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On Wilson's theorem about domains of attraction and tubular neighborhoods

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ABSTRACT

In this paper, we show that the domain of attraction of a compact asymptotically stable submanifold in a finite-dimensional smooth manifold of an autonomous system is homeomorphic to the submanifold's tubular neighborhood. The compactness of the submanifold is crucial, without which this result is false; two counterexamples are provided to demonstrate this.

1. Introduction

The domain of attraction of an attractor of a continuous dynamical system has been widely studied. An *attractor* is a closed invariant set of which there exists an open neighborhood such that every trajectory of the dynamical system starting within the neighborhood eventually converges to the attractor, in the sense that the distance between the trajectory and the attractor converges to zero; namely, the attractor is *attractive*. And the set of all initial conditions rendering the corresponding trajectories to converge to the attractor is called the *domain of attraction* of the attractor [1,2]. Generally, it is difficult or sometimes impossible to find analytically the domain of attraction of an attractor. Since an attractor is attractive, if additionally it is *Lyapunov stable* [1, Chapter 4], then it is called an *asymptotically stable attractor*; sometimes Lyapunov functions can be utilized to estimate its domain of attraction, but the estimate can be conservative [1, Chapters 4 and 8].

Partly due to the difficulty of calculating the domain of attraction of an attractor, some studies in the literature instead investigate the "shapes" or "sizes" of domains of attraction in the topological sense [2–6]. In particular, in the simplest case where the attractor is an asymptotically stable equilibrium point, it has been shown in [3, Theorem 21] that the domain of attraction is contractible. This result characterizes the "shape" of the domain of attraction, and it also implies the "size" of the domain of attraction. Namely, it leads to the topological obstruction that if the state space of the system is not contractible, then an equilibrium point cannot be stabilized globally [3, Corollary 5.9.3]. Another topological obstruction is shown in [6], which states that the domain of attraction of an asymptotically stable equilibrium point cannot be the whole state space (i.e., global asymptotic stability of an equilibrium is impossible) if the state space of the continuous dynamical system has the structure of a vector bundle over a compact manifold. Some studies partly generalize these results to asymptotically stable attractors that are not necessarily equilibrium points. In [5], it is proved that a compact, asymptotically stable attractor defined on a manifold (or more generally, on a locally compact metric space) is a *weak deformation retract* of its domain of attraction. The conclusion is further developed in [4], which shows that if the considered manifold is the Euclidean space $\mathbb{R}^n$, then the compact asymptotically stable attractor is a *strong deformation retract* of its domain of attraction.

Assuming that the asymptotically stable attractors are *compact submanifolds* of some ambient finite-dimensional smooth manifolds, stronger conclusions can be made about the domains of attraction. For example, it is proved in [2, Chapter V, Lemma 3.2] that the intersection of an ϵ-neighborhood of the attractor and some sublevel set of a corresponding Lyapunov function (of which the existence is automatically guaranteed [7]) is a deformation retract of the domain of attraction of the attractor. This result is refined in [5,8], which conclude that the attractor itself is a strong deformation retract of its domain of attraction. Therefore, the attractor and its domain of attraction are homotopy equivalent. This result has practical significance. For example, it facilitates the analysis regarding the existence of singular points and the possibility of global convergence of trajectories to desired paths in the vector-field guided path-following problem for robotic control

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systems [8]. Furthermore, Theorem 3.4 in [9] claims that, the domain of attraction of a uniformly asymptotically stable attractor is diffeomorphic to a tubular neighborhood of the attractor. Note that the attractor is assumed to be an embedded submanifold which can be compact or non-compact. However, in this paper, we will show that the compactness of the attractor is crucial, without which such a claim becomes inaccurate.[1] In addition, the proof of [9, Theorem 3.4] is very brief, only indicating the method without giving sufficient detail. In this paper, we will detail the proof for a corrected version of this theorem, where the attractor is required to be compact. For convenience, we will henceforth refer to [9, Theorem 3.4] as Wilson's theorem.

Contributions: Throughout the paper, manifolds or submanifolds are *without* boundaries, and they are second countable and paracompact. We assume that the attractor is compact, (uniformly) asymptotically stable and it is a submanifold of some finite-dimensional smooth manifold. We show that the compactness of the attractor is crucial by providing counterexamples where Wilson's theorem no longer holds if the attractor is *not* compact. Taking the compactness of the attractor into account, we will prove the following theorem:

Theorem 1. *The domain of attraction of a compact asymptotically stable submanifold $\mathcal{S}$ in a finite-dimensional smooth manifold $\mathcal{M}$ of an autonomous system is diffeomorphic to the tubular neighborhood of $\mathcal{S}$.*

In this paper, we will give a complete and detailed proof of Theorem 1, along with some auxiliary results to gain more insights into the theorem.

The remainder of the paper is organized as follows. Section 2 provides some preparatory results for the convenience of proving Theorem 1. Then the detailed proof of Theorem 1 is elaborated in Section 3. To justify the importance of the compactness of the attractor in this theorem, we provide two counterexamples where the attractor is *not* compact and hence Theorem 1 fails to hold in Section 4. Finally, Section 5 concludes the paper.

2. Preparatory results

In this section, we recall some preliminaries. Let $\mathcal{M}$ and $\mathcal{N}$ be smooth manifolds, and $\mathcal{S}$ be a submanifold of $\mathcal{M}$. Note that in this section, the submanifold $\mathcal{S}$ can be compact or non-compact unless its compactness is specified explicitly. The notation $:=$ means "defined to be", and the notation $\circ$ denotes the composition of functions. For example, $f \circ g$ is the composition of the functions f and g. The map id is the identity map where the domain and codomain are clear from the context.

For convenience, throughout this paper we consider autonomous systems with complete vector fields on a Riemannian manifold $\mathcal{M}$ with the distance function d. Note that the assumption on the completeness of vector fields can be dropped (see Remark 21).

Denote by φ the flow of the system, and by $\mathcal{U}_\epsilon$, $\mathcal{U}_\delta$ and $\mathcal{U}_r$ the neighborhoods consisting of all points in $\mathcal{M}$ from which the distances to $\mathcal{S}$ are not larger than ϵ, δ and r respectively. If an attractor $\mathcal{S}$ is uniformly asymptotically stable, then (i) it is stable; namely, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\varphi^t(\mathcal{U}_\delta) \subseteq \mathcal{U}_\epsilon$ for $t \geq 0$; (ii) there exists $r > 0$, such that for any $\epsilon > 0$, there is some positive number T_ϵ with $\varphi^t(\mathcal{U}_r) \subseteq \mathcal{U}_\epsilon$ for $t \geq T_\epsilon$ [7]. Uniform asymptotic stability is stronger than asymptotic stability, but if $\mathcal{A}$ is compact, then these two notions are equivalent.

Now we recall the definitions of topological and smooth embeddings.

[1] If the attractor is compact, then asymptotic stability automatically implies uniform asymptotic stability.

Definition 2 (*Topological and Smooth Embeddings, [10, p. 85]*). A *(topological) embedding* is an injective continuous map that is a homeomorphism onto its image (with the subspace topology). A *smooth embedding* is a smooth immersion that is also a (topological) embedding.

If $f : \mathcal{M} \to \mathcal{N}$ is an embedding, the image $f(\mathcal{M})$ can be regarded as a homeomorphic copy of $\mathcal{M}$ inside $\mathcal{N}$. If $f : \mathcal{M} \to \mathcal{N}$ is a smooth embedding, then it is both a topological embedding and a smooth immersion.

For each $p \in \mathcal{M}$, denote by $T_p\mathcal{M}$ and $T_p\mathcal{S}$ the tangent spaces respectively of $\mathcal{M}$ and $\mathcal{S}$ at p, and by $T\mathcal{M}$ and $T\mathcal{S}$ the tangent bundles. Note that $T\mathcal{S}$ can be regarded as a subbundle of $T\mathcal{M}$ in a natural way.

Definition 3 (*Normal Bundle*). The normal bundle $\mathcal{N}_\mathcal{S}$ of $\mathcal{S}$ in $\mathcal{M}$ is the quotient bundle $T_\mathcal{S}\mathcal{M}\big/T\mathcal{S} := \bigsqcup_{p\in\mathcal{S}}(T_p\mathcal{M}/T_p\mathcal{S})$, where $\bigsqcup$ denotes the disjoint union.

Fact 1 (*[11, Sections 6.1 and 7.1]*). Let g be any Riemannian metric on $\mathcal{M}$. For each $p \in \mathcal{M}$, let $\mathcal{N}_p$ be the orthogonal complement of $T_p\mathcal{S}$ in $T_p\mathcal{M}$ with respect to g. Then $\bigsqcup_{p\in\mathcal{S}}\mathcal{N}_p$ is a subbundle of $T_\mathcal{S}\mathcal{M}$ and it is isomorphic to $T_\mathcal{S}\mathcal{M}\big/T\mathcal{S}$. This gives another way of defining the normal bundle of $\mathcal{S}$ in $\mathcal{M}$.

Fact 2 (*[11, Section 5.1]*). For any vector bundle $\mathcal{E}$ over $\mathcal{S}$, (the image of) the zero section of $\mathcal{E}$ can be canonically identified with $\mathcal{S}$ via

$$\begin{aligned} \iota_\mathcal{S} : \bar{0}_\mathcal{S} \subseteq \mathcal{E} &\to \mathcal{S} \\ 0_x &\mapsto x \end{aligned}$$

where $\bar{0}_\mathcal{S} \subseteq \mathcal{E}$ denotes (the image of) the zero section of $\mathcal{E}$, and 0_x denotes the zero vector in the vector space $\mathcal{E}_x$ for $x \in \mathcal{S}$. Therefore, $\iota_\mathcal{S}$ is a diffeomorphism from $\bar{0}_\mathcal{S}$ to $\mathcal{S}$. Note that viewing $\mathcal{S}$ as a submanifold of $\mathcal{M}$, $\iota_\mathcal{S}$ can also be regarded as an embedding of $\bar{0}_\mathcal{S}$ into $\mathcal{M}$.

Definition 4 (*Tubular Neighborhood*). A tubular neighborhood of $\mathcal{S}$ is an open embedding $\tau : \mathcal{E} \to \mathcal{M}$ from some vector bundle $\mathcal{E}$ over $\mathcal{S}$ to $\mathcal{M}$ satisfying

$$\tau\big|_{\bar{0}_\mathcal{S}} = \iota_\mathcal{S}.$$

More loosely, we often call the open set $\mathcal{W} := \tau(\mathcal{E})$ a tubular neighborhood of $\mathcal{S}$.

Whether we refer to a tubular neighborhood as an embedding or an open set should be clear from the context.

Theorem 5 (*Existence of Tubular Neighborhood, [11, Proposition 7.1.3]*). *Suppose that $\mathcal{S}$ is a submanifold of $\mathcal{M}$. Then there exists an embedding $\tau : N_\mathcal{S} \to \mathcal{M}$ from the normal bundle $N_\mathcal{S}$ of $\mathcal{S}$ into $\mathcal{M}$ such that τ keeps the zero section of $N_\mathcal{S}$ (i.e., $\tau(0_x) = x$ for all $x \in \mathcal{S}$, or $\tau\big|_{\bar{0}_\mathcal{S}} = \iota_\mathcal{S}$).*

Remark 6. This means that $\tau : N_\mathcal{S} \to \mathcal{M}$ is a tubular neighborhood of $\mathcal{S}$, and τ is a diffeomorphism between $N_\mathcal{S}$ and $\tau(N_\mathcal{S})$. $\triangleleft$

Before presenting the uniqueness result of tubular neighborhoods, we first recall the definitions of *isotopy* and *diffeotopy*.

Definition 7 (*Isotopy and Diffeotopy, [12, pp. 177–178]*). An *isotopy* from $\mathcal{M}$ to $\mathcal{N}$ is a map $F : \mathcal{M} \times \mathcal{I} \to \mathcal{N}$, where $\mathcal{I} \subseteq \mathbb{R}$ is an interval, such that for each $t \in \mathcal{I}$, the map $F_t : \mathcal{M} \to \mathcal{N}$ defined by $x \mapsto F(x, t)$ is an embedding. We also say F is an isotopy from F_0 to F_1, and F_0 and F_1 are called *isotopic*. If each F_t is a smooth embedding, then F is a *smooth isotopy* from $\mathcal{M}$ to $\mathcal{N}$. If each F_t is a diffeomorphism, then F is called a *diffeotopy*.

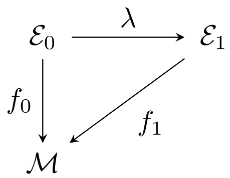

Fig. 1. Relations in Theorem 8. If f_0 and f_1 are two tubular neighborhoods, then there exists a bundle map $\lambda : \mathcal{E}_0 \to \mathcal{E}_1$ such that f_0 and $f_1 \circ \lambda$ are isotopic.

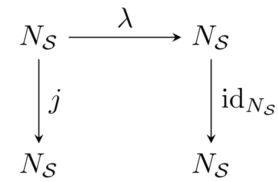

Fig. 2. Proof of Lemma 17.

Throughout the paper, whenever we mention an *isotopy*, we mean a *smooth isotopy*. Now we show the uniqueness result of the tubular neighborhood as follows.

Theorem 8 (*Uniqueness of Tubular Neighborhood I, [11, Theorem 7.4.4]*)**.** *Suppose that $f_i : \mathcal{E}_i \to \mathcal{M}$, $i = 0, 1$, are tubular neighborhoods of $\mathcal{S}$. Then there exists a bundle map*[2] *$\lambda : \mathcal{E}_0 \to \mathcal{E}_1$ such that f_0 and $f_1 \circ \lambda$ are isotopic (see Fig. 1).*

Denote by $G : \mathcal{E}_0 \times (-\delta, 1 + \delta) \to \mathcal{M}$ the isotopy from f_0 to $f_1 \circ \lambda$. Then Theorem 8 implies that $G_t : \mathcal{E}_0 \to \mathcal{M}$ is a tubular neighborhood for any $t \in (-\delta, 1 + \delta)$. Now let

$$h(x, t) = G(f_0^{-1}(x), t)$$

for $(x, t) \in f_0(\mathcal{E}_0) \times (-\delta, 1 + \delta)$. We have the following corollary.

Corollary 9 (*Uniqueness of Tubular Neighborhood II, [11, Theorem 7.4.4]*)**.** *Suppose that $\mathcal{S}$ is a submanifold of $\mathcal{M}$, and $\mathcal{W}_0$ and $\mathcal{W}_1$ are two tubular neighborhoods (as open sets) of $\mathcal{S}$ in $\mathcal{M}$, then there exists an isotopy $h : \mathcal{W}_0 \times (-\delta, 1 + \delta) \to \mathcal{M}$ such that*

$$h_0 = j_{\mathcal{W}_0}, \quad h_1(\mathcal{W}_0) = \mathcal{W}_1, \quad h_t\big|_\mathcal{S} = j_\mathcal{S}$$

for every $t \in (-\delta, 1 + \delta)$, where $h_t := h(\cdot, t)$, $j_{\mathcal{W}_0}$ and $j_\mathcal{S}$ are the inclusions of $\mathcal{W}_0$ and $\mathcal{S}$ into $\mathcal{M}$ respectively.

Therefore, any two tubular neighborhoods $\mathcal{W}_0$ and $\mathcal{W}_1$ are homeomorphic.

Definition 10 (*Closed Tubular Neighborhood*)**.** Fix a Euclidean metric g on the vector bundle $\mathcal{E}$ over $\mathcal{S}$, and for any $r > 0$, let[3]

$$\mathcal{BE}_r = \{v \in \mathcal{E} : g(v, v) \leq r^2\}.$$

A *closed tubular neighborhood* $\mathcal{K}$ of $\mathcal{S}$ is a closed neighborhood of $\mathcal{S}$ in $\mathcal{M}$ such that there is an embedding $\phi : \mathcal{BE}_r \to \mathcal{M}$ satisfying

$$\phi(\mathcal{BE}_r) = \mathcal{K}, \quad \phi\big|_{\bar{0}_\mathcal{S}} = \iota_\mathcal{S}.$$

Remark 11. If $\mathcal{S}$ is compact, then $\mathcal{BE}_r$ is by definition a closed tubular neighborhood of $\bar{0}_\mathcal{S}$ in $\mathcal{E}$ and that it is compact. Since $\mathcal{E}_r = \{v \in \mathcal{E} : g(v, v) < r^2\}$ can homeomorphically map to $\mathcal{E}$ while keeping the zero section, it is an (open) tubular neighborhood of $\bar{0}_\mathcal{S}$ in $\mathcal{E}$. In particular, $\mathcal{E}$ itself is a tubular neighborhood of $\bar{0}_\mathcal{S}$ in $\mathcal{E}$. ◃

Due to Remark 11, the following proposition holds.

Proposition 12. *If $\mathcal{S}$ is compact, then there exists some tubular neighborhood $\mathcal{W}$ of $\mathcal{S}$ such that its closure $\bar{\mathcal{W}}$ is a closed tubular neighborhood which is also compact.*

We will use a technique which relies on the following results to prove Theorem 1 later.

Lemma 13 (*[12, Chapter 8, Theorem 1.4]*)**.** *Suppose that $\mathcal{U}$ is an open set of the manifold $\mathcal{N}$ and that $\mathcal{C}$ is a compact subset of $\mathcal{N}$ contained in $\mathcal{U}$. Suppose that $h : \mathcal{U} \times (-\delta, 1+\delta) \to \mathcal{N}$ is an isotopy with $h_0 : \mathcal{U} \to \mathcal{N}$ being the inclusion. Then for any $\delta' \in (0, \delta)$, there exists a diffeotopy $H : \mathcal{N} \times (-\delta', 1 + \delta') \to \mathcal{N}$ with some open neighborhood $\mathcal{U}_0$ of $\mathcal{C}$ in $\mathcal{U}$ such that*

$$H\big|_{\mathcal{U}_0 \times (-\delta', 1+\delta')} = h\big|_{\mathcal{U}_0 \times (-\delta', 1+\delta')}.$$

Remark 14. Let $\tilde{h}$ be the level preserving map[4]:

$$\tilde{h} : \mathcal{U} \times (-\delta, 1 + \delta) \to \mathcal{N} \times (-\delta, 1 + \delta)$$
$$(p, t) \mapsto \big(h_t(p), t\big).$$

Note that Theorem 1.4 in Chapter 8 of [12] requires $\tilde{h}\big(\mathcal{U} \times (-\delta, 1+\delta)\big)$ to be open in $\mathcal{N} \times (-\delta, 1 + \delta)$. However, this requirement is unnecessary at least in our case, since it can be easily checked that $\tilde{h}$ is a submersion[5] and hence an open map. ◃

Corollary 15. *Suppose that $\mathcal{U}$ is an open set of the manifold $\mathcal{N}$ and that $\mathcal{C}$ is a compact subset of $\mathcal{N}$ contained in $\mathcal{U}$. Suppose that $h' : \mathcal{U} \times (-\delta, 1 + \delta) \to \mathcal{N}$ is an isotopy, and there exists a diffeomorphism $f_0 : \mathcal{N} \to \mathcal{N}$ that agrees with h'_0 on $\mathcal{U}$; i.e.,*

$$f_0|_\mathcal{U} = h'_0. \tag{1}$$

Then for any $\delta' \in (0, \delta)$, there is a diffeotopy $F : \mathcal{N} \times (-\delta', 1+\delta') \to \mathcal{N}$ with some open neighborhood $\mathcal{U}_0$ of $\mathcal{C}$ in $\mathcal{U}$ such that

$$F\big|_{\mathcal{U}_0 \times (-\delta', 1+\delta')} = h'\big|_{\mathcal{U}_0 \times (-\delta', 1+\delta')}.$$

Proof. Let $h = f_0^{-1} \circ h'$. Therefore, from (1), we have $h_0 = f_0^{-1} \circ h'_0 = j_\mathcal{U}$, where $j_\mathcal{U} : \mathcal{U} \to \mathcal{N}$ is the inclusion map from $\mathcal{U}$ to $\mathcal{N}$. According to Lemma 13, there is a diffeotopy H such that $H\big|_{\mathcal{U}_0 \times (-\delta', 1+\delta')} = h\big|_{\mathcal{U}_0 \times (-\delta', 1+\delta')}$. Then let $F = f_0 \circ H$. □

Remark 16. Note that the open set $\mathcal{U}$ in the theorems above may be $\mathcal{N}$ itself, which is the case in Lemma 17 to be discussed later. ◃

Now we prove a lemma concerning tubular neighborhoods of the submanifold $\mathcal{S}$ of $\mathcal{M}$. This lemma greatly facilitates the arguments in Section 3.

Note that $(N_\mathcal{S}, \pi, \bar{0}_\mathcal{S})$, where $\pi : N_\mathcal{S} \to \bar{0}_\mathcal{S}$ defined by $p \mapsto 0_x$ for any $p \in N_x$ and $x \in \mathcal{S}$, is a vector bundle over $\bar{0}_\mathcal{S}$. Though this might be trivial since $\bar{0}_\mathcal{S}$ is identical to $\mathcal{S}$ in a canonical way, we still point it out as follows for the sake of clarity from the set-theoretic perspective.

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2 More specifically, the bundle map γ is a bundle isomorphism. This is because f_0 and f_1 are embeddings and their images are open sets in $\mathcal{M}$; therefore, $\mathcal{E}_0$ and $\mathcal{E}_1$ are vector bundles which, as manifolds, have the same dimensions as $\mathcal{M}$ does.

3 Note that $\mathcal{BE}_r$ is a submanifold of $\mathcal{E}$ with boundary $\partial(\mathcal{BE}_r) = \{v \in \mathcal{E} : g(v, v) = r^2\}$.

4 The map $\tilde{h}$ is called the *track* of h [12, p. 111].

5 This is because $\tilde{h}$ is an immersion and the dimensions of $\mathcal{U}$ and $\mathcal{N}$ are the same.

Lemma 17 (*Extension of Tubular Neighborhoods*). *Suppose that $j : N_{\mathcal{S}} \to N_{\mathcal{S}}$ is a tubular neighborhood of $\bar{0}_{\mathcal{S}}$; i.e., j is an embedding and $j(0_x) = 0_x$ for all $x \in \mathcal{S}$. Then for any compact set $\mathcal{K}$ in $N_{\mathcal{S}}$, there is a diffeomorphism β on $N_{\mathcal{S}}$ such that β agrees with j on some neighborhood of $\mathcal{K}$.*

Proof. The idea is to use Corollary 15. To this end, we seek an isotopy $h : N_{\mathcal{S}} \times (-\delta, 1+\delta) \to N_{\mathcal{S}}$ such that $h_1 = j$ and h_0 is a diffeomorphism on $N_{\mathcal{S}}$.

Note that both $\mathrm{id}_{N_{\mathcal{S}}} : N_{\mathcal{S}} \to N_{\mathcal{S}}$ and $j : N_{\mathcal{S}} \to N_{\mathcal{S}}$ are tubular neighborhoods of $\bar{0}_{\mathcal{S}}$ in $N_{\mathcal{S}}$. Hence, according to Theorem 8, there exists a bundle isomorphism $\lambda : N_{\mathcal{S}} \to N_{\mathcal{S}}$ such that there exists an isotopy h from $\mathrm{id}_{N_{\mathcal{S}}} \circ \lambda$ to j (see Fig. 2). Since $\mathrm{id}_{N_{\mathcal{S}}} \circ \lambda$ is a diffeomorphism, according to Corollary 15, there exists a diffeotopy $H : N_{\mathcal{S}} \times (-\delta, 1+\delta) \to N_{\mathcal{S}}$ such that H agrees with h on some neighborhood of $\mathcal{K}$. Let $\beta = H_1$ and then it is a diffeomorphism and agrees with $h_1 = j$ on such a neighborhood. □

3. Proof of Theorem 1

The proof of Theorem 1 is based on [9, Lemma 3.3]. For clarity, we decompose the proof into several propositions. Denote by $\mathcal{M}$ the state space with a vector field X. Denote by φ the flow of X and assume that $\mathcal{S}$ is a *compact* boundaryless submanifold of $\mathcal{M}$ and is an asymptotic stable attractor of φ. Denote by $\mathcal{D}_A$ the domain of attraction of $\mathcal{S}$.

We start by fixing a precompact tubular neighborhood

$$f_o : N_{\mathcal{S}} \to \mathcal{W}$$

of $\mathcal{S}$ in $\mathcal{D}_A$, where $\mathcal{W} := f_o(N_{\mathcal{S}})$. The existence of f_o is guaranteed by Proposition 12.

Proposition 18. *For each compact set $\mathcal{K}$ in the domain of attraction $\mathcal{D}_A$, there exists some $T_K > 0$, such that $\varphi^T(\mathcal{K}) \subseteq \mathcal{W}$ for any $T > T_K$. Consequently, $\mathcal{K} \subseteq \varphi^{-T}(\mathcal{W})$ for any $T > T_K$.*

Proof. Due to the asymptotic stability of $\mathcal{S}$, there is some neighborhood $\mathcal{U}$ of $\mathcal{S}$ in $\mathcal{W}$ such that $\varphi^{[0,\infty)}(\mathcal{U}) \subseteq \mathcal{W}$. For any $x \in \mathcal{K}$, there is some $T_x > 0$ with some neighborhood $\mathcal{B}_x$ of x such that $\varphi^{T_x}(\mathcal{B}_x) \subseteq \mathcal{U}$. Since $\mathcal{K}$ is compact, there is $\{\mathcal{B}_{x_i}\}_{i=1,\dots,k}$, where $k < \infty$, such that $\bigcup_i \mathcal{B}_{x_i} \supseteq \mathcal{K}$. Let $T_K := \max_{i=1,\dots,k} T_{x_i}$ and the proof is completed. □

Note that $\mathcal{S}$ is invariant under φ, and hence $\mathcal{S} \subseteq \varphi^{-T}(\mathcal{W})$ for any $T \in \mathbb{R}$. Since $\varphi^{-T} : \mathcal{W} \to \mathcal{W}_T := \varphi^{-T}(\mathcal{W})$ is a diffeomorphism and $\mathcal{W}$ is a tubular neighborhood of $\mathcal{S}$, it is natural to conjecture that $\mathcal{W}_T$ should also be a tubular neighborhood of $\mathcal{S}$. This is indeed true as shown in the next proposition, but it is not straightforward. According to Definition 4, we still need to find a diffeomorphism f_T from $N_{\mathcal{S}}$ to $\mathcal{W}_T$ such that $f_T|_{\bar{0}_{\mathcal{S}}} = \iota_{\mathcal{S}}$. Although $f = \varphi^{-T} \circ f_o$ is a diffeomorphism from $N_{\mathcal{S}}$ to $\mathcal{W}_T$, we have $f|_{\bar{0}_{\mathcal{S}}} = \varphi^{-T} \circ \iota_{\mathcal{S}}$, which is not necessarily equal to $\iota_{\mathcal{S}}$, and hence $f : N_{\mathcal{S}} \to \mathcal{W}_T$ is not necessarily a tubular neighborhood. Yet $f|_{\bar{0}_{\mathcal{S}}} = \varphi^{-T} \circ \iota_{\mathcal{S}}$ and $\iota_{\mathcal{S}}$ are isotopic as maps from $\bar{0}_{\mathcal{S}}$ to $\mathcal{W}_T$ while f and φ^T are both diffeomorphisms. This makes it possible to use Lemma 13.

Proposition 19. *For any $T > 0$, $\mathcal{W}_T := \varphi^{-T}(\mathcal{W})$ is a tubular neighborhood of $\mathcal{S}$ in $\mathcal{D}_A$. That is, there exists a diffeomorphism $f_T : N_{\mathcal{S}} \to \mathcal{W}_T$ such that $f_T|_{\bar{0}_{\mathcal{S}}} = \iota_{\mathcal{S}}$.*

Proof. Obviously $f = \varphi^{-T} \circ f_o$ is a diffeomorphism from $N_{\mathcal{S}}$ to $\mathcal{W}_T$ with $0_x \in \bar{0}_{\mathcal{S}} \mapsto \varphi^{-T}(x)$. Now we need to "rectify" the map. Denote by $f_{\mathcal{S}}$ the restriction of f on $\bar{0}_{\mathcal{S}}$. Then $j_1 = f^{-1} \circ \varphi^T \circ f_{\mathcal{S}}$ is a map mapping $\bar{0}_{\mathcal{S}}$ diffeomorphically to $\bar{0}_{\mathcal{S}}$. Let $j_s = f^{-1} \circ \varphi^{s \cdot T} \circ f_{\mathcal{S}}$ for $s \in (-\delta, 1+\delta)$ and then $j : \bar{0}_{\mathcal{S}} \times (-\delta, 1+\delta) \to N_{\mathcal{S}}$ is an isotopy such that j_0 is the inclusion map, and $f \circ j_1 = \iota_{\mathcal{S}}$ on $\bar{0}_{\mathcal{S}}$.

Note that $g = \varphi \circ f$ with $g(x,t) = \varphi^t \circ f(x)$ is a smooth map from $N_{\mathcal{S}} \times \mathbb{R}$ to $\mathcal{D}_A$. Since $\varphi^{[-\delta,1+\delta]\cdot T} \circ f(\bar{0}_{\mathcal{S}}) = \mathcal{S} \subseteq \mathcal{W}_T$ and $[-\delta, 1+\delta] \cdot T$ is compact, there exists an open neighborhood $\mathcal{U}$ of $\bar{0}_{\mathcal{S}}$ in $N_{\mathcal{S}}$ such that $\varphi^{[-\delta,1+\delta]\cdot T} \circ f(\mathcal{U}) \subseteq \mathcal{W}_T$. Moreover, for any fixed $s \in [-\delta, 1+\delta]$, $\varphi^{s\cdot T} \circ f(\cdot)$ is an injective submersion, and hence a smooth embedding. Define

$$h : \mathcal{U} \times (-\delta, 1+\delta) \to N_{\mathcal{S}}$$
$$h(x,s) = f^{-1} \circ \varphi^{s\cdot T} \circ f(x),$$

which is an isotopy with h_0 being the inclusion map of $\mathcal{U}$ into $N_{\mathcal{S}}$ and $h_s|_{\bar{0}_{\mathcal{S}}} = j_s$. Then by Lemma 13, there exists a diffeotopy $H : N_{\mathcal{S}} \times (-\delta', 1+\delta') \to N_{\mathcal{S}}$ for $\delta' \in (0,\delta)$ such that H agrees with h on $\mathcal{U}_0 \times (-\delta', 1+\delta')$ for some open neighborhood $\mathcal{U}_0$ of $\mathcal{S}$.

Let $f_T = f \circ H_1$ and this is a diffeomorphism between $N_{\mathcal{S}}$ and $\mathcal{W}_T$. Moreover, restricted on $\mathcal{S}$, $f_T = f \circ h_1 = f \circ j_1 = \iota_{\mathcal{S}}$. Hence, $f_T : N_{\mathcal{S}} \to \mathcal{W}_T$ is a tubular neighborhood. □

Since the domain of attraction $\mathcal{D}_A$ is a smooth manifold with the second countability, there exists an ascending chain of compact subsets $\mathcal{K}_0 \subseteq \mathcal{K}_1 \subseteq \cdots$ such that $\bigcup_{i\in\mathbb{N}} \mathcal{K}_i = \mathcal{D}_A$. Choose $0 < T_0 < T_1 < \cdots$ such that

$$\mathcal{W}_i := \varphi^{-T_i}(\mathcal{W})$$

contains $\mathcal{K}_i$ for each i and that $\overline{\mathcal{W}}_i \subseteq \mathcal{W}_{i+1}$. This is possible due to the precompactness of $\mathcal{W}$. By Proposition 19, there exist tubular neighborhoods $f_i : N_{\mathcal{S}} \to \mathcal{W}_i$ for all $i \in \mathbb{N}$. The strategy to prove Theorem 1 is to construct by induction an ascending chain of compact subsets $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \cdots$ with tubular neighborhoods $g_i : N_{\mathcal{S}} \to \mathcal{W}_i$ "rectified" from f_i such that $g_i(\mathcal{C}_i) \supseteq \mathcal{K}_i$, g_{i+1} agrees with g_i on $\mathcal{C}_i$ and $\bigcup_i \mathcal{C}_i = N_{\mathcal{S}}$. Then the theorem follows by defining a map $g : N_{\mathcal{S}} \to \mathcal{D}_A$ with $g = g_i$ on $\mathcal{C}_i$.

Theorem 20. *There exists a diffeomorphism $g : N_{\mathcal{S}} \to \mathcal{D}_A$ such that $g|_{\bar{0}_{\mathcal{S}}} = \iota_{\mathcal{S}}$.*

Proof. Let $\mathcal{K}_0 \subseteq \mathcal{K}_1 \subseteq \cdots$ be an ascending chain of compact subsets such that $\bigcup_{i\in\mathbb{N}} \mathcal{K}_i = \mathcal{D}_A$ and $\mathcal{K}_0 \supseteq \mathcal{S}$. Since $\mathcal{W}$ is precompact in $\mathcal{D}_A$, $\varphi^{-T}(\mathcal{W})$ is precompact for any $T > 0$ in $\mathcal{D}_A$. Then by Proposition 18 we can choose inductively $0 < T_0 < T_1 < \cdots$ such that $\overline{\mathcal{W}}_i \cup \mathcal{K}_{i+1} \subseteq \mathcal{W}_{i+1}$. According to Proposition 19, for each $i \in \mathbb{N}$, there is a diffeomorphism $f_i : N_{\mathcal{S}} \to \mathcal{W}_i$ such that $f_i(0_x) = x$ for all $x \in \mathcal{S}$. Now we construct $\{(g_i, \mathcal{C}_i) : i \in \mathbb{N}\}$ with $\mathcal{C}_i$ being compact sets in $N_{\mathcal{S}}$ and $g_i : N_{\mathcal{S}} \to \mathcal{W}_i$ being tubular neighborhoods such that

(i) $\mathcal{C}_i \subseteq \operatorname{int} \mathcal{C}_{i+1}$;
(ii) $g_i(\mathcal{C}_i) \supseteq \mathcal{K}_i$;
(iii) $g_{i+1}|_{\mathcal{C}_i} = g_i|_{\mathcal{C}_i}$;
(iv) $\bigcup_{i\in\mathbb{N}} \mathcal{C}_i = N_{\mathcal{S}}$;

Take $g_0 = f_0$ and $\mathcal{C}_0 = \mathcal{BE}_{r_0}$ with r_0 large enough such that $\mathcal{BE}_{r_0} \supseteq g_0^{-1}(\mathcal{K}_0)$. Let $j_1 = f_1^{-1} \circ g_0$. Then $j_1 : N_{\mathcal{S}} \to N_{\mathcal{S}}$ is a tubular neighborhood of $\bar{0}_{\mathcal{S}}$ in $N_{\mathcal{S}}$ and $f_1 \circ j_1 = g_0$. According to Lemma 17, there is a bundle isomorphism $\beta_1 : N_{\mathcal{S}} \to N_{\mathcal{S}}$ such that β_1 agrees with j_1 on $\mathcal{C}_0$. Let $g_1 = f_1 \circ \beta_1$. Then $g_1 : N_{\mathcal{S}} \to \mathcal{W}_1$ is a diffeomorphism and $g_1 = g_0$ on $\mathcal{C}_0$. Take r_1 large enough such that $r_1 > 2r_0$ and $\mathcal{C}_1 = \mathcal{BE}_{r_1}$contains $g_1^{-1}(\mathcal{K}_1)$.

Suppose that for $n \in \mathbb{N}$, $\mathcal{A}_n = \{(g_i, \mathcal{C}_i) : 0 \le i \le n\}$ such that (i), (ii), (iii) are satisfied and $\mathcal{C}_n = \mathcal{BE}_{r_n}$ with $r_n > 2^n r_0$. Let $j_{n+1} = f_{n+1}^{-1} \circ g_n$. Then $j_{n+1} : N_{\mathcal{S}} \to N_{\mathcal{S}}$ is a tubular neighborhood of $\bar{0}_{\mathcal{S}}$ in $N_{\mathcal{S}}$. Again, according to Lemma 17, there exists a diffeomorphism β_{n+1} on $N_{\mathcal{S}}$ such that $\beta_{n+1} = j_{n+1}$ on $\mathcal{C}_n$. Set $g_{n+1} = f_{n+1} \circ \beta_{n+1}$ and then $g_{n+1} = g_n$ on $\mathcal{C}_n$. Pick a positive number r_{n+1} such that $r_{n+1} > 2r_n$ and $\mathcal{C}_{n+1} = \mathcal{BE}_{r_{n+1}} \supseteq g_{n+1}^{-1}(\mathcal{K}_{n+1})$. Then $\mathcal{A}_{n+1} =$

$\mathcal{A}_n \cup \{(g_{n+1}, \mathcal{C}_{n+1})\}$ again satisfies (i), (ii), (iii) with $r_{n+1} > 2^{n+1} r_0$. By induction we have $\{(g_k, \mathcal{C}_k) : k \in \mathbb{N}\}$ satisfying (i), (ii), (iii) with $r_k > 2^k r_0$ for all $k \in \mathbb{N}$.

Define $g : N_{\mathcal{S}} \to \mathcal{D}_A$ with $g\big|_{\mathcal{C}_i} = g_i\big|_{\mathcal{C}_i}$ for all $i \in \mathbb{N}$. Then g is well defined and $\operatorname{Im} g = \mathcal{D}_A$ due to (ii) and (iii) respectively. Moreover, since the maps g_i are diffeomorphisms and $\bigcup_i \operatorname{int} \mathcal{C}_i = N_{\mathcal{S}}$, g is a local diffeomorphism. It is also obvious that (iv) is satisfied. For any $p, q \in N_{\mathcal{S}}$, there exists i such that $\mathcal{C}_i$ contains p and q. Then $g(p) = g(q) \implies g_i(p) = g_i(q) \implies p = q$. Hence g is also injective. Therefore, g is a diffeomorphism from $N_{\mathcal{S}}$ onto $\mathcal{D}_A$. Moreover, since $\mathcal{K}_0$ is chosen to contain $\mathcal{S}$ in the beginning and g_0 keeps the zero section (i.e., $g(0_x) = x$ for all $x \in \mathcal{S}$), $\mathcal{C}_0 \supseteq \bar{0}_{\mathcal{S}}$. Therefore $g\big|_{\bar{0}_{\mathcal{S}}} = g_0\big|_{\bar{0}_{\mathcal{S}}} = \iota_{\mathcal{S}}$, which concludes the proof. □

Remark 21. In the argument above, the vector field X is assumed to be complete and then the flow γ exists for all $t \in \mathbb{R}$. In the general case where X is not necessarily complete, we can resort to the existence of a complete Riemannian metric on the manifold $\mathcal{M}$ [13]. For any point $p \in \mathcal{M}$, denote by $|X_p|$ the norm of the vector X_p at p with respect to such a Riemannian metric. Then the vector field $Y = \frac{X}{1+|X|^2}$ is a complete vector field on $\mathcal{M}$ since $|Y|$ is bounded and the Riemannian metric is complete. Moreover, since the phase portraits of X and Y are the same [14, Proposition 1.14], $\mathcal{S}$ is still a compact asymptotically stable attractor under the flow of Y with the same domain of attraction as that under the flow of X. ◃

4. Two counterexamples

In this section, we illustrate two counterexamples to show that the original claim of Wilson's theorem is inaccurate in the cases with non-compact attractors. The first counterexample is presented in Section 4.1. The idea of constructing the counterexample is straightforward, but it usually involves an *incomplete* Riemannian manifold as the ambient space. Nevertheless, another counterexample in Section 4.2 involves a *complete* Riemannian manifold as the ambient space. The idea of the counterexample is to present two topologically equivalent dynamical systems, where the domains of attraction of the non-compact attractor are not homotopy equivalent. As a result, the domain of attraction of the non-compact attractor of either of the system is of a different homotopy type from its tubular neighborhood. Note that all the vector fields of the dynamical systems in this section are complete; i.e., solutions exist for all $t \in \mathbb{R}$.

4.1. $\mathcal{M}$ is an incomplete Riemannian manifold

(The original) Wilson's theorem states that the domain of attraction of a uniformly asymptotically stable attractor, be it a compact or non-compact manifold, of a complete autonomous system is diffeomorphic to its tubular neighborhood. While the argument in Section 3 holds for a compact attractor $\mathcal{S}$, it does not hold for a *non-compact* attractor, since Proposition 18 may be invalid when the attractor is non-compact. More specifically, when $\mathcal{S}$ is non-compact, it is possible that none of its tubular neighborhood contains any ϵ-neighborhood of $\mathcal{S}$. To see this, note that if we take out one point from a submanifold, the ϵ-neighborhood of the new submanifold will only miss one point compared to that of the original submanifold, while its tubular neighborhood (viewed as a vector bundle) would lose the whole fiber over the missing point. Exploiting this observation, we can construct a counterexample by starting with a compact asymptotically stable attractor and then taking one fixed point out of it.

Example 22. Start with the smooth function $\bar{f}(x) = (\operatorname{dist}(x, \mathbb{S}^1))^2$ on $\mathbb{R}^2$ and let $\bar{X} = -\operatorname{grad} \bar{f}$. This system has the unit circle $\mathbb{S}^1 \subseteq \mathbb{R}^2$ as the asymptotically stable attractor, and all points on $\mathbb{S}^1$ are fixed points. Now consider the state space $\mathcal{M} = \mathbb{R}^2 - \{(1, 0)\}$. Let $\mathcal{S} = \mathbb{S}^1 - \{(1, 0)\}$. It is a closed set and also a submanifold of $\mathcal{M}$, but it is non-compact. Let f be the function on $\mathcal{M}$ such that $f(x) = (\operatorname{dist}(x, \mathcal{S}))^2$. The function f is the restriction of $\bar{f}$ on $\mathcal{M}$, and hence it is smooth. The vector field $X = -\operatorname{grad} f$ is then the restriction of $\bar{X}$ on $\mathcal{M}$, and it has $\mathcal{S}$ as an attractor, which is uniformly asymptotically stable. The domain of attraction is $\mathcal{M} - \{(0, 0)\}$, which is not contractible. However a tubular neighborhood of $\mathcal{S}$ is homeomorphic to $\mathcal{S} \times \mathbb{R}$, which is contractible. ◃

4.2. $\mathcal{M}$ is a complete Riemannian manifold

In this section we demonstrate a dynamical system $(\mathcal{M}, \varphi)$ where the state space $\mathcal{M}$ is a complete Riemannian manifold and the asymptotically stable attraction $\mathcal{S}$ is *not* compact. Instead of directly showing the construction of the flow map φ on $\mathcal{M}$, we first construct an auxiliary system $(\mathcal{M}_0, \varphi_0)$, and then obtain $(\mathcal{M}, \varphi)$ via a topological conjugacy [15, Chapter 2] $h : \mathcal{M}_0 \to \mathcal{M}$. As an extra benefit to be seen later, such a demonstration shows that when the attractor is non-compact, its uniform asymptotic stability is rather a "geometric" concept than a "topological" one. Namely, even if two dynamical systems are topologically conjugate, properties concerning the uniform asymptotic stability of the systems may not be (fully) preserved by the conjugacy.

4.2.1. The auxiliary system $(\mathcal{M}_0, \varphi_0)$

Let

$$\mathcal{M}_0 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = 1\}$$

and

$$\mathcal{S}_0 = \{(x, y, z) \in \mathcal{M}_0 : x = 0, z = 1\}.$$

Endow $\mathcal{M}_0$ with the Riemannian metric g_0 induced by the standard Riemannian metric $(dx)^2 + (dy)^2 + (dz)^2$ on $\mathbb{R}^3$. Then $(\mathcal{M}_0, g_0)$ is a complete Riemannian manifold with the distance $d_{\mathcal{M}_0}$.

Let Y_0, Z_0 be the vector fields on $\mathcal{M}_0$ defined by

$$Y_0(x, y, z) = \begin{cases} e^{-\frac{1}{y}} \frac{\partial}{\partial y}\Big|_{(x,y,z)} & y > 0 \\ 0 & y \le 0 \end{cases}$$

and

$$Z_0(x, y, z) = x \cdot \left(x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} \right) \Big|_{\mathcal{M}_0}.$$

Let $X_0 = Y_0 + Z_0$ and denote by φ_0 the flow of X_0 on $\mathcal{M}_0$. Then $\mathcal{S}_0$ is a uniformly asymptotically stable manifold of the dynamical system $(\mathcal{M}_0, \varphi_0)$ with its domain of attraction being

$$\mathcal{D}_0 = \{(x, y, z) \in \mathcal{M}_0 : z > -1\}.$$

The following characterization of the stability of $\mathcal{S}_0$ will be needed later. Namely, given any $a > -1$ with $\mathcal{W}'_{z>a} = \mathcal{M}_0 \cap \{(x, y, z) \in \mathcal{M}_0 : z > a\}$, corresponding to each $\epsilon > 0$, there exists some $T_\epsilon > 0$ such that $d_{\mathcal{M}_0}\big(\varphi_0^{[T_\epsilon, +\infty)}(\mathcal{W}'_{z>a}), \mathcal{S}_0\big) < \epsilon$. To see this, denote by (x'_t, y'_t, z'_t) the orbit $\varphi_0^t(p')$ for $p' = (x', y', z') \in \mathcal{M}_0$. Then $(x'_t, z'_t) \subseteq \mathbb{S}^1$ is subject to the equation

$$\frac{d}{dt}(x'_t, z'_t) = \left(-x'_t z'_t, {x'_t}^2\right). \tag{2}$$

Note that the dynamical system (2) on $\mathbb{S}^1$ has the point $q'_0 = (0, 1)$ as an asymptotically stable equilibrium with the domain of attraction $\{(x, z) \in \mathbb{S}^1 : z \neq -1\}$. Hence for any $\epsilon > 0$, there exists $T' > 0$ such that for any $t \ge T'$ and $q' = (x', z') \in \mathbb{S}^1$ with $z' \ge a$,

$\mathrm{dist}\big(\phi^t(q'), q'_0\big) < \epsilon$, where dist is the distance on $\mathbb{S}^1$ measured by lengths of minor arcs, and ϕ is the flow of (2). Therefore,

$$d_{\mathcal{M}_0}\big(\varphi_0^t(x', y', z'), \mathcal{S}_0\big) \le \mathrm{dist}\big(\phi^t(x', z'), q'_0\big) < \epsilon$$

for all $t > T'$ and $(x', y', z') \in W'_{z>a}$. Moreover, $\mathrm{dist}(\phi^t(q'), q'_0) \le \mathrm{dist}(q', q'_0)$, and then

$$d_{\mathcal{M}_0}(\varphi_0^t(x', y', z'), \mathcal{S}_0) \le \mathrm{dist}\big((x', z'), q'_0\big) = d_{\mathcal{M}_0}\big((x', y', z'), \mathcal{S}_0\big). \tag{3}$$

For a point $(0, y, -1) \in \mathcal{M}_0 - \mathcal{D}_0$ with $y \le 0$, it holds that $X_0\big|_{(x,y,z)} = 0$. For any $y > 0$,

$$X_0\big|_{(0,y,-1)} = Y_0\big|_{(0,y,-1)} = e^{-\frac{1}{y}} \frac{\partial}{\partial y}\Big|_{(0,y,-1)}, \tag{4}$$

implying

$$\varphi_0^t(0, y, -1) = (0, \gamma(t), -1) \tag{5}$$

with

$$\dot{\gamma}(t) = e^{-\frac{1}{\gamma(t)}} > 0. \tag{6}$$

Therefore, both $\gamma(t)$ and $\dot{\gamma}(t)$ increase strictly with respect to $t > 0$.

4.2.2. The system $(\mathcal{M}, \varphi)$

Now we construct the dynamical system $(\mathcal{M}, \varphi)$ which will serve as a counterexample. More specifically, a vector field X on some Riemannian manifold $(\mathcal{M}, g)$ is to be constructed with a uniformly asymptotically stable submanifold $\mathcal{S}$ of which the domain of attraction $\mathcal{D}$ is not homotopy equivalent to $\mathcal{S}$ itself.

Let

$$r(y) = \begin{cases} 1 - e^{-\frac{1}{y}} & y > 0 \\ 1 & y \le 0. \end{cases}$$

Let $\mathcal{M}$ be the two-dimensional cylinder embedded in $\mathbb{R}^3$ defined by

$$\mathcal{M} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = r(y)\},$$

and let

$$\mathcal{S} = \{(x, y, z) \in \mathcal{M} : x = 0, z = \sqrt{r(y)}\}.$$

Then $\mathcal{S}$ is an embedded submanifold and a closed subset in $\mathcal{M}$. Endowed with the Riemannian metric $g_{\mathcal{M}}$ induced by the standard Riemannian metric $g = (dx)^2 + (dy)^2 + (dz)^2$ on $\mathbb{R}^3$, $\mathcal{M}$ is a complete Riemannian manifold. Note that although the Riemannian metric $g_{\mathcal{M}}$ is induced by g, the corresponding distance $d_{\mathcal{M}}$ on $\mathcal{M}$ is not the restriction on $\mathcal{M}$ of the Euclidean distance d on $\mathbb{R}^3$. Generally speaking, it holds that $d_{\mathcal{M}}(p, q) \ge d(p, q)$ for $p, q \in \mathcal{M}$. However, the topology $\tau_{\mathcal{M}}$ induced by $d_{\mathcal{M}}$ on $\mathcal{M}$ is exactly the subspace topology inherited from $\mathbb{R}^3$, meaning that $\tau_{\mathcal{M}}$ is also the same as the topology induced by (the restriction of) d. Then, if a sequence $\{p_n\}$ on $\mathcal{M}$ is a Cauchy sequence with respect to $d_{\mathcal{M}}$, it is also a Cauchy sequence with respect to d. Due to the completeness of $\mathbb{R}^3$ and the closedness of $\mathcal{M}$ in $\mathbb{R}^3$, there exists $\bar{p} \in \mathcal{M}$ such that $p_n \xrightarrow{d} \bar{p}$ (i.e., the sequence $\{p_n\}$ converges to $\bar{p}$ with respect to the metric d). Since $d_{\mathcal{M}}$ and d induce the same topology on $\mathcal{M}$, this implies that $p_n \xrightarrow{d_{\mathcal{M}}} \bar{p}$ (i.e., the sequence $\{p_n\}$ converges to $\bar{p}$ with respect to the metric $d_{\mathcal{M}}$), ensuring the completeness of $(\mathcal{M}, d_{\mathcal{M}})$.

The map $h : \mathcal{M}_0 \to \mathcal{M}$ defined by

$$h(x, y, z) = \Big(\sqrt{r(y)} \cdot x,\ y,\ \sqrt{r(y)} \cdot z\Big)$$

is a diffeomorphism between the pairs $(\mathcal{M}_0, \mathcal{S}_0)$ and $(\mathcal{M}, \mathcal{S})$. Here, we define X to be the vector field on $\mathcal{M}$ related to X_0 by h. That is, $X = h_*(X_0)$, where $h_* : T\mathcal{M}_0 \to T\mathcal{M}$ is the tangent map. Let φ be the flow of X on $\mathcal{M}$. Then h is a conjugacy between the flows φ_0 and φ. That is, the identity $h \circ \varphi_0 = \varphi \circ h$ holds, or equivalently,

$$\varphi^t(p'') = h \circ \varphi_0^t \circ h^{-1}(p'') \tag{7}$$

for all $p'' \in \mathcal{M}$.

Note that for a point $p' = (x', y', z')$ on $\mathcal{M}_0$, the distance $d_{\mathcal{M}_0}(p', \mathcal{S}_0)$ is *exactly* the length of the minor arc on the circle $\mathcal{M}_0 \cap \{(x, y, z) \in \mathbb{R}^3 : y = y'\}$ between p' and $(0, y', 1)$. Meanwhile, for a point $p'' = (x'', y'', z'') = h(p')$ on $\mathcal{M}$, the distance $d_{\mathcal{M}}(p'', \mathcal{S})$ is *no larger than* the length of the minor arc on the circle $\mathcal{M} \cap \{(x, y, z) \in \mathbb{R}^3 : y = y''\}$ between p'' and $\big(0, y'', \sqrt{r(y'')}\big)$. With $r(y) \le 1$, this implies

$$d_{\mathcal{M}}\big(h(p'), \mathcal{S}\big) \le d_{\mathcal{M}_0}(p', \mathcal{S}_0)$$

for all $p' \in \mathcal{M}_0$. Combined with (7), it yields the following inequality:

$$d_{\mathcal{M}}\big(\varphi^t(p''), \mathcal{S}\big) = d_{\mathcal{M}}\big(h \circ \varphi_0^t \circ h^{-1}(p''), \mathcal{S}\big) \le d_{\mathcal{M}_0}\big(\varphi_0^t \circ h^{-1}(p''), \mathcal{S}_0\big). \tag{8}$$

Since h^{-1} maps $\tilde{\mathcal{D}}_0 := \{(x, y, z) \in \mathcal{M} : z > -1\}$ diffeomorphically to $\mathcal{D}_0$, it implies that as $t \to +\infty$, $d_{\mathcal{M}}\big(\varphi^t(p), \mathcal{S}\big) \to 0$ for all $p \in \tilde{\mathcal{D}}_0$. However, if $\mathcal{S}$ is an attractor, then the domain of attraction of $\mathcal{S}$ should be

$$\mathcal{D} = \tilde{\mathcal{D}}_0 \cup \{\big(0, y, -\sqrt{r(y)}\big) : y > 0\}.$$

To see this, first note that for any point $p'' = (x'', y'', z'')$ in $\{\big(0, y, -\sqrt{r(y)}\big) : y > 0\}$,

$$\begin{aligned} \varphi^t(p'') &= h \circ \varphi_0^t \circ h^{-1}(p'') \\ &= h \circ \varphi_0^t(0, y'', -1) \\ &= h(0, \gamma''(t), -1) \\ &= \Big(0, \gamma''(t), \sqrt{r \circ \gamma''(t)}\Big), \end{aligned}$$

where $\frac{d\gamma''}{dt} > 0$. Then from (6) we can deduce that $\gamma''(t)$ and $\frac{d\gamma''}{dt}$ both strictly increase with respect to t. Hence $d_{\mathcal{M}}(\varphi^t(p''), \mathcal{S}) \le \pi\sqrt{r \circ \gamma''(t)} \to 0$ as $t \to +\infty$. Meanwhile, for any point $p \in \mathcal{M} - \mathcal{D}$, i.e. $p = (0, y, -1)$ with $y \le 0$, $X|_p = h_*(X_0|_p) = 0$, and hence p stays stationary under the flow φ. Therefore, if $p'' \in \mathcal{M}$, then $\varphi^t(p) \xrightarrow{d_{\mathcal{M}}} \mathcal{S}$ as $t \to \infty$ if and only if $p'' \in \mathcal{D}$. Since $\mathcal{D}$ contains circles in the form of $\{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = r(y), y > 0\}$ in $\mathcal{M}$, its fundamental group is non-zero and hence is not homotopy equivalent to $\mathcal{S}$.

To show that this is a counterexample, it remains to prove that $\mathcal{S}$ is indeed a uniformly asymptotically stable manifold of the system $(\mathcal{M}, \varphi)$.

We first show the uniform attractiveness of $\mathcal{S}$. Let

$$\mathcal{W} := \{(x, y, z) \in \mathcal{M} : z > 0\} \cup \{(x, y, z) \in \mathcal{M} : y > 1\}.$$

We will first show that $\mathcal{W}$ contains some α-neighborhood $\mathcal{N}_\alpha$ of $\mathcal{S}$ for some $\alpha > 0$, and then show that for each $\epsilon > 0$, there exists some $T_\epsilon > 0$ such that $d_{\mathcal{M}}\big(\varphi^{[T_\epsilon, +\infty)}(\mathcal{W}), \mathcal{S}\big) < \epsilon$.

To see that $\mathcal{W}$ contains some α-neighborhood of $\mathcal{S}$, we only need to show that there is a positive distance between its complement $\mathcal{W}^c$ and $\mathcal{S}$. Note that

$$\mathcal{W}^c = \{(x, y, z) \in \mathcal{M} : z \le 0, y \le 1\} = \mathcal{C} \cup \mathcal{K}$$

with

$$\mathcal{C} := \{(x, y, z) \in \mathcal{M} : z \le 0, y \le -\pi\}$$

and

$$\mathcal{K} := \{(x, y, z) \in \mathcal{M} : z \le 0, -\pi \le y \le 1\}.$$

Then $\mathcal{K}$ is compact and $\mathcal{C}$ is closed in $\mathcal{M}$, and $\mathcal{C}\cap\mathcal{S}$, $\mathcal{K}\cap\mathcal{S}$ are both empty. Since $(\mathcal{M}, g_{\mathcal{M}})$ is a complete Riemannian manifold with the distance $d_{\mathcal{M}}$, it holds that $d_{\mathcal{M}}(\mathcal{S},\mathcal{K}) > 0$ as a consequence of the disjointedness of a closed subset and a compact subset. To see that $d_{\mathcal{M}}(\mathcal{S},\mathcal{C}) > 0$, note that $d_{\mathcal{M}}(\mathcal{S}_{y\le 0},\mathcal{C}) = \pi/2$ and $d_{\mathcal{M}}(\mathcal{S}_{y\ge 0},\mathcal{C}) \ge \pi$, where $\mathcal{S}_{y\le 0} := \mathcal{S}\cap\{(x,y,z)\in\mathbb{R}^3 : y\le 0\}$ and $\mathcal{S}_{y\ge 0} := \mathcal{S}\cap\{(x,y,z)\in\mathbb{R}^3 : y\ge 0\}$. Then for any $0<\alpha<\min\{d_{\mathcal{M}}(\mathcal{S},\mathcal{K}), d_{\mathcal{M}}(\mathcal{S},\mathcal{C})\}$, there holds $\mathcal{N}_\alpha\subset\mathcal{W}$.

Now we proceed to show that for any $\epsilon > 0$, there exists $T_\epsilon > 0$. Denote by $\mathcal{W}_{z>0}$ the set $\{(x,y,z)\in\mathcal{M} : z>0\}$ and by $\mathcal{W}_{y>1}$ the set $\{(x,y,z)\in\mathcal{M} : y>1\}$. Then $\mathcal{W} = \mathcal{W}_{z>0}\cup\mathcal{W}_{y>1}$. Note that for each point $p=(x,y,z)\in\mathcal{W}_{y>1}$, $X|_p$ takes the form $a_p\frac{\partial}{\partial x} + e^{-\frac{1}{y}}\frac{\partial}{\partial y} + c_p\frac{\partial}{\partial z}$, and therefore, $\mathcal{W}_{y>1}$ is an invariant open set of the system $(\mathcal{M},\varphi)$. It holds that $\varphi^t(p) = (x_t, y_t, z_t)$ with $\frac{dy_t}{dt} > e^{-1}$ for any $p\in\mathcal{W}_{y>1}$. Choose T'' to be some positive number large enough such that $r(e^{-1}\cdot T'') < (\epsilon/\pi)^2$. Then for any $t\ge T''$ and $p\in\mathcal{W}_{y>1}$, it holds that $r(y_t) < r(e^{-1}\cdot T'')$ and therefore $d_{\mathcal{M}}(\varphi^t(p),\mathcal{S}) \le \pi\sqrt{r(y_t)} < \epsilon$. To see that the points in $\mathcal{W}_{z>0}$ converge uniformly towards $\mathcal{S}$, first note that $h^{-1}(\mathcal{W}_{z>0}) = \mathcal{W}'_{z>0} = \mathcal{M}_0\cap\{(x,y,z)\in\mathbb{R}^3 : z>0\}$. Then combined with (8), it holds $d_{\mathcal{M}}(\varphi^{[T',+\infty)}(\mathcal{W}_{z>0}),\mathcal{S}) < \epsilon$ for some $T'>0$. Finally, one only needs to choose T_ϵ to be $\min\{T',T''\}$ and the argument for the uniform attractiveness is complete.

Now it remains to prove the stability of $\mathcal{S}$, for which we need to show that $\forall\epsilon>0, \exists\delta>0$ such that $\varphi^T(\mathcal{U}_\delta)\subseteq\mathcal{U}_\epsilon$ for any $T>0$. Here, $\mathcal{U}_\epsilon$ and $\mathcal{U}_\delta$ stand for the $\epsilon-$ and the $\delta-$ neighborhoods of $\mathcal{S}$ respectively.

Note that for any $\delta'>0$, there is some $y_0>0$ large enough, such that all points $(x,y,z)\in\mathcal{M}$ with $y\ge y_0$ are in $\mathcal{U}_{\delta'}$. Moreover, the subset $\mathcal{V}_{\ge y_0} := \{(x,y,z)\in\mathcal{M} : y\ge y_0\}$ is forward invariant under the flow φ. Therefore, we choose $\delta'<\epsilon$ and then $\varphi^t(\mathcal{V}_{\ge y_0})\subseteq\mathcal{U}_\epsilon$ for all $t\ge 0$.

For the compact subset $\mathcal{V}_{[0,y_0]} := \{(x,y,z)\in\mathcal{M} : y\in[0,y_0]\}$, note that h maps $h^{-1}(\mathcal{V}_{[0,y_0]})$ homeomorphically to $\mathcal{V}_{[0,y_0]}$, and then there is some constant $c>1$ such that $d_{\mathcal{M}_0}(p,q) < c\cdot d_{\mathcal{M}}(h(p),h(q))$ holds for any $p,q\in h^{-1}(\mathcal{V}_{[0,y_0]})$. For the subset $\mathcal{V}_{\le 0} := \{(x,y,z)\in\mathcal{M} : y\le 0\}$, the subset $h^{-1}(\mathcal{V}_{\le 0})$ of $\mathcal{M}_0$ is identical to the set $\mathcal{V}_{\le 0}$ and h (restricted on $\mathcal{V}_{\le 0}$) is merely the identity. Hence $d_{\mathcal{M}_0}(p,q) = d_{\mathcal{M}}(p,q)$ for all $p,q\in\mathcal{V}_{\le 0}$. Now we choose $\delta''>0$ such that $c\cdot\delta''<\epsilon$. Then for any $p'\in\mathcal{V}_{\le 0}\cup\mathcal{V}_{[0,y_0]}$ with $d_{\mathcal{M}}(p',\mathcal{S})<\delta''$, by combining (3), (7) and (8), we have

$$d_{\mathcal{M}}(\varphi^t(p'),\mathcal{S}) \le d_{\mathcal{M}_0}(\varphi_0^t(h^{-1}(p')),\mathcal{S}_0) \le d_{\mathcal{M}_0}(h^{-1}(p'),\mathcal{S}_0) \le c\cdot d_{\mathcal{M}}(p',\mathcal{S}) \le \epsilon.$$

Take $\delta = \min\{\delta',\delta''\}$ and then the proof is done.

5. Conclusion

In this paper, we have revisited Wilson's theorem about the relation between the domain of attraction of an attractor and its tubular neighborhood. Specifically, we show with detailed and rigorous proofs that the domain of attraction of a *compact* asymptotically submanifold of a finite-dimensional smooth manifold of a continuous autonomous system is homeomorphic to its tubular neighborhood. We emphasize that the compactness of the attractor is crucial, without which Wilson's theorem cannot hold. This is shown by two counterexamples where the attractor is not compact and the state space is either complete or incomplete. This work is of great interest to the studies on characterizing domains of attraction. For example, the main results may be used for refining the topological conclusions or showing the existence of singular points in the robotic path following problem [8,16,17].

CRediT authorship contribution statement

Bohuan Lin: Conceptualization, Formal analysis, Writing – original draft. **Weijia Yao:** Conceptualization, Formal analysis, Writing – original draft. **Ming Cao:** Conceptualization, Writing – review & editing, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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