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The vanishing viscosity limit for Hamilton–Jacobi equations on networks ☆

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ABSTRACT

For a Hamilton–Jacobi equation defined on a network, we introduce its vanishing viscosity approximation. The elliptic equation is given on the edges and coupled with Kirchhoff-type conditions at the transition vertices. We prove that there exists exactly one solution of this elliptic approximation and mainly that, as the viscosity vanishes, it converges to the unique solution of the original problem.

1. Introduction

The study of partial differential equations on networks arises in several applications as information networks (internet, social networks, email exchange), economical networks (business relation between companies, postal delivery and traffic routes), biological networks (neural networks, food web, blood vessel, disease transmission).

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Starting with the seminal work of Lumer [16], a fairly complete theory for linear and semilinear equations on networks has been developed in the last 30 years (for instance, see: Lagnese et al. [15], Von Below et al. [4], Engel et al. [7], Freidlin et al. [9,10]). Only in recent times it has been initiated the study of some classes of fully nonlinear equations, such as conservation laws (see [6,11] and reference therein) or Hamilton–Jacobi equations (see [1,5,12,13,20]).

All the approaches to Hamilton–Jacobi equations aim to extend the concept of viscosity solution (see [2,3]) to networks, but they differ for the assumptions made on the Hamiltonians at the vertices. Hence, different frameworks reflect in different definitions of viscosity solutions, even if all of them give existence and uniqueness of the solution. However, any generalization of viscosity solution should preserve the other main features of existing theory such as stability with respect to uniform convergence and the method of vanishing viscosity.

In this paper we aim to show that the definition of solution introduced in [20] is consistent with vanishing viscosity method, which consists in approximating the original nonlinear problem by a family of semilinear ones. The difficulty is thus transferred to the question, whether the approximating family of solutions converges.

The first step establishes uniqueness and, for some cases, existence of classical solutions to the viscous Hamilton–Jacobi equation on networks. In doing so, the necessity of an extra condition at transition vertices becomes clear. We impose the classical Kirchhoff condition which establishes a relation among the outer normal derivatives of the solution along the edges incident the same vertex. The Kirchhoff condition can be thought of as an extension of the "averaging effect" of the viscosity term on the vertices.

The second step is to prove some a priori estimates, uniform in the viscosity parameter. These estimates are obtained by explicit arguments which take advantage of the intrinsic one dimensional nature of the problem.

The final step is the convergence of the solution of viscous approximation to the one of the starting problem. Obviously this issue requires a special care at the vertices, while it follows by classical arguments inside the edges.

The paper is organized as follows. In Section 2, we introduce some notations, the standing assumptions and recall the definition of viscosity solution. In Section 3 we study existence and uniqueness of the solution to the second order problem. Section 4 is devoted to the proof of the a priori estimates, whereas in Section 5 we show the convergence of the vanishing viscosity method; we work out in detail the Eikonal problem in Section 5.1. In Appendix A we prove some technical lemmas.

2. Notations and preliminary definitions

2.1. Topological network

A topological network is a collection of points in $\mathbb{R}^n$ connected by continuous, non-self-intersecting curves. More precisely (see [16,20]):

Definition 2.1. Let $V=\{v_i,\ i\in I\}$ be a finite collection of points in $\mathbb{R}^n$ and let $\{\pi_j,\ j\in J\}$ be a finite collection of smooth, non-self-intersecting curves in $\mathbb{R}^n$ given by $\pi_j:[0,l_j]\to\mathbb{R}^n$, $l_j>0$. For $e_j:=\pi_j((0,l_j))$ and $\bar e_j:=\pi_j([0,l_j])$, assume that

i) $\pi_j(0),\pi_j(l_j)\in V$, and $\#(\bar e_j\cap V)=2$ for all $j\in J$,
ii) $\bar e_j\cap\bar e_k\subset V$, and $\#(\bar e_j\cap\bar e_k)\leqslant 1$ for all $j,k\in J$, $j\neq k$.
iii) For all $v,w\in V$ there is a path with endpoints v and w (i.e. a sequence of edges $\{e_j\}_{j=1}^N$ such that $\#(\bar e_j\cap\bar e_{j+1})=1$ and $v\in\bar e_1$, $w\in\bar e_N$).

Then $\Gamma:=\bigcup_{j\in J}\bar e_j\subset\mathbb{R}^n$ is called a (finite) *topological network* in $\mathbb{R}^n$.

In the following we always identify $x\in\bar e_j$ with $y=\pi_j^{-1}(x)\in[0,l_j]$. For $i\in I$ we set $Inc_i:=\{j\in J$: e_j is incident to $v_i\}$, moreover two vertices v_i, v_j are said adjacent (in symbols $v_i\,\mathrm{adj}\,v_j$) if there exists $k\in J$ such that $v_i,v_j\in e_k$.

Observe that the parametrization of the arcs e_j induces an orientation which can be expressed by the *signed incidence matrix* $A = \{a_{ij}\}$ with

$$a_{ij} := \begin{cases} 1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(0) = v_i, \\ -1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(l_j) = v_i, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

We denote: $I_B := \{i \in I \mid \#Inc_i = 1\}$, $I_T := I \setminus I_B$ and $\partial\Gamma := \{v_i \in V \mid i \in I_B\}$. We call $\partial\Gamma$ the set of boundary vertices and $\{v_i \mid i \in I_T\}$ the set of transition vertices.

2.2. Function spaces

For any function $u : \Gamma \to \mathbb{R}$ and each $j \in J$ we denote by $u^j : [0, l_j] \to \mathbb{R}$ the restriction of u to $\bar{e}_j$, i.e. $u^j(y) = u(\pi_j(y))$ for $y \in [0, l_j]$. For $\alpha \in \mathbb{N}$, we define differentiation along an edge e_j by

$$\partial_j^\alpha u(x) := \frac{d^\alpha u^j}{dy^\alpha}(y) \quad \text{for } y = \pi_j^{-1}(x),\ x \in e_j$$

and at a vertex v_i by

$$\partial_j^\alpha u(v_i) := \frac{d^\alpha u^j}{dy^\alpha}(y) \quad \text{for } y = \pi_j^{-1}(v_i),\ j \in Inc_i.$$

Definition 2.2.

i) We say that a function u belongs to $USC(\Gamma)$ (respectively, to $LSC(\Gamma)$) if it is upper (resp., lower) semicontinuous with respect to the topology induced by $\mathbb{R}^n$ on Γ. In other words, $u \in USC(\Gamma)$ if and only if $u^j \in USC([0, l_j])$ for every $j \in J$ and $u^j(\pi_j^{-1}(v_i)) = u^k(\pi_k^{-1}(v_i))$ for every $i \in I$, $j, k \in Inc_i$; an analogous property holds for $u \in LSC(\Gamma)$.

ii) We say that a function u is continuous in Γ and we write $u \in C(\Gamma)$ if it is continuous with respect to the subspace topology of Γ, namely, $u^j \in C([0, l_j])$ for any $j \in J$ and $u^j(\pi_j^{-1}(v_i)) = u^k(\pi_k^{-1}(v_i))$ for any $i \in I$, $j, k \in Inc_i$.

iii) We say that $u \in C^k(\Gamma)$ if $u \in C(\Gamma)$ and if $u^j \in C^k([0, l_j])$ for any $j \in J$.

iv) For any collection $\beta = (\beta_{ij})_{i \in I_T,\, j \in Inc_i}$ with $\beta_{ij} \geqslant 0$, we say that $u \in C^k_{*,\beta}(\Gamma)$ if $u \in C^k(\Gamma)$, $k \geqslant 1$, and there holds

$$S^i_\beta u := \sum_{j \in Inc_i} \beta_{ij} a_{ij} \partial_j u(v_i) = 0 \quad \forall i \in I_T. \tag{2.2}$$

Remark 2.1. Condition (2.2) is known in the literature as the Kirchhoff condition. In a way, differentiability of a function along the edges means that the slopes in outward (or inward) direction with respect to each given point add up to zero. At vertices, this condition naturally generalizes to the Kirchhoff condition.

2.3. Viscosity solutions

A Hamiltonian $H : \Gamma \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a collection of operators $(H^j)_{j \in J}$ with $H^j : [0, l_j] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Along the paper we will consider the following conditions

$$H^j \in C^0\big([0, l_j] \times \mathbb{R} \times \mathbb{R}\big), \quad j \in J; \tag{2.3}$$

$$H^j(x, \cdot, p) \quad \text{is nondecreasing for all } (x, p) \in [0, l_j] \times \mathbb{R},\ j \in J; \tag{2.4}$$

$$H^j(v_i, r, \cdot) \quad \text{is nondecreasing in } (0, +\infty) \text{ for any } i \in I_T,\ r \in \mathbb{R},\ j \in J; \tag{2.5}$$

$$H^j(x, r, \cdot) \to +\infty \quad \text{as } |p| \to \infty \text{ uniformly in } (x, r) \in [0, l_j] \times [-R, R],\ j \in J; \tag{2.6}$$

$$H^j\big(\pi_j^{-1}(v_i), r, p\big) = H^k\big(\pi_k^{-1}(v_i), r, p\big) \quad \text{for any } r \in \mathbb{R},\ p \in \mathbb{R},\ i \in I_T,\ j, k \in Inc_i; \tag{2.7}$$

$$H^j\big(\pi_j^{-1}(v_i), r, p\big) = H^j\big(\pi_j^{-1}(v_i), r, -p\big) \quad \text{for any } r \in \mathbb{R},\ p \in \mathbb{R},\ i \in I_T,\ j \in Inc_i. \tag{2.8}$$

Remark 2.2. Assumptions (2.7)–(2.8) represent compatibility conditions of H at the vertices of Γ, i.e. continuity at the vertices and independence of the orientation of the incident arc, respectively (the network is not oriented).

Example 2.1. The operator $H(x, r, p) := |p|^\alpha + b(x)r + f(x)$ satisfies assumptions (2.3)–(2.8) provided that $\alpha > 0$, $b, f \in C^0(\Gamma)$ and $b(x) \geqslant 0$ for every $x \in \Gamma$.

On the graph Γ, we consider the Hamilton–Jacobi equation

$$H(x, u, \partial u) = 0, \quad x \in \Gamma, \tag{2.9}$$

namely, on each edge e_j, we address the Hamilton–Jacobi equation

$$H^j\big(y, u^j(y), \partial_j u\big) = 0, \quad y \in [0, l_j].$$

In the next definitions we introduce the class of test functions and solution of (2.9).

Definition 2.3. Let $\phi \in C(\Gamma)$.

i) Let $x \in e_j$, $j \in J$. We say that ϕ is a test function at x, if ϕ^j is differentiable at $\pi_j^{-1}(x)$.
ii) Let $x = v_i$, $i \in I_T$, $j, k \in Inc_i$, $j \neq k$. We say that ϕ is a (j,k)-test function at x, if ϕ^j and ϕ^k are differentiable at $\pi_j^{-1}(x)$ and $\pi_k^{-1}(x)$, respectively and

$$a_{ij}\partial_j\phi\big(\pi_j^{-1}(x)\big) + a_{ik}\partial_k\phi\big(\pi_k^{-1}(x)\big) = 0, \tag{2.10}$$

where (a_{ij}) is as in (2.1).

Definition 2.4. A function $u \in USC(\Gamma)$ is called a (viscosity) subsolution of (2.9) in Γ if the following hold:

i) If $x \in e_j$, $j \in J$, for any test function ϕ for which $u - \phi$ attains a local maximum at x, we have

$$H^j\big(\pi_j^{-1}(x), u^j\big(\pi_j^{-1}(x)\big), \partial_j\phi\big(\pi_j^{-1}(x)\big)\big) \leqslant 0.$$

ii) If $x = v_i$, $i \in I_T$, for any $j, k \in Inc_i$ and any (j,k)-test function ϕ for which $u - \phi$ attains a local maximum at x relatively to $\bar{e}_j \cup \bar{e}_k$, we have

$$H^j\big(\pi_j^{-1}(x), u^j\big(\pi_j^{-1}(x)\big), \partial_j\phi\big(\pi_j^{-1}(x)\big)\big) \leqslant 0.$$

A function $u \in LSC(\Gamma)$ is called a (viscosity) supersolution of (2.9) in Γ if the following hold:

i) If $x \in e_j$, $j \in J$, for any test function ϕ for which $u - \phi$ attains a local minimum at x, we have

$$H^j\big(\pi_j^{-1}(x), u^j\big(\pi_j^{-1}(x)\big), \partial_j\phi\big(\pi_j^{-1}(x)\big)\big) \geqslant 0.$$

ii) If $x = v_i$, $i \in I_T$, for any $j \in Inc_i$, there exists $k \in Inc_i$, $k \neq j$, (said i-feasible for j at x) such that for any (j,k)-test function ϕ for which $u - \phi$ attains a local minimum at x relatively to $\bar{e}_j \cup \bar{e}_k$, we have

$$H^j\big(\pi_j^{-1}(x), u^j\big(\pi_j^{-1}(x)\big), \partial_j\phi\big(\pi_j^{-1}(x)\big)\big) \geqslant 0.$$

A continuous function $u \in C(\Gamma)$ is called a (viscosity) solution of (2.9) if it is both a viscosity subsolution and a viscosity supersolution of (2.9).

Remark 2.3. It is important to observe that the definitions of subsolution and supersolution are not symmetric at the vertices. As observed in [20] for the equation $|\partial u|^2 = 1$, a definition of supersolution similar to the one of subsolution would not characterize the correct solution, i.e. the distance from the boundary.

Remark 2.4. The definition of solution does not involve the vertices $v_i \in \partial\Gamma$: at these points no "transition" condition is required. Let us note that the condition "$\#Inc_i = 1$" for $i \in I_B$ can be omitted, namely I_B can be an arbitrary subset of I. In this case, whenever $i \in I_B$ and $\#Inc_i > 1$, the problem will be equivalent to the one obtained by splitting the common endpoints of the edges incident v_i.

2.4. Perron method and comparison principle

In this section we collect some known results on the well posedness of the Hamilton–Jacobi equations (2.9). In fact, the original papers only concern Hamiltonians independent of u; however, the proofs can be easily adapted to operators depending on u as in (2.4). Concerning the existence of a solution we have the following result; for the proof, obtained via Perron's method, we refer the reader to [5, Thm6.1].

Theorem 2.1. *Assume (2.3)–(2.8) and that there is a viscosity subsolution $w \in USC(\Gamma)$ and a viscosity supersolution $W \in LSC(\Gamma)$ of (2.9) such that $w \leqslant W$ and $w_*(x) = W^*(x) = g(x)$ for $x \in \partial\Gamma$. Let the function $u : \Gamma \to \mathbb{R}$ be defined by $u(x) := \sup_{v \in X} v(x)$ where*

$$X = \big\{v \in USC(\Gamma)\colon\ v \text{ is a viscosity subsolution of (2.9) with } w \leqslant v \leqslant W \text{ on } \Gamma\big\}.$$

Then, u^* and u_* are respectively a sub- and a supersolution to problem (2.9) with $u = g$ on $\partial\Gamma$.

The proof of the following theorem relies on the classical doubling of variable argument; for the detailed proof, we refer to [20, Thm5.1] and to [19, Lem5.2].

Theorem 2.2. *Assume (2.3)–(2.8).*

(a) *Assume*

$$H^j(y, \cdot, p) \text{ is strictly increasing for any } y \in [0, l_j],\ p \in \mathbb{R},\ j \in J. \tag{2.11}$$

Let u_1 and u_2 be respectively a bounded super- and a bounded subsolution of (2.9) such that $u_1(v_i) \geqslant u_2(v_i)$ for all $i \in I_B$. Then $u_1 \geqslant u_2$ in Γ.

(b) *Let u_1 and u_2 be respectively a supersolution to (2.9) and a subsolution to*

$$H(x, u, \partial u) = g(x) \quad x \in \Gamma$$

with $g \in C(\Gamma)$, $g < 0$. Then there holds $u_1 \geqslant u_2$ in Γ, provided that $u_1(v_i) \geqslant u_2(v_i)$ for all $i \in I_B$.

Finally, let us state a stability result (see [20, Prp3.2]):

Proposition 2.1. *Assume (2.3)–(2.8). Let u_n be a solution of*

$$H_n(x, u_n, \partial u_n) = 0 \quad x \in \Gamma,\ n \in \mathbb{N}.$$

Assume that, as $n \to \infty$, $H_n(x, r, p) \to H(x, r, p)$ locally uniformly and $u_n \to u$ uniformly in Γ. Then u is a solution of (2.9).

Remark 2.5. For the Hamilton–Jacobi equation (2.9) it is well known that a smooth solution will not exist in general. Furthermore it is equally easy to see that the Kirchhoff condition (2.2) is not satisfied. Continuity is the only property of a solution to (2.9) which is reasonable to expect.

3. The viscous Eikonal equation on networks

In this section we study the existence and the uniqueness of a classical solution to second order equations coupled with Kirchhoff condition.

3.1. Linear problems

We consider the following class of linear problems on Γ

$$L^j w(x) + g^j(x) = 0 \quad x \in e_j,\ j \in J, \qquad w(v_i) = \gamma_i \quad \forall i \in I_B, \tag{3.1}$$

where $L = (L^j)_{j\in J}$ is a collection of elliptic linear operators of the form

$$L^j w(x) := a^j(x)\partial_j^2 w(x) + b^j(x)\partial_j w(x) - c^j(x)w(x) \quad x \in e_j,\ j \in J. \tag{3.2}$$

We assume the following hypotheses

$$a^j, b^j, c^j, g^j \in C\big([0, l_j]\big), \quad a^j(x) \geqslant \lambda > 0, \qquad c^j(x) \geqslant 0 \quad \forall x \in [0, l_j],\ j \in J. \tag{3.3}$$

Let us now state a maximum principle for problem (3.1).

Theorem 3.1. *Let $L = (L^j)_{j\in J}$, $S_\beta = (S^i_\beta)_{i\in I_T}$ be defined as in (3.2)–(3.3) and respectively in (2.2) with $\sum_{j\in Inc_i} \beta_{ij} > 0$ for each $i \in I_T$. Assume that the function $w \in C^2(\Gamma)$ satisfies*

$$L^j w(x) \geqslant 0 \quad x \in e_j,\ j \in J \quad \textit{and} \quad S^i_\beta w \geqslant 0 \quad i \in I_T. \tag{3.4}$$

Then w attains a nonnegative maximum in $\Gamma \setminus \partial\Gamma$ if and only if, it is constant. A similar result holds for the minimum of w if we revert the inequalities in (3.4).

Proof. We set $M := \max w$ and $A := \{x \in \Gamma \setminus \partial\Gamma\colon\ w(x) = M\}$. We proceed by contradiction assuming $M \geqslant 0$ and $A \neq \emptyset$. For the sake of clarity, we split the arguments in two cases.

Case (I). We assume that $Lw > 0$, $S_\beta w > 0$ and $x_0 \in A$. If $x_0 \in e_j$ for some $j \in J$, then we have: $\partial_j w(x_0) = 0$ and $\partial_j^2 w(x_0) \leqslant 0$, a contradiction to $L^j w > 0$. If $x_0 = v_i$ for some $i \in I_T$, then we have $a_{ij}\partial_j w(v_i) \leqslant 0$ for all $j \in Inc_i$, hence $S^i_\beta w \leqslant 0$, a contradiction.

Case (II). We assume that $Lw \geqslant 0$, $S_\beta w \geqslant 0$ and $x_0 \in A$. By the continuity of w, one of the following two cases must occur somewhere in Γ

(i) for some $j \in J$, $x_0 \in e_j$ and $w(y) < w(x_0)$ for some $y \in e_j$,
(ii) for some $i \in I$, $x_0 = v_i$ and $w(y) < w(x_0)$ for some $y \in e_j$ with $j \in Inc_i$.

In case (i), the (nonconstant) function w^j solves $L^j w^j \geqslant 0$ in $(0, l_j)$ and it attains a nonnegative maximum inside $(0, l_j)$. This situation is impossible by classical results (see [18, Ch.1]).

Let us consider case (ii). Now it suffices to prove the statement in the network $\Gamma_0 := \bigcup_{j \in Inc_i} \bar{e}_j$. Moreover, wlog, we shall assume $\pi_j(0) = v_i$ for any $j \in Inc_i$ and $y \in e_{\bar{j}}$, for some $\bar{j} \in J$. We claim that there exists a function $\phi \in C^2(\Gamma_0)$ such that

$$L^j \phi^j(x) > 0 \quad \forall x \in e_j,\ j \in Inc_i, \qquad S^i_\beta \phi > 0, \qquad \phi \geqslant 0, \qquad \phi(v_i) = 0. \tag{3.5}$$

To this end, we define $\phi^j(x) := e^{\alpha_j x} - 1$ (for $j \in Inc_i$) with a parameter α_j such that $L^j \phi^j > 0$. In order to have this inequality, it suffices to choose $\alpha_j > 0$ such that there holds

$$\lambda \alpha_j^2 - \|b^j\|_\infty \alpha_j - \|c^j\|_\infty > 0.$$

Moreover, we have: $S^i_\beta \phi = \sum_{j \in Inc_i} \beta_{ij} \partial_j \phi^j = \sum_{j \in Inc_i} \beta_{ij} \alpha_j > 0$. Hence, our claim (3.5) is completely proved.

Fix $\eta := (w(v_i) - w(y))(e^{\alpha_{\bar{j}} l_{\bar{j}}} - 1)^{-1}$ (note $\eta > 0$ by our assumptions) and introduce the function $\tilde{w}(x) := w(x) + \eta\phi(x)$, $x \in \Gamma_0$. We observe that there hold

$$S^i_\beta \tilde{w} = S^i_\beta w + \eta S^i_\beta \phi > 0, \qquad L^j \tilde{w}^j = L^j w^j + \eta L^j \phi^j > 0 \quad \forall j \in Inc_i,$$
$$\tilde{w}(v_i) = w(v_i), \qquad \tilde{w}(y) = w(y) + \big(w(v_i) - w(y)\big)\frac{\phi(y)}{e^{\alpha_{\bar{j}} l_{\bar{j}}} - 1} < w(v_i).$$

Invoking case (i) we obtain a contradiction. □

Theorem 3.2. *There exists a unique solution $u \in C^2_{*,\beta}(\Gamma)$ to problem* (3.1).

Proof. By standard arguments (see [18, Ch.1]), uniqueness is an immediate consequence of Theorem 3.1. Existence of a solution to (3.1) is proved in [10, Thm3.3] (see also the related comments and [9]) via a probabilistic representation formula. In fact a solution of (3.1) can be represented as

$$u(x) = \mathbb{E}_x \left\{ \int_0^\tau e^{-c(Y(s))} g\big(Y(s)\big)\, ds + e^{-c(Y(\tau))} \gamma_{i(\tau)} \right\}$$

where $Y(s)$ is a Markov process defined on the graph which on each edge e_j solves the stochastic differential equation

$$dY(s) = b^j\big(Y(s)\big)\, ds + a^j\big(Y(s)\big)\, dW(s),$$

$\tau = \inf\{t > 0\colon\ Y(t) \in \partial\Gamma\}$ and $i(\tau) \in I_B$ is such that $Y(\tau) = v_{i(\tau)} \in \partial\Gamma$. In this interpretation the Kirchhoff condition (2.2) implies that the process almost surely spends zero time at each transition vertex v_i, (see [10, Thm3.1]) while the term $\beta_{ij}/(\sum_{j\in Inc_i}\beta_{ij})$ is the probability that $Y(t)$ enters in the edge e_j when it is in v_i. □

3.2. Semilinear problems

Theorem 3.3. *For any $\varepsilon > 0$, there exists a unique solution $u_\varepsilon \in C^2_{*,\beta}(\Gamma)$ of*

$$-\varepsilon\partial_j^2 u + |\partial_j u|^2 - f(x) = 0 \quad x \in e_j,\ j \in J, \qquad u(v_i) = g_i \quad i \in I_B \tag{3.6}$$

where f is a continuous, nonnegative function on Γ.

Proof. We consider the logarithmic transformation (see [8]): $u_\varepsilon = -\varepsilon\ln(w_\varepsilon + 1)$. Invoking Theorem 3.2, we have that for any $\varepsilon > 0$ there exists a unique solution $w_\varepsilon \in C^2_{*,\beta}(\Gamma)$ to the linear problem

$$\varepsilon^2\partial_j^2 w_\varepsilon - f(x)w_\varepsilon - f(x) = 0 \quad x \in e_j,\ j \in J, \qquad w_\varepsilon(v_i) = e^{-\frac{g_i}{\varepsilon}} - 1 \quad i \in I_B.$$

Hence, reversing the logarithmic transformation, we conclude that there exists a unique solution to (3.6). □

Another consequence of Theorem 3.1 is the following comparison principle

Corollary 3.1. *Assume that $H = (H^j)_{j\in J}$ satisfies (2.3)–(2.4) and*

$$H^j(x,\cdot,\cdot) \in C^1(\mathbb{R}\times\mathbb{R}) \quad \text{for any } x \in (0, l_j),\ j \in J. \tag{3.7}$$

Let $w_1, w_2 \in C^2(\Gamma)$ be such that

$$\begin{cases} -\varepsilon\partial_j^2 w_1 + H^j(x, w_1, \partial_j w_1) \geqslant -\varepsilon\partial_j^2 w_2 + H^j(x, w_2, \partial_j w_2) & x \in e_j,\ j \in J, \\ S^i_\beta w_1 \leqslant S^i_\beta w_2 & i \in I_T, \\ w_1(v_i) \geqslant w_2(v_i) & i \in I_B. \end{cases} \tag{3.8}$$

Then $w_1 \geqslant w_2$ on Γ.

Proof. Set $A = \{w_2 > w_1\} \subset \Gamma$; the function $w := w_2 - w_1$ is a solution to

$$\begin{cases} \varepsilon\partial_j^2 w + \tilde b^j(x)\partial_j w - \tilde c^j(x) w \geqslant 0 & x \in e_j \cap A,\ j \in J, \\ S^i_\beta w \geqslant 0 & i \in I_T \cap A, \\ w(v_i) \leqslant 0 & i \in I_B \cap A \end{cases}$$

where

$$\tilde b^j(x) = -\int_0^1 \frac{\partial H^j}{\partial p}\big(x, \theta w_1 + (1-\theta)w_2, \theta\partial_j w_1 + (1-\theta)\partial_j w_2\big)\,d\theta,$$

$$\tilde c^j(x) = \int_0^1 \frac{\partial H^j}{\partial r}\big(x, \theta w_1 + (1-\theta)w_2, \theta\partial_j w_1 + (1-\theta)\partial_j w_2\big)\,d\theta.$$

By Theorem 3.1, w cannot attain a local nonnegative maximum inside the open set A. As we have $A \cap \partial\Gamma = \emptyset$, it follows that A is empty and $w_1 \geqslant w_2$ in Γ. □

3.3. Other comparison principles for (3.8)

For the sake of completeness, we establish some comparison principles for problem (3.8) under assumptions different from Corollary 3.1; especially, in both of them we shall drop the regularity condition (3.7). In the former we require the strict monotonicity of H with respect to u, while in the latter we require a linear growth of H with respect to u and ∂u.

Proposition 3.1. *Assume that $H = (H^j)_{j\in J}$ satisfies (2.3)–(2.4) and (2.11). Let the functions $w_1, w_2 \in C^2(\Gamma)$ satisfy (3.8) with $\beta_{ij} > 0$ for any $i \in I_T$, $j \in Inc_i$. Then $w_1 \geqslant w_2$ on Γ.*

Proof. We argue by contradiction assuming $\max_\Gamma(w_2 - w_1) =: \delta > 0$. Let x_0 be a point where $w_2 - w_1$ attains its maximum; whence $x_0 \in \Gamma$. The point x_0 either belongs to some edge or it coincides with a transition vertex. Assume that x_0 belongs to some edge e_j. By their regularity, the functions w_1 and w_2 fulfill

$$w_2(x_0) = w_1(x_0) + \delta, \qquad \partial_j w_2(x_0) = \partial_j w_1(x_0), \qquad \partial_j^2 w_2(x_0) \leqslant \partial_j^2 w_1(x_0).$$

In particular, we deduce

$$\begin{aligned} -\varepsilon\partial_j^2 w_1(x_0) + H\big(x_0, w_1(x_0), \partial_j w_1(x_0)\big) &\leqslant -\varepsilon\partial_j^2 w_2(x_0) + H\big(x_0, w_2(x_0) - \delta, \partial_j w_2(x_0)\big) \\ &< -\varepsilon\partial_j^2 w_2(x_0) + H\big(x_0, w_2(x_0), \partial_j w_2(x_0)\big) \end{aligned}$$

which contradicts the first relation in (3.8).

Assume that $x_0 = v_i$ for some $i \in I_T$. Being regular, the functions w_1 and w_2 fulfill $a_{ij}\partial_j w_2(v_i) \leqslant a_{ij}\partial_j w_1(v_i)$. We claim $\partial_j w_2(v_i) = \partial_j w_1(v_i)$ for each $j \in Inc_i$. In order to prove this equality we proceed by contradiction and we assume that $a_{ij}\partial_j w_2(v_i) < a_{ij}\partial_j w_1(v_i)$ for some $j \in Inc_i$. In this case we get $S^i_\beta w_2 < S^i_\beta w_1$ which contradicts the second hypothesis in (3.8); therefore, our claim is proved. Moreover, since $w_1(x_0) = w_2(x_0) - \delta$, we deduce

$$H\big(x_0, w_1(x_0), \partial_j w_1(x_0)\big) = H\big(x_0, w_2(x_0) - \delta, \partial_j w_2(x_0)\big) < H\big(x_0, w_2(x_0), \partial_j w_2(x_0)\big).$$

Taking into account the regularity of H and of w_i $(i = 1, 2)$, we infer that in a sufficiently small neighborhood $B_\eta(v_i)$ there holds

$$H\big(x, w_1(x), \partial w_1(x)\big) < H\big(x, w_2(x), \partial w_2(x)\big) \quad \forall x \in B_\eta(v_i).$$

This inequality and the first relation in (3.8) entail

$$\varepsilon\partial_j^2(w_2 - w_1) \geqslant H\big(x, w_2(x), \partial_j w_2(x)\big) - H\big(x, w_1(x), \partial_j w_1(x)\big) > 0$$

which, together with $\partial_j w_2(v_i) = \partial_j w_1(v_i)$, contradicts that $w_2 - w_1$ attains a maximum in v_i. □

Proposition 3.2. *Assume that $H = (H^j)_{j\in J}$ satisfies (2.3)–(2.4) and that*

$$\big|H^j(x, r, p) - H^j(x, s, q)\big| \leqslant K\big(|r - s| + |p - q|\big) \quad \forall r, s, p, q \in \mathbb{R}. \tag{3.9}$$

Assume also that $\beta_{ij} > 0$ *for any* $i \in I_T$, $j \in Inc_i$. *Let the functions* $w_1, w_2 \in C^2(\Gamma)$ *satisfy* (3.8). *Then* $w_1 \geqslant w_2$ *on* Γ.

Proof. We proceed by contradiction assuming $\max_\Gamma(w_2 - w_1) =: \delta > 0$. We need the following result whose proof is postponed to Appendix A.

Lemma 3.1. *Assume the hypotheses of Proposition* 3.2. *For every* $\eta > 0$, *there exists a function* $\phi_\eta \in C^2(\Gamma)$, *with* $\|\phi_\eta\|_\infty \leqslant \eta$, *such that the function* $\bar{w}_\eta := w_2 + \phi_\eta$ *satisfies*

$$-\varepsilon\partial_j^2\bar{w}_\eta + H^j(x, \bar{w}_\eta, \partial_j\bar{w}_\eta) < -\varepsilon\partial_j^2 w_1 + H^j(x, w_1, \partial_j w_1), \qquad S_\beta\bar{w}_\eta > S_\beta w_2.$$

Set $\phi := \phi_{\delta/3}$ and $\bar{w} := \bar{w}_{\delta/3}$ (here, the functions ϕ_η and $\bar{w}_\eta$ are those introduced in Lemma 3.1). We note that $\bar{\delta} := \max_\Gamma(\bar{w} - w_1) > 2\delta/3$ and $\bar{w}(v_i) - w_1(v_i) \leqslant \delta/3$ for every $i \in I_B$; therefore, for $B := \{x \in \Gamma\colon\ \bar{w}(x) - w_1(x) = \bar{\delta}\}$, there holds $B \cap \Gamma \neq \emptyset$. In fact, we claim that $B \subset \bigcup_{j\in J} e_j$, namely

$$v_i \notin B \quad \forall i \in I_T. \tag{3.10}$$

In order to prove this relation, we assume by contradiction that $v_i \in B$ for some $i \in I_T$. By Lemma 3.1, we have $S^i_\beta(\bar{w} - w_1) > 0$; in particular, there exists $j \in Inc_i$ such that $\beta_{ij}a_{ij}\partial_j(\bar{w} - w_1) > 0$. This inequality contradicts the presence of a maximum at v_i; whence, our claim (3.10) is established.

Fix $\hat{x} \in B$. Relation (3.10) guarantees that $\hat{x}$ belongs to some e_j and that both the extremities of e_j do not belong to B. This is impossible by standard arguments; we refer the reader to [14, Prp3.3] for a detailed proof. □

4. A priori estimates for viscous equations

This section is devoted to some a priori bounds for the viscous equation

$$\begin{cases} -\varepsilon\partial_j^2 w + H^j\big(x, w(x), \partial_j w\big) = 0 & x \in e_j,\ \text{for all } j \in J,\\ S^i_\beta w = 0 & i \in I_T,\\ w(v_i) = g_i & i \in I_B. \end{cases} \tag{4.1}$$

We assume that

- $H = (H^j)_{j\in J}$ satisfies (2.3)–(2.6) and either (3.7) or (2.11) or (3.9);
- there exist $\delta > 0$ and $\psi \in C^2(\Gamma)$ such that

$$H(x, \psi, \partial\psi) \leqslant -\delta \quad \text{on } \Gamma \setminus V, \qquad S^i_\beta\psi \geqslant 0 \quad i \in I_T, \qquad \psi(v_i) = g_i \quad i \in I_B; \tag{4.2}$$

- $\beta_{ij} > 0$ for every $i \in I_T$, $j \in Inc_i$.

The proofs of the next two lemmas are postponed to Appendix A.

Lemma 4.1. *Let* $\theta, \eta \in \mathbb{R}$, $\theta > 0$. *Then there exists a number* $M_{\theta,\eta} > 0$ *such that*

$$H^j(x, r, p) > \theta \quad \text{for all } p \in \mathbb{R},\ |p| > M_{\theta,\eta},\ r \geqslant \eta,\ x \in [0, l_j],\ j \in J. \tag{4.3}$$

Lemma 4.2. *There are a function* $\phi \in C^2(\Gamma)$ *and a vector* $(\alpha_j)_{j\in J}$, *with* $\alpha_j \neq 0$ *for all* $j \in J$, *for which*

$$\partial_j\phi(x) = \alpha_j \quad \forall x \in e_j,\ j \in J, \qquad S^i_\beta\phi < 0 \quad \forall i \in I_T.$$

Theorem 4.1. *Assume that, for each ε, there is a solution $u_\varepsilon \in C^2_{*,\beta}(\Gamma)$ of (4.1). Then there is $\bar{\varepsilon}$ sufficiently small such that, for any $0 < \varepsilon < \bar{\varepsilon}$, the functions u_ε are uniformly bounded and equi-Lipschitz continuous on $\bar{\Gamma}$.*

Proof. Bound on $|u_\varepsilon|$. For ε_1 sufficiently small, the function ψ in (4.2) satisfies $\varepsilon\partial^2\psi \geqslant -\delta$ for every $\varepsilon < \varepsilon_1$ and also

$$-\varepsilon\partial^2\psi + H(x, \psi, \partial\psi) \leqslant \delta + H(x, \psi, \partial\psi) \leqslant 0 \quad \forall\varepsilon < \varepsilon_1.$$

On the other hand, it fulfills $S^i_\beta\psi \geqslant 0$ for any $i \in I_T$ and $\psi(v_i) = g_i$ for any $i \in I_B$. By Corollary 3.1 (or Proposition 3.1 or Proposition 3.2), we get the lower bound

$$\psi \leqslant u_\varepsilon \quad \text{on } \Gamma,\ \forall\varepsilon < \varepsilon_1. \tag{4.4}$$

To get the upper bound, we consider a function ϕ as in Lemma 4.2 and we set $\alpha := \min_{j\in J}|\alpha_j|$. Define a function $W \in C^2(\Gamma)$ by $W := M_{0,0}\phi/\alpha + C$, where $M_{0,0}$ as in Lemma 4.1 and choose the constant C in such a way that

$$W(x) > \max\Big\{0, \max_{i\in I_B} g_i\Big\} \quad \text{for } x \in \Gamma.$$

By construction we have

$$W^j(x) \geqslant 0, \qquad \big|\partial_j W^j(x)\big| > M_{0,0}, \qquad \partial_j^2 W(x) = 0 \quad \text{for } x \in e_j,\ j \in J. \tag{4.5}$$

By (4.3) and (4.5), we infer

$$-\varepsilon\partial_j^2 W + H^j(x, W, \partial_j W) = H^j(x, W, \partial_j W) > 0 \quad \text{for } x \in e_j,\ j \in J.$$

Moreover $S^i_\beta W < 0$ for all $i \in I_T$ and $W(v_i) \geqslant g_i$ for all $i \in I_B$. Invoking again Corollary 3.1 (or Proposition 3.1 or Proposition 3.2) we get the upper bound: $u_\varepsilon \leqslant W$ on Γ, for any $\varepsilon > 0$. We conclude that there is a constant C_1, independent of ε, such that, for $\varepsilon < \varepsilon_1$, there holds

$$\max_{\bar{\Gamma}} |u_\varepsilon| \leqslant C_1. \tag{4.6}$$

Bound on $|\partial^j u_\varepsilon|$. We split the proof in three steps devoted respectively to boundary vertices, to transition vertices and to interior of edges.

Step 1: *Bound on $|\partial_j u_\varepsilon(v_i)|$, for $i \in I_B$, $j \in Inc_i$.* Let $d_{\partial\Gamma} : \Gamma \to \mathbb{R}$ be the distance from the boundary of Γ, i.e. $d_{\partial\Gamma}(x) := \min\{d(x, v_i):\ i \in I_B\}$ where d is the path distance on the network. For $\beta > 0$ set $\Gamma_\beta := \{x \in \Gamma:\ d_{\partial\Gamma}(x) \leqslant \beta\}$. We show that there are constants $K > 0$, $\beta > 0$ such that

$$\psi \leqslant u_\varepsilon \leqslant \psi + K d_{\partial\Gamma} \quad \text{on } \Gamma_\beta,\ \forall 0 < \varepsilon < \varepsilon_1, \tag{4.7}$$

where ψ is as in (4.2) while ε_1 is defined before. The former inequality has been established in (4.4). In order to prove the latter inequality, let β be such that $d_{\partial\Gamma}$ does not obtain a local maximum on the interior of Γ_β and such that there is no $i \in I_T$ for which $v_i \in \Gamma_\beta$. It follows that for any $i \in I_B$ and $j \in Inc_i$, $|\partial_j d^j_{\partial\Gamma}| \equiv 1$ and $|\partial_j^2 d^j_{\partial\Gamma}| \equiv 0$ on Γ_β. Let

$$\theta := \varepsilon_1 \max_{j\in J} \max_{e_j} \partial_j^2 \psi^j, \qquad \eta := \min_{j\in J} \min_{e_j} \psi^j$$

and define $M_{\theta,\eta}$ as in Lemma 4.1. Set $K := M_{\theta,\eta} + \max_{j\in J} \max_{e_j} |\partial_j \psi^j(x)|$ and $\bar\psi := \psi + K d_{\partial\Gamma}$. Hence $|\partial_j \bar\psi(x)| > M_{\theta,\eta}$ for $x \in \Gamma_\beta$ and by (4.3)

$$-\varepsilon\partial_j^2 \bar\psi + H(x, \bar\psi, \partial_j \bar\psi) \geqslant -\theta + H(x, \bar\psi, \partial_j \bar\psi) > 0 \quad x \in \Gamma_\beta.$$

By possible enlarging K, we can assume that

$$\bar\psi(x) \geqslant u_\varepsilon(x) \quad \text{for } x \in \partial\Gamma_\beta \cap (\Gamma \setminus \partial\Gamma).$$

By Corollary 3.1 (or Proposition 3.1 or Proposition 3.2), on each segment $e_j \cap \Gamma_\beta$ (recall that $\Gamma_\beta \cap \{v_i\}_{i\in I_T}$ is empty) we get that $\bar\psi \geqslant u_\varepsilon$ for any $0 < \varepsilon < \varepsilon_1$; hence relation (4.7) is completely proved.

By (4.7) it follows that there exists a constant C_2, independent of ε, such that

$$\left|\partial_j u_\varepsilon(v_i)\right| \leqslant C_2 \quad \forall i \in I_B,\ \forall 0 < \varepsilon < \varepsilon_1. \tag{4.8}$$

Step 2: *Bound on* $|\partial_j u_\varepsilon(v_i)|$, *for* $i \in I_T$, $j \in Inc_i$. We claim that there exists a constant C_3 such that, for $\bar\varepsilon$ sufficiently small, there holds

$$\left|\partial_j u_\varepsilon(v_i)\right| \leqslant C_3 \quad \forall i \in I_T,\ j \in Inc_i,\ 0 < \varepsilon < \bar\varepsilon. \tag{4.9}$$

If the claim is false, there exist $i \in I_T$, $k \in Inc_i$ and a sequence $\varepsilon_n \to 0$ such that, for $u_n := u_{\varepsilon_n}$, we have

$$\lim_{n\to\infty} \left|\partial_k u_n(v_i)\right| = +\infty.$$

Let us recall: $S^i_\beta u_n = \sum_{j\in Inc_i} \beta_{ij} a_{ij} \partial_j u_n(v_i) = 0$ for any $n \in \mathbb{N}$. Hence, by passing to a subsequence, there exists $j \in Inc_i$ such that $\lim_n a_{ij}\partial_j u_n(v_i) = +\infty$. Wlog, assume $a_{ij} = 1$. Hence, there exists a sequence $x_n \in e_j$ with $x_n \to v_i$ such that

$$\lim_{n\to\infty} \partial_j u_n(x_n) = +\infty. \tag{4.10}$$

Set $y_n := \pi_j^{-1}(x_n)$ and fix $t_0 > 0$ such that $y_n + t \in [0, l_j]$ for all $t \in [0, t_0]$ and $n \in \mathbb{N}$. (Note that t_0 is independent of n; indeed, as $n \to +\infty$, y_n converges to 0.) For $f_n(t) := u_n^j(y_n + t)$, relation (4.10) is equivalent to

$$\lim_{n\to\infty} f_n'(0) = +\infty. \tag{4.11}$$

Substituting in (4.1) (recall: $f_n \in C^2([0, t_0])$), we get

$$f_n''(t) = \varepsilon_n^{-1} H^j\bigl(y_n + t, f_n(t), f_n'(t)\bigr) \quad \text{for all } t \in [0, t_0],\ n \in \mathbb{N}. \tag{4.12}$$

For C_1 as in (4.6), set

$$\theta := 2C_1/t_0^2 \quad \text{and} \quad \eta := -C_1. \tag{4.13}$$

Let $M_{\theta,\eta}$ be as in (4.3). Then by (4.11) there is $n \in \mathbb{N}$ such that $|f_n'(0)| = f_n'(0) > M_{\theta,\eta}$. By (4.6), (4.12) and Lemma 4.1, we have for $\varepsilon_n < 1$

$$f_n''(0) > \varepsilon_n^{-1}\theta > \theta. \tag{4.14}$$

We claim that there holds

$$f_n''(t) > \theta \quad \text{for all } t \in [0, t_0]. \tag{4.15}$$

For this purpose we set $A := \{t \in [0, t_0]\colon\ f_n''(t) \geqslant \theta\}$. By (4.14) there is a connected subset A_0 of A which contains 0. Since $f_n \in C^2([0, t_0])$, A_0 is closed, hence there is a maximal $\bar{t} \in A_0$. If (4.15) is false, then $\bar{t} < t_0$. Since $f_n'(0) > M_{\theta,\eta}$ and $f_n''(s) \geqslant \theta > 0$ for $s \in A_0$ and therefore f_n' is increasing in A_0, there is a neighborhood $U \subset [0, t_0]$ of $\bar{t}$ such that $f_n'(s) > M_{\theta,\eta}$ for all $s \in U$. Then Lemma 4.1 and (4.12) imply that $f_n''(s) > \theta$ for all $s \in U$, contradicting the maximality of $\bar{t}$ so claim (4.15) is proved.

Relation (4.15) entails the inequality

$$f_n(t) \geqslant \theta t^2 + f_n'(0)t + f_n(0) \quad \forall t \in [0, t_0].$$

Taking into account $f_n'(0) > 0$ and (4.13), we estimate

$$u_n^j(y_n + t_0) = f_n(t_0) > f_n(0) + \theta t_0^2 \geqslant -C_1 + \theta t_0^2 = C_1.$$

This relation contradicts the definition of C_1, hence (4.9) is proved.

Step 3: *Bound on* $|\partial^j u_\varepsilon|$ *on* Γ. By later contradiction, let us assume that $|\partial_j u_\varepsilon|$ are not uniformly bounded in Γ, namely, there exist two sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$, with $x_n \in \Gamma \setminus V$, such that $|\partial_j u_{\varepsilon_n}(x_n)| \to +\infty$. Possibly passing to a subsequence, by the compactness of Γ, there exist $j \in J$ and $\hat{x} \in \bar{e}_j$ such that $x_n \to \hat{x}$ and $|\partial_j u_n(x_n)| \to +\infty$ for $u_n := u_{\varepsilon_n}$.

Case (a): $\hat{x} \in e_j$ *and* $\partial_j u_n(x_n) \to +\infty$. We shall argue as in Step 2; for $y_n := \pi_j^{-1}(x_n)$, we fix $t_0 > 0$ such that $y_n + t \in [0, l_j]$ for all $t \in [0, t_0]$ and $n \in \mathbb{N}$. (Note that such a t_0 exists since $\hat{x} \in e_j$.) The functions $f_n(t) := u_n^j(x_n + t)$ satisfy relations (4.11) and (4.12). For θ and η as in (4.13), we can fix n sufficiently large to have $|f_n'(0)| = f_n'(0) > M_{\theta,\eta}$. By (4.6), (4.12) and Lemma 4.1, we have $f_n''(0) > \theta$. We obtain relation (4.15) and then we conclude the proof following the same arguments as before.

Case (b): $\hat{x} \in e_j$ *and* $\partial_j u_n(x_n) \to -\infty$. We shall use arguments analogous to those of previous case. Fix $t_0 > 0$ such that $y_n - t \in [0, l_j]$ for all $t \in [0, t_0]$ and $n \in \mathbb{N}$. (Note that such a t_0 exists since $\hat{x} \in e_j$.) The functions $f_n(t) := u_n^j(y_n - t)$ satisfy relation (4.11) and

$$f_n''(t) = \varepsilon_n^{-1} H^j\big(y_n - t, f_n(t), -f_n'(t)\big) \quad \text{for all } t \in [0, t_0],\ n \in \mathbb{N}. \tag{4.16}$$

Fix θ and η as in (4.13); fix n sufficiently large to have $-f_n'(0) < -M_{\theta,\eta}$. By (4.6), (4.16) and Lemma 4.1, we have $f_n''(0) > \theta$. We obtain relation (4.15) and then we conclude the proof following the same arguments as before.

Case (c): $\hat{x} = v_i \in V$ *and* $\partial_j u_n(x_n) \to -\infty$. Wlog, we assume $a_{ij} = 1$ (recall that e_j is the edge containing all the x_n). Fix n sufficiently large to have

$$\partial_j u_n(x_n) < -\max\{C_2, C_3, \bar{C}\} \tag{4.17}$$

where C_2 and C_3 are respectively the constant introduced in (4.8) and in (4.9) while $\bar{C}$ is such that

$$H(x, -C_1, p) > 0 \quad \forall x \in \Gamma,\ |p| > \bar{C} \tag{4.18}$$

(assumption (2.6) ensures the existence of the constant $\bar{C}$). For each $n \in \mathbb{N}$, let $t_n \in (0, l_j)$ be such that $y_n - t \in [0, l_j]$ for all $t \in [0, t_n]$. Observe that in this case t_n depends on n and that $\pi_j(y_n - t_n) = v_i$. By assumption (2.4), for every $n \in \mathbb{N}$, the function $f_n(t) := u_n^j(y_n - t)$ satisfies relation (4.11) and also

$$f_n''(t) = \varepsilon_n^{-1} H^j\big(y_n - t, f_n(t), -f_n'(t)\big) \geqslant \varepsilon_n^{-1} H^j\big(y_n - t, -C_1, -f_n'(t)\big) \tag{4.19}$$

for every $t \in [0, t_n]$. Taking into account relations (4.6), (4.17), (4.18) and (4.19), we infer: $f_n''(0) > 0$. In fact, let us prove

$$f_n''(t) > 0 \quad \forall t \in [0, t_n]. \tag{4.20}$$

In order to prove this inequality, we introduce the set $A := \{t \in [0, t_n]\colon\ f_n''(t) \geqslant 0\}$ and the set A_0 as its connected component containing $t = 0$. Let $\bar{t}$ be the maximal point of A_0; for later contradiction, assume that $\bar{t} < t_n$. We observe that the function f_n' is increasing in $(0, \bar{t})$ and, by (4.17), $f_n'(0) > \max\{C_2, C_3, \bar{C}\}$. Hence, it follows that: $f_n'(\bar{t}) > \max\{C_2, C_3, \bar{C}\}$ and, by (4.19), $f_n''(\bar{t}) > 0$. A contradiction to the maximality of $\bar{t}$ is obtained so inequality (4.20) is completely proved.

Relations (4.17) and (4.20) entail

$$\partial_j u_n(v_i) = -f_n'(t_n) < -f'(0) = \partial_j u_n(x_n) < -\max\{C_2, C_3, \bar{C}\}$$

which contradicts the definition either of C_2 or of C_3.

Case (d): $\hat{x} = v_i \in V$ *and* $\partial_j u_n(x_n) \to +\infty$. In this case, it suffices to follow the same arguments of Step 2. □

Remark 4.1. This theorem applies to problem (3.6). In fact, a priori estimates for this problem could be obtained by [17, Thm2, App1]. However, for the sake of completeness, a direct proof has been given.

5. The vanishing viscosity limit

In this section we prove the vanishing viscosity result, i.e. the convergence of the solution of (4.1) to the one of (2.9). We observe that assumptions (2.7)–(2.8) are not necessary for (4.1) but they play a crucial role for the uniqueness of (2.9). Moreover the specific form of the Hamiltonian in (3.6) is only used to prove the existence of a solution, while a priori estimates in Section 4 and the convergence of the vanishing viscosity limit in this section hold for the more general class of Hamiltonians.

Theorem 5.1. *Assume that $H = (H^j)_{j \in J}$ satisfies (2.3)–(2.8). Let $u_n := u_{\varepsilon_n} \in C^2_{*,\beta}(\Gamma)$ be a sequence of solutions of (4.1) such that u_n and ∂u_n are uniformly bounded on Γ. If u_n converges uniformly to a function $u \in C(\Gamma)$, then u is a solution of (2.9).*

For the proof we need two lemmas: the former is an immediate consequence of (2.3)–(2.8) while the proof of the latter is postponed to Appendix A.

Lemma 5.1. *Under the hypotheses of Theorem 5.1, for $i \in I_T$, define a function $h_i : \mathbb{R} \to \mathbb{R}$ by $h_i(p) := H^j(v_i, 0, p)$, $j \in Inc_i$ (by (2.7) the definition is independent of j). Then, $h_i(0) = \min h_i$, h_i is symmetric and nondecreasing on $(0, +\infty)$. In particular, either it is strictly positive or there are two positive numbers $a_1 \leqslant a_2$, such that $\{h = 0\} = [a_1, a_2] \cup [-a_2, -a_1]$.*

Lemma 5.2. *Assume the hypotheses of Theorem 5.1. Let $i \in I_T$, $j \in Inc_i$ and $\xi > 0$. Furthermore let $x_m \in e_j$, $m \in \mathbb{N}$, such that $\lim_m x_m = v_i$. Then there is a number $m_\xi \in \mathbb{N}$ such that for all $m > m_\xi$*

$$H^j\left(v_i, u(v_i), \frac{u(x_m) - u(v_i)}{d(x_m, v_i)}\right) \leqslant \xi. \tag{5.1}$$

Proof of Theorem 5.1. *Step* 1: *u is a subsolution of* (2.9). For $x \in e_j$ (for some $j \in J$), the proof is standard and we skip it (see [2, Thm2.3]). Consider v_i, for some $i \in I_T$. Let $j, k \in Inc_i$, $j \neq k$ and let ϕ be a (j,k)-test function such that $u - \phi$ has a local maximum at v_i. We shall assume $u(v_i) = 0$; the general case can be dealt with by similar arguments and we shall omit it. Then

$$h\big(\partial_j\phi(v_i)\big) = H^j\big(v_i, u(v_i), \partial_j\phi(v_i)\big)$$

where $h = h_i$ as in Lemma 5.1. We claim that h is not strictly positive; actually, by contradiction, let us assume $h > 0$. In particular, we have $h(0) > 0$ and, by the continuity of H^j, we infer $H(x, u(x), 0) > 0$ in some $B_\eta(v_i)$. By Lemma 5.1, we get $H(x, u(x), \partial_j\phi(x)) > 0$ for every test function at some points in $B_\eta(v_i)$. This inequality contradicts that u is a subsolution in e_j.

We now want to prove: $h(\partial_j\phi(v_i)) \leqslant 0$; to this end, let us suppose by contradiction that $h(\partial_j\phi(v_i)) > 0$. Since ϕ is (j,k)-differentiable at v_i and therefore $a_{ij}\partial_j\phi(v_i) + a_{ik}\partial_k\phi(v_i) = 0$, for one of the indices j, k, say for j, there is a number $\delta_0 > 0$ such that

$$a_{ij}\partial_j\phi(v_i) = -(a_2 + \delta_0) \tag{5.2}$$

where a_2 is the positive constant defined in Lemma 5.1. Let x_m be a sequence with $x_m \in e_j$ with $\lim_{m\to\infty} x_m = v_i$. As $u - \phi$ attains a local maximum at v_i, by (5.2) we get

$$p_m := \frac{u(x_m) - u(v_i)}{d(x_m, v_i)} \leqslant \frac{\phi(x_m) - \phi(v_i)}{d(x_m, v_i)} < -\left(a_2 + \frac{\delta_0}{2}\right)$$

for m sufficiently large. By the properties of h it follows that there exists $\delta_1 > 0$ such that

$$\delta_1 < h(p_m) = H^j\big(x, u(x), p_m\big)$$

for m sufficiently large, a contradiction to Lemma 5.2. Hence

$$h\big(\partial_j\phi(v_i)\big) = H^j\big(v_i, u(v_i), \partial_j\phi(v_i)\big) \leqslant 0.$$

Step 2: *u is a supersolution of* (2.9). For $x \in e_j$ (for some $j \in J$), the proof is standard and we skip it (see [2, Thm2.3]). Assume that $x = v_i$, for some $i \in I_T$. The proof is based on the following lemma (the proof is in Appendix A).

Lemma 5.3. *Assume the hypotheses of Theorem 5.1. Let $i \in I_T$ and assume that, for $j \in Inc_i$, there holds $a_{ij}\partial_j u_n(v_i) \leqslant 0$ for infinitely many $n \in \mathbb{N}$. Furthermore assume that there is a function $\phi \in C^2(\Gamma)$ such that $u - \phi$ has a local minimum at v_i. Then $H^j(v_i, u(v_i), \partial^j\phi(v_i)) \geqslant 0$.*

Since u_n satisfies (2.2) at $x = v_i$, there is an index $j \in Inc_i$ such that

$$a_{ij}\partial_j u_n(v_i) \leqslant 0 \tag{5.3}$$

for infinite many $n \in \mathbb{N}$. We show that j is a k-feasible index for each $k \in Inc_i \setminus \{j\}$. We assume wlog that $a_{ij} = 1$ and we fix a (j,k)-test function ϕ such that $u - \phi$ has a strict minimum point at

$0 = \pi_j^{-1}(v_i)$ relatively to $\bar{e}_j \cup \bar{e}_k$. Let $\phi_m \in C^2([0, l_j])$ $(m \in \mathbb{N})$, be such that ϕ_m converges to ϕ with respect to the topology of $C^1([0, l_j])$. Let $z_m \in \bar{e}_j \cup \bar{e}_k$ be such that $u - \phi_m$ attains a local minimum with respect to $\bar{e}_j \cup \bar{e}_k$. Then, by standard arguments, the point z_m converges to x and either by the case $x \in e_j$ if $z_m \in e_j$ or by Lemma 5.3 if $z_m = v_i$, we conclude that

$$H^j\big(z_m, u(z_m), \partial_j \phi_m(z_m)\big) \geqslant 0.$$

Since $\lim_{m\to\infty} \partial_j \phi_m(z_m) = \partial_j \phi(v_i)$, we obtain

$$H^j\big(v_i, u(v_i), \partial_j \phi(v_i)\big) = \lim_{m\to\infty} H^j\big(z_m, u(z_m), \partial_j \phi_m(z_m)\big) \geqslant 0.$$

Hence j is i-feasible for k and by symmetry k is i-feasible for j at x. $\square$

5.1. Example: the Eikonal equation

We consider the Eikonal equation on the network Γ with null boundary condition

$$|\partial u| = f(x) \quad \text{on } \Gamma, \qquad u(v_i) = 0 \quad \forall i \in I_B \tag{5.4}$$

where f is a Lipschitz continuous function with $f \geqslant \alpha > 0$.

Fact 1. *There exists a unique viscosity solution u to (5.4).*

For the proof, we refer the reader to [20] (see also [5] for the generalization to LEP spaces); in fact, u can be written as a *weighted* distance from $\partial\Gamma$.

We observe that a function u solves (5.4) if and only if, it solves

$$|\partial u|^2 = f^2(x) \quad \text{on } \Gamma, \qquad u(v_i) = 0 \quad \forall i \in I_B. \tag{5.5}$$

For any collection $\beta = (\beta_{ij})$ $(i \in I_T,\ j \in Inc_i)$ with $\beta_{ij} > 0$, we introduce the viscous approximation to (5.5):

$$-\varepsilon \partial^2 u + |\partial u|^2 = f^2(x) \quad \text{on } \Gamma, \qquad S^i_\beta u = 0 \quad \forall i \in I_T, \qquad u(v_i) = 0 \quad \forall i \in I_B. \tag{5.6}$$

Fact 2. *By Theorem 3.3, there exists a unique classical solution u_ε to (5.6).*

Fact 3. *By Theorem 4.1, the functions u_ε are equibounded and equilipschitz continuous.*

Fact 4. *The sequence $\{u_\varepsilon\}$ uniformly converges to u.*

Actually, by Fact 3, Ascoli's Theorem ensures that there exists a subsequence $\{u_{\varepsilon_n}\}$ uniformly convergent to some function v. By Theorem 5.1, v is a solution to (5.5). By the uniqueness of the solution to (5.5), we deduce that the whole sequence $\{u_\varepsilon\}$ converges to the unique solution u to (5.5).

Appendix A

Proof of Lemma 3.1. Fix two functions $w_1, w_2 \in C^2(\Gamma)$ such that relations (3.8) hold. By the regularity of w_1, we can introduce $\tilde{H}^j(x,r,p) := H^j(x,r,p) + \varepsilon\partial_j^2 w_1 - H^j(x, w_1, \partial_j w_1)$. For $\bar{w}_\eta := w_2 + \phi_\eta$, assumptions (3.8) and (3.9) entail

$$-\varepsilon\partial_j^2\bar{w}_\eta + \tilde{H}^j(x, \bar{w}_\eta, \partial_j\bar{w}_\eta) \leqslant -\varepsilon\partial_j^2\phi_\eta + K\left(\left\|\phi_\eta^j\right\|_\infty + \left\|\partial_j\phi_\eta^j\right\|_\infty\right).$$

Therefore, it is enough to prove that, for every $\eta > 0$ there exists ϕ_η such that

$$\|\phi_\eta\|_\infty \leqslant \eta, \qquad S_\beta\phi_\eta > 0, \qquad -\varepsilon\partial_j^2\phi_\eta + K\left(\left\|\phi_\eta^j\right\|_\infty + \left\|\partial_j\phi_\eta^j\right\|_\infty\right) < 0. \tag{A.1}$$

Let $\delta : I \times I \to \mathbb{N}$ be the metric given by the smallest number $\delta(i,j)$ of the edges a path connecting v_i and v_j can consist of. It induces a partition $I_l := \{i \in I\colon\ \delta(i, I_B) = l\}$. Observe that $I_0 = I_B$ and set $m := \max\{l \in \mathbb{N}\colon\ I_l \neq \emptyset\}$.

For simplicity, we address only the case $m = 1$ with $l_j = l$ for $j \in J$; the general case can be dealt with in a similar manner and we shall omit it. In this case, each vertex belongs either to $\Gamma_0 := \{v_i\colon i \in I_0\}$ or to $\Gamma_1 := \{v_i\colon\ i \in I_1\}$; furthermore, each edge connects either two vertices in Γ_1 or a vertex in Γ_0 and one in Γ_1 (namely, it does not connect two vertices in Γ_0).

Let us enumerate the elements in Γ_1 as $\{v_{i_1}, \dots, v_{i_n}\}$. Wlog, we assume that: when e_j connects $v_{i_s}, v_{i_t} \in \Gamma_1$, with $1 \leqslant s < t \leqslant n$, its parametrization is $\pi_j(0) = v_{i_s}$, $\pi_j(l_j) = v_{i_t}$ while, for e_j connecting $v_{i_s} \in \Gamma_1$ and $v_k \in \Gamma_0$, its parametrization is $\pi_j(0) = v_{i_s}$, $\pi_j(l_j) = v_k$. Let us now define a function $\phi \in C(\Gamma)$ in the following manner: on the vertices, we set

$$\phi(v_{i_s}) := e^{2K(s-1)l\varepsilon^{-1}} \quad \forall v_{i_s} \in \Gamma_1, \qquad \phi(v_k) := e^{2K(n+1)^2\beta_0 l\varepsilon^{-1}} \quad \forall v_k \in \Gamma_0$$

with $\beta_0 := \max\beta_{ij}/\min\beta_{ij}$; moreover, on the edge e_j, we set

$$\phi^j(x) := e^{2K(s-1)l\varepsilon^{-1}} e^{2K(t-s)\varepsilon^{-1}x} \quad \text{if } e_j \text{ connects } v_{i_s} \text{ and } v_{i_t},\ s < t,$$
$$\phi^j(x) := e^{2K(s-1)l\varepsilon^{-1}} e^{2K[(n+1)^2\beta_0 - s + 1]\varepsilon^{-1}x} \quad \text{if } e_j \text{ connects } v_{i_s} \in \Gamma_1 \text{ and } v_k \in \Gamma_0.$$

One can easily check that, on each edge e_j, the last relation of (A.1) is satisfied. On the other hand, for $J_1 := \{j \in Inc_{i_s}\colon\ e_j \text{ connects } v_{i_s} \text{ with some } v_{i_t} \in \Gamma_1\}$ and $J_2 := \{j \in Inc_{i_s}\colon\ e_j \text{ connects } v_{i_s} \text{ with some } v_k \in \Gamma_0\}$, we have

$$S_\beta^{i_s}\phi = \sum_{j\in J_1}\beta_{i_s j}a_{i_s j}\partial_j\phi(v_{i_s}) + \sum_{j\in J_2}\beta_{i_s j}a_{i_s j}\partial_j\phi(v_{i_s}) \equiv S_1 + S_2.$$

Since $\#J_2 \geqslant 1$ and $a_{i_s j} = 1$ for $j \in J_2$, we infer

$$S_2 \geqslant 2K(\min\beta_{ij})e^{2K(s-1)l}\left[(n+1)^2\beta_0 - s + 1\right]\varepsilon^{-1} \geqslant 2K(\max\beta_{ij})e^{2K(s-1)l}\left[(n+1)^2 - n\right]\varepsilon^{-1}.$$

On the other hand, since $\#J_1 \leqslant n - 1$, we get

$$S_1 \geqslant -\sum_{t=1}^{n}\beta_{i_s j}e^{2K(s-1)l}(t-s)\varepsilon^{-1} \geqslant -K(\max\beta_{ij})e^{2K(s-1)l}(n+1)^2\varepsilon^{-1}.$$

Owing to the last three relations, we have $S_\beta^{i_s}\phi > 0$ for $s = 1, \dots, n$.

Finally, we observe that relations in (A.1) are linear; whence, the function $\phi_\eta := \eta \frac{\phi}{\|\phi\|}$ is a desired function. □

Proof of Lemma 4.1. Fix θ and η as in the statement. By (2.4), we have: $H^j(x, r, p) \geqslant H^j(x, \eta, p)$ for every $x \in e_j$, $r \geqslant \eta$, $p \in \mathbb{R}$, $j \in J$. By (2.6), there exists $M_{\theta,\eta} > 0$ such that: $H^j(x, \eta, p) > \theta$ for every $x \in e_j$, $r \geqslant \eta$, $|p| > M_{\theta,\eta}$, $j \in J$. Substituting the previous inequality in the last one, we accomplish the proof. □

Proof of Lemma 4.2. Define the set

$$M := \{\xi \in \mathbb{R}^I \colon \xi_i \neq \xi_j \text{ for all } i, j \in I \text{ with } v_i \operatorname{adj} v_j\}$$

and observe that there is an injective map $\Phi : M \to D$ with

$$D := \{\phi \in C^2(\Gamma) \colon \text{there exists } (\alpha_j)_{j \in J} \text{ s.t. } \alpha_j \neq 0 \text{ and } \partial_j \phi \equiv \alpha_j \text{ on } e_j,\ j \in J\}$$

such that $\Phi[\xi](v_i) = \xi_i$, $i \in I$. It suffices to show that there is a $\xi \in M$ such that $S^i_\beta(\Phi[\xi]) < 0$ for all $i \in I_T$. To this end, we define I_l and m as in the proof of Lemma 3.1 and, for $i \in I_T$, we introduce the map $T_i := S^i_\beta \circ \Phi$ which is: (a) continuous, unbounded and strictly decreasing in the component ξ_i, (b) continuous, unbounded and strictly increasing in each component ξ_j, $j \in A_i := \{j \in I \colon v_j \operatorname{adj} v_i\}$, (c) independent of ξ_j for any $j \in I \setminus (\{i\} \cup A_i)$.

Let us now construct $\xi \in M$ such that $T_i(\xi) < 0$ for all $i \in I_T$. We first choose $\xi \in M$. Let $i \in I_m$, by property (b) and by $I_{m-1} \cap A_i \neq \emptyset$, we may decrease the value of ξ_j, $j \in I_{m-1}$, such that we obtain $T_i(\xi) < 0$ for all $i \in I_m$ and such that ξ remains in M. Analogously, we can decrease ξ_j, $j \in I_{m-2}$, such that $T_i(\xi) < 0$ for all $i \in I_{m-1}$ and such that ξ_j, $j \in J$, remain pairwise different. For $k = 3, \ldots, m$ we continue this procedure by sufficiently decreasing ξ_j, $j \in I_{m-k}$, in order to ensure that $T_i(\xi) < 0$ for all $i \in I_{m-k+1}$, ending up with a choice for $\xi \in M$ such that $T_i(\xi) < 0$ for all $i \in \bigcup_{l=1}^m I_l = I_T$. □

Proof of Lemma 5.2. Let us recall that our hypotheses entail: $\|u_n\|_\infty \leqslant C_1$, $\|\partial u_n\|_\infty \leqslant C_2$, $\|u_n - u\|_\infty \to 0$, $\varepsilon_n \to 0$ and u is Lipschitz continuous with a Lipschitz constant not greater than C_2. For the sake of clarity, we split the proof in several steps.

Step 1. Fix $\varepsilon > 0$. We introduce N^x_θ as

$$H^j(x, u(x), p) > \theta \quad \forall p,\ |p| \geqslant N^x_\theta$$

and we observe that, by (2.5), for any $x \in \Gamma$, N^x_θ is nondecreasing in θ. Consider $\eta > 0$ such that, for n sufficiently large, there holds

$$|u_n(x) - u(y)| \leqslant 2C_1\eta \quad \forall x, y \in B_\eta(v_i), \qquad \omega(2C_1\eta) \leqslant \varepsilon/2,$$
$$|H^j(x, u_n(x), p) - H^j(v_i, u(v_i) - 2C_1\eta, p)| \leqslant \varepsilon/2 \quad \forall x \in B_\eta(v_i),\ |p| < C_2$$

where ω is the modulus of continuity of H on $\Gamma \times [-2C_1, 2C_1] \times [-2C_2, 2C_2]$. Fix $\bar{x} \in (0, \eta/2)$ and $\bar{\eta} < \bar{x}$; our claim is to prove that, there exists $N \in \mathbb{N}$ such that

$$H^j(v_i, u(v_i), \partial_j u_n(x)) < 2\varepsilon \quad \forall x \in (\bar{x} - \bar{\eta}, \bar{x} + \bar{\eta}),\ n > N. \tag{A.2}$$

In order to prove this relation, we proceed by contradiction assuming that (possibly passing to a subsequence) there is a sequence $x_n \in (\bar{x} - \bar{\eta}, \bar{x} + \bar{\eta})$ such that $H^j(v_i, u(v_i), \partial_j u_n(x_n)) \geqslant 2\varepsilon$ for every n. By assumption (2.4) and the equation in (4.1), for n sufficiently large, we deduce

$$\begin{aligned}\varepsilon_n \partial_j^2 u_n(x_n) &= H^j\big(x_n, u_n(x_n), \partial_j u_n(x_n)\big) \geqslant H^j\big(v_i, u(v_i) - 2C_1\eta, \partial_j u_n(x_n)\big) - \varepsilon/2 \\ &> H^j\big(v_i, u(v_i), \partial_j u_n(x_n)\big) - \omega(2C_1\eta) - \varepsilon/2 \geqslant \varepsilon.\end{aligned}$$

Therefore, we have: $u_n(x_n) \geqslant u(v_i) - 2C_1\eta$, $|\partial_j u_n(x_n)| \geqslant N_{2\varepsilon}^{v_i}$ and $\varepsilon_n \partial_j^2 u_n(x_n) > \varepsilon$. Assume $\partial_j u_n(x_n) \geqslant N_{2\varepsilon}^{v_i} > 0$. We claim that, for n sufficiently large (it suffices to have $\varepsilon_n < \varepsilon C_2^{-1}(\eta/2 - \bar{\eta})$), these inequalities still hold in $[x_n, \bar{x} + \eta/2]$, namely

$$u_n(y) \geqslant u(v_i) - 2C_1\eta, \qquad \partial_j u_n(y) \geqslant N_{2\varepsilon}^{v_i}, \qquad \varepsilon_n \partial_j^2 u_n(y) \geqslant \varepsilon \quad \forall y \in [x_n, \bar{x} + \eta/2]. \tag{A.3}$$

Indeed, let A and $\bar{t}$ be respectively the connect set containing x where they hold and its maximum point. If $\bar{t} < \bar{x} + \eta/2$, since u_n and ∂u_n are both strictly increasing on $[x_n, \bar{t}]$, we have $u_n(\bar{t}) > u(v_i) - 2C_1\eta$, $\partial_j u_n(\bar{t}) > N_{2\varepsilon}^{v_i}$; by (4.1), we get

$$\varepsilon_n \partial_j^2 u_n(\bar{t}) = H^j\big(\bar{t}, u_n(\bar{t}), \partial_j u_n(\bar{t})\big) \geqslant H^j\big(v_i, u(v_i) - 2C_1\eta, \partial_j u_n(\bar{t})\big) - \varepsilon/2 > \varepsilon.$$

Hence by continuity there is a neighborhood of $\bar{t}$ contained in A; this fact contradicts the definition of $\bar{t}$. Claim (A.3) is completely proved.

Relations (A.3) and our choice of ε_n ensure the following relation

$$\partial_j u_n(\bar{x} + \eta/2) \geqslant \partial_j u_n(x_n) + \varepsilon\varepsilon_n^{-1}(\bar{x} + \eta/2 - x_n) > C_2$$

which contradicts our bound on ∂u_n.

Assume $\partial_j u_n(x_n) \leqslant -N_{2\varepsilon}^{v_i} < 0$. In this case, arguing as before, we get $u_n(y) \geqslant u(v_i) - 2C_1\eta$, $\partial_j u_n(y) \leqslant -N_{2\varepsilon}^{v_i}$ and $\varepsilon_n \partial_j^2 u_n(y) \geqslant \varepsilon$ for any $y \in [0, x_n]$. For n so large to have $\varepsilon_n \leqslant \varepsilon C_2^{-1}(\bar{x} - \bar{\eta})$, we infer $\partial_j u_n(0) \leqslant \partial_j u_n(x) - \varepsilon\varepsilon_n^{-1}(x_n) < -C_2$ which contradicts our bound on ∂u_n. Hence, we get: $H^j(x, u_n(x), \partial_j u_n(x)) \leqslant 2\varepsilon$.

Step 2. Assume wlog $a_{ij} = 1$. The aim is to prove that, for each $\xi > 0$, there exists $\eta > 0$ such that

$$H^j\left(v_i, u(v_i), \frac{u^j(y) - u^j(0)}{y}\right) \leqslant \xi \quad \forall y \in (0, \eta). \tag{A.4}$$

In order to prove this relation, for each $\varepsilon > 0$, consider η as before. Fix $y \in (0, \eta/2]$ and $x := y\varepsilon C_2^{-1}$. By the Lipschitz continuity of u, our choice of x and the uniform convergence, for n sufficiently large, we infer

$$\begin{aligned}&\left|\frac{u^j(y) - u^j(0)}{y} - \frac{u_n^j(y) - u_n^j(x)}{y - x}\right| \\ &\quad \leqslant \left|\frac{u^j(y) - u^j(0)}{y} - \frac{u^j(y) - u^j(x)}{y}\right| + \left|\frac{u^j(y) - u^j(x)}{y} - \frac{u_n^j(y) - u_n^j(x)}{y - x}\right| \\ &\quad \leqslant C_2\frac{x}{y} + \left|\frac{(u^j(y) - u_n^j(y)) - (u^j(x) - u_n^j(x))}{y - x} - \frac{x}{y} \cdot \frac{u^j(y) - u^j(x)}{y - x}\right| \\ &\quad \leqslant 2C_2\frac{x}{y} + 2\frac{\|u_n - u\|_\infty}{y - x} \leqslant 2\varepsilon + 2\frac{\|u_n - u\|_\infty}{y - x}.\end{aligned}$$

By the mean value theorem, we deduce for any $n \in \mathbb{N}$

$$
\begin{aligned}
&H^j\left(v_i, u^j(v_i), \frac{u^j(y)-u^j(0)}{y}\right)\\
&\quad\leqslant H^j\left(v_i, u^j(v_i), \frac{u_n^j(y)-u_n^j(x)}{y-x}\right)+\omega\left(2\frac{\|u_n-u\|_\infty}{|y-x|}+2\varepsilon\right)\\
&\quad\leqslant H^j\big(v_i, u^j(v_i), \partial_j u_n\big(x_n'\big)\big)+\omega\left(2\frac{\|u_n-u\|_\infty}{|y-x|}+2\varepsilon\right)
\end{aligned}
$$

for some $x_n' \in (x, y)$, (recall that ω is the modulus of continuity of H on $\Gamma \times [-2C_1, 2C_1] \times [-2C_2, 2C_2]$). Letting $n \to +\infty$, by Step 1 and our choice of x, we infer

$$
H^j\left(v_i, u^j(v_i), \frac{u^j(y)-u^j(0)}{y}\right) \leqslant 2\varepsilon + \omega(\varepsilon).
$$

In conclusion, it suffices to choose ε such that $2\varepsilon + \omega(\varepsilon) < \xi$. $\square$

Proof of Lemma 5.3. Wlog, we assume that $u(v_i) = \phi(v_i) = 0$ and $a_{ij} = 1$. By the assumptions, we can choose a subsequence of $(u_n)_{n\in\mathbb{N}}$ (still denoted by $(u_n)_{n\in\mathbb{N}}$) such that $\partial_j u_n(v_i) \leqslant 0$ for all $n \in \mathbb{N}$. Our aim is to prove that

$$
h\big(\partial\phi(x)\big) := h_i\big(\partial\phi(x)\big) \geqslant 0
$$

where h_i is the function introduced in Lemma 5.1. For $h(p) \geqslant 0$ for every p, there is nothing to prove. By Lemma 5.1, let us assume that there exists $a > 0$ such that $\{h < 0\} = (-a, a)$. We want to show that $\partial_j\phi(v_i) \leqslant -a$. To this end we assume the contrary, i.e. there is $\delta \in (0, 2a)$ such that

$$
\partial_j\phi(v_i) = -a + \delta \tag{A.5}
$$

and we set $H^j(v_i, u(v_i), \partial_j\phi(v_i)) = -\alpha < 0$. We claim that for $n \in \mathbb{N}$ sufficiently large, there is $r_n > 0$ such that

$$
u_n^j(x) < u_n^j(0), \qquad \partial^j u_n(x) < 0 \quad \text{for } x \in (0, r_n]. \tag{A.6}
$$

This is clear if $\partial_j u_n(v_i) < 0$. Assume $\partial_j u_n(v_i) = 0$. In order to prove (A.6), it is enough to prove that, for n sufficiently large, there exists $r_n > 0$ such that

$$
\partial_j^2 u_n(x) < -\alpha/2 \quad \forall x \in (0, r_n].
$$

To this end, we argue by contradiction and we assume that there exists a sequence $x_m \in e_j$, with $x_m \to v_i$ as $m \to +\infty$, such that $\partial_j^2 u_n(x_m) > -\alpha/2$. The continuity of $\partial_j u_n$ ensures: $\partial_j u_n(x_m) \to 0$ as $m \to +\infty$. Moreover, we have

$$
\begin{aligned}
\varepsilon_n\partial_j^2 u_n(x_m) &= H^j\big(x_m, u_n(x_m), \partial_j u_n(x_m)\big)\\
&= H^j\big(v_i, u(v_i), 0\big) + \omega\big(|x_m - v_i| + \big|u_n(x_m) - u(v_i)\big| + \big|\partial_j u_n(x_m)\big|\big)
\end{aligned}
$$

where ω is the modulus of continuity of H in $\Gamma \times [-C, C] \times [-C, C]$ and C is a constant such that $\|u_n\|_\infty, \|\partial u_n\|_\infty \leqslant C$ (its existence is ensured by the hypotheses of Theorem 5.1). Owing to its monotonicity in $|p|$, H^j fulfills

$$
H^j\big(v_i, u(v_i), 0\big) \leqslant H^j\big(v_i, u(v_i), \partial_j\phi(v_i)\big) \leqslant -\alpha.
$$

Taking into account the last two relations, we infer

$$\varepsilon_n \partial_j^2 u_n(x_m) \leqslant -\alpha + \omega\big(|x_m - v_i| + \big|u_n(x_m) - u(v_i)\big| + \big|\partial_j u_n(x_m)\big|\big)$$

which gives the desired contradiction for m sufficiently large; hence, (A.6) is proved.

Let us now show that there exists $r > 0$ such that, for n sufficiently large, u_n^j cannot obtain a local minimum in $(0, r]$. In fact, if u_n^j has a minimum at x, we get by (2.3) and the uniform bound on $\partial_j u_n$

$$\begin{aligned}0 \leqslant \varepsilon_n \partial_j^2 u_n(x) &= H^j\big(x, u_n^j(x), 0\big) \leqslant H^j\big(v_i, u_n^j(v_i), 0\big) + \omega\big(|v_i - x|(1+C)\big)\\ &\leqslant -\alpha + \omega\big(|v_i - x|(1+C)\big) < 0\end{aligned}$$

for n sufficiently large and $|v_i - x|$ small, hence a contradiction. Therefore $u_n^j(x) \leqslant u_n^j(0)$ for $x \in [0, r]$. It follows that

$$u^j(y) = \lim_{n\to\infty} u_n^j(y) \leqslant \lim_{n\to\infty} u_n^j(0) = u(v_i) = 0 \quad \forall y \in [0, r] \tag{A.7}$$

namely, u^j attains in $0 = \pi_j^{-1}(v_i)$ its maximum with respect to $[0, r]$. Since $u - \phi$ attains a local minimum at v_i, (A.7) implies that we may restrict to consider the case $\delta \leqslant a$ in (A.5).

By the continuity of H^j, (2.4) and Lemma 4.1, it follows that there are $\eta, \gamma > 0$ with $\eta < \min\{\delta, r\}$ such that

$$H^j(x, z, p) \leqslant -\gamma \quad \text{for all } p \in [-\beta, 0],\ z \in (-\infty, \eta] \text{ and } x \in [0, \eta] \tag{A.8}$$

where $\beta := a - \delta + \eta$. Choose n_0 such that $\varepsilon_{n_0}\beta/\gamma < \eta$ and $u_n^j(0) < \eta$ for all $n \geqslant n_0$. For $n \geqslant n_0$ set $v_n(x) := \partial_j u_n(x)$ for $x \in (0, r)$. By (4.1), (A.6), $u_n^j(x) \leqslant u_n^j(0)$ for $x \in [0, r]$ and (A.8), we get

$$\partial_j v_n(x) = H^j\big(x, u_n(x), v_n(x)\big)/\varepsilon_n \leqslant -\gamma/\varepsilon_n \tag{A.9}$$

for all $x \in [0, \eta)$ and $-\beta \leqslant v_n(x) \leqslant 0$. Moreover, since $u - \phi$ attains a minimum at v_i, by relation (A.5), we infer $-\beta \leqslant v_n(0) \leqslant 0$ for n sufficiently large. In particular, owing to (A.9), we derive that there is x_n with

$$0 \leqslant x_n \leqslant \varepsilon_n \beta/\gamma \leqslant \varepsilon_{n_0}\beta/\gamma < \eta \tag{A.10}$$

such that $v_n(x_n) = -\beta$. We furthermore claim that

$$v_n(x) \leqslant -\beta \quad \text{for all } x_n < x \leqslant \eta. \tag{A.11}$$

Actually, if the claim were not true, there would be x_0 with $x_n < x_0 < \eta$ such that $v_n(x_0) \geqslant -\beta$ and $\partial_j v_n(x_0) \geqslant 0$. This contradicts (A.9).

Now, (A.11) and $u_n^j(x) \leqslant u_n^j(0)$ for $x \in e_j$, $n \in \mathbb{N}$ imply

$$u_n^j(y) = u_n^j(x_n) + \int_{x_n}^{y} v_n(s)\,ds \leqslant u_n^j(x_n) - \beta(y - x_n) \leqslant u_n^j(0) - \beta(y - x_n)$$

for all y with $x_n \leqslant y \leqslant \eta$. Using (A.10) we conclude

$$u^j(y) = \lim_{n\to\infty} u_n^j(y) \leqslant -y\beta = y(-a+\delta-\eta) \quad \forall y \in [0,\eta].$$

As $u^j - \phi^j$ has a local minimum at $0 = \pi_j^{-1}(v_i)$, it follows that there is $\rho > 0$ such that $\phi^j(y) \leqslant y(-a+\delta-\eta)$ for all $0 \leqslant y \leqslant \rho$, a contradiction to (A.5). $\square$

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