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Random Walks in the Quarter Plane: Solvable Models with an Analytical Approach

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ABSTRACT Initially, an urn contains 3 blue balls and 1 red ball. A ball is randomly chosen from the urn. The ball is returned to the urn, together with one additional ball of the same type (red or blue). When the urn has twenty balls in it, what is the probability that exactly ten balls are blue? This is a model for a random process. This urn model has been extended in various ways and we consider some of these generalizations. Urn models can be formulated as random walks in the quarter plane. Our findings indicate that for a specific type of random walk, we can calculate the generating function explicitly. Instead of using Markov chains, our approach is to use analytical techniques from differential equations.

INTRODUCTION

Imagine there are four people in a classroom, with three people at the front of the room and the other person sitting at the back. A person arrives and she randomly chooses one of the four people already in the room. If she chooses a person at the front of the room, then she will sit at the front. If she chooses a person from the back of the room, then she will sit at the back. After one minute another person arrives and he randomly chooses one of the five people already in the room. Again, he will sit at the front of the room if the randomly chosen person is from the front, and otherwise, he will sit at the back. As more people come, this process repeats. That means people arrive; one person at a time. Each new person sits at the front, or at the back, depending on whether the new person selects someone from the front or the back of the room.

This describes a random process of people entering the room, and there are many questions we could ask about the process. We will direct our attention to the following type of question: When the room has N people, what is the probability that there are a persons at the front and b persons at the back?

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We can formulate this problem in terms of an urn that contains red and blue balls. People at the front and the back of the room correspond to the blue and red balls, respectively. This example, when formulated as an urn process, is known as the Polya urn model.

An urn process is a model for a random process in computer science. It can be a model for random allocation for two types of tasks. The balls in the urn represent two types of tasks waiting to be processed, and each new task that arrives belongs to one of the two types, according to the random allocation scheme.

In the generalized Polya urn model, the urn initially contains a_0 blue balls and b_0 red balls. Balls are drawn at random in succession. If a blue ball is drawn, then we return it to the urn, together with *a* blue and *b* red balls. If a red ball is drawn, then we put it back to the urn, together with *c* blue and *d* red balls.

Below we describe several variations on the urn model, which we will return to throughout the paper. For each case, we can formulate the following problem: When there are N balls in the urn, what is the probability that there are *a* red balls and *b* blue balls?

Example 1

(Polya Urn Model) Initially, an urn contains 3 blue balls and 1 red ball. A ball is randomly drawn from the urn. The ball is returned to the urn, together with one additional ball of the same color. In this example, (illustrated in Figure 1), the probability is 3/4 that a blue ball is drawn. The Polya urn model is a particular case of the generalized Polya urn model, with parameters a = 1, b = 0, c = 0, and d = 1.

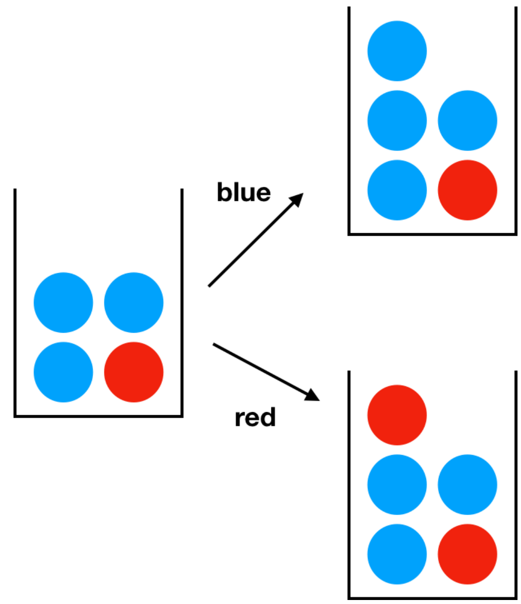

Figure 1. The urn initially has 3 blue balls and 1 red ball. A ball is randomly chosen. If it is blue, we return it with an extra blue ball to the urn. If it is red, we return it with an extra red ball.

Example 2

(Two-chamber Urn Model) Initially, an urn contains 4 blue balls and 4 red balls. A ball is randomly drawn from the urn. If a blue ball is drawn, then we throw it away and replace it by a red ball in the urn. Otherwise, if a red ball is drawn, we discard it and replace it by a blue ball in the urn. Initially, the urn has 8 balls. Each time after a ball is drawn and being replaced, there are still 8 balls in total. This is also known as the Erhenfest model.

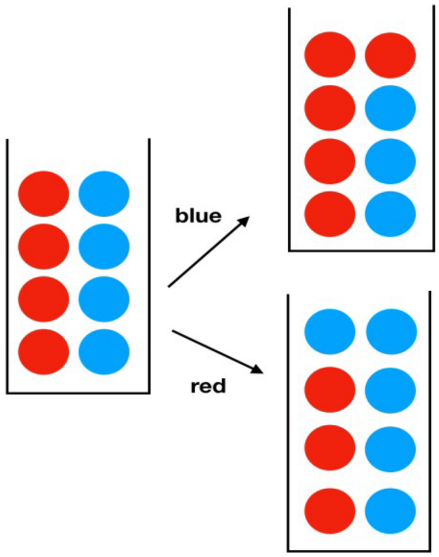

Figure 2. The urn initially has 4 blue balls and 4 red balls. A ball is randomly chosen. If it is blue, we replace it with a red ball. If it is red, we replace it with a blue ball.

The balls represent molecules in a room. Each time, a molecule is chosen at random from either the left or right chamber and being moved to the other chamber. The molecules can be regarded as the two types of balls, with blue balls in the left chamber and red balls in the right chamber.

We recognize this is a particular case of the generalized Polya model, with parameters $a = -1$, $b = 1$, $c = 1$, and $d = -1$.

Example 3

(The coupon-collector's urn) An urn contains 4 blue balls and 2 red balls. If a blue ball is drawn, then it is taken out, and a red ball is placed back into the urn. If a red ball is drawn, then it is placed back into the urn. In this example, (illustrated in Figure 3), the probability is 4/6 that a blue ball is drawn. After we have made N draws, find the probability that all the balls are red.

This is a particular case of the generalized Polya urn model, with parameters $a = -1$, $b = 1$, $c = 0$, and $d = 0$.

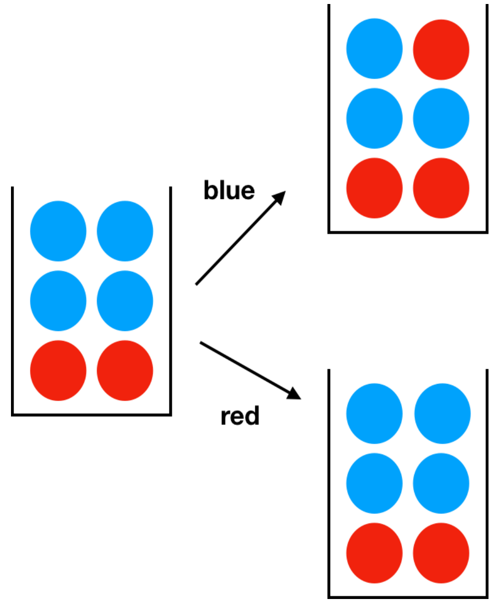

Figure 3. The urn initially has 4 blue balls and 2 red balls. A ball is randomly chosen. If it is blue, we replace it with a red ball. If it is red, we return it to the urn.

This is the Woodbury random allocation model analyzed in [4]. This can be a model for contagion (the spread of bacteria) in a population. The two types of balls represent people in a population: some are healthy, and some are infected. Each time, if a heathy person is in contact with someone infected, the healthy person becomes infected. If an infected person is in contact with someone infected, there is no change in the population because the person is already infected.

Main Contribution

The main contribution of this work is to provide an explicit formula for the generating function for a specific type of model. The generating function encodes all the information needed to compute the probabilities for specific number of blue balls in the urn.

In the paper by Flajolet, Dumas, and Puyhaubert [1], they provide a unifying framework to analyze a large class of urn models. In section 1 of their paper, they develop a general theory by using analytical techniques of differential equations. In section 2 of their paper, they apply the theory to seven examples of urn models. Each example is a particular instance of the generalized Polya urn model.

We will focus on three of these urn models: the Polya urn model, Two-chamber Urn Model, and the coupon collector's urn. They are illustrated by Example 1, 2, and 3, respectively.

In the next section, we describe the connection between urn processes and random walks in the quarter plane. Then in section 3 and 4, we will summarize the key insight from the theory developed in the paper by Flajolet et al. Section 5 illustrates how the theory applies to three urn models. In section 6, we provide our analysis on a new example, with details in the Appendix.

Random Walks

The probability process associated with an urn model is called an urn process. There is a connection between an urn process and a random walk in the quarter plane. Suppose there are x blue balls and y red balls in the urn. Consider the Two-chamber Urn Model, which is Example 2 from the Introduction. If a blue ball is drawn and replaced with a red ball, there are now $x - 1$ blue balls and $y + 1$ red balls. If instead, a red ball is drawn and replaced with a blue ball, then there is $x+1$ blue balls and $y - 1$ red balls. We imagine a person is walking in the XY-Plane. The location of the person is specified by the coordinates (x, y). This corresponds to how many blue balls and red balls are in the urn. Suppose the current location of the person is at (x, y). The next location is at $(x - 1, y + 1)$, with probability $\frac{x}{x+y}$,

or is at $(x + 1, y - 1)$, with probability $\frac{y}{x+y}$.

Since the number of blue and red balls in an urn are never negative numbers, it means the x and y coordinates are never negative. The person never walks below the horizontal axis or to the left side of the vertical axis, i.e., the person is restricted to walk in the quarter plane.

Generating Function

The generating function is a mathematical tool that determines the composition of the urn at any given time n. We will first define what a generating function is, in the context of urn models. Then we summarize the key insight in the first section of the paper by Flajolet et al.

The initial configuration (a_0, b_0) indicates the urn begins with a_0 blue and b_0 red balls. A history of length n specifies the evolution of an urn from time 0 to time n. As an illustration, consider the Two-Chamber Urn Model (Example 2 from the Introduction). Suppose $a_0 = 1$, $b_0 = 1$. This means at time $n = 0$, there is one blue and one red ball.

Using x for blue and y for red, we can write down one history from time 0 to 4.

$$\underline{x}\,y \to y\,\underline{y} \to x\,\underline{y} \to x\,\underline{x} \to x\,y$$

To indicate which ball is drawn, we underline it. The length of a history is the number of arrows that it comprises, so that a history of length n indicates one possible evolution of an urn from time 0 to time n.

Let $G_n(a_0, b_0)$ be the set of histories of length n when the urn has initial configuration (a_0, b_0). Let $G_n(a, b; a_0, b_0)$ be the subset of those histories which, at time n, correspond to an urn with a red balls and b red balls. To continue our illustration with the Two Chamber Urn Model, we can list the set of all histories from $n = 0$ to 2. There are 4 possible histories of length 2:

$$\underline{x}\,y \to y\,\underline{y} \to x\,y$$
$$\underline{x}\,y \to \underline{y}\,y \to x\,y$$
$$x\,\underline{y} \to x\,\underline{x} \to x\,y$$
$$x\,\underline{y} \to \underline{x}\,x \to x\,y$$

There are 8 possible histories of length 3 because from xy at time 2, we can either pick blue ball (x)

or red ball (y), and there are 4 possible histories leading to xy at time 2. We see that there are 2^n possible histories from time 0 to n.

Given a set, we can ask: how many elements are there in the set? $G_n(a_0, b_0)$ is a set, where each element of the set is one history of length n. The number of elements in $G_n(a_0, b_0)$ is denoted by $H_n(a_0, b_0)$. Since $G_n(a, b; a_0, b_0)$ is also a set, we can ask: how many elements are in this set?

Let $H_n(a, b; a_0, b_0)$ be the number of elements in $G_n(a, b; a_0, b_0)$.

We define the generating functions (GFs) for an urn model by

$$H(z|a_0, b_0) = \sum_n H_n(a_0, b_0)\frac{z^n}{n!}$$

and

$$H(x, y, z|a_0, b_0)$$

$$= \sum_n \sum_a \sum_b H_n\ (a, b;\ a_0, b_0)\ x^a y^b\ \frac{z^n}{n!}.$$

The trivariate GF is also called the complete generating function of urn histories. When there is no ambiguity, we will write H(z) for $H(z|a_0, b_0)$ and H (x, y, z) for $H(x, y, z|a_0, b_0)$. In the next section, we will see the significance of GFs. If we treat the function H (x, y, z) as a power series, then the coefficient of the term $x^a y^b z^n$ is denoted by $[x^a y^b z^n]$ H (x, y, z).

General Theory for Urn Model

An urn model is specified by a matrix with the integer entries,

$$M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}. \qquad (1)$$

At time 0, the urn contains a_0 blue balls and b_0 red balls. The urn evolves according to the following rule: A ball is chosen from the urn, where each ball is equally likely to be chosen. If a blue ball is chosen, then α new blue balls and β new red balls are placed into the urn; if a red ball is chosen, then α new blue balls and β new red balls are placed into the urn. We allow negative entries. In particular, if α = −1, that means a blue ball is removed from the urn. We consider only those

urns that are balanced, in the sense that the sum of entries in both rows of M are equal and we let σ be this sum,

$$\sigma = \alpha + \beta = \gamma + \delta. \qquad (2)$$

We use Greek letters for the parameters to emphasize the condition of equation (2), i.e.,

there is a crucial condition imposed on the generalized Polya urn model; the urn must be balanced. The condition implies that at any given time, the total number of balls in the urn must equal $a_0 + b_0 + n\sigma$.

Proposition. Let A_n and B_n be the number of blue balls and red balls, respectively, of a balanced urn process at time n. Then, we have

$$P(A_n = a, B_n = b) = \frac{[x^a y^b z^n]\ H(x,y,z)}{[z^n]\ H(1,1,z)} \qquad (3)$$

■

This proposition shows that the GF's encode all the information that is needed to compute the probabilities for an urn process. If the urn is not balanced, the conclusion of the proposition can fail to hold, because not all paths are equally likely for an unbalanced urn. We provide a concrete illustration with an example in Appendix C.

Every urn process is associated with a random walk in the quarter plane, as we explain in Section 2 of this article. We can state the proposition in terms of GF's of a random walk. It is a matter of personal preference whether to use the language of an urn process or a random walk. We explain everything in terms of blue and red balls because of the colorful language. Finding an explicit formula for the generating function of a random walk in the quarter plane, in general, is a notoriously difficult problem (see, e.g. [2], [3]).

For a balanced urn model with matrix *M*, we associate the system of ordinary differential equations (ODEs),

$$\frac{dx}{dt} = x^{\alpha+1} y^{\beta} \qquad (4)$$

and

$$\frac{dy}{dt} = x^{\gamma}\, y^{\delta+1} \qquad (5)$$

The insight from [1] is that the solution to the system of ODEs is equivalent to the complete generating function H (x, y, z) of the urn process, in the following sense.

The following theorem is taken from [1].

Theorem. (Isomorphism between Systems of ODEs and Balanced Urns)

Consider a balanced urn with the Matrix *M*, specified by equations (1) and (2), with the initial configuration (a_0, b_0). Let *X (t)* and *Y (t)* be the solution of the pair of differential equations (4) and (5), with the initial conditions X (0) = x_0 and Y (0) = y_0. Then the complete generating function of urn histories satisfies the relation,

$$\mathrm{H}\,(x, \mathrm{y}, \mathrm{z}) = X(z)^{a0}\, Y(z)^{b0}, \quad (6)$$

where the symbols x, y on the left side has the association: x = X(t) and y = Y (t).

■

We stated the theorem slightly differently for the clarity. We refer the interested reader to [1] for a precise statement of the Theorem and its Proof.

The theorem tells us that the problem of calculating the function H $(x, \mathrm{y}, \mathrm{z})$ for a balanced urn process becomes the problem of solving a system of ODE. If we can solve the pair of differential equations (4) and (5), then we can use equation (6) to obtain the complete GF for the urn model. Then, based on equation (3), to find the probability the urn has a blue ball, we compute the coefficient of $x^a y^b z^n$ in the series expansion of H $(x, \mathrm{y}, \mathrm{z})$ and the coefficient of z^n in the series expansion of H (1, 1, z).

Applications to Three Models

We now illustrate how to apply the theory developed in Section 4 to three balanced urn models described in the introduction.

Model 1. The Polya urn model is specified by the matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The associated system of ODE is given by,

$$\frac{dx}{dt} = x^2,\ x(0) = x_0 \quad (7)$$

and

$$\frac{dy}{dt} = y^2,\ y(0) = y_0. \quad (8)$$

This pair of differential equations can be solved by solving each one individually, using separation of variables. By applying equation (6) from the theorem, we obtain the complete generating function,

$$\mathrm{H}\,(x, \mathrm{y}, \mathrm{z}) = \frac{x^{a_0}\, y^{b_0}}{(1-zx)^{a_0}\,(1-zy)^{b_0}}.$$

Model 2. (Two-chamber Urn Model) The model is specified by the matrix

$$M = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

We have the associated system of ODE,

$$\frac{dx}{dt} = y,\ x\,(0) = x_0 \quad (9)$$

and

$$\frac{dy}{dt} = x,\ y\,(0) = y_0 \quad (10)$$

To solve this system of ODE, we first solve for y(t) by observing that

$$\frac{d^2y}{dt} = y,\ y\,(0) = y_0.$$

Once we obtain y(t), we use equation (9) to solve for x(t). In the standard interpretation of the model, all the balls (molecules) are initially in the left chamber, which means a_0 = N and b_0 = 0. We apply equation (6) from the theorem to obtain

$$\mathrm{H(x, y, z)} = \left(\left(\frac{x+y}{2}\right)e^{z} + \left(\frac{x-y}{2}\right)e^{-z}\right)^{N}.$$

Model 3. (The coupon-collector's urn) The model is specified by

$$M = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}.$$

We have the associated system of ODE,

$$\frac{dx}{dt} = y, \qquad x(0) = x_0$$

and

$$\frac{dy}{dt} = y, \qquad y(0) = y_0.$$

The solution of this system is

$$X(t) = x_0 - y_0 + y_0 e^t, \; Y(t) = y_0 e^t$$

and the complete generating function is

$$H(x, y, z) = (x_0 - y_0 + y_0 e^z)^{a_0} y^{b_0} e^{b_0 z}$$

New Results

In each of the three models presented in the last section, we can solve the system of ODE explicitly because it is a linear system. In general, we have a pair of nonlinear differential equations in (4) and (5). Suppose the urn model is specified by the matrix

$$M = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}.$$

The associated system of ODE is given by

$$\frac{dx}{dt} = y^4 \tag{11}$$

and

$$\frac{dy}{dt} = x^2 y^2. \tag{12}$$

To deal with those urn models, in which we cannot explicitly find the solution of the resulting pair of nonlinear DE, we need a new tool. We first summarize this tool in the next proposition. Then, we apply this technique to two examples. The first example is to illustrate the technique with the Two-chamber Urn Model and confirm that the result agrees with the one before. The second example is to apply this technique to an urn model, which is associated with the system of ODE given by (11) and (12). Note that this means the technique allows us to obtain the generating function of the urn process, without solving explicitly for x(t) and y(t).

Proposition. Consider a balanced urn with matrix M, with $\alpha = -1$, $\delta = 1$ or -1, and with initial configuration (a_0, b_0). Define the parameters

$$p = \beta - \delta, \lambda = \frac{\beta}{p}$$

and

$$\sigma = \alpha + \beta, s_0 = a_0 + b_0$$

Define the integral,

$$J(u) = \int_0^u \frac{d\zeta}{(1+\zeta^p)^\lambda}$$

Let $\Delta = (1 - x^p)^{1/p}$. Then, we have

$$H(x, 1, z) = \Delta^{s_0} \left[S\left(z\Delta^\sigma + J\left(\frac{x}{\Delta}\right)\right)\right]^{a_0} . \left[C\left(z\Delta^\sigma + J\left(\frac{x}{\Delta}\right)\right)\right]^{-b_0}.$$

The function S and C are defined to satisfy

S (J(u)) = u,
J (S(u)) = u,
$C(z)^{-p} = 1 + S(z)^r$, when $\delta = 1$.

Remark. When $a_0 = N$ and $b_0 = 0$,
$$H(x, 1, z) = \Delta^{s_0} \left[S\left(z\Delta^\sigma + J\left(\frac{x}{\Delta}\right)\right)\right]^N$$

and the function C does not matter.

We include a heuristic for this proportion in Appendix B. It provides a motivation of why the function J(u) appears. Next, we highlight three applications with two examples. We provide full detail of all calculations for these two examples in the Appendix A.

First Example

Consider the Two-chamber Urn Model.

Let $M = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ with $a_0 = N$, $b_0 = 0$.

The generating function is,

$$H(x, 1, z) = \left(\left(\frac{x+1}{2}\right) e^{z} - \left(\frac{x-1}{2}\right) e^{-z} \right)^{N}.$$

Second Example

If a blue ball is drawn, then it is removed from the urn and 4 red balls are placed into the urn. If a red ball is drawn, then it is returned to the urn, together with 2 blue balls and 1 new red ball. The urn initially has N red balls. The model is specified by the matrix

$$M = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}.$$

In the terminology of a random walk in the quarter plane, when the current location is at (x, y), the next location is either at ($x - 1$, y + 4) or at (x + 2, y + 1). The former occurs with probability $\frac{x}{x+y}$. The walk begins at the location (0, N).

The generating function is

$$H(x, 1, z) = (1 - x^{3})^{\frac{N}{3}} \cdot \frac{1}{[1 - (z(1-x^{3}) + x)^{3}]^{\frac{N}{3}}}.$$

Remark. For both examples above, we compute the bivariate generating function *H (x, 1, z).* The urn is balanced, so the total number of balls at time n is $s_0 + n\sigma$, where $s_0 = a_0 + b_0$. All terms that appear in *H (x, 1, z)* are of the form

$$x^{a} y^{b} z^{n} = x^{a} y^{s_0 + n\sigma - a} z^{n}.$$

This implies we have the relation,

$$H(x, 1, z) = y^{s_0} H\left(\frac{x}{y}, 1, z y^{\sigma}\right).$$

Therefore, the variable y is redundant, from a strictly logical viewpoint, and we may freely set y = 1, without loss of generality. For this reason, it is sufficient to compute *H (x, 1, z).*

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APPENDIX

In this appendix, in the first part, we detail the calculations for the two examples in Section 6. In the second part, we provide a heuristic for the proposition in Section 6. Then, in the third part, we show an example of an unbalanced urn to show that the conclusion of the Proposition in Section 4 can fail to hold if the condition of an unbalanced urn is not satisfied.

APPENDIX A

Solutions for Two Examples

First Example

Let M = $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ which is a model with σ = 0.

For this model, the parameters are σ = 0, α = -1, β = 1, p = β - δ = 2.

The integral J(u) is given by

$$J(u) = \int_0^u \frac{d\zeta}{\sqrt{1+\zeta^2}}.$$

This integral can be evaluated in closed form. Let $\zeta = \tan\theta$, $d\zeta = sec^2\,\theta d\theta$, *so that,*

$$\int \frac{d\zeta}{\sqrt{1+\zeta^2}} = \int \frac{sec^2\,\theta d\theta}{sec\theta} = \int sec\theta d\theta = ln(sec\theta + tan\theta)$$

This implies J(u) has the closed-form expression,

$$J(u) = ln\,(\sqrt{1+u^2} + u).$$

Let a_0= N and b_0 = 0. That means initially all the balls are blue. By definition, we have S(J(z)) = z.

Let $x = \ln\sqrt{1+y^2} + y$

$$\Rightarrow e^x = \sqrt{1+y^2} + y \Rightarrow (e^x - y)^2 = 1 + y^2$$

$$\Rightarrow e^{2x} + y^2 - 2ye^x = 1 + y^2$$

$$\Rightarrow e^{2x} - 2ye^x = 1$$

$$\Rightarrow e^x - e^{-x} = 2y \qquad \Rightarrow y = \frac{e^x - e^{-x}}{2}$$

The above calculation shows that,

$$S(z) = \frac{1}{2}(e^z - e^{-z}).$$

We want to calculate

$$H(x, 1, z) = \Delta^{s0}\left[S\left(z + J\left(\frac{x}{\Delta}\right)\right)\right]^{a0} \quad (13)$$

Now, to determine the function H (x, 1, z), we need to expand the expression $S\,(z + J\,(\frac{x}{\Delta}))$.

From the formulas for J(u) and S(z), we now expand,

$$\frac{1}{2}\exp\left(z + J\left(\frac{x}{\Delta}\right)\right)$$

$$= \frac{1}{2}\left(exp(z)\; exp\left(ln\left(\sqrt{1+\left(\frac{x}{\Delta}\right)^2} + \frac{x}{\Delta}\right)\right)\right)$$

$$= \frac{1}{2}\left(e^z\left(\sqrt{1+\left(\frac{x}{\Delta}\right)^2} + \frac{x}{\Delta}\right)\right).$$

The expression in the square bracket can be simplified.

$$\sqrt{1+\left(\frac{x}{\Delta}\right)^2} + \frac{x}{\Delta} = \sqrt{\frac{1+x}{1-x}}.$$

We provide the detail for the calculation of the function H (x, 1, z). From (∗), we have

$$H(x, 1, \mathrm{z}) = (1-x^2)^{N/2} \cdot \frac{1}{2}\left(\left(\frac{1+x}{1-x}\right)^{1/2} e^z - \left(\frac{1-x}{1+x}\right)^{1/2} e^{-z}\right)^N$$

This implies that,

$$\begin{aligned}
&H(x, 1, \mathrm{z})\\
&= \frac{1}{2}\left((1-x^2)^{1/2} \cdot \left(\frac{1+x}{1-x}\right)^{1/2} e^z - (1-x^2)^{1/2} \cdot \left(\frac{1-x}{1+x}\right)^{1/2} e^{-z}\right)^N\\
&= \frac{1}{2}\left((1+x)^{1/2}(1-x)^{1/2} \cdot \left(\frac{1+x}{1-x}\right)^{1/2} e^z - (1+x)^{1/2}(1-x)^{1/2} \cdot \left(\frac{1-x}{1+x}\right)^{1/2} e^{-z}\right)^N\\
&= \left(\left(\frac{1+x}{2}\right) e^z - \left(\frac{1-x}{2}\right) e^{-z}\right)^N\\
&= \left(\left(\frac{x+1}{2}\right) e^z - \left(\frac{x-1}{2}\right) e^{-z}\right)^N
\end{aligned}$$

Compare this to the model when,

$M = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ we have σ = 0, α = -1, β = 1, p = 2 and s_0 = N.

The previous calculation from Section 5 gives the complete generating function,

$$H(x, 1, \mathrm{z}) = \left(\left(\frac{x+y}{2}\right) e^z + \left(\frac{x-y}{2}\right) e^{-z}\right)^N;$$

when a_0 = N, b_0 = 0. Note that this agrees with our above calculation for $H(x, 1, \mathrm{z})$. This completes the example.

Second Example

Let $M = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ which is a model with σ = 3.

For this model, the parameters are σ = 3, β = 4, p = β - δ = 3, $\lambda = \frac{\beta}{p} = \frac{4}{3}$.

The integral J(u) is given by

$$J(u) = \int_0^u \frac{d\zeta}{(1+\zeta^3)^{4/3}}$$

and the function *S*(z)) is defined by *J*(S(z)) = z, i.e., *S* is the inverse function of *J.*

By applying the Fundamental Theorem of Calculus,

$$\frac{dJ}{du} = \frac{1}{(1+u^3)^{4/3}}$$

We note that *J* (0) = 0. Let

$$\text{g(u)} = \frac{u}{(1+u^3)^{1/3}}$$

We express this as g(u) = $\text{u}(1+u^3)^{-1/3}$ and calculate the derivative of g.

$$\begin{aligned} \text{g'(u)} &= (1+u^3)^{-1/3} + (-\tfrac{1}{2})\,(1+u^3)^{-4/3} * 3u^2\,u \\ &= (1+u^3)^{-1/3} - u^3(1+u^3)^{-4/3} \\ &= (1+u^3)^{-4/3}\,(1+u^3) - u^3\,(1+u^3)^{-4/3} \\ &= (1+u^3)^{-4/3}\,[(1+u^3) - u^3] \\ &= (1+u^3)^{-4/3}. \end{aligned}$$

Our calculations show that *J'(u) = g'(u)*. Note that *J* (0) = 0 and *g* (0) = 0. Therefore, we conclude that *J(u) = g(u).*

Let $\Delta = (1-x^3)^{1/3}$. We next determine an expression for the inverse function of J.

$$J(\text{S(z)}) = \text{z}$$

$$J(\text{u}) = \frac{u}{(1+u^3)^{1/3}} \quad \Rightarrow \frac{S}{(1+S^3)^{1/3}} = \text{z}$$

$$\Rightarrow \text{S} = \text{z}(1+S^3)^{1/3}$$

$$\Rightarrow S^3 = z^3(1+S^3) \;\Rightarrow S^3 = z^3 + z^3S^3$$

$$\Rightarrow S^3(1-z^3) \;=\; z^3 \qquad \Rightarrow \mathrm{S(z)} = \frac{z}{(1-z^3)^{1/3}}$$

Using the relation,

$$\mathrm{C}\,(z)^{-3} = 1 + S(z)^3,$$

We can write,

$$\mathrm{C(z)} = (1-z^3)^{1/3}.$$

Let $a_0 = 0$ and $b_0 = \mathrm{N}$.

$$\mathrm{C}\,(z\Delta^\sigma \;+\; J\,(\tfrac{x}{\Delta}))^N$$

$$= \mathrm{C}\left(\mathrm{z}\,(1-x^3) \;+\; \frac{\frac{x}{\Delta}}{[1+\left(\frac{x}{\Delta}\right)^3]^{1/3}}\right)^N \;;\, 1+\left(\frac{x}{\Delta}\right)^3 \;=\; \frac{1}{1-x^3}$$

$$= \mathrm{C}\left(\mathrm{z}\,(1-x^3) \;+\; \frac{\frac{x}{\Delta}}{(\frac{1}{1-x^3})^{1/3}}\right)^N \;;\, \frac{x}{\Delta}(1-x^3)^{1/3} = x$$

$$= \mathrm{C}\,(\mathrm{z}\,(1-x^3) + \;x)^{\;N}$$
$$= [1 - \;(\mathrm{z}\,(1-x^3) \;+\; x)^3]^{\frac{N}{3}}$$

That means,

$$\mathrm{C}(z\Delta^3 \;+\; J\,(\tfrac{x}{\Delta}))^{-\,N} = [1 - \;(\mathrm{z}\,(1-x^3) \;+\; x)^3]^{\frac{-N}{3}}$$

Finally, putting all the terms together, the generating function is,

$$H(x,1,\mathrm{z}) = (1-x^3)^{\frac{N}{3}} \;\cdot\; \frac{1}{[1 - (\mathrm{z}\,(1-x^3) + \mathrm{x})^3]^{\frac{N}{3}}}.$$

This completes the example.

APPENDIX B

Heuristic for the Proposition

Let us recall that we let X (t) and Y (t) be the solution of the pair of differential equations (4) and (5), with initial conditions X (0) = x_0 and Y (0) = y_0.

Let $\Delta \;= (x_0{}^p - y_0{}^p)^{1/p}$, where we define p = γ - α = β – δ and assume p > 0.

We think of $x_0 \approx 0$ and $y_0 \approx 1$.

It can be shown that $X^p - Y^p = C_1$ (for some constant C_1), which means we can write

$$X\,(t)^p - Y\,(t)^p = x_0{}^p - y_0{}^p$$

and moreover,

$$\frac{dx}{dt} = x^{\alpha+1}y^{\beta}$$
$$= x^{\alpha+1}(x^p - x_0{}^p + y_0{}^p)^{\beta/p}$$
$$= x^{\alpha+1}(x^p + \Delta^p)^{\beta/p}.$$

Let X (t, x_0, y_0) = $\Delta\zeta(t\Delta^{\alpha})$ and Y (t, x_0, y_0) = $\Delta\eta(t\Delta^{\alpha})$.

Then the transformed system of pair (δ, η) is

$$\frac{d\zeta}{dt} = \zeta^{\alpha+1}(\zeta^p + 1)^{\beta/p}, \text{ where } \zeta(0) = \frac{x_0}{\Delta}$$

$$\frac{d\eta}{dt} = \eta^{\alpha+1}(\eta^p - 1)^{\gamma/p}, \text{ where } \eta(0) = \frac{y_0}{\Delta}.$$

By hypothesis, α = -1. We can express t formally as

$$t = \int_{x_0/\Delta}^{\zeta(t)} \frac{dw}{(1+w^p)^{\beta/p}}.$$

We introduce two parameters $\lambda = \frac{\beta}{p}$ and $r = \frac{-p}{\alpha}$.

Define the function

$$J(u) = \int_0^u \frac{d\zeta}{(1+\zeta^r)^{\lambda}}.$$

Then,

$$t = J(\zeta(t)) - J\left(\frac{x_0}{\Delta}\right).$$

We can write ζ(t) formally as

$$\Rightarrow \zeta(t) = S\left(t + J\left(\frac{x_0}{\Delta}\right)\right); \text{ where } S(J(u)) = u.$$

This completes our heuristic, which illustrates the role of the function J(u). The heuristic is a series of formal calculations and is not intended to be a proof of the proposition.

$$H(X(t), Y(t)) = X(t)^{a_0}\, Y(t)^{b_0}.$$

APPENDIX C

Example of an Unbalanced Urn

We require our urns to be balanced. Here, we give a counterexample to show that the conclusion of the Proposition in Section 4 can fail to hold if the urn is not balanced.

$$\text{Let } M = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}.$$

The condition of a balanced urn is not satisfied because the first-row sums to zero, but the second-row sums to 2. It can be shown that there are 11 histories for time n = 2. Using the notation $H_n(x, y, z)$ for the terms that involve z^n in $H(x, y, z)$, we have

$$H_2(x, y, z) = 2y^3z^2 + 6xy^4z^2 + 3x^2y^5z^2.$$

This is consistent with having 11 histories for n = 2 since the sum of the coefficients is 2 + 6 + 3 = 11. Ignoring the $\frac{1}{n!}$ term in this discussion, since it plays no role, we can write

$$[xy^4z^2]\, H(x, y, z) = 6.$$

Since there are 11 histories for n = 2, if we apply the Proposition, then we have

$$P(A_n = 1, B_n = 4) = \frac{6}{11}.$$

However, a detailed calculation reveals that when n = 2,

$$P(A_n = 1, B_n = 4) \neq \frac{6}{11}.$$

When the urn model does not satisfy the balanced condition, it is not true that each history occurs equally likely, with probability of $\frac{1}{11}$. By enumerating all possible cases, we find that

$$P(A_n = 1, B_n = 4) = 2\left(\frac{1}{3}\right)\left(\frac{1}{5}\right) + 4\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{26}{45}.$$

This completes our counterexample to the Proposition in Section 4.