Results on data-driven controllers for unknown nonlinear systems

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Results on data-driven controllers for unknown nonlinear systems

Monica Rotulo

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To my beloved family,
my parents Enza and Alfonso,
and my brother Marco.

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1

Introduction

Torture the data, and it will confess to anything.

Ronald Coase,
Nobel Prize in Economics in 1991

We live in a complex and rapidly changing world, where humanity has to face many global challenges, from climate change to the current and future pandemics. Solutions to many - if not all - of these challenges rely on understanding of the underlying phenomena. However, our planet is a complex dynamical system and empirical models or derivations from first-principles are not always a viable path for addressing the biggest scientific and engineering problems of the modern era.

Fortunately, we are also living in a *data-rich* era. In fact, the amount of available data is growing exponentially every year. In comparison to 2010, for instance, the volume of generated data in 2021 increased 48 times, and this number is estimated to become 90 times in 2025[1]. The availability of large datasets is already reshaping scientific and engineering fields, giving scientists and engineers new ways to tackle a diverse range of complex problems related to energy systems, climate, additive manufacturing, and aerospace [2–7], to name a few.

Generally speaking, data-driven methods aim to advance our scientific understanding of physical processes by learning unforeseen patterns from data. This approach is at the core of machine learning and artificial intelligence for modeling unknown phenomena from observations [8]. The learning from data paradigm has been also adopted in the control engineering domain to cope with complex systems whose dynamics are poorly known [9–12]. Classical system identification is a notable example of this domain [13, 14], and more recent examples are based on optimization in high-dimensional spaces [15–17].

Apart from the system identification problem, an active line of research is related to using observed data to design controllers for unknown plants. One strategy is to use the

[1] Today, the digital universe is expected to contain more than 64 *zettabytes* [1], which is around 64 trillion gigabytes. This means that there are 60 times more bytes than there are stars in the observable universe.

aforementioned identification methods to find a representative model of the unknown system, and then design a suitable controller based on such model. Nevertheless, there are many situations where identifying a *good* model which is able to capture the full complexity of the system can be difficult or impossible. In such cases, learning control laws *directly* from data is a solution.

Over the past few years, different techniques are explored such that effective control laws can be learned directly from data, skipping altogether any attempt of identifying the unknown system's model. This is a rapidly developing field, with methods ranging from reinforcement learning [18] to deep learning for control [19]. Recently, an idea from system theory [20], remained unnoticed since 2005, has turned out be a game changer for developing control algorithms in a data-based fashion. The key feature behind this idea is that a dynamical system is represented as a set of trajectories, rather than as a specific set of equations modeling the underlying phenomenon. This concept may sound simple, yet its implications are of significant importance in system analysis and control, as it allows to replace the usual mathematical model with a representation of the system directly on the basis of time-series data. This data-centric perspective lays down the foundations of a new class of direct *data-driven* methods — the topic of this thesis.

1.1. From data to controllers

Data-driven control refers to the procedure of designing controllers for an unknown system starting solely from measurements collected from the plant and some priors about the plant itself (linear vs. nonlinear parametrization, nature of the noise, etc.). This approach does not require any intermediate system identification step, making it a viable and practical alternative to the conventional model-based approach when we need to cope with complex dynamical plants whose dynamics is poorly known. Figure 1.1 provides a sketch of the two paradigms.

The history of data-driven control can be traced back to the 40's with the pioneer self-tuning method by Ziegler and Nichols [21]. This work allowed for the first time to tune the parameters of a PID controller according to experimental measurements. Alongside, McCulloch and Pitts introduced the concept of the artificial neural cell as a mathematical logic system [22]. By the late 80's, microprocessor-based systems were dominating the market. This pushed further developments in self-tuning regulation and direct adaptive control [23] and extended the use of multilayer neural networks to control purposes [24]. In the same years, unfalsified control [25], reinforcement learning [26], and iterative feedback tuning [27] were also developed and can be considered as classical data-driven control techniques. Since then, this topic has been attracting more and more researchers. Many techniques have been developed under different names, such as virtual reference feedback tuning [28], iterative learning control [29], and model-free control [30]. Additional methods are listed in [31–33], and various efforts have also been made in connection with other control problems, including nonlinear [34–36] and predictive control [37, 38]. We refer the interested reader to the survey [10] for a complete classification of data-driven techniques and earlier contributions on the subject.

For linear time-invariant systems, a result from Willems et *al.* [20] has been given renewed interest due to its deep implications for data-driven control. Essentially, [20] stipulates that the whole behavior of a linear time-invariant system can be captured by a finite set of data, provided that the system is sufficiently excited. This way, the data implicitly give a non-parametric system representation which can be directly used for control purposes. This key result, which became known as the *fundamental lemma*, inspired works in diverse areas of data-driven system analysis and control.

It was used in [39] to develop *data-enabled*, rather than model-based, predictive control. The proposed approach uses the aforementioned behavioral settings to learn a non-parametric system model and is proven to be equivalent to model predictive control. In [40], the behavioral approach is translated to the *state-space* framework, based on which a data-dependent representation for linear feedback systems is formulated. The state-space description is particularly appealing, as it allows to use linear matrix inequalities solely based on data to provide solutions to various problems, such as state- and output-feedback stabilization as well as the linear quadratic regulator synthesis. Optimal and robust controllers are further explored in [41–46], all of which are based directly on measured data without any model knowledge. Other applications of such data-based methods include stabilizing linear time-varying systems [47, 48] and delay systems [49], controlling linear network systems [50, 51], estimating the state in presence of unknown inputs [52], and model reduction [53, 54]. In [55], the informativity of data and its role in data-driven analysis and control is investigated. The work [56] provides a concise and comprehensive review of the literature centered around the fundamental lemma.

When the data are generated by *nonlinear* dynamics, deriving direct data-driven methods for unknown nonlinear systems can be extremely challenging. To address this difficulty, a common approach in the literature is exploiting some knowledge on the model structure. In this context, [40] provides a nonlinear extension of the linear result, where the unknown nonlinear dynamics are represented as the sum of a linear model and a noise term containing the higher-order terms of the nonlinearity. When it is known a priori the class to which the system belongs, different data-driven control methods can be derived to stabilize special classes of nonlinear systems. For instance, switched linear systems are studied in [57], where data-driven methods are derived by using polynomial optimization techniques. The data-driven approach is extended in [58] to second-order Volterra systems, where an internal model control to achieve output-tracking is proposed. Furthermore, in [59, 60], stabilization of nonlinear polynomial systems is achieved by using data-based sum of square programming, and in [61, 62] the stabilization of bilinear systems is investigated.

Finally, an increasing number of data-driven control results focuses on how to deal with *perturbations* and *noise* affecting the data and the resulting noise-induced uncertainty. In [40], the presence of deterministic noise with bounded energy affecting the data is dealt with a matrix elimination result to get rid of the resulting noise-induced uncertainty in the representation. Furthermore, the work of [57] defines a set of system's matrices pairs consistent with the data and, using an extended Farkas' lemma, derives conditions under which stability of all systems in the set hold. These conditions can be checked using polynomial optimization techniques. If the samples of process noise are i.i.d. and Gaussian, then [63] provides a quantification in probability of the confidence region, which [64] exploits to give data-dependent conditions for minimizing the worst case cost of the LQ problem over all the system's matrices in the confidence region. The technical tool for this

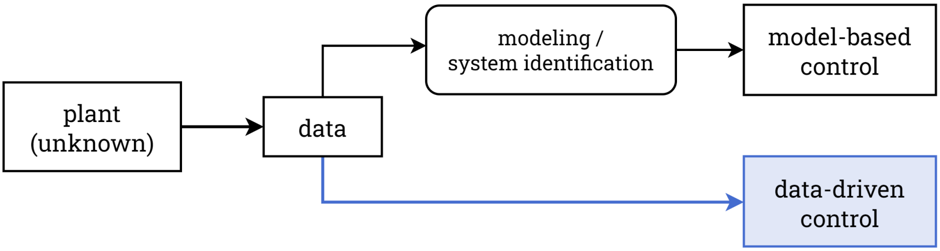

Figure 1.1: In the conventional model-based approach, the control design process is a two-steps procedure consisting of sequential system identification and control design for the identified system. However, this method may not work well if the derived model is not accurate enough. A direct data-driven design (in blue) has the advantage that the knowledge of the model is not required. Rather, the controller is obtained directly from the data. This way, the controller is not affected by possible modeling errors/uncertainties and the overall design process is simplified by removing the intermediate identification step.

study is an extension of the S-lemma provided in [65]. A new matrix S-lemma is introduced in [43] to provide non-conservative conditions for designing controllers from data affected by disturbances satisfying quadratic bounds. Other results to deal with disturbances use a full-block S-procedure and linear fractional representations [66], the classical S-procedure [67] and Petersen's lemma [68].

1.2. Contributions

In this thesis, we propose novel data-driven algorithms and data-dependent optimization methods to control and stabilize different families of unknown dynamical systems solely via the use of data. These methods provide control-theoretic guarantees, they are computationally tractable, and in comparison with machine learning methods require small amount of data. The main contributions of this thesis are the followings:

- **Development of data-dependent convex programs for applications in linear quadratic regulation problems.**
  We propose a data-based formulation of the (finite-horizon) linear quadratic regulation problem (LQR). We show that the optimal feedback controller can be directly parametrized through data. This is possible by combining a direct data-based parametrization of the closed-loop system with the convex optimization formulation (semidefinite programming) of the LQR problem. This makes it possible to determine the optimal control law in *one shot*, with no intermediate identification step.

- **Development of an online algorithm to learn stabilizing controllers from data for switched systems.**
  We present a novel data-driven algorithm for learning controllers applied to complex systems whose dynamics change over time. Our approach is *online*, meaning

that the data are collected over time while the system is evolving in closed-loop, and are directly used to iteratively update the controller. We demonstrate the capabilities of our approach in stabilizing switched linear systems with unknown subsystems dynamics and unknown switching signals.

- **Design of stabilizing controllers from data for nonlinear systems.**
  We design data-driven controllers from data for unknown systems with nonlinear dynamics. In this scenario, we make use of a *dictionary* of nonlinear terms that includes the nonlinearities of the unknown system. We derive conditions to design controllers via (approximate) nonlinearity cancellation, where the designed controllers discover the nonlinear terms and cancel them out automatically. We show that the resulting controllers can be certified to stabilize the system even when it is affected by perturbations and neglected nonlinearities.

1.3. Thesis outline

The work of this thesis is presented in four chapters, Chapters 2-5, and is finalized with some conclusions and final suggestions for future research, Chapter 6. In Chapter 2 we introduce the notation and some mathematical concepts that are at the basis of the results achieved in this thesis. The adopted data-driven framework is also introduced, which form the foundation of this thesis. Chapters 3-5 are the core of this thesis, whose contents can be summarized as follows:

Chapter 3
Building up on the data-driven framework in [40], we first consider the *finite-horizon* LQR problem for linear time-invariant discrete-time (unknown) systems. In this context, we first formulate the LQR problem as a covariance optimization problem. Then, we show how this formulation can be restated in terms of a semidefinite program. By considering a parametrization of the feedback system, we obtain a pure data-driven formulation. This procedure allows the development of semidefinite programming formulations only depending on data with no intermediate identification step.

The contents of this chapter have been published in:

[69] M. Rotulo, C. De Persis, and P. Tesi. "Data-driven linear quadratic regulation via semidefinite programming." *IFAC-PapersOnLine* 53.2: 3995-4000, Elsevier, 2020.

Chapter 4
Here, we shift our attention to *nonlinear* scenarios. Specifically, in this chapter, we develop novel data-dependent algorithms to learn controllers for complex unknown system whose dynamics change over time. The problem poses non-trivial challenges that are hard to address with conventional schemes. In fact, a major challenge is how to capture any changes in the dynamics of the system and adjust the controller accordingly to achieve stabilization of the unknown system. To this end, we utilize the concept of *online learning*. Specifically, we do not design controllers using a finite number of pre-collected data; rather, we propose a novel scheme which uses data generated online. By collecting the data in an online fashion, the controller is able to infer from data any changes in the operating condition

of the plant and adjust itself accordingly to stabilize the running dynamics. We formally demonstrate the capabilities of our approach in stabilizing switched linear systems with unknown subsystems dynamics and unknown switching signals.

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[70] M. Rotulo, C. De Persis, and P. Tesi. "Online data-driven stabilization of switched linear systems." 2021 *European Control Conference* (ECC), IEEE, 2021.

[71] M. Rotulo, C. De Persis, and P. Tesi. "Online learning of data-driven controllers for unknown switched linear systems." *Automatica*, 2022, accepted for publication.

Chapter 5

In this chapter, we derive a data-driven control method for stabilizing *nonlinear* systems *via nonlinearity cancellation*. To this end, we assume to know a vector-based function that contains the nonlinear terms of the unknown system dynamics. This allows us to consider an equivalent representation of the system, and in turn, provide a data-based representation of it. We then use such data-based formulation to design controllers that stabilize the closed-loop dynamics by canceling out the nonlinearities. The derived conditions take the compact form of data-dependent linear matrix inequalities. This way, the proposed approach returns formulas for controller design which retain the same simplicity of the formulas for linear systems. When exact nonlinearity cancellation is not achievable, the controller design is approached as a minimization problem, which consists in finding a controller that minimizes the nonlinearity with respect to some chosen norm. We then show that the proposed results can be extended in different directions. Firstly, we consider the case in which data are corrupted by noise. In particular, we study both deterministic and stochastic perturbations on the data. Secondly, we consider the case in which the combination of known nonlinear functions does not include all the nonlinearities present in the system. Finally, the results are also extended to systems with nonlinearities that are not expressible as combination of known functions, thus significantly enlarging the class of nonlinear systems the approach can cope with.

The contents of this chapter have been submitted for publication as:

[72] C. De Persis, M. Rotulo, and P. Tesi. "Learning Controllers from Data via Approximate Nonlinearity Cancellation." *IEEE Transactions on Automatic Control*, 2022, under review.

2

Preliminaries on data-driven control

In this chapter, we first briefly present some mathematical concepts and preliminary results about the adopted data-driven framework, which form the foundation of the results presented in this thesis.

2.1. General notation

In this section we define the general notation that we will use throughout the thesis.

We denote the set of integers, non-negative and positive integers by $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_{>0}$, respectively. Vectors and matrices are denoted by $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$, respectively. The transpose operator is denoted by $x^\top$, the inverse of a matrix is denoted by A^{-1}, and the Moore-Penrose inverse of a matrix is denoted by $A^\dagger$. For a square matrix $A \in \mathbb{R}^{n\times n}$, we denote by $\mathbf{Tr}(A)$ its trace. In addition, given a matrix A, the notations $A \succ 0$ and $A \succeq 0$ respectively denote that $A = A^\top \in \mathbb{R}^{n\times n}$ is positive definite and semi-definite. We denote with $\mathbb{S}^{n\times n}$ the set of real-valued symmetric matrices of dimension $n \times n$.

We use the notation $w \sim \mathcal{N}(0, W)$ to represent a zero-mean Gaussian random vector such that $\mathbf{E}[w] = 0$ and $\mathbf{E}[w\,w^\top] = W$, where $\mathbf{E}$ denotes the expectation. Furthermore, note that $\mathbf{E}[v^\top v] = \mathbf{Tr}(\mathbf{E}[v\,v^\top])$ for some vector v.

For simplicity of the notation, we write $[k, r]$ to denote the discrete interval $[k, r] \cap \mathbb{Z}$. The standard Euclidean norm is denoted by $\|\cdot\|$.

2.2. Preliminaries

This section provides some preliminaries on linear algebra and presents the notion of persistence of excitation. These are the two pillars of data-driven control.

Linear algebra

The data-driven framework allows approaching system and control problems using basic linear algebra. It leads to general, simple, and practical solution methods. In this section, we review few results from linear algebra useful for our work. The interested reader is referred to the book [14] for a complete overview of the subject.

Given a matrix A, we define its *rank*, denoted by $\operatorname{rank}(A)$, as the number of linearly independent columns of A. It follows that the number of linearly independent columns must equal the number of linearly independent rows. Hence, for any matrix $A \in R^{m\times n}$, $\operatorname{rank}(A) = \operatorname{rank}(A^\top)$ and $\operatorname{rank}(A) \leq \min(m,n)$. If $\operatorname{rank}(A) = \min(m,n)$, then A is said to be full rank. An important property of the rank of the product of two matrices is stated in the following lemma.

Lemma 2.1 (Sylvester's inequality) *Consider the matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, then*

$$\operatorname{rank}(A) + \operatorname{rank}(B) - n \leq \operatorname{rank}(AB) \leq \min\big(\operatorname{rank}(A), \operatorname{rank}(B)\big).$$

•

With the definition of the rank of a matrix we can now define the four fundamental subspaces related to a matrix. The *image* space (range) of a matrix $A \in \mathbb{R}^{m\times n}$ is denoted by $\operatorname{im} A$ and defined as

$$\operatorname{im} A := \{v \in \mathbb{R}^m : \ v = Ax, \ x \in \mathbb{R}^n\}.$$

Similarly, the *row* space of A is denoted by $\operatorname{im} A^\top$. The *kernel* (null) space of A is denoted by $\ker A$ and defined as

$$\ker A := \{x \in \mathbb{R}^n : \ Ax = 0\}.$$

The *left kernel* of A is defined as $\ker A^\top$. The kernel space answers the question of uniqueness of solutions to $Ax = b$. Given vectors $x, y \in \mathbb{R}^n$, if $Ax = b$ and $Ay = b$ then $A(x-y) = 0$ and thus $(x-y) \in \operatorname{im} A$. Hence, a solution to $Ax = b$ is unique if and only if $\operatorname{im} A = \{0\}$.

Persistence of excitation

A central question in data-driven control is how to replace process models with data. For linear systems, there is actually a key result, which answers this question, proposed by Willems et *al.* [20]. An important implication of this result is that a single, *sufficiently exciting* trajectory of a linear system can be used to parameterize *all* trajectories that the system can produce. Central to this implication is the notion of *persistence of excitation*, which is recalled via the following definitions.

Definition 2.1 (see [20]) *Given a signal $z : \mathbb{Z} \to \mathbb{R}^\sigma$ and a positive integer T, we define the following matrix as*

$$Z_{i,\ell,j} := \begin{bmatrix} z(i) & \cdots & z(i+j-1) \\ \vdots & \ddots & \vdots \\ z(i+\ell-1) & \cdots & z(i+\ell+j-2) \end{bmatrix}$$

with $\ell \in \mathbb{N}_{>0}$ and $j := T - \ell + 1$. If $\ell = 1$, we denote the matrix as

$$Z_{i,T} := \begin{bmatrix} z(i) & z(i+1) & \cdots & z(i+T-1) \end{bmatrix}.$$

•

Definition 2.2 (see [20]) *A signal* $\{z(0),\ldots,z(T-1)\} \in \mathbb{R}^\sigma$ *is said to be persistently exciting of order* $L \in \mathbb{N}_{>0}$ *if the matrix* $Z_{0,L,T-L+1}$ *has full rank* σL. •

For a signal to be persistently exciting of order L, it must be sufficiently long in the sense that $T \geq (\sigma+1)L-1$. For further discussion on the types of persistently exciting signals the interested reader is referred to [14, Section 10].

2.3. Data-driven modeling and control

In this section, we introduce fundamental concepts and results about data-driven modeling and control, that are used as a basis for obtaining some of the results in this thesis.

Based on the fundamental results in [20, 40], we introduce the data-based framework for the representation and control of linear time-invariant systems.

Consider the discrete-time linear system

$$x(k+1) = Ax(k) + Bu(k), \quad k \in \mathbb{N} \tag{2.1}$$

where $x(k) \in \mathbb{R}^n$ is the state and $u(k) \in \mathbb{R}^m$ is the control input. Let the pair (A, B) be controllable. During an experiment of duration $T > 0$, a sequence $\{u(0),\ldots,u(T-1)\}$ of inputs is applied to the system and the corresponding values $\{x(0),\ldots,x(T)\}$ of the state response are measured. Bear in mind that these are *offline* data. These input-state data are organized in data matrices as

$$\begin{aligned} U_{0,T} &:= \begin{bmatrix} u(0) & u(1) & \ldots & u(T-1) \end{bmatrix}, \\ X_{0,T} &:= \begin{bmatrix} x(0) & x(1) & \ldots & x(T-1) \end{bmatrix}, \\ X_{1,T} &:= \begin{bmatrix} x(1) & x(2) & \ldots & x(T) \end{bmatrix}. \end{aligned}$$

A main observation that emerges from [40] is that system (2.1) can be fully parametrized in terms of data provided the following condition is satisfied:

$$\operatorname{rank} \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} = m + n. \tag{2.2}$$

Condition (2.2) guarantees that any T-long input-state trajectory of the system can be expressed as a linear combination of the collected input-state data. It is possible to guarantee (2.2) when persistently exciting inputs are injected to the system. We introduce the following results, which are key for the developments of the formulas for data-driven control.

Lemma 2.2 [20, Corollary 1] *Suppose that system* (2.1) *is controllable. If the input signal* $\{u(0),\ldots,u(T-1)\}$ *is persistently exciting of order* $n+1$*, then condition* (2.2) *holds.* •

The next result gives a data-based representation of a linear system.

Theorem 2.1 [40, Theorem 1] *Let condition* (2.2) *hold. Then, system* (2.1) *has the following equivalent representation*

$$x(k+1) = X_{1,T} \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix}^\dagger \begin{bmatrix} u(k) \\ x(k) \end{bmatrix}. \tag{2.3}$$

•

It turns out that condition (2.2) also enables a data-based parametrization of all stabilizing state feedback controllers in the form $u = Kx$.

Theorem 2.2 [40, Theorem 2] *Let condition* (2.2) *hold. Then, system* (2.1) *in closed loop with a state feedback $u = Kx$ has the following equivalent representation:*

$$x(k+1) = X_{1,T} G_K x(k) \tag{2.4}$$

where $G_K \in \mathbb{R}^{T \times n}$ satisfies

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} G_K. \tag{2.5}$$

In particular,

$$u(k) = U_{0,T} G_K x(k). \tag{2.6}$$

•

Theorem 2.2 is very appealing from the perspective of computing the control law from data, as it enables the design of controllers without the intermediate step of estimating the model. In fact, one can regard G_K as a *decision variable*, and search for the matrix G_K that guarantees stability and performance specifications.

In the context of *linear quadratic regulation* (LQR), we can formulate the LQR problem as an $\mathcal{H}_2$ problem and derives a data-based solution based on convex programming. Specifically, consider the problem of designing a state feedback controller K that renders $A + BK$ Schur and minimizes

$$\mathbf{Tr}(W_x P) + \mathbf{Tr}(W_u K P K^\top), \tag{2.7}$$

where $W_x, W_u \succ 0$ are weighting matrices and P is the unique solution to

$$(A + BK)P(A + BK)^\top - P + I = 0. \tag{2.8}$$

It is known [73, Sec. 6.4] that the state feedback controller minimizing (2.7), here denoted by K_{opt}, is unique. The work in [41] establishes that K_{opt} can be parametrized directly in terms of data. Specifically, the following semidefinite program (SDP)[1] is formulated:

$$\begin{array}{rl} \min_{\gamma,Q,P,L} & \gamma \\ \text{subject to} & X_{1,T}\, Q P^{-1} Q^\top\, X_{1,T}^\top - P + I \preceq 0 \\ & P \succeq I \\ & X_{0,T} Q = P \\ & L - U_{0,T}\, Q P^{-1} Q^\top\, U_{0,T}^\top \succeq 0 \\ & \mathbf{Tr}(W_x P) + \mathbf{Tr}(W_u L) \leq \gamma \end{array} \tag{2.9}$$

which is only based on data.

[1] With some abuse of terminology we refer to (2.9) and subsequent derivations as an SDP, with the understanding that by using standard manipulations they can be written as SDP.

Lemma 2.3 [41, Theorem 1] *Let condition* (2.2) *holds. Then problem* (2.9) *is feasible. Also, any optimal solution* $(\gamma_o, Q_o, P_o, L_o)$ *satisfies* $K_{opt} = U_{0,T} Q_o P_o^{-1}$*, where* K_{opt} *is the unique state feedback controller that minimizes* (2.7). •

Lemma 2.3 establishes that problem (2.9) is an equivalent data-based formulation of the classic LQR problem, formulated as an $\mathcal{H}_2$-norm minimization problem. We remark that by "equivalent" we mean both problems yield the same solution. For a discussion on the properties related to this formulation the interested reader is referred to [41].

3

Data-driven linear quadratic regulation via semidefinite programming

A core problem of learning control is to determine *optimal* feedback controllers for unknown systems from experimental data. In this context, the canonical linear quadratic regulator (LQR) problem in control theory has reemerged as an important theoretical benchmark for this kind of problem. In this chapter, we focus on the finite-horizon linear quadratic regulation problem. In our problem setup, the dynamics of the system are assumed to be unknown and the state is accessible. Information on the system is given by a finite set of input-state data, where the input injected in the system is persistently exciting of a sufficiently high order. Using data, the optimal control law is then obtained as the solution of a suitable data-based semidefinite program. The effectiveness of the approach is illustrated via numerical examples.

3.1. Introduction

Optimization and control have always been closely related when there is need for operating a dynamical system at minimum cost. When the system is linear and the cost function is quadratic, the optimal control problem amounts to solving the popular Linear-Quadratic Regulation (LQR) problem [74], whose duality with convex optimization is shown in [75] for continuous-time systems and more recently in [76, 77] for discrete-time systems. The LQR problem is one of the most fundamental and well-studied problems in optimal control, and has recently aroused renewed interest in the context of data-driven control.

The common assumption for deriving the solution for optimal control problems is that an exact model of the system is available. To cope with the lack of prior knowledge of the system dynamics, various control techniques have been developed. The classic approach is *model-based*: a model is first determined from data, and then the control law is designed using the model. It is worth to mention that the control objectives are not taken into account in the identification step and, once the model is derived from data, the data is not used in the synthesis of the control law. More recently, as opposed to the model-based paradigm, *data-driven* control has become increasingly popular. In the context of optimal control, various efforts have been made [39, 78–80]. Data-driven optimal control has also been approached using popular machine learning tools such as Reinforcement Learning. In Reinforcement Learning [18, 26, 81], also referred to as approximate dynamic programming [82, 83] within the control community, the controller learns an optimal policy through *trial-and-error*, trying to estimate a long-term value function. Reinforcement Learning has proven to be a promising framework [84, 85], yet often requires performing many experiments on the physical system to even find suitable controllers, which limits the applicability of such techniques.

In this chapter, we consider the *finite-horizon* LQR problem for linear time-invariant discrete-time systems. Different from the mainstream approaches based on Reinforcement Learning-based techniques [86–88], the approach we propose does not involve iterations. We formulate the LQR problem as a *one-shot* semidefinite program in which the model of the system is replaced by a finite number of data collected from the system. The method proposed in this chapter recovers the infinite-horizon solution in a very natural way, which was investigated in [40].

Our approach is based on the framework developed in [40], whose foundation lies on the *fundamental lemma* by [20]. Roughly speaking, the *fundamental lemma* stipulates that one can describe all possible trajectories of a linear time-invariant system using any given finite set of its input-output data, provided that these data come from sufficiently excited dynamics. This result thus establishes that data implicitly give a *non-parametric* system representation which can be directly used for control design.

The chapter is organized as follows: Section 3.2 briefly reviews the model-based finite-horizon LQR solution. Then, the LQR problem is reformulated as a convex optimization problem involving linear matrix inequalities [89], resulting in a semidefinite program (SDP). In Section 3.3, we show that the data-based parametrization introduced in [40] combined with the SDP formulation of the LQR problem results in a direct parametrization of the feedback system through data. In turn, this makes it possible to determine the optimal control law in one-shot, with no intermediate identification step. Numerical examples are discussed in Section 3.4. The chapter ends with some concluding remarks in Section 3.5.

3.2. Framework

We consider a discrete-time linear system

$$x(k+1) = Ax(k) + Bu(k) \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state while $u \in \mathbb{R}^m$ is the control input, and where A and B are matrices of an appropriate dimension. It is assumed throughout the chapter that (A, B) is controllable and the state is available for measurements.

3.2.1. Finite-horizon LQR problem

Given an initial condition $x(0) = x_0$ and a control sequence $\{u(0),\ \ldots,\ u(N-1)\}$ over the horizon $N \in \mathbb{N}$, we consider the quadratic cost J associated to system (3.1) starting at

x_0,

$$J := x(N)^\top Q_f x(N) + \sum_{k=0}^{N-1} \rho(u(k), x(k)) \tag{3.2}$$

where

$$\rho(u, x) = x^\top Q_x x + u^\top R u,$$

where $Q_x, Q_f \succeq 0$ and $R \succ 0$. The finite-horizon linear quadratic regulator (LQR) problem is as follows:

Problem 3.1 *Given system* (3.1) *with initial condition x_0, and given a time horizon of length N, find an input sequence such that the cost function* (3.2) *is minimized, i.e. solve the minimization problem:*

$$\begin{aligned} \min_{\mu_k} \quad & J \\ \text{subject to} \quad & x(k+1) = Ax(k) + Bu(k) \\ & u(k) = \mu_k(x(0), x(1), \ldots, x(k)). \end{aligned}$$

Here, the last constraint means that the control input is a causal function of the system state.

The following result holds.

Lemma 3.1 *For Problem 3.1, the optimal control sequence*

$$\{u^*(0), \ \ldots, \ u^*(N-1)\}$$

is unique, and it is generated by the feedback law

$$u^*(k) = K^*(k)x(k) \tag{3.3}$$

where

$$K^*(k) := -(R + B^\top P(k+1)B)^{-1} B^\top P(k+1)A \tag{3.4}$$

where $P(k)$ is the solution to the so-called discrete-time difference Riccati equation

$$P(k) = Q_x + A^\top P(k+1)A - A^\top P(k+1)B(R + B^\top P(k+1)B)^{-1} B^\top P(k+1)A$$

initialized from $P(N) = Q_f$.

Proof. See for instance [90]. ∎

The computed control law (3.4) is time-varying and defined in the interval $[0, N]$. However, the computation of the gain $K^*(k)$ does not require the knowledge of the current state, and can be computed offline. For $N \to \infty$, if the pair (Q_x, A) is observable, the sequence of the matrices $P(k)$ converges to a matrix P, which is the so-called *stabilizing* solution of the discrete-time algebraic Riccati equation

$$P = Q_x + A^\top PA - A^\top PB(R + B^\top PB)^{-1} B^\top PA.$$

In this case, the optimal control for the infinite-horizon problem is a time-invariant state-feedback $u(k) = K^* x(k)$ with $K^* = -(R + B^\top PB)^{-1} B^\top PA$.

Remark 3.1 *Here, similarly to the data-driven infinite-horizon LQR problem studied in [40], we have assumed that the pair (A, B) is controllable. This assumption ensures that we can always collect sufficiently rich data by applying exciting input signals. As shown in [55], except for pathological cases, data richness is indeed necessary for the data-driven solution of the LQR problem. On the other hand, data richness is also necessary for reconstructing the system matrices A and B from data, thus necessary also for the model-based solution whenever A and B have to be identified from data.*

3.2.2. Solution as a covariance optimization problem

We introduce an equivalent formulation of the LQR problem, where by "equivalent" we mean that the corresponding optimal solution is still given by (3.3).

Consider the linear quadratic stochastic control problem [76]:

$$\begin{aligned} \min_{\mu_k} \quad & \mathbf{E}[J] \\ \text{subject to} \quad & x(k+1) = Ax(k) + Bu(k) + w(k) \\ & x(0) \sim \mathcal{N}(0, I_n) \\ & w(k) \sim \mathcal{N}(0, I_n) \\ & \mathbf{E}[w(k)x^\top(l)] = 0, \quad \forall l \leq k \\ & u(k) = \mu_k(x(0), x(1), \ldots, x(k)) \end{aligned} \tag{3.5}$$

with J as in (3.2). As detailed in [76], this problem is equivalent to the covariance selection problem

$$\begin{aligned} \min_{V(0),\ldots,V(N)\succeq 0} \quad & \mathbf{Tr}\left(Q_f S(N)\right) + \textstyle\sum_{k=0}^{N-1} \mathbf{Tr}(Q_x S(k) + RU(k)) \\ \text{subject to} \quad & S(0) = I_n \\ & S(k+1) - \begin{bmatrix} A & B \end{bmatrix} V(k) \begin{bmatrix} A & B \end{bmatrix}^\top - I_n = 0 \end{aligned} \tag{3.6}$$

for $k = 0, \ldots, N-1$, where

$$V(k) = \begin{bmatrix} S(k) & Y(k) \\ Y(k)^\top & U(k) \end{bmatrix} := \mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^\top. \tag{3.7}$$

In particular, the following result holds.

Theorem 3.1 [76, Theorem 2] *The optimal solution of the covariance selection problem* (3.6) *is given by*

$$\begin{aligned} & S(0) = I_n \\ & Y(k)^\top = L(k)S(k) \\ & U(k) = Y(k)^\top S^{-1}(k) Y(k) \\ & S(k+1) = \begin{bmatrix} A & B \end{bmatrix} V(k) \begin{bmatrix} A & B \end{bmatrix}^\top + I_n. \end{aligned}$$

The corresponding optimal control law is $u(k) = L(k)x(k)$ with $L(k) = K^*(k)$ as in (3.3).

3.3. DATA-DRIVEN LQR VIA SEMIDEFINITE PROGRAMMING

Building on the formulation described in Section 3.2.2, it is possible to derive a simple solution to the LQR problem where the system matrices A and B are replaced by data. We first show how the covariance optimization problem can be restated in terms of a semidefinite program. Then, we consider a parametrization of the feedback system which results in a pure data-driven formulation.

3.3.1. SEMIDEFINITE PROGRAMMING FORMULATION

The following result holds.

Theorem 3.2 *The optimal control law for problem* (3.6) *can be computed as the solution* $\mathcal{K}$ *to the problem*

$$\begin{array}{ll} \min_{\mathcal{S},\mathcal{K},\mathcal{Z}} & \mathbf{Tr}\left(Q_f S(N)\right) + \sum_{k=0}^{N-1} \mathbf{Tr}(Q_x S(k) + Z(k)) \\ \text{subject to} & S(0) \succeq I_n \\ & S(k+1) - (A+BK(k))S(k)(A+BK(k))^\top - I_n \succeq 0 \\ & Z(k) - R^{1/2}K(k)S(k)K(k)^\top R^{1/2} \succeq 0 \end{array} \tag{3.8}$$

for $k = 0, \ldots, N-1$, *where*

$$\begin{aligned} \mathcal{S} &:= \{S(1), \ldots, S(N)\}, \\ \mathcal{K} &:= \{K(0), \ldots, K(N-1)\}, \\ \mathcal{Z} &:= \{Z(0), \ldots, Z(N-1)\}. \end{aligned}$$

Proof. Exploiting the fact that the optimal control law takes the form $u(k) = K(k)x(k)$, the term $\mathbf{Tr}(RU(k))$ appearing in the objective function of (3.6) can be written as

$$\begin{aligned} \mathbf{Tr}(RU(k)) &= \mathbf{Tr}\left(R\,\mathbf{E}[K(k)x(k)x(k)^\top K(k)^\top]\right) \\ &= \mathbf{Tr}\left(R^{1/2}K(k)S(k)K(k)^\top R^{1/2}\right). \end{aligned}$$

In addition,

$$V(k) = \begin{bmatrix} I \\ K(k) \end{bmatrix} S(k) \begin{bmatrix} I \\ K(k) \end{bmatrix}^\top$$

so that the second constraint of (3.6) becomes

$$S(k+1) - (A+BK(k))S(k)(A+BK(k))^\top + I_n = 0.$$

Accordingly, the optimization problem (3.6) is equivalent to the following problem:

$$\begin{array}{ll} \min_{\mathcal{S},\mathcal{K},\mathcal{Z}} & \mathbf{Tr}(Q_f S(N)) + \sum_{k=0}^{N-1} \mathbf{Tr}(Q_x S(k) + Z(k)) \\ \text{subject to} & S(0) = I_n \\ & S(k+1) - (A+BK(k))S(k)(A+BK(k))^\top - I_n = 0 \\ & Z(k) - R^{1/2}K(k)S(k)K(k)^\top R^{1/2} = 0 \end{array} \tag{3.9}$$

Finally, let $(\overline{\mathcal{S}}, \overline{\mathcal{K}}, \overline{\mathcal{Z}})$ be an optimal solution to problem (3.8) and let $(\mathcal{S}^*, \mathcal{K}^*, \mathcal{Z}^*)$ be an optimal solution to (3.9), where $\mathcal{K}^* = \{K^*(0), \dots, K^*(N-1)\}$ is the optimal sequence of state-feedback matrices given by (3.3). Also, denote by $\overline{J}$ and J^* the corresponding costs. Clearly $\overline{J} \leq J^*$ since (3.8) has a larger feasible set than (3.9). To prove the converse inequality, first note that $\overline{S}(k) \succeq S^*(k)$ and $\overline{Z}(k) \succeq Z^*(k)$ for all $k \geq 0$. This follows because $\overline{S}(0) \succeq S^*(0)$ and since $\overline{S}(k) \succeq S^*(k)$ implies

$$
\begin{aligned}
&(A + BK(k)) \left(\overline{S}(k) - S^*(k)\right) (A + BK(k))^\top \succeq 0 \\
&R^{1/2} K(k) \left(\overline{S}(k) - S^*(k)\right) K(k)^\top R^{1/2} \succeq 0.
\end{aligned}
$$

Substituting $(\overline{\mathcal{S}}, \overline{\mathcal{Z}})$ in (3.8) and $(\mathcal{S}^*, \mathcal{Z}^*)$ in (3.9), we thus have $\overline{J} \geq J^*$ so that $\overline{J} = J_*$. In turn, this implies $\overline{\mathcal{K}} = \mathcal{K}^*$ since the optimal control law achieving J^* is unique. ■

Defining $H(k) = K(k)S(k)$ and using the property that $S(k) \succeq I_n$ for every $k \geq 0$, problem (3.8) can be converted into a semidefinite program. The idea of resorting to SDP formulations has been originally proposed in [91] in the context of model-based LQR, and considered in [40] in the context of data-driven infinite-horizon LQR.

3.3.2. Data-driven parametrization of LQR

The formulation (3.8) is very appealing from the perspective of computing the control law using data only since the decision variables $\mathcal{S}$, $\mathcal{K}$ and $\mathcal{Z}$ appearing in (3.8) enter the problem in a form which permits to write the constraints as data-dependent linear matrix inequalities.

Our approach uses the concept of persistence of excitation, whose notion has been introduced in Definition 2.2. Consider system (3.1),

$$
x(k+1) = Ax(k) + Bu(k)
$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Suppose that we carried out an experiment of duration $T > 0$, where a sequence $\{u(0), \dots, u(T-1)\}$ of inputs is applied to the system and the corresponding values $\{x(0), \dots, x(T)\}$ of the state response are measured. These input-state data are organized in data matrices as

$$
\begin{aligned}
U_{0,T} &:= \begin{bmatrix} u(0) & u(1) & \dots & u(T-1) \end{bmatrix} \\
X_{0,T} &:= \begin{bmatrix} x(0) & x(1) & \dots & x(T-1) \end{bmatrix} \\
X_{1,T} &:= \begin{bmatrix} x(1) & x(2) & \dots & x(T) \end{bmatrix}.
\end{aligned}
\tag{3.10}
$$

Suppose that system (3.1) is controllable. Assume now that the input sequence $\{u(0), \dots, u(T-1)\}$ is persistently exciting of order $n+1$. Notice that the only requirement on T is that $T \geq (m+1)n + m$, which is necessary for the persistence of excitation condition to hold. By Lemma 2.2,

$$
\operatorname{rank} \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} = n + m. \tag{3.11}
$$

We remark that condition (3.11) expresses the property that the data content is sufficiently rich. This enables the data-driven solution of the LQR problem.

As shown in Chapter 2, one can use condition (3.11) for parametrizing an arbitrary feedback interconnection. In fact, consider an arbitrary matrix $K(k)$, possibly time-varying, of dimension $m \times n$. By the Rouché-Capelli theorem, there exists a $T \times n$ matrix $G(k)$ solution to

$$\begin{bmatrix} K(k) \\ I_n \end{bmatrix} = \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} G(k). \tag{3.12}$$

Accordingly the closed-loop system formed by system (3.1) with $u(k) = K(k)x(k)$ is such that

$$\begin{aligned} A + BK(k) &= \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K(k) \\ I_n \end{bmatrix} \\ &= \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} G(k) \\ &= X_{1,T} G(k) \end{aligned} \tag{3.13}$$

where we used the identity $X_{1,T} = AX_{0,T} + BU_{0,T}$.

Using this result, one can provide a data-based formulation of problem (3.8).

Theorem 3.3 *Consider system* (3.1) *along with an experiment of length* $T > 0$ *resulting in input and state data* $\{u(0), \ldots, u(T-1)\}$ *and* $\{x(0), \ldots, x(T)\}$*, respectively. Let the matrices* $U_{0,T}$*,* $X_{0,T}$ *and* $X_{1,T}$ *be as in* (3.10)*, and suppose that the rank condition* (3.11) *holds. Then, the optimal solution to problem* (3.8)*, hence to Problem 1, is given by*

$$\mathcal{K} := \{K(0), \ldots, K(N-1)\}$$

with

$$K(k) = U_{0,T} Q(k) S^{-1}(k)$$

where the matrices $Q(k)$ *and* $S(k)$ *solve the optimization problem*

$$\begin{aligned} \min_{\mathcal{S},\mathcal{Q},\mathcal{Z}} \quad & \mathbf{Tr}(Q_f S(N)) + \textstyle\sum_{k=0}^{N-1} \mathbf{Tr}(Q_x S(k) + Z(k)) \\ \text{subject to} \quad & S(0) \succeq I_n \\ & S(k) = X_{0,T} Q(k) \\ & \begin{bmatrix} S(k+1) - I_n & X_{1,T} Q(k) \\ Q^\top(k) X_{1,T}^\top & S(k) \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} Z(k) & R^{1/2} U_{0,T} Q(k) \\ Q(k)^\top U_{0,T}^\top R^{1/2} & S(k) \end{bmatrix} \succeq 0 \end{aligned} \tag{3.14}$$

for $k = 0, \ldots, N-1$, *where*

$$\begin{aligned} \mathcal{S} &:= \{S(1), \ldots, S(N)\} \\ \mathcal{Q} &:= \{Q(0), \ldots, Q(N-1)\} \\ \mathcal{Z} &:= \{Z(0), \ldots, Z(N-1)\}. \end{aligned} \tag{3.15}$$

Proof. We show that the constraints of (3.8) can be written as in (3.14). To this end, first note that the parametrization (3.13) implies that the second constraint of (3.8) can also be written as

$$S(k+1) - X_{1,T} G(k) S(k) G(k)^\top X_{1,T}^\top - I_n \succeq 0$$

where $G(k)$ satisfies (3.12). Let now

$$Q(k) := G(k) S(k). \tag{3.16}$$

Exploiting the fact that $S(k) \succeq I_n$ for every $k \geq 0$ the second constraint of (3.8) becomes

$$S(k+1) - X_{1,T} Q(k) S^{-1}(k) Q(k)^\top X_{1,T}^\top - I_n \succeq 0$$

which is equivalent to the third constraint in (3.14). Along the same lines, the third constraint in (3.8) can be written as the fourth constraint in (3.14). Finally, the optimal solution $\mathcal{K} := \{K(0), \ldots, K(N-1)\}$, with $K(k) = U_{0,T} Q(k) S^{-1}(k)$, is obtained from the first one of (3.12) and (3.16). ■

Remark 3.2 *As* $N \to \infty$ *the solution converges to the infinite-horizon steady-state solution, which is stabilizing.*

3.4. Monte Carlo simulations

We consider both the semidefinite programs described in Theorem 3.2 (model-based) and Theorem 3.3 (data-driven) and compare their performance for randomly generated systems and for the batch reactor system.

3.4.1. Collection of randomly generated systems

We perform Monte Carlo simulations with $N_{trials} = 1000$ random systems with 3 states and 1 input. Simulations are performed in MATLAB. For each trial, the entries of the system matrices are generated using the command `randn` (normally distributed random number). For each trial, the data are generated by applying a random input sequence of length $T = 15$ and random initial conditions, again using the command `randn`. Using CVX [92], we solve the model-based program (3.8) and the data-driven program (3.14) for $N = 10$ steps with $Q_x = Q_f = I_3$ and $R = 1$, and measure the resulting optimal costs.

A shown in Figure 3.1, for each trial the data-driven solution achieves the same cost as the model-based solution, with an average error of order 10^{-7}. Also the sequence $\mathcal{K}_{dd}^*$ of data-driven feedback gains coincides with the sequence $\mathcal{K}_{mb}^*$ obtained by solving the model-based formulation, with an average error over the various gains of order 10^{-6} (Figure 3.2).

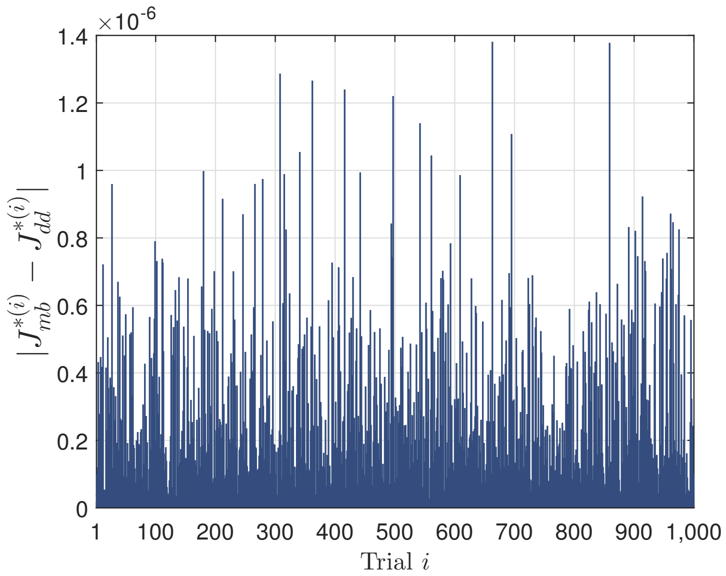

Figure 3.1: Absolute error between optimal cost $J^{*(i)}_{mb}$ achieved in model-based and optimal cost $J^{*(i)}_{dd}$ achieved in data-driven for the i-th trial.

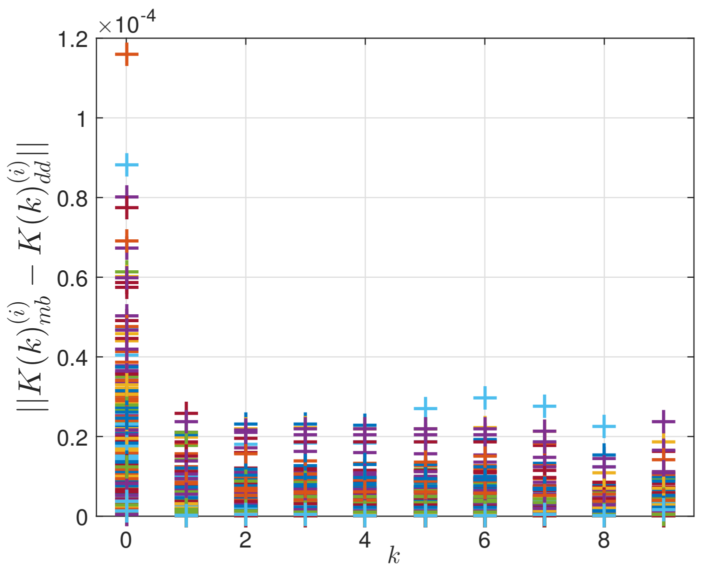

Figure 3.2: Values of the error (Euclidean norm) between the optimal model-based solutions $K(k)^{(i)}_{mb}$ and data-based solutions $K(k)^{(i)}_{dd}$ for the i-th trial, with $k = 0, \dots,\ N-1$.

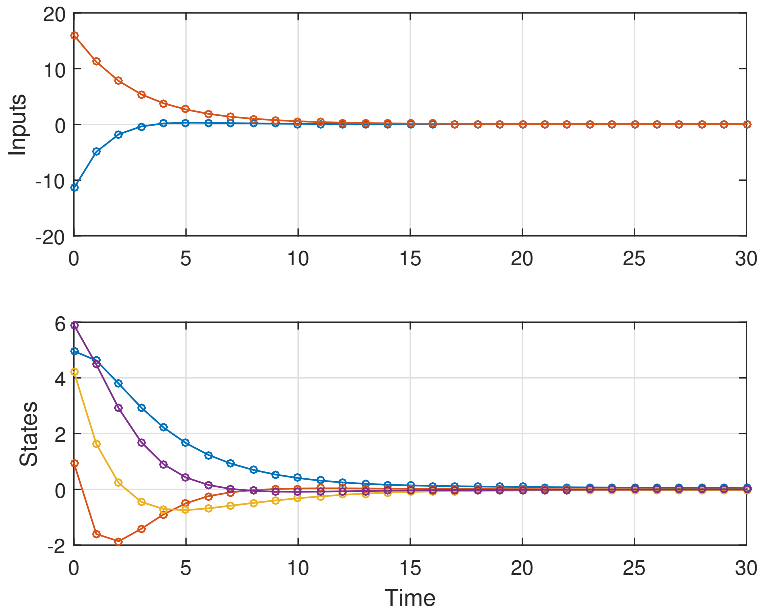

Figure 3.3: Optimal input sequence and state response with $N = 30$ for the batch reactor system.

3.4.2. Batch reactor

As a second example, we consider the discretized version of the batch reactor system [93], using a sampling time of $0.1s$,

$$\left[\begin{array}{c|c} A & B \end{array}\right] = \left[\begin{array}{rrrr|rr} 1.178 & 0.001 & 0.511 & -0.403 & 0.004 & -0.087 \\ -0.051 & 0.661 & -0.011 & 0.061 & 0.467 & 0.001 \\ 0.076 & 0.335 & 0.560 & 0.382 & 0.213 & -0.235 \\ 0 & 0.335 & 0.089 & 0.849 & 0.213 & -0.016 \end{array}\right]$$

which is open-loop unstable.

Under the same experimental conditions as in the previous example ($T = 15$, $N = 10$, $N_{trials} = 1000$), and taking cost weights $Q_x = Q_f = I_4$ and $R = I_2$, Monte Carlo simulations return an average error of order 10^{-3} for what concerns the discrepancy $|J^*_{dd} - J^*_{mb}|$, and an average error of order 10^{-3} for what concerns $\|\mathcal{K}^*_{dd} - \mathcal{K}^*_{mb}\|$.

As N grows the solution approximates the infinite-horizon steady-state solution, which is stabilizing. Figure 3.3 shows the closed-loop response with the data-driven solution for one experiment carried out with $N = 30$.

3.5. Conclusions

In this chapter, we considered a finite-horizon linear quadratic regulation problem, where the knowledge about the dynamics of the system is replaced by a finite set of input and state data collected from an experiment. We have shown that if the experiment is carried out with a sufficiently exciting input signal then the optimal solution can be computed only using the data, with no intermediate identification step, as the result of a data-dependent semidefinite programming problem.

An important continuation of this research line involves the extension of these results to the case where data are affected by noise, also in comparison with techniques based on system identification [63]. Concerning data-driven methods, previous efforts in this direction include [40] for stabilization in the presence of input disturbances and/or measurement noise, and [42], which considers robust performance (including H_∞ control as a special case) in the presence of input disturbances.

4

Online learning of data-driven controllers for switched linear systems

Motivated by the goal of learning controllers for complex systems whose dynamics change over time, we consider the problem of designing control laws for systems that switch among a finite set of unknown discrete-time linear subsystems under unknown switching signals. To this end, we propose a method that uses data to directly design a control mechanism without any explicit identification step. Our approach is *online*, meaning that the data are collected over time while the system is evolving in closed-loop, and are directly used to iteratively update the controller. A major benefit of the proposed online implementation is therefore the ability of the controller to automatically adjust to changes in the operating mode of the system. We show that the proposed control mechanism guarantees stability of the closed-loop switched linear system provided that the switching is slow enough. Effectiveness of the proposed design technique is illustrated for two aerospace applications.

4.1. Introduction

In its original formulation, the data-driven paradigm addresses the problem of learning controllers from data for unknown time-invariant linear systems, a family of dynamical systems that could be fully represented by a finite set of data. This batch of data is collected in some open-loop experiments before the system becomes operational, and it is later exploited in the design of the controller. This method is commonly implemented offline, and is particularly important for those systems in which real-time data collection is limited or too costly. In contrast, *online* data-driven techniques exploit real-time data obtained during controller operation to improve the representation of the system and the performance of the controller each time new information on the unknown system is collected. Several works have considered online approaches in various contexts, including self-tuning regulators [25, 94], policy iteration schemes [26, 95, 96] and the online linear quadratic regulation problem [97].

Online approaches are particularly suitable for controlling more complex scenarios in which a finite set of data may not capture the full complexity of the system. This is the case for systems with dynamics that evolve over time, where changes in a system's dynamics can occur due to faults or different operating modes and we have limited information about those changes. Unknown *switched systems* are a notable example of such a type of systems.

As a special class of switched systems, switched linear systems provide an attractive framework which bridges the gap between linear systems and more complex systems. Switched linear systems consist of a finite number of subsystems described by linear dynamics, together with a switching signal that coordinates the switching between these subsystems. Control and stabilization of this type of systems have been extensively studied in the literature [98–101], and the interested reader is referred to the survey paper [102] for a in-depth overview on the field of stability analysis and switching stabilization for switched systems.

When it comes to controlling *unknown* switched linear systems, the existing literature follows several directions based on the available information and characteristics of the switching signal. A line of research focuses on controlling unknown switched systems by assuming the switching signal to be part of the control law, see e.g. [103] and references therein. In another line of research, the switching signal is not assumed to be a free design variable, but determined by external commands. In this setup, the work [100] addresses the stabilization problem when the dynamics of the subsystems as well as the switching instances are known, but the switching signal is unknown.

Very recently, control of unknown switched linear systems has been approached by the data-driven control community [57, 104, 105]. Specifically, [104] proposes offline design of data-driven controllers based on input-output data. However, the proposed work is limited to a certain class of systems and closed-loop stability cannot be formally guaranteed. The work [105] presents a data-driven framework to design a controller for switched systems under arbitrary switching. There are two main limitations of this framework. First, it assumes having access to experimental data collected at different operating points of the system. Second, stabilization is ensured at the expense of assuming existence of a common polyhedral Lyapunov function. This leads to the formulation of computationally expensive non-convex optimization problems, whose computational burden can be reduced via an heuristic approach.

In this chapter, we present a novel *online* data-driven approach for learning controllers applied to complex systems whose dynamics change over time. Changes in a system's dynamics can occur due to faults or different operating modes, and switched systems effectively capture such behavior. Therefore, we demonstrate the capabilities of our approach in stabilizing switched linear systems with unknown modes and unknown switching signals. For these systems, data collected in an offline experiment may be inadequate to inform about all the modes of the switched systems and a mechanism that collect data online is better suited for the control task. In this way, the control mechanism is directly parametrized through these online data and iteratively updated via computationally tractable data-dependent semidefinite programs.

The main features of the proposed online implementation are as follows:

(i) The data generated online are *persistently exciting*. This is a key property in data-driven control which allows the designer to parametrize controllers as solutions of data-dependent semidefinite programs. In general, when a controller is placed in the feedback loop, the closed-loop data are not necessarily sufficiently exciting. This issue is addressed by including an additive term in the controller, and we formally show that suitable selection of this term can always preserve the persistence of excitation condition.

(ii) The proposed control mechanism is able to capture any changes in the dynamics of the system and adjust the controller accordingly. After a new operating mode is learned from data, the controller stabilizes the running subsystem. This results in *stabilization* of the overall closed-loop system, provided that the changes in the dynamics do not occur too frequently. This result shows the potential of the data-driven paradigm in solving problems that could not be solved using conventional control schemes.

The rest of the chapter is organized as follows. Section 4.2 introduces the problem under consideration. In Section 4.3, the online data-based control mechanism is presented and guarantees on the persistence excitation condition are established. In Section 4.4, stability results of the closed-loop switched system are presented. Various practical case studies are discussed in Section 4.5. The chapter ends with some concluding remarks in Section 4.6. Proofs of Lemmas are provided in Section 4.7

4.2. Framework

We consider the discrete-time switched linear system

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{4.1}$$

where $x(k) \in \mathbb{R}^n$ is the state and $u(k) \in \mathbb{R}^m$ is the control input. The switching signal $\sigma : \mathbb{N} \to \mathcal{I}$ is a piecewise constant function of time that selects its values in the finite set $\mathcal{I} := \{1, 2, \dots, M\}$, with $M > 1$ being the number of modes. Here, $(A_{\sigma(k)}, B_{\sigma(k)})$ are constant matrices of appropriate dimensions which are allowed to take values, at an arbitrary discrete time, in the finite set $\big\{(A_i, B_i) : i \in \mathcal{I}\big\}$.

Throughout this note, the following assumption holds.

Assumption 4.1 *The pairs (A_i, B_i) for $i \in \mathcal{I}$ are controllable.* •

Without loss of generality, we assume that at time $k_0 := 0$ the system undergoes no switching and we denote by k_s the time instant of the s-th switching, i.e $k_{s+1} := \min\{k > k_s : \sigma(k) \neq \sigma(k_s)\}$, where $s \in \mathbb{N}$. The active mode selected by $\sigma(k_s)$ is indicated by the index i, i.e.

$$i = \sigma(k), \quad k \in [k_s, k_{s+1} - 1].$$

We now formulate the following problem.

Problem 4.1 *Consider the switched system* (4.1). *The pairs (A_i, B_i), for all $i \in \mathcal{I}$, the switching signal $\sigma(\cdot)$ and the switching instants k_s with $s \in \mathbb{N}$ are all assumed to be unknown. Design a data-based feedback control law to ensure exponential stability of the closed-loop switched system.*

4.3. Online data-driven control

Inspired by the data-driven stabilization of linear systems, we aim to design a data-driven control mechanism for switched linear systems in the form of (4.1). Intuitively, one would naturally collect data from the system performing offline experiments at different modes of operation. However, this approach is not directly applicable when the number of modes as well as the switching signal are not available to the designer. Thus, we address Problem 4.1 by applying the data-driven framework in an *online* setting. By "online" we refer to the operation of collecting new data and accordingly modifying the control law while the system is evolving. In this way, the data-driven framework is used as a tool to adjust the controller to changes in the operating condition of the plant while running online.

We propose the following feedback control law:

$$u(k) = K(k)x(k) + \varepsilon(k)\|x(k)\|, \tag{4.2}$$

where $K(k) \in \mathbb{R}^{m\times n}$ is the state feedback gain and $\varepsilon(k) \in \mathbb{R}^m$ is an auxiliary input signal that belongs to the ball

$$B_\delta := \{\varepsilon \in \mathbb{R}^m : \|\varepsilon\| \leq \delta\}$$

for every k and some arbitrary $\delta > 0$.

At each time $k \geq 0$, we collect the measurements of the system in appropriate matrices of data. Note that the state response is generated according to (4.1) interconnected with (4.2). As it is not practically appealing, we do not want to increase the size of the data matrices every time new samples are measured, but we aim to keep the size fixed to a suitable length T. To this purpose, at each time $k \geq 0$, the following matrices of data are available:

$$\begin{aligned}
U_{k-1} :=& U_{k-T,T} \\
=& \begin{bmatrix} u(k-T) & u(k-T+1) & \dots & u(k-1) \end{bmatrix}, \\
X_{k-1} :=& X_{k-T,T} \\
=& \begin{bmatrix} x(k-T) & x(k-T+1) & \dots & x(k-1) \end{bmatrix}, \\
X_k :=& X_{k-T+1,T} \\
=& \begin{bmatrix} x(k-T+1) & x(k-T+2) & \dots & x(k) \end{bmatrix}.
\end{aligned}$$

In the above definitions, we shift the window of the dataset one-step ahead, where an old data sample is discarded each time a new one is added. Note that if the index of the sample is negative, it refers to data obtained from some offline *open-loop* experiments, that is without having (4.2) in the loop. In particular, we apply to system (4.1) an initial input sequence $\{u(-T), \dots, u(-1)\}$ and collect the corresponding state sequence $\{x(-T), \dots, x(0)\}$. Hence, at time $k = 0$ we construct the initial matrices of data X_{-1}, U_{-1}, X_0.

4.3.1. Persistence of excitation

Throughout the Chapter, the following condition plays an important role:

$$\operatorname{rank}\begin{bmatrix} U_{k-1} \\ X_{k-1} \end{bmatrix} = m + n. \tag{4.3}$$

Condition (4.3) guarantees that as long as the T-long data matrices U_{k-1}, X_{k-1} are generated by a single controllable subsystem, they encode all the information regarding the dynamics of that subsystem. On the path of guaranteeing this rank condition, inspired by Lemma 2.2, we require the sequence $\{u(k-T), \ldots, u(k-1)\}$ to be persistently exciting of order $n+1$ for any k. Note that, in general, without the auxiliary input ε in the structure of (4.2), the persistence of excitation condition on the input sequence would not necessarily hold. The reason is that the input signal at each time k would be merely restricted to $u(k) = K(k)x(k)$. This relation can result in loosing the persistence of excitation condition since the role of $K(k)$ is solely to stabilize the closed-loop system. Therefore, the auxiliary input ε is added to overcome the possible lack of excitation caused by the feedback. This is stated in the following lemma.

Lemma 4.1 *For any $k \geq 0$ let the input sequence $\{u(k-N), \ldots, u(k-1)\}$ with $N := (m+1)n+m$ be persistently exciting of order $n+1$ and $\|x(k)\| \neq 0$. Then, there exists some $\varepsilon(k) \in B_\delta$ such that the sequence $\{u(k-N+1), \ldots, u(k)\}$ with $u(k) = K(k)x(k) + \varepsilon(k)\|x(k)\|$ is persistently exciting of order $n+1$.*

Proof. See Section 4.7. ∎

Lemma 4.1 shows that for any $k \geq 1$ there exists some $\varepsilon(k-1) \in B_\delta$ such that the input sequence $\{u(k-N), \ldots, u(k-1)\}$ is persistently exciting of order $n+1$.[1] Note that this also guarantees that the input sequence $\{u(k-T), \ldots, u(k-1)\}$ with $T \geq N$ is also sufficiently exciting of the same order. Note finally that the condition $\|x(k)\| \neq 0$ is not restrictive since the origin is the equilibrium of the closed-loop system.

Remark 4.1 *Note that the above lemma not only does guarantee existence of an ε such that the persistence of excitation condition is satisfied, but also provides a tool to select such signal. In particular, let the input sequence $\{u(k-N), \ldots, u(k-1)\}$ be persistently exciting of order $n+1$. This means that the corresponding Hankel matrix has full row rank (see Definition 2.2). At time k, a new sample $K(k)x(k)$ is generated. Consider the new sequence $\{u(k-N+1), \ldots, K(k)x(k)\}$. If such sequence is persistently exciting (the corresponding Hankel matrix has full rank), then $\varepsilon(k)$ can be set to zero and $u(k) = K(k)x(k)$. Otherwise, $\varepsilon(k)$ should be properly selected such that $u(k) = K(k)x(k) + \varepsilon(k)\|x(k)\|$ preserves the persistence of excitation condition.*

4.3.2. Online semidefinite programming formulation

We now exploit the rank condition (4.3) for designing the state feedback gain at every step. For any $k \geq 0$, the matrices of data U_{k-1}, X_{k-1}, X_k are available and one can define the

[1]For related work on selecting a suitable input sequence so as to preserve persistence of excitation, see [106].

program:

$$
\begin{aligned}
\min_{\gamma,Q,P,L} \quad & \gamma \\
\text{subject to} \quad & X_k\, Q P^{-1} Q^\top X_k^\top - P + I \preceq 0 \\
& P \succeq I \\
& X_{k-1} Q = P \\
& L - U_{k-1}\, Q P^{-1} Q^\top U_{k-1}^\top \succeq 0 \\
& \mathbf{Tr}(P) + \mathbf{Tr}(L) \leq \gamma
\end{aligned}
\tag{4.4}
$$

The control at time k is defined as

$$
K(k) = U_{k-1} Q^*(k) P^*(k)^{-1} \tag{4.5}
$$

where the tuple $(\gamma^*(k), Q^*(k), P^*(k), L^*(k))$ is any optimal solution to (4.4). In particular, for $k = 0$, the matrices of data U_{-1}, X_{-1}, X_0 are available. Since the system undergoes no switching during the interval $[-T, -1]$, and the sequence $\{u(-T), \dots, u(-1)\}$ is persistently exciting of order $n+1$, the condition (4.3) holds. Then the program (4.4) is feasible and returns $K(0)$. Feasibility of problem (4.4) for any $k \geq 0$ will be discussed in the next section.

Note that by using the LQR formulation we make sure that each discrete mode is associated with only one feedback gain. In addition, this allows us to simplify the analysis of the closed-loop system, which will be discussed in the next section. Furthermore, we point out that robust formulations of the LQR problem have been addressed in [61, 67], which provide computationally tractable results for handling noisy data. However, we opt for the classical LQR formulation in order to provide a more explicit analysis and to highlight the underlying intuition behind our online framework. Hence, in the present work we will not consider noisy data.

Remark 4.2 *The matrices U_{k-1} that appear in* (4.4) *are generated over time by the control signal $u(k) = K(k)x(k) + \varepsilon(k)\|x(k)\|$. We remark that the term $\varepsilon(k)\|x(k)\|$ is needed to have persistence of excitation, a property which might be lost with a pure feedback law $u(k) = K(k)x(k)$. We also stress that the addition of the signal $\varepsilon(k)\|x(k)\|$ does not affect the feasibility of* (4.4)*, as we will formally show in Lemma 4.2, nor the ability of the controller to stabilize the system, as we will formally prove in Theorem 4.2 in the next section.*

Remark 4.3 (Implementation of (4.4)) *Problem* (4.4) *can be written in the equivalent SDP form:*

$$
\begin{aligned}
\min_{\gamma,Q,P,L} \quad & \gamma \\
\text{subject to} \quad & \begin{bmatrix} I - P & X_k Q \\ Q^\top X_k^\top & -P \end{bmatrix} \preceq 0 \\
& \begin{bmatrix} L & U_{k-1} Q \\ Q^\top U_{k-1}^\top & P \end{bmatrix} \succeq 0 \\
& X_{k-1} Q = P \\
& \mathbf{Tr}(P) + \mathbf{Tr}(L) \leq \gamma
\end{aligned}
\tag{4.6}
$$

Remark 4.4 (Complexity of (4.4)) *Our design scheme involves the solution of a semi-definite program (SDP), whose complexity is roughly $\mathcal{O}(n^3T^3)$. As long as persistence of excitation is satisfied we can take $T = (m+1)n + m$. Thus the computation complexity is roughly $\mathcal{O}(n^6)$. The reader may refer to [12], where this point is discussed in more details.*

We have presented the two main ingredients of the proposed online framework, i.e. the notion of persistence of excitation and the linear quadratic formulation in the form of data-dependent semidefinite program. Persistence of excitation enables the solution of programs (4.4), which generates the time-varying controller gain $K(k)$. We base our design on LQR formulation (4.4) because it allows us to show uniform boundedness of the data-generated gains $K(k)$ over time, a feature exploited in the next section to conclude our stabilization result.

4.4. STABILITY ANALYSIS

In this section, we investigate the stability of the switched system (4.1) under the feedback law (4.2) with control gain as in (4.5).

We conduct our analysis in two steps. In the first step, we observe that after each switching instance the data matrices contain a mixture of measurements coming from the active subsystem and the subsystem active at the previous switching interval. In general, due to the inconsistent data collection, we do not have any guarantees that feasible solutions to problem (4.4) result in stabilizing controllers. This may lead the state trajectory to grow unboundedly over time. In this regard, we show that the rate of growth of the state trajectory is limited by proving that problem (4.4) returns uniformly bounded controller gains $K(k)$ over $k \in \mathbb{N}$.

In the second step, we show that the closed-loop switched system is exponentially stable under sufficiently slow switching. For that, we consider the following definition.

Definition 4.1 [99] *For a sequence of switching times k_s with $s \geq 0$, the dwell time τ is the minimum interval between any two consecutive switching times, that is*

$$\tau := \min_{s \geq 0} k_{s+1} - k_s.$$

•

To guarantee that during each switching interval we collect T samples of the active subsystem, we will assume that $\tau > T$. Later in the analysis we will show that, if the dwell time τ is sufficiently large, then stabilization of the overall closed-loop system is achieved.

To prove uniform boundedness of $K(k)$, we first consider any time interval $[k_s + 1, k_{s+1}]$ with $s \geq 0$ partitioned into two sub-intervals $[k_s + 1, k_s + T - 1]$ and $[k_s + T, k_{s+1}]$. The motivation behind this partitioning is clarified in Figure 4.1. For the rest of our Chapter, the following assumption holds.

Assumption 4.2 *The length of the data matrices satisfies $T \geq 2N - 1$, where $N = (m + 1)n + m$ is necessary for the persistence of excitation to hold.* •

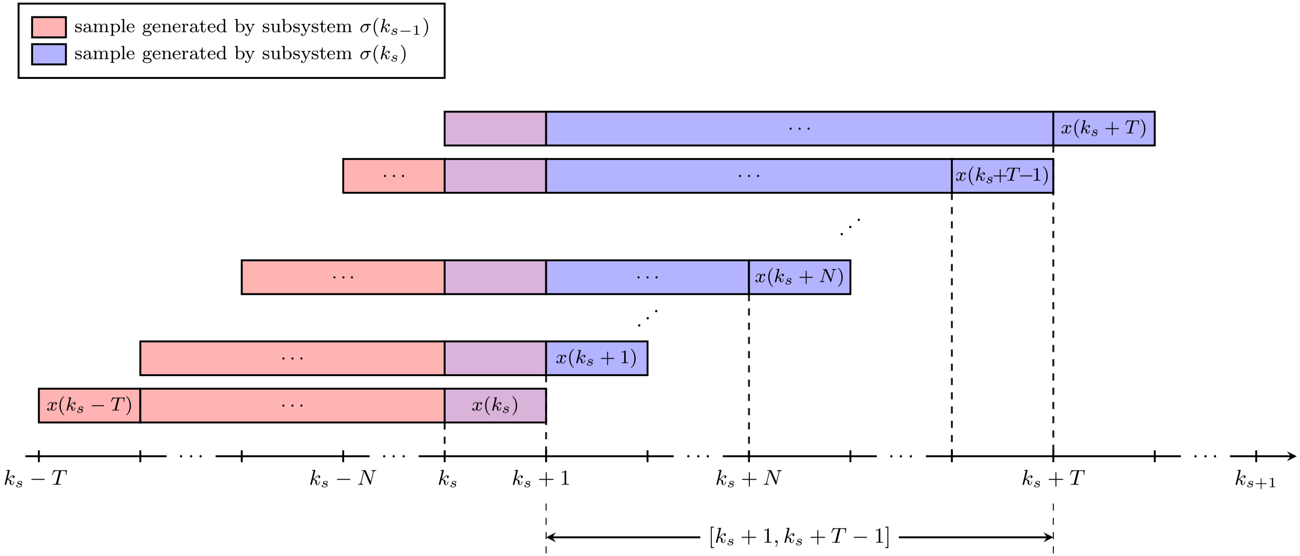

Figure 4.1: Illustration of the timing scheme adopted in the feasibility analysis, where the colored blocks represent the state sequence $\{x(k-T), \ldots, x(k)\}$ at different time instants k. The different colors indicate which subsystem has generated the data. Notice that the sequence is used to construct the data matrices X_{k-1} and X_k. The violet block represents the sample $x(k_s)$, which is generated by $\sigma(k_{s-1})$ and serves as initial condition for subsystem $\sigma(k_s)$. We can observe that in the time interval $[k_s + 1, k_s + T - 1]$ the available information is a mixture of data generated by the two different subsystems so that problem (4.4) may not be feasible. By choosing $T \geq 2N - 1$ one can guarantee that during this transient interval the matrix X_{k-1} always contains at least N samples generated by the same subsystem.

We proceed the analysis by discussing the feasibility of problem (4.4) in the aforementioned sub-intervals. We provide the analysis for the latter sub-interval in the next lemma. The former sub-interval, i.e., $[k_s + 1, k_s + T - 1]$, will be discussed afterwards. These auxiliary results are later used to derive an uniform bound on the controller gain.

Lemma 4.2 *Let $i \in \mathcal{I}$ denote the subsystem selected by $\sigma(k_s)$, i.e. $i = \sigma(k_s)$, and consider $k \in [k_s + T, k_{s+1}]$. Then, problem* (4.4) *is feasible and any optimal solution $(\gamma_i^*(k), Q_i^*(k), P_i^*(k), L_i^*(k))$ satisfies $K_{opt}^i = U_{k-1} Q_i^*(k) P_i^*(k)^{-1}$, where K_{opt}^i is the unique LQR controller of subsystem i.*

Proof. See Section 4.7. ■

Lemma 4.2 shows that in the interval $[k_s+T, k_{s+1}]$ the solution of problem (4.4) returns the unique LQR controller for the active subsystem $i = \sigma(k_s)$. Hence, for $k \in [k_s + T, k_{s+1}]$ the control signal (4.2) turns out to be equal to

$$u(k) = K_{opt}^i \, x(k) + \varepsilon(k)\|x(k)\|. \tag{4.7}$$

Consider now $k \in [k_s + 1, k_s + T - 1]$. We refer to $[k_s + 1, k_s + T - 1]$ as the *transient* interval. Within the transient interval, recalling the definition of the data matrices, the matrices X_{k-1} and X_k contain samples generated by both the active subsystem $\sigma(k_s)$ and the subsystem active at the previous switching interval, i.e. subsystem $\sigma(k_{s-1})$. Therefore, in such interval there is no guarantee that the controller $K(k)$ computed as (4.5) is stabilizing. In the following lemma, we discuss feasibility of problem (4.4) in this transient interval.

Lemma 4.3 *Let $z \in \mathcal{I}$ denote the subsystem selected by $\sigma(k_{s-1})$, i.e. $z = \sigma(k_{s-1})$, and $j \in \mathcal{I}$ denote the subsystem selected by $\sigma(k_s)$, i.e. $j = \sigma(k_s)$. Then, for each $k \in [k_s + 1, k_s + T - 1]$, problem* (4.4) *is feasible.*

Proof. See Section 4.7. ■

Building on the the results of Lemmas 4.2 and 4.3, we now show in Theorem 4.1 that problem (4.4) provides uniformly bounded controllers for all times.

Theorem 4.1 *Consider the switched system* (4.1) *consisting of a finite number of unknown subsystems and unknown switching law $\sigma(\cdot)$. Also, consider problem* (4.4) *whose solution computes the state feedback gain $K(k)$ as in* (4.5)*. Then, there exists some $\kappa > 0$ such that*

$$\|K(k)\| \leq \kappa \tag{4.8}$$

for all $k \geq 0$.

Proof. We start by partitioning any interval $[k_s + 1, k_{s+1}]$ into the sub-intervals $[k_s + 1, k_s + T - 1]$ and $[k_s + T, k_{s+1}]$. Hence, based on these time partitions, we break the proof into three parts. First, we exploit the result established in Lemma 4.2 to deduce that the controllers $K(k)$ are uniformly bounded over $k \in \bigcup_{s\geq 0}[k_s + T, k_{s+1}]$. Second, building on the result of Lemma 4.3, we show that $K(k)$ are uniformly bounded over $k \in \bigcup_{s\geq 0}[k_s + 1, k_s + T - 1]$. Finally, we conclude the proof by combining the first two parts to prove our claim for all $k \geq 0$.

Consider $k \in [k_0, k_1]$ with $k_0 = 0$ and k_1 the first switching instant. At time $k_0 = 0$ the matrices of data X_{-1}, U_{-1}, X_0 are available and contain samples generated by some subsystem $z \in \mathcal{I}$. Since the system undergoes no switching during the interval $[k_0, k_1]$, by Lemma 2.3 the program (4.4) is feasible and any optimal solution $(\gamma_z^*(k), Q_z^*(k), P_z^*(k), L_z^*(k))$ to problem (4.4) satisfies $K(k) = K_{opt}^z$ with $K(k) = U_{k-1}Q_z^*(k)P_z^*(k)^{-1}$, where K_{opt}^z is the unique LQR controller stabilizing subsystem z. Hence, it follows that $\|K(k)\| = \|K_{opt}^z\|$.

Consider now the sub-interval $[k_1 + T, k_2]$. Let $j \in \mathcal{I}$ denote the active subsystem selected by $\sigma(k_1)$, i.e. $j = \sigma(k_1)$. Observe that during the interval $[k_1 + T, k_2]$ the data matrices completely parametrize subsystem j. Then, in view of Lemma 4.2, any optimal solution $(\gamma_j^*(k), Q_j^*(k), P_j^*(k), L_j^*(k))$ to problem (4.4) satisfies $K_{opt}^j = K(k)$ with $K(k) = U_{k-1}Q_j^*(k)P_j^*(k)^{-1}$, where K_{opt}^j is the unique LQR controller stabilizing subsystem j. Hence, it follows that $\|K(k)\| = \|K_{opt}^j\|$.

Thus, by considering $k \in [k_0, k_1] \cup [k_1 + T, k_2]$ the following bound holds

$$\|K(k)\| \le \max(\|K_{opt}^z\|, \|K_{opt}^j\|).$$

We therefore deduce by induction that

$$\|K(k)\| \le \max_{i \in \mathcal{I}} \|K_{opt}^i\|, \tag{4.9}$$

for all

$$k \in [k_0, k_1] \cup \bigcup_{s \ge 1} [k_s + T, k_{s+1}].$$

In the second part of the proof, we demonstrate that $K(k)$ are uniformly bounded over

$$k \in \bigcup_{s \ge 1} [k_s + 1, k_s + T - 1].$$

Consider $k \in [k_1 + 1, k_1 + T - 1]$. During this interval, the data matrices are made of samples coming from two different subsystems, i.e. the subsystems z and j. By Lemma 4.3, problem (4.4) is feasible. Let $(\gamma^*(k), Q^*(k), P^*(k), L^*(k))$ be an optimal solution and $K(k) = U_{k-1}Q^*(k)P^*(k)^{-1}$ be the corresponding controller. Furthermore, as discussed in the proof of Lemma 4.3, we can define the tuples $(\gamma_z, Y_z(k), P_z, L_z)$ and $(\gamma_j, Y_j(k), P_j, L_j)$ which are feasible for (4.4). Suppose that the tuple $(\gamma_z, Y_z(k), P_z, L_z)$ is feasible for (4.4). Since $(\gamma^*(k), Q^*(k), P^*(k), L^*(k))$ is by definition optimal for (4.4) we must therefore have

$$\mathbf{Tr}(P^*(k)) + \mathbf{Tr}(L^*(k)) \le \gamma_z,$$

where $\gamma_z := \mathbf{Tr}(P_z) + \mathbf{Tr}(L_z)$ has been defined in (4.29) as shown in the proof of Lemma 4.3. Bearing in mind that $\mathbf{Tr}(P^*(k)) \ge n$, the previous inequality can be rewritten as $\mathbf{Tr}(L^*(k)) \le \gamma_z - n$. As $\mathbf{Tr}(K(k)P^*(k)K(k)^\top) \le \mathbf{Tr}(L^*(k))$ and $P^*(k) \succeq I$, it holds that $\mathbf{Tr}(K(k)K(k)^\top) \le \gamma_z - n$. Recalling that $\|K(k)\|^2 \le \mathbf{Tr}(K(k)K(k)^\top)$, we finally obtain

$$\|K(k)\| \le c_z, \quad c_z := \sqrt{\gamma_z - n}.$$

Assume now that the tuple $(\gamma_j, Y_j(k), P_j, L_j)$ is feasible to (4.4). Then, by the same arguments, $\|K(k)\| \leq c_j$, with $c_j := \sqrt{\gamma_j - n}$. Thus, we deduce that for $k \in [k_1+1, k_1+T]$ it holds that $\|K(k)\| \leq \max(c_z, c_j)$. More generally, given a feasible tuple $(\gamma_i, Y_i(k), P_i, L_i)$ constructed based on subsystem $i \in \mathcal{I}$, it holds that

$$\|K(k)\| \leq \max_{i \in \mathcal{I}} c_i, \quad c_i := \sqrt{\gamma_i - n} \tag{4.10}$$

for every $k \in \bigcup_{s \geq 1} [k_s + 1, k_s + T - 1]$.

We conclude the proof by combining (4.10) and (4.9), which gives that for every $k \geq 0$ the following relation holds

$$\|K(k)\| \leq \max_{i \in \mathcal{I}} (\|K^i_{opt}\|, c_i).$$

On the other hand, consider $\sqrt{\gamma_i - n} = c_i$. By definition, $\gamma_i := \mathbf{Tr}(P_i) + \mathbf{Tr}(L_i)$. Therefore it holds that $\sqrt{\mathbf{Tr}(L_i)} \leq c_i$, where we have used $\mathbf{Tr}(P_i) \geq n$. Bearing in mind that $\|K^i_{opt}\| \leq \sqrt{\mathbf{Tr}(L_i)}$, we therefore deduce that

$$\|K^i_{opt}\| \leq c_i,$$

which implies that for all $k \geq 0$ the following bound holds

$$\|K(k)\| \leq \kappa, \quad \kappa := \max_{i \in \mathcal{I}} c_i$$

which proves our claim. ∎

By the result established in Theorem 4.1, we guarantee existence of some $\kappa > 0$ such that $\|K(k)\| \leq \kappa$. Thus, the system state remains bounded during the transient interval. In particular, the system is evolving as

$$x(k+1) = (A_{\sigma(k)} + B_{\sigma(k)} K(k))x(k) + B_{\sigma(k)} \varepsilon(k) \|x(k)\|,$$

which implies

$$\|x(k+1)\| \leq \Big(\|A_{\sigma(k)} + B_{\sigma(k)} K(k)\| + \|B_{\sigma(k)} \varepsilon(k)\| \Big) \|x(k)\|.$$

Let

$$C_0 := \max_{i \in \mathcal{I}} \Big(\|A_i\| + \|B_i\|(\kappa + \delta) \Big).$$

It then follows from $\|K(k)\| \leq \kappa$ and $\varepsilon(k) \in B_\delta$ that

$$\|x(k+1)\| \leq C \|x(k)\|, \tag{4.11}$$

where $C := \max\{C_0, 1\}$. We can now tackle the stability analysis of the closed-loop system. In particular, the finite set $\{K^i_{opt} : i \in \mathcal{I}\}$ allows us to approach the stability analysis by using multiple Lyapunov functions [98]. The key point of this approach is to construct a set of Lyapunov functions $\{V_i : i \in \mathcal{I}\}$ such that, considering suitable choice of design parameters, the value of V_i decreases on each time interval where the i-th subsystem is active. Then, the closed-loop switched system is exponentially stable under sufficiently slow switching. This is stated in the following Theorem.

Theorem 4.2 *Consider the switched system* (4.1) *with unknown* $(A_{\sigma(k)}, B_{\sigma(k)})$ *and unknown switching law* $\sigma(\cdot)$ *with dwell time* τ. *Also, consider the feedback law* (4.2) *with the state feedback gain* $K(k)$ *as in* (4.5) *and with* $\varepsilon(k) \in B_\delta$ *for all* k. *Then, there exist some* $\bar{\delta} > 0$ *and* $\bar{\tau} > 0$ *such that, if* $\delta \leq \bar{\delta}$ *and* $\tau > \bar{\tau}$, *the closed-loop system is exponentially stable.*

Proof. Consider system (4.1) on any switching interval $[k_s, k_{s+1} - 1]$, and apply the feedback law (4.2). Let $i \in \mathcal{I}$ denote the active subsystem selected by $\sigma(k_s)$, i.e.

$$i = \sigma(k), \quad k \in [k_s, k_{s+1} - 1].$$

We first show stability of this subsystem on the time interval $[k_s + T, k_{s+1} - 1]$. For all k in this interval, we know from Lemma 4.2 that the control law is (4.7). Hence, the closed-loop system can be written as

$$x(k+1) = \mathcal{A}_i x(k) + g_i(x(k)), \tag{4.12}$$

where $\mathcal{A}_i := A_i + B_i K^i_{opt}$ and $g_i(x(k)) := B_i \varepsilon(k) \|x(k)\|$. Since $\mathcal{A}_i$ is stable, there exists a positive definite matrix P_i satisfying

$$\mathcal{A}_i^\top P_i \mathcal{A}_i - P_i = -I. \tag{4.13}$$

Let $\underline{\lambda}_P := \min_{i \in \mathcal{I}} \lambda_{\min}(P_i)$ and $\overline{\lambda}_P := \max_{i \in \mathcal{I}} \lambda_{\max}(P_i)$, where $\lambda_{\min}(P_i)$ and $\lambda_{\max}(P_i)$ stand for the minimal and maximum eigenvalue of P_i, respectively.

We consider the Lyapunov candidate $V_i(x) = x^\top P_i x$ such that

$$\underline{\lambda}_P \|x\|^2 \leq V_i(x) \leq \overline{\lambda}_P \|x\|^2. \tag{4.14}$$

The evolution $\Delta V_i(x(k)) := V_i(x(k+1)) - V_i(x(k))$ along the trajectories of (4.12) satisfies

$$\Delta V_i(x(k)) = x(k)^\top (\mathcal{A}_i^\top P_i \mathcal{A}_i - P_i) x(k) + 2x(k)^\top \mathcal{A}_i^\top P_i g_i(x(k)) + g_i(x(k))^\top P_i g_i(x(k)).$$

Using (4.13) and the definition of $g_i(x(k))$, we get

$$\Delta V_i(x(k)) \leq -\frac{1}{2}\|x(k)\|^2 + \psi(\|\varepsilon(k)\|)\|x(k)\|^2,$$

with

$$\psi(\|\varepsilon\|) := \overline{\lambda}_P \|B_i\|^2 \|\varepsilon\|^2 + 2\overline{\lambda}_P \|\mathcal{A}_i\| \|B_i\| \|\varepsilon\| - \frac{1}{2}.$$

We proceed by showing that $\psi(\|\varepsilon\|)$ becomes non-positive when $\|\varepsilon\|$ is small enough. Let $\delta \leq \bar{\delta}$ where $\bar{\delta} := \min_{i \in \mathcal{I}} \delta_i$ and

$$\delta_i := \frac{-\overline{\lambda}_P \|\mathcal{A}_i\| + \sqrt{\overline{\lambda}_P^2 \|\mathcal{A}_i\|^2 + \frac{1}{2}\overline{\lambda}_P}}{\overline{\lambda}_P \|B_i\|}.$$

Then, it follows from $\varepsilon \in B_\delta$ that $\psi(\|\varepsilon\|) \leq 0$. In particular, it follows that

$$V_i(x(k+1)) - V_i(x(k)) \leq -\frac{1}{2}\|x(k)\|^2$$

for $k \in [k_s + T, k_{s+1} - 1]$. By considering (4.14), the previous expression can be written as

$$V_i(x(k+1)) - V_i(x(k)) \leq -\frac{1}{2\overline{\lambda}_P} V_i(x(k)).$$

The above expression implies that for $k \in [k_s + T, k_{s+1} - 1]$ the following relation holds

$$V_i(x(k+1)) \leq \alpha^2 \, V_i(x(k)), \tag{4.15}$$

where $\alpha := ((\overline{\lambda}_P - 0.5)/\overline{\lambda}_P)^{1/2}$. Note that it follows from the Lyapunov equation and [107, Thm. 5.D6] that $\overline{\lambda}_P \geq 1$. Hence, $0 < \alpha < 1$ and each subsystem is stable in the interval $[k_s + T, k_{s+1} - 1]$.

The rest of the proof establishes exponential stability of the switched system using standard arguments that are reported for the sake of completeness. We show that there exist some $\mu > 0$ and $0 < \lambda < 1$ such that the following relation is satisfied

$$\|x(k_s + t)\| \leq \mu \, \lambda^{k_s + t - k_0} \|x(k_0)\| \tag{4.16}$$

for every $s \geq 0$ and $t \in [1, k_{s+1} - k_s]$. For $t \in [1, T]$, based on the definition of C in (4.11), it follows that

$$\|x(k_s + t)\| \leq C \|x(k_s + t - 1)\|. \tag{4.17}$$

For $t \in [T + 1, k_{s+1} - k_s]$, from (4.15) it yields

$$V_i(x(k_s + t)) \leq \alpha^2 \, V_i(x(k_s + t - 1)),$$

and, in particular,

$$V_i(x(k_s + t)) \leq \alpha^{2(t-T)} \, V_i(x(k_s + T)).$$

Hence, for $t \in [T + 1, k_{s+1} - k_s]$, the evolution of the states satisfies

$$\|x(k_s + t)\| \leq \varphi \, \alpha^{t-T} \, \|x(k_s + T)\|, \tag{4.18}$$

where $\varphi := (\overline{\lambda}_P / \underline{\lambda}_P)^{1/2}$. By iterating (4.17) and (4.18), it results in

$$\|x(k_s + t)\| \leq \begin{cases} C^t \alpha^{k_s - k_0} \mu^s \, \|x(k_0)\|, & t \in [1, T] \\ \alpha^t \alpha^{k_s - k_0} \mu^{s+1} \, \|x(k_0)\|, & t \in [T + 1, k_{s+1} - k_s], \end{cases} \tag{4.19}$$

where $\mu := \varphi \left(\frac{C}{\alpha}\right)^T$. Let $0 < \alpha < \lambda < 1$ and the dwell time be sufficiently large, i.e. $\tau > \bar{\tau}$ where

$$\bar{\tau} := \frac{\ln(\mu)}{\ln(\lambda/\alpha)}. \tag{4.20}$$

Then, we conclude the proof by establishing that (4.16) can be derived from (4.19). For $t \in [1, T]$, it holds that

$$\begin{aligned} C^t \alpha^{k_s - k_0} \mu^s &= \lambda^{k_s + t - k_0} \left(\frac{C}{\lambda}\right)^t \left(\frac{\alpha}{\lambda}\right)^{k_s - k_0} \mu^s \\ &\leq \lambda^{k_s + t - k_0} \left(\frac{C}{\lambda}\right)^T \left(\frac{\alpha}{\lambda}\right)^{s\tau} \varphi \mu^s \\ &\leq \lambda^{k_s + t - k_0} \left(\frac{\alpha}{\lambda}\right)^{s\tau} \mu^{s+1} \leq \mu \, \lambda^{k_s + t - k_0}, \end{aligned}$$

where the first inequality holds since $C/\lambda \geq 1$, $t \leq T$, $\alpha/\lambda < 1$, $k_s - k_0 \geq s\tau$ and $\varphi \geq 1$, while the second inequality follows from $\alpha < \lambda$ and the definition of μ. The last inequality is satisfied as long as $\tau > \bar{\tau}$, with $\bar{\tau}$ as in (4.20). By similar arguments, for $t \in [T+1, k_{s+1} - k_s]$, it holds that

$$\begin{aligned}\alpha^{k_s+t-k_0}\mu^{s+1} &= \lambda^{k_s+t-k_0}\left(\frac{\alpha}{\lambda}\right)^{k_s+t-k_0}\mu^{s+1} \\ &\leq \lambda^{k_s+t-k_0}\left(\frac{\alpha}{\lambda}\right)^{s\tau}\mu^{s+1} \leq \mu\,\lambda^{k_s+t-k_0},\end{aligned}$$

where the first inequality holds since $k_s+t-k_0 \geq s\tau$ and $\alpha/\lambda < 1$, and the last inequality is satisfied as long as $\tau > \bar{\tau}$, with $\bar{\tau}$ as in (4.20). ■

Remark 4.5 *While Theorem 4.2 guarantees existence of a sufficiently small $\bar{\delta}$, computing its value requires the knowledge of the norms of the system matrices. If such norms are not available, then one can estimate their values from the collected input-state data set, which provides an equivalent data-based representation of the system. Note that the design parameter δ can take any values below the upper bound $\delta \leq \bar{\delta}$, and therefore its selection is oblivious of the exact value of $\bar{\delta}$.*

4.5. Case studies

In this section, two examples are presented to show the effectiveness of the proposed control approach.

4.5.1. Flight control system

We consider the problem of stabilizing the linearized longitudinal dynamics of a F-18 aircraft operating on two different heights [108, 109]. Using a sampling rate of $h = 0.1$s, we write the aircraft model in the form of a discrete-time switched linear system (4.1). Without causing confusion, we will refer to the time instant k instead of kh. The discretized system matrices are given by

$$\begin{aligned}A_1 &= \begin{bmatrix} 0.977 & 0.097 \\ 0.002 & 0.981 \end{bmatrix}, B_1 = \begin{bmatrix} -0.013 & -0.004 \\ -0.171 & -0.051 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.852 & 0.088 \\ -0.753 & 0.878 \end{bmatrix}, B_2 = \begin{bmatrix} -0.106 & -0.021 \\ -1.8143 & -0.358 \end{bmatrix}.\end{aligned}$$

where A_1 is the longitudinal state matrix at Mach 0.3 and altitude 26 kft and A_2 is the longitudinal state matrix at Mach 0.7 and 14 kft. Both the pairs (A_1, B_1) and (A_2, B_2) are controllable. In this simplified model, the state variables represent the angle of attack and the pitch rate. We mention that the model is reported *only* for illustrative purposes, because our design does not rely on the knowledge of the model, but *only* on data collected while the system is evolving in closed-loop. Furthermore, although we know that the flying modes are based on the altitude and the speed of the aircraft, the switching signal cannot be observed a priori.

An initial data set is obtained offline by using $T = 15$ samples generated by applying to the subsystem (A_1, B_1) an input signal u distributed uniformly in $[-0.3, 0.3]$ (by Lemma 4.1 condition (4.3) requires a minimum of $N = 8$ samples). The collected samples are organized into appropriate data matrices of length T. At time $k = 0$ we run our algorithm online. At every iteration $k \geq 0$, the controller gain $K(k)$ is computed by solving the data-based convex program (4.4) using CVX [92]. The control signal $u(k)$ is then applied to the system in the form of (4.2), where we choose $\varepsilon(k)$ as a random variable uniformly distributed in $[-0.001, 0.001]$. The new data are then measured and saved in the data matrices, which are updated by removing the oldest sample each time a new one is added. Based on this moving window of data, the controller can be updated at every iteration.

We simulate the system response for arbitrary switching signal σ with $\tau = 1.5$s. Figure 4.2 depicts the corresponding input and state responses. As it can be seen from the Figure, at every switching instant the flying mode of the aircraft changes and hence the algorithm needs to learn the changing dynamics. After T samples of the current operating mode are collected, the stabilizing controller can be computed and applied until the next switch.

Our theoretical results show that stability of the overall closed-loop system is guaranteed for $\tau > \bar{\tau}$ with $\bar{\tau}$ given by (4.20). By carrying out the calculation for this specific case study[2], we get $\bar{\tau} = 4$min. However, our numerical investigation has shown that stability is achieved for smaller values of τ as demonstrated in Figure 4.2. Furthermore, the input variables remain bounded during the transient phase, which is consistent with our theoretical results.

4.5.2. AIRCRAFT ENGINE SYSTEM

We investigate the use of our online mechanism in the area of fault tolerant control. A fault is an unexpected event that changes the characteristic property of a component or the whole plant.

We consider a linearized model of an F-404 aircraft engine system [110] subject to system and actuator fault. In the engine model, the state variables denote the sideslip angle, the roll rate and the yaw rate. The control inputs represent the engine thrust and the flight path angle. Setting the sampling time at $h = 0.1$s, the discretized nominal system matrices are expressed as

$$A = \begin{bmatrix} 0.867 & 0 & 0.202 \\ 0.015 & 0.961 & -0.032 \\ 0.026 & 0 & 0.803 \end{bmatrix}, \ B = \begin{bmatrix} 0.011 & 0 \\ 0.014 & -0.039 \\ 0.009 & 0 \end{bmatrix}.$$

As in the previous example, we will refer to the time instant k instead of kh. In our case study, no previous models of the system or faults are available, but only real-time input-state data streams.

We generate an initial T-long set of data by applying to the nominal system a T-long sequence of input u uniformly distributed in $[-3.5, 3.5]$. We choose $T = 21$ (by Lemma 4.1 condition (4.3) requires a minimum of 11 samples) as a size for our matrices of data. Then, we run our algorithm online. At every iteration $k \geq 0$, the program (4.4) is solved

[2]We note that the controller design is *independent* of the knowledge of $\bar{\tau}$, therefore explicit calculation of $\bar{\tau}$ is reported *only* for illustrative purposes.

using CVX and the controller gain $K(k)$ is computed based on the available data set. The control signal $u(k)$ is applied to the system in the form of (4.2), where we have chosen $\varepsilon(k)$ as a random variable uniformly distributed in $[-0.001, 0.001]$. As in the previous case study, the data are collected in the data matrices over time by removing the oldest sample each time a new one is added.

The movement of the aircraft is commonly affected by some external disturbance and unknown fault, such as wind gusts or structural vibrations which will degrade the stability of the system. Similarly to [110, 111], we simulate various system faults leading to changes in the system matrix as $\tilde{A} = A + \beta(k)D$ with

$$D = \begin{bmatrix} 0.075 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & -0.75 \end{bmatrix}, \quad \beta(k) = \begin{cases} 0.1 & k \in [0, 2.7) \\ 0.05 & k \in [2.7, 5.2) \\ -0.5 & k \in [5.2, 9.5) \\ 0 & \text{else.} \end{cases}$$

Next, failure of the engine generating thrust and the motor moving the path angle are simulated for $k \in [2.7, 5.2)$ and $k \geq 5.2$, respectively. The component failure is expressed by setting to zero the corresponding column in the matrix B to reflect the command outage. The effectiveness of the proposed online approach is illustrated in Figure 4.3, which shows the state trajectories of the controlled system, together with the control input. As we can see from the figure, after each fault, we can observe the state growing due to the changing in the dynamics. This behavior occurs during the transient interval, during which the algorithm is learning the new dynamics. Once T samples of the faulty system are collected, a stabilizing controller can be computed and applied until the next fault occurs. This shows that the controller is able to automatically adjust whenever a fault occurs. Furthermore, the controller manages to stabilize the closed-loop system, provided that the faults do not occur too frequently, which is consistent with our theoretical results. As a final consideration, we observe that the process of switching introduces some jumps in the control action. This is a common phenomenon in the context of switching control and various schemes have been proposed to enable bumpless transfer between controllers [112, 113].

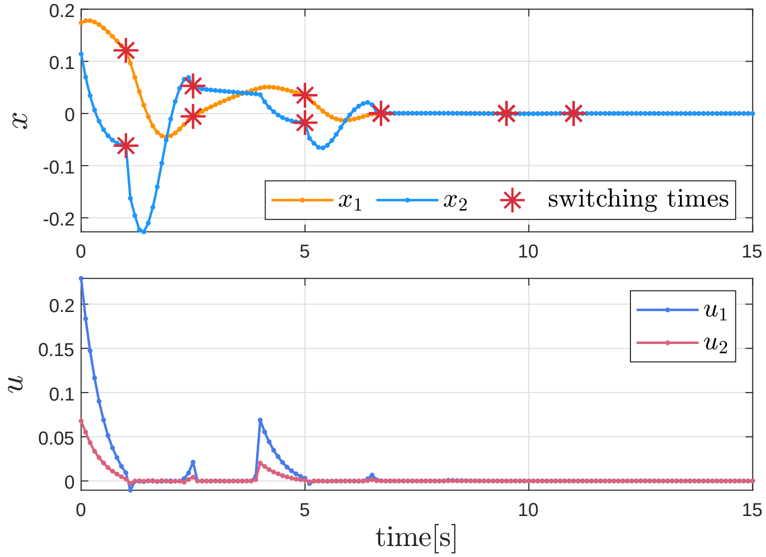

Figure 4.2: State and control trajectories of the F-18 aircraft system switching between two operating modes.

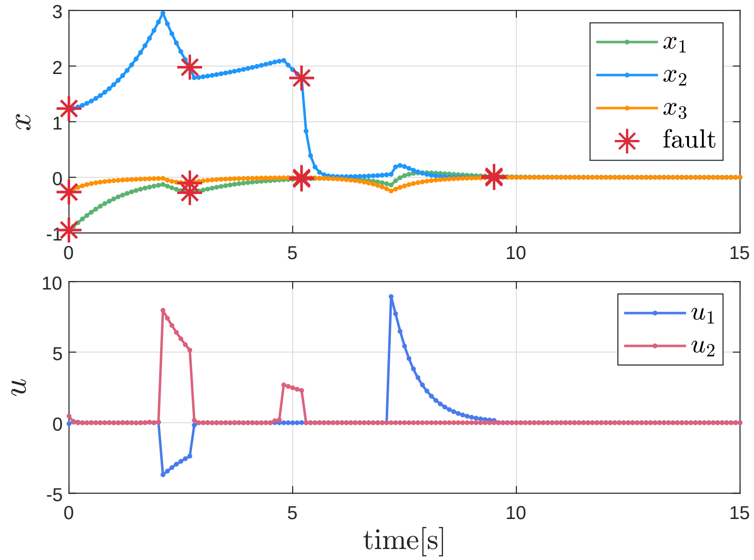

Figure 4.3: State and control trajectories of the aircraft engine system subject to system and actuator fault. We observe failure of the first actuator for $k \in [2.7, 5.2)$ and failure of the second component for time $k \geq 5.2$.

4.6. Conclusions

In this Chapter, we have considered the design of a data-based feedback controller for switched discrete-time linear systems. Both the dynamics of each subsystem and the switching signal are assumed to be unknown. We have proposed a data-based framework which requires no intermediate identification steps and provides stability guarantees. The key idea relies on an online scheme where input-state data are collected over time as the system is evolving. While in general closed-loop data are not necessarily sufficiently exciting, we have formally shown that by adding a suitable term in the control scheme, the persistence of excitation condition can be preserved. The control mechanism is directly parametrized through data and iteratively updated via a computationally tractable data-dependent semidefinite program. The resulting controller is guaranteed to exponentially stabilize the closed-loop system under sufficiently slow switching.

Future works include extension of the current framework to cope with noisy data. Robust data-driven design has been previously addressed in [61, 67] and its extension to unknown switched systems can be studied. Moreover, the computational complexity of the online algorithm can be studied, and its recursive implementation, which may be more suitable for real-time applications, can be investigated.

4.7. Proofs

4.7.1. Persistence of Excitation

Proof of Lemma 4.1

Without loss of generality, we consider $k = 0$. Let $\{u(-N), \ldots, u(-1)\}$ be persistently exciting of order $n+1$, in the sense that the corresponding Hankel matrix $U_{-N,n+1,N-n}$ has full rank $m(n+1)$. We partition this matrix as follows

$$\begin{aligned} U_{-N,n+1,N-n} &= \left[\begin{array}{c|c} U_{-N,n+1,1} & S \end{array}\right] \\ &= \left[\begin{array}{c} U_{-N,1,N-n} \\ \hline R \end{array}\right] \end{aligned} \tag{4.21}$$

where

$$S := \left[\begin{array}{c} U_{1-N,n,N-n-1} \\ \hline U_{1-N+n,1,N-n-1} \end{array}\right], \quad R := \left[\begin{array}{c|c} U_{1-N,n,N-n-1} & U_{-n,n,1} \end{array}\right].$$

By Sylvester's inequality, it follows from

$$\operatorname{rank}(U_{-N,n+1,N-n}) = m(n+1)$$

and the above definitions that

$$\operatorname{rank}(S) = m(n+1) - 1 \tag{4.22}$$

$$\operatorname{rank}(R) = mn. \tag{4.23}$$

Given some initial $K(0)$ and $x(0)$, consider $u(0) = K(0)x(0) + \varepsilon(0)\|x(0)\|$ with $\varepsilon(0) \in B_\delta$. We aim to show that there exists some $\varepsilon(0) \in B_\delta$ such that the Hankel matrix

$U_{1-N,n+1,N-n}$ is also full rank, i.e., $\operatorname{rank}(U_{1-N,n+1,N-n}) = m(n+1)$. We use the definition of S and partition this matrix as

$$U_{1-N,n+1,N-n} = \left[\begin{array}{c|c} S & \dfrac{U_{-n,n,1}}{u(0)} \end{array}\right].$$

Then, it follows from the above equation and (4.22) that

$$m(n+1) - 1 \leq \operatorname{rank}(U_{1-N,n+1,N-n}) \leq m(n+1).$$

We now proceed by contradiction. Suppose that $U_{1-N,n+1,N-n}$ has rank $m(n+1)-1$ for all $\varepsilon(0) \in B_\delta$. This means that for all points inside the ball B_δ, the last column of $U_{1-N,n+1,N-n}$ must lie inside the column space of the matrix S, i.e.,

$$\left[\frac{U_{-n,n,1}}{f_0 + \varepsilon_0\|x(0)\|}\right] \in \operatorname{im} S, \quad \forall \varepsilon_0 \in B_\delta, \tag{4.24}$$

where $\operatorname{im} S$ denotes the image of S and $f_0 := K(0)x(0)$, which implies in particular

$$\left[\frac{U_{-n,n,1}}{f_0}\right] \in \operatorname{im} S.$$

Let some $0 < \rho \leq \delta\|x(0)\|$, then any point $\frac{\rho}{\|x(0)\|}e_i$ with e_i the i-th unit vector of $\mathbb{R}^m$ belongs to the ball B_δ. Therefore, it follows from (4.24) that

$$\left[\frac{U_{-n,n,1}}{f_0 + \rho e_i}\right] \in \operatorname{im} S, \quad \forall i = 1, \ldots, m.$$

We then deduce that the augmented matrix

$$\left[\begin{array}{c|c|ccc} S & \dfrac{U_{-n,n,1}}{f_0} & \dfrac{U_{-n,n,1}}{f_0 + \rho e_1} & \dfrac{\cdots}{\cdots} & \dfrac{U_{-n,n,1}}{f_0 + \rho e_m} \end{array}\right]$$

has rank equal to $m(n+1)-1$. By elementary column operations, the rank of the following matrix

$$M := \left[\begin{array}{c|c|c} S & \dfrac{U_{-n,n,1}}{f_0} & \dfrac{\mathbb{0}}{\rho I_m} \end{array}\right]$$

is equal to $m(n+1)-1$ as well. We use the definitions of S and R to get

$$M = \left[\begin{array}{c|c} R & \mathbb{0} \\ \hline \left[\begin{array}{c|c} U_{1-N+n,1,N-n-1} & f_0 \end{array}\right] & \rho I_m \end{array}\right].$$

Note that the above matrix is block lower triangular, and using (4.23), we have $\operatorname{rank}(M) = \operatorname{rank}(R) + m = m(n+1)$. Thus we have reached to a contradiction, which means that $U_{1-N,n+1,N-n}$ is full rank for some values of $\varepsilon(0) \in B_\delta$. By similar reasoning, it holds that for any $k > 0$ and any input sequence $\{u(k-N), \ldots, u(k-1)\}$ such that $U_{k-N,n+1,N-n}$ has full rank, there exists some $\varepsilon(k) \in B_\delta$ such that the Hankel matrix $U_{k-N+1,n+1,N-n}$ has full row rank, i.e. the input sequence $\{u(k-N+1), \ldots, u(k)\}$ with $u(k) = K(k)x(k) + \varepsilon(k)\|x(k)\|$ is persistently exciting of order $n+1$ which concludes the lemma. ∎

4.7.2. Feasibility of the online SDP I

Proof of Lemma 4.2

Consider $k \in [k_s + T, k_{s+1}]$. In this time interval, the matrices X_{k-1}, U_{k-1}, X_k are made of T input-state samples generated by subsystem $i = \sigma(k_s)$ interconnected with (4.2). Since the input sequence $\{u(k-T), \dots, u(k-1)\}$ is persistently exciting (see Lemma 4.1) and the subsystem $i = \sigma(k_s)$ is controllable (Assumption 4.1), it follows that condition

$$\operatorname{rank} \begin{bmatrix} U_{k-1} \\ X_{k-1} \end{bmatrix} = m + n$$

holds. We then conclude from [41, lemma 3], that problem (4.4) is feasible. Also, any optimal solution $(\gamma_i^*(k), Q_i^*(k), P_i^*(k), L_i^*(k))$ satisfies $K_i(k) = K_{opt}^i$ with $K_i(k) := U_{k-1} Q_i^*(k) P_i^*(k)^{-1}$ and K_{opt}^i is the unique LQR controller of subsystem i. ■

4.7.3. Feasibility of the online SDP II

Proof of Lemma 4.3

Consider the interval $k \in [k_s + 1, k_s + T - 1]$. We recall that such time interval is called transient as the matrices of data X_{k-1} and X_k contain samples generated by both the active subsystem $j = \sigma(k_s)$ and the subsystem active at the previous switching interval, i.e., subsystem $z = \sigma(k_{s-1})$.

We partition the time interval $[k_s+1, k_s+T-1]$ into two sub-intervals $[k_s+1, k_s+T_0]$ and $[k_s + 1 + T_0, k_s + T - 1]$, where T_0 is chosen such that

$$N - 1 \leq T_0 \leq T - N + 1.$$

We remark that T_0 is chosen such that in the above sub-intervals the data matrix X_{k-1} contains at least N samples from the same subsystem. This feature is later used in the proof. Furthermore, we recall the reader that $T \geq 2N - 1$ and $N = (m+1)n + m$ is the minimum length required for the persistence of excitation condition to hold. Based on this partition, we organize the proof in two parts. First, we consider $k \in [k_s + 1, k_s + T_0]$ and show that given subsystem z it is possible to construct a tuple, which we denote with $(\gamma_z, Y_z(k), P_z, L_z)$, feasible for (4.4). Then, we consider $k \in [k_s + T_0 + 1, k_s + T - 1]$ and argues the existence of a tuple $(\gamma_j, Y_j(k), P_j, L_j)$ feasible for (4.4) given subsystem j.

Feasibility for $k \in [k_s + 1, k_s + T_0]$: Intuitively, as we are at the beginning of the transient interval, most of the samples collected in the data matrices have been generated by the subsystem active at the previous switching interval, i.e. subsystem $z = \sigma(k_{s-1})$. Hence, we write the following data equation which relates the data matrices X_{k-1}, U_{k-1}, X_k and the subsystem z:

$$X_k = \begin{bmatrix} B_z & A_z \end{bmatrix} \begin{bmatrix} U_{k-1} \\ X_{k-1} \end{bmatrix} + \Delta \begin{bmatrix} U_{k-1} \\ X_{k-1} \end{bmatrix} E_k, \tag{4.25}$$

where $\Delta := \begin{bmatrix} B_j - B_z & A_j - A_z \end{bmatrix}$ and $E_k \in \mathbb{R}^{T \times T}$ is an auxiliary matrix defined as follows

$$E_k := \begin{bmatrix} \mathbb{0}_{T-t \times T-t} & \mathbb{0}_{T-t \times t} \\ \mathbb{0}_{t \times T-t} & I_t \end{bmatrix}, \quad t := k - k_s. \tag{4.26}$$

Note that $t \in [1, T_0]$. We remark that the matrix E_k is constructed to select the last t columns of $\begin{bmatrix} U_{k-1}^\top & X_{k-1}^\top \end{bmatrix}^\top$. Moreover, we define

$$W_{k-1} := \begin{bmatrix} U_{k-1} \\ X_{k-1} \end{bmatrix}$$

and we argue that W_{k-1} is full row rank for $k \in [k_s + 1, k_s + T_0]$. In fact, note that the input sequence $\{u(k-N), \dots, u(k-1)\}$ is persistently exciting of order $n+1$ (see Lemma 4.1). Also, due to the choice of T_0 and the lower bound on T, the first N columns of W_{k-1} are generated by subsystem z. Consequently, it follows from [20, Cor. 2] that W_{k-1} is full rank. Consider now the LQR controller K^z_{opt} stabilizing subsystem z and denote with P_z the solution of

$$\mathcal{A}_z P_z \mathcal{A}_z^\top - P_z + I = 0 \tag{4.27}$$

where $\mathcal{A}_z := A_z + B_z K^z_{opt}$. Let

$$Q_z(k) := W_{k-1}^\dagger \begin{bmatrix} K^z_{opt} \\ I \end{bmatrix} P_z, \tag{4.28}$$

where $\dagger$ denotes the right inverse. From the above definition we note that $K^z_{opt} = U_{k-1} Q_z(k) P_z^{-1}$. Now, define $L_z := U_{k-1} Q_z(k) P_z^{-1} Q_z(k)^\top U_{k-1}^\top$ and

$$\gamma_z := \mathbf{Tr}(P_z) + \mathbf{Tr}(L_z). \tag{4.29}$$

We will next show that there exists a matrix $S(k) \in \ker W_{k-1}$ such that the tuple $(\gamma_z, Y_z(k), P_z, L_z)$ with $Y_z(k) := Q_z(k) + S(k)$ and $K^z_{opt} = U_{k-1} Q_z(k) P_z^{-1}$ is feasible for (4.4) for $k \in [k_s + 1, k_s + T_0]$.

Consider the constraints of problem (4.4). We observe that the tuple $(\gamma_z, Y_z(k), P_z, L_z)$ satisfies the last four constraints for any $S(k) \in \ker W_{k-1}$. We proceed then by verifying the first constraint, that is

$$X_k \, Y_z(k) P_z^{-1} Y_z(k)^\top \, X_k^\top - P_z + I \preceq 0. \tag{4.30}$$

On the other hand, by writing X_k as (4.25), it is possible to notice that

$$X_k \, Y_z(k) = \mathcal{A}_z P_z + \Sigma_z(k),$$

where $\Sigma_z(k) := \Delta W_{k-1} E_k (Q_z(k) + S(k))$. Note that we have used $Y_z(k) = Q_z(k) + S(k)$ and $W_{k-1} S(k) = 0$. The term $\mathcal{A}_z P_z$ follows by the definition of $Q_z(k)$ in (4.28). Hence, the constraint in (4.30) can be written as

$$\mathcal{A}_z P_z \mathcal{A}_z^\top - P_z + I + \Sigma_z(k) P_z^{-1} \Sigma_z(k)^\top + \mathcal{A}_z \Sigma_z(k)^\top + \Sigma_z(k) \mathcal{A}_z^\top \preceq 0$$

which, considering (4.27), it can be simplified to

$$\Sigma_z(k) P_z^{-1} \Sigma_z(k)^\top + \mathcal{A}_z \Sigma_z(k)^\top + \Sigma_z(k) \mathcal{A}_z^\top \preceq 0.$$

Consequently, checking the feasibility of (4.30) is equivalent to solve the following problem:

$$\begin{aligned} \text{find} \quad & \Sigma_z,\ S \\ \text{subject to} \quad & W_{k-1}S = 0 \\ & \Sigma_z = \Delta W_{k-1}E_k(Q_z(k)+S) \\ & \Sigma_z P_z^{-1}\Sigma_z^\top + \mathcal{A}_z\Sigma_z^\top + \Sigma_z\mathcal{A}_z^\top \preceq 0 \end{aligned} \tag{4.31}$$

We approach the feasibility problem (4.31) by partitioning

$$W_{k-1} = \begin{bmatrix} W_{k-1}^1 & W_{k-1}^2 \end{bmatrix}$$

with $W_{k-1}^1 \in \mathbb{R}^{(n+m)\times(T-t)}$ and $W_{k-1}^2 \in \mathbb{R}^{(n+m)\times t}$, where t is defined in (4.26). Hence, we write the first constraint of (4.31) as

$$\begin{aligned} 0 &= W_{k-1}S \\ &= \begin{bmatrix} W_{k-1}^1 & W_{k-1}^2 \end{bmatrix}\begin{bmatrix} S^1 \\ S^2 \end{bmatrix}, \end{aligned} \tag{4.32}$$

which implies $-W_{k-1}^1 S^1 = W_{k-1}^2 S^2$. Note that at each time instant $k \in [k_s+1, k_s+T_0]$, the dimensions of W_{k-1}^1 and W_{k-1}^2 change. On the other hand, it follows from the definitions of T and T_0 that W_{k-1}^1 has at least N columns generated by subsystem z for $t \in [1, T_0]$. Thus, W_{k-1}^1 is full row rank for $k \in [k_s+1, k_s+T_0]$. This implies that for any S^2, we can find some S^1 to satisfy (4.32) (since W_{k-1}^1 is full row rank) and hence the variable S^2 is free. Then, by using the structure of E_k the second constraint of (4.31) can be rewritten as

$$\Sigma_z = \Delta W_{k-1}^2(Q_z^2(k) + S^2),$$

where $Q_z^2(k)$ is a suitable partition of

$$Q_z(k) = \begin{bmatrix} Q_z^1(k) \\ Q_z^2(k) \end{bmatrix}.$$

As S^2 is free, one can choose $S^2 = -Q_z^2(k)$ to get $\Sigma_z = 0$ and satisfy the last constraint of (4.31). Hence, it is possible to find some Σ_z, S such that all the constraints of (4.31) are satisfied. In other words, this means that the constraint (4.30) is also satisfied. This proves that for $k \in [k_s+1, k_s+T_0]$ it is possible to construct a tuple $(\gamma_z, Y_z(k), P_z, L_z)$ with $Y_z(k) = Q_z(k) + S(k)$ and $K_{opt}^z = U_{k-1}Q_z(k)P_z^{-1}$ feasible to (4.4).

Feasibility for $k \in [k_s+T_0+1, k_s+T-1]$: The second part of the proof follows along the same lines as that of the first part and is reported for the sake of completeness. Roughly speaking, when $k \in [k_s+T_0+1, k_s+T-1]$, the data matrices contains more samples from the currently active subsystem $j = \sigma(k_s)$. In particular, the following equation holds

$$X_k = \begin{bmatrix} B_j & A_j \end{bmatrix} W_{k-1} + \Delta W_{k-1}(E_k - I_T), \tag{4.33}$$

where the matrix $E_k - I_T$ is constructed to select the first $T-t$ columns of W_{k-1} with $t \in [T_0+1, T-1]$. Furthermore, W_{k-1} is full rank. This is due to Lemma 4.1 and to the definition of T_0 and T. In fact, the last N columns of W_{k-1} are generated by subsystem j, and thus, it follows from [20, Cor. 2] that the last N columns of W_{k-1} span $\mathbb{R}^{n+m}$, thus W_{k-1} is full rank. Consider now the corresponding controller K_{opt}^j stabilizing subsystem j and denote with P_j the solution of

$$\mathcal{A}_j P_j \mathcal{A}_j^\top - P_j + I = 0 \tag{4.34}$$

with $\mathcal{A}_j := A_j + B_j K_{opt}^j$. Let

$$Q_j(k) := W_{k-1}^\dagger \begin{bmatrix} K_{opt}^j \\ I \end{bmatrix} P_j.$$

It is clear from above that $K_{opt}^j = U_{k-1} Q_j(k) P_j^{-1}$.
We define $L_j := U_{k-1} Q_j(k) P_j^{-1} Q_j(k)^\top U_{k-1}^\top$ and $\gamma_j := \mathbf{Tr}(P_j) + \mathbf{Tr}(L_j)$, and we show that there exists a matrix $S(k) \in \ker W_{k-1}$ such that the tuple $(\gamma_j, Y_j(k), P_j, L_j)$ with $Y_j(k) := Q_j(k) + S(k)$ and $K_{opt}^j = U_{k-1} Q_j(k) P_j^{-1}$ is feasible to (4.4) for $k \in [k_s + T_0 + 1, k_s + T - 1]$. As in the previous part of the proof, we notice that the last four constraints of problem (4.4) are satisfied. Then, we analyze the first constraint

$$X_k\, Y_j(k) P_j^{-1} Y_j(k)^\top\, X_k^\top - P_j + I \preceq 0. \tag{4.35}$$

By (4.33) we can write $X_k\, Y_j(k) = \mathcal{A}_j P_j + \Sigma_j(k)$ where

$$\Sigma_j(k) := \Delta W_{k-1}(E_k - I_T)(Q_j(k) + S(k)).$$

By substituting the above expression in (4.35) we obtain

$$\Sigma_j(k) P_j^{-1} \Sigma_j(k)^\top + \mathcal{A}_j \Sigma_j(k)^\top + \Sigma_j(k) \mathcal{A}_j^\top \preceq 0,$$

where we used (4.34). Then, we solve:

$$\begin{aligned} \text{find} \quad & \Sigma_j,\ S \\ \text{subject to} \quad & W_{k-1} S = 0 \\ & \Sigma_j = \Delta W_{k-1}(E_k - I_T)(Q_j(k) + S) \\ & \Sigma_j P_j^{-1} \Sigma_j^\top + \mathcal{A}_j \Sigma_j^\top + \Sigma_j \mathcal{A}_j^\top \preceq 0 \end{aligned} \tag{4.36}$$

To solve the above feasibility problem, we write the first constraint as in (4.32). In this case, we note that W_{k-1}^2 is full rank, i.e. $\operatorname{rank} W_{k-1}^2 = n + m$ for all $t \in [T_0+1, T-1]$. This implies that (4.32) admits solutions and S^1 is a free variable. Then, by writing the second constraint of (4.36) as $\Sigma_j = \Delta W_{k-1}^1 (Q_j^1(k) + S^1)$, we can choose $S^1 = -Q_j^1(k)$ to get $\Sigma_j = 0$ and satisfy the last constraint of (4.36). Hence, it follows that we can find Σ_j, S to solve problem (4.36), and thus show feasibility of (4.35). This also shows that for $k \in [k_s + T_0 + 1, k_s + T - 1]$ it is possible to construct a tuple $(\gamma_j, Y_j(k), P_j, L_j)$ with $Y_j(k) = Q_j(k) + S(k)$ and $K_{opt}^j = U_{k-1} Q_j(k) P_j^{-1}$ feasible to (4.4), as we claimed. Hence, this concludes the proof. ■

5

Learning controllers from data via nonlinearity cancellation

In the previous chapters we have considered the design of data-driven controllers for linear systems and a very special class of nonlinear systems, that is switched linear systems. Unsurprisingly, deriving solutions for a more general class of nonlinear systems is harder. In this chapter, we introduce a method to deal with the data-driven control design of *nonlinear* systems. We derive conditions to design controllers via (approximate) nonlinearity cancellation. These conditions take the compact form of data-dependent semi-definite programs. The method returns controllers that can be certified to stabilize the system even when data are perturbed and disturbances affect the dynamics of the system during the execution of the control task, in which case an estimate of the robustly positively invariant set is provided.

5.1. Introduction

Most physical systems are inherently *nonlinear* in nature, making nonlinear systems one of the most interesting research areas for engineers, physicists, mathematicians and other scientists. However, identification and control of nonlinear systems has always been a challenging and non-trivial task. For this reason, various results on direct data-driven control have been proposed and much attention has been devoted to the specific study of nonlinear systems. Earlier representative results of data-driven control of nonlinear systems include the nonlinear extension of the virtual reference feedback tuning (VRFT) approach [114], the design of controllers in the form of kernel functions tuned using data via set-membership identification techniques [36], and the so-called model-free control [30, 33]. A way to deal with nonlinear systems is to exploit some structure, when it is a priori known the class to which the system belongs. Data-driven control of second-order Volterra systems is studied in [58] and data-dependent LMI-based stabilization of bilinear systems in [61], the latter being motivated by Carleman bilinearization of general nonlinear systems. A point-to-point optimal control problem for bilinear systems is formulated in the recent work [62]. The data-driven control design for polynomial systems is the subject of [60, 115]. While [115] uses Rantzer's dual Lyapunov's theory and moments based techniques, [60] uses Lyapunov second method and a particular parametrization of the Lyapunov function to obtain SOS programs whose feasibility directly provide stabilizing controllers. See [68] for additional results on the data-driven control design of polynomial systems based on Petersen's lemma. When the system is not polynomial, the approach in [60] returns a state-dependent matrix condition rather than an SOS condition. If such a state-dependent matrix condition can be solved at each time step along a trajectory of the system, then a control sequence that steers that trajectory to the origin is obtained. This idea is pursued in [116].

In this chapter, we introduce a method to deal with the data-driven control design of nonlinear systems. For doing so, we build up on and strengthen the results of [40] in several directions.

We first consider nonlinear vector fields that are expressed as combinations of known nonlinear functions (not necessarily polynomials). We then derive conditions to design from data controllers that stabilize the closed-loop system via nonlinearity cancellations. This approach returns formulas for controller design which retain the same simplicity and compactness of the formulas established in [40] for linear systems, namely semidefinite programs (SDP) only depending on data.

We then make the crucial observation that, were exact nonlinearity cancellation unfeasible, we can instead formulate an optimization program, i.e. semidefinite program (SDP), that minimizes the norm of the matrix by which the nonlinearities enter the dynamics. This idea is suggested by a regularization procedure in which the hard constraint of the first approach, corresponding to an exact nonlinearity cancellation, is lifted to an objective function, corresponding to an *approximate* nonlinearity cancellation. In different contexts, this "lifting" idea has been pursued in [117–119]. In general the design based on an approximate nonlinearity cancellation does not return globally stabilizing controllers, whence the need to explicitly characterize the region of attraction of the closed-loop system. We show that this is indeed possible by bounding the Lyapunov decrement via functions which are obtainable from the data. We remark here that, although we focus on nonlinear discrete-time systems, analogous results can be derived for continuous-time systems too.

To present the main ideas, we choose to give the results first for data that are not perturbed. The results are then extended to the case is which data are perturbed by process disturbances. In doing so, we show how our approach can accommodate the presence of process disturbances not only during the collection of data used in the controller design, but also during the execution of the control task and provide estimates of *robustly positively invariant sets* [120] for the closed-loop system. The results are also extended to systems with nonlinearities that are not expressible as combination of known functions. By doing so, we significantly enlarge the class of nonlinear systems the approach can cope with.

The framework is presented in Section 5.2. The main results are discussed in Sections 5.3 and 5.4, with some extensions in Section 5.5. Control design in the presence of disturbances and neglected nonlinearities is studied in Section 5.6. Some additional discussion is finally provided in Section 5.7.

5.2. FRAMEWORK

We consider a discrete-time system in the form

$$x^+ = A_\star Z_\star(x) + Bu \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control input, $A_\star \in \mathbb{R}^{n\times R}$, $B \in \mathbb{R}^{n\times m}$ are constant matrices, $Z_\star : \mathbb{R}^n \to \mathbb{R}^R$ is a vector-valued function. Any nonlinear system in the form $x^+ = f(x) + Bu$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ being an arbitrary function can be written as in (5.1); we adopt the representation (5.1) for convenience. In this chapter, $A_\star$ and B are regarded unknown while the following standing assumption is made for $Z_\star$.

Assumption 5.1 *We know a function $Z : \mathbb{R}^n \to \mathbb{R}^S$ such that any element of $Z_\star$ is also an element of Z.* •

Under Assumption 5.1, system (5.1) reads equivalently as

$$x^+ = AZ(x) + Bu \tag{5.2}$$

with $A \in \mathbb{R}^{n\times S}$, and A, B unknown. Assumption 5.1 amounts to considering systems with known *type* of dynamics (but possibly unknown parameters). This assumption is satisfied in many practical cases such as with mechanical and electrical systems where information about the dynamics can be derived from first principles, but the exact systems parameters may be unknown. We allow Z to contain terms not present in $Z_\star$, which may arise from an imprecise knowledge of the system dynamics. In this Chapter, we will directly consider the case where Z contains *both* linear and nonlinear functions, *i.e.*,

$$Z(x) = \begin{bmatrix} x \\ \hline Q(x) \end{bmatrix}, \tag{5.3}$$

with $Q : \mathbb{R}^n \to \mathbb{R}^{S-n}$ containing only nonlinear functions. The special case where $Z(x) = x$ reduces the analysis to that of linear systems. In contrast, $Z(x) = Q(x)$ accounts for purely nonlinear systems, and just leads to simplified algorithms and results. We will exemplify this point in connection with Theorem 5.1. Let

$$\mathbb{D} := \{x(k), u(k)\}_{k=0}^{T} \tag{5.4}$$

be a dataset collected from the system with an experiment, meaning that we have a set of state and input samples that satisfy $x(k+1) = AZ(x(k)) + Bu(k)$ for $k = 0, \dots, T-1$, with $T > 0$. The problem of interest is to determine, using $\mathbb{D}$, a control law

$$u = KZ(x)$$

that stabilizes the system around the origin (globally or locally, both cases will be considered). Note that we might consider a control law $u = KH(x)$ with H different from Z. As it will become clear soon, we focus on $u = KZ(x)$ as our approach is based on nonlinearity cancellation / minimization.

In the course of this Chapter, we will extend the framework in several directions:

(i) We analyze the case of continuous-time systems, which we handle with similar arguments (Section 5.5.1).

(ii) We extend the analysis to a more general class of nonlinear systems (Section 5.5.2).

(iii) We consider noisy data and neglected nonlinearities in Section 5.6.

5.3. Exact nonlinearity cancellation

We start by considering the scenario in which there exists a controller K that linearizes the closed-loop dynamics, namely the scenario in which there exists a controller K such that

$$u = KZ(x) \quad \Longrightarrow \quad x^+ = Mx \tag{5.5}$$

for some matrix M (which we will also require to be Schur[1]).

5.3.1. Data-based closed-loop representation and control design

Consider the dataset $\mathbb{D}$ in (5.4), and define

$$U_0 := \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix} \in \mathbb{R}^{m\times T}\,, \tag{5.6a}$$

$$X_0 := \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix} \in \mathbb{R}^{n\times T}\,, \tag{5.6b}$$

$$X_1 := \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix} \in \mathbb{R}^{n\times T}\,, \tag{5.6c}$$

$$Z_0 := \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \\ Q(x(0)) & Q(x(1)) & \cdots & Q(x(T-1)) \end{bmatrix} \in \mathbb{R}^{S\times T}\,, \tag{5.6d}$$

All the results of this chapter rest on the following lemma. An analogous result was established in [59, Lemma 1] for the case of polynomial systems.

Lemma 5.1 *Consider any matrices $K \in \mathbb{R}^{m\times S}$, $G \in \mathbb{R}^{T\times S}$ such that*

$$\begin{bmatrix} K \\ I_S \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G\,. \tag{5.7}$$

Let G be partitioned as $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$, where $G_1 \in \mathbb{R}^{T\times n}$ and $G_2 \in \mathbb{R}^{T\times (S-n)}$. Then, system (5.1) *under the control law $u = KZ(x)$ results in the closed-loop dynamics*

$$x^+ = Mx + NQ(x) \tag{5.8}$$

where $M := X_1 G_1$ and $N := X_1 G_2$.

Proof. The closed-loop dynamics resulting from the control law $u = KZ(x)$ is given by

$$x^+ = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_S \end{bmatrix} Z(x) \tag{5.9a}$$

$$= \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} GZ(x) \;=\; X_1 GZ(x)\,. \tag{5.9b}$$

[1]A matrix M is said to be Schur if all its eigenvalues have modulus less than one. For continuous-time systems, a matrix M is said to be Hurwitz if all its eigenvalues have negative real part.

The second identity follows from (5.7) while the last one follows because the elements of X_1, Z_0 and U_0 satisfy the relation $x(k+1) = AZ(x(k)) + Bu(k)$, $k = 0, \dots, T-1$, which, in compact form, gives $X_1 = AZ_0 + BU_0$. ■

Arrived at this stage, it is simple to derive a convex program (specifically a semi-definite program (SDP)) that searches for a controller K that cancels out the nonlinearities and renders the closed-loop system (globally) asymptotically stable. Note that in next Theorem 5.1 the decision variable G_2 represents the same quantity that appears in Lemma 5.1. The decision variables Y_1, P_1 are instead related to G_1 in Lemma 5.1 via $Y_1 = G_1 P_1$ with P_1 a positive definite matrix, that is Y_1 defines a change of variable relative to G_1. As it becomes clear from the proof of Theorem 5.1, this change of variable is instrumental to arrive at a convex formulation of the design program.

Theorem 5.1 *Consider a nonlinear system as in* (5.1)*, along with the following SDP in the decision variables $P_1 \in \mathbb{S}^{n\times n}$, $Y_1 \in \mathbb{R}^{T\times n}$, and $G_2 \in \mathbb{R}^{T\times(S-n)}$:*

$$Z_0 Y_1 = \begin{bmatrix} P_1 \\ 0_{(S-n)\times n} \end{bmatrix}, \tag{5.10a}$$

$$\begin{bmatrix} P_1 & (X_1 Y_1)^\top \\ X_1 Y_1 & P_1 \end{bmatrix} \succ 0\,, \tag{5.10b}$$

$$Z_0 G_2 = \begin{bmatrix} 0_{n\times(S-n)} \\ I_{S-n} \end{bmatrix}, \tag{5.10c}$$

$$X_1 G_2 = 0_{n\times(S-n)}\,. \tag{5.10d}$$

If the SDP is feasible then the control law $u = KZ(x)$ with

$$K = U_0 \begin{bmatrix} Y_1 & G_2 \end{bmatrix} \begin{bmatrix} P_1 & 0_{n\times(S-n)} \\ 0_{(S-n)\times n} & I_{S-n} \end{bmatrix}^{-1} \tag{5.11}$$

linearizes the closed-loop dynamics, and renders the origin a globally asymptotically stable equilibrium.

Proof. Suppose that (5.10) is feasible. Let $G_1 = Y_1 P_1^{-1}$ and note that the two constraints (5.10a) and (5.10c) together yield

$$Z_0 \begin{bmatrix} G_1 & G_2 \end{bmatrix} = I_S\,. \tag{5.12}$$

This relation, combined with (5.11), gives

$$\begin{bmatrix} K \\ I_S \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \end{bmatrix}, \tag{5.13}$$

which is (5.7). By Lemma 5.1, we conclude that the closed-loop dynamics satisfies $x^+ = Mx + NQ(x)$ with $M = X_1 G_1$ and $N = X_1 G_2$. By (5.10d), $N = 0$. Hence, K linearizes the closed-loop dynamics. Finally, note that (5.10b) is equivalent to $P_1 \succ 0$ and $(X_1 Y_1)^\top P_1^{-1} (X_1 Y_1) - P_1 \prec 0$. The latter, in turn, is equivalent to $(X_1 Y_1 P_1^{-1})^\top P_1^{-1} (X_1 Y_1 P_1^{-1}) - P_1^{-1} \prec 0$. By recalling that $Y_1 P_1^{-1} = G_1$ and

$X_1 G_1 = M$, we conclude that M is Schur. (This also shows that $V(x) = x^\top P_1^{-1} x$ is a Lyapunov function for the closed-loop system.) ■

Theorem 5.1 gives an extension to nonlinear systems of the results in [40]. In fact, in the limit case where $Z(x) = x$ we have $S = n$ and (5.10) reduces to the first two constraints (5.10a)-(5.10b). In general, (5.10c)-(5.10d) implement the linearization constraint, and (5.10a)-(5.10b) ensure a stable behavior for the linear dynamics. Note in particular that (5.10c), together with (5.10a), forms a consistency relation which makes it possible to parametrize the closed-loop dynamics through data alone. The other extreme case occurs when Z contains only nonlinear functions, *i.e.,* when $Z(x) = Q(x)$. In this case, (5.10) reduces to the two constraints (5.10c)-(5.10d). This corresponds to a situation where the system has stable open-loop linear dynamics and the controller is only responsible for canceling out all the nonlinearities.

As a second remark, we observe that a necessary condition for the SDP (5.10) to be feasible is that Z_0 has full row rank. This is indeed necessary to have both (5.10a) and (5.10c) fulfilled. This requirement can be viewed as a condition on the *richness* of the data, and is the natural generalization of the condition on the rank of X_0 that appears in the linear case. This condition is weaker than having $\left[\begin{smallmatrix} U_0 \\ Z_0 \end{smallmatrix}\right]$ full row rank, which is instead necessary to identify A, B from data, and this shows that learning a control law is in general easier than identifying the dynamics of the system. Note that Lemma 5.1 indeed gives a data-based *closed-loop* representation of the system dynamics, without any explicit estimate of the system matrices.

Having $\left[\begin{smallmatrix} U_0 \\ Z_0 \end{smallmatrix}\right]$ full row rank brings certain advantages, though. In fact, in this case, *any* controller that linearizes the closed-loop dynamics can be parametrized through the data. In particular, in this situation we obtain an "*if and only if*" result, meaning that (5.10) is feasible and returns a stabilizing and linearizing controller whenever such a controller exists.

Theorem 5.2 *Suppose there exists a stabilizing and linearizing feedback controller,* i.e.*, a controller $K = [\overline{K} \quad \hat{K}]$ such that*

$$A + BK = \left[\overline{A} + B\overline{K} \quad 0_{n\times(S-n)}\right] \tag{5.14a}$$

$$\overline{A} + B\overline{K} \text{ is Schur} \tag{5.14b}$$

having partitioned $A = [\overline{A} \quad \hat{A}]$ with $\overline{A} \in \mathbb{R}^{n\times n}$. Let $\left[\begin{smallmatrix} U_0 \\ Z_0 \end{smallmatrix}\right]$ have full row rank. Then (5.10) *is feasible and K can be written as in* (5.11) *for some Y_1, P_1, G_2 satisfying* (5.10).

Proof. See Section 5.9. ■

Example 1

Consider the Euler discretization of an inverted pendulum

$$x_1^+ = x_1 + T_s x_2 \tag{5.15a}$$

$$x_2^+ = \frac{T_s g}{\ell} \sin x_1 + \left(1 - \frac{T_s \mu}{m\ell^2}\right) x_2 + \frac{T_s}{m\ell^2} u\,, \tag{5.15b}$$

where T_s is the sampling time, m is the mass to be balanced, ℓ is the distance from the base to the center of mass of the balanced body, μ is the coefficient of rotational friction, and g is the acceleration due to gravity. The states x_1, x_2 are the angular position and velocity, respectively, u is the applied torque. The system has an unstable equilibrium in $(x,u) = (0,0)$, corresponding to the pendulum upright position, which we want to stabilize. Suppose that the parameters are $T_s = 0.1$, $m = 1$, $\ell = 1$, $g = 9.8$ and $\mu = 0.01$.

We choose $Z(x) = \begin{bmatrix} x_1 & x_2 & \sin(x_1) \end{bmatrix}^\top$, and regard all the parameters T_s, m, ℓ, g, μ as unknown (here, a correct choice for $Z(x)$ simply derives from physical considerations, namely Lagrange's equations of motion). We collect data by running an experiment with input uniformly distributed in $[-0.5, 0.5]$, and with an initial state within the same interval. We collect $T = 10$ samples (corresponding to the motion of the pendulum that oscillates around the upright position). The SDP (5.10) is feasible and we obtain $K = \begin{bmatrix} -23.5641 & -10.3901 & -9.8 \end{bmatrix}$. The resulting control law indeed cancels out the nonlinearity ensuring global asymptotic stability. •

Example 2

Consider the polynomial system

$$x_1^+ = x_2 + x_1^3 + u \tag{5.16a}$$

$$x_2^+ = 0.5 x_1 \,. \tag{5.16b}$$

Suppose that we choose

$$Z(x) = \begin{bmatrix} x^\top & x_1^2 & x_2^2 & x_1 x_2 & x_1^3 & x_2^3 & x_1 x_2^2 & x_1^2 x_2 \end{bmatrix}^\top , \tag{5.17}$$

i.e., we capture the nonlinearity by including all the possible monomials up to degree 3. The equilibrium of the unforced system ($u = 0$) is only locally asymptotically stable (*e.g.*, any initial condition such that $x_1(0) > 1$ and $x_2(0) \geq 0$ leads to a divergent solution). We collect data by running an experiment with input uniformly distributed in $[-0.5, 0.5]$, and with an initial state within the same interval. We collect $T = 10$ samples. The SDP is feasible and returns the controller

$$K = [\, \underbrace{0}_{x_1} \;\; \underbrace{-1.0007}_{x_2} \;\; \underbrace{0}_{x_1^2} \;\; \underbrace{0}_{x_2^2} \;\; \underbrace{0}_{x_1 x_2} \;\; \underbrace{-1}_{x_1^3} \;\; \underbrace{0}_{x_2^3} \;\; \underbrace{0}_{x_1 x_2^2} \;\; \underbrace{0}_{x_1^2 x_2} \,] \tag{5.18}$$

The SDP correctly assigns the value -1 to the sixth entry of K, and automatically discovers that no other nonlinearities are present. The resulting control law is $u = -1.0007 x_2 - x_1^3$ and ensures global asymptotic stability. •

The examples show that even a few samples may suffice to learn a stabilizing control policy. In fact, in terms of number of data points, the only necessary condition in (5.10) comes from having Z_0 full row rank, and this condition can be met even with $T = S$ samples. The situation may be different with noisy data as we discuss in Section 5.6. As a second remark, note that this approach considers *nonlinear* control laws; this is indeed essential to achieve nonlinearity cancellation (or nonlinearity minimization, if cancellation is impossible, as we discuss in Section 5.4).

5.3.2. Nonlinearity cancellation as a minimization problem

A variant of (5.10) consists in approaching the design problem as a *minimization* problem, namely as the problem of finding a controller that minimizes the nonlinearity in closed loop with respect to some chosen norm.

Theorem 5.3 *Consider a nonlinear system as in* (5.1) *along with the following SDP in the decision variables* $P_1 \in \mathbb{S}^{n\times n}$, $Y_1 \in \mathbb{R}^{T\times n}$, *and* $G_2 \in \mathbb{R}^{T\times(S-n)}$*:*

$$\min_{P_1,Y_1,G_2} \quad \|X_1 G_2\| \tag{5.19a}$$

$$\text{subject to} \quad Z_0 Y_1 = \begin{bmatrix} P_1 \\ 0_{(S-n)\times n} \end{bmatrix} \tag{5.19b}$$

$$\begin{bmatrix} P_1 & (X_1 Y_1)^\top \\ X_1 Y_1 & P_1 \end{bmatrix} \succ 0 \tag{5.19c}$$

$$Z_0 G_2 = \begin{bmatrix} 0_{n\times(S-n)} \\ I_{S-n} \end{bmatrix} \tag{5.19d}$$

*If this SDP is feasible and the solution achieves zero cost (*i.e., $\|X_1 G_2\| = 0$*) then the control law* $u = KZ(x)$ *with* K *given by* (5.11) *linearizes the closed-loop dynamics, and renders the origin a globally asymptotically stable equilibrium. (Here,* $\|\cdot\|$ *is any norm.)*

Proof. The proof is analogous to the proof of Theorem 5.1 and therefore omitted. ■

Example 3

Consider again system (5.16) under the same experimental setting as before. The SDP (5.19) is feasible and we obtain (we use the induced 2-norm in (5.19a))

$$K = \begin{bmatrix} 0.0001 & -1.0007 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{5.20}$$

As before, the program correctly assigns the value -1 to the sixth entry of K. Note that when nonlinearity cancellation is possible, (5.10) and (5.19) are *equivalent* in the sense that their feasible sets coincide. The controller in (5.20) differs from the one in (5.18) simply because there are infinitely many stabilizing and linearizing controllers and neither (5.10) nor (5.19) involve constraints other than stability and linearization. •

5.4. Approximate nonlinearity cancellation

There is a simple yet important difference between (5.10) and its lifted version (5.19). The difference is that the latter is always feasible when the former is feasible and this implies that we can always use (5.19) in place of (5.10) when exact nonlinearity cancellation is possible. In the following, we see that (5.19) can be adopted even when exact cancellation is impossible, in which case (5.10) is instead infeasible.

5.4.1. CONTROL DESIGN FOR APPROXIMATE NONLINEARITY CANCELLATION

The next result indeed addresses the scenario where exact cancellation is impossible. It shows in particular that, in this case, we can still have stability guarantees.

Theorem 5.4 *Consider a nonlinear system as in* (5.1)*, along with the SDP* (5.19)*. Assume that*

$$\lim_{|x|\to 0} \frac{|Q(x)|}{|x|} = 0\,. \tag{5.21}$$

If the SDP is feasible then $u = KZ(x)$, with K as in (5.11)*, renders the origin an asymptotically stable equilibrium.*

Proof. The first part of the proof is analogous to that of Theorem 5.1. Suppose that (5.19) is feasible. Let $G_1 = Y_1 P_1^{-1}$, and note that the two constraints (5.19b) and (5.19d) together yield $Z_0 \begin{bmatrix} G_1 & G_2 \end{bmatrix} = I_S$. This identity, along with (5.11), gives (5.7). By Lemma 5.1, we have that the closed-loop dynamics satisfies $x^+ = Mx + NQ(x)$, where $M = X_1 G_1$ and $N = X_1 G_2$. Although N might be different from zero, (5.19c) ensures that M is Schur. Asymptotic stability thus follows from (5.21). ■

In Theorem 5.4, the condition $\lim_{|x|\to 0} \frac{|Q(x)|}{|x|} = 0$ ensures that the linear dynamics dominates the nonlinear dynamics around the origin. In turn, as shown in the next subsection, this guarantees that we can obtain an estimate of the *region of attraction*. This condition is satisfied for many systems of practical relevance, for instance is satisfied by *any* polynomial system. More generally, the condition $\lim_{|x|\to 0} \frac{|Q(x)|}{|x|} = 0$ can be rephrased by asking that Z is differentiable at $x = 0$ and satisfies $Z(0) = 0$. In fact, in this case Q is differentiable at $x = 0$ and satisfies $Q(0) = 0$, hence it admits a Taylor's expansion at $x = 0$, namely we have

$$Q(x) = \left[\frac{\partial Q}{\partial x}\right]_{x=0} x + r(x) \tag{5.22a}$$

with $r : \mathbb{R}^n \to \mathbb{R}^{S-n}$ a differentiable function of the state such that $\lim_{|x|\to 0} \frac{|r(x)|}{|x|} = 0$. Thus, system (5.1) can be equivalently represented as

$$\begin{aligned} x^+ &= \overline{A}x + \hat{A}Q(x) + Bu && (5.23a)\\ &= (\overline{A} + \hat{A}F)x + \hat{A}r(x) + Bu && (5.23b) \end{aligned}$$

where we have partitioned A as $A = \begin{bmatrix} \overline{A} & \hat{A} \end{bmatrix}$ with $\overline{A} \in \mathbb{R}^{n\times n}$. Hence, Theorem 5.4 becomes applicable with Q replaced by r, where r can be determined from Q. As an example, for the inverted pendulum this reasoning leads to $r(x) = \sin(x_1) - x_1$, which gives $\lim_{|x|\to 0} \frac{|r(x)|}{|x|} = 0$ (for the inverted pendulum Theorem 5.4 reduces in any case to Theorem 5.3 because exact cancellation is possible).

We point out that there exists a counterpart of Theorem 5.2, which provides conditions under which we can parametrize *all* feedback controllers that ensure local stability through a stable linear dynamics. We state the result in the following Theorem.

Theorem 5.5 *Suppose that there exists a feedback controller,* $K = [\overline{K} \quad \hat{K}]$ *such that* $\overline{A} + B\overline{K}$ *is Schur, having partitioned* $A = [\overline{A} \quad \hat{A}]$ *with* $\overline{A} \in \mathbb{R}^{n\times n}$*. Let* $\left[\begin{smallmatrix} U_0 \\ Z_0 \end{smallmatrix}\right]$ *have full row rank. Then* (5.19) *is feasible and* K *can be written as in* (5.11) *for some* P_1, Y_1, G_2 *satisfying* (5.19).

Proof. See Section 5.9 ■

5.4.2. Estimating the region of attraction

We provide the following definition.

Definition 5.1 *A set* $\mathcal{S}$ *is called* positively invariant *(PI) for the system* $x^+ = f(x)$ *if for every* $x(0) \in \mathcal{S}$ *the solution is such that* $x(t) \in \mathcal{S}$ *for* $t > 0$*. Let* $\overline{x}$ *be an asymptotically stable equilibrium point for the system* $x^+ = f(x)$*. A set* $\mathcal{R}$ *defines a* region of attraction *(ROA) for the system relative to* $\overline{x}$ *if for every* $x(0) \in \mathcal{R}$ *we have* $\lim_{t\to\infty} x(t) = \overline{x}$*.* •

Building on Theorem 5.4, we can give estimates of the ROA for the closed-loop system relative to the equilibrium $\overline{x} = 0$. Consider the same conditions as in Theorem 5.4 and note that $V(x) := x^\top P_1^{-1} x$ is a Lyapunov function for the linear part of the dynamics. In particular,

$$V(x^+) - V(x) = \underbrace{(Mx + NQ(x))^\top P_1^{-1}(Mx + NQ(x)) - x^\top P_1^{-1} x}_{=:h(x)}. \tag{5.24}$$

where the matrices M, N and P_1 are all computable from data. We immediately obtain the following result.

Proposition 5.1 *Consider the same setting as in Theorem 5.4. Let* $\mathcal{V} := \{x : h(x) < 0\}$ *with* $h(x)$ *as in* (5.24)*, and consider the Lyapunov function* $V(x) = x^\top P_1^{-1} x$*. Then, any sub-level set* $\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$ *of* V *contained in* $\mathcal{V} \cup \{0\}$ *is a PI set for the closed-loop system and defines an estimate of the ROA relative to* $\overline{x} = 0$*.* •

We close this section with an example that illustrates both Theorem 5.4 and Proposition 5.1.

EXAMPLE 4
Consider the nonlinear system

$$x_1^+ = x_2 + x_1^3 + u \tag{5.25a}$$
$$x_2^+ = 0.5x_1 + 0.2x_2^2 \tag{5.25b}$$

under the same experimental setting as before, in particular $Z(x)$ is as in (5.17). Exact nonlinearity cancellation is now impossible. Nonetheless, the SDP (5.19) is feasible and returns the controller K (we take the induced 2-norm in the objective function):

$$K = \big[\underbrace{-0.0113}_{x_1} \ \underbrace{-1.0862}_{x_2} \ \underbrace{0.0005}_{x_1^2} \ \underbrace{0}_{x_2^2} \ \underbrace{0.0039}_{x_1x_2} \ \underbrace{-1.0010}_{x_1^3} \ \underbrace{-0.0130}_{x_2^3} \ \underbrace{0.0119}_{x_1x_2^2} \ \underbrace{-0.0010}_{x_1^2x_2}\big] \tag{5.26}$$

For this controller, we numerically determine the set $\mathcal{V} = \{x : h(x) < 0\}$ over which the Lyapunov function $V(x) = x^\top P_1^{-1} x$ decreases and a sub-level set $\mathcal{R}_\gamma$ of V contained in $\mathcal{V} \cup \{0\}$ which gives a valid estimate of the ROA. These two sets are displayed in Figure 5.1. We note that the SDP (5.19) almost assigns the value -1 to the sixth entry of K, thus reducing the effect of the nonlinearity on the first state component. Specifically, this controller results in the matrices M and N given by

$$M = \begin{bmatrix} -0.0113 & -0.0862 \\ 0.5000 & 0 \end{bmatrix},$$

$$N = \begin{bmatrix} 0.0005 & 0 & 0.0039 & -0.0010 & -0.0130 & 0.0119 & -0.0010 \\ 0 & 0.2000 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and the matrix N has indeed minimum norm $\|N\| = 0.2$ (this value cannot be further reduced because the term $0.2x_2^2$ cannot be canceled out).

The approach that we just described for estimating the ROA is fully automatic and is generically applicable. Note, however, that once we compute a controller K then we can pursue *any* approach (data- or model-based) to estimate the ROA. In fact, the SPD (5.19) returns the *exact* description of the closed-loop dynamics: $x^+ = \begin{bmatrix} M & N \end{bmatrix} Z(x)$ (we stress that this expression does not correspond to identifying open-loop dynamics of the system). From this description, we can then indeed apply any technique to find Lyapunov functions and estimate the ROA, see for instance [121, Section 8.2].

To illustrate this point in a simple manner, suppose that (5.19) returns

$$K = [\, \underbrace{0}_{x_1} \;\; \underbrace{-1}_{x_2} \;\; \underbrace{0}_{x_1^2} \;\; \underbrace{0}_{x_2^2} \;\; \underbrace{0}_{x_1 x_2} \;\; \underbrace{-1}_{x_1^3} \;\; \underbrace{0}_{x_2^2} \;\; \underbrace{0}_{x_1 x_2^2} \;\; \underbrace{0}_{x_1^2 x_2} \,] \tag{5.27}$$

(this is indeed what we obtain with a variant of (5.19), see next (5.29)), from which we have

$$M = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix}, \quad N = \left[\begin{array}{cc|c} 0 & 0 & 0_{1\times 5} \\ 0.2 & 0 & 0_{1\times 5} \end{array}\right],$$

or, equivalently,

$$x_1^+ = 0 \tag{5.28a}$$

$$x_2^+ = 0.5x_1 + 0.2x_2^2\,. \tag{5.28b}$$

From the closed-loop dynamics we conclude that the exact ROA is given by the set $\mathcal{R} := \{x : |0.5x_1 + 0.2x_2^2| < 5\}$. In fact, the solution to system (5.28) is given by $x_1(t) = 0$ for $t \geq 1$ e $x_2(t) = b^{-1}(b(ax_1(0) + bx_2(0)^2))^{2^{t-1}}$ for $t \geq 2$, with $a = 0.5$ and $b = 0.2$. Hence, the solution converges asymptotically if and only if $|b(ax_1(0) + bx_2(0)^2)| < 1$, from which one infers the ROA $\mathcal{R}$ specified above. This is a situation where it is simple to exactly compute by inspection the ROA, which gives a better result with respect to the automatic procedure, *cf.* Figures 5.2, 5.3. The automatic procedure, however, is applicable even when an exact description of the closed-loop dynamics is not available, as it is the case when noisy data are being measured, a case examined in Section 5.6. •

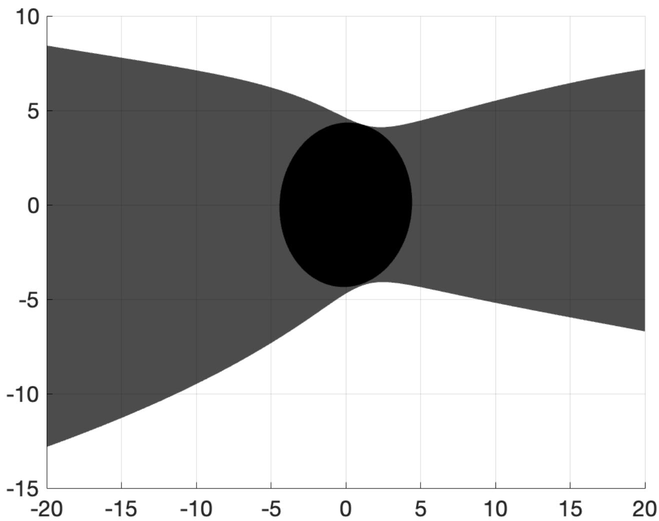

Figure 5.1: Results for Example 4. Sets $\mathcal{V}$ and $\mathcal{R}_\gamma$ in grey and black color, respectively, for the controller K in (5.26) (we recall that $\mathcal{R}_\gamma$ is a valid estimate for the ROA).

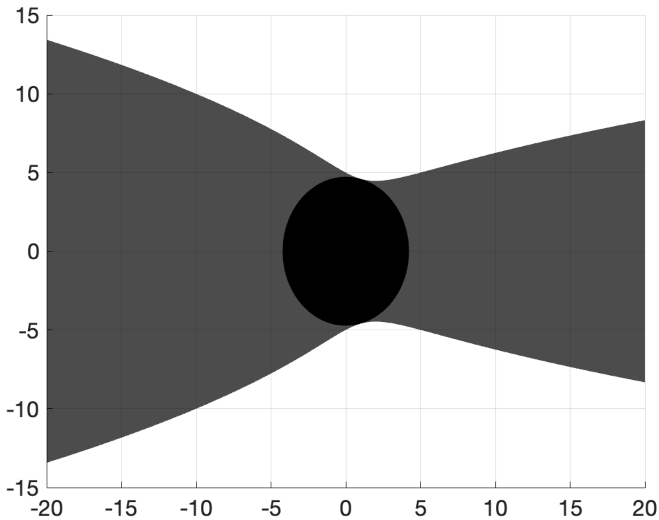

Figure 5.2: Results for Example 4. Sets $\mathcal{V}$ and $\mathcal{R}_\gamma$ in grey and black color, respectively, for the controller K in (5.27).

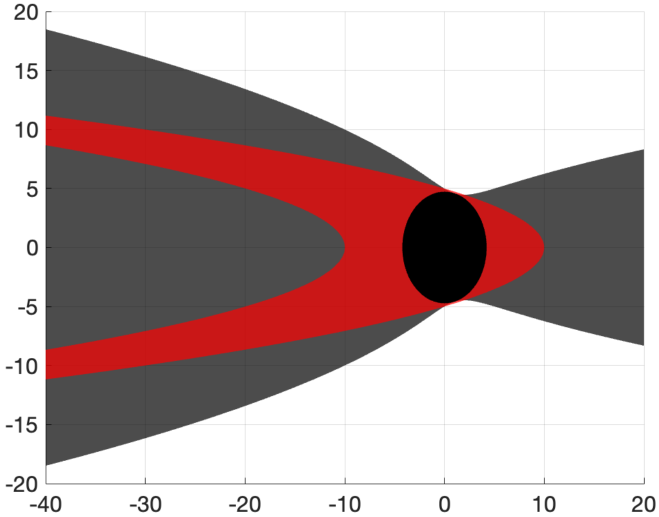

Figure 5.3: Results for Example 4. Sets $\mathcal{V}$, $\mathcal{R}_\gamma$ and $\mathcal{R}$ (exact ROA) for the controller K in (5.27). The set $\mathcal{R}$ is displayed in red color.

We conclude this section with a few additional remarks.

As a first comment, note that the SDP (5.19) can also be used to infer the stability properties of any controller K for which a solution to (5.7) exists. This can be done by regarding (5.11) as an additional constraint to (5.19), *i.e.*, by adding the constraint

$$U_0 \begin{bmatrix} Y_1 & G_2 \end{bmatrix} = K \begin{bmatrix} P_1 & 0_{n\times(S-n)} \\ 0_{(S-n)\times n} & I_{S-n} \end{bmatrix}$$

which is convex. This can be useful whenever a controller is inferred based on physical intuition and we want to determine closed-loop stability properties *before* inserting the controller into the loop. For the same reason, by adding the constraint $U_0 \begin{bmatrix} Y_1 & G_2 \end{bmatrix} = 0$ we infer the ROA for the open-loop system.

As a final observation, we mention a particularly effective variant of (5.19):

$$\min_{P_1,Y_1,G_2,X,V} \quad \mathbf{Tr}(X) + \mathbf{Tr}(V) \tag{5.29a}$$

$$\text{subject to} \quad (5.19\text{b}) - (5.19\text{d}) \tag{5.29b}$$

$$\begin{bmatrix} X & X_1 G_2 \\ (X_1 G_2)^\top & V \end{bmatrix} \succeq 0 \tag{5.29c}$$

This SDP uses the trace as a convex envelope of the rank [122], hence it searches for solutions yielding a *sparse* nonlinear term $N = X_1 G_2$, which can be useful to analyse properties of the closed-loop system, including the ROA. Applied to Example 4, this SDP indeed systematically returns a controller with third-to-ninth entries as in (5.27). If we further regularize (5.29) by enforcing a sparsity term for X_1Y_1, the SDP exactly returns (5.27) (systematically for different datasets). In a sense, the cost function in (5.29) is analogous to regularization terms used in regression algorithms to penalize complex models [17]. The difference is that (5.29) promotes low-complexity (sparse) *closed-loop* systems (the matrix X_1G_2), and this favours low-complexity (sparse) control laws.

5.5. Extensions

The proposed approach can be extended in many directions. In this section, we discuss two of them.

5.5.1. Continuous-time systems

Continuous-time systems can be treated in a similar way to the discrete-time case, we will report the main differences. Suppose that we have a continuous-time system

$$\dot{x} = AZ(x) + Bu \tag{5.30}$$

and that we make an experiment on it. Sampling the observed trajectory with sampling time $T_s > 0$ we collect data matrices U_0, X_0, Z_0, X_1 with U_0, X_0 and Z_0 as in (5.6a), (5.6b) and (5.6d), respectively, and with

$$X_1 := \begin{bmatrix} \dot{x}(0) & \dot{x}(T_s) & \dots & \dot{x}((T-1)T_s) \end{bmatrix}.$$

It is readily seen that these data matrices satisfy the relation $X_1 = AZ_0 + BU_0$. As a consequence, the same analysis carried out in Section 5.3 and 5.4 carries over to the present case. The only modification occurs in the Lyapunov stability condition which reads

$$X_1Y_1 + (X_1Y_1)^\top \prec 0$$

instead of (5.19c) (or (5.10b)). In fact, recalling that the matrix M that dictates the linear dynamics in closed loop is given by

$$M = X_1Y_1P_1^{-1},$$

the above Lyapunov inequality gives

$$P_1^{-1}M + M^\top P_1^{-1} \prec 0$$

and this implies that M is Hurwitz (with Lyapunov function $V(x) = x^\top P_1^{-1}x$). Hence, (5.19) ((5.10) is analogous) becomes

$$\begin{aligned} \min_{P_1,Y_1,G_2} \quad & \|X_1G_2\| && (5.31a)\\ \text{subject to} \quad & (5.19b), (5.19d) && (5.31b)\\ & X_1Y_1 + (X_1Y_1)^\top \prec 0\,, && (5.31c) \end{aligned}$$

and the (continuous-time) control law is given by $u = KZ(x)$ with K as in (5.11).

For estimating the ROA we can proceed as in Section 5.4.2, we omit the details since they are straightforward.

5.5.2. A MORE GENERAL CLASS OF NONLINEAR SYSTEMS

We now turn our attention to the case of systems

$$x^+ = A_\star \mathcal{Z}_\star(\xi) \tag{5.32}$$

where $\xi := \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$, $A_\star \in \mathbb{R}^{n\times R}$ is an unknown constant matrix and where $\mathcal{Z}_\star : \mathbb{R}^{n+m} \to \mathbb{R}^R$ is a vector-valued function of the state and the input. System (5.32) is more general than (5.1) for it allows *both* the state x and the input u to enter the dynamics nonlinearly. We rephrase Assumption 5.1 as follows:

Assumption 5.2 *We know a function $\mathcal{Z} : \mathbb{R}^{n+m} \to \mathbb{R}^S$ such that any element of $\mathcal{Z}_\star$ is also an element of $\mathcal{Z}$.* •

Under this assumption, (5.32) can be equivalently written as $x^+ = A\mathcal{Z}(\xi)$ with $A \in \mathbb{R}^{n\times S}$ an unknown matrix. As before, we allow $\mathcal{Z}(\xi)$ to contain both ξ and the nonlinear function $\mathcal{Q} : \mathbb{R}^{n+m} \to \mathbb{R}^{S-n-m}$, namely we consider

$$\mathcal{Z}(\xi) = \begin{bmatrix} \xi \\ \mathcal{Q}(\xi) \end{bmatrix}. \tag{5.33}$$

The presence of $\mathcal{Q}(\xi)$ makes it difficult to adopt a similar design as in the previous sections, unless one regards the control input u as a state variable and extends the dynamics to include the controller dynamics. This "adding one integrator" tool, which has been widely used in control theory, reduces the design of the controller for (5.32) to the case with constant input vector fields previously studied, as we detail below.

Let us add the controller dynamics in the form $u^+ = v$, with $v \in \mathbb{R}^m$ a new control input. This extension leads to the system

$$\xi^+ = \mathcal{A}\mathcal{Z}(\xi) + \mathcal{B}v, \tag{5.34}$$

where

$$\mathcal{A} := \begin{bmatrix} \overline{A} & \hat{A} \\ 0_{m\times(n+m)} & 0_{m\times(S-n-m)} \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} 0_{n\times m} \\ I_m \end{bmatrix} \tag{5.35}$$

having partitioned A as $A = \begin{bmatrix} \overline{A} & \hat{A} \end{bmatrix}$ with $\overline{A} \in \mathbb{R}^{n\times(n+m)}$. We therefore arrived at a representation which allows us to proceed as in the previous sections. We collect the dataset $\{x(k), u(k), v(k)\}_{k=0}^T$ from the system and define the data matrices

$$\begin{aligned}
V_0 &:= \begin{bmatrix} v(0) & v(1) & \dots & v(T-1) \end{bmatrix} \in \mathbb{R}^{m\times T} \\
\Xi_0 &:= \begin{bmatrix} \xi(0) & \xi(1) & \dots & \xi(T-1) \end{bmatrix} \in \mathbb{R}^{(n+m)\times T} \\
\Xi_1 &:= \begin{bmatrix} \xi(1) & \xi(2) & \dots & \xi(T) \end{bmatrix} \in \mathbb{R}^{(n+m)\times T} \\
\mathcal{Z}_0 &:= \begin{bmatrix} \xi(0) & \dots & \xi(T-1) \\ Q(\xi(0)) & \dots & Q(\xi(T-1)) \end{bmatrix} \in \mathbb{R}^{S\times T}
\end{aligned}$$

which satisfy the identity $\Xi_1 = \mathcal{A}Z_0 + \mathcal{B}V_0$.

The following result parallels Theorem 5.4.

Corollary 5.1 *Consider a nonlinear system as in* (5.32), *and assume that* $\lim_{|\xi|\to 0}\frac{|\mathcal{Q}(\xi)|}{|\xi|}=0$. *Consider the following SDP in the decision variables* $Y_1\in\mathbb{R}^{T\times(n+m)}$, $G_2\in\mathbb{R}^{T\times(S-n-m)}$, $P_1\in\mathbb{S}^{(n+m)\times(n+m)}$*:*

$$\min_{P_1,Y_1,G_2}\quad \|\Xi_1 G_2\| \tag{5.36a}$$

$$\text{subject to}\quad \mathcal{Z}_0 Y_1=\begin{bmatrix}P_1\\ 0_{(S-n-m)\times(n+m)}\end{bmatrix} \tag{5.36b}$$

$$\begin{bmatrix}P_1 & (\Xi_1 Y_1)^\top\\ \Xi_1 Y_1 & P_1\end{bmatrix}\succ 0 \tag{5.36c}$$

$$\mathcal{Z}_0 G_2=\begin{bmatrix}0_{(n+m)\times(S-n-m)}\\ I_{S-n-m}\end{bmatrix} \tag{5.36d}$$

If this SDP is feasible then the dynamical controller

$$\begin{aligned}u^+&=\left[\ \overline{\mathcal{K}}\ \middle|\ \hat{\mathcal{K}}\ \right]\left[\frac{\xi}{\mathcal{Q}(\xi)}\right]\quad \textit{with}\\ \left[\ \overline{\mathcal{K}}\ \middle|\ \hat{\mathcal{K}}\ \right]&=V_0\left[\ Y_1P_1^{-1}\ \middle|\ G_2\ \right]\end{aligned} \tag{5.37}$$

renders the origin of the closed-loop system an asymptotically stable equilibrium.

Proof. The proof follows that of Theorem 5.4. The constraints (5.36b), (5.36d), along with $P_1\succ 0$ guaranteed by (5.36c), imply that $\mathcal{Z}_0\begin{bmatrix}G_1 & G_2\end{bmatrix}=I_S$, having set $G_1:=Y_1P_1^{-1}$. Bearing in mind the expression of $\mathcal{K}:=\begin{bmatrix}\overline{\mathcal{K}} & \hat{\mathcal{K}}\end{bmatrix}$ in (5.37), we obtain

$$\begin{bmatrix}V_0\\ \mathcal{Z}_0\end{bmatrix}\begin{bmatrix}G_1 & G_2\end{bmatrix}=\begin{bmatrix}\mathcal{K}\\ I_S\end{bmatrix}. \tag{5.38}$$

Finally, system (5.32) with the control law (5.37) can be written as $\xi^+=(\mathcal{A}+\mathcal{B}\mathcal{K})\mathcal{Z}(\xi)$, or, in view of the identities (5.38) and $\Xi_1=\mathcal{A}\mathcal{Z}_0+\mathcal{B}V_0$, as $\xi^+=\begin{bmatrix}\Xi_1G_1 & \Xi_1G_2\end{bmatrix}\mathcal{Z}(\xi)$. The constraint (5.36c) ensures that Ξ_1G_1, the matrix describing the linear dynamics of the closed-loop system, is Schur, and the thesis follows because by hypothesis $\mathcal{Q}(\xi)$ decays faster than linearly as ξ goes to zero. ■

As before, we can replace the property $\lim_{|\xi|\to 0}\frac{|\mathcal{Q}(\xi)|}{|\xi|}=0$ by requiring $\mathcal{Q}(\xi)$ to be differentiable at $\xi=0$ and $\mathcal{Q}(0)=0$, so that $\mathcal{Q}(\xi)=\left[\frac{\partial\mathcal{Q}}{\partial\xi}\right]_{\xi=0}\xi+r(\xi)$, with $r(\xi)$ differentiable and such that $\lim_{|\xi|\to 0}\frac{|r(\xi)|}{|\xi|}=0$. In such a way, one can take $\mathcal{Z}(\xi)=\begin{bmatrix}\xi\\ r(\xi)\end{bmatrix}$ instead of (5.33). Further, the Lyapunov function $V(\xi)=\xi^\top P_1^{-1}\xi$ in Corollary 5.1 can be used to estimate the ROA of the closed-loop system (5.32), (5.37), similarly to what has been done to establish Proposition 5.1.

Example 5

Consider the Euler discretization of an inverted pendulum

$$x_1^+=x_1+T_sx_2 \tag{5.39a}$$

$$x_2^+=\frac{T_sg}{\ell}\sin x_1+\left(1-\frac{T_s\mu}{m\ell^2}\right)x_2+\frac{T_s}{m\ell}\cos x_1\,u\,, \tag{5.39b}$$

where now the force is applied at the base, and this results in a state-dependent input vector field $\begin{bmatrix} 0 & \frac{T_s}{m\ell}\cos x_1 \end{bmatrix}^\top$. The parameters T_s, m, ℓ, μ, g and the states x_1, x_2 are the same as in Example 1. The problem is again that of stabilizing the unstable equilibrium in $(x, u) = (0, 0)$.

The vector $\mathcal{Q}(\xi)$ suggested by physical considerations is $\left[\begin{smallmatrix} \sin\xi_1 & \cos\xi_1\,\xi_3 \end{smallmatrix}\right]^\top$, which is zero at $\xi = 0$ and differentiable. Hence, the function $r(\xi) = \left[\begin{smallmatrix} \sin\xi_1 - \xi_1 & (\cos\xi_1 - 1)\,\xi_3 \end{smallmatrix}\right]^\top$ satisfies $\lim_{|\xi|\to 0} \frac{|r(\xi)|}{|\xi|} = 0$. Here, $r(\xi)$ is a preferred choice over $\mathcal{Q}(\xi)$ because it yields a controllable linear part, which is necessary for the feasibility of the SDP. We collect data by running an experiment with input uniformly distributed in $[-0.5, 0.5]$, and with an initial state within the same interval. We collect $T = 10$ samples corresponding to the motion of the pendulum that oscillates around the upright position. The SDP (5.36) is feasible and we obtain

$$\mathcal{K} = \begin{bmatrix} -17.6197 & -5.6815 & -0.3012 & 0 & 0 \end{bmatrix}.$$

The controller locally asymptotically stabilizes the closed-loop system around the origin. For this controller, we numerically determine the set

$$\mathcal{V} = \{\xi : V(\xi^+) - V(\xi) = H(\xi) < 0\},$$

with

$$H(\xi) := (\Xi_1 G_1 \xi + \Xi_1 G_2 \mathcal{Q}(\xi))^\top P_1^{-1} (\Xi_1 G_1 \xi + \Xi_1 G_2 \mathcal{Q}(\xi)) - \xi^\top P_1^{-1} \xi$$

over which the Lyapunov function $V(\xi) = \xi^\top P_1^{-1} \xi$ decreases. Any sub-level set $\mathcal{R}_\gamma$ of V contained in $\mathcal{V} \cup \{0\}$ gives an estimate of the ROA for the closed-loop system. The set $\mathcal{V}$ and a sublevel set of V are displayed in Figure 5.4.

The zero values taken on by the last two entries of $\mathcal{K}$ (which correspond to the sub-vector $\hat{\mathcal{K}}$ in (5.37)) is a byproduct of the minimization of $\|\Xi_1 G_2\|$, which in turn imposes a small value of $\|V_0 G_2\|$, in view of the addition of the integrator (V_0 equals the last m rows of Ξ_1, therefore $\Xi_1 G_2 = \left[\begin{smallmatrix} X_1 G_2 \\ V_0 G_2 \end{smallmatrix}\right]$). •

Corollary 5.1 is a direct extension of Theorem 5.4 and allows the designer to deal with a more general class of nonlinear systems, including systems with state-dependent input vector fields. Nevertheless, if it is known that the input vector field is state-independent, it is preferable to use the design proposed by Theorem 5.4, which might guarantee a global stabilization result by a static feedback in case the solution attains a zero cost, as formalized in Theorem 5.3.

5.6. Robustness

In this section, we discuss robustness to disturbances and/or neglected nonlinearities. Consider a system in the form

$$x^+ = AZ(x) + Bu + Ed \tag{5.40}$$

where $d \in \mathbb{R}^s$ is an unknown signal that accounts for process disturbances and/or neglected nonlinearities (when Z does *not* include all the nonlinearities present in the system), whereas $E \in \mathbb{R}^{n\times s}$ is a known matrix that specifies which channel the signal d

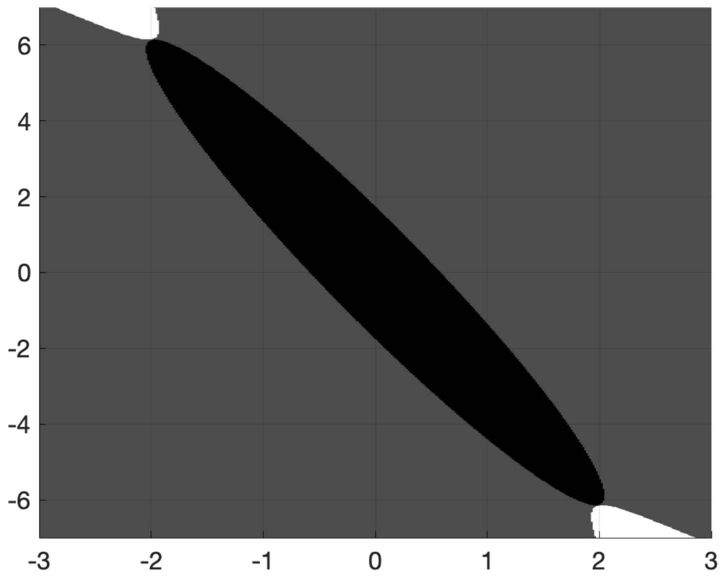

Figure 5.4: Results for Example 5. The grey set represents the set $\mathcal{V}$ where $V(\xi^+) - V(\xi)$ is negative. Here, $\mathcal{Z}(\xi) = \begin{bmatrix}\xi^\top & \sin\xi_1 - \xi_1 & (\cos\xi_1 - 1)\,\xi_3\end{bmatrix}^\top$ and $V(\xi) = \xi^\top P_1^{-1}\xi$, with

$$P_1^{-1} = \begin{bmatrix} 0.2159 & 0.0689 & 0.0123 \\ 0.0689 & 0.0240 & 0.0039 \\ 0.0123 & 0.0039 & 0.0009 \end{bmatrix}.$$

The black set is a Lyapunov sublevel set $\mathcal{R}_\gamma$, with $\gamma = 0.076$, contained in $\mathcal{V}$, hence it provides an estimate of the ROA for the system. Both sets $\mathcal{V}$ and $\mathcal{R}_\gamma$ are projected onto the plane $\{\xi : \xi_3 = 0\}$.

enters. If such information is not available then we simply let $E = I_n$. Because of d, the previous tools must be modified to maintain stability guarantees. While the tools we use to study process disturbances and neglected nonlinearities are similar, we will tackle the two cases separately.

5.6.1. DISTURBANCES: NOISY DATA AND ROBUST INVARIANCE

We start with the case where d is a process disturbance. The presence of d affects the analysis in two different directions. First, it affects controller design since it corrupts the data.[2] Second, it leads to notions other than Lyapunov stability and ROA. We will address both the questions.

Similarly to the disturbance-free case, suppose we perform an experiment on the system, and we collect state and input samples satisfying

$$x(k+1) = AZ(x(k)) + Bu(k) + Ed(k),$$

with $k = 0, \ldots, T-1$. These samples are then grouped into the data matrices U_0, X_0, X_1, Z_0 as in (5.6). Furthermore, let

$$D_0 := \begin{bmatrix} d(0) & d(1) & \cdots & d(T-1) \end{bmatrix} \tag{5.41}$$

be the (unknown) data matrix that collects the samples of d. Our first step is to establish an analogue of Lemma 5.1.

Lemma 5.2 *Consider any matrices $K \in \mathbb{R}^{m \times S}$, $G \in \mathbb{R}^{T \times S}$ satisfying* (5.7). *Let G be partitioned as $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$, where $G_1 \in \mathbb{R}^{T \times n}$. System* (5.40) *under the control law $u = KZ(x)$ results in the closed-loop dynamics*

$$x^+ = \Psi x + \Xi Q(x) + Ed \tag{5.42}$$

where $\Psi := (X_1 - ED_0)G_1$ and $\Xi := (X_1 - ED_0)G_2$,

Proof. Similarly to (5.9), we have

$$x^+ = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_S \end{bmatrix} Z(x) + Ed \tag{5.43a}$$

$$= \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} GZ(x) + Ed \tag{5.43b}$$

$$= (X_1 - ED_0)GZ(x) + Ed\,. \tag{5.43c}$$

The last identity follows as X_1, U_0, Z_0, D_0 satisfy the relation

$$x(k+1) = AZ(x(k)) + Bu(k) + Ed(k), \quad k = 0, \ldots, T-1,$$

which gives $X_1 = AZ_0 + BU_0 + ED_0$. ■

[2]By following [40, Section V-A], the analysis can be extended to the case of measurement noise.

By looking at (5.43) we note that the closed-loop dynamics now depends on the unknown matrix D_0, and (5.19) no longer provides stability guarantees. In fact, the constraint (5.19c) ensures that $M = X_1 G_1$ is Schur. By Lemma 5.2, however, the matrix of interest is now $\Psi = (X_1 - ED_0)G_1$, and stability of M does not ensure that also Ψ is stable. To have stability, we need to modify (5.19c) accounting for the uncertainty induced by D_0. A simple and effective way to achieve this is to ensure that $(X_1 - ED)G_1$ is stable *for all* the matrices D in a given set $\mathcal{D}$ to which D_0 is deemed to belong (this approach can in fact be viewed as a *robust control* approach). We will consider the set

$$\mathcal{D} := \{D \in \mathbb{R}^{s \times T} : DD^\top \preceq \Delta\Delta^\top\} \tag{5.44}$$

with Δ a design parameter, and enforce, in place of (5.19c),

$$Y_1^\top (X_1 - ED)^\top P_1^{-1}(X_1 - ED)Y_1 - P_1 + \Omega \prec 0 \quad \forall D \in \mathcal{D} \tag{5.45}$$

where Y_1 and $P_1 \succ 0$ are decision variables which satisfy the identity $Y_1 P_1^{-1} = G_1$, while $\Omega \succ 0$ is a free design parameter we will comment on shortly. By enforcing (5.45) we guarantee that $(X_1 - ED)G_1$ is stable for all $D \in \mathcal{D}$, hence we ensure stability of $(X_1 - ED_0)G_1$ if $D_0 \in \mathcal{D}$. The choice of the set $\mathcal{D}$ clearly reflects our prior information or guess about d. For instance, if we know that $|d| \leq \delta$ for some $\delta > 0$ then we let $\Delta := \delta\sqrt{T} I_s$. Stochastic disturbances can also be accounted for (possibly, with other choices of Δ), see Section 5.6.3. In general, large sets $\mathcal{D}$ make condition $D_0 \in \mathcal{D}$ easier to hold but make (5.45) more difficult to satisfy. We proceed by making the assumption $D_0 \in \mathcal{D}$ explicit.

Assumption 5.3 $D_0 \in \mathcal{D}$. •

A final comment regards the matrix Ω. This matrix ensures that

$$Y_1^\top (X_1 - ED)^\top P_1^{-1}(X_1 - ED)Y_1 - P_1$$

is bounded away from singularity, as we vary D, by a *known* quantity, and this is key to have an explicit expression for the ROA. There is no loss of generality in considering (5.45) instead of

$$Y_1^\top (X_1 - ED)^\top P_1^{-1}(X_1 - ED)Y_1 - P_1 \prec 0 \quad \forall D \in \mathcal{D}. \tag{5.46}$$

Indeed, for any $\Omega \succ 0$ there exist $Y_1, P_1 \succ 0$ that satisfy (5.45) if and only if there exist $Y_1, P_1 \succ 0$ that satisfy (5.46).

Condition (5.45) cannot be implemented directly as it involves *infinitely* many constraints. The next result provide a tractable (and convex) condition for (5.45). Following [123, Lemma A.4][3], we could actually establish the *equivalence* between the next (5.47) and (5.45). Here, we will only show that (5.47) implies (5.45), which is enough for our purposes.

[3]Lemma A.4 in [123], also known as the *Petersen's lemma*, permits to study matrix inequalities which involve uncertainty, like (5.45), and gives conditions under which such inequalities can be equivalently assessed considering only the 'boundary' of the uncertainty, like (5.47) does. We refer the reader to [68] for a recent discussion on the use of Petersen's lemma in data-driven control of linear and polynomial systems.

Lemma 5.3 *Suppose that there exist* $Y_1 \in \mathbb{R}^{T\times n}$, $P_1 \in \mathbb{S}^{n\times n}$, *and a scalar* $\epsilon > 0$ *such that*

$$\begin{bmatrix} P_1 - \Omega & (X_1Y_1)^\top & Y_1^\top \\ X_1Y_1 & P_1 - \epsilon E\Delta\Delta^\top E^\top & 0_{n\times T} \\ Y_1 & 0_{T\times n} & \epsilon I_T \end{bmatrix} \succ 0 \tag{5.47}$$

with $\Omega \succ 0$ *and* Δ *given. Then,* (5.45) *holds.*

Proof. See Section 5.9. ■

We arrive at the following main result.

Theorem 5.6 *Consider a nonlinear system as in* (5.40) *with* Z *satisfying the condition* (5.21) *and with* d *a process disturbance. For a given* $\Omega \succ 0$ *and* Δ*, suppose that the following SDP*

$$\min_{P_1,Y_1,G_2} \quad \|X_1G_2\| \tag{5.48a}$$

$$\text{subject to} \quad (5.19\text{b}), (5.47), (5.19\text{d}) \tag{5.48b}$$

is feasible. If Assumption 5.3 holds then the control law $u = KZ(x)$ *with* K *in* (5.11) *renders the origin an asymptotically stable equilibrium for the closed-loop system.*

Proof. The SDP (5.48) follows from (5.19) with (5.19c) replaced by (5.47) to account for robust stability. Suppose that (5.48) is feasible. Let $G_1 = Y_1P_1^{-1}$ and note that the two constraints (5.19b) and (5.19d) together yield $Z_0 \begin{bmatrix} G_1 & G_2 \end{bmatrix} = I_S$. This relation, combined with (5.11), gives (5.7). In view of Lemma 5.2, the closed-loop dynamics satisfies $x^+ = \Psi x + \Xi Q(x) + Ed$, with $\Psi = (X_1 - ED_0)G_1$. Next, we prove that Ψ is Schur. By Lemma 5.3 and since $D_0 \in \mathcal{D}$ by hypothesis, (5.45) holds for $D = D_0$. We have in particular

$$P_1^{-1}Y_1^\top(X_1 - ED_0)^\top P_1^{-1}(X_1 - ED_0)Y_1P_1^{-1} - P_1^{-1} \prec 0.$$

By recalling that $Y_1P_1^{-1} = G_1$, we conclude that Ψ is Schur. The result follows from (5.21). ■

Building on Theorem 5.6 it is possible to characterize regions of attractions as well as robust invariant sets [120]. We start with the ROA as a preliminary step for robust invariance. Consider the closed-loop dynamics $x^+ = \Psi x + \Xi Q(x)$ where we set $d \equiv 0$ since we consider the ROA, and let $V(x) := x^\top P_1^{-1}x$.

We have

$$V(x^+) - V(x) = \underbrace{(\Psi x + \Xi Q(x))^\top P_1^{-1}(\Psi x + \Xi Q(x)) - x^\top P_1^{-1}x}_{=:s(x)} \tag{5.49}$$

with $\Psi = (X_1 - ED_0)G_1, \Xi = (X_1 - ED_0)G_2$. We cannot proceed as in the disturbance-free case because Ψ and Ξ are unknown. Nonetheless, we can upper bound $s(x)$ with a quantity that is computable from data alone. First, we tackle $x^\top \Phi x$ where

$$\Phi := P_1^{-1} - \Psi^\top P_1^{-1}\Psi.$$

By Theorem 5.6, (5.45) holds for $D = D_0$, namely $P_1 \Phi P_1 - \Omega \succ 0$. Pre-multiplying this inequality left and right by P_1^{-1} gives

$$\Phi - P_1^{-1} \Omega P_1^{-1} \succ 0,$$

and hence

$$x^\top \Phi x \geq x^\top \underline{\Phi} x$$

for all x, where $\underline{\Phi} := P_1^{-1} \Omega P_1^{-1}$. Accordingly, we have

$$V(x^+) - V(x) \leq -x^\top \underline{\Phi} x + (2\Psi x + \Xi Q(x))^\top P_1^{-1} \Xi Q(x).$$

Bearing in mind the expressions of Ψ and Ξ, and the fact that $\|D_0\|_2 \leq \|\Delta\|_2$, we can write

$$V(x^+) - V(x) \leq \underbrace{-x^\top \underline{\Phi} x + \ell_1(x) + \ell_2(x) + \ell_3(x) + \ell_4(x)}_{=: \ell(x)} \tag{5.50}$$

having set

$$\begin{aligned}
\ell_1(x) &:= (2X_1 G_1 x + X_1 G_2 Q(x))^\top P_1^{-1} X_1 G_2 Q(x), \\
\ell_2(x) &:= \|\Delta\|_2 |(2X_1 G_1 x + X_1 G_2 Q(x))^\top P_1^{-1} E| |G_2 Q(x)|, \\
\ell_3(x) &:= \|\Delta\|_2 |2G_1 x + G_2 Q(x)| |E^\top P_1^{-1} X_1 G_2 Q(x)|, \\
\ell_4(x) &:= \|\Delta\|_2^2 \|E^\top P_1^{-1} E\|_2 |2G_1 x + G_2 Q(x)| |G_2 Q(x)|,
\end{aligned}$$

which are all computable from data alone.

Proposition 5.2 *Consider the same setting as in Theorem 5.6. Let*

$$\mathcal{L} := \{x : \ell(x) < 0\},$$

with $\ell(x)$ as in (5.50)*, and consider the Lyapunov function $V(x) = x^\top P_1^{-1} x$. Then, any sub-level set*

$$\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$$

of V contained in $\mathcal{L} \cup \{0\}$ is a PI set for the closed-loop system with $d \equiv 0$ and defines an estimate of the ROA relative to $\overline{x} = 0$. •

We now consider robust invariance [120, Definition 2.2].

Definition 5.2 *A set $\mathcal{S}$ is called* robustly positively invariant *(RPI) for the system $x^+ = f(x, d)$ if for every $x(0) \in \mathcal{S}$ and all $d(t) \in \mathcal{I}$, with $\mathcal{I}$ a compact set, the solution is such that $x(t) \in \mathcal{S}$ for $t > 0$.* •

Unlike local stability and invariance, which pose conditions on the disturbance only relatively to the data collection phase (Assumption 5.3, i.e. the condition $D_0 \in \mathcal{D}$), robust invariance constrains d for *all* times $t \geq 0$. This calls for strengthening Assumption 5.3 in the sense of Definition 5.2. [4]

[4] As an example, a Gaussian disturbance may satisfy the condition $D_0 \in \mathcal{D}$ but is not bounded in the sense of Definition 5.2. Set invariance for unbounded disturbances is studied in [124]. We will not pursue this problem here.

Assumption 5.4 $|d| \leq \delta$ *for some known* $\delta > 0$. •

Assumption 5.4 is indeed stronger than Assumption 5.3 in the sense that it implies Assumption 5.3 once we set $\Delta := \delta\sqrt{T} I_s$. We can now proceed with the analysis of robust invariance. Consider the closed-loop system $x^+ = \Psi x + \Xi Q(x) + Ed$ with d satisfying Assumption 5.4, and let $V(x) := x^\top P_1^{-1} x$. It is simple to verify that we now have

$$V(x^+) - V(x) \leq \ell(x) + g(x, \delta), \tag{5.52}$$

where $\ell(x)$ is as in (5.50), and where

$$g(x, \delta) := r_1(x)\delta + r_2(x)\delta + r_3\delta^2, \tag{5.53a}$$

$$r_1(x) := 2|(X_1 G_1 x + X_1 G_2 Q(x))^\top P_1^{-1} E|, \tag{5.53b}$$

$$r_2(x) := 2\|\Delta\|_2 \|E^\top P_1^{-1} E\|_2 |G_1 x + G_2 Q(x)|, \tag{5.53c}$$

$$r_3 := \|E^\top P_1^{-1} E\|_2. \tag{5.53d}$$

Let

$$\mathcal{X} := \{x : \ell(x) + g(x, \delta) \leq 0\} \tag{5.54}$$

and let $\mathcal{X}^c$ be its complement.

Theorem 5.7 *Consider a nonlinear system as in* (5.40) *with* Z *satisfying* (5.21) *and with* d *a process disturbance for which Assumption 5.4 holds. For a given* $\Omega \succ 0$, *suppose that* (5.48) *is feasible with* $\Delta := \delta\sqrt{T} I_s$, *and consider the control law* $u = KZ(x)$ *where* K *is as in* (5.11). *Let* $V(x) := x^\top P_1^{-1} x$, *and define*

$$\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\},$$

where $\gamma > 0$ *is arbitrary. Finally, let*

$$\mathcal{Z} := \mathcal{R}_\gamma \cap \mathcal{X}^c$$

($\mathcal{Z}$ *defines all the points* x *of* $\mathcal{R}_\gamma$ *for which the Lyapunov difference* $V(x^+) - V(x)$ *can be positive; it is nonempty for any choice of* $\gamma > 0$*). If*

$$V(x) + \ell(x) + g(x, \delta) \leq \gamma \quad \forall x \in \mathcal{Z} \tag{5.55}$$

then $\mathcal{R}_\gamma$ *is an RPI set for the closed-loop system.*

Proof. As shown in Theorem 5.6, feasibility of (5.48), along with $D_0 \in \mathcal{D}$, ensures that $V(x) = x^\top P_1^{-1} x$ is a Lyapunov function for the linear part of the dynamics, and (5.21) ensures that $\mathcal{L} = \{x : \ell(x) < 0\}$, with $\ell(x)$ as in (5.50), is nonempty (if $\mathcal{L}$ is empty then (5.55) never holds). Then, assume that (5.55) holds and let $x \in \mathcal{R}_\gamma$. We divide the analysis in two cases. First assume that $x \notin \mathcal{Z}$. Since $x \in \mathcal{R}_\gamma$ then $x \notin \mathcal{X}^c$. Then $x \in \mathcal{X}$, so that $V(x^+) - V(x) \leq \ell(x) + g(x, \delta) \leq 0$, and this implies $x^+ \in \mathcal{R}_\gamma$. Next, assume that $x \in \mathcal{Z}$. In view of (5.55) we have $V(x^+) \leq \gamma$, thus $x^+ \in \mathcal{R}_\gamma$. ■

Equations (5.50) and (5.52) suggest that from a practical point of view it might be convenient to *regularize* the objective function in (5.48) so as to mitigate the effect of the disturbance. As shown in the subsequent numerical examples, a convenient choice is the following one:

$$\min_{P_1, Y_1, G_2} \quad \|X_1 G_2\| + \lambda_1 \|P_1\| + \lambda_2 \|G_2\| \tag{5.56a}$$

$$\text{subject to} \quad (5.19\text{b}), (5.47), (5.19\text{d}) \tag{5.56b}$$

where $\lambda_1, \lambda_2 \geq 0$ are weighting parameters. Penalizing $\|P_1\|$ increases the smallest eigenvalue of $\underline{\Phi}$, while penalizing $\|G_2\|$ decreases the various terms ℓ_i and r_i in (5.50) and (5.52). Notice that penalizing $\|P_1\|$ might increase the terms ℓ_i and r_i, but while these quantities depend on P_1^{-1}, $\underline{\Phi}$ depends on P_1^{-2}, so penalizing $\|P_1\|$ can still be advantageous.

Since (5.56) has the same feasible set as (5.48) it is understood that all the results of this section as well as those to follow remain true if (5.48) is replaced with (5.56).

Example 6

We consider again the inverted pendulum of Example 1, this time assuming that a disturbance d acts on the control channel, namely we have $E = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and the second equation is modified as

$$x_2^+ = \frac{T_s g}{\ell} \sin x_1 + \left(1 - \frac{T_s \mu}{m\ell^2}\right) x_2 + \frac{T_s}{m\ell^2} u + d.$$

We collect data by running an experiment with input uniformly distributed in $[-0.5, 0.5]$, and with an initial state within the same interval. We consider a disturbance uniformly distributed in $[-\delta, \delta]$. We collect $T = 30$ samples and solve (5.56) with $\lambda_1 = \lambda_2 = 0.1$, $\Omega = I_2$ and $\Delta = \delta\sqrt{T}$. Figure 5.5 reports the simulation results for $\delta = 0.01$.

We observe the following:

(i) The program (5.56) remains feasible up to $\delta \approx 0.1$ but for such large values we get empty estimates of ROA/RPI.

(ii) The regularization is in fact needed to get nonempty estimates of ROA/RPI, and even small values for λ_1, λ_2 suffice. This permits to preserve the baseline strategy of nonlinearity minimization. In fact, the controller we obtain is

$$K = \begin{bmatrix} -23.9436 & -11.4581 & -9.8564 \end{bmatrix},$$

which generates the term $-9.8564 \sin(x_1)$ that approximately cancels out the nonlinearity.

(iii) Compared with the disturbance-free case, here we need a larger number of samples to get nonempty estimates of ROA/RPI, although (5.56) remains feasible even for $T = 10$. Intuitively, collecting more samples can indeed help to get more information on the system's dynamics; we will elaborate on this point in Section 5.6.3.

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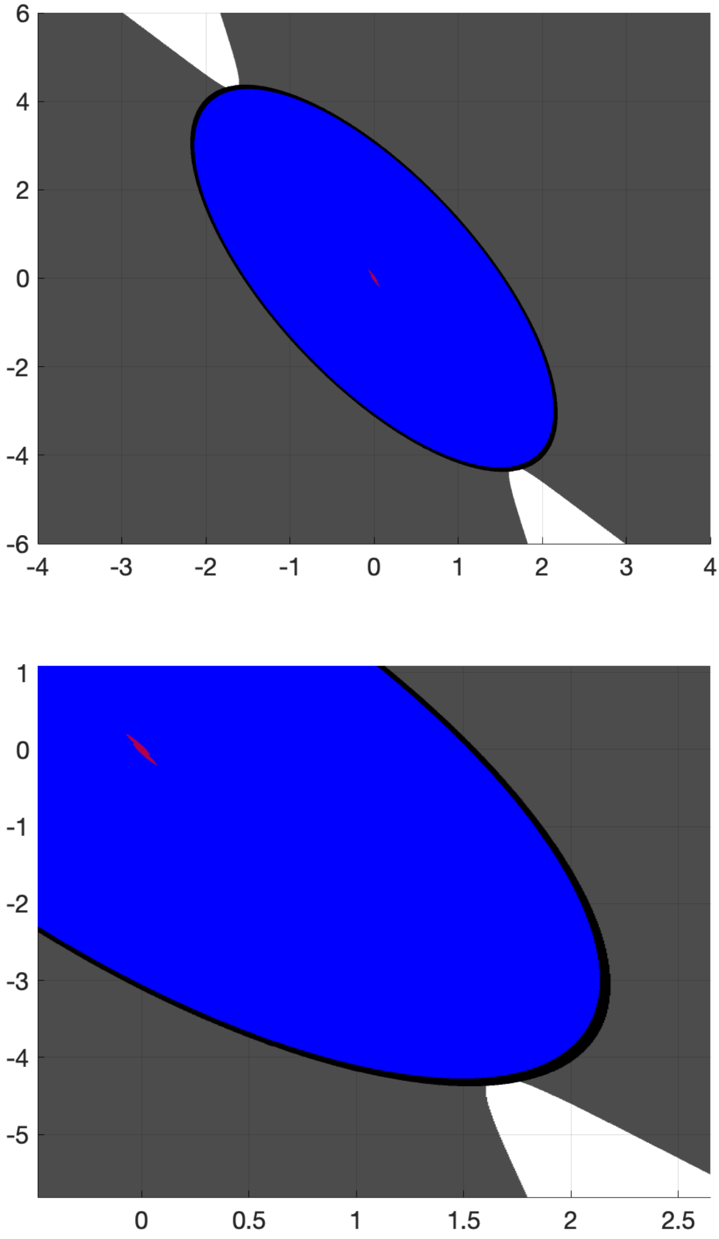

Figure 5.5: Results for Example 6. Simulation result for (5.56) with $Z(x) = \begin{bmatrix} x_1 & x_2 & \sin(x_1) - x_1 \end{bmatrix}^\top$, $\lambda_1 = \lambda_2 = 0.1$, and $\delta = 0.01$. Top: the grey set represents the set $\mathcal{X}$ in (5.54), while the blue set is the RPI set $\mathcal{R}_\gamma$. Here, $P_1^{-1} = \begin{bmatrix} 0.1901 & 0.0664 \\ 0.0664 & 0.0475 \end{bmatrix}$, and $\gamma = 0.4440$. The black set wrapping $\mathcal{R}_\gamma$ is the ROA, which is larger than the RPI set. Finally, the red set around the origin corresponds to the set $\mathcal{Z}$. Here, $\max_{x \in \mathcal{Z}} V(x) + \ell(x) + g(x, \delta) = 0.001$. States originating in $\mathcal{Z}$ do not exit $\mathcal{R}_\gamma$. In particular, any sub-level set $\mathcal{R}_\gamma = \{x : V(x) \leq \gamma\}$ with $\gamma \in [0.0010, 0.4440]$ is a RPI set for the closed-loop system. Bottom: zoom showing $\mathcal{R}_\gamma$ close to the border of $\mathcal{X}$.

5.6.2. Neglected nonlinearities

A similar analysis can be carried out in case of neglected nonlinearities. The difference is that now d will be a function of the state x, say $d = d(x)$. The *combination* of neglected nonlinearities and genuine disturbances is also possible, but we omit the details for brevity. Thus, the analysis which follows only considers invariance instead of *robust* invariance.

In order to handle the case of neglected nonlinearities, we assume some knowledge on the *strength* of such nonlinearities (Assumption 5.5 is essentially the counterpart of Assumption 5.4).

Assumption 5.5 *We know a set $\mathcal{Q} \subseteq \mathbb{R}^n$ and a scalar $\delta > 0$ such that $|d(x)| \leq \delta$ for all $x \in \mathcal{Q}$.* •

Theorem 5.8 *Consider a nonlinear system as in* (5.40) *with Z satisfying* (5.21) *and with $d = d(x)$ a nonlinear function of the state for which Assumption 5.5 holds. Consider an experiment on the system such that $x(k) \in \mathcal{Q}$ for $k = 0, \ldots, T-1$. For a given $\Omega \succ 0$, suppose that* (5.48) *is feasible with $\Delta = \delta\sqrt{T} I_s$. Let $V(x) := x^\top P_1^{-1} x$ and*

$$\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$$

where $\gamma > 0$ is arbitrary. Finally, let $\mathcal{X}$ be as in (5.54) *and*

$$\mathcal{Z} := \mathcal{R}_\gamma \cap \mathcal{X}^c.$$

If $\mathcal{R}_\gamma \subseteq \mathcal{Q}$ and

$$V(x) + \ell(x) + g(x,\delta) \leq \gamma \quad \forall x \in \mathcal{Z} \tag{5.57}$$

then $\mathcal{R}_\gamma$ is a PI set for the closed-loop system.

Proof. Under the stated conditions we have $D_0 \in \mathcal{D}$. Thus, the feasibility of (5.48) guarantees that $V(x) = x^\top P_1^{-1} x$ is a Lyapunov function for the linear part of the dynamics, and (5.21) ensures that

$$\mathcal{L} = \{x : \ell(x) < 0\},$$

with $\ell(x)$ as in (5.50), is nonempty (otherwise (5.57) would never hold). Then, assume that (5.57) holds and let $x \in \mathcal{R}_\gamma$. Since $x \in \mathcal{R}_\gamma$ then $x \in \mathcal{Q}$, and therefore $|d(x)| \leq \delta$. Hence, exactly as in (5.52), we have

$$V(x^+) - V(x) \leq \ell(x) + g(x,\delta)$$

where $g(x,\delta)$ is as in (5.53). The rest of the proof is analogous to that of Theorem 5.7. Assume that $x \notin \mathcal{Z}$. Since $x \in \mathcal{R}_\gamma$ then $x \notin \mathcal{X}^c$. Thus $x \in \mathcal{X}$, and hence

$$V(x^+) - V(x) \leq \ell(x) + g(x,\delta) \leq 0,$$

which implies $x^+ \in \mathcal{R}_\gamma$. Next, assume that $x \in \mathcal{Z}$. In view of (5.57), we have $V(x^+) \leq \gamma$, thus $x^+ \in \mathcal{R}_\gamma$. ■

We can also have asymptotic stability under a strengthened Assumption 5.5. Here we report a prototypical result.

Theorem 5.9 *Consider the same setting as in Theorem 5.8, and suppose that* $|d(x)| \leq \delta(x)$ *for all* x*, where* $\delta(x) : \mathbb{R}^n \to \mathbb{R}_+$ *is some known function such that*

$$\lim_{|x| \to 0} \frac{\delta(x)}{|x|} = 0.$$

Let $\ell(x)$ *be as in* (5.50)*, and let* $g(x, \delta(x))$ *be as in* (5.53) *with* δ *replaced by* $\delta(x)$*. Finally, define*

$$\mathcal{W} := \{x : \ell(x) + g(x, \delta(x)) < 0\}.$$

Then, the origin is an asymptotically stable equilibrium for the closed-loop system, and any set

$$\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$$

of V *contained in* $\mathcal{W} \cup \{0\}$ *is a PI set and defines an estimate of the ROA relative to* $\overline{x} = 0$.

Proof. Analogously to (5.52), the Lyapunov function satisfies

$$V(x^+) - V(x) \leq \ell(x) + g(x, \delta(x))$$

for all x. Then the result follows immediately. ∎

EXAMPLE 7

Consider the previous example, but this time assume that we purposely neglect the nonlinearity and design a *linear* control law. Specifically, the dynamics of the inverted pendulum can be written as

$$\begin{aligned}
x_1^+ &= x_1 + T_s x_2, \\
x_2^+ &= \frac{T_s g}{\ell} x_1 + \left(1 - \frac{T_s \mu}{m\ell^2}\right) x_2 + \frac{T_s}{m\ell^2} u + d, \\
d &= \frac{T_s g}{\ell} (\sin x_1 - x_1).
\end{aligned}$$

In this case, the type of dynamics is known, hence we focus on Theorem 5.9. We consider $\delta(x) = 2|\sin x_1 - x_1|$, thus $|d(x)| \leq \delta(x)$ for all x (we over-approximate d by more than 100%). We run an experiment with input and initial state uniformly distributed in $[-0.1, 0.1]$. This ensures that up to $T = 10$ the state x_1 remains close to the equilibrium, so that d remains small. In particular, with this choice, x_1 never exceeds ± 0.06 ($\approx \pm 3.5°$), and $\delta(x) \leq 3 \cdot 10^{-5} =: c$. Thus we take $T = 10$, set $\Omega = I_2$, $\Delta = c\sqrt{T}$ and solve (5.56) (by the same arguments in Example 6 on the impact of noise on the estimate of the ROA/RPI, we solve the regularized version of (5.48)).

Note that (5.56) now involves only the variables P_1, Y_1, thus only the two constraints (5.19b) and (5.47) are present. We get

$$K = \begin{bmatrix} -19.0204 & -10.7947 \end{bmatrix}$$

and the ROA in Figure 5.6. As expected, the outcome is worse than the one obtained when we exploit the knowledge of the nonlinearities and we use a nonlinear control law. In particular, the main shortcoming is that we now need to run the experiment close to the equilibrium in order to keep d small, which is not needed when we take the nonlinearity into account. •

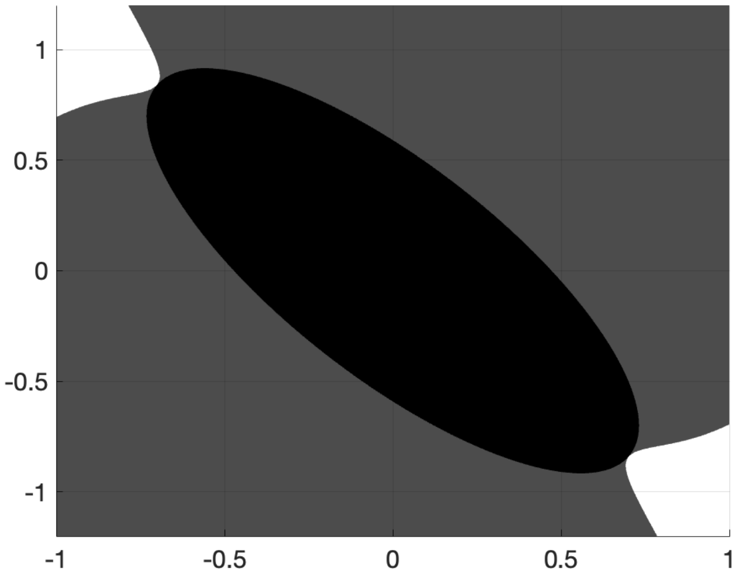

Figure 5.6: Results for Example 7 when we consider a linear control law. The grey set represents the set $\mathcal{W}$, while the black set represents the set $\mathcal{R}_\gamma$ which defines the ROA. Here, $\gamma = 0.0473$ and $P_1^{-1} = \begin{bmatrix} 0.2116 & 0.1291 \\ 0.1291 & 0.1351 \end{bmatrix}$.

5.6.3. RESULTS IN PROBABILITY

All previous results rest on the assumption that $D_0 \in \mathcal{D}$. Clearly, once the experiment is performed and the data are collected, whether $D_0 \in \mathcal{D}$ or not is a *deterministic* property (*yes* or *no*). Yet, certifying that D_0 actually belongs to $\mathcal{D}$ can be a difficult task. It turns out that we can establish results that relate closed-loop stability with the *probability* that $D_0 \in \mathcal{D}$. We focus on the case of process disturbances, in particular we give a probabilistic version of Theorem 5.6.

Theorem 5.10 *Consider a nonlinear system as in* (5.40) *with Z satisfying* (5.21) *and with d a process disturbance. For a given $\Omega \succ 0$ and Δ, suppose that* (5.48) *is feasible. If $D_0 \in \mathcal{D}$ with probability at least p then the control law $u = KZ(x)$, with K as in* (5.11)*, renders the origin an asymptotically stable equilibrium with probability at least p.*

Proof. The result is a direct consequence of the *law of total probability* [125, Theorem 3, pp. 28]. Given two events $\mathcal{E}_1$ and $\mathcal{E}_2$, let $P(\mathcal{E}_1)$ and $P(\mathcal{E}_1|\mathcal{E}_2)$ denote the probability of $\mathcal{E}_1$ and the conditional probability of $\mathcal{E}_1$ given $\mathcal{E}_2$. Let $\mathcal{E}_1$ denote the event that K is stabilizing and $\mathcal{E}_2$ denote the event $D_0 \in \mathcal{D}$. We have $P(\mathcal{E}_1) = P(\mathcal{E}_1|\mathcal{E}_2)P(\mathcal{E}_2) + P(\mathcal{E}_1|\mathcal{E}_2^c)P(\mathcal{E}_2^c)$, with $\mathcal{E}^c$ the complement of $\mathcal{E}$. Then, $P(\mathcal{E}_1) \geq P(\mathcal{E}_1|\mathcal{E}_2)P(\mathcal{E}_2)$ and the result follows because $P(\mathcal{E}_1|\mathcal{E}_2) = 1$ by Theorem 5.6. ■

Theorem 5.10 allows us to extend our range of application to cases where bounds on d are known only with a limited accuracy, as exemplified in next Proposition 5.3. Theorem 5.10 has another interesting implication. For disturbances obeying the *law of large numbers* [125, Section 5] we can repeat the same experiment multiple times and average the data so as to filter out noise. Specifically, suppose we make N experiments on system (5.40), each of length T, and let $(U_0^{(r)}, D_0^{(r)}, Z_0^{(r)}, X_1^{(r)})$, with $r = 1, \ldots, N$, be the dataset resulting from the r-th experiment. Given N matrices $S^{(r)}$, with $r = 1, \ldots, N$, let $\underline{S} := \frac{1}{N}\sum_{r=1}^N S^{(r)}$ denote their average. Since each dataset satisfies the relation $X_1^{(r)} = AZ_0^{(r)} + BU_0^{(r)} + ED_0^{(r)}$, if we average N datasets we obtain the relation

$$\underline{X}_1 = A\underline{Z}_0 + B\underline{U}_0 + E\underline{D}_0 \tag{5.59}$$

Because the dynamics are nonlinear, (5.59) does *not* represent a valid trajectory of the system in the sense that it cannot result from a single experiment on (5.40). Yet, and this is the crucial point, the dataset $(\underline{U}_0, \underline{D}_0, \underline{Z}_0, \underline{X}_1)$ still provides a data-based parametrization of the closed loop in the sense of Lemma 5.2. Specifically, for any K, G satisfying

$$\begin{bmatrix} K \\ I_S \end{bmatrix} = \begin{bmatrix} \underline{U}_0 \\ \underline{Z}_0 \end{bmatrix} G \tag{5.60}$$

we have (*cf.* (5.9))

$$A + BK = (\underline{X}_1 - E\underline{D}_0)G. \tag{5.61}$$

Hence, Lemma 5.2, and consequently Theorems 5.6 and 5.10, apply to $(\underline{U}_0, \underline{D}_0, \underline{Z}_0, \underline{X}_1)$ with no modifications, with the advantage that $\underline{D}_0$ will have a reduced norm in expectation thanks to the law of large numbers.

While the law of large numbers gives an asymptotic result, there are recent results in non-asymptotic statistics that permit us, for relevant classes of disturbance, to get high-confidence bounds on $\|\underline{D}_0\|_2$ even with a *finite* number of experiments. As an example, we give the following result.[5]

Proposition 5.3 *Consider N experiments, each of length T, on system* (5.40)*, and assume that the disturbances $d(k) \in \mathbb{R}^s$ are i.i.d. zero-mean random vectors with covariance matrix Σ such that $|d(k)| \leq \delta$* almost surely *(i.e., with probability 1). Then, for all $\mu > 0$,*

$$\|\underline{D}_0\|_2 \leq \sqrt{T\left(\frac{\|\Sigma\|_2}{N} + \mu\right)} \tag{5.62}$$

with probability at least $1 - 2s\, exp\left(-\frac{TN\mu^2}{2\delta^2(\|\Sigma\|_2 + N\mu)}\right)$.

Let instead the disturbances $d(k)$ be i.i.d. random vectors drawn from $\mathcal{N}(0,\Sigma)$. Then, for all $\mu > 0$,

$$\|\underline{D}_0\|_2 \leq \sqrt{\frac{T}{N}}\left(\lambda_{max}(\Sigma^{1/2})(1+\mu) + \sqrt{\frac{trace(\Sigma)}{T}}\right) \tag{5.63}$$

with probability at least $1 - exp(-T\mu^2/2)$. where λ_{max} denotes the maximum eigenvalue.

Proof. Since the disturbances $d(k)$ are independent then the vectors which form the columns of $\underline{D}_0$ are also independent. This can be easily verified, for instance, through the so-called *characteristic function*, *e.g.*, see [125, Theorem 28, pp. 131]. It is also easy to verify that these vectors have zero mean and covariance matrix Σ/N. The bounds (5.62) and (5.63) follow from Corollary 6.20 and Theorem 6.1 in [126], respectively. ■

Under the assumption on the disturbances stated in Proposition 5.3, we can choose $\Delta = \eta I_s$ with η equal to the right-hand side of (5.62) or (5.63), and control η via T, μ and N. This may lead us to satisfy, with a certain probability, the condition $\|\underline{D}_0\|_2 \leq \eta$ (thus $\underline{D}_0 \in \mathcal{D}$) with η small. As a result, we may render (5.48) easier to satisfy and have stability guarantees (in probability). Specifically, by applying Theorem 5.10, if (5.48), with X_1, Z_0 replaced by $\underline{X}_1, \underline{Z}_0$, is feasible then the control law $u = KZ(x)$, where K is given by (5.11) with U_0 replaced by $\underline{U}_0$, will asymptotically stabilize the origin with the same probability as condition $\|\underline{D}_0\|_2 \leq \eta$ is satisfied.

A second advantage of having $\|\underline{D}_0\|_2 \leq \eta$ with η small is that, by virtue of (5.50) and (5.52), we may have (in probability) less conservative estimates for the ROA and RPI sets compared to the ones obtained with deterministic (worst-case) bounds for the disturbance.

Example 8

We consider again Example 6 under the same experimental setup for the disturbance, but now we repeat the experiment $N = 100$ times, each time using the same input pattern. For the uniform distribution it holds that $\Sigma = \delta^2/3$. With $\mu = 4 \cdot 10^{-5}$, Proposition 5.3 implies $\|\underline{D}_0\|_2 \leq 0.0348$ with probability at least 99.48%. The bound is much tighter

[5] The notation used in the sequel is standard, *e.g.*, see [125]. Independent and identically distributed random vectors are abbreviated as i.i.d.. We will denote by $\mathcal{N}(\mu,\Sigma)$ the multivariate normal (Gaussian) distribution with mean μ and covariance matrix Σ.

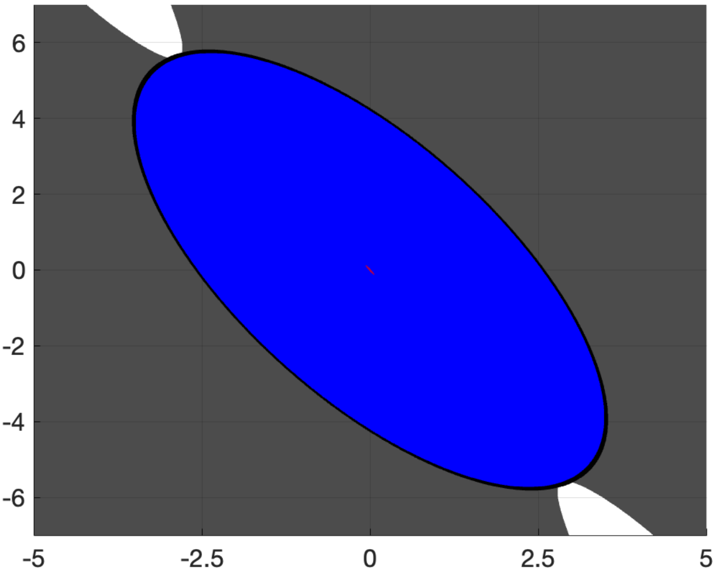

Figure 5.7: Results for Example 8 for the pendulum in case of repeated experiments. See the caption of Figure 5.5 for a description of the various sets.

compared to the worst-case bound $\|\underline{D}_0\|_2 \le \delta\sqrt{T} = 0.0548$ obtained by only exploiting the property $|d| \le \delta$.

We solve (5.56) (recall that (5.56) has the same feasible set as (5.48)) using the same parameters as in Example 6 but now with the average matrices $\underline{U}_0, \underline{Z}_0, \underline{X}_1$, and $\Delta = 0.0348$. We obtain

$$K = \begin{bmatrix} -20.9897 & -11.1369 & -9.8222 \end{bmatrix}.$$

Theorem 5.10 implies that K is stabilizing with probability at least 99.48% (K is indeed stabilizing as $\|\underline{D}_0\|_2 = 0.0050 < \Delta$). The RPI set obtained with $\Delta = 0.0348$ is much larger than the one obtained in Example 6 with the worst-case value $\Delta = \delta\sqrt{T}$; compare the new Figure 5.7 with Figure 5.5. •

EXAMPLE 9

We conclude the section with some simulation results for the polynomial system of Example 4. The system has "more unstable" dynamics than the pendulum system, and we obtain non-negligible RPI sets only for $|d| \le 0.001$. For the same setting as in Example 4 and a disturbance uniformly distributed the SDP (5.56) returns the RPI set in Figure 5.8 (Top). With averaging, we already improve the estimate for $N = 10$, see Figure 5.8 (Bottom). With averaging, we also systematically obtain non-negligible RPI sets up to $|d| \le 0.01$. •

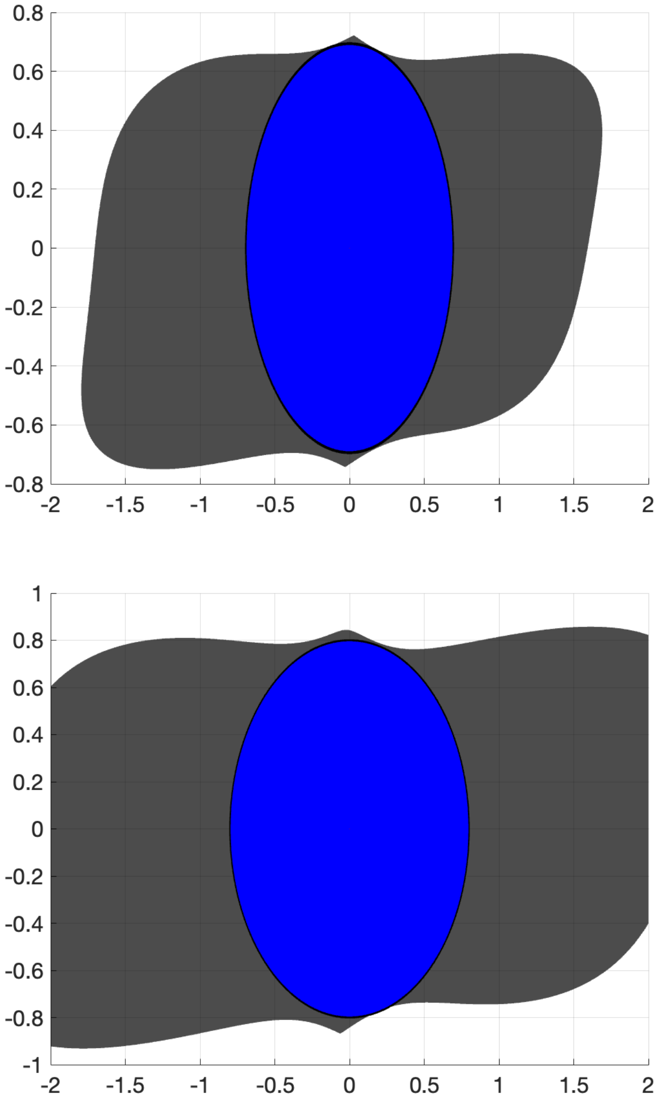

Figure 5.8: Results for Example 9, where we consider the polynomial system of Example 4 with a disturbance uniformly distributed in $[-0.001, 0.001]$ which affects both the states. We consider trajectories of length $T = 50$ and solve (5.56) with $\lambda_1 = \lambda_2 = 0.1$. Top: results without averaging. The grey set represents the set $\mathcal{X}$ in (5.54), while the blue set is the RPI set. Bottom: results with averaging ($N = 10$). We took $\mu = 5 \cdot 10^{-7}$ which gives $\Delta = 0.0052 I_2$ and certifies stability with 98.86% probability.

5.7. Discussion

We provide in the following a pair of additional discussion points.

5.7.1. Approximate nonlinearity cancellation and ROA size

Exact nonlinearity cancellation leads to global asymptotic controllers in the case no noise is affecting the data used in the design (Theorem 5.1). When an exact cancellation of the nonlinearities is not possible, an approximate one should be considered, as studied in Theorem 5.4. In general this result returns a local asymptotic stabilizer. Here, we would like to stress that this does not imply that it does not exist a global stabilizer attaining the same cost as the feasible solutions of the SDP (5.19) appearing in Theorem 5.4. We illustrate this point by revisiting system (5.25) in Example 4, which was used to demonstrate Theorem 5.4 and its follow-up, Proposition 5.1.

We observe that, were the model of the system known, one could design a global asymptotic stabilizer given by $u = -x_2 - 0.1x_1^2 - x_1^3 - 0.08x_1x_2^2 - 0.016x_2^4$. This controller returns a closed-loop system whose linear part M is Schur and whose nonlinear part N has norm equal to 0.2, the optimal value attained by the SDP (5.19). Hence, if one would include quartic monomials in $Z(x)$, it could be numerically verified whether or not the global asymptotic stabilizer is a feasible solution to the SDP (5.19). However, there is no analytic guarantee that the SDP will return exactly the global stabilizer, and in general it will not. This is because the SDP is obtained adopting a quadratic Lyapunov function and does not currently include a constraint to select a controller that maximizes the region of attraction, topics which are left for future research.

5.7.2. Nonlinearity cancellation and coordinate transformations

In model-based design, the possibility of cancelling the nonlinearity is eased by the existence of a normal form revealed by a suitable coordinate transformation. In this section we comment on how the techniques investigated so far lend themselves to be used along with such coordinate transformations obtainable for systems having a uniform relative degree equal to the dimension of the state space.

Consider the discrete time nonlinear system with output

$$x^+ = f(x,u) \tag{5.64a}$$
$$y = h(x) \tag{5.64b}$$

where $u, y \in \mathbb{R}$ for the sake of simplicity. We assume that both the state x and the output y are available for measurements. A prior information about the system is that it satisfies

$$\begin{aligned} \frac{\partial h \circ f_0^i \circ f(x,u)}{\partial u} &= 0, \ \forall (x,u) \in \mathbb{R}^{n+1}, \quad 0 \le i \le n-2 \\ \frac{\partial h \circ f_0^{n-1} \circ f(x,u)}{\partial u} &\neq 0, \quad \forall (x,u) \in \mathbb{R}^{n+1} \end{aligned} \tag{5.65}$$

where $f_0(x) = f(x,0)$, $f_0^d = \underbrace{f_0 \circ f_0 \circ \ldots \circ f_0}_{d \text{ times}}$,

$$\begin{bmatrix} h(x) \\ h \circ f_0(x) \\ \vdots \\ h \circ f_0^{n-1}(x) \end{bmatrix} =: \Phi_0(x) \tag{5.66}$$

is a global coordinate transformation [127, 128]. The transformation Φ_0 depends on the system's dynamics, which is not available; nevertheless it can be implemented bearing in mind the interpretation of its entries as the value of the output at a given time and at future time instants, namely, at any time k, we have that

$$w(k) := \begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+n-1) \end{bmatrix} = \Phi_0(x(k)),$$

so that in the coordinates w the system's dynamics can be written as

$$w(k+1) = \begin{bmatrix} w_2(k) \\ w_3(k) \\ \vdots \\ w_n(k) \\ h \circ f_0^{n-1} \circ f(x(k), u(k)) \end{bmatrix}, \; y(k) = w_1(k) \tag{5.67}$$

Note that the last entry of the vector field on the right-hand side has been deliberately left to depend on the original state x rather on the new one z, which turns out to be useful to obtain a *causal* control policy. The point of this transformation is that, were the system's dynamics known, one could design a static feedback controller that stabilizes the system via exact nonlinearity cancellation. When the dynamics are unknown, one can still achieve exact nonlinearity cancellation by modifying the techniques proposed in Section 5.3.1, provided that the following assumption holds:

Assumption 5.6 *A vector-valued function $Q : \mathbb{R}^n \to \mathbb{R}^{S-n}$ is known for which $h \circ f_0^{n-1} \circ f(x,u) = a^\top Q(x) + bu$ for some (unknown) quantities $a \in \mathbb{R}^S$, $b \in \mathbb{R} \setminus \{0\}$.* •

Asking for $h \circ f_0^{n-1} \circ f(x,u)$ to take this specific form is clearly demanding, but one can in principle collect the discrepancy between $h \circ f_0^{n-1} \circ f(x,u)$ and $a^\top Q(x) + bu$ into a mismatch function and treat it as a disturbance, analogously to what has been discussed in Section 5.6.2.

Under the assumption above, a controller can be designed following the construction in the previous subsection with suitable modifications. We start defining the matrix of input samples U_0 as in (5.6a), and

$$\begin{aligned}
W_0 &:= \begin{bmatrix} w(0) & w(1) & \cdots & w(T-1) \end{bmatrix} \in \mathbb{R}^{n\times T}, \\
W_1 &:= \begin{bmatrix} w(1) & w(2) & \cdots & w(T) \end{bmatrix} \in \mathbb{R}^{n\times T}, \\
Q_0 &:= \begin{bmatrix} Q(x(0)) & Q(x(1)) & \cdots & Q(x(T-1)) \end{bmatrix} \in \mathbb{R}^{(S-n)\times T}, \\
Z_0 &:= \begin{bmatrix} W_0^\top & Q_0^\top \end{bmatrix}^\top \in \mathbb{R}^{S\times T}, \qquad (5.68a)
\end{aligned}$$

which satisfy the identity $W_1 = A_c W_0 + B_c(a^\top Q_0 + bU_0)$, where the pair (A_c, B_c) is in the Brunovsky canonical form [129]. Note that since both the state x and the output y are assumed to be available for measurements, the matrices of data W_0, W_1, Q_0 are known. In particular, the matrix W_0 (similarly for W_1) comprises output samples:

$$W_0 = \begin{bmatrix} y(0) & y(1) & \dots & y(T-1) \\ y(1) & y(2) & \dots & y(T) \\ \vdots & \vdots & \ddots & \vdots \\ y(n-1) & y(n) & \dots & y(n+T-2) \end{bmatrix}.$$

We have the following result.

Corollary 5.2 *Consider the nonlinear system with output* (5.64). *Assume that conditions* (5.65) *hold and that the map Φ_0 in* (5.66) *is a global coordinate transformation. If there exist decision variables $G_1 \in \mathbb{R}^{T\times n}$, $k_1 \in \mathbb{R}$, and $G_2 \in \mathbb{R}^{T\times(S-n)}$ such that*

$$Z_0 G_1 = \begin{bmatrix} I_n \\ 0_{(S-n)\times n} \end{bmatrix}, \qquad (5.69a)$$

$$W_1 G_1 = A_c + B_c \begin{bmatrix} k_1 & \underbrace{0 \ \cdots \ 0}_{n-1 \text{ times}} \end{bmatrix}, \qquad (5.69b)$$

$$k_1 \in (-1, 1), \qquad (5.69c)$$

$$Z_0 G_2 = \begin{bmatrix} 0_{n\times(S-n)} \\ I_{S-n} \end{bmatrix}, \qquad (5.69d)$$

$$W_1 G_2 = 0_{n\times(S-n)}, \qquad (5.69e)$$

then $u = K \left[\begin{smallmatrix} w \\ Q(x) \end{smallmatrix}\right]$, with $K = U_0 G$, linearizes the closed-loop system and renders the origin a globally asymptotically stable equilibrium.

Proof. Conditions (5.69a), (5.69d) along with the definition of the controller gain K, show that the identity (5.7) holds. Thus, the closed-loop system is of the form

$$\begin{aligned}
w^+ &= A_c w + B_c(a^\top Q(x) + bu) && (5.70a)\\
&= A_c w + B_c(a^\top Q(x) + bU_0 G \left[\begin{smallmatrix} w \\ Q(x) \end{smallmatrix}\right]) && (5.70b)\\
&= W_1 G \left[\begin{smallmatrix} w \\ Q(x) \end{smallmatrix}\right] = W_1 G_1 w && (5.70c)
\end{aligned}$$

where the third equality follows from the identities $B_c b U_0 G = W_1 G - A_c W_0 G - B_c a^\top Q_0 G$, (5.69a) and (5.69d), and the last one from (5.69e). Hence, the controller

$u = K \begin{bmatrix} w \\ Q(x) \end{bmatrix}$ linearizes the closed-loop system. Finally, by (5.69b), the closed-loop system coincides with $w^+ = (A_c + B_c \begin{bmatrix} k_1 & 0 & \cdots & 0 \end{bmatrix})w$, where the matrix $A_c + B_c \begin{bmatrix} k_1 & 0 & \cdots & 0 \end{bmatrix}$ is Schur since all its eigenvalues are given by the solutions of the equation $\lambda^n = (-1)^n k_1$ and $|k_1| < 1$. ■

The control law only uses the variables y, x and as such it is implementable. In fact, bearing in mind (5.69a) and (5.69d), the identity

$$W_1 G = A_c W_0 G + B_c (a^\top Q_0 G + b\, U_0 G)$$

is equivalent to

$$\begin{aligned} &\begin{bmatrix} A_c + B_c \begin{bmatrix} k_1 & 0 & \cdots & 0 \end{bmatrix} & 0_{n\times(S-n)} \end{bmatrix} = \begin{bmatrix} A_c & 0_{n\times(S-n)} \end{bmatrix} \\ &+ B_c \begin{bmatrix} 0_{n\times n} & a^\top \end{bmatrix} + B_c\, b\, U_0 G \end{aligned}$$

from which we deduce that

$$U_0 G = b^{-1} \begin{bmatrix} \begin{bmatrix} k_1 & 0 & \cdots & 0 \end{bmatrix} & -a^\top \end{bmatrix},$$

that is $U_0 G_1 w$ only depends on the first component of w, which is the output y.

Example 10

Consider the polynomial system

$$x_1^+ = x_2^2 + x_1^3 + u \tag{5.71a}$$

$$x_2^+ = 0.5x_1 + 0.2x_2^2 \tag{5.71b}$$

$$y = x_2 \tag{5.71c}$$

Exact cancellation based on Theorem 5.1 is not possible for this system. On the other hand, the conditions of Corollary 5.2 hold.

In particular, notice that

$$h \circ f_0^{n-1} \circ f(x,u) = \frac{1}{20}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_1^3 + \frac{1}{25}x_1 x_2^2 + \frac{1}{125}x_2^4 + \frac{1}{2}u.$$

Hence, if we choose

$$Q(x) = \begin{bmatrix} x_1^2 & x_2^2 & x_1x_2 & x_1^3 & x_2^3 & x_1x_2^2 & x_1^2x_2 & x_1^4 & x_2^4 & x_1x_2^3 & x_1^2x_2^2 & x_1^3x_2 \end{bmatrix}$$

then Assumption 5.6 is satisfied. The choice of such a $Q(x)$ can be guided by some prior knowledge, namely that the nonlinearity in the last equation of the system in the new coordinates is a polynomial of degree no larger than 4. On the other hand, the exclusion of x from $Q(x)$ is suggested by the fact that, if this were not the case, then the matrix Z_0 would be rank deficient (this is a test that can be carried out from the collected data). This is because each column i of W_0 is equal to

$$[y(i-1) \;\; y(i)]^\top = [x_2(i-1) \;\; 0.5x_1(i-1) + 0.2x_2(i-1)^2]^\top$$

and it would be expressible as a linear combination of the entries of column i of Q_0 if the latter would include x.

Applying Corollary 5.2, we find that the SDP (5.69) is feasible and returns the solution $k_1 = 0.372$ and

$$K = [0.7423 \;\; 0 \;\; -0.1 \;\; -1 \;\; 0 \;\; -1 \;\; 0 \;\; -0.08 \;\; 0 \;\; 0 - 0.016 \;\; 0 \;\; 0 \;\; 0]$$

which linearizes the closed-loop system in the coordinates w, and renders the origin a globally asymptotically stable equilibrium. •

5.8. Conclusions

In this extensive chapter, we have introduced a method to design Lyapunov-based stabilizing controllers for nonlinear systems from data, which reduces the design to the solution of data-dependent SDP. The method is certified to provide a solution in the presence of perturbed data as well as estimates of the region of attraction of the closed-loop system. Both deterministic and stochastic perturbations on the data are studied. We also extended the results to deal with the presence of neglected nonlinearities. Possible future research should focus on output feedback control design, the inclusion of criteria to maximize the region of attraction and the design of more general (non quadratic) Lyapunov functions.

5.9. Proofs

5.9.1. Parametrization of all stabilizing and linearizing feedback controllers

Suppose that $\left[\begin{smallmatrix} U_0 \\ Z_0 \end{smallmatrix}\right]$ has full row rank. In this case, we can prove that any stabilizing and linearizing feedback controller can be parametrized as in (5.11) for some Y_1, P_1, G_2 satisfying (5.10). Note in particular that this implies that the SDP is feasible. This result is as a generalization of [40, Theorem 3] where an analogous result for linear system is provided under the condition that $\left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]$ has full row rank. In the linear case, the latter condition reduces to a design condition for controllable dynamics, see [20, Theorem 1], [130, Theorem 1]. To the best of our knowledge, no analogous design conditions exists for nonlinear systems.

Proof of Theorem 5.2.
Consider any stabilizing and linearizing feedback controller K. We have

$$A + BK = X_1 G \tag{5.72}$$

for some $G \in \mathbb{R}^{T\times S}$ satisfying (5.7). Note that G exists as $\left[\begin{smallmatrix} U_0 \\ Z_0 \end{smallmatrix}\right]$ has full row rank by hypothesis. By partitioning $K = [\overline{K} \quad \hat{K}]$ with $\overline{K} \in \mathbb{R}^{m\times n}$ and $G = [G_1 \quad G_2]$ with $G_1 \in \mathbb{R}^{T\times n}$, we have $X_1 G_1 = \overline{A} + B\overline{K}$ and $X_1 G_2 = \hat{A} + B\hat{K} = 0$, where the matrix $X_1 G_1$ is Schur and $X_1 G_2 = 0$ by the assumption that K is stabilizing and linearizing. Hence, there exists a matrix $P_1 \succ 0$ such that $(X_1 G_1)^\top P_1^{-1} X_1 G_1 - P_1^{-1} \prec 0$. This implies $(X_1 Y_1)^\top P_1^{-1} X_1 Y_1 - P_1 \prec 0$ with $Y_1 = G_1 P_1$, which is the stability constraint in (5.10b). Since $Z_0 G = I_S$ and $Y_1 = G_1 P_1$ we have

$$Z_0 \begin{bmatrix} Y_1 & G_2 \end{bmatrix} = \begin{bmatrix} P_1 & 0_{n\times(S-n)} \\ 0_{(S-n)\times n} & I_{S-n} \end{bmatrix}, \tag{5.73}$$

which matches the constraints (5.10a) and (5.10c). Thus, all the constraints in (5.10) are satisfied, hence the program is feasible.

As for the form of the controller, by (5.7) we have $K = U_0 G$ which in terms of Y_1, G_2 reads as (5.11). ■

5.9.2. Parametrization of all (locally) stabilizing feedback controllers

Proof of Theorem 5.5.
The identity (5.72) is still valid because independent of the properties of K. Furthermore, we can still write $X_1 G_1 = \overline{A} + B\overline{K}$ and $X_1 G_2 = \hat{A} + B\hat{K}$. (The only difference with respect to Theorem 5.2 is that now $X_1 G_2$ might be different from zero.) Observe now that, by assumption, $X_1 G_1$ is Schur. Hence, there exists a matrix $P_1 \succ 0$ such that $(X_1 G_1)^\top P_1^{-1} X_1 G_1 - P_1^{-1} \prec 0$. By defining $Y_1 = G_1 P_1$, this is equivalent to (5.19c). Finally, recalling that $Z_0 G = I_S$, we have again the identity (5.73). Thus, all the constraints in (5.19) are satisfied and the program is feasible.

As for the form of the controller, by (5.7) we have $K = U_0 G$ which in terms of Y_1, G_2 reads as (5.11). ■

5.9.3. Proof of Lemma 5.3

Lemma 5.3 is a direct consequence of the following result.

Lemma 5.4 *Let $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$ be given matrices, and let $\mathcal{D} := \{D \in \mathbb{R}^{q\times p} : DD^\top \preceq \Delta\Delta^\top\}$. Then, for arbitrary $\epsilon > 0$ it holds that*

$$BD^\top C + C^\top D B^\top \preceq \epsilon^{-1} BB^\top + \epsilon C^\top \Delta\Delta^\top C \quad \forall D \in \mathcal{D}$$

Proof. A completion of squares

$$\left(\sqrt{\epsilon^{-1}}B - \sqrt{\epsilon}C^\top D\right)\left(\sqrt{\epsilon^{-1}}B - \sqrt{\epsilon}C^\top D\right)^\top \succeq 0$$

gives the result. ■

Proof of Lemma 5.3.
Let (5.47) hold. By a Schur complement, this is equivalent to

$$\begin{bmatrix} P_1 - \Omega & (X_1 Y_1)^\top \\ X_1 Y_1 & P_1 \end{bmatrix} - \epsilon^{-1} \underbrace{\begin{bmatrix} Y_1^\top \\ 0_{n\times T} \end{bmatrix}}_{:=B} \begin{bmatrix} Y_1 & 0_{T\times n} \end{bmatrix}$$
$$-\epsilon \underbrace{\begin{bmatrix} 0_{n\times s} \\ E \end{bmatrix}}_{:=C^\top} \Delta\Delta^\top \begin{bmatrix} 0_{s\times n} & E^\top \end{bmatrix} \succ 0$$

An application of Lemma 5.4 gives

$$\begin{aligned}\begin{bmatrix} P_1 - \Omega & (X_1 Y_1)^\top \\ X_1 Y_1 & P_1 \end{bmatrix} - \begin{bmatrix} Y_1^\top \\ 0_{n\times T} \end{bmatrix} D^\top \begin{bmatrix} 0_{s\times n} & E^\top \end{bmatrix} \\ - \begin{bmatrix} 0_{n\times s} \\ E \end{bmatrix} D \begin{bmatrix} Y_1 & 0_{T\times n} \end{bmatrix} \succ 0 \\ \forall D \in \mathcal{D}\end{aligned}$$

or, equivalently,

$$\begin{bmatrix} P_1 - \Omega & Y_1^\top (X_1 - ED)^\top \\ (X_1 - ED) Y_1 & P_1 \end{bmatrix} \succ 0 \quad \forall D \in \mathcal{D} \tag{5.74}$$

This is equivalent to (5.45) after another Schur complement, and this gives the result. ■

6
Conclusions

The big data revolution is deeply changing the way we understand and analyze natural phenomena around us. In the field of control engineering, learning from data enables researchers to explore new intelligent algorithms to model, predict, and control various dynamical systems. The work presented in this thesis aims to discuss and investigate the potentialities of this ever growing field in control design applications. In particular, the main questions addressed here are the following: **How can we apply data-oriented techniques in a control engineering context? How to derive data-based stabilizing (and optimal) control algorithms which are computationally tractable and require small amount of data?** We address these questions by developing a framework from which we can design suitable controllers to stabilize, optimize and linearize complex systems when the underlying dynamics are unknown. The methods proposed in this thesis build up and extend the data-driven literature and have the following desirable features:

- Simplicity — they are based on concepts from control theory and linear algebra.
- Low complexity — they lead to algorithms that are easily implementable as convex programs with low computational effort and require small amount of data.
- Theoretical guarantees — they return controllers with stability guarantees.

We list the main contributions in Section 6.1, and we conclude this thesis by suggesting some possible extensions to the current work in Section 6.2.

6.1. Conclusions

This section is devoted to giving an overall summary of the main contributions in this thesis. In our work, we have considered the development of data-driven algorithms to model, optimally control, and stabilize different families of unknown dynamical systems via the use of data. The main contributions are listed as follows.

- We have extended the data-driven framework to reformulate the finite-horizon linear quadratic regulation (LQR) problem as convex optimization problems involving linear matrix inequalities. In particular, by using data, the optimal control law is then obtained as the solution of a suitable semidefinite program.

- We have developed a data-based control method for switched systems with unknown modes, unknown switching signal and unknown switching instants. In this context, we have proposed a framework which requires no intermediate identification steps and provides stability guarantees. The key idea relies on an online scheme where input-state data are collected over time as the system is evolving. This scenario poses various challenges that required us to establish non-trivial results, such as the satisfaction of the persistence of excitation condition and the feasibility of the optimization problem in the online implementation. The control mechanism is then directly parametrized through these online data and iteratively updated via a computationally tractable data-dependent semidefinite program. The resulting controller has shown to provide stability guarantees.

- We have introduced a method to design stabilizing controllers for unknown nonlinear systems from data. To achieve so, we have expressed the nonlinear vector-based function as combinations of known nonlinear functions. This made it possible to derive an equivalent representation of the system, and in turn, to provide a data-based parametrization of it. This data-dependent formulations was used to derive controllers able to automatically capture and cancel out all the nonlinearities of the system, without any identification of its dynamics. The developed design methods are formulated in the compact form of data-dependent linear matrix inequalities, which are computationally inexpensive and retain the same simplicity of the formulations established for linear systems. We have provided solutions in the presence of perturbed data as well as neglected nonlinearities. Regarding noisy data, we have considered both deterministic and stochastic perturbations. We have also extended the results to a more general class of nonlinear systems.

6.2. FUTURE WORKS

In this section, we provide some recommendation for future research that can extend and improve the work presented in this thesis.

- In the current work, the LQR problem was considered in the uncertainty-free form. An interesting extension concerns robustness against perturbed data. A natural way to address this uncertainty induced by noisy data is via Petersen's lemma [131, 132], which has been proven to be a powerful tool for data-driven control [68].

- The proposed data-based formulation of the LQR problem was obtained via the use of linear matrix inequalities. This formulation can be further extended to incorporate *safety* constraints. In this context, the work of [133] presents a computationally tractable method for robust MPC synthesis involving linear matrix inequalities constraints.

- The online learning algorithm presented in Chapter 4 can be extended in several directions. One possible extension is incorporating robustness techniques to deal with noisy data. For this we advocate once again the use of Petersen's lemma. As another direction, the investigation of resource-constrained control systems represents an interesting research venue. In fact, the online mechanism can be integrated with event-triggering and self-triggering transmission schemes able to optimally schedule the update of the data-based controllers only when the dynamics of the systems are changing.

- With respect to the control of nonlinear systems, when an exact cancellation of the nonlinearities is not possible, approximate cancellation should be considered, as studied in Chapter 5. In general this result returns a local stabilizer, but this does not exclude existence of a global stabilizer. However, there is no analytic guarantee that our formulation will return such global stabilizer. This is because the proposed convex formulation (i) is restricted to quadratic Lyapunov functions, and (ii) does not include a criteria for maximizing the region of attraction. In this respect, a possible research line could be the inclusion of criteria to maximize the region of attraction and the design of more general Lyapunov functions.

- Finally, the data-driven control framework can be further extended to tackle various control engineering problems, ranging from power systems to applications in the field of fault detection.

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Summary

The *big data* revolution is deeply changing the way we understand and analyze natural phenomena around us. In the field of control engineering, learning from data enables researchers to explore new intelligent algorithms to model, predict, and control complex dynamical systems. In this context, data-driven control has become increasingly popular. Data-driven control is based on the paradigm of learning controllers of an unknown dynamical system by directly using data. By data, we mean measurements of a dynamical system, typically of its inputs and states/outputs. The underlying idea is that, information about the model can be gathered from experiments, bypassing completely the identification step, which can be impractical or too costly.

This thesis presents data-driven control solutions for different families of unknown dynamical systems, with a focus on both linear and special classes of nonlinear ones. The thesis is divided into three parts. Firstly, we consider the design of data-based linear quadratic regulators for unknown linear systems. Secondly, we present an online algorithm for learning controllers applied to switched linear systems. The algorithm collects data over time and iteratively updates the control rule via computationally inexpensive data-dependent convex programs. Lastly, we focus on a more general class of nonlinear systems, and we derive conditions to design controllers via nonlinearity cancellation.

In the first part of the thesis, we consider the finite-horizon linear quadratic regulator problem for linear time-invariant discrete-time systems. The system is assumed to be unknown and information on the system is given by a finite set of input-state data. This finite collection of data allows to determine the optimal control law in one shot, with no intermediate identification step.

In the second part of the thesis, we turn our attention to more complex scenarios, and we consider a special class of nonlinear systems, namely switched linear systems. We do not assume any knowledge of the system: the different modes of the switched system, the switching signal, and the switching instants are all assumed to be unknown. This scenario poses non-trivial challenges that are hard to address with conventional approaches. In fact, a major challenge is how to capture any changes in the dynamics of the system and adjust the controller accordingly to achieve stabilization of the unknown system. We address this challenge by developing an online scheme that can collect data over time while the system is evolving. By collecting data *on the fly*, we show that the control mechanism can capture any changes in the dynamics of the plant and adapt itself accordingly to achieve stabilization of the running dynamics.

Lastly, in the third part, we derive data-driven methods for a more general class of nonlinear systems via nonlinearity cancellation. To this end, we make use of a "dictionary" of nonlinear terms that includes the nonlinearities of the unknown system. This allows us to consider an equivalent representation of the system, and in turn, provide a

data-based representation of it. This data-based representation allows us to devise stabilizing controllers via nonlinearity cancellations, where the designed controllers discover the nonlinear terms and cancel them out automatically. When exact nonlinearity cancellation is not achievable, the controller design is approached as a computationally tractable minimization problem. In this case, the hard constraint of exact nonlinearity cancellation is relaxed to a condition corresponding to an approximate nonlinearity cancellation. In general, the design based on an approximate nonlinearity cancellation does not return globally stabilizing controllers, hence we explicitly characterize the region of attraction of the closed-loop system. We then show that the proposed results can be extended in different directions, namely in the case of continuous dynamics, noisy data, and neglected nonlinearities.

Sommario

La rivoluzione dei *big data* sta cambiando profondamente il modo in cui comprendiamo e analizziamo i fenomeni naturali che ci circondano. Nel campo dell'ingegneria del controllo, l'apprendimento dai dati consente ai ricercatori di esplorare nuovi algoritmi intelligenti per modellare, prevedere e controllare sistemi dinamici complessi. In questo contesto, il controllo basato sui dati è diventato sempre più popolare. Il controllo guidato dai dati si basa sull'apprendere le leggi di controllo di un sistema dinamico ignoto utilizzando direttamente i dati. Per dati si intendono le misurazioni di un sistema dinamico, tipicamente misure dei suoi input e stati/uscite. L'idea generale è che le informazioni sul modello possono essere raccolte dagli esperimenti, bypassando completamente la fase di identificazione, che può essere poco pratica o troppo costosa.

Questa tesi presenta soluzioni di controllo basate sui dati per diverse famiglie di sistemi dinamici ignoti, con particolare attenzione ai sistemi lineari e certe classi speciali di sistemi non-lineari. La tesi è divisa in tre parti. Primo, consideriamo la progettazione di regolatori quadratici lineari basati su dati per sistemi lineari ignoti. Secondo, presentiamo un algoritmo online per l'apprendimento di leggi di controllo applicati a sistemi lineari a commutazione. L'algoritmo raccoglie i dati nel tempo e aggiorna in modo iterativo la regola di controllo tramite programmi convessi dipendenti dai dati e poco costosi dal punto di vista computazionale. Infine, ci concentriamo su una classe più generale di sistemi non-lineari e deriviamo le condizioni per progettare i controllori tramite la cancellazione della non-linearità.

Nella seconda parte della tesi, rivolgiamo la nostra attenzione a scenari più complessi e consideriamo una classe speciale di sistemi non-lineari, ovvero i sistemi lineari a commutazione. Non assumiamo alcuna conoscenza del sistema: si presume che le diverse modalità del sistema a commutazione, il segnale di commutazione e gli istanti di commutazione siano tutti sconosciuti. Questo scenario pone sfide non banali difficili da affrontare con gli approcci convenzionali. In effetti, una sfida importante è come catturare eventuali cambiamenti nella dinamica del sistema e regolare di conseguenza il controllore per ottenere la stabilizzazione del sistema ignoto. Affrontiamo questa sfida sviluppando uno schema online in grado di raccogliere dati nel tempo mentre il sistema è in evoluzione. Raccogliendo i dati *al volo*, dimostriamo che il meccanismo di controllo è in grado di catturare eventuali cambiamenti nella dinamica dell'impianto e di adattarsi di conseguenza per ottenere la stabilizzazione della dinamica corrente.

Infine, nella terza parte, deriviamo metodi basati sui dati per una classe più generale di sistemi non-lineari tramite la cancellazione della non-linearità. A tal fine, utilizziamo un "dizionario" di termini non-lineari che include le non-linearità del sistema ignoto. Questo ci permette di considerare una rappresentazione equivalente del sistema e, a sua volta, fornirne una rappresentazione basata sui dati. Questa rappresentazione ci consente di

ideare leggi di controllo stabilizzanti tramite cancellazioni delle non-linearità, in cui i controllori progettati scoprono i termini non-lineari e li annullano automaticamente. Quando non è possibile ottenere l'esatta cancellazione della non-linearità, la progettazione del controllore viene affrontata come un problema di minimizzazione computazionalmente trattabile. In questo caso, il vincolo rigido dell'annullamento della non-linearità esatta viene rilassato a una condizione corrispondente a un annullamento della non-linearità approssimativa. In generale, la progettazione basata su una cancellazione approssimativa della non-linearità non restituisce controllori globalmente stabilizzanti, per cui procediamo a caratterizzare esplicitamente la regione di attrazione del sistema ad anello chiuso. Mostriamo successivamente che i risultati proposti possono essere estesi in diverse direzioni, vale a dire il caso della dinamica continua, dati rumorosi e non-linearità trascurate.

Samenvatting

De *big data* revolutie verandert ingrijpend de manier waarop we natuurlijke fenomenen om ons heen begrijpen en analyseren. Op het gebied van regeltechniek stelt het leren van gegevens onderzoekers in staat om nieuwe intelligente algoritmen te verkennen voor het modelleren, voorspellen en besturen van complexe dynamische systemen. In deze context is datagedreven besturing steeds populairder geworden. Datagestuurde besturing is gebaseerd op het paradigma van het leren van controllers van een onbekend dynamisch systeem door direct gebruik te maken van data. Met gegevens bedoelen we metingen van een dynamisch systeem, meestal van zijn ingangen en toestanden/uitgangen. Het achterliggende idee is dat informatie over het model kan worden verzameld uit experimenten, waarbij de identificatiestap volledig wordt omzeild, wat onpraktisch of te duur kan zijn.

Dit proefschrift presenteert datagestuurde regeloplossingen voor verschillende families van onbekende dynamische systemen, met een focus op zowel lineaire als speciale klassen van niet-lineaire systemen. Het proefschrift is opgedeeld in drie delen. Ten eerste beschouwen we het ontwerp van op data gebaseerde lineaire kwadratische regelaars voor onbekende lineaire systemen. Ten tweede presenteren we een online algoritme voor lerende controllers toegepast op geschakelde lineaire systemen. Het algoritme verzamelt gegevens in de loop van de tijd en werkt de controller iteratief bij via rekenkundig goedkope gegevens afhankelijk convexe programma's. Ten slotte richten we ons op een meer algemene klasse van niet-lineaire systemen, en leiden we voorwaarden af om controllers te ontwerpen via niet-lineariteit annulering.

In het eerste deel van het proefschrift beschouwen we het eindige-horizon lineaire kwadratische regulator probleem voor lineaire tijdsinvariante discrete-tijdsystemen. Er wordt aangenomen dat het systeem onbekend is en informatie over het systeem wordt gegeven door een eindige verzameling invoer statusgegevens. Deze eindige verzameling gegevens maakt het mogelijk om in één keer de optimale controlewet te bepalen, zonder tussenliggende identificatiestap.

In het tweede deel van het proefschrift richten we onze aandacht op complexere scenario's en beschouwen we een speciale klasse van niet-lineaire systemen, namelijk geschakelde lineaire systemen. We veronderstellen geen kennis van het systeem: de verschillende modi van het geschakelde systeem, het schakelsignaal en de schakelmomenten worden allemaal als onbekend verondersteld. Dit scenario stelt niet-triviale uitdagingen die moeilijk aan te pakken zijn met conventionele benaderingen. In feite is het een grote uitdaging om eventuele veranderingen in de dynamiek van het systeem vast te leggen en de controller dienovereenkomstig aan te passen om stabilisatie van het onbekende systeem te bereiken. We pakken deze uitdaging aan door een online schema te ontwikkelen dat in de loop van de tijd gegevens kan verzamelen terwijl het systeem evolueert. Door gegevens *on the fly* te verzamelen, laten we zien dat het regelmechanisme eventuele ver-

anderingen in de dynamiek van de plant kan zien en zichzelf dienovereenkomstig kan aanpassen om stabilisatie van de evoluerende dynamiek te bereiken.

Ten slotte leiden we in het derde deel gegevensgestuurde methoden af voor een meer algemene klasse van niet-lineaire systemen via niet-lineariteits annulering. Hiertoe maken we gebruik van een "woordenboek" van niet-lineaire termen die de niet-lineariteiten van het onbekende systeem bevat. Dit stelt ons in staat om een equivalente representatie van het systeem te overwegen en op zijn beurt een op gegevens gebaseerde representatie ervan te bieden. Deze op gegevens gebaseerde representatie stelt ons in staat stabiliserende controllers te bedenken via niet-lineariteits annuleringen, waarbij de ontworpen controllers de niet-lineaire termen ontdekken en deze automatisch opheffen. Wanneer exacte niet-lineariteits annulering niet haalbaar is, wordt het controller ontwerp benaderd als een rekenkundig goedkoop minimalisering probleem. In dit geval wordt de harde beperking van exacte niet-lineariteits opheffing versoepeld tot een voorwaarde die overeenkomt met een benaderde niet-lineariteits opheffing. Over het algemeen levert het ontwerp op basis van een geschatte niet-lineariteits annulering geen globaal stabiliserende controllers op, daarom karakteriseren we expliciet het aantrekkingsgebied van het gesloten-lussysteem. Vervolgens laten we zien dat de voorgestelde resultaten in verschillende richtingen kunnen worden uitgebreid, namelijk in het geval van continue dynamica, data met ruis en verwaarloosde niet-lineariteiten.