, j \in \{1, 2, \ldots, n\}$ and $h \in \{1, 2, \ldots, m\}$, now denotes the *probability density function* that the particle of the ith functional subsystem with state $(\mathbf{u}_*)_h = ((u_*)_1, (u_*)_2, \ldots, (u_*)_h, \ldots, (u_*)_m)$ acquires the state $\mathbf{u}$ after interacting with the particle $(\mathbf{u}^*)_h = ((u^*)_1, (u^*)_2, \ldots, (u^*)_h\ \ldots, (u^*)_m)$ of the jth functional subsystem.

The Cauchy Problem

Let $D_{\mathbf{u}} = D_{u_1} \times D_{u_2} \times \cdots \times D_{u_m}$, the *vectorial Cauchy problem* related to the kinetic theory framework (9) reads:

$$\begin{cases} \partial_t \mathbf{f}(t, \mathbf{u}) = \mathbf{J}[\mathbf{f}](t, \mathbf{u}) & (t, \mathbf{u}) \in [0, +\infty[\times D_{\mathbf{u}} \\ \\ \mathbf{f}(0, \mathbf{u}) = \mathbf{f}^0(\mathbf{u}) & u \in D_{\mathbf{u}} \end{cases} \tag{12}$$

where $\mathbf{f}^0(\mathbf{u}) = \left(f_1^0(\mathbf{u}), f_2^0(\mathbf{u}), \ldots, f_n^0(\mathbf{u})\right) : D_{\mathbf{u}} \to (\mathbb{R}^+)^n$ denotes the *initial data function*.

The Cauchy problem (12) is analyzed under the following assumptions:

H1. The interaction rate η_{ij} is a bounded function of its arguments, namely there exists $\eta > 0$ such that $\eta_{ij} \leq \eta$;

H2. Let $h \in \{1, 2, \ldots, m\}$, the transition probability function $\mathcal{A}_{ij} \geq 0$ is such that:

$$\int_{D_{\mathbf{u}}} \mathcal{A}_{ij} \left((\mathbf{u}_*)_h, (\mathbf{u}^*)_h, \mathbf{u}\right) d\mathbf{u} = 1 \quad \forall (\mathbf{u}^*)_h, (\mathbf{u}_*)_h \in D_{\mathbf{u}}.$$

Remark 2. *The assumption **H2** ensures that the framework* (9) *is conservative, namely* $\mathbb{E}_0[f_i](t)$*, for* $i \in \{1, 2, \ldots, n\}$*, is constant. In particular, it is assumed that* $\mathbb{E}_0[f_i] = 1$.

Let $i \in \{1, 2, \ldots, n\}$. The *integral Volterra formulation* of (12) is:

$$\begin{aligned} f_i(t, \mathbf{u}) = & \int_0^t \left(\sum_{h=1}^m \sum_{j=1}^n \int_{D_{\mathbf{u}} \times D_{\mathbf{u}}} \eta_{ij}((\mathbf{u}_*)_h, (\mathbf{u}^*)_h) \, \mathcal{A}_{ij} \left((\mathbf{u}_*)_h, (\mathbf{u}^*)_h, \mathbf{u}\right) \right. \\ & \left. \times f_i(\tau, (\mathbf{u}_*)_h) f_j(\tau, (\mathbf{u}^*)_h) \, d(\mathbf{u}_*)_h \, d(\mathbf{u}^*)_h \right) d\tau \\ & - \int_0^t f_i(\tau, \mathbf{u}) \left(\sum_{h=1}^m \sum_{j=1}^n \int_{D_{\mathbf{u}}} \eta_{ij}(\mathbf{u}, (\mathbf{u}^*)_h) f_j(\tau, (\mathbf{u}^*)_h) \, d(\mathbf{u}^*)_h \right) d\tau \\ & + f_i^0(\mathbf{u}). \end{aligned} \tag{13}$$

Definition 1. *Let* $f(t, \mathbf{u}) : [0, +\infty[\times D_{\mathbf{u}} \to \mathbb{R}^+$*. The set* $\mathcal{E}_0(D_{\mathbf{u}})$ *is defined as follows:*

$$\mathcal{E}_0(D_{\mathbf{u}}) = \left\{ f(t, \mathbf{u}) \in C\left([0, +\infty[; L^1(D_{\mathbf{u}})\right) : \mathbb{E}_0[f](t) = 1 \right\}. \tag{14}$$

Theorem 1. *Assume that **H1** and **H2** hold true. If* $f_i^0(\mathbf{u}) \in \mathcal{E}_0(D_{\mathbf{u}})$*, for* $i \in \{1, 2, \ldots, n\}$*, then there exists a unique function* $f_i(t, \mathbf{u}) \in \mathcal{E}_0(D_{\mathbf{u}})$*, for* $i \in \{1, 2, \ldots, n\}$*, which is solution of the Cauchy problem* (12).

Proof. Using Equations (10) and (11), (9) becomes:

$$\partial_t f_i(t, \mathbf{u}) + P_i[\mathbf{f}] f_i(t, \mathbf{u}) = G_i[\mathbf{f}], \tag{15}$$

where

$$P_i[\mathbf{f}] = \sum_{h=1}^m \sum_{j=1}^n \int_{D_{\mathbf{u}}} \eta_{ij}(\mathbf{u}, (\mathbf{u}^*)_h) f_j(t, (\mathbf{u}^*)_h) \, d(\mathbf{u}^*)_h.$$

Setting

$$\gamma_{\mathbf{u}}(t) = \int_0^t P_i[\mathbf{f}](s, \mathbf{u}) \, ds,$$

if there exists a solution of the Cauchy problem (12) then (15) becomes:

$$f_i(t, \mathbf{u}) = f_i^0(\mathbf{u}) \exp\left(-\gamma_{\mathbf{u}}(t)\right) + \int_0^t \exp\left(\gamma_{\mathbf{u}}(s)\right) G_i[\mathbf{f}](s, \mathbf{u}) \, ds.$$

Accordingly f_i, for $i \in \{1, 2, \ldots, n\}$, is a non negative function.

Let $T > 0$ and $\mathbf{T}[\mathbf{f}] = (T_1[f_1], T_2[f_2], \ldots, T_n[f_n])$ the following operator:

$$
\begin{aligned}
T_i\left[f_i\right](t,\mathbf{u}) = \int_0^t \Bigg(\sum_{h=1}^{m}\sum_{j=1}^{n} \int_{D_\mathbf{u}\times D_\mathbf{u}} & \eta_{ij}((\mathbf{u}_*)_h,(\mathbf{u}^*)_h)\,\mathcal{A}_{ij}\left((\mathbf{u}_*)_h,(\mathbf{u}^*)_h,\mathbf{u}\right) \\
& \times f_i(\tau,(\mathbf{u}_*)_h) f_j(\tau,(\mathbf{u}^*)_h)\, d(\mathbf{u}_*)_h\, d(\mathbf{u}^*)_h \Bigg)\, d\tau \\
& - \int_0^t f_i(\tau,\mathbf{u}) \left(\sum_{h=1}^{m}\sum_{j=1}^{n} \int_{D_\mathbf{u}} \eta_{ij}(\mathbf{u},(\mathbf{u}^*)_h) f_j(\tau,(\mathbf{u}^*)_h)\, d(\mathbf{u}^*)_h \right) d\tau \\
& + f_i^0(\mathbf{u}).
\end{aligned} \tag{16}
$$

The main step is to prove that the operator $\mathbf{T}[\mathbf{f}](t,\mathbf{u})$ is a contraction in the Banach space $\left(C\left([0,T];L^1\left(D_\mathbf{u}\right)\right)\right)^n$.

Let $\mathbf{f} \in \left(C\left([0,T];L^1\left(D_\mathbf{u}\right)\right)\right)^n$, by assumption **H1** one has:

$$
\begin{aligned}
\|T_i[f_i]\|_{L^1(D_\mathbf{u})} = \int_{D_\mathbf{u}} \Bigg| \int_0^t \Bigg(& \sum_{h=1}^{m}\sum_{j=1}^{n} \int_{D_\mathbf{u}\times D_\mathbf{u}} \eta_{ij}((\mathbf{u}_*)_h,(\mathbf{u}^*)_h)\mathcal{A}_{ij}\left((\mathbf{u}_*)_h,(\mathbf{u}^*)_h,\mathbf{u}\right) \\
& \times f_i(\tau,(\mathbf{u}_*)_h) f_j(\tau,(\mathbf{u}^*)_h)\, d(\mathbf{u}_*)_h\, d(\mathbf{u}^*)_h \Bigg)\, d\tau \\
& - \int_0^t f_i(\tau,\mathbf{u}) \left(\sum_{h=1}^{m}\sum_{j=1}^{n} \int_{D_\mathbf{u}} \eta_{ij}(\mathbf{u},(\mathbf{u}^*)_h) f_j(\tau,(\mathbf{u}^*)_h)\, d(\mathbf{u}^*)_h \right) d\tau + f_i^0(\mathbf{u}) \Bigg|\, d\mathbf{u} \\
& \leq \eta \int_{D_\mathbf{u}} \left[\int_0^t \left(\sum_{h=1}^{m}\sum_{j=1}^{n} \int_{D_\mathbf{u}\times D_\mathbf{u}} \mathcal{A}_{ij}\, |f_i(\tau,(\mathbf{u}_*)_h)|\, \left|f_j(\tau,(\mathbf{u}^*)_h)\right|\, d(\mathbf{u}_*)_h\, d(\mathbf{u}^*)_h \right) \right. \\
& \left. \times\, d\tau \right] d\mathbf{u} + \eta \int_{D_\mathbf{u}} \left[\int_0^t |f_i(\tau,\mathbf{u})| \left(\sum_{h=1}^{m}\sum_{j=1}^{n} \int_{D_\mathbf{u}} \left|f_j(\tau,(\mathbf{u}^*)_h)\right|\, d(\mathbf{u}^*)_h \right) d\tau \right] d\mathbf{u} \\
& + \int_{D_\mathbf{u}} \left|f_i^0(\mathbf{u})\right|\, d\mathbf{u}.
\end{aligned} \tag{17}
$$

By assumptions **H1** and **H2** and the fact that $f_i^0(\mathbf{u}) \in L^1\left(D_u\right), \forall i \in \{1,2,\ldots,n\}$, and $\mathbb{E}_0[f_i] = 1$, Equation (17) becomes:

$$
\begin{aligned}
\|T_i[f_i]\|_{L^1(D_\mathbf{u})} \leq \eta \int_0^t \sum_{h=1}^{m} \Bigg(& \sum_{j=1}^{n} \int_{D_\mathbf{u}\times D_\mathbf{u}} |f_i(\tau,((\mathbf{u}_*)_h))|\, \left|f_j(\tau,(\mathbf{u}^*)_h)\right| \\
& \times \left(\int_{D_\mathbf{u}} \mathcal{A}_{ij}\left((\mathbf{u}_*)_h,(\mathbf{u}^*)_h,\mathbf{u}\right)\, d\mathbf{u} \right) d(\mathbf{u}_*)_h\, d(\mathbf{u}^*)_h \Bigg)\, d\tau \\
& + \eta \int_0^t |f_i(\tau,\mathbf{u})| \left(\sum_{h=1}^{m}\sum_{j=1}^{n} \int_{D_\mathbf{u}} \left|f_j(\tau,(\mathbf{u}^*)_h)\right|\, d(\mathbf{u}^*)_h \right) d\tau\, d\mathbf{u} \\
& + \int_{D_\mathbf{u}} \left|f_i^0(\mathbf{u})\right|\, d\mathbf{u} \\
& \leq \eta \int_0^t \left(\int_{D_\mathbf{u}} |f_i(\tau,((\mathbf{u}_*)_h))|\, d(\mathbf{u}_*)_h \right) \\
& \times \left(\sum_{j=1}^{n}\sum_{h=1}^{m} \int_{D_\mathbf{u}} \left|f_j(\tau,(\mathbf{u}^*)_h)\right|\, d(\mathbf{u}^*)_h \right) d\tau \\
& + \eta m n \int_0^t \left(\int_{D_\mathbf{u}} |f_i(\tau,\mathbf{u})|\, d\mathbf{u} \right) d\tau + \left\|f_i^0\right\|_{L^1(D_\mathbf{u})} \\
& \leq 2\eta m n \int_0^T \|f_i(t,\mathbf{u})\|_{L^1(D_\mathbf{u})}\, dt + 1.
\end{aligned} \tag{18}
$$

Then, by using Equation (18), one has:

$$\begin{aligned}\|\mathbf{T}\,[\mathbf{f}]\|_{\left(C\left([0,T];L^1(D_\mathbf{u})\right)\right)^n} &= \max_{i\in\{1,2,\ldots,n\}}\left(\max_{t\in[0,T]}\|T_i[f_i]\|_{L^1(D_\mathbf{u})}\right)\\ &\le 2\eta mnT+1.\end{aligned} \tag{19}$$

$\mathbf{T}$ is thus an operator of $\left(C\left([0,T];L^1\left(D_\mathbf{u}\right)\right)\right)^n$ into itself.

Let $\mathbf{f}^1,\mathbf{f}^2 \in \left(C\left([0,T];L^1\left(D_\mathbf{u}\right)\right)\right)^n$ such that $f_i^1(t,\mathbf{u}), f_i^2(t,\mathbf{u}) \in C\left([0,T];L^1\left(D_\mathbf{u}\right)\right)$, for $i \in \{1,2,\ldots,n\}$. Bearing the expression (16) of the operator $\mathbf{T}$ in mind and the fact that $\mathbb{E}_0[f_i]=1$, by straightforward calculations one has:

$$\begin{aligned}\|T_i[f_i^1]-T_i[f_i^2]\|_{L^1(D_\mathbf{u})} &= \int_{D_\mathbf{u}}\Bigg|\int_0^t\left(\sum_{h=1}^m\sum_{j=1}^n\int_{D_\mathbf{u}\times D_\mathbf{u}}\eta_{ij}((\mathbf{u}_*)_h,(\mathbf{u}^*)_h)\right.\\ &\left.\mathcal{A}_{ij}\left((\mathbf{u}_*)_h,(\mathbf{u}^*)_h,\mathbf{u}\right)f_i^1(\tau,(\mathbf{u}_*)_h)f_j^1(\tau,(\mathbf{u}^*)_h)\,d(\mathbf{u}_*)_h\,d(\mathbf{u}^*)_h\right)d\tau\\ &-\int_0^t f_i^1(\tau,\mathbf{u})\left(\sum_{h=1}^m\sum_{j=1}^n\int_{D_\mathbf{u}}\eta_{ij}(\mathbf{u},(\mathbf{u}^*)_h)f_j^1(\tau,(\mathbf{u}^*)_h)\,d(\mathbf{u}^*)_h\right)d\tau+f_i^0(\mathbf{u})\\ &-\int_0^t\left(\sum_{h=1}^m\sum_{j=1}^n\int_{D_\mathbf{u}\times D_\mathbf{u}}\eta_{ij}((\mathbf{u}_*)_h,(\mathbf{u}^*)_h)\,\mathcal{A}_{ij}\left((\mathbf{u}_*)_h,(\mathbf{u}^*)_h,\mathbf{u}\right)\right.\\ &\left.\times f_i^2(\tau,(\mathbf{u}_*)_h)f_j^2(\tau,(\mathbf{u}^*)_h)\,d(\mathbf{u}_*)_h\,d(\mathbf{u}^*)_h\right)d\tau+\int_0^t f_i^2(\tau,\mathbf{u})\\ &\times\left(\sum_{h=1}^m\sum_{j=1}^n\int_{D_\mathbf{u}}\eta_{ij}(\mathbf{u},(\mathbf{u}^*)_h)f_j^2(\tau,(\mathbf{u}^*)_h)\,d(\mathbf{u}^*)_h\right)d\tau-f_i^0(\mathbf{u})\Bigg|\,d\mathbf{u}\\ &\le \eta\int_{D_\mathbf{u}}\Bigg|\int_0^t\sum_{h=1}^m\sum_{j=1}^n\left(\int_{D_\mathbf{u}\times D_\mathbf{u}}\mathcal{A}_{ij}\left((\mathbf{u}_*)_h,(\mathbf{u}^*)_h,\mathbf{u}\right)\right.\\ &\left.\times\left(f_i^1(\tau,(\mathbf{u}_*)_h)f_j^1(\tau,(\mathbf{u}^*)_h)-f_i^2(\tau,(\mathbf{u}_*)_h)f_j^2(\tau,(\mathbf{u}^*)_h)\right)d(\mathbf{u}_*)_h\,d(\mathbf{u}^*)_h\right)d\tau\Bigg|\,d\mathbf{u}\\ &+\eta\int_{D_\mathbf{u}}\Bigg|\int_0^t f_i^1(\tau,\mathbf{u})\sum_{h=1}^m\sum_{j=1}^n\int_{D_\mathbf{u}}f_j^1(\tau,(\mathbf{u}^*)_h)\,d(\mathbf{u}^*)_h\\ &-f_i^2(\tau,\mathbf{u})\sum_{h=1}^m\sum_{j=1}^n\int_{D_\mathbf{u}}f_j^2(\tau,(\mathbf{u}^*)_h)\,d(\mathbf{u}^*)_h\,d\tau\Bigg|\,d\mathbf{u}.\end{aligned} \tag{20}$$

By using the triangular inequality, (20) is:

$$
\begin{aligned}
&\left\|T_i[f_i^1] - T_i[f_i^2]\right\|_{L^1(D_{\mathbf{u}})} \\
&\leq \eta \int_{D_{\mathbf{u}}} \left| \int_0^t \sum_{h=1}^m \sum_{j=1}^n \left(\int_{D_{\mathbf{u}} \times D_{\mathbf{u}}} \mathcal{A}_{ij}\left((\mathbf{u}_*)_h, (\mathbf{u}^*)_h, \mathbf{u}\right) \Big(f_i^1(\tau, (\mathbf{u}_*)_h) \right.\right. \\
&\times f_j^1(\tau, (\mathbf{u}^*)_h) - f_i^1(\tau, (\mathbf{u}_*)_h) f_j^2(\tau, (\mathbf{u}^*)_h) + f_i^1(\tau, (\mathbf{u}_*)_h) f_j^2(\tau, (\mathbf{u}^*)_h) \\
&\left.\left. - f_i^2(\tau, (\mathbf{u}_*)_h) f_j^2(\tau, (\mathbf{u}^*)_h) \Big) \, d(\mathbf{u}_*)_h \, d(\mathbf{u}^*)_h \right) d\tau \right| d\mathbf{u} \\
&+ \eta m n \int_0^T \int_{D_{\mathbf{u}}} \left| f_i^1(t, \mathbf{u}) - f_i^2(t, \mathbf{u}) \right| \, d\mathbf{u} \, dt \\
&\leq \eta \int_{D_{\mathbf{u}}} \left| \int_0^t \sum_{h=1}^m \sum_{j=1}^n \left(\int_{D_{\mathbf{u}} \times D_{\mathbf{u}}} \mathcal{A}_{ij}\left((\mathbf{u}_*)_h, (\mathbf{u}^*)_h, \mathbf{u}\right) f_i^1(\tau, (\mathbf{u}_*)_h) \right.\right. \\
&\left.\left. \times \Big(f_j^1(\tau, (\mathbf{u}^*)_h) - f_j^2(\tau, (\mathbf{u}^*)_h) \Big) \, d(\mathbf{u}_*)_h \, d(\mathbf{u}^*)_h \right) d\tau \right| d\mathbf{u} \\
&+ \eta \int_{D_{\mathbf{u}}} \left| \int_0^t \sum_{h=1}^m \sum_{j=1}^n \left(\int_{D_{\mathbf{u}} \times D_{\mathbf{u}}} \mathcal{A}_{ij}\left((\mathbf{u}_*)_h, (\mathbf{u}^*)_h, \mathbf{u}\right) f_j^2(\tau, (\mathbf{u}^*)_h) \right.\right. \\
&\left.\left. \times \Big(f_i^1(\tau, (\mathbf{u}_*)_h) - f_i^2(\tau, (\mathbf{u}_*)_h) \Big) \, d(\mathbf{u}_*)_h \, d(\mathbf{u}^*)_h \right) d\tau \right| d\mathbf{u} \\
&+ \eta m n \int_0^T \int_{D_{\mathbf{u}}} \left| f_i^1(t, \mathbf{u}) - f_i^2(t, \mathbf{u}) \right| \, d\mathbf{u} \, dt.
\end{aligned} \tag{21}
$$

By assumption **H2** and the fact that $\mathbb{E}_0[f_i] = 1$, for $i \in \{1, 2, \ldots, n\}$, for the first term of (21) one has:

$$
\begin{aligned}
&\int_{D_{\mathbf{u}}} \left| \int_0^t \sum_{h=1}^m \sum_{j=1}^n \left(\int_{D_{\mathbf{u}} \times D_{\mathbf{u}}} \mathcal{A}_{ij}\left((\mathbf{u}_*)_h, (\mathbf{u}^*)_h, \mathbf{u}\right) f_i^1(\tau, (\mathbf{u}_*)_h) \right.\right. \\
&\qquad \left.\left. \times \Big(f_j^1(\tau, (\mathbf{u}^*)_h) - f_j^2(\tau, (\mathbf{u}^*)_h) \Big) \, d(\mathbf{u}_*)_h \, d(\mathbf{u}^*)_h \right) d\tau \right| d\mathbf{u} \\
&\qquad = \int_{D_{\mathbf{u}}} \left| \int_0^t \left(\int_{D_{\mathbf{u}} \times D_{\mathbf{u}}} \mathcal{A}_{ij}\left((\mathbf{u}_*)_h, (\mathbf{u}^*)_h, \mathbf{u}\right) f_i^1(\tau, (\mathbf{u}_*)_h) \right.\right. \\
&\qquad \left.\left. \times \sum_{h=1}^m \sum_{j=1}^n \Big(f_j^1(\tau, (\mathbf{u}^*)_h) - f_j^2(\tau, (\mathbf{u}^*)_h) \Big) \, d(\mathbf{u}_*)_h \, d(\mathbf{u}^*)_h \right) d\tau \right| d\mathbf{u} \\
&\qquad \leq \int_0^t \left| \left(\int_{D_{\mathbf{u}}} f_i^1(\tau, (\mathbf{u}_*)_h) \, d(\mathbf{u}_*)_h \right) \right. \\
&\qquad \times \left[\sum_{h=1}^m \sum_{j=1}^n \left(\int_{D_{\mathbf{u}}} f_j^1(\tau, (\mathbf{u}^*)_h) \, d(\mathbf{u}^*)_h \right) \right. \\
&\qquad \left.\left. - \sum_{h=1}^m \sum_{j=1}^n \left(\int_{D_{\mathbf{u}}} f_j^2(\tau, (\mathbf{u}^*)_h) \, d(\mathbf{u}^*)_h \right) \right] \right| d\tau \\
&\qquad \leq 0.
\end{aligned} \tag{22}
$$

Then by (22), the relation (21) becomes:

$$
\begin{aligned}
&\|T_i[f_i^1] - T_i[f_i^2]\|_{L^1(D_{\mathbf{u}})} \\
&\le \eta \int_{D_{\mathbf{u}}} \left| \int_0^t \sum_{h=1}^m \sum_{j=1}^n \left(\int_{D_{\mathbf{u}} \times D_{\mathbf{u}}} \mathcal{A}_{ij}\left((\mathbf{u}_*)_h, (\mathbf{u}^*)_h, \mathbf{u}\right) f_j^2(\tau, (\mathbf{u}^*)_h) \right.\right. \\
&\times \left.\left. \left(f_i^1(\tau, (\mathbf{u}_*)_h) - f_i^2(\tau, (\mathbf{u}_*)_h) \right) d(\mathbf{u}_*)_h \, d(\mathbf{u}^*)_h \right) d\tau \right| d\mathbf{u} \\
&+ \eta mn \int_0^T \int_{D_{\mathbf{u}}} \left| f_i^1(t, \mathbf{u}) - f_i^2(t, \mathbf{u}) \right| d\mathbf{u} \, dt \\
&\le \eta \int_0^t \left(\left(\sum_{h=1}^m \sum_{j=1}^n \int_{D_{\mathbf{u}}} f_j^2(\tau, (\mathbf{u}^*)_h) \, d(\mathbf{u}^*)_h \right) \right. \\
&\times \left. \left(\int_{D_{\mathbf{u}}} \left| f_i^1(\tau, (\mathbf{u}_*)_h) - f_i^2(\tau, (\mathbf{u}_*)_h) \right| \right) d(\mathbf{u}_*)_h \right) d\tau \\
&+ \eta mn \int_0^T \int_{D_{\mathbf{u}}} \left| f_i^1(t, \mathbf{u}) - f_i^2(t, \mathbf{u}) \right| d\mathbf{u} \, dt \\
&\le 2\eta mn \int_0^T \int_{D_{\mathbf{u}}} \left| f_i^1(t, \mathbf{u}) - f_i^2(t, \mathbf{u}) \right| d\mathbf{u} \, dt.
\end{aligned} \tag{23}
$$

Bearing all of the above in mind, one has:

$$
\begin{aligned}
&\left\| \mathbf{T}[\mathbf{f}_1] - \mathbf{T}[\mathbf{f}_2] \right\|_{\left(C\left([0,T];L^1(D_{\mathbf{u}})\right)\right)^n} \\
&\quad = \max_{i \in \{1,2,\ldots,n\}} \left(\max_{t \in [0,T]} \int_{D_{\mathbf{u}}} \left| T_i[f_i^1] - T_i[f_i^2] \right| d\mathbf{u} \right) \\
&\quad \le \max_{i \in \{1,2,\ldots,n\}} \left(\max_{t \in [0,T]} 2\eta nm \int_0^T \int_{D_{\mathbf{u}}} \left| f_i^1(t, \mathbf{u}) - f_i^2(t, \mathbf{u}) \right| d\mathbf{u} \, dt \right) \\
&\quad \le 2\eta mnT \left\| \mathbf{f}^1 - \mathbf{f}^2 \right\|_{\left(C\left([0,T];L^1(D_{\mathbf{u}})\right)\right)^n}.
\end{aligned} \tag{24}
$$

Thus, the operator $\mathbf{T}$ is a contraction for $T < \dfrac{1}{2\eta mn}$. The local existence and uniqueness of a non negative solution of problem (12) is proved by a Banach fixed-point theorem, see ([19] Chapter 1).

The local solution of the Cauchy problem (12) can be extended, by prolongation, for all $T \in [0, +\infty[$, see ([20] Chapter 2). □

4. A Vector Activity Thermostatted Kinetic Framework

This section is devoted to the derivation of a thermostatted kinetic theory framework in the case of a vector activity variable $\mathbf{u} = (u_1, u_2, \ldots, u_m) \in D_{\mathbf{u}} = D_{u_1} \times D_{u_2} \times \cdots \times D_{u_m}$. The system is composed by n functional subsystems and subjected to an *external force field* $\mathbf{F}(\mathbf{u}) : D_{\mathbf{u}} \to (\mathbb{R}^+)^n$, such that $F_i(\mathbf{u})$ is the ith component of the external force field acting on the ith function subsystem. In what follows $\mathbf{F}$ is assumed constant, i.e., $F_i(\mathbf{u}) = F$, for all $i \in \{1, 2, \ldots, n\}$. The *thermostatted kinetic framework* reads:

$$
\partial_t f_i(t, \mathbf{u}) + \sum_{j=1}^m \partial_{u_j} \left(F f_i(t, \mathbf{u}) - \alpha u_j f_i(t, \mathbf{u}) \right) = J_i[\mathbf{f}](t, \mathbf{u}), \tag{25}
$$

where α, called the *thermostat term*, is obtained by conservation of the following moment (*global activation energy*):

$$
\mathbb{E}_2[\mathbf{f}](t) := \sum_{i=1}^n \sum_{h=1}^m \int_{D_{\mathbf{u}}} u_h^2 f_i(t, \mathbf{u}) \, d\mathbf{u}. \tag{26}
$$

The conservation of (26) implies that:

$$\frac{d\mathbb{E}_2[\mathbf{f}]}{dt}(t) = 0, \tag{27}$$

and then:

$$\sum_{i=1}^{n}\sum_{h=1}^{m}\int_{D_{\mathbf{u}}} u_h^2 \frac{\partial f_i}{\partial t}(t,\mathbf{u})\, d\mathbf{u} = 0. \tag{28}$$

The following assumption is added.

H3. Let $h \in \{1,2,\ldots,m\}$, it is assumed that:

$$\int_{D_{\mathbf{u}}} u_h^2 \mathcal{A}_{ij}\left((\mathbf{u}_*)_h, (\mathbf{u}^*)_h, \mathbf{u}\right) \mathbf{du} = (u_*)_h^2, \quad \forall (\mathbf{u}^*)_h (\mathbf{u}_*)_h \in D_{\mathbf{u}}.$$

Multiplying Equation (25) for u_h^2, integrating on $D_{\mathbf{u}}$ and summing on $i \in \{1,2,\ldots,n\}$ and $h \in \{1,2,\ldots,m\}$, using (28) and the following consequence of the assumption **H3**:

$$\int_{D_{\mathbf{u}}} u_h^2 J_i[\mathbf{f}](t,\mathbf{u})\, d\mathbf{u} = 0, \quad \forall h \in \{1,2,\ldots,m\}, i \in \{1,2,\ldots,n\},$$

one has (integrating by parts):

$$\sum_{h=1}^{m}\sum_{i=1}^{n}\int_{D_{\mathbf{u}}} \left(F u_h f_i(t,\mathbf{u}) - \alpha u_h^2 f_i(t,\mathbf{u})\right) d\mathbf{u} = 0.$$

Without loss of generality $\mathbb{E}_2[\mathbf{f}](t) = 1$ is assumed. Defining $\mathbb{E}_1[\mathbf{f}](t) := \mathbb{E}_{1,1,\ldots,1}[\mathbf{f}](t)$, straightforward calculations show:

$$\begin{aligned}
\alpha &= \sum_{h=1}^{m}\sum_{i=1}^{n}\int_{D_{\mathbf{u}}} u_h F f_i(t,\mathbf{u})\, d\mathbf{u} \\
&= F\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\int_{D_{\mathbf{u}}} u_h f_i(t,\mathbf{u})\, d\mathbf{u}\right) \\
&= F\mathbb{E}_1[\mathbf{f}](t),
\end{aligned}$$

where

$$\mathbb{E}_1[\mathbf{f}](t) = \sum_{j=1}^{m}\sum_{i=1}^{n}\int_{D_{\mathbf{u}}} u_h f_i(t,\mathbf{u})\, d\mathbf{u}.$$

Bearing all of the above in mind, the thermostatted kinetic framework for the functional subsystem $f_i(t,\mathbf{u})$, for $i \in \{1,2,\ldots,n\}$, reads:

$$\partial_t f_i(t,\mathbf{u}) + \sum_{j=1}^{m} \partial_{u_j}\left(F\left(1 - u_j \mathbb{E}_1[\mathbf{f}](t)\right) f_i(t,\mathbf{u})\right) = J_i[\mathbf{f}](t,\mathbf{u}). \tag{29}$$

The Cauchy Problem

The *vectorial Cauchy problem for the vector activity thermostatted framework* (29) reads:

$$\begin{cases}
\partial_t \mathbf{f}(t,\mathbf{u}) + \mathbf{F}\cdot\nabla_{\mathbf{u}}\left(\left(1 - \mathbf{u}\mathbb{E}_1[\mathbf{f}](t)\right)\mathbf{f}(t,\mathbf{u})\right) = \mathbf{J}[\mathbf{f}](t,\mathbf{u}) & (t,\mathbf{u}) \in [0,+\infty[\times D_{\mathbf{u}} \\
\mathbf{f}(0,\mathbf{u}) = \mathbf{f}^0(\mathbf{u}) & \mathbf{u} \in D_{\mathbf{u}}
\end{cases} \tag{30}$$

where $\mathbf{f}^0(\mathbf{u})$ is the *initial data function*.

Definition 2. *Let* $\mathbf{f}(t,\mathbf{u}) = (f_1(t,\mathbf{u}), f_2(t,\mathbf{u}), \dots, f_n(t,\mathbf{u})) \in \mathbb{R}^n$*, then one defines the following set:*

$$\mathcal{K}(D_\mathbf{u}) = \left\{ \mathbf{f}(t,\mathbf{u}) : [0,+\infty[\times D_\mathbf{u} \to \left(\mathbb{R}^+\right)^n : \mathbb{E}_0[\mathbf{f}](t) = \mathbb{E}_2[\mathbf{f}](t) = 1 \right\}. \tag{31}$$

The existence and uniqueness theorem for the Cauchy problem (30) is obtained by a generalization of the method employed in [6].

Theorem 2. *Assume that **H1**, **H2** and **H3** hold true. If* $\mathbf{F}$ *is constant and* $\mathbf{f}^0 \in \left(L^1(D_\mathbf{u})\right)^n \cap \mathcal{K}(D_\mathbf{u})$*, then there exists a unique function* $\mathbf{f} \in \left(C\left([0,+\infty[; L^1(D_\mathbf{u})\right)\right)^n \cap \mathcal{K}(D_\mathbf{u})$ *which is solution of the Cauchy problem (30).*

Proof. Let $\mathbf{f}$ be a solution of the Cauchy problem (30) and

$$\mathbb{E}_1^+ = \frac{-\eta + \sqrt{\eta^2 + 4mF^2}}{2F},$$

$$\mathbb{E}_1^- = \frac{-\eta - \sqrt{\eta^2 + 4mF^2}}{2F},$$

$$\mathbb{E}_1^0 = \mathbb{E}_1[\mathbf{f}](0) = \mathbb{E}_1[\mathbf{f}^0] = \sum_{j=1}^{m}\sum_{i=1}^{n}\int_{D_\mathbf{u}} u_j f_i^0(\mathbf{u})\, d\mathbf{u}.$$

Using the same arguments of Theorem 2.3 of [6], one has:

$$\mathbb{E}_1[\mathbf{f}](t) = \frac{\mathbb{E}_1^+\left(\mathbb{E}_1^- - \mathbb{E}_1^0\right) - \mathbb{E}_1^-\left(\mathbb{E}_1^+ - \mathbb{E}_1^0\right) e^{-\frac{\sqrt{\eta^2+4mF^2}}{F}t}}{\left(\mathbb{E}_1^- - \mathbb{E}_1^0\right) - \left(\mathbb{E}_1^+ - \mathbb{E}_1^0\right) e^{-\frac{\sqrt{\eta^2+4mF^2}}{F}t}}. \tag{32}$$

The right hand side of (32) is denoted by $\bar{\mathbb{E}}_1[\mathbf{f}](t)$.

The Cauchy problem (30) can be written as follows:

$$\partial_t f_i(t,\mathbf{u}) + \sum_{j=1}^{m} F\partial_{u_j}\left(\left(1 - u_j\bar{\mathbb{E}}_1[\mathbf{f}](t)\right) f_i(t,\mathbf{u})\right) = J_i[\mathbf{f}](t,\mathbf{u}),$$

and by straightforward calculations, one has:

$$\begin{aligned}\partial_t f_i(t,\mathbf{u}) + \sum_{j=1}^{m} F\left(1 - u_j\bar{\mathbb{E}}_1[\mathbf{f}](t)\right)\partial_{u_j} f_i(t,u) + \left(\eta - mF\bar{\mathbb{E}}_1[\mathbf{f}](t)\right) f_i(t,u) \\ = G_i[\mathbf{f}](t,\mathbf{u}).\end{aligned} \tag{33}$$

Let $U_j(t,u_j)$, $\forall j \in \{1,2,\dots,m\}$, the following characteristic curve:

$$U_j(t,u_j) = \varphi(u_j) = u_j e^{-\lambda(t)} + F e^{-\lambda(t)} \int_0^t e^{\lambda(s)}\, ds, \tag{34}$$

where

$$\lambda(t) = F\int_0^t \bar{\mathbb{E}}_1[\mathbf{f}](s)\, ds.$$

The family $\mathbf{U} = \left(U_j\right)_j$ is the collection of characteristic curves along which Equation (33) is integrated and it becomes:

$$\frac{d}{dt}f_{i_{\mathbf{U}}} + \left(\eta - mF\bar{\mathbb{E}}_1[\mathbf{f}](t)\right) f_{i_{\mathbf{U}}} = G_{i_{\mathbf{U}}}[\mathbf{f}], \tag{35}$$

where

$$f_{i_{\mathbf{U}}}(t,\mathbf{u}) = f_i\left(t, \mathbf{U}(t,\mathbf{u})\right),$$

and

$$G_{i_U} = G[\mathbf{f}]\left(t, \mathbf{U}(t,\mathbf{u})\right).$$

The function $e^{-\lambda(t)}$ is the Jacobian of the transformation (34); then u_j can be seen as function of U_j as follows:

$$u_j = \varphi_t^{-1}\left(U_j\right) = U_j e^{\lambda(t)} - mF\int_0^t e^{\lambda(s)}\,ds, \quad j \in \{1,2,\ldots,m\}.$$

Let $\Lambda(t)$ be the following function:

$$\Lambda(t) = \int_0^t \left(\eta - mF\bar{\mathbb{E}}_1[\mathbf{f}](t)\right) ds = m\left(\eta t - m\lambda(t)\right).$$

Equation (35) can be written as follows:

$$f_{i_{\mathbf{U}}}(t,\mathbf{u}) = e^{-\Lambda(t)} f_{i_{\mathbf{U}}}(0,\mathbf{u}) + e^{-\Lambda(t)}\int_0^t e^{\Lambda(\tau)} G_{i_{\mathbf{U}}}[\mathbf{f}](\tau,\mathbf{u})\,d\tau.$$

Let

$$\Phi_{f_0^i}[\mathbf{f}] = e^{-\Lambda(t)} f_{0_i}\left(\varphi_t^{-1}(\mathbf{u})\right) + e^{-\Lambda(t)}\int_0^t e^{\Lambda(t)} G_i[\mathbf{f}]\left(\tau, \varphi_\tau \circ \varphi_\tau^{-1}(\mathbf{u})\right),$$

where

$$\varphi_t(\mathbf{u}) = \left(\varphi_t(u_j)\right)_j, \quad \varphi_t^{-1}(\mathbf{u}) = \left(\varphi_t^{-1}(u_j)\right)_j,$$

and

$$\varphi_t \circ \varphi_t^{-1}(\mathbf{u}) = \left(\varphi_t \circ \varphi_t^{-1}(u_j)\right).$$

Equation (35) becomes:

$$f_i(t,\mathbf{u}) = \Phi_{f_i^0}[\mathbf{f}](t,\mathbf{u}), \quad \forall i \in \{1,2,\ldots,n\}. \tag{36}$$

Bearing all above in mind, using the same arguments of Theorem 2.3 of [6], since $\mathbb{E}_0[\mathbf{f}^0] = 1$, the following successive approximations sequence

$$\begin{cases} f_i^{(1)}(t,\mathbf{u}) = 0 \\ f_i^{(n)}(t,\mathbf{u}) = \Phi_{f_i^0}[f_i^{(n-1)}](t,\mathbf{u}) \quad n > 1 \end{cases}$$

converges to a non negative function $f_i(t,\cdot)$ in $L^1\left(D_{\mathbf{u}}\right)$ for $i \in \{1,2,\ldots,n\}$, and the limit function $\mathbf{f}(t,\cdot) = \left(f_i(t,\cdot)\right)_i$ is such that $\mathbb{E}_0[\mathbf{f}](t) = 1$.

Consider now, for all $i \in \{1,2,\ldots,n\}$, the following successive approximations sequence:

$$\begin{cases} g_i^{(1)}(t,\mathbf{u}) = f_{0_i}(\mathbf{u}) \\ g_i^{(n)} = \Phi_{f_i^0}[g_i^{(n-1)}](t,\mathbf{u}), \quad n > 1 \end{cases}$$

such that $\mathbb{E}_0[\mathbf{g}^{(n)}] = \mathbb{E}_2[\mathbf{g}^{(n)}] = 1$.

Then $\mathbf{g}^n \in (\mathcal{K}(D_\mathbf{u}))^n$ for all $n \geq 1$ and $\mathbb{E}_1[\mathbf{g}^{(n)}] = \bar{\mathbb{E}}_1[\mathbf{f}](t)$.

The last sequence converges to the previous function $\mathbf{f}$ which is solution of problem (30) in $\left(L^1(D_\mathbf{u})\right)^n$.

Let $\hat{\mathbf{f}}$ be another solution of problem (30). Since G_i and $\Phi_{f_i^0}$ are positive operators, one has:

$$f_i^{(n)}(t,\mathbf{u}) \leq \hat{f}_i(t,\mathbf{u}), \quad \forall i \in \{1,2,\ldots,n\}, \quad \forall n > 1,$$

then by convergence

$$f_i(t,\mathbf{u}) \leq \hat{f}_i(t,\mathbf{u}), \quad \forall i \in \{1,2,\ldots,n\}.$$

The $\mathbf{f}$ and $\hat{\mathbf{f}}$ as solutions of problem (30) are such that

$$\mathbb{E}_0[\mathbf{f}](t) = \mathbb{E}_0[\hat{\mathbf{f}}](t) = 1,$$

it means that

$$f_i(t,\mathbf{u}) = \hat{f}_i(t,\mathbf{u}), \quad \forall i \in \{1,2,\ldots,n\},$$

and then $\mathbf{f}(t,\mathbf{u}) = \hat{\mathbf{f}}(t,\mathbf{u})$. □

5. The Vectorial Discrete Thermostatted Kinetic Framework

This section is devoted to the definition and analysis of a *vectorial discrete thermostatted kinetic framework*. Specifically, let $\mathcal{C}$ be a homogeneous complex system with respect to the mechanical variables. The system is assumed to be composed of particles whose microscopic state consists of a *vectorial activity variable* which can attain a *discrete vector* value $\mathbf{u} \in \mathbf{I} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, where $\mathbf{u}_i = \left(u_i^1, u_i^2, \ldots, u_i^m\right) \in I_i \subseteq \mathbb{R}^m$, for $i \in \{1,2,\ldots,n\}$.

The overall *distribution function* of the system is denoted by $\mathbf{f}(t) = \left[f_{ij}(t)\right]_{i,j}$, where $f_{ij}(t) := f(t, u_i^j)$, for $i \in \{1,2,\ldots,n\}$ and $j \in \{1,2,\ldots,m\}$.

The *discrete pth-order moment* is:

$$\mathbb{E}_p[\mathbf{f}](t) = \sum_{i=1}^{n}\sum_{j=1}^{m} \left(u_i^j\right)^p f_{ij}(t), \quad p \in \mathbb{N}. \tag{37}$$

The *evolution equation* of the (i,j)th functional subsystem, for $i \in \{1,2,\ldots,n\}$ and $j \in \{1,2,\ldots,m\}$, reads:

$$\frac{df_{ij}}{dt}(t) = J_{ij}[\mathbf{f}](t) = G_{ij}[\mathbf{f}](t) - L_{ij}[\mathbf{f}](t), \tag{38}$$

where

$$G_{ij}[\mathbf{f}](t) = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{j_1=1}^{m}\sum_{j_2=1}^{m} \eta_{i_1j_1,i_2j_2}\mathcal{B}_{i_1j_1,i_2j_2}^{ij} f_{i_1j_1}(t) f_{i_2j_2}(t), \tag{39}$$

denotes the *gain-term* particles and

$$L_{ij}[\mathbf{f}](t) = f_{ij}(t)\sum_{i_2=1}^{n}\sum_{j_2=1}^{m} \eta_{ij,i_2j_2} f_{i_2j_2}(t), \tag{40}$$

denotes the *loss-term* particles.

The function $\mathcal{B}_{i_1j_1,i_2j_2}^{ij}$ denotes, for $j, j_1, j_2 \in \{1,2,\ldots,m\}$ and $i, i_1, i_2 \in \{1,2,\ldots,n\}$, the *transition probability density* that the particle with state $u_{i_1}^{j_1}$ acquires the state u_i^j after interacting with the particle $u_{i_2}^{j_2}$.

The *vectorial discrete thermostatted kinetic framework* is:

$$\frac{d\mathbf{f}}{dt}(t) = \mathbf{J}[\mathbf{f}](t) = \mathbf{G}[\mathbf{f}](t) - \mathbf{L}[\mathbf{f}](t). \tag{41}$$

The complex system $\mathcal{C}$ evolves under the action of the *external force field* $\mathbf{F}(t) = \left[F_{ij}(t)\right]_{i\in\{1,2,\dots,n\},j\in\{1,2,\dots,m\}}$, and a dissipative term (*discrete thermostat term*) is introduced in order to keep the pth-order moment constant. Accordingly, the *evolution equation of the distribution function* f_{ij} now reads:

$$\frac{df_{ij}}{dt}(t) = J_{ij}[\mathbf{f}](t) + F_{ij}(t) - \alpha f_{ij}(t). \tag{42}$$

The *thermostat term* α is obtained by imposing the conservation of the following pth-order moment:

$$\mathbf{E}_p[\mathbf{f}](t) = \mathbb{E}_p^0 \neq 0, \quad \forall t > 0.$$

Accordingly

$$\frac{d}{dt}\mathbf{E}_p[\mathbf{f}](t) = 0,$$

which means that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(u_i^j\right)^p \frac{df_{ij}}{dt}(t) = 0. \tag{43}$$

Equations (42) and (43) and straightforward calculations and show that:

$$\begin{aligned}
\alpha = \alpha\left(\mathbf{J}[\mathbf{f}], \mathbb{E}_p[\mathbf{f}], \mathbf{f}\right) &= \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left((u_i^j)^p J_{ij}[\mathbf{f}]\right) + \left((u_i^j)^p F_{ij}\right)\right)}{\sum_{i=1}^{n}\sum_{j=1}^{m}\left(u_i^j\right)^p} \\
&= \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\left((u_i^j)^p\left(J_{ij}[\mathbf{f}] + F_{ij}\right)\right)}{\mathbb{E}_p[\mathbf{f}]} \\
&= \left(\frac{\langle \mathbf{U}^p, \mathbf{J}[\mathbf{f}] + \mathbf{F}\rangle_F}{\mathbb{E}_p[\mathbf{f}]}\right),
\end{aligned} \tag{44}$$

where $\mathbf{U}^p := \left[(u_i^j)^p\right]_{i\in\{1,2,\dots,n\},j\in\{1,2,\dots,m\}}$ and $\langle \mathbf{U}^p, \mathbf{J}[\mathbf{f}] + \mathbf{F}\rangle_F$ denotes the *Frobenius inner product*. Given two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m,n}$, the *Frobenius inner product* of $\mathbf{A}$ and $\mathbf{B}$ is defined as follows:

$$\langle \mathbf{A}, \mathbf{B}\rangle_F = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ij}.$$

between $\mathbf{U}^p$ and $\mathbf{J}[\mathbf{f}] + \mathbf{F}$. Accordingly, for $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, m\}$, the vectorial discrete thermostatted kinetic theory framework (42) is written as follows:

$$\frac{df_{ij}}{dt}(t) = J_{ij}[\mathbf{f}](t) + F_{ij}(t) - \frac{\langle \mathbf{U}^p, \mathbf{J}[\mathbf{f}] + \mathbf{F}\rangle_F}{\mathbb{E}_p[\mathbf{f}]} f_{ij}(t). \tag{45}$$

Equation (45) consists of a system of $n \times m$ nonlinear ODEs with quadratic nonlinearities.

The Cauchy Problem

The *vectorial Cauchy problem* for *the vector activity discrete thermostatted framework* (45) reads:

$$\begin{cases} \frac{d\mathbf{f}}{dt}(t) = \mathbf{J}[\mathbf{f}](t) + \mathbf{T_F}[\mathbf{f}](t) \quad t \in [0, +\infty[\\ \mathbf{f}(0) = \mathbf{f}^0 \end{cases} \tag{46}$$

where $\mathbf{f}^0 = \left[f_{ij}^0\right]_{i\in\{1,2,\dots,n\},j\in\{1,2,\dots,m\}}$ is the *initial data* and $\mathbf{T_F}[\mathbf{f}]$ the following operator:

$$\mathbf{T_F}[\mathbf{f}] = \mathbf{F} - \left(\frac{\langle \mathbf{U}^p, \mathbf{J}[\mathbf{f}] + \mathbf{F}\rangle_F}{\mathbb{E}_p[\mathbf{f}]}\right)\mathbf{f}. \tag{47}$$

Definition 3. *Let $f(t) = \left[f_{ij}(t)\right] \in \mathbb{R}^{n,m}$, then*

$$\mathcal{R}_{\mathbf{f}}^p \equiv \mathcal{R}_{\mathbf{f}}^p\left(\mathbb{R}^+; \mathbb{E}_p^0\right) = \left\{\mathbf{f}(t) = \left[f_{ij}(t)\right] \in \left(C\left([0, +\infty[; \mathbb{R}^+\right)\right)^{n\times m} : \mathbb{E}_p[\mathbf{f}](t) = \mathbb{E}_p^0\right\}. \tag{48}$$

The mathematical analysis of the Cauchy problem (46) is performed under the following assumptions (see [21] and references therein):

H′1. There exists a constant $k > 0$ such that $\eta_{ij,i_1j_1} \leq k$, for all $i, i_1 \in \{1,2,\dots,n\}$ and $j, j_1 \in \{1,2,\dots,m\}$;
H′2. Let $i_1, i_2 \in \{1,2,\dots,n\}$ and $j_1, j_2 \in \{1,2,\dots,m\}$, then:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \mathcal{B}_{i_1j_1,i_2j_2}^{ij} = 1, \quad \forall i \in \{1,2,\dots,n\}, j \in \{1,2,\dots,m\}; \tag{49}$$

H′3. $u_i^j \geq 1$, for $i \in \{1,2,\dots,n\}$ and $j \in \{1,2,\dots,m\}$;
H′4. There exists $F > 0$ such that $F_{ij}(t) \leq F$, for $i \in \{1,2,\dots,n\}$ and $j \in \{1,2,\dots,m\}$.

Let $\mathbf{A} \in \mathbb{R}^{n,n}$, in what follows the quantity $\|A\|_1$ denotes the following norm:

$$\|A\|_1 := \sum_{i=1}^{n}\sum_{j=1}^{m} |A_{ij}|. \tag{50}$$

Theorem 3. *Assume that* **H′1**, **H′2**, **H′3** *and* **H′4** *hold true. If $\mathbf{f}^0 \in \mathcal{R}_{\mathbf{f}}^p$, then there exists a unique non negative function $f \in \mathcal{R}_{\mathbf{f}}^p$ which is solution of the Cauchy problem* (46).

Proof. By using the same arguments of theorem 4.1 of [21], there exists a constant $c_1 > 0$, depending on the parameters of the system (46) such that:

$$\|\mathbf{J}[\mathbf{f}] - \mathbf{J}[\mathbf{g}]\|_1 \leq c_1\mathbb{E}_p^0 \|\mathbf{f} - \mathbf{g}\|_1, \forall \mathbf{f}, \mathbf{g} \in \mathcal{R}_{\mathbf{f}}^p, \tag{51}$$

and there exists another constant c_2, also depending on the parameters of the system (46), such that:

$$\|\mathbf{T}[\mathbf{f}] - \mathbf{T}[\mathbf{g}]\|_1 \leq \left(F\frac{\|\mathbf{U}^p\|_1}{\mathbb{E}_p^0} + c_2\mathbb{E}_p^0 + c_1\mathbb{E}_p^0\|\mathbf{U}^p\|_1\right)\|\mathbf{f} - \mathbf{g}\|_1, \tag{52}$$

for all $\mathbf{f}, \mathbf{g} \in \mathbb{R}_{\mathbf{f}}^p$. By (51) and (52), it is possible to conclude that the operators $\mathbf{J}[\mathbf{f}]$ and $\mathbf{T}[\mathbf{f}]$ are locally Lipschitz in $\mathbf{f}$, uniformly with respect to t. The existence and uniqueness of a function $\mathbf{f}(t) = \left[f_{ij}(t)\right] \in \mathbb{R}^{n,m}$ local solution of the Cauchy problem (46) is thus ensured. The local solution of the Cauchy problem (46) can be extended for all $t \in [0, +\infty[$, see ([20] Chapter 2).

It remains to prove that $\mathbf{f} \in \mathcal{R}^p_{\mathbf{f}}$. Rewriting Equation (46) in integral form, for $0 < t \leq T$, one has:

$$\begin{aligned} \mathbf{f}(t) &= \mathbf{f}^0 + \int_0^t \frac{d\mathbf{f}}{ds}(s)\, ds \\ &= \mathbf{f}^0 + \int_0^t \left(\mathbf{J}[\mathbf{f}] + \mathbf{F}(s) - \left(\frac{\langle \mathbf{U}^p, \mathbf{J}[\mathbf{f}] + \mathbf{F} \rangle_F}{\mathbb{E}_p[\mathbf{f}]} \right) \mathbf{f}(s) \right) ds, \end{aligned} \tag{53}$$

and

$$\begin{aligned} f_{ij}(t) &= f^0_{ij} + \int_0^t \frac{d f_{ij}}{ds}(s)\, ds \\ &= f^0_{ij} + \int_0^t \left(J_{ij}[\mathbf{f}] + F_{ij}(s) - \left(\frac{\langle \mathbf{U}^p, \mathbf{J}[\mathbf{f}] + \mathbf{F} \rangle_F}{\mathbb{E}_p[\mathbf{f}]} \right) f_{ij}(s) \right) ds. \end{aligned} \tag{54}$$

By multiplying both sides of (54) by $(u^j_i)^p$ and taking the sum on $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, m\}$, one has:

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m (u^j_i)^p f_{ij}(t) &= \sum_{i=1}^n \sum_{j=1}^m (u^j_i)^p f^0_{ij} + \int_0^t \sum_{i=1}^n \sum_{j=1}^m (u^j_i)^p J_{ij}[\mathbf{f}](s)\, ds \\ &+ \int_0^t \sum_{i=1}^n \sum_{j=1}^m (u^j_i)^p F_{ij}\, ds - \int_0^t \sum_{i=1}^n \sum_{j=1}^m \left(\frac{\langle \mathbf{U}^p, \mathbf{J}[\mathbf{f}] + \mathbf{F} \rangle_F}{\mathbb{E}_p[\mathbf{f}]} \right) (u^j_i)^p f_{ij}(s)\, ds. \end{aligned} \tag{55}$$

Now, straightforward calculations yield:

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m (u^j_i)^p f_{ij} &= \mathbb{E}^0_p + \int_0^t \sum_{i=1}^n \sum_{j=1}^m \left((u^j_i)^p \left(J_{ij}[\mathbf{f}](s) + F_{ij}(s) \right) \right) ds \\ &- \int_0^t \sum_{i=1}^n \sum_{j=1}^m \left((u^j_i)^p \left(\frac{\langle \mathbf{U}^p, \mathbf{J}[\mathbf{f}] + \mathbf{F} \rangle_F}{\mathbb{E}_p[\mathbf{f}]} \right) \right) f_{ij}(s)\, ds \\ &= \mathbb{E}^0_p + \int_0^t \sum_{i=1}^n \sum_{j=1}^m \left((u^j_i)^p \left(J_{ij}[\mathbf{f}](s) + F_{ij}(s) \right) \right) ds \\ &- \int_0^t \left(\sum_{i=1}^n \sum_{j=1}^m \left((u^j_i)^p f_{ij}(s) \right) \left(\frac{\langle \mathbf{U}^p, \mathbf{J}[\mathbf{f}] + \mathbf{F} \rangle_F}{\mathbb{E}_p[\mathbf{f}]} \right) \right) ds \\ &= \mathbb{E}^0_p + \int_0^t \left(\sum_{i=1}^n \sum_{j=1}^m \left((u^j_i)^p \left(J_{ij}[\mathbf{f}](s) + F_{ij}(s) \right) \right) - \langle \mathbf{U}^p, \mathbf{J}[\mathbf{f}] + \mathbf{F} \rangle_F \right) ds. \end{aligned} \tag{56}$$

Since the integrand function of Equation (56) vanishes for all s and the pth-order moment is conserved, the following relation holds true:

$$\sum_{i=1}^n \sum_{j=1}^m (u^j_i)^p f_{ij}(t) = \mathbb{E}_p[\mathbf{f}](t) = \mathbb{E}^0_p.$$

Then $\mathbf{f}(t) \in \mathcal{R}^p_{\mathbf{f}}$. Using the same arguments of theorem 4.1 of [21] it is possible to conclude the non-negativity of the solution. Indeed, let

$$Q_{ij}[\mathbf{f}, \mathbf{F}](t) = G_{ij}[\mathbf{f}](t) + F_{ij}(t), \quad P_{ij}[\mathbf{f}, \alpha] = \sum_{i_2=1}^n \sum_{j_2=1}^m \eta_{ij, i_2 j_2} f_{i_2 j_2}(t) + \alpha,$$

then the Cauchy problem Equation $(46)_1$ becomes:

$$\frac{df_{ij}}{dt}(t) + f_{ij}(t)P_{ij}[\mathbf{f}, \alpha](t) = Q_{ij}[\mathbf{f}, \mathbf{F}](t). \tag{57}$$

Finally, setting

$$\gamma_{ij}(t) = \int_0^t P_{ij}[\mathbf{f}, \alpha](s)\, ds,$$

the solution of Equation (57), for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$, is:

$$f_{ij}(t) = f_{ij}^0 \exp\left(-\gamma_{ij}(t)\right) + \int_0^t \exp\left(\gamma_{ij}(s) - \gamma_{ij}(t)\right) Q^{ij}[\mathbf{f}, \mathbf{F}](s)\, ds,$$

which shows the non negativity of the solution of the Cauchy problem (46). □

6. Conclusions and Research Perspectives

The present paper has been devoted to enhance the capability of the thermostatted kinetic framework to model real world complex systems by introducing a vector-valued activity variable structure. As shown in the paper, the introduction of such a vector-valued variable does not affect the validity of well-known existence and uniqueness theorems of the Cauchy problem, both in the continuous and in the discrete framework. In particular, the main results have been obtained by assumptions on the transition probability density, the interaction rate and the external force field. The proposed mathematical frameworks can be thus considered as a general paradigm for the derivation of specific models for complex living systems.

As already mentioned, the main interest for introducing a vectorial structure in the thermostatted kinetic theory framework refers to the applications. Specifically, the applications deal with the modeling of complex living systems where each particle (cell, pedestrian, animal) expresses different functions, e.g., in criminality [22,23], social dynamics [24,25], crowd modeling [26,27], virus spread [28,29] and animal behaviors [30].

From the research perspectives point of view, different theoretical and applied issues could be investigated. Firstly, the evolution of the vectorial activity variable has been chosen additive, namely for each variable a transition probability density is defined and the overall contribution to each functional subsystem is considered as the sum of every transition density function. A research perspective can be addressed to a multiplicative evolution, thus increasing the nonlinearity in the framework. Another research perspective refers to the introduction of a time-fractional derivative, in this case some singularity can appear as in [31], and time delays [32]. In particular, an open problem is the mathematical proof of the existence of the nonequilibrium stationary state [33] and the convergence of stationary solutions to the nonequilibrium state.

It is worth stating that the proposed vectorial thermostatted kinetic theory frameworks are based on the assumption that the system is homogeneous with respect to the space and velocity variables. Even if this assumption is robust with respect to the envisaged applications, from the theoretical viewpoint the introduction of the space and velocity dynamics needs to be carefully introduced. The vectorial thermostatted frameworks proposed in this paper can be further generalized in order to include the role of nonconservative and mutative interactions. From the mathematical viewpoint, the distribution function may blow up or decay to zero in finite time, thus losing the global existence of the solution. The vectorial frameworks proposed in this paper can also be generalized to open complex living systems, namely systems where the external actions are introduced at the same scale of the particles. Another research perspective is to investigate the type of interactions and/or external action which lead a to desired macroscopic behavior; the formulation of a problem in the inverse theory framework can be thus envisaged [34].

Finally, an important research direction is the possibility to derive the macroscopic equations by means of hydrodynamic limits, see [35] and the references cited therein.

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