Topics in algebra, geometry and differential equations

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DOI:
10.33612/diss.225415066

Publication date:
2022

Propositions

belonging to the thesis

Topics in algebra, geometry and differential equations

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1. Differential equations of the form $P(u, u') = 0$, where $P \in C[X, Y]$ is an irreducible polynomial involving both variables and C is an algebraically closed field of constants of characteristic 0, naturally fall into four classes. For the classes of exact, exponential type and Weierstrass type equations, algebraic relations between non-constant solutions are common, while for the class of general type equations, such relations are rare. Moreover, algebraic relations between non-constant solutions of different, pairwise non-isogenous differential equations of general type do not occur.

2. The study of differential equations of the form indicated in Proposition 1 is greatly aided by tools from algebraic geometry – in particular, the theory of differential forms and generalized Jacobian varieties attached to smooth curves.

3. Given a field C of characteristic $p > 0$, a formal group law $F \in C[[X, Y]]$ of height ≥ 2, a separable extension $K/C(t)$ and a C-linear derivation $D\colon K \to K$ such that $D^p = 0$, there exists an F-iterative derivation $\partial\colon K \to K[[T]]$ such that $\partial_1 = D$. Moreover, the components of such a derivation can be explicitly constructed.

4. The Zassenhaus formula, which describes for non-commuting variables A and B the expression $(A + B)^n$ in terms of iterated commutators $\mathrm{ad}_A^n(B)$ (where $\mathrm{ad}_X(Y) = [X, Y] = XY - YX$), can be regarded as a special case, corresponding to the formal group law of the additive group, of a more general formula attached to any formal group law in characteristic 0.

5. Let $G \subseteq K[[t_1, \ldots, t_m]]\{x_1, \ldots, x_n\}$ be a differential ideal, where K is an uncountable algebraically closed field of characteristic 0. The support of any power series solution of G is constrained by the need for cancellations to occur. If for a potential support these constraints are satisfied, then a power series solution with that support indeed exists. These cancellation constraints can be equivalently formulated in terms of monomial-freeness of certain initial ideals. This leads to a "tropical" description of the set of supports of power series solutions of G.

6. The tropical description of the set of supports of power series solutions in Proposition 5 does not hold in general when $K = \overline{\mathbb{Q}}$, nor when one replaces power series with Laurent series or Puiseux series.

7. Siegel's theorem on the finiteness of the set of integral points on an elliptic curve can be reproven using the method of Lawrence and Venkatesh. For this purpose one can use an explicitly constructed finite-by-abelian cover of a punctured elliptic curve obtained from the Legendre family of elliptic curves over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

8. While proving theorems in the most general setting formally provides the most powerful versions of these theorems, such generality comes at a practical cost when readers need to invest a large amount of time and effort in order to understand the theory. A researcher interested in producing results that are useful to the research community at large should carefully weigh the advantages of a more general setting against this cost.

Propositions 1 and 2 are based on joined work with Marius van der Put and Jaap Top. Propositions 5 and 6 are based on joined work with François Boulier, Sebastian Falkensteiner, Cristhian Garay-López, Mercedes Haiech and Zeinab Toghani.