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Diejen, Jan Felipe van; Görbe, Tamás

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Elliptic Racah polynomials

Jan Felipe van Diejen[1] · Tamás Görbe[2]

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Abstract
Upon solving a finite discrete reduction in the difference Heun equation, we arrive at an elliptic generalization of the Racah polynomials. We exhibit the three-term recurrence relation and the orthogonality relations for these elliptic Racah polynomials. The well-known q-Racah polynomials of Askey and Wilson are recovered as a trigonometric limit.

Keywords Difference Heun equation · Racah polynomials · Exactly solvable quantum models · Eigenfunctions · Tridiagonal matrices

Mathematics Subject Classification Primary 42C05; Secondary 33C47 · 33E10 · 47B36 · 81Q80

1 Introduction

The Askey–Wilson polynomials [2] constitute a master family from which all other members listed in Askey's celebrated scheme of (basic) hypergeometric orthogonal polynomials can be recovered via parameter specializations and limit transitions [13]. In particular, for parameters subject to a suitable truncation condition the Askey–Wilson polynomials reduce to q-Racah polynomials [1], a finite-dimensional discrete orthogonal family that is known to express the $6j$ symbols associated with the $SL_q(2)$ quantum group [12]. In the limit $q \to 1$, this reproduces a previously observed interpretation of the classical $6j$ symbols for the Lie group $SL(2)$ in terms of a hyper-

✉ Jan Felipe van Diejen
diejen@inst-mat.utalca.cl

Tamás Görbe
t.f.gorbe@rug.nl

1 Instituto de Matemáticas, Universidad de Talca, Casilla 747, Talca, Chile

2 Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, PO Box 407, 9700 AK Groningen, The Netherlands

geometric orthogonal family known as Racah polynomials, which arises similarly as a finite discrete truncation of Wilson's master family of hypergeometric orthogonal polynomials.

A remarkable elliptic hypergeometric generalization of the $6j$ symbols originating from the Yang–Baxter equation for exactly solvable lattice models [7, 8] has been identified and studied by Frenkel and Turaev [9]. It was pointed out by Spiridonov and Zhedanov [31, 32] that rather than expressing orthogonal polynomials, these elliptic $6j$ symbols constitute in fact an elliptic hypergeometric counterpart of biorthogonal rational functions that had been found previously at the basic hypergeometric level by Wilson as a (non-polynomial) generalization of the q-Racah polynomials [38]. From the point of view of representation theory, the elliptic hypergeometric biorthogonal rational functions in question arise, respectively, as matrix elements for (co)representations of an elliptic quantum group associated with $U(2)$ [14] or as a transition matrix between two different solutions for generalized eigenvalue problems in a finite-dimensional representation of the Sklyanin algebra [26, 28]. A corresponding extension of the Askey scheme to the case of (basic) hypergeometric biorthogonal rational functions has been worked out in [39, 40].

The hallmark duality symmetry [1, 18] between the orthogonality relations and the dual orthogonality relations for the (q-)Racah polynomials and the corresponding $6j$ symbols is known to persist at the level of the elliptic hypergeometric biorthogonal rational functions and the elliptic $6j$ symbols [31] (cf. also [14]). The purpose of the present note, however, is to point out an elliptic generalization of the (q-)Racah orthogonal polynomials that avoids the transition to biorthogonal rational functions, at the expense of sacrificing this manifest duality symmetry. To this end, we start from a difference Heun equation that is obtained from the eigenvalue problem for a quantum Ruijsenaars–Schneider-type particle Hamiltonian introduced in [41] (cf. also [15, 16] for a proof of the integrability), upon specializing to the case of just a single particle. Systematic studies of the solutions of this difference Heun equation were performed in [6] for integral values of the (coupling) parameters and in [27] for parameters pertaining to a much larger domain of orthogonality. Particular solutions for special parameter instances of the difference Heun equation can be found in [36] (within the framework of the finite-gap integration of soliton equations) and in [29, 30] (through elliptic hypergeometry). Moreover, the difference Heun equation arises in the context of the representation theory of the Sklyanin algebra [24, 25, 30], as a linear problem associated with the elliptic Painlevé VI equation [21], and it turns out to describe the introduction of surface defects to the index computation of certain four-dimensional compactifications of the six-dimensional E string theory on a Riemann surface [20].

The difference Heun equation admits a rich hierarchy of degenerations generalizing the Askey scheme (cf. [42]), the solutions of which are currently under active investigation [3–5, 34, 35, 37]. In this same spirit, we will introduce below a finite-dimensional reduction in the difference Heun equation that is obtained by means of a truncation procedure that should be viewed as an elliptic counterpart of the truncation yielding the q-Racah polynomials from the Askey–Wilson polynomials. We thus end up with a finite discrete Heun equation describing the eigenvalue problem for a finite-dimensional tridiagonal matrix with explicit entries given by theta functions. By means of standard techniques from the theory of tridiagonal matrices, we will solve the corresponding spectral problem in terms of an orthogonal system of discrete Heun functions given by an elliptic generalization of the (q-)Racah polynomials. This elliptic Racah polynomial is defined by a tridiagonal determinant giving rise to a three-term recurrence relation and explicit orthogonality relations determined by the Christoffel–Darboux identities. In the trigonometric limit, one recovers the q-Racah polynomial of Askey and Wilson.

Let us now outline the precise layout of this note. In Sect. 2, we recall the definition of the difference Heun equation; by implementing a truncation condition on the parameters the finite discrete Heun equation is introduced. This finite discrete Heun equation encodes the eigenvalue problem for a finite-dimensional tridiagonal matrix with simple spectrum. In Sect. 3, we construct the corresponding eigenvectors, which entails the elliptic Racah polynomials together with their three-term recurrence relation and orthogonality relations. In Sect. 4, it is verified that in the trigonometric limit, the elliptic Racah polynomials recuperate the q-Racah polynomials together with the corresponding recurrence relation and orthogonality relations. We also point out a Lamé type parameter reduction in the elliptic Racah polynomials that diagonalizes a recently found elliptic generalization of the Kac–Sylvester matrix [43]. This is a discrete elliptic counterpart of a well-known parameter reduction retrieving Rogers' q-ultraspherical polynomials from the Askey–Wilson polynomials [13]. The main text exploits some standard formulas involving the construction of eigenvectors for tridiagonal matrices via the theory of orthogonal polynomials. For the reader's convenience, the pertinent formulas are recalled in Appendix A at the end.

2 Finite discrete Heun equation

2.1 Difference Heun equation

The difference Heun equation is an eigenvalue equation for a complex function $f(z)$:

$$Hf = \mathrm{E} f, \tag{2.1a}$$

which is determined by a linear second-order difference operator of the form

$$(Hf)(z) = \mathrm{A}(z) f(z+1) + \mathrm{A}(-z) f(z-1) + \mathrm{B}(z) f(z) \tag{2.1b}$$

and a spectral parameter $\mathrm{E} \in \mathbb{C}$; notice that we have scaled the independent variable z such that the steps of the difference equation take unit values. The coefficients $\mathrm{A}(z)$ and $\mathrm{B}(z)$ denote meromorphic functions that are given explicitly by

$$\begin{aligned} \mathrm{A}(z) &= \prod_{1\le r\le 4} \frac{[z+u_r]_r}{[z]_r} \frac{[z+\frac{1}{2}+v_r]_r}{[z+\frac{1}{2}]_r}, \\ \mathrm{B}(z) &= \sum_{1\le r\le 4} c_r \frac{[z+\frac{1}{2}+u]_r}{[z+\frac{1}{2}]_r} \frac{[z-\frac{1}{2}-u]_r}{[z-\frac{1}{2}]_r}, \end{aligned} \tag{2.1c}$$

with

$$c_r = \frac{2}{[u]_1[u+1]_1} \prod_{1\le s\le 4} \left[u_{\pi_r(s)} - \frac{1}{2}\right]_s [v_{\pi_r(s)}]_s. \tag{2.1d}$$

In these formulas, $\pi_1, \ldots, \pi_4$ stand for permutations that act on (the indices of) the parameters representing translations over the half-periods of the elliptic functions: $\pi_1 = \mathrm{id}, \pi_2 = (12)(34), \pi_3 = (13)(24), \pi_4 = (14)(23)$. Moreover, we have employed the following rescaled and normalized variants:

$$[z]_1 = \frac{\theta_1(\frac{\alpha}{2}z)}{\frac{\alpha}{2}\theta_1'(0)}, \quad [z]_2 = \frac{\theta_2(\frac{\alpha}{2}z)}{\theta_2(0)}, \quad [z]_3 = \frac{\theta_3(\frac{\alpha}{2}z)}{\theta_3(0)}, \quad [z]_4 = \frac{\theta_4(\frac{\alpha}{2}z)}{\theta_4(0)}, \tag{2.2}$$

of the Jacobi theta functions

$$\begin{aligned}
\theta_1(z) = \theta_1(z;p) &= 2\sum_{n=0}^{\infty}(-1)^n p^{(n+\frac{1}{2})^2}\sin(2n+1)z \\
&= 2p^{1/4}\sin(z)\prod_{n=1}^{\infty}(1-p^{2n})(1-2p^{2n}\cos(2z)+p^{4n}), \\
\theta_2(z) = \theta_2(z;p) &= 2\sum_{n=0}^{\infty} p^{(n+\frac{1}{2})^2}\cos(2n+1)z \\
&= 2p^{1/4}\cos(z)\prod_{n=1}^{\infty}(1-p^{2n})(1+2p^{2n}\cos(2z)+p^{4n}), \\
\theta_3(z) = \theta_3(z;p) &= 1+2\sum_{n=1}^{\infty} p^{n^2}\cos(2nz) \\
&= \prod_{n=1}^{\infty}(1-p^{2n})(1+2p^{2n-1}\cos(2z)+p^{4n-2}), \\
\theta_4(z) = \theta_4(z;p) &= 1+2\sum_{n=1}^{\infty}(-1)^n p^{n^2}\cos(2nz) \\
&= \prod_{n=1}^{\infty}(1-p^{2n})(1-2p^{2n-1}\cos(2z)+p^{4n-2}),
\end{aligned}$$

where $0 < p < 1$ stands for the elliptic nome and the scaling parameter $\alpha > 0$ regulates the real period $\frac{2\pi}{\alpha}$ of the coefficients of H. The difference Heun equation depends on eight coupling parameters $u_1, \ldots, u_4, v_1, \ldots, v_4$ and a virtual regularization parameter u; this last parameter merely shifts the spectrum of H (because the elliptic function $\mathrm{B}(z)$ has only simple poles with positions and residues that do not depend on u).

The difference Heun operator H (2.1b)–(2.1d) goes back to a difference operator introduced in [41, Eqs. (4.1)–(4.3)] (upon specialization to the case $n = 1$). Notice that for a precise comparison between the above formulas and those in [41], it is needed to pass from Jacobi theta functions to Weierstrass sigma functions associated with the period lattice $\Omega = 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$ (cf. e.g., [17, Chapter 6]):

$$[z]_r = \sigma_{r-1}(z)e^{-\frac{\alpha\eta_1}{2\pi}z^2}, \quad r = 1, \ldots, 4, \tag{2.3}$$

where $\omega_1 = \frac{\pi}{\alpha}$, $p = e^{i\pi\tau}$ with $\tau = \omega_3/\omega_1$, $\omega_3 = -\omega_1 - \omega_2$ and

$$\sigma_0(z) = \sigma(z), \quad \sigma_s(z) = e^{-\eta_s z}\frac{\sigma(z + \omega_s)}{\sigma(\omega_s)} \quad \text{with } \eta_s = \zeta(\omega_s) \quad s = 1, 2, 3.$$

Here, $\sigma(z)$ and $\zeta(z) = \sigma'(z)/\sigma(z)$ stand for the Weierstrass sigma and zeta functions, respectively. Indeed, it is readily seen by means of the relation in Eq. (2.3) that—upon conjugation with a Gaussian and multiplication by an overall constant—our difference Heun operator H can be converted into a difference operator $\hat{H}$ of the same form as in Eqs. (2.1b)–(2.1d) but with all rescaled theta functions $[\cdot]_r$ being replaced by sigma functions $\sigma_{r-1}(\cdot)$ $(r = 1, \ldots, 4)$:

$$\hat{H} = e^{-a+b}e^{-az^2}He^{az^2},$$

where $a = \frac{\alpha\eta_1}{2\pi}\sum_{1\leq r\leq 4}(u_r + v_r)$ and $b = \frac{\alpha\eta_1}{2\pi}\sum_{1\leq r\leq 4}(u_r^2 + v_r^2 + v_r)$, i.e.,

$$H \to \hat{H} \iff [z]_r \to \sigma_{r-1}(z).$$

The gauged and normalized difference Heun operator $\hat{H}$ thus obtained coincides therefore with the difference operator in [41, Eqs. (4.1)–(4.3)] (with $n = 1$, $\beta\hbar = 1/i$, $\gamma = 1/2$, $\mu = -u$, and $\mu_{r-1} = u_r$, $\mu'_{r-1} = v_r$ for $r = 1, \ldots, 4$).

For the case of nonpositive integral values of the coupling parameters $u_1, \ldots, u_4$ and $v_1, \ldots, v_4$, eigenfunctions for $\hat{H}$ were computed in [6]. A more general construction of the difference Heun eigenfunctions covering a much larger domain of parameter values can be found in [27].

2.2 Finite-dimensional reduction

From now on, we will pick real-valued coupling parameters $u_1, \ldots u_4, v_1, \ldots v_4$ from the domain

$$\boxed{u_r > 0,\ |v_r| < u_r + \frac{1}{2}\ (r = 1, 2) \quad \text{and} \quad u_r, v_r \in \mathbb{R}\ (r = 3, 4),} \tag{2.4a}$$

while throughout it will be assumed that the virtual parameter u is chosen in $\mathbb{R}$ such that $u, u + 1 \neq 0 \mod \frac{2\pi}{\alpha}\mathbb{Z}$. To truncate the difference Heun equation, we adjust the real period $\frac{2\pi}{\alpha}$ in terms of the coupling parameters in the following way:

$$\boxed{\alpha = \frac{\pi}{u_1 + u_2 + \textsc{m}} \quad \text{with } \textsc{m} \in \mathbb{N}} \tag{2.4b}$$

(so $u_1 + u_2 + \textsc{m} = \frac{\pi}{\alpha}$). Indeed, the conditions on the parameters ensure that the $(\textsc{m} + 1)$-dimensional space of functions $f : \Lambda_\textsc{m} \to \mathbb{C}$ over the shifted finite integer lattice

$$\Lambda_\textsc{m} = \{u_1, u_1 + 1, u_1 + 2, \ldots, u_1 + \textsc{m}\}$$

is stable for the action of the difference operator H (2.1b)–(2.1d), because $\textsc{a}(-u_1) = \textsc{a}(u_1 + \textsc{m}) = 0$ in view of the zeros of $[z]_1$ and $[z]_2$ at $z = 0$ and $z = \frac{\pi}{\alpha}$, respectively. This gives rise to the following finite-dimensional reduction in the difference Heun equation:

$$\tilde{a}_{\textsc{m}-k} f_{k+1} + a_k f_{k-1} + b_k f_k = \textsc{e} f_k \tag{2.5a}$$

for $k = 0, 1, \ldots, \textsc{m}$, where $f_k = f(u_1 + k)$ and

$$\tilde{a}_k = \textsc{a}(u_1 + \textsc{m} - k), \quad a_k = \textsc{a}(-u_1 - k), \quad b_k = \textsc{b}(u_1 + k) \tag{2.5b}$$

(so $\tilde{a}_0 = a_0 = 0$).

It is helpful to write out the coefficients in question explicitly:

$$a_k = \prod_{1 \le r \le 4} \frac{[u_1 - u_r + k]_r}{[u_1 + k]_r} \frac{[u_1 - v_r - \frac{1}{2} + k]_r}{[u_1 - \frac{1}{2} + k]_r} \tag{2.6a}$$

(because $[-z]_1 = -[z]_1$ and $[-z]_r = [z]_r$ if $r \neq 1$),

$$\tilde{a}_k = \prod_{1 \le r \le 4} \frac{[u_2 - u_{\pi_2(r)} + k]_r}{[u_2 + k]_r} \frac{[u_2 - v_{\pi_2(r)} - \frac{1}{2} + k]_r}{[u_2 - \frac{1}{2} + k]_r} = \pi_2(a_k) \tag{2.6b}$$

(because $[z + \frac{\pi}{\alpha}]_r = [-z]_{\pi_2(r)}$), and similarly

$$b_k = \sum_{1 \le r \le 4} c_r \frac{[u_1 + k + \frac{1}{2} + u]_r}{[u_1 + k + \frac{1}{2}]_r} \frac{[u_1 + k - \frac{1}{2} - u]_r}{[u_1 + k - \frac{1}{2}]_r}, \tag{2.6c}$$

so

$$b_{\textsc{m}-k} = \sum_{1 \le r \le 4} c_{\pi_2(r)} \frac{[u_2 + k + \frac{1}{2} + u]_r}{[u_2 + k + \frac{1}{2}]_r} \frac{[u_2 + k - \frac{1}{2} - u]_r}{[u_2 + k - \frac{1}{2}]_r} = \pi_2(b_k) \tag{2.6d}$$

(because $\pi_r \circ \pi_s = \pi_{\pi_r(s)}$), where π_r is understood to act on the coefficients a_k and b_k by permuting the parameters: $\pi_r(u_s) = u_{\pi_r(s)}$ and $\pi_r(v_s) = v_{\pi_r(s)}$. With the aid of these formulas, one readily checks the positivity of the off-diagonal coefficients in the finite discrete Heun equation.

Lemma 2.1 (Positivity) *The off-diagonal coefficients* $a_1, \ldots, a_{\mathrm{M}}$ *and* $\tilde{a}_1, \ldots, \tilde{a}_{\mathrm{M}}$ *of the finite discrete Heun equation (2.5a), (2.5b) are all positive.*

Proof Since $\tilde{a}_k = \pi_2(a_k)$ (cf. Eq. (2.6b)) and the parameter restrictions in Eqs. (2.4a), (2.4b) are invariant with respect to the action of π_2, it suffices to verify the positivity of a_k. To this end, we observe from the product expansions for the theta functions that the sign of a_k (2.6a) coincides with the overall sign of the principal trigonometric factor:

$$\frac{\sin(\frac{\alpha}{2}k)}{\sin(\frac{\alpha}{2}(u_1+k))}\,\frac{\cos(\frac{\alpha}{2}(u_1-u_2+k))}{\cos(\frac{\alpha}{2}(u_1+k))}\,\frac{\sin(\frac{\alpha}{2}(u_1-v_1-\frac{1}{2}+k)}{\sin(\frac{\alpha}{2}(u_1-\frac{1}{2}+k))}\,\frac{\cos(\frac{\alpha}{2}(u_1-v_2-\frac{1}{2}+k))}{\cos(\frac{\alpha}{2}(u_1-\frac{1}{2}+k))}.$$

For parameters in accordance with Eqs. (2.4a), (2.4b) and $1 \leq k \leq \mathrm{M}$, the positivity of this trigonometric factor is clear because all sine functions are evaluated at angles between 0 and π:

$$0 < k < u_1 + k < u_1 + u_2 + \mathrm{M} = \frac{\pi}{\alpha}$$

and

$$0 < u_1 - |v_1| - \frac{1}{2} + k \leq u_1 - \frac{1}{2} + k \leq u_1 + |v_1| - \frac{1}{2} + k < 2u_1 + \mathrm{M} < \frac{2\pi}{\alpha},$$

whereas all cosine functions are evaluated at angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$:

$$-\frac{\pi}{\alpha} < u_1 - u_2 + k < u_1 + k < \frac{\pi}{\alpha}$$

and

$$-\frac{\pi}{\alpha} < u_1 - u_2 < u_1 - |v_2| - \frac{1}{2} + k \leq u_1 - \frac{1}{2} + k \leq u_1 + |v_2| - \frac{1}{2} + k < \frac{\pi}{\alpha}.$$

□

The upshot is that the finite discrete Heun Eqs. (2.5a), (2.5b) encode the spectral problem for a real-valued finite-dimensional tridiagonal matrix of the form

$$\mathbf{Hf} = \mathrm{E}\mathbf{f}, \tag{2.7a}$$

with

$$\mathbf{H} = \begin{bmatrix} b_0 & \tilde{a}_{\mathrm{M}} & 0 & \cdots & 0 \\ a_1 & b_1 & \ddots & & \vdots \\ 0 & a_2 & \ddots & \tilde{a}_2 & 0 \\ \vdots & & \ddots & b_{\mathrm{M}-1} & \tilde{a}_1 \\ 0 & \cdots & 0 & a_{\mathrm{M}} & b_{\mathrm{M}} \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{\mathrm{M}-1} \\ f_{\mathrm{M}} \end{bmatrix}. \tag{2.7b}$$

In view of the positivity of the matrix elements on the sub- and superdiagonal by virtue of Lemma 2.1, it is clear that the spectrum of $\mathbf{H}$ is given by $\mathrm{M}+1$ distinct and real eigenvalues (cf. e.g., [23, Chapter III.11.4]):

$$\mathrm{E}_0 > \mathrm{E}_1 > \cdots > \mathrm{E}_{\mathrm{M}}. \tag{2.8}$$

Moreover, the tridiagonal matrix $\mathbf{H}$ in Eqs. (2.7a), (2.7b) is quasi-centrosymmetric in the sense that its matrix elements $H_{j,k}$ obey the relation

$$H_{\mathrm{M}-j,\mathrm{M}-k} = \pi_2\left(H_{j,k}\right) \quad \text{for } 0 \leq j, k \leq \mathrm{M}. \tag{2.9}$$

3 Elliptic Racah polynomials

3.1 Diagonalization

We will now solve the eigenvalue problem in Eqs. (2.7a), (2.7b) in terms of orthogonal polynomials on the spectrum by applying standard techniques involving the interplay between tridiagonal matrices and orthogonal polynomials, cf. e.g., [33, Chapter III] and [10, Chapter 2]. To keep our presentation self-contained, a minimal compendium of the pertinent formulas from the literature has been collected in Appendix A.

Specifically, let $p_0(\mathrm{E}) = 1$ and

$$p_k(\mathrm{E}) = \det \begin{bmatrix} \mathrm{E}-b_0 & -\tilde{a}_{\mathrm{M}} & 0 & \cdots & 0 \\ -a_1 & \mathrm{E}-b_1 & \ddots & & \vdots \\ 0 & -a_2 & \ddots & -\tilde{a}_{\mathrm{M}+3-k} & 0 \\ \vdots & & \ddots & \mathrm{E}-b_{k-2} & -\tilde{a}_{\mathrm{M}+2-k} \\ 0 & \cdots & 0 & -a_{k-1} & \mathrm{E}-b_{k-1} \end{bmatrix} \tag{3.1}$$

for $k = 1, \ldots, \mathrm{M}+1$. In other words, $p_k(\mathrm{E})$ is given by the kth leading principal minor of the matrix $(\mathrm{E}\mathbf{I}_{\mathrm{M}+1} - \mathbf{H})$ governing the characteristic polynomial of $\mathbf{H}$ (2.7b). (Here $\mathbf{I}_{\mathrm{M}+1}$ denotes the $(\mathrm{M}+1)$-dimensional identity matrix.) We will refer to the polynomials $p_0(\mathrm{E}), p_1(\mathrm{E}), \ldots, p_{\mathrm{M}+1}(\mathrm{E})$ as (monic) *elliptic Racah polynomials*. By construction, these polynomials capture the characteristic polynomial of $\mathbf{H}$ at the top degree $k = \mathrm{M}+1$:

$$p_{\mathrm{M}+1}(\mathrm{E}) = \det(\mathrm{E}\mathbf{I}_{\mathrm{M}+1} - \mathbf{H}) = (\mathrm{E}-\mathrm{E}_0)(\mathrm{E}-\mathrm{E}_1)\cdots(\mathrm{E}-\mathrm{E}_{\mathrm{M}}) \tag{3.2}$$

(cf. Eq. (2.8)).

The following three-term recurrence relation is manifest from the definition (cf. Appendix A, Lemma A.1):

$$p_{k+1}(\mathrm{E}) = (\mathrm{E}-b_k)p_k(\mathrm{E}) - a_k\tilde{a}_{\mathrm{M}+1-k}p_{k-1}(\mathrm{E}) \quad \text{for } k = 0,\ldots,\mathrm{M} \tag{3.3}$$

(where $a_0 = \tilde{a}_{\mathrm{M}+1} := 0$). Moreover, by expanding the determinant $p_k(\mathrm{E})$ (3.1) as an alternating sum of products of matrix elements pulled from the distinct rows/columns, one arrives at the following explicit expansion for the elliptic Racah polynomials (cf. Appendix A, Lemma A.2):

$$p_k(\mathrm{E}) = \sum_{l=0}^{\lfloor k/2\rfloor}(-1)^l \sum_{\substack{1\le j_1<j_2<\cdots<j_l<k\\ j_{s+1}-j_s>1\\ \text{for } s=1,\ldots,l-1}} a_{j_1}\tilde{a}_{\mathrm{M}+1-j_1}\cdots a_{j_l}\tilde{a}_{\mathrm{M}+1-j_l} \prod_{\substack{1\le j\le k\\ j\notin\{j_s,j_s+1\}\\ \text{for } s=1,\ldots,l}}(\mathrm{E}-b_{j-1}) \tag{3.4}$$

(with the convention that empty factors are equal to 1).

This brings us in position to solve the finite discrete Heun equation in terms of elliptic Racah polynomials (cf. Appendix 1, Lemma A.4).

Proposition 3.1 (Eigenvectors) *For any eigenvalue* E *in the spectrum* $\{\mathrm{E}_0 > \mathrm{E}_1 > \cdots > \mathrm{E}_{\mathrm{M}}\}$ *of* $\mathbf{H}$ *(2.7b) (i.e., on shell), the* $(\mathrm{M}+1)$*-dimensional (column) vector* $\mathbf{f}(\mathrm{E})$ *with components given by normalized elliptic Racah polynomials of the form*

$$f_k(\mathrm{E}) = c_k p_k(\mathrm{E}), \quad k = 0, 1, \ldots, \mathrm{M}, \tag{3.5}$$

where

$$c_k = \prod_{0\le l<k}\tilde{a}^{-1}_{\mathrm{M}-l} = \prod_{\substack{1\le l<k\\ 1\le r\le 4}} \frac{[u_2+\mathrm{M}-l]_r[u_2-\frac{1}{2}+\mathrm{M}-l]_r}{[u_2-u_{\pi_2(r)}+\mathrm{M}-l]_r[u_2-v_{\pi_2(r)}-\frac{1}{2}+\mathrm{M}-l]_r},$$

solves the corresponding eigenvalue Eq. (2.7a).

Proof Since on shell we assume that E belongs to the spectrum of $\mathbf{H}$ and $p_{\mathrm{M}+1}(\mathrm{E}) = \det(\mathrm{E}\mathbf{I}_{\mathrm{M}+1}-\mathbf{H})$, it is clear that $p_{\mathrm{M}+1}(\mathrm{E}) = 0$ in this situation. The three-term recurrence relation (3.3) for the elliptic Racah polynomials then affirms that on shell:

$$\mathrm{E}p_k(\mathrm{E}) = \begin{cases} p_{k+1}(\mathrm{E}) + a_k\tilde{a}_{\mathrm{M}+1-k}p_{k-1}(\mathrm{E}) + b_kp_k(\mathrm{E}) & \text{for } k = 0,\ldots,\mathrm{M}-1,\\ a_{\mathrm{M}}\tilde{a}_1p_{\mathrm{M}-1}(\mathrm{E}) + b_{\mathrm{M}}p_{\mathrm{M}}(\mathrm{E}) & \text{for } k = \mathrm{M}. \end{cases}$$

Multiplication of the kth equation by $c_k = \prod_{0\le l<k}\tilde{a}^{-1}_{\mathrm{M}-l}$ on both sides and rewriting the result in terms of $f_k(\mathrm{E})$ for $k = 0,\ldots,\mathrm{M}$, verifies that on shell the components of the vector $\mathbf{f}(\mathrm{E})$ solve the finite discrete Heun equation (2.5a), (2.5b). □

Since all eigenvalues (2.8) are simple and $\mathbf{f}(\mathrm{E})$ is not a null vector (because $f_0(\mathrm{E}) = 1$), it is clear that the corresponding eigenvectors in Proposition 3.1 provide an eigenbasis diagonalizing $\mathbf{H}$:

$$\mathbf{F}^{-1}\mathbf{H}\mathbf{F} = \mathbf{E} \tag{3.6a}$$

with

$$\mathbf{F} = \big[\mathbf{f}(\mathrm{E}_0), \mathbf{f}(\mathrm{E}_1), \ldots, \mathbf{f}(\mathrm{E}_{\mathrm{M}})\big] \quad \text{and} \quad \mathbf{E} = \mathrm{diag}(\mathrm{E}_0, \mathrm{E}_1, \ldots, \mathrm{E}_{\mathrm{M}}). \tag{3.6b}$$

Moreover, upon pulling out the normalization constants from the rows of $\mathbf{F}$ and bringing the resulting $(\mathrm{M}+1) \times (\mathrm{M}+1)$ matrix of monic elliptic Racah polynomials to Vandermonde form via unitriangular row operations, it is readily seen that

$$\det(\mathbf{F}) = (-1)^{\frac{1}{2}\mathrm{M}(\mathrm{M}+1)} \prod_{1\le l\le \mathrm{M}} \tilde{a}_l^{-l} \prod_{0\le j<k\le \mathrm{M}} (\mathrm{E}_j - \mathrm{E}_k). \tag{3.7}$$

3.2 Orthogonality relation

Let

$$\tilde{\mathbf{H}} = \pi_2\big(\mathbf{H}\big), \quad \tilde{\mathbf{f}}(\mathrm{E}) = \pi_2\big(\mathbf{f}(\mathrm{E})\big), \quad \tilde{p}_k(\mathrm{E}) = \pi_2\big(p_k(\mathrm{E})\big), \quad \text{and } \tilde{\mathrm{E}}_j = \pi_2(\mathrm{E}_j). \tag{3.8}$$

From Eq. (2.9), one learns that the matrices $\tilde{\mathbf{H}}$ and $\mathbf{H}$ are related by conjugation to the $(\mathrm{M}+1) \times (\mathrm{M}+1)$ palindromic involution matrix:

$$\tilde{\mathbf{H}} = \mathbf{J}\mathbf{H}\mathbf{J} \quad \text{with} \quad \mathbf{J} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}. \tag{3.9}$$

Lemma 3.2 (Palindromic quasi-symmetry) (i) *For any* $0 \le j \le \mathrm{M}$, *one has that*

$$\tilde{\mathrm{E}}_j = \mathrm{E}_j \quad \textit{and} \quad \tilde{\mathbf{f}}(\mathrm{E}_j) = \epsilon_j \mathbf{J}\mathbf{f}(\mathrm{E}_j) \quad \textit{with } \epsilon_j = \frac{\tilde{p}_{\mathrm{M}}(\mathrm{E}_j)}{a_1 \cdots a_{\mathrm{M}}}. \tag{3.10a}$$

(ii) *Let* $\tilde{\epsilon}_j = \pi_2(\epsilon_j) = p_{\mathrm{M}}(\mathrm{E}_j)/(\tilde{a}_1 \cdots \tilde{a}_{\mathrm{M}})$. *Then*

$$\epsilon_j \tilde{\epsilon}_j = 1 \quad \textit{and} \quad \mathrm{Sign}\,(\epsilon_j) = \mathrm{Sign}\,(\tilde{\epsilon}_j) = (-1)^j. \tag{3.10b}$$

Proof Because $\tilde{\mathbf{H}}$ and $\mathbf{H}$ are related by a similarity transformation, it is clear that $\tilde{\mathrm{E}}_j = \mathrm{E}_j$ and

$$\mathbf{H}\mathbf{J}\tilde{\mathbf{f}}(\mathrm{E}_j) = \mathbf{J}\tilde{\mathbf{H}}\tilde{\mathbf{f}}(\mathrm{E}_j) = \mathrm{E}_j\mathbf{J}\tilde{\mathbf{f}}(\mathrm{E}_j).$$

Since the eigenvalue E_j is simple, this implies that $\mathbf{J}\tilde{\mathbf{f}}(\mathrm{E}_j) = \epsilon_j \mathbf{f}(\mathrm{E}_j)$ for some constant $\epsilon_j = \epsilon_j(u_r, v_r)$ in $\mathbb{R}$. Upon comparing the last components on both sides of the second equality in Eq. (3.10a), we see that $\epsilon_j = \epsilon_j f_0(\mathrm{E}_j) = \tilde{f}_{\mathrm{M}}(\mathrm{E}_j) = \tilde{p}_{\mathrm{M}}(\mathrm{E}_j)/(a_1 \cdots a_{\mathrm{M}})$, which proves *(i)*.

Twice iterated application of Eq. (3.10a) shows that $\tilde{\mathbf{f}}(\mathrm{E}_j) = \epsilon_j \mathbf{J}\mathbf{f}(\mathrm{E}_j) = \epsilon_j \tilde{\epsilon}_j \tilde{\mathbf{f}}(\mathrm{E}_j)$ (since $\mathbf{J}^2 = \mathbf{I}_{\mathrm{M}+1}$), so $\epsilon_j \tilde{\epsilon}_j = 1$ (as $\tilde{\mathbf{f}}(\mathrm{E}_j)$ is not a null vector). To compute the sign of ϵ_j, we evaluate the confluent Christoffel–Darboux identity in Eq. (A.5b) of Lemma A.3 for $n = \mathrm{M}$ at $\mathrm{X} = \mathrm{E}_j$ by means of the factorization $p_{\mathrm{M}+1}(\mathrm{X}) = (\mathrm{X}-\mathrm{E}_0)(\mathrm{X}-\mathrm{E}_1) \cdots (\mathrm{X}-\mathrm{E}_{\mathrm{M}})$; this reveals that $p_{\mathrm{M}}(\mathrm{E}_j) \prod_{\substack{0 \le l \le \mathrm{M} \\ l \neq j}} (\mathrm{E}_j - \mathrm{E}_l) = p_{\mathrm{M}}(\mathrm{E}_j) p'_{\mathrm{M}+1}(\mathrm{E}_j) > 0$, so

$$\mathrm{Sign}(\epsilon_j) = \mathrm{Sign}(\tilde{\epsilon}_j) = \mathrm{Sign}\big(p_{\mathrm{M}}(\mathrm{E}_j)\big) = \mathrm{Sign}\left(\prod_{\substack{0 \le l \le \mathrm{M} \\ l \neq j}} (\mathrm{E}_j - \mathrm{E}_l) \right) = (-1)^j$$

(upon recalling the ordering of the eigenvalues from Eq. (2.8)). This completes the proof of *(ii)*. □

Notice that it follows from the relation $\epsilon_j \tilde{\epsilon}_j = 1$ in Lemma 3.2 that

$$p_{\mathrm{M}}(\mathrm{E}_j) \tilde{p}_{\mathrm{M}}(\mathrm{E}_j) = \prod_{1 \le k \le \mathrm{M}} a_k \tilde{a}_k. \tag{3.11}$$

Moreover, if

$$(u_1, v_1) = (u_2, v_2) \quad \text{and} \quad (u_3, v_3) = (u_4, v_4), \tag{3.12a}$$

then our matrix becomes centrosymmetric: $\mathbf{J}\mathbf{H}\mathbf{J} = \tilde{\mathbf{H}} = \mathbf{H}$. We then have that $\tilde{\epsilon}_j = \epsilon_j$ with $\epsilon_j^2 = 1$, so

$$\epsilon_j = (-1)^j \tag{3.12b}$$

on this particular parameter manifold enjoying palindromic symmetry.

The orthogonality relations for the elliptic Racah polynomials are governed by the positive weights

$$\Delta_k = \prod_{1 \le l \le k} \frac{\tilde{a}_{\mathrm{M}+1-l}}{a_l} \qquad (\text{for } k = 0, 1 \ldots, \mathrm{M})$$

$$= \frac{[2u_1+2k]_1}{[2u_1]_1} \prod_{\substack{1\le l\le k\\ 1\le r\le 4}} \frac{[u_2-u_{\pi_2(r)}+\mathrm{M}+1-l]_r[u_2-v_{\pi_2(r)}+\mathrm{M}+\frac{1}{2}-l]_r}{[u_1-u_r+l]_r[u_1-v_r-\frac{1}{2}+l]_r} \tag{3.13}$$

(where the expression was simplified using the duplication formula $[2z]_1 = 2\prod_{1\le r\le 4}[z]_r$ for the scaled theta functions).

Proposition 3.3 (Orthogonality relation) *The normalized elliptic Racah polynomials* $f_k(\mathrm{E})$ *(3.5) satisfy the orthogonality relation*

$$\sum_{k=0}^{\mathrm{M}} f_k(\mathrm{E}_i)f_k(\mathrm{E}_j)\Delta_k = \begin{cases} N_j & \text{if } i=j,\\ 0 & \text{if } i\neq j,\end{cases} \tag{3.14a}$$

for $0\le i,j\le \mathrm{M}$, *with*

$$\begin{aligned} N_j &= \frac{1}{|\epsilon_j|\,a_1\cdots a_{\mathrm{M}}} \prod_{\substack{0\le l\le \mathrm{M}\\ l\neq j}} |\mathrm{E}_j-\mathrm{E}_l| \\ &= \frac{1}{|\epsilon_j|} \prod_{\substack{1\le k\le \mathrm{M}\\ 1\le r\le 4}} \frac{[u_1+k]_r}{[u_1-u_r+k]_r}\frac{[u_1-\frac{1}{2}+k]_r}{[u_1-v_r-\frac{1}{2}+k]_r} \prod_{\substack{0\le l\le \mathrm{M}\\ l\neq j}} |\mathrm{E}_j-\mathrm{E}_l|. \end{aligned} \tag{3.14b}$$

Proof The asserted orthogonality follows by combining the Christoffel–Darboux formulas in Lemma A.3 for $n=\mathrm{M}$ with Eq. (3.2) (cf. Appendix 1, Remark A.5):

$$\sum_{k=0}^{\mathrm{M}} f_k(\mathrm{E}_i)f_k(\mathrm{E}_j)\Delta_k = \sum_{k=0}^{\mathrm{M}} \frac{p_k(\mathrm{E}_i)p_k(\mathrm{E}_j)}{\prod_{1\le l\le k} a_l\tilde{a}_{\mathrm{M}+1-l}} = \begin{cases} \dfrac{p'_{\mathrm{M}+1}(\mathrm{E}_j)p_{\mathrm{M}}(\mathrm{E}_j)}{\prod_{1\le l\le \mathrm{M}} a_l\tilde{a}_{\mathrm{M}+1-l}} & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases}$$

Indeed, since $p'_{\mathrm{M}+1}(\mathrm{E}_j) = \prod_{\substack{0\le l\le \mathrm{M}\\ l\neq j}}(\mathrm{E}_j-\mathrm{E}_l)$ and $p_{\mathrm{M}}(\mathrm{E}_j) = \tilde{a}_1\cdots\tilde{a}_{\mathrm{M}}/\epsilon_j$ (by Eqs. (3.10a), (3.10b)), the expression for the quadratic norm readily simplifies to the formula stated in the proposition. □

The orthogonality relation in Proposition 3.3 supplies the following expressions for the inverse and the determinant of the elliptic Racah matrix **F** from Eqs. (3.6a), (3.6b).

Corollary 3.4 (Inverse Elliptic Racah Matrix) *The inverse and the determinant of the elliptic Racah matrix* **F** *(3.6b) are given by*

$$\mathbf{F}^{-1} = \mathbf{N}^{-1}\mathbf{F}^T\boldsymbol{\Delta} \tag{3.15a}$$

and

$$\det(\mathbf{F}) = (-1)^{\frac{1}{2}\mathrm{M}(\mathrm{M}+1)} \prod_{0\le l\le \mathrm{M}} \left(\frac{N_l}{\Delta_l}\right)^{1/2}, \tag{3.15b}$$

where $\mathbf{N} = \mathrm{diag}\,(N_0, N_1, \ldots, N_{\mathrm{M}})$, $\boldsymbol{\Delta} = \mathrm{diag}\,(\Delta_0, \Delta_1, \ldots, \Delta_{\mathrm{M}})$ *and*

$$\mathbf{F}^T = \begin{bmatrix} f_0(\mathrm{E}_0) & f_1(\mathrm{E}_0) & \cdots & f_{\mathrm{M}}(\mathrm{E}_0) \\ f_0(\mathrm{E}_1) & f_1(\mathrm{E}_1) & \cdots & f_{\mathrm{M}}(\mathrm{E}_1) \\ \vdots & \vdots & \vdots & \vdots \\ f_0(\mathrm{E}_{\mathrm{M}-1}) & f_1(\mathrm{E}_{\mathrm{M}-1}) & \cdots & f_{\mathrm{M}}(\mathrm{E}_{\mathrm{M}-1}) \\ f_0(\mathrm{E}_{\mathrm{M}}) & f_1(\mathrm{E}_{\mathrm{M}}) & \cdots & f_{\mathrm{M}}(\mathrm{E}_{\mathrm{M}}) \end{bmatrix}.$$

If one compares Eq. (3.15b) with the evaluation of the determinant in Eq. (3.7), then it follows that

$$\epsilon_0\epsilon_1\cdots\epsilon_{\mathrm{M}} = (-1)^{\frac{1}{2}\mathrm{M}(\mathrm{M}+1)} \prod_{1\le l\le \mathrm{M}} \left(\frac{\tilde{a}_l}{a_l}\right)^l, \tag{3.16a}$$

or equivalently (cf. Eq. (3.11))

$$p_{\mathrm{M}}(\mathrm{E}_0)p_{\mathrm{M}}(\mathrm{E}_1)\cdots p_{\mathrm{M}}(\mathrm{E}_{\mathrm{M}}) = (-1)^{\frac{1}{2}\mathrm{M}(\mathrm{M}+1)} \prod_{1\le l\le \mathrm{M}} (a_l\tilde{a}_{\mathrm{M}+1-l})^l. \tag{3.16b}$$

Remark 3.5 It is clear from the Christoffel–Darboux formulas (cf. Lemma A.3 of the appendix) that the general form of the orthogonality relation in Proposition 3.3 holds for an arbitrary real tridiagonal matrix $\mathbf{H}$ of the form in Eq. (A.1) with $a_l\tilde{a}_{\mathrm{M}+1-l} > 0$ for $l = 1, \ldots, \mathrm{M}$ (upon reading $\mathbf{JHJ}$ for $\tilde{\mathbf{H}}$). The same is then true for Eq. (3.11), Corollary 3.4 and Eqs. (3.16a), (3.16b). If in addition $a_l = \tilde{a}_l$ for $l = 1, \ldots, \mathrm{M}$, i.e., if we are in the centrosymmetric situation that $\mathbf{JHJ} = \mathbf{H}$, then the norm formulas in question simplify as in this case $\epsilon_j = (-1)^j$ (cf. Eq. (3.12b)).

3.3 Finite discrete Heun function

In order to describe the complete solution of the finite discrete Heun equation (2.5a), (2.5b) in terms of elliptic Racah polynomials, the following theorem summarizes the main findings of this section.

Theorem 3.6 (Finite Discrete Heun Function)

(i) *The finite discrete Heun Eq. (2.5a), (2.5b) only possesses nontrivial solutions for* $\mathrm{E} \in \{\mathrm{E}_0, \ldots, \mathrm{E}_{\mathrm{M}}\}$, *where* $\mathrm{E}_0 > \mathrm{E}_1 > \cdots > \mathrm{E}_{\mathrm{M}}$ *denote the roots of the top-degree elliptic Racah polynomial* $p_{\mathrm{M}+1}(\mathrm{E})$ *(3.1).*

(ii) *The solutions of the finite discrete Heun equation from part* (i) *are given by*

$$h_k(\mathrm{E}_j) = h_0(\mathrm{E}_j)p_k(\mathrm{E}_j) \prod_{0\le l<k} \tilde{a}^{-1}_{\mathrm{M}-l} \qquad (k = 0, \ldots, \mathrm{M}), \tag{3.17a}$$

where it is convenient to fix the normalization picking

$$h_0(\mathrm{E}_j) = |\epsilon_j|^{1/2} = \left|\frac{\tilde{a}_1\cdots\tilde{a}_{\mathrm{M}}}{p_{\mathrm{M}}(\mathrm{E}_j)}\right|^{1/2} \tag{3.17b}$$

(cf. Proposition 3.1).

(iii) *The finite discrete Heun function (3.17a), (3.17b) satisfies the palindromic quasi-symmetry*

$$\tilde{h}_k(\tilde{\mathrm{E}}_j) = (-1)^j h_{\mathrm{M}-k}(\mathrm{E}_j) \quad \textit{with} \quad \tilde{\mathrm{E}}_j = \mathrm{E}_j \quad (j, k = 0, \ldots, \mathrm{M}), \tag{3.17c}$$

where $\tilde{h}_k(\tilde{\mathrm{E}}_j)$ denotes the finite discrete Heun function with permuted coupling parameters: $(u_1, v_1) \leftrightarrow (u_2, v_2)$ and $(u_3, v_3) \leftrightarrow (u_4, v_4)$ (cf. Lemma 3.2).

(iv) *The finite discrete Heun functions satisfy the orthogonality relation*

$$\sum_{k=0}^{\mathrm{M}} h_k(\mathrm{E}_i) h_k(\mathrm{E}_j) \prod_{1 \le j \le k} \tilde{a}_{\mathrm{M}+1-j} \prod_{k+1 \le j \le \mathrm{M}} a_j = \begin{cases} \prod_{\substack{0 \le l \le \mathrm{M} \\ l \ne j}} |\mathrm{E}_j - \mathrm{E}_l| & \textit{if } i = j, \\ 0 & \textit{if } i \ne j \end{cases} \tag{3.17d}$$

$(0 \le i, j \le \mathrm{M})$, and the dual orthogonality relation

$$\sum_{j=0}^{\mathrm{M}} \frac{h_l(\mathrm{E}_j) h_k(\mathrm{E}_j)}{\prod_{\substack{0 \le i \le \mathrm{M} \\ i \ne j}} |\mathrm{E}_j - \mathrm{E}_i|} = \begin{cases} \left(\prod_{1 \le j \le k} \tilde{a}_{\mathrm{M}+1-j} \prod_{k+1 \le j \le \mathrm{M}} a_j \right)^{-1} & \textit{if } l = k, \\ 0 & \textit{if } l \ne k \end{cases} \tag{3.17e}$$

$(0 \le l, k \le \mathrm{M})$, which encode, respectively, the column and row orthogonality of the elliptic Racah matrix **F** *(3.6b) (cf. Proposition 3.3).*

4 Degenerations

4.1 Finite discrete Lamé equation

If all parameters v_r tend to zero, then the difference Heun operator H (2.1a)–(2.1d) reduces to a second-order difference operator stemming from the Sklyanin algebra [24, 25, 28, 30]:

$$(Hf)(z) = f(z+1) \prod_{1 \le r \le 4} \frac{[z + u_r]_r}{[z]_r} + f(z-1) \prod_{1 \le r \le 4} \frac{[z - u_r]_r}{[z]_r}. \tag{4.1}$$

The corresponding parameter degeneration of Theorem 3.6 solves a finite discrete Heun equation of the form

$$\tilde{a}_{\mathrm{M}-k} f_{k+1} + a_k f_{k-1} = \mathrm{E} f_k \quad \text{for } k = 0, \ldots, \mathrm{M}, \tag{4.2a}$$

with

$$a_k = \prod_{1 \le r \le 4} \frac{[u_1 - u_r + k]_r}{[u_1 + k]_r} \quad \text{and} \quad \tilde{a}_k = \prod_{1 \le r \le 4} \frac{[u_2 - u_{\pi_2(r)} + k]_r}{[u_2 + k]_r}. \tag{4.2b}$$

After this reduction, the diagonal of the discrete Heun matrix $\mathbf{H}$ in Eqs. (2.7a), (2.7b) vanishes, so the corresponding elliptic Racah polynomials $p_k(\mathrm{E})$ (3.1) are even in E if k is even and odd in E if k is odd. This implies that in this situation the eigenvalues $\mathrm{E}_0 > \mathrm{E}_1 > \cdots > \mathrm{E}_\mathrm{M}$ of $\mathbf{H}$ are distributed symmetrically around the origin:

$$\mathrm{E}_{\mathrm{M}-j} = -\mathrm{E}_j \quad (j = 0, \ldots, \mathrm{M}). \tag{4.3}$$

Moreover—via the duplication formula $[2z]_1 = 2\prod_{1\le r\le 4}[z]_r$—it is seen that if all parameters u_r are equal (to $u > 0$ say), then the eigenvalue problem in Eqs. (4.2a), (4.2b) reduces to a finite discrete Lamé equation of the form studied in [43]:

$$\frac{[\mathrm{M}-k]_1}{[u+\mathrm{M}-k]_1} f_{k+1} + \frac{[k]_1}{[u+k]_1} f_{k-1} = \mathrm{E} f_k \quad \text{for } k = 0, \ldots, \mathrm{M}, \tag{4.4}$$

with $\alpha = \frac{2\pi}{2u+\mathrm{M}}$ (so $2u + \mathrm{M} = \frac{2\pi}{\alpha}$).

4.2 Trigonometric limit: *q*-Racah polynomials

In [43], it was shown that the solutions of the finite discrete Lamé Eq. (4.4) can be expressed in terms of Rogers' q-ultraspherical polynomials in the trigonometric limit $p \to 0$. Here we finish by checking that the elliptic Racah polynomials degenerate in turn to the q-Racah polynomials of Askey and Wilson in the limit $p \to 0$. To this end, let us first observe that in the trigonometric limit the scaled theta functions degenerate as follows:

$$\lim_{p\to 0}[z]_1 = \tfrac{2}{\alpha}\sin(\tfrac{\alpha}{2}z), \quad \lim_{p\to 0}[z]_2 = \cos(\tfrac{\alpha}{2}z), \quad \lim_{p\to 0}[z]_3 = 1, \quad \lim_{p\to 0}[z]_4 = 1.$$

The corresponding coefficients of the difference Heun equation thus become

$$\begin{aligned}\mathrm{A}_t(z) &= \lim_{p\to 0}\mathrm{A}(z)\\ &= \frac{\sin\left(\frac{\alpha}{2}(z+u_1)\right)}{\sin\left(\frac{\alpha}{2}z\right)}\frac{\cos\left(\frac{\alpha}{2}(z+u_2)\right)}{\cos\left(\frac{\alpha}{2}z\right)}\frac{\sin\left(\frac{\alpha}{2}(z+\frac{1}{2}+v_1)\right)}{\sin\left(\frac{\alpha}{2}(z+\frac{1}{2})\right)}\frac{\cos\left(\frac{\alpha}{2}(z+\frac{1}{2}+v_2)\right)}{\cos\left(\frac{\alpha}{2}(z+\frac{1}{2})\right)}\end{aligned} \tag{4.5a}$$

and

$$\begin{aligned}\mathrm{B}_t(z) = \lim_{p\to 0}\mathrm{B}(z) &= c_{t,1}\frac{\sin\left(\frac{\alpha}{2}(z+\frac{1}{2}+u)\right)}{\sin\left(\frac{\alpha}{2}(z+\frac{1}{2})\right)}\frac{\sin\left(\frac{\alpha}{2}(z-\frac{1}{2}-u)\right)}{\sin\left(\frac{\alpha}{2}(z-\frac{1}{2})\right)}\\ &\quad + c_{t,2}\frac{\cos\left(\frac{\alpha}{2}(z+\frac{1}{2}+u)\right)}{\cos\left(\frac{\alpha}{2}(z+\frac{1}{2})\right)}\frac{\cos\left(\frac{\alpha}{2}(z-\frac{1}{2}-u)\right)}{\cos\left(\frac{\alpha}{2}(z-\frac{1}{2})\right)} + c_{t,3} + c_{t,4}\end{aligned} \tag{4.5b}$$

with

$$c_{t,r} = \frac{2\sin\left(\frac{\alpha}{2}(u_{\pi_r(1)} - \frac{1}{2})\right)\sin\left(\frac{\alpha}{2}v_{\pi_r(1)}\right)\cos\left(\frac{\alpha}{2}(u_{\pi_r(2)} - \frac{1}{2})\right)\cos\left(\frac{\alpha}{2}v_{\pi_r(2)}\right)}{\sin\left(\frac{\alpha}{2}u\right)\sin\left(\frac{\alpha}{2}(u+1)\right)}.$$

We now have that

$$\mathrm{A}_t(z) + \mathrm{A}_t(-z) + \mathrm{B}_t(z) = \mathrm{C}_t = 2\cos\frac{\alpha}{2}(u_1 + u_2 + v_1 + v_2) + \sum_{1\le r\le 4} c_{t,r}. \tag{4.6}$$

Indeed, as a periodic function of z all poles on the LHS of Eq. (4.6) are seen to cancel, while for $\mathrm{Im}\,(z) \to \infty$ the expression in question tends to the constant value on the RHS.

The q-Racah polynomials [1] are basic hypergeometric orthogonal polynomials of the form [13, Chapter 4.2]:

$$R_k(\mathrm{X}) = R_k(\mathrm{X}(x); a, b, c, d|q) = {}_4\phi_3\left({q^{-k}, abq^{k+1}, q^{-x}, cdq^{x+1} \atop aq, bdq, cq}; q, q\right) \tag{4.7a}$$

with

$$\mathrm{X} = \mathrm{X}(x) = cdq^{x+1} + q^{-x}. \tag{4.7b}$$

From the basic hypergeometric representation, the $k \leftrightarrow j$, $a \leftrightarrow c$, $b \leftrightarrow d$ duality symmetry [1, 18] of the q-Racah polynomial $R_k(\mathrm{X}(j); a, b, c, d|q)$ is immediate:

$$R_k(\mathrm{X}(j); a, b, c, d|q) = R_j(\hat{\mathrm{X}}(k); \hat{a}, \hat{b}, \hat{c}, \hat{d}|q), \tag{4.8a}$$

with

$$\hat{\mathrm{X}}(x) = \hat{c}\hat{d}q^{x+1} + q^{-x} \quad \text{and} \quad (\hat{a}, \hat{b}, \hat{c}, \hat{d}) = (c, d, a, b). \tag{4.8b}$$

The following proposition recovers the q-Racah polynomials as a trigonometric limit of the elliptic Racah polynomials. The proof hinges on the observation that the trigonometric degeneration of the recurrence in Eq. (3.3) can be identified with the three-term recurrence relation for the q-Racah polynomials found by Askey and Wilson [1, Section 3]. Throughout, we will implicitly exploit that the normalized theta functions $[\cdot]_r$ (2.2) extend analytically in p to the domain $-1 < p < 1$. The matrix elements of $\mathbf{H}$ (2.7a), (2.7b) inherit this analyticity in p and thus so do the eigenvalues $\mathrm{E}_0, \cdots, \mathrm{E}_\mathrm{M}$ (cf. e.g., [11, Chapter II, Theorem 6.1]).

Proposition 4.1 (q-Racah limit) *For $1 \le j, k \le \mathrm{M}$ and parameters in accordance with Eqs. (2.4a), (2.4b), one has that*

$$\lim_{p\to 0} \mathrm{E}_j = \mathrm{E}_{t,j} = 2\cos\frac{\alpha}{2}(2j + u_1 + u_2 + v_1 + v_2) + \sum_{1\le r\le 4} c_{t,r} \tag{4.9a}$$

and

$$\lim_{p\to 0} f_k(\mathrm{E}) = f_{t,k}(\mathrm{E}) = R_k(\mathrm{X}(x); a, b, c, d|q) \tag{4.9b}$$
$$= {}_4\phi_3\left(\begin{matrix} q^{-k}, q^{2u_1+k}, q^{-x}, q^{u_1+u_2+v_1+v_2+x} \\ -q^{u_1+u_2}, q^{u_1+v_1+1/2}, -q^{u_1+v_2+1/2} \end{matrix}; q, q\right),$$

where

$$q = e^{i\alpha}, \quad \mathrm{E} = 2\cos\frac{\alpha}{2}(2x + u_1 + u_2 + v_1 + v_2) + \sum_{1\le r\le 4} c_{t,r} \tag{4.9c}$$

and

$$a = -q^{u_1+u_2-1},\ b = -q^{u_1-u_2},\ c = -q^{u_1+v_2-1/2}, d = -q^{u_2+v_1-1/2}. \tag{4.9d}$$

Proof From the recurrence relation for the normalized elliptic Racah polynomials, it follows that in the trigonometric limit:

$$\tilde{a}_{t,\mathrm{M}-k} f_{t,k+1}(\mathrm{E}) + a_{t,k} f_{t,k-1}(\mathrm{E}) + (\mathrm{C}_t - \tilde{a}_{t,\mathrm{M}-k} - a_{t,k}) f_{t,k}(\mathrm{E}) = \mathrm{E} f_{t,k}(\mathrm{E}) \tag{4.10}$$

for $0 \le k < \mathrm{M}$ as a polynomial identity in E, where

$$a_{t,k} = \mathrm{A}_t(-u_1 - k)$$
$$= \frac{\sin\frac{\alpha}{2}(k)}{\sin\frac{\alpha}{2}(u_1+k)} \frac{\sin\frac{\alpha}{2}(u_1 - v_1 - \frac{1}{2} + k)}{\sin\frac{\alpha}{2}(u_1 - \frac{1}{2} + k)} \frac{\cos\frac{\alpha}{2}(u_1 - u_2 + k)}{\cos\frac{\alpha}{2}(u_1 + k)} \frac{\cos\frac{\alpha}{2}(u_1 - v_2 - \frac{1}{2} + k)}{\cos\frac{\alpha}{2}(u_1 - \frac{1}{2} + k)},$$
$$\tilde{a}_{t,k} = \mathrm{A}_t(u_1 + \mathrm{M} - k)$$
$$= \frac{\sin\frac{\alpha}{2}(k)}{\sin\frac{\alpha}{2}(u_2+k)} \frac{\sin\frac{\alpha}{2}(u_2 - v_2 - \frac{1}{2} + k)}{\sin\frac{\alpha}{2}(u_2 - \frac{1}{2} + k)} \frac{\cos\frac{\alpha}{2}(u_2 - u_1 + k)}{\cos\frac{\alpha}{2}(u_2 + k)} \frac{\cos\frac{\alpha}{2}(u_2 - v_1 - \frac{1}{2} + k)}{\cos\frac{\alpha}{2}(u_2 - \frac{1}{2} + k)},$$

and the coefficient $b_{t,k} = \mathrm{B}_t(u_1 + k)$ has been rewritten with the aid of the identity in Eq. (4.6). Upon comparing with the three-term recurrence relation for the q-Racah polynomials [13, Eq. (14.2.3)] for $0 \le k < \mathrm{M}$:

$$A_k R_{k+1}(\mathrm{x}) + C_k R_{k-1}(\mathrm{x}) + (cdq + 1 - A_k - C_k) R_k(\mathrm{x}) = \mathrm{x} R_k(\mathrm{x}), \quad (4.11)$$

with

$$A_k = \frac{(1 - aq^{k+1})(1 - abq^{k+1})(1 - bdq^{k+1})(1 - cq^{k+1})}{(1 - abq^{2k+1})(1 - abq^{2k+2})}$$

and

$$C_k = \frac{q(1 - q^k)(1 - bq^k)(c - abq^k)(d - aq^k)}{(1 - abq^{2k})(1 - abq^{2k+1})},$$

one observes that $\tilde{a}_{t,\mathrm{M}-k} = (qcd)^{-1/2} A_k$, $a_{t,k} = (qcd)^{-1/2} C_k$, and $\mathrm{E} - \mathrm{C}_t = (qcd)^{-1/2}(\mathrm{x}(x) - cdq - 1)$ provided the variables and parameters are identified in accordance with Eqs. (4.9c), (4.9d). The upshot is that the recurrences in Eqs. (4.10) and (4.11) coincide while $f_{t,0}(\mathrm{E}) = R_0(\mathrm{x}) = 1$, so Eq. (4.9b) follows.

To infer the limit in Eq. (4.9a), it suffices to check that the eigenvalues of the trigonometric degeneration of our finite discrete Heun operator are indeed given by the asserted formulas on the RHS. In principle, this can be deduced via the explicit formula for the roots of the top degree $\mathrm{M}+1$ q-Racah polynomial found by Askey and Wilson, cf. [1, Section 3]. Here, however, we prefer to rather determine the eigenvalues at the trigonometric level directly from the q-difference equation [13, Eq. (14.2.6)] for the dual q-Racah polynomial $\hat{R}_j(\hat{\mathrm{x}}(k)) = R_j(\hat{\mathrm{x}}(k); \hat{a}, \hat{b}, \hat{c}, \hat{c}|q)$ (4.8a), (4.8b):

$$A_k \hat{R}_j(\hat{\mathrm{x}}(k+1)) + C_k \hat{R}_j(\hat{\mathrm{x}}(k-1)) + (cdq + 1 - A_k - C_k)\hat{R}_j(\hat{\mathrm{x}}(k)) \\ = \mathrm{x}(j)\hat{R}_j(\hat{\mathrm{x}}(k))$$

for $0 \le j, k \le \mathrm{M}$, where $A_\mathrm{M} = 0$ (because $1 - aq^{\mathrm{M}+1} = 1 + q^{u_1+u_2+\mathrm{M}} = 0$) and $C_0 = 0$. Indeed, with the aid of the duality symmetry in Eqs. (4.8a), (4.8b) this yields Eq. (4.11) evaluated at $\mathrm{x} = \mathrm{x}(j)$ for $0 \le j, k \le \mathrm{M}$. Upon rewriting the latter formula in terms of $f_{t,k}(\mathrm{E})$ by means of Eqs. (4.9b)–(4.9d), we conclude that Eq. (4.10) holds at $\mathrm{E} = \mathrm{E}_{t,j}$ (4.9a) for $0 \le j, k \le \mathrm{M}$. The upshot is that $\mathrm{E}_{t,j}$ (4.9a) must be an eigenvalue of the trigonometric degeneration of the matrix $\mathbf{H}$ in Eqs. (2.7a), (2.7b). Since our parameter restrictions (2.4a), (2.4b) guarantee that the ordering of the eigenvalues in question agrees with the convention in Eq. (2.8): $\mathrm{E}_{t,0} > \mathrm{E}_{t,1} > \cdots > \mathrm{E}_{t,\mathrm{M}}$, the trigonometric limit asserted in Eq. (4.9a) now follows. □

Proposition 4.1 reveals that on shell the trigonometric degeneration of the normalized elliptic Racah polynomial $f_k(\mathrm{E}_j)$ is given by the following q-Racah polynomial

$$f_{t,k}(\mathrm{E}_{t,j}) = R_k\left(\mathrm{x}(j); -q^{u_1+u_2-1}, -q^{u_1-u_2}, -q^{u_1+v_2-1/2}, -q^{u_2+v_1-1/2}|q\right) \\ = {}_4\phi_3\left({q^{-k}, q^{2u_1+k}, q^{-j}, q^{u_1+u_2+v_1+v_2+j} \atop -q^{u_1+u_2}, q^{u_1+v_1+1/2}, -q^{u_1+v_2+1/2}}; q, q\right), \quad (4.12)$$

with $\mathrm{X}(j) = q^{u_1+u_2+v_1+v_2+j} + q^{-j}$. The corresponding degeneration of the orthogonality relation from Proposition 3.3 becomes

$$\sum_{k=0}^{\mathrm{M}} f_{t,k}(\mathrm{E}_{t,i}) f_{t,k}(\mathrm{E}_{t,j}) \Delta_{t,k} = \begin{cases} N_{t,j} & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{4.13a}$$

for $0 \le i, j \le \mathrm{M}$, with

$$\begin{aligned} \Delta_{t,k} =& \frac{\sin \alpha(u_1+k)}{\sin(\alpha u_1)} \prod_{1\le l\le k} \frac{\sin \frac{\alpha}{2}(\mathrm{M}+1-l) \sin \frac{\alpha}{2}(u_2 - v_2 + \mathrm{M} + \frac{1}{2} - l)}{\sin\left(\frac{\alpha}{2} l\right) \sin \frac{\alpha}{2}(u_1 - v_1 - \frac{1}{2} + l)} \\ & \times \prod_{1\le l\le k} \frac{\cos \frac{\alpha}{2}(u_2 - u_1 + \mathrm{M} + 1 - l) \cos \frac{\alpha}{2}(u_2 - v_1 + \mathrm{M} + \frac{1}{2} - l)}{\cos \frac{\alpha}{2}(u_1 - u_2 + l) \cos \frac{\alpha}{2}(u_1 - v_2 - \frac{1}{2} + l)} \end{aligned} \tag{4.13b}$$

and

$$\begin{aligned} N_{t,j} =& \frac{1}{\epsilon_{t,j}\, a_{t,1}\cdots a_{t,\mathrm{M}}} \prod_{\substack{0\le l\le \mathrm{M} \\ l\neq j}} (\mathrm{E}_{t,j} - \mathrm{E}_{t,l}) \\ =& \frac{1}{\epsilon_{t,j}} \prod_{1\le k\le \mathrm{M}} \frac{\sin \frac{\alpha}{2}(u_1+k)}{\sin \frac{\alpha}{2} k} \frac{\sin \frac{\alpha}{2}(u_1 - \frac{1}{2} + k)}{\sin \frac{\alpha}{2}(u_1 - v_1 - \frac{1}{2} + k)} \\ & \times \prod_{1\le k\le \mathrm{M}} \frac{\cos \frac{\alpha}{2}(u_1+k)}{\cos \frac{\alpha}{2}(u_1 - u_2 + k)} \frac{\cos \frac{\alpha}{2}(u_1 - \frac{1}{2} + k)}{\cos \frac{\alpha}{2}(u_1 - v_2 - \frac{1}{2} + k)} \\ & \times \prod_{\substack{0\le l\le \mathrm{M} \\ l\neq j}} \Big(2\cos\frac{\alpha}{2}(2j + u_1 + u_2 + v_1 + v_2) \\ & \qquad - 2\cos\frac{\alpha}{2}(2l + u_1 + u_2 + v_1 + v_2)\Big), \end{aligned} \tag{4.13c}$$

where

$$\epsilon_{t,j}^{-1} = \frac{p_{t,\mathrm{M}}(\mathrm{E}_{t,j})}{\tilde{a}_{t,1}\cdots\tilde{a}_{t,\mathrm{M}}} = f_{t,\mathrm{M}}(\mathrm{E}_{t,j}) = {}_4\phi_3\left({q^{-\mathrm{M}}, q^{2u_1+\mathrm{M}}, q^{-j}, q^{u_1+u_2+v_1+v_2+j} \atop -q^{u_1+u_2}, q^{u_1+v_1+1/2}, -q^{u_1+v_2+1/2}} ; q, q \right).$$

By means of the relation $q^{u_1+u_2+\mathrm{M}} = -1$, we reduce the latter ${}_4\phi_3$ series to a ${}_3\phi_2$ series that can be evaluated via Jackson's q-Pfaff-Saalschütz sum [22, Eq. (17.7.4)]:

$$\epsilon_{t,j}^{-1} = {}_3\phi_2\left({-q^{u_1-u_2}, q^{-j}, q^{u_1+u_2+v_1+v_2+j} \atop q^{u_1+v_1+1/2}, -q^{u_1+v_2+1/2}} ; q, q \right) = \frac{(-q^{u_2+v_1+1/2}, q^{-u_2-v_2+1/2-j}; q)_j}{(q^{u_1+v_1+1/2}, -q^{-u_1-v_2+1/2-j}; q)_j}$$

$$= (-1)^j \prod_{0\le l<j} \frac{\sin\frac{\alpha}{2}(u_2+v_2+1/2+l)}{\sin\frac{\alpha}{2}(u_1+v_1+1/2+l)} \frac{\cos\frac{\alpha}{2}(u_2+v_1+1/2+l)}{\cos\frac{\alpha}{2}(u_1+v_2+1/2+l)} \tag{4.14}$$

(where $(z;q)_j = \prod_{0\le l<j}(1-zq^l)$ and $(z_1,\ldots,z_s;q)_j = (z_1;q)_j\cdots(z_s;q)_j$).

It is instructive to compare the orthogonality in Eqs. (4.13a)–(4.14) with the (dual) orthogonality relations for the q-Racah polynomials subject to the truncation condition $aq^{\mathrm{M}+1} = 1$ (cf. [13, Eq. (14.2.2)]):

$$\sum_{k=0}^{\mathrm{M}} R_k(\mathrm{x}(i);a,b,c,d|q) R_k(\mathrm{x}(j);a,b,c,d|q)\,\Delta_k(a,b,c,d;q)$$
$$= \begin{cases} N_0/\Delta_j(c,d,a,b;q) & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \tag{4.15a}$$

with

$$\Delta_k(a,b,c,d;q) = \frac{(cq,bdq,aq,abq;q)_k}{(q,c^{-1}abq,d^{-1}aq,bq;q)_k} \frac{(1-abq^{2k+1})}{(cdq)^k(1-abq)} \tag{4.15b}$$

and

$$N_0 = \sum_{k=0}^{\mathrm{M}} \Delta_k(a,b,c,d;q) = \sum_{j=0}^{\mathrm{M}} \Delta_j(c,d,a,b;q) = \frac{(b^{-1},cdq^2)_{\mathrm{M}}}{(b^{-1}cq,dq;q)_{\mathrm{M}}}. \tag{4.15c}$$

Indeed, the orthogonality weights in Eq. (4.13b) and Eq. (4.15b) coincide for parameters in accordance with Eq. (4.9c), (4.9d) (subject to the truncation condition (2.4b)):

$$\Delta_{t,k} = \Delta_k(-q^{u_1+u_2-1}, -q^{u_1-u_2}, -q^{u_1+v_2-1/2}, -q^{u_2+v_1-1/2}; q)$$

when $q = e^{\mathrm{i}\alpha}$ with $\alpha = \frac{\pi}{u_1+u_2+\mathrm{M}}$. This implies that the quadratic norms $N_{t,j}$ (4.13c), (4.14) can be rewritten in the form:

$$N_{t,j} = N_{t,0}/\hat{\Delta}_{t,j} \tag{4.16a}$$

with

$$\hat{\Delta}_{t,j} = \Delta_j(-q^{u_1+v_2-1/2}, -q^{u_2+v_1-1/2}, -q^{u_1+u_2-1}, -q^{u_1-u_2}; q)$$
$$= \frac{\sin\frac{\alpha}{2}(u_1+u_2+v_1+v_2+2j)}{\sin\frac{\alpha}{2}(u_1+u_2+v_1+v_2)}$$
$$\times \prod_{1\le l\le j} \frac{\sin\frac{\alpha}{2}(\mathrm{M}+1-l)\sin\frac{\alpha}{2}(u_2-v_2+\mathrm{M}+\frac{1}{2}-l)}{\sin(\frac{\alpha}{2}l)\sin\frac{\alpha}{2}(u_2+v_2-\frac{1}{2}+l)}$$

$$\times \prod_{1\le l\le j} \frac{\cos\frac{\alpha}{2}(-v_1-v_2+\mathrm{M}+1-l)\cos\frac{\alpha}{2}(u_2-v_1+\mathrm{M}+\frac{1}{2}-l)}{\cos\frac{\alpha}{2}(v_1+v_2+l)\cos\frac{\alpha}{2}(u_2+v_1-\frac{1}{2}+l)} \tag{4.16b}$$

and

$$\begin{aligned} N_{t,0} &= \sum_{k=0}^{\mathrm{M}} \Delta_{t,k} = \sum_{j=0}^{\mathrm{M}} \hat{\Delta}_{t,j} \\ &= \prod_{1\le l\le \mathrm{M}} \frac{\sin\frac{\alpha}{2}(2u_1+l)\ \sin\frac{\alpha}{2}(u_1+u_2+v_1+v_2+l)}{\sin\frac{\alpha}{2}(u_1-v_1-\frac{1}{2}+l)\sin\frac{\alpha}{2}(u_2+v_2-\frac{1}{2}+l)}. \end{aligned} \tag{4.16c}$$

We thus conclude that the inverse of the matrix

$$\mathbf{F}_t = \begin{bmatrix} f_{t,0}(\mathrm{E}_{t,0}) & f_{t,0}(\mathrm{E}_{t,1}) & \cdots & f_{t,0}(\mathrm{E}_{t,\mathrm{M}}) \\ f_{t,1}(\mathrm{E}_{t,0}) & f_{t,1}(\mathrm{E}_{t,1}) & \cdots & f_{t,1}(\mathrm{E}_{t,\mathrm{M}}) \\ \vdots & \vdots & \vdots & \vdots \\ f_{t,\mathrm{M}-1}(\mathrm{E}_{t,0}) & f_{t,\mathrm{M}-1}(\mathrm{E}_{t,1}) & \cdots & f_{t,\mathrm{M}-1}(\mathrm{E}_{t,\mathrm{M}}) \\ f_{t,\mathrm{M}}(\mathrm{E}_{t,0}) & f_{t,\mathrm{M}}(\mathrm{E}_{t,1}) & \cdots & f_{t,\mathrm{M}}(\mathrm{E}_{t,\mathrm{M}}) \end{bmatrix}$$

is given by (cf. Corollary 3.4)

$$\mathbf{F}_t^{-1} = N_{t,0}^{-1}\hat{\boldsymbol{\Delta}}_t \mathbf{F}_t^T \boldsymbol{\Delta}_t \tag{4.17a}$$

with

$$\boldsymbol{\Delta}_t = \operatorname{diag}\big(\Delta_{t,0}, \Delta_{t,1}, \ldots, \Delta_{t,\mathrm{M}}\big), \quad \hat{\boldsymbol{\Delta}}_t = \operatorname{diag}\big(\hat{\Delta}_{t,0}, \hat{\Delta}_{t,1}, \ldots, \hat{\Delta}_{t,\mathrm{M}}\big), \tag{4.17b}$$

while its determinant is given by

$$\det(\mathbf{F}_t) = \frac{(-1)^{\frac{1}{2}\mathrm{M}(\mathrm{M}+1)} N_{t,0}^{\frac{1}{2}(\mathrm{M}+1)}}{\sqrt{\prod_{0\le l\le \mathrm{M}} \Delta_{t,l}\hat{\Delta}_{t,l}}}. \tag{4.17c}$$

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Declarations

Conflicts of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix A. Eigenvectors of tridiagonal matrices via orthogonal polynomials

This appendix collects a few standard formulas concerning the construction of eigenvectors for tridiagonal matrices in terms of orthogonal polynomials. These formulas were used in Sect. 3 to build the solutions of our finite discrete Heun equation and to derive their orthogonality relations. For the broader context in which formulas of this type arise in connection with the theory of orthogonal polynomials, the reader is referred to, e.g., [33, Chapter III] and [10, Chapter 2]. Explicit expansions for the determinants of tridiagonal matrices are discussed in further detail in [19, Chapter XIII].

For *any* tridiagonal matrices

$$\begin{bmatrix} b_0 & \tilde{a}_{\mathrm{M}} & 0 & \cdots & 0 \\ a_1 & b_1 & \ddots & & \vdots \\ 0 & a_2 & \ddots & \tilde{a}_2 & 0 \\ \vdots & & \ddots & b_{\mathrm{M}-1} & \tilde{a}_1 \\ 0 & \cdots & 0 & a_{\mathrm{M}} & b_{\mathrm{M}} \end{bmatrix}, \tag{A.1}$$

with $a_1, \ldots, a_{\mathrm{M}}$, $\tilde{a}_1, \ldots, \tilde{a}_{\mathrm{M}}$ and $b_0, \ldots, b_{\mathrm{M}}$ in $\mathbb{C}$, let $p_k(\mathrm{x})$ denote the polynomial given by the kth leading principal minor stemming from the characteristic polynomial:

$$p_k(\mathrm{x}) = \det \begin{bmatrix} \mathrm{x} - b_0 & -\tilde{a}_{\mathrm{M}} & 0 & \cdots & 0 \\ -a_1 & \mathrm{x} - b_1 & \ddots & & \vdots \\ 0 & -a_2 & \ddots & -\tilde{a}_{\mathrm{M}+3-k} & 0 \\ \vdots & & \ddots & \mathrm{x} - b_{k-2} & -\tilde{a}_{\mathrm{M}+2-k} \\ 0 & \cdots & 0 & -a_{k-1} & \mathrm{x} - b_{k-1} \end{bmatrix}. \tag{A.2}$$

Lemma A.1 (Three-term recurrence relation) *The polynomials $p_k(\mathrm{X})$ obey the following three-term recurrence relation*

$$p_{k+1}(\mathrm{X}) = (\mathrm{X} - b_k)p_k(\mathrm{X}) - a_k\tilde{a}_{\mathrm{M}+1-k}\,p_{k-1}(\mathrm{X}) \quad \textit{for } k = 0, \ldots, \mathrm{M}, \tag{A.3}$$

with the convention that $p_0(\mathrm{X}) = 1$ and $a_0 = \tilde{a}_{\mathrm{M}+1} = 0$.

Proof Immediate upon expanding the determinant for $p_{k+1}(\mathrm{X})$ with respect to the last row/column. □

Lemma A.2 (Explicit expansion) *The polynomial $p_k(\mathrm{X})$ is given explicitly by*

$$p_k(\mathrm{X}) = \sum_{l=0}^{\lfloor k/2 \rfloor}(-1)^l \sum_{\substack{1\le j_1<j_2<\cdots<j_l<k \\ j_{s+1}-j_s>1 \\ \textit{for } s=1,\ldots,l-1}} a_{j_1}\tilde{a}_{\mathrm{M}+1-j_1}\cdots a_{j_l}\tilde{a}_{\mathrm{M}+1-j_l} \prod_{\substack{1\le j\le k \\ j\notin\{j_s,j_s+1\} \\ \textit{for } s=1,\ldots,l}} (\mathrm{X} - b_{j-1}), \tag{A.4}$$

with the convention that empty factors are equal to 1.

Proof Let us recall that the determinant of any $k \times k$ matrix $[A_{i,j}]_{1\le i,j\le k}$ is given by an alternating sum of terms $(-1)^\tau A_{1,\tau(1)}A_{2,\tau(2)}\ldots A_{k,\tau(k)}$ summed over all permutations $\tau = \left(\begin{smallmatrix} 1 & 2 & \cdots & k \\ \tau(1) & \tau(2) & \cdots & \tau(k)\end{smallmatrix}\right)$ of the symmetric group S_k (where $(-1)^\tau$ refers to the sign of τ). In the case of a tridiagonal matrix, nonvanishing products can occur only when τ decomposes as a product of $0 \le l \le \lfloor k/2 \rfloor$ commuting simple transpositions:

$$\tau = (j_1, j_1+1)(j_2, j_2+1)\cdots(j_l, j_l+1)$$

with

$$1 \le j_1 < j_1+1 < j_2 < j_2+1 < \cdots < j_l < j_l+1 \le k$$

(so the sign of τ is equal to $(-1)^l$). In the case of $p_k(\mathrm{E})$ (A.2), each transposition (j_s, j_s+1) contributes a factor $a_{j_s}\tilde{a}_{\mathrm{M}+1-j_s}$ to the product, while the indices j that are fixed by τ each contribute a factor of the form $(\mathrm{X} - b_{j-1})$. By collecting the contributions

$$(-1)^l a_{j_1}\tilde{a}_{\mathrm{M}+1-j_1}\cdots a_{j_l}\tilde{a}_{\mathrm{M}+1-j_l} \prod_{\substack{1\le j\le k \\ j\notin\{j_s,j_s+1\} \\ \text{for } s=1,\ldots,l}} (\mathrm{X} - b_{j-1})$$

from all such permutations τ, the asserted formula for $p_k(\mathrm{X})$ follows. □

Lemma A.3 (Christoffel–Darboux formulas) *For any $0 \le n \le \mathrm{M}$, the polynomials $p_0, \ldots, p_{n+1}$ enjoy the following Christoffel–Darboux identities:*

$$\sum_{k=0}^{n} \frac{p_k(\mathrm{X})p_k(\mathrm{Y})}{\prod_{j=1}^{k} a_j\tilde{a}_{\mathrm{M}+1-j}} = \frac{p_{n+1}(\mathrm{X})p_n(\mathrm{Y}) - p_n(\mathrm{X})p_{n+1}(\mathrm{Y})}{(\mathrm{X}-\mathrm{Y})\prod_{j=1}^{n} a_j\tilde{a}_{\mathrm{M}+1-j}}, \tag{A.5a}$$

and

$$\sum_{k=0}^{n} \frac{p_k^2(\mathrm{X})}{\prod_{j=1}^{k} a_j \tilde{a}_{\mathrm{M}+1-j}} = \frac{p'_{n+1}(\mathrm{X}) p_n(\mathrm{X}) - p'_n(\mathrm{X}) p_{n+1}(\mathrm{X})}{\prod_{j=1}^{n} a_j \tilde{a}_{\mathrm{M}+1-j}}, \tag{A.5b}$$

assuming $a_1 \cdots a_{\mathrm{M}} \cdot \tilde{a}_1 \cdots \tilde{a}_{\mathrm{M}} \neq 0$.

Proof From the three-term recurrence (A.3), it follows that

$$\begin{aligned} &p_{n+1}(\mathrm{X}) p_n(\mathrm{Y}) - p_n(\mathrm{X}) p_{n+1}(\mathrm{Y}) \\ &= (\mathrm{X} - \mathrm{Y}) p_n(\mathrm{X}) p_n(\mathrm{Y}) + a_n \tilde{a}_{\mathrm{M}+1-n} \big(p_n(\mathrm{X}) p_{n-1}(\mathrm{Y}) - p_{n-1}(\mathrm{X}) p_n(\mathrm{Y}) \big). \end{aligned}$$

Downward iteration entails that

$$p_{n+1}(\mathrm{X}) p_n(\mathrm{Y}) - p_n(\mathrm{X}) p_{n+1}(\mathrm{Y}) = (\mathrm{X} - \mathrm{Y}) \sum_{k=0}^{n} \left(p_k(\mathrm{X}) p_k(\mathrm{Y}) \prod_{j=k+1}^{n} a_j \tilde{a}_{\mathrm{M}+1-j} \right).$$

Upon dividing both sides by $(\mathrm{X}-\mathrm{Y}) \prod_{j=1}^{n} a_j \tilde{a}_{\mathrm{M}+1-j}$, the Christoffel–Darboux formula in Eq. (A.5a) is immediate, while Eq. (A.5b) follows subsequently via the confluent limit $\mathrm{Y} \to \mathrm{X}$. □

Lemma A.4 (Eigenvector) *If* E *is a root of* $p_{\mathrm{M}+1}(\mathrm{X})$ *and* $\tilde{a}_1 \cdots \tilde{a}_{\mathrm{M}} \neq 0$, *then*

$$\begin{bmatrix} b_0 & \tilde{a}_{\mathrm{M}} & 0 & \cdots & 0 \\ a_1 & b_1 & \ddots & & \vdots \\ 0 & a_2 & \ddots & \tilde{a}_2 & 0 \\ \vdots & & \ddots & b_{\mathrm{M}-1} & \tilde{a}_1 \\ 0 & \cdots & 0 & a_{\mathrm{M}} & b_{\mathrm{M}} \end{bmatrix} \begin{bmatrix} c_0 p_0(\mathrm{E}) \\ c_1 p_1(\mathrm{E}) \\ c_2 p_2(\mathrm{E}) \\ \vdots \\ \vdots \\ c_{\mathrm{M}} p_{\mathrm{M}}(\mathrm{E}) \end{bmatrix} = \mathrm{E} \begin{bmatrix} c_0 p_0(\mathrm{E}) \\ c_1 p_1(\mathrm{E}) \\ c_2 p_2(\mathrm{E}) \\ \vdots \\ \vdots \\ c_{\mathrm{M}} p_{\mathrm{M}}(\mathrm{E}) \end{bmatrix}, \tag{A.6}$$

where $c_k = \prod_{0 \leq l < k} \tilde{a}_{\mathrm{M}-l}^{-1}$ *with the convention that* $c_0 = 1$ *(and* $p_0(\mathrm{E}) = 1$*).*

Proof Since $p_{\mathrm{M}+1}(\mathrm{X})$ encodes the characteristic polynomial of the matrix in Eq. (A.1), it is clear that the root E is an eigenvalue. The asserted eigenvalue equation in Eq. (A.6) amounts in turn to the recurrence relation of Lemma A.1 evaluated at $\mathrm{X} = \mathrm{E}$. □

Remark A.5 If the upper diagonal matrix elements are nonzero, Lemma A.4 produces a complete basis of eigenvectors *provided* the spectrum of the matrix in Eq. (A.1) is simple. If both the matrix elements on the upper and lower diagonals are nonzero, then the eigenvectors in question satisfy (generally complex) orthogonality relations stemming from the Christoffel–Darboux identities in Lemma A.3 with $n = \mathrm{M}$.

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