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Partial Exponential Stability Analysis of Slow-fast Systems via Periodic Averaging

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***Abstract*— This letter presents some new criteria for partial exponential stability of a slow-fast nonlinear system with a fast scalar variable using periodic averaging methods. Unlike classical averaging techniques, we construct an averaged system by averaging over this fast scalar variable instead of the time variable. We show that partial exponential stability of the averaged system implies that of the original one. We then apply the obtained criteria to the study of remote synchronization of Kuramoto-Sakaguchi oscillators coupled by a star network with two peripheral nodes. We show that detuning the natural frequency of the central mediating oscillator increases the robustness of the remote synchronization against phase shifts. This work appears to be the first-known attempt to analytically study phase-unlocked remote synchronization.**

***Index Terms*— Partial exponential stability, averaging, remote synchronization, Kuramoto-Sakaguchi.**

I. Introduction

Partial stability describes a property of dynamical systems that only a portion, instead of all, of its states are stable. Different from standard full-state stability theory which usually deals with stability of point-wise equilibria, partial stability is more associated with stability of motions lying in a subspace [1], [2]. It provides a powerful framework to study a range of application-motivated theoretical problems, such as spacecraft stabilization by rotating masses [1], inertial navigation systems [3], transient stability of power systems [4], and synchronization in complex networks [5], [6].

Some Lyapunov criteria have been established to study partial stability of nonlinear systems [1], [2, Chap. 4], [7], [8]. However, when it comes to the analysis of systems with multiple timescales, the existing results are usually difficult to apply. In fact, time scale separation is ubiquitous in physical, biological, and ecological systems [9, Chap. 20], and one often needs to study their partial stability. Therefore, there is a great need to further develop new criteria for partial stability analysis, in particular in the setting of slow-fast systems. In this letter, we aim at developing new criteria for exponential partial stability of a particular type of slow-fast systems, wherein the fast variable is a scalar. Various practical systems can be modeled by this type of slow-fast systems, such as semiconductor lasers [10], and mixed-mode oscillations in chemical systems [11], where the fast scalar variables are the photon density and a chemical concentration, respectively. In particular, fast time-varying systems can always be modeled in this way since the time variable t can be taken as the fast scalar [12].

In full-state stability analysis of fast time-varying systems, averaging methods are widely used to establish criteria for exponential or asymptotic stability [13, Chap. 10], [14], [15]. Inspired by these works, we utilize periodic averaging techniques and obtain some criteria for partial exponential stability of the considered type of slow-fast systems. We then show that partial exponential stability of the averaged system implies partial exponential stability of the original one. The proposed averaging techniques are more general than the classical one (e.g., [14], [15]) in the sense that the system is averaged over the fast scalar variable, but not necessarily the time variable.

Next, we apply our obtained results to the study of partial exponential stability of a concrete slow-fast system that arises from the remote synchronization problem of coupled oscillators. Remote synchronization describes the phenomenon wherein oscillators coupled indirectly become synchronized, but the ones connecting them are not synchronized with them [5]. It is ubiquitous in nature, e.g, distant cortical regions without apparent direct neural links in the brain exhibit coherent behaviors [16]. Unlike the emergence of the classical synchronization, for which strong connections are required [17]–[19], remote synchronization is more associated with morphological symmetry. For example, nodes located remotely in a network might be able to swap their positions without changing the functioning of the overall system [20], [21]. In this letter, we study remote synchronization of oscillators coupled by a star network with two peripheral nodes. Despite the simple structure, this network has been shown to render zero-lag synchronization of remotely separated neuronal populations [22], coupled semiconductor lasers [23], or chaotic electronic circuits [24], even in the presence of considerable delays. The central node in this network is believed to play an essential role in dynamically relaying or mediating the dynamics of the peripheral ones. Some recent studies show that the remote synchronization can be enhanced if some parameter mismatch or heterogeneity is introduced to the central element [25], [26]. We seek to analytically study this interesting experimental finding.

Towards this end, we employ the Kuramoto-Sakaguchi model to describe the dynamics of the oscillators [27], wherein the phase shift term is often used to model small time delays such as synaptic connection delays [28], [29]. We first show that a large phase shift can destabilize remote synchronization. We then detune the natural frequency of the central oscillator to introduce some parameter heterogeneity. Modeling the problem into a slow-fast system and using our obtained criteria for partial stability, we rigorously prove that this natural frequency detuning strengthens the remote synchronization by making it more robust against phase shifts. Notice that the remote synchronization emerges in the absence of frequency synchronization (or phase locking) of the network, in sharp contrast to the classical synchronization of coupled Kuramoto oscillators (see [30] for a survey). It is the first-known attempt to analytical study phase-unlocked remote synchronization.

The rest of the letter is structured as follows. Section II introduces the model of a slow-fast system and formulates the problem. The

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main results are provided in Section III. As an application, remote synchronization of Kuramoto-Sakaguchi oscillators is studied in Section IV. Concluding remarks appear in Section V.

Notations: Let $\mathbb{R}$ denote the set of real numbers. For any $\delta > 0$, let $\mathcal{B}_\delta := \{x \in \mathbb{R}^n : \|x\| < \delta\}$. Given two vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, denote $\mathrm{col}(x, y) = (x^\top, y^\top)^\top$. Denote the unit circle by $\mathbb{S}^1$, and a point on it is called a *phase* since the point can be used to indicate the phase angle of an oscillator. For any two phases $\theta_1, \theta_2 \in \mathbb{S}^1$, the geodesic distance between them is the minimum of the lengths of the counter-clockwise and clockwise arcs connecting them, which is denoted by $|\theta_1 - \theta_2|_{\mathbb{S}}$; the geodesic difference between θ_1 and θ_2 is $\langle\theta_1, \theta_2\rangle := \theta_1 - \theta_2 + 2n\pi$, where n is the integer such that $|\theta_1 - \theta_2 + 2n\pi| = |\theta_1 - \theta_2|_{\mathbb{S}}$. Note that $\langle\theta_1, \theta_2\rangle \in \mathbb{S}^1$ for any $\theta_1, \theta_2 \in \mathbb{S}^1$. Let $\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ denote the n-torus. Given $\gamma, \psi \in \mathbb{T}^n$, $\|\gamma - \psi\|_{\mathbb{S}} := \sqrt{|\gamma_1 - \psi_1|_{\mathbb{S}}^2 + \cdots + |\gamma_n - \psi_n|_{\mathbb{S}}^2}$.

II. Problem Formulation

Consider a class of slow-fast systems described by

$$\dot{x} = f_1(x, y, z), \quad \dot{y} = f_2(x, y, z), \quad \varepsilon\dot{z} = f_3(x, y, z), \tag{1}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^m, z \in \mathbb{R}$, and $\varepsilon > 0$ is a small constant. That is, x, y are the states of slow dynamics, and z of the fast dynamics. All the maps, $f_1 : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$, $f_2 : \mathbb{R}^{n+m+1} \to \mathbb{R}^m$, $f_3 : \mathbb{R}^{n+m+1} \to \mathbb{R}$, are continuously differentiable, and T-periodic in z, i.e., $f_i(x, y, z + T) = f_i(x, y, z)$ for all $i = 1, 2, 3$. We further assume: a) the f_i's are such that the solution to the system (1) with any initial condition, (x_0, y_0, z_0), exists for all $t \geq 0$; b) $x = 0$ is a partial equilibrium point of the system (1), i.e., $f_1(0, y, z) = 0$ for any $y \in \mathbb{R}^m$ and $z \in \mathbb{R}$; and c) $f_2(0, y, z) = 0$ for any y and z.

We are interested in studying uniform partial exponential stability of the system (1). Let us first provide the definition.

Definition 1 ([2, Chap. 4]): A partial equilibrium point $x = 0$ of the system (1) is *exponentially x-stable* uniformly in y and z if there exist $c_1, c_2, \delta > 0$ such that $\|x_0\| < \delta$ implies that $\|x(t)\| \leq c_1\|x_0\|e^{-c_2 t}$ for any $t \geq 0$ and $(y_0, z_0) \in \mathbb{R}^m \times \mathbb{R}$.

In this note, when we refer to this definition, we also say that $x = 0$ of the system (1) is partially exponentially stable or the system (1) is partially exponentially stable with respect to x. Some noticeable efforts to study partial stability of nonlinear systems are found in [1], [2, Chap. 4], [8], where Lyapunov criteria have been established. However, it is not always easy to verify partial stability by using those criteria. As a motivating example, we consider the following academic but suggestive model.

Example 1: Consider a nonlinear system whose dynamics are described by $\dot{x} = -x - 0.2x\sin y - 2x\cos z, \dot{y} = 2x\cos y + x\sin z, \varepsilon\dot{z} = 3 - \sin x + \cos y$. As it will be shown later, for sufficiently small $\varepsilon > 0$, it is possible to prove that the partial equilibrium point $x = 0$ is exponentially stable uniformly in y and z. However, it is difficult to construct a Lyapunov function using the existing criteria. For example, one might choose $V = x^2$ as a Lyapunov function candidate. Its time derivative $\dot{V} = -2(1 + 0.2\sin y + 2\cos z)x^2$ can be positive for some y and z, while it is required by [2, Theorem 1, Chap. 4] to be negative for any $x \neq 0$, y, and z in order to show the partial exponential stability. △

Motivated by the above example, in the next section we aim at further developing Lyapunov theory for partial stability analysis of slow-fast systems.

III. Partial Stability of Slow-Fast Dynamics

In this section, our goal is to provide a new Lyapunov criterion for partial stability of the system (1). First, we construct reduced dynamics for it. Under some practically reasonable assumptions, the partial stability of the reduced system and that of the original (1) are shown to be equivalent. That is, analysis reduces to the study of partial stability of the reduced dynamics. Second, we will develop a new criterion for partial stability of the reduced system via averaging. This new criterion can be then used to deduce the partial stability of the original system (1).

A. Reduced Dynamics

In this subsection, we construct a reduced dynamics. We make the following assumption first.

Assumption 1: We assume that

$$f_3(x, y, z) \geq \vartheta, \qquad \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m, z \in \mathbb{R}, \tag{2}$$

or $f_3(x, y, z) \leq -\vartheta$, where $\vartheta > 0$.

Note that we only consider $f_3 \geq \vartheta$ in this letter since the other case $f_3 \leq -\vartheta$ is essentially the same. Assumption 1 is naturally satisfied for some practical problems such as vibration suppression of rotating machinery where f_3 is the angular velocity [31], and spin stabilization of spacecrafts where f_3 describes the spin rate [32].

Assumption 1 implies that $t \mapsto z(t)$ can be interpreted as a change of time (recall that z is a scalar). That is, $t \mapsto z(t)$ is a global diffeomorphism[1] from $[0, \infty)$ to $[0, \infty)$. In the new time axis $z(t)$, the slow-fast system becomes

$$\begin{aligned}\frac{dx(t)}{dz(t)} &= \frac{dx(t)}{dt}\frac{dt}{dz(t)} = \varepsilon\frac{f_1(x(t), y(t), z(t))}{f_3(x(t), y(t), z(t))},\\ \frac{dy(t)}{dz(t)} &= \frac{dy(t)}{dt}\frac{dt}{dz(t)} = \varepsilon\frac{f_2(x(t), y(t), z(t))}{f_3(x(t), y(t), z(t))}.\end{aligned}$$

This system can be viewed as a time-varying system with the new time variable $z(t)$. Note that, since $t \mapsto z(t)$ is a global diffeomorphism, the partial stability with respect to x of the system in this new time axis is equivalent to that of the system (1) in the original time axis. Therefore, hereafter we focus on the system in the new time axis. For the sake of simplicity of description, this system is rewritten into

$$\frac{dx}{dz} = \varepsilon h_1(x, y, z), \qquad \frac{dy}{dz} = \varepsilon h_2(x, y, z), \tag{3}$$

where $h_1 = f_1/f_3$, and $h_2 = f_2/f_3$. From the properties of f_1, f_2, and f_3, it holds that $h_1(0, y, z) = 0$ and $h_2(0, y, z) = 0$ for any $y \in \mathbb{R}^m$ and $z \in \mathbb{R}$, and both h_1 and h_2 are T-periodic in z. Also, the solution to the system (3) exists for all $z \geq z_0$.

B. Partial Stability Conditions via Averaging

In order to study partial stability with respect to x of the constructed periodic slow dynamics (3), we resort to a periodic averaging technique. Different from existing results, we partially average the system (3), and the (partially) averaged system is given by

$$\frac{dw}{dz} = \varepsilon h_{\mathrm{av}}(w, v), \qquad \frac{dv}{dz} = \varepsilon h_2(w, v, z), \tag{4}$$

where the function h_{av} is defined by

$$h_{\mathrm{av}}(w, v) = \frac{1}{T}\int_0^T h_1(w, v, \tau)d\tau, \tag{5}$$

where $h_{\mathrm{av}}(0, v) = 0$ for any $v \in \mathbb{R}^m$ as $h_1(0, v, z) = 0$. Denote the initial condition of (4) by (w_0, v_0, z_0).

The following theorem (with proof deferred to later subsections) shows how the averaged system (4) can be used to infer the partial stability of the system (3), and, in turn, of the original system (1).

[1]This follows from the earlier assumption that for any initial condition the solution to the system (1) exists for all $t \geq 0$ (which implies f_3 is upper bounded).

Theorem 1: Suppose that $w = 0$ of the averaged system (4) is partially exponentially stable uniformly in v, i.e., there exist $\delta, k, \lambda > 0$ such that for any $(v_0, z_0) \in \mathbb{R}^m \times \mathbb{R}$ and $w_0 \in \mathcal{B}_\delta$, $\|w(z)\| \leq k\|w_0\|e^{-\lambda(z-z_0)}$ for all $z \geq z_0$. Assume that there are $L_1, L_2 > 0$ such that for any $x \in \mathcal{B}_\delta, y \in \mathbb{R}^m, z \in \mathbb{R}$, the functions h_1 and h_2 in (3) satisfy

$$\left\|\frac{\partial h_1}{\partial \mathbf{x}}(x,y,z)\right\| \leq L_1, \qquad \left\|\frac{\partial h_2}{\partial \mathbf{x}}(x,y,z)\right\| \leq L_2, \tag{6}$$

where $\mathbf{x} = \text{col}(x, y)$. Then, there exists $\varepsilon_1 > 0$ such that, for any $\varepsilon < \varepsilon_1$, the partial equilibrium point $x = 0$ of the system (3) is exponentially stable uniformly in y. Consequently, $x = 0$ of the system (1) is also exponentially stable uniformly in y and z. △

Averaging methods have been used to study (full-state) exponential stability [14], asymptotic stability [15], and practical asymptotic stability [33], but not for partial stability. As clarified by Theorem 1, for partial exponential stability analysis, not all states need to be averaged. This is the core distinguishing feature of partial stability analysis. Notice that Theorem 1 states that if a ball of initial states for the averaged system associated with the partial exponential stability exists, then some corresponding ball exists for the original system, but the two balls can have different radii.

We illustrate the utility of Theorem 1 by revisiting Example 1, and see how the obtained results can be applied.

Continuation of Example 1: As $3 - \sin x + \cos y \geq 1$ for any x, y, Assumption 1 holds. Then, for $T = 2\pi$ one can construct the averaged system (4) of the system in Example 1 as

$$\frac{dw}{dz} = \varepsilon\frac{-w - 0.2w\sin v}{3 - \sin w + \cos v}, \qquad \frac{dv}{dz} = \varepsilon\frac{2w\cos v}{3 - \sin w + \cos v}.$$

Choose a Lyapunov function candidate $V(w, v, z) = w^2$. Then, it holds that $\frac{dV}{dz} = -2\varepsilon \cdot \frac{1+0.2\sin v}{3-\sin w+\cos v}w^2 \leq -\frac{8}{15}\varepsilon w^2$. According to [2, Theorem 1], $w = 0$ of the averaged system is partially exponentially stable. From Theorem 1, one can conclude that $x = 0$ of the original system in Example 1 is partially exponentially stable if $\varepsilon > 0$ is sufficiently small. △

By using averaging techniques, Theorem 1 provides a new way to study partially stability of slow-fast systems for which the existing criteria are difficult to apply. Next, we provide a corollary of this theorem for a simpler version of the system (1).

Consider the following system with respect to x,

$$\dot{x} = f_1(x, z), \qquad \varepsilon\dot{z} = f_3(x, z), \tag{7}$$

where f_1 and f_3 satisfy all the assumptions for the system (1), but without the variable y. To study partial stability with respect to x, we apply the change of time-axis, $t \to z$. Then, we have

$$\frac{dx}{dz} = \varepsilon h_1(x, z), \tag{8}$$

where $h_1 := f_1/f_3$. Next, compute the averaged system of the fast subsystem

$$\frac{dw}{dz} = \varepsilon\hat{h}_{\text{av}}(w), \tag{9}$$

where the function $\hat{h}_{\text{av}}$ is defined by $\hat{h}_{\text{av}}(w) = \frac{1}{T}\int_0^T h_1(w, \tau)d\tau$.

As expected, if the averaged system (9) is exponentially stable, then the partial stability of (7) is ensured as long as $\varepsilon > 0$ is sufficiently small; this is formally stated in the following corollary.

Corollary 1: Suppose that $w = 0$ is exponentially stable for the averaged system (9). Assume that there is $L > 0$ such that for any $x \in \mathcal{B}_\delta, z \in \mathbb{R}$ the function h_1 in (8) satisfies

$$\left\|\frac{\partial h_1}{\partial x}(x, z)\right\| \leq L. \tag{10}$$

Then, there exists $\varepsilon_1 > 0$ such that, for any $\varepsilon < \varepsilon_1$, the partial equilibrium $x = 0$ of the system (7) is partially exponentially stable uniformly in z. △

If $f_3(x, z) = 1$ for all $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$, this corollary reduces to the criteria in [13, Chap. 10] and [14] for exponential stability of fast time-varying systems since z and t are the same.

In the next section, we apply our results on partial exponential stability to the study of remote synchronization of coupled oscillators. Before that, we construct the proof of Theorem 1 in the following subsections. As typically done in averaging methods, the system (3) can be taken as a perturbed system of (4), where the perturbation decreases as ε does. We construct a Lyapunov function for (4), and use this Lyapunov function to show that the system (3) is also partially exponentially stable with the aid of perturbation theory. Towards this end, we next provide a converse Lyapunov result and a perturbation inequality for partially exponentially stable systems.

C. A Converse Lyapunov Result

This subsection is dedicated to constructing a Lyapunov function. As a generalized form of (4), we consider the following time-varying systems

$$\frac{dw}{dz} = \varphi_1(w, v, z), \qquad \frac{dv}{dz} = \varphi_2(w, v, z), \tag{11}$$

where $w \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, $z \in \mathbb{R}$, and the functions, $\varphi_1 : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$, $\varphi_2 : \mathbb{R}^{n+m+1} \to \mathbb{R}^m$ are continuously differentiable, and satisfy that $\varphi_1(0, v, z) = 0$ and $\varphi_2(0, v, z) = 0$ for any $v \in \mathbb{R}^m$. We further assume that for any z_0 the solution to the system (11) exists for all $z \geq z_0$.

Now, we provide a converse result that is directly applicable to the averaged system (4).

Proposition 1: Suppose that $w = 0$ is partially exponentially stable uniformly in v for the system (11), i.e., there exists $\delta > 0$ such that for any $z_0 \in \mathbb{R}$ and $w(0) \in \mathcal{B}_\delta$, there are $k, \lambda > 0$ such that $\|w(z)\| \leq k\|w(0)\|e^{-\lambda(z-z_0)}$ for all $z \geq z_0$. Also, assume that there are $L_1, L_2 > 0$ such that

$$\left\|\frac{\partial\varphi_1}{\partial \mathbf{w}}(w, v, z)\right\| \leq L_1, \qquad \left\|\frac{\partial\varphi_2}{\partial \mathbf{w}}(w, v, z)\right\| \leq L_2, \tag{12}$$

for any $w \in \mathcal{B}_\delta, v \in \mathbb{R}^m, z \in \mathbb{R}$, where $\mathbf{w} = \text{col}(w, v)$. Then, there exists a function $V : \mathcal{B}_\delta \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ that satisfies the following inequalities: 1) $c_1\|w\|^2 \leq V(w, v, z) \leq c_2\|w\|^2$, 2) $\frac{\partial V}{\partial z} + \frac{\partial V}{\partial w}\varphi_1(w, v, z) + \frac{\partial V}{\partial v}\varphi_2(w, v, z) \leq -c_3\|w\|^2$, 3) $\left\|\frac{\partial V}{\partial w}\right\| \leq c_4\|w\|$, and 4) $\left\|\frac{\partial V}{\partial v}\right\| \leq c_5\|w\|$ for some constants $c_1, c_2, c_3, c_4, c_5 > 0$. △

If we let $\mathcal{A} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : x = 0\}$, Proposition 1 can be equivalently taken as a converse result for exponential stability of the unbounded closed set $\mathcal{A}$. Studies of converse Lyapunov theorems for stability of sets date back to decades ago [34], [35, Chap. V]. Those early results have later been developed for stability of compact sets and general closed sets [36], [37] (see also a comprehensive survey [38]). The proof of the proposition is similar to those converse theorems in the literature, and is thus omitted. For a complete proof, we refer to the extended arXiv version of this letter [39].

D. Analysis of Perturbed Systems

In this subsection, following some classical results, we provide a perturbation inequality for the following perturbed version of the system (11):

$$\frac{dw_p}{dz} = \varphi_1(w_p, v_p, z) + g_1(w_p, v_p, z), \tag{13a}$$

$$\frac{dv_p}{dz} = \varphi_2(w_p, v_p, z) + g_2(w_p, v_p, z), \tag{13b}$$

where $g_1 : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$ and $g_2 : \mathbb{R}^{n+m+1} \to \mathbb{R}^m$ are piecewise continuous in z and locally Lipschitz in (w_p, v_p). In particular, we assume that the perturbation terms satisfy the bounds

$$\|g_1\| \le \gamma_1(z)\|w_p\| + \psi_1(z), \quad \|g_2\| \le \gamma_2(z)\|w_p\| + \psi_2(z), \tag{14}$$

where $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}$ are nonnegative and continuous for all $z \in \mathbb{R}$, and $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$ are nonnegative, continuous and bounded for all $z \in \mathbb{R}$. Notice that the bounds on the right are independent of v_p.

The following proposition presents some results on the behavior of the perturbed system (13) when the nominal system (11) has a partially exponentially stable equilibrium $w_p = 0$.

Proposition 2: Suppose that the nominal system (11) satisfies all the assumptions in Proposition 1. Also, assume that the perturbation terms $g_1(w_p, v_p, z)$ and $g_2(w_p, v_p, z)$ are respectively bounded as in (14) for γ_1, γ_2 and ψ_1, ψ_2 satisfying

$$c_4 \int_{z_0}^{z} \gamma_1(\tau)d\tau + c_5 \int_{z_0}^{z} \gamma_2(\tau)d\tau \le \kappa(z - z_0) + \eta, \tag{15}$$

where $0 \le \kappa < c_1c_3/c_2, \eta \ge 0$, and $c_4\psi_1(z) + c_5\psi_2(z) < 2c_1k_1\delta/k_2$ for all $z \ge z_0$ with $k_1 = c_3/2c_2 - \kappa/2c_1$ and $k_2 = e^{\eta/2c_1}$. Then for all $z \ge z_0$, the solution to the perturbed (13) satisfies that $\|w_p(z)\| \le k_2\sqrt{\frac{c_2}{c_1}}\|w_p(z_0)\|e^{-k_1(z-z_0)} + \frac{k_2}{2c_1}\int_{z_0}^{z} e^{-k_1(z-\tau)}\psi(\tau)d\tau$ for any initial time $z_0 \in \mathbb{R}$ and any initial state $w_p(z_0) \in \mathbb{R}^n$ and $v_p(z_0) \in \mathbb{R}^m$ such that $\|w_p(z_0)\| < \frac{\delta}{k_2}\sqrt{\frac{c_1}{c_2}}$. △

If the perturbations g_1 and g_2 in (13) are vanishing, we obtain the next Corollary, which will be used to prove Theorem 1.

Corollary 2: Suppose that the nominal system (11) satisfies all the assumptions in Proposition 2. In addition, assume that $\psi_1(\cdot) = 0$ and $\psi_2(\cdot) = 0$. Then, $w_p = 0$ of the system (13) is partially exponentially stable uniformly in v_p. Moreover, the solution to (13) satisfies that for all $z \ge z_0$ it holds that $\|w_p(z)\| \le k_2\sqrt{\frac{c_2}{c_1}}\|w_p(z_0)\|e^{-k_1(z-z_0)}$ for any initial time $z_0 \in \mathbb{R}$ and any initial condition $w_p(z_0) \in \mathcal{B}_\delta$ and $v_p(z_0) \in \mathbb{R}^m$ satisfying $\|w_p(z_0)\| < \frac{\delta}{k_2}\sqrt{\frac{c_1}{c_2}}$.

Some perturbation theorems for full-state, but not partial, stability analysis can be found in [13]. Our results are slightly more general since Proposition 2 and Corollary 2, respectively, reduce to Lemma 9.4 and Corollary 9.1 in [13] if $v_p \in \mathbb{R}$, $\varphi_2 = 1$, and $g_2 = 0$. The proofs are similar to those in [13], and are thus omitted.

E. Proof of Theorem 1

Now, we are ready to provide the proof of Theorem 1.

Proof: First, we introduce the following change of variables to the original slow system (3):

$$x = w_p + \varepsilon u(w_p, v_p, z), \qquad y = v_p, \tag{16}$$

where

$$u(w_p, v_p, z) = \int_0^z \Delta(w_p, v_p, \tau)d\tau, \tag{17}$$

with $\Delta(w_p, v_p, z) = h_1(w_p, v_p, z) - h_{\rm av}(w_p, v_p)$. From the definition of $h_{\rm av}$ in (5), it holds that $\int_0^T \Delta(w_p, v_p, \tau)d\tau = 0$. After substituting (16) into (3), we obtain

$$\frac{dx}{dz} = \frac{dw_p}{dz} + \varepsilon\frac{\partial u}{\partial z} + \varepsilon\frac{\partial u}{\partial w_p}\frac{dw_p}{dz} + \varepsilon\frac{\partial u}{\partial v_p}\frac{dv_p}{dz}, \quad \frac{dy}{dz} = \frac{dv_p}{dz}.$$

Substituting (3) and (16) into the above equations yields

$$P(\varepsilon)\begin{bmatrix}\frac{dw_p}{dz}\\ \frac{dv_p}{dz}\end{bmatrix} = \begin{bmatrix}\varepsilon h_1(w_p + \varepsilon u, v_p, z) - \varepsilon h_1(w_p, v_p, z) + \varepsilon h_{av}(w_p, v_p)\\ \varepsilon h_2(w_p + \varepsilon u, v_p, z)\end{bmatrix}, \tag{18}$$

where $P = \begin{bmatrix} I + \varepsilon\frac{\partial u}{\partial w_p} & \varepsilon\frac{\partial u}{\partial y}\\ 0 & I\end{bmatrix}$. We then show that the obtained dynamics (18) can be viewed as a perturbation of the averaged system (4). Therefore, Corollary 2 can be used to show the partial stability of the obtained dynamics from that of the averaged system. Our goal is to show that the partial stability of the obtained dynamics (18) implies that of the original slow system (3).

Let us represent the obtained dynamics (18) by a perturbation of the averaged system (4). For $k = 1, 2$, let h_k^i be the ith component of h_k. From the mean value theorem, for each $k = 1, 2$, there exists $\lambda_k^i = \lambda_k^i(w_p, v_p, z, \varepsilon) > 0$ such that $h_k^i(w_p + \varepsilon u, v_p, z) - h_k^i(w_p, v_p, z) = \frac{\partial h_k^i}{\partial w_p}(w_p + \varepsilon\lambda_k^i u, v_p, z) \cdot \varepsilon u$. Let us denote

$$H_1 = \left[\frac{\partial h_1^1}{\partial w_p}(w_p + \varepsilon\lambda_1^1 u, v_p, z), \dots, \frac{\partial h_1^n}{\partial w_p}(w_p + \varepsilon\lambda_1^n u, v_p, z)\right]^\top,$$

$$H_2 = \left[\frac{\partial h_2^1}{\partial w_p}(w_p + \varepsilon\lambda_2^1 u, v_p, z), \dots, \frac{\partial h_2^n}{\partial w_p}(w_p + \varepsilon\lambda_2^n u, v_p, z)\right]^\top.$$

Then we have

$$h_1(w_p + \varepsilon u, v_p, z) - h_1(w_p, v_p, z) = H_1 \cdot \varepsilon u, \tag{19}$$

$$h_2(w_p + \varepsilon u, v_p, z) - h_2(w_p, v_p, z) = H_2 \cdot \varepsilon u, \tag{20}$$

where both H_1 and H_2 are bounded since from (6) each $\partial h_k^i/\partial w$ is. Due to the boundedness of $\|\partial u/\partial z\|$, $\|\partial u/\partial w_p\|$, and $\|\partial u/\partial v_p\|$ from Proposition 3 in Appendix VI-A, it is clear that the matrix $P(\varepsilon)$ is nonsingular for sufficiently small $\varepsilon > 0$, and its inverse can be described as $P^{-1}(\varepsilon) = I + \mathcal{O}(\varepsilon)$ with some $\mathcal{O}(\varepsilon)$. Applying this fact together with the equalities (19) and (20) to (18), one can show that there are bounded $H_1'(w_p, v_p, z, \varepsilon u)$ and $H_2'(w_p, v_p, z, \varepsilon u)$ such that

$$\frac{dw_p}{dz} = \varepsilon h_{av}(w_p, v_p) + \varepsilon^2 H_1'(w_p, v_p, z, \varepsilon u)u, \tag{21a}$$

$$\frac{dv_p}{dz} = \varepsilon h_2(w_p, v_p, z) + \varepsilon^2 H_2'(w_p, v_p, z, \varepsilon u)u. \tag{21b}$$

This is a perturbation of the averaged system (4).

Next, we apply Corollary 2 to show that the partial exponential stability of the averaged system (4) implies that of its perturbation (21) for sufficiently small $\varepsilon > 0$. From the definition of $h_{\rm av}$, we have $\left\|\frac{\partial h_{\rm av}}{\partial \mathbf{w}_p}(w_p, v_p)\right\| = \left\|\frac{1}{T}\int_0^T \frac{\partial h_1}{\partial \mathbf{w}_p}(w_p, v_p, \tau)d\tau\right\| \le L_1$ for any $w_p \in \mathcal{B}_\delta, v \in \mathbb{R}^m$, where $\mathbf{w}_p = {\rm col}(w_p, v_p)$. By the assumptions in (6), $\|\partial h_2/\partial \mathbf{w}_p\| \le L_2$. Therefore, both inequalities in (12) are satisfied. Since the system (4) is assumed to be partially exponentially stable, all the assumptions in Proposition 2 are satisfied. To apply Corollary 2, it remains to show that the perturbation terms are bounded linearly in $\|w_p\|$. Let $b_1 > 0$ and $b_2 > 0$ be constants such that $\|H_1'(w, v, z, \varepsilon u)\| \le b_1$ and $\|H_2'(w_p, v_p, z, \varepsilon u)\| \le b_2$. From (28) in Appendix VI-A, it holds that $\|u(w_p, v_p, s)\| \le 2TL_1\|w\|$, and then the perturbation terms satisfy $\|\varepsilon^2 H_1'(w_p, v_p, z, \varepsilon u)u\| \le 2\varepsilon^2 b_1 TL_1\|w_p\|, \|\varepsilon^2 H_2'(w_p, v_p, z, \varepsilon u)u\| \le 2\varepsilon^2 b_2 TL_1\|w_p\|$. Moreover, for sufficiently small $\varepsilon_1 > 0$, any $\varepsilon < \varepsilon_1$ ensures the inequalities in Corollary 2 is satisfied. Therefore, Corollary 2 implies that $w_p = 0$ is partially exponentially stable for the perturbed system (21), or, equivalently, the system (18). In other words, there are $\delta', k', \lambda' > 0$ such that $w_p(z_0) \in \mathcal{B}_{\delta'}$ implies $\|w_p(z)\| \le k'\|w_p(z_0)\|e^{-\lambda'(z-z_0)}$, for all $z \ge z_0$.

Finally, we show that the partial exponential stability of the system (18) implies that of the slow dynamics (3). From (16) and (28) in the Appendix, one obtains $|1 - 2\varepsilon TL_1| \cdot \|w_p(z)\| \le \|x(z)\| \le |1 + 2\varepsilon TL_1| \cdot \|w_p(z)\|$ for all $z \ge z_0$. Then, it follows that

$$\|x(z)\| \le k'\frac{|1 + 2\varepsilon TL_1|}{|1 - 2\varepsilon TL_1|}\|x(z_0)\|e^{-\lambda'(z-z_0)}, \qquad \forall z \ge z_0,$$

proving the partial exponential stability of $x = 0$ for the system (3) for sufficiently small $\varepsilon > 0$. Finally, one can conclude that $x = 0$ is also partially exponentially stable for the original slow-fast system (1) uniformly in y and z under Asssumption 1. ■

IV. Remote Synchronization in a Network of Kuramoto Oscillators

In this section, we apply the results on partial stability to the study of remote synchronization in a network motif depicted in Fig. 1.

A. Problem Statements and Preliminary

The dynamics of the oscillators are described by

$$\dot{\theta}_i = \omega + A_i \sin(\theta_0 - \theta_i - \alpha), i = 1, 2; \tag{22a}$$

$$\dot{\theta}_0 = \omega + \sum_{j=1}^{2} A_j \sin(\theta_j - \theta_0 - \alpha) + u, \tag{22b}$$

where $\theta_i \in \mathbb{S}^1$ is the phase of the ith oscillator; $\omega > 0$ is the uniform natural frequency of each oscillator; $A_i > 0$ is the coupling strength between the central node 0 and the peripheral node i; α is the phase shift, and we assume $\alpha \in (0, \pi/2)$ as generally done in the literature (e.g., [20]); and $u \geq 0$ is a constant representing the natural frequency detuning (for the case $u < 0$, one can obtain virtually identical results to those obtained below). Note that, the phase shift α is often used to model delays arising in synaptic connections in neural networks [28].

Let $\theta = (\theta_0, \theta_1, \theta_2)^\top \in \mathbb{T}^3$. To study the remote synchronization in our considered network, we define the remote synchronization manifold as follows.

Definition 2: The remote synchronization manifold is defined by $\mathcal{M} := \left\{\theta \in \mathbb{T}^3 : \theta_1 = \theta_2\right\}$.

A solution $\theta(t)$ to (22) is said to be *remotely synchronized* if it satisfies $\theta(t) \in \mathcal{M}$ for all $t \geq 0$. The simulation results in [20] has shown that when $u = 0$ the remote synchronization is stable for small α, but becomes unstable if α is large. However, an analytical characterization of the threshold for α is still missing. The first and secondary goal in the section is to identify this threshold as a measure of robustness of the remote synchronization against phase shifts or time delays. More interestingly, it is unclear how the introduction of a natural frequency detuning $u > 0$ affects this robustness. The second and main goal is then to study this problem.

In fact, remote synchronization can be categorized into two different types, i.e., *phase-locked* and *phase-unlocked* remote synchronization, depending on whether phase locking occurs or not. Phase locking is a phenomenon wherein every pairwise phase difference is a constant, i.e., $\theta_i - \theta_j = c_{ij}$ for all i, j (the special case wherein $c_{ij} = 0$ is called phase synchronization). Phase locking is also called frequency synchronization because it is equivalent to the case of all oscillators' frequencies being synchronized. In our considered case, for the phase-locked remote synchronization, it holds that $\dot{\theta}_0 = \dot{\theta}_1 = \dot{\theta}_2$, while for the phase-unlocked case, $\dot{\theta}_0$ is distinct from $\dot{\theta}_1 = \dot{\theta}_2 \neq \dot{\theta}_0$.

As shown in [20], network symmetries are critical to the emergence of remote synchronization. In our considered network in Fig. 1, we say that the oscillators 1 and 2 are *symmetric* if $A_1 = A_2$. In what follows, we assume that the oscillators 1 and 2 are symmetric[2].

[2]This assumption requires the two coupling strengths, A_1 and A_2, to be strictly identical. This is somewhat demanding since it is not easy to be fulfilled in practical situations. Numerical studies show that the phase difference between the peripheral oscillators remains bounded if A_1 and A_2 are only approximately the same, although exact phase synchronization cannot take place. It is quite interesting to study this problem in the future, though we only consider $A_1 = A_2$ in this letter.

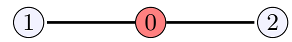

Fig. 1. A simple network motif: central node **0** and peripherals **1** and **2**.

Assumption 2 (Symmetric coupling): We assume that the coupling strengths satisfy $A_1 = A_2 = A$.

We make this assumption to ensure that both phase-locked and phase-unlock remote synchronization can emerge in (22). It is noteworthy that this assumption is necessary for phase-unlocked remote synchronization, which can be seen by the following argument by contradiction. Suppose that if $A_1 \neq A_2$, then phase-unlocked remote synchronization happens. It follows from (22a) that $\dot{\theta}_1 - \dot{\theta}_2 = A_1 \sin(\theta_0 - \theta_1 - \alpha) - A_2 \sin(\theta_0 - \theta_1 - \alpha) = 0$. Solving this equation yields $\theta_0 - \theta_1 = n\pi + \alpha$, and subsequently, it holds that $\dot{\theta}_0 = \dot{\theta}_1 = \dot{\theta}_2$, which implies a contradiction.

To study the stability of the remote synchronization, we study the stability of the corresponding manifolds. Given a manifold $\mathcal{C} \in \mathbb{T}^n$, define a δ-neighborhood of $\mathcal{C}$ by $U_\delta(\mathcal{C}) = \{\theta \in \mathbb{T}^n : \mathrm{dist}(\theta, \mathcal{C}) < \delta\}$, where $\mathrm{dist}(\theta, \mathcal{C})$ is the minimum distance from θ to a point on $\mathcal{C}$, that is, $\mathrm{dist}(\theta, \mathcal{C}) = \inf_{y \in \mathcal{C}} \|\theta - y\|_{\mathbb{S}}$. Let us define the exponential stability of manifolds.

Definition 3: For the system (22), a manifold $\mathcal{C} \in \mathbb{T}^3$ is said to be exponentially stable along the system (22) if there is $\delta > 0$ such that for any initial phase $\theta(0) \in \mathbb{T}^3$ satisfying $\theta(0) \in U_\delta(\mathcal{C})$ it holds that for all $t \geq 0$, $\mathrm{dist}(\theta(t), \mathcal{C}) = k \cdot \mathrm{dist}(\theta(0), \mathcal{C}) \cdot e^{-\lambda t}$. for some $k > 0$ and $\lambda > 0$.

B. Robustness of Remote synchronization when $u = 0$

Let us first define the following phase-locked remote synchronization manifolds: $\mathcal{M}_1 := \{\theta \in \mathcal{M} : \theta_0 - \theta_1 = c(\alpha)\}$ and $\mathcal{M}_1' := \{\theta \in \mathcal{M} : \theta_0 - \theta_1 = c'(\alpha)\}$, where $c(\alpha) := -\arctan\left(\frac{\sin\alpha}{3\cos\alpha}\right)$, $c'(\alpha) := \pi + c(\alpha)$. The main result in this subsection is presented as follows (the proof is in Appendix VI-B).

Theorem 2: Suppose that Assumption 2 is satisfied, and $u = 0$. For any $A > 0$, the following statements hold:

1) if $\alpha < \arctan\left(\sqrt{3}\right)$, there exists a unique exponentially stable phase-locked remote synchronization manifold, that is $\mathcal{M}_1$;
2) if $\alpha > \arctan\left(\sqrt{3}\right)$, there does not exist an exponentially stable phase-locked remote synchronization manifold. △

This theorem captures the robustness of the phase-locked remote synchronization against phase shifts (quantified by the threshold $\arctan\left(\sqrt{3}\right)$). The phase-locked remote synchronization is exponentially stable for a range of α, but becomes unstable if α is greater than the threshold (note that a bifurcation occurs when $\alpha = \arctan\sqrt{3}$, but the problem of whether remote synchronization is stable remains open). We further put forth that stable remote synchronization can only appear in the form of phase locking if $u = 0$. This is because for any $\alpha \in (0, \pi/2)$, the solution $\theta(t)$ to (22) converges to $\mathcal{M}_1$ for any initial condition $\theta(0) \in \mathcal{M} - (\mathcal{M}_1 \cup \mathcal{M}_1')$, which implies phase-unlocked remote synchronization manifold cannot be stable (see a brief analysis at the end of Appendix VI-B).

C. Robustness Improvement by the Detuning $u > 0$

In this subsection, we detune the natural frequency of the central oscillator (i.e., let $u > 0$) and show that a sufficiently large u actually stabilizes the remote synchronization even for α larger than the threshold in Theorem 2, making the remote synchronization much more robust. The main result of this section is presented in the following theorem.

Theorem 3: Suppose that Assumption 2 is satisfied. There is a positive constant $u_1 > 3A$ such that for any u satisfying $u > u_1$, the remote synchronization manifold $\mathcal{M}$ is exponentially stable along the system (22) for any phase shift $\alpha \in (0, \pi/2)$. △

Notice that $u > u_1 > 3A$ in this theorem implies that a phase-locked solution to (22) does not exist. This is because phase locking requires $\dot{x}_1 = \dot{x}_2 = 0$, i.e., $u + \sum_{j=1}^{2} A\sin(-x_j - \alpha) - A\sin(x_i - \alpha) = 0$ for $i = 1, 2$, but these equations do not have a solution. Consequently, the manifold $\mathcal{M}$ corresponds to phase-unlocked remote synchronization only. Note that given $\alpha > \arctan\sqrt{3}$, there exists $0 < u < 3A$, whose value depends on α, such that stable phase-locked remote synchronization may occur. However, this case is not of concern to us since we are more interested in providing a u that ensures the stability of the remote synchronization for any α. Particularly, we have shown that there exists a natural frequency detuning $u > 3A$ such that the remote synchronization is exponentially stable for any $\alpha \in (0, \pi/2)$, although phase locking becomes impossible. Most of the existing results on synchronization of Kuramoto (or generalized-Kuramoto) oscillators (e.g., [19], [40], and a survey [30]) are built on phase locking. When the phases are unlocked, the analysis is more challenging. It is noteworthy that Theorem 3 is the first-known result that has analytically studied phase-unlocked remote synchronization.

We next prove Theorem 3 using the results on partial stability.

Proof of Theorem 3: First, we define some new variables, and analyze the system defined on them. Specifically, let ϕ and z be

$$\phi = \frac{\theta_1 - \theta_2}{2}, \qquad z = \theta_0 - \frac{\theta_1 + \theta_2}{2} + \eta. \tag{23}$$

Here $\eta \in \mathbb{S}^1$ is the phase angle such that

$$\cos\eta = \frac{3\cos\alpha}{D}, \qquad \sin\eta = \frac{\sin\alpha}{D}, \tag{24}$$

where $D = \sqrt{9\cos^2\alpha + \sin^2\alpha}$. In fact, it can be observed from $\alpha \in (0, \frac{\pi}{2})$ that $\eta \in (0, \frac{\pi}{2})$. In addition, $\eta \to 0$ as $\alpha \to 0$ and $\eta \to \frac{\pi}{2}$ as $\alpha \to \frac{\pi}{2}$.

According to Proposition 4 in the Appendix VI-C, the time derivatives of ϕ and z are respectively given by

$$\dot{\phi} = -A\sin\phi\cos(z - \alpha - \eta), \tag{25a}$$

$$\dot{z} = u - DA\cos\phi\sin z. \tag{25b}$$

Notice that $3 \geq D = \sqrt{8\cos^2\alpha + 1}$. This and $u > 3A$ by assumption imply $u - DA\cos\phi\sin z > u - DA =: \vartheta > 0$ for any ϕ and z, i.e., Assumption 1 holds for (25). Following similar lines as Step 3 in the proof of Theorem 2, one can reason that the remote synchronization manifold $\mathcal{M}$ is exponentially stable if one can prove that the system (25) is partially exponentially stable with respect to ϕ.

Since $u > 3A$ by assumption, $u - DA\cos\phi\sin z > 0$ for any ϕ and z. Then,

$$\frac{d\phi}{dz} = -\frac{A\sin\phi\cos(z - \alpha - \eta)}{u - DA\cos\phi\sin z} := f(\phi, z). \tag{26}$$

To show the partial exponential stability of (25), it is sufficient to prove the exponential stability of (26) as argued in subsection II-A. We then associate the system (26) with the averaged system

$$\dot{\phi} = \varepsilon f_{\mathrm{av}}(\phi), \tag{27}$$

with $\varepsilon = \frac{1}{u}$ and

$$f_{\mathrm{av}}(\phi) = \int_0^{2\pi} f(\phi, z)dz = -A\sin\phi \underbrace{\int_0^{2\pi} \frac{\cos(z - \alpha - \eta)}{1 - \frac{1}{u}DA\cos\phi\sin z}dz}_{g(\phi)},$$

where the fact that 2π is the period has been used.

From (33) in Appendix VI-D, it holds with $a := DA\cos\phi$ that

$$\begin{aligned} g(\phi) &= \frac{2u\pi\sin(\alpha+\eta)}{a\sqrt{u^2 - a^2}}\left(u - \sqrt{u^2 - a^2}\right) \\ &\geq \frac{2\pi\sin(\alpha+\eta)}{DA}\left(u - \sqrt{u^2 - a^2}\right), \end{aligned}$$

where $\sqrt{u^2 - a^2} \leq u$ and $a = DA\cos\phi \leq DA$ are used. Consider a constant $\epsilon \in [0, \pi/2)$. For any ϕ satisfying $|\phi| \leq \epsilon$, it holds that $DA\cos\epsilon \leq DA\cos\phi = a$. This implies

$$g(\phi) \geq \frac{2\pi\sin(\alpha+\eta)}{DA}\left(u - \sqrt{u^2 - (DA)^2\cos^2\epsilon}\right) := C.$$

As mentioned, $\alpha, \eta \in (0, \frac{\pi}{2})$, which means $\alpha + \eta \in (0, \pi)$ and consequently $\sin(\alpha + \eta) > 0$. Therefore, C is a positive constant.

In summary, it follows that $\dot{\phi} \leq -\varepsilon CA\sin\phi$ for any ϕ satisfying $|\phi| \leq \epsilon$. The comparison principle and $-\varepsilon CA\cos(0) = -\varepsilon CA < 0$ imply the exponential stability of the averaged system at $\phi = 0$.

According to Corollary 1, one can prove that there exists $\varepsilon_1 > 0$ such that for any $\varepsilon < \varepsilon_1$ the system (25) is partially exponentially stable. Since $\varepsilon = \frac{1}{u}$, there exists $u_1 > 3A$ such that for any $u > u_1$ the system (25) is partially exponentially stable, which implies that the remote synchronization manifold $\mathcal{M}$ is exponentially stable. ■

V. Concluding Remarks

Using periodic averaging methods, we have obtained some criteria for partial exponential stability of a type of slow-fast nonlinear systems in this letter. Associating the original system with an averaged one by averaging over the fast varying variable, we have shown that the partial exponential stability of the averaged system implies that of the original one. We have applied our results to the stability analysis of remote synchronization in a network of Kuramoto oscillators with a phase shift. In the future, we are interested in developing new criteria for partial asymptotic stability of nonlinear systems using averaging techniques. Moreover, it is interesting to study stability of remote synchronization in more complex networks.

VI. Appendix

A. Proposition 3

Proposition 3: Consider the function $u(w_p, v_p, z)$ defined in (17). For any $w_p \in \mathcal{B}_\delta$, $v_p \in \mathbb{R}^m$, and $z \in \mathbb{R}$, $\|u(w_p, v_p, z)\|$, $\|\partial u/\partial w_p\|$, and $\|\partial u/\partial v_p\|$ are all bounded, and particularly

$$\|u(w_p, v_p, z)\| \leq 2TL_1\|w_p\|. \tag{28}$$

Proof: First, we prove that $\|u(w_p, v_p, z)\|$ is bounded. One can observe that $u(w_p, v_p, z)$ is T-periodic in z since $\Delta(w_p, v_p, z)$ is. For any $z \geq 0$, there exists a nonnegative integer N_1 and z' satisfying $0 \leq z' < T$ such that $z = N_1 T + z'$. Then, from $\int_0^T \Delta(w_p, v_p, \tau)d\tau = 0$ we have $\int_0^z \Delta(w_p, v_p, \tau)d\tau = \int_0^{z'} \Delta(w_p, v_p, \tau)d\tau$. Next, the partial derivative of Δ with respect to w satisfies

$$\left\|\frac{\partial\Delta}{\partial w_p}\right\| = \left\|\frac{\partial h_1}{\partial w_p} - \frac{1}{T}\int_0^T \frac{\partial h_1}{\partial w_p}(w_p, v_p, \tau)d\tau\right\| \leq 2L_1,$$

where the inequalities $\|\partial h_1/\partial w_p\| \leq \|[\partial h_1/\partial w_p, \partial h_1/\partial v_p]\|$ and (6) have been used. This inequality and $\Delta(0, v_p, z) = 0$ in (17) yield $\|u\| \leq \int_0^{z'} \|\Delta(w_p, v_p, \tau) - \Delta(0, v_p, \tau)\|d\tau \leq 2z'L_1\|w_p\|$, where implies (28). For any $w_p \in \mathcal{B}_\delta$ and $v_p \in \mathbb{R}^m$, $z \in \mathbb{R}$, it is clear that $\|u(w_p, v_p, z)\| \leq 2TL_1\delta$. ■

B. Analysis of Section IV-B

Proof of Theorem 2: (Step 1) First, we confirm that for any $\alpha \in (0, \pi/2)$, any phase-locked and remotely synchronized solution to (22) belongs to either $\mathcal{M}_1$ or $\mathcal{M}_1'$. In other words, $\mathcal{M}_1$ or $\mathcal{M}_1'$ are the only two phase-locked remote synchronization manifolds.

A solution $(\theta_0, \theta_1, \theta_2) \in \mathbb{T}^3$ is phase-locked and remotely synchronized if and only if there exists $\gamma \in \mathbb{S}^1$ such that $\theta_0 + \gamma = \theta_1 = \theta_2$. Then, from (22) with $A = A_1 = A_2$ and $u = 0$, the solution satisfies $2\sin(\gamma - \alpha) + \sin(\gamma + \alpha) = 0$. In fact, only $\gamma = c(\alpha)$ and $\beta = c'(\alpha)$ satisfy this equation in $\mathbb{S}^1$.

(Step 2) Next, we study stability of $x = c(\alpha)\mathbf{1}_2$ by viewing the system (22) as a system defined on $\mathbb{R}^3$ with $\theta_i \in \mathbb{R}^3$, $i = 1, 2, 3$. Then we apply the change of coordinates $x_i = \theta_0 - \theta_i$, $i = 1, 2$ (and $\theta_0 = \theta_0$). In the new coordinates, the system with $A = A_1 = A_2$ and $u = 0$ becomes

$$\dot{x}_i = \sum_{j=1}^{2} A\sin(-x_j - \alpha) - A\sin(x_i - \alpha),\ i = 1, 2, \tag{29a}$$

$$\dot{\theta}_0 = \omega + A\sum_{j=1}^{2}\sin(x_j - \alpha). \tag{29b}$$

Note that $\dot{x}_i$, $i = 1, 2$ does not depend on θ_0, i.e., the subsystem (29a) is in the closed form. From Step 1, both $x = c(\alpha)\mathbf{1}_2 \in (-\pi/2, \pi/2)^2$ and $x = c'(\alpha)\mathbf{1}_2 \in (\pi/2, 3\pi/2)^2$ are equilibria of this subsystem.

The Jacobian matrix corresponding to this subsystem at $x = (x_1, x_2)^\top \in \mathbb{R}^2$ is

$$J(x) = -A\begin{bmatrix}\cos(x_1+\alpha) + \cos(x_1 - \alpha) & \cos(x_2 + \alpha) \\ \cos(x_1+\alpha) & \cos(x_2+\alpha) + \cos(x_2 - \alpha)\end{bmatrix}. \tag{30}$$

If $0 < \alpha < \arctan(\sqrt{3})$, all the eigenvalues of $J(c(\alpha)\mathbf{1}_2)$ are negative, and thus the subsystem is exponentially stable at $c(\alpha)$. Moreover, since $\dot{x}_i$, $i = 1, 2$ is independent from θ_0, the system (29) is exponentially x-stable uniformly in θ_0 at $c(\alpha)\mathbf{1}_2$. In other words, if $0 < \alpha < \arctan(\sqrt{3})$, there exist $c_1, c_2, \delta > 0$ such that

$$\|x(t) - c(\alpha)\mathbf{1}_2\|_2 \le c_1 e^{-c_2 t}\|x(0) - c(\alpha)\mathbf{1}_2\|_2 \tag{31}$$

for any $\theta_0(t)$ and $t > 0$, as long as $x(0) \in \mathbb{R}^2$ satisfies $\|x(0) - c(\alpha)\mathbf{1}_2\|_2 < \delta$.

(Step 3) Now, we take θ as an element in $\mathbb{T}^3$. Since $\mathcal{M}_1 := \{\theta \in \mathbb{T}^3 : \theta_0 - \theta_i = c(\alpha), i = 1, 2\}$, given θ, its distance from $\mathcal{M}_1$ is

$$\operatorname{dist}(\theta, \mathcal{M}_1) = \sqrt{|\langle\theta_0, \theta_1\rangle - \mathrm{c}(\alpha)|_{\mathbb{S}}^2 + |\langle\theta_0, \theta_2\rangle - \mathrm{c}(\alpha)|_{\mathbb{S}}^2},$$

where $\langle a, b\rangle$ and $|a - b|_{\mathbb{S}}$ denote the geodesic difference and distance (defined at the end of Section I) between angles $a, b \in \mathbb{S}^1$, respectively. For any θ such that $\|(\theta_0 - \theta_1, \theta_0 - \theta_2)^\top - c(\alpha)\mathbf{1}_2\|_2 < \pi$, it holds that $|\langle\theta_0, \theta_i\rangle - c(\alpha)|_{\mathbb{S}} = |\theta_0 - \theta_1 - c(\alpha)|$ for $i = 1, 2$, which yields that

$$\begin{aligned}\operatorname{dist}(\theta, \mathcal{M}_1) &= \sqrt{|\theta_0 - \theta_1 - c(\alpha)|^2 + |\theta_0 - \theta_1 - c(\alpha)|^2} \\ &= \|x - c(\alpha)\mathbf{1}_2\|_2.\end{aligned}$$

Consequently, from (31) there exists $\delta < \pi$ such that for any $\theta(0) \in \mathbb{T}^3$ satisfying $\operatorname{dist}(\theta(0), \mathcal{M}_1) < \delta$, it holds that

$$\begin{aligned}\operatorname{dist}(\theta(t), \mathcal{M}_1) &\le c_1 e^{-c_2 t}\|x(0) - c(\alpha)\mathbf{1}_2\|_2 \\ &= c_1 e^{-c_2 t}\operatorname{dist}(\theta(0), \mathcal{M}_1),\end{aligned}$$

which proves the exponential stability of the remote synchronization $\mathcal{M}_1$.

(Step 4) Finally, we prove the instability of: (a) the manifold $\mathcal{M}_1$ for $\alpha \in (\arctan\sqrt{3}, \pi/2)$, and (b) the manifold $\mathcal{M}_1'$ for any $\alpha \in (0, \pi/2)$. We first prove (a). One can calculate that the Jacobian matrix $J(x)$ has a positive eigenvalue. Let $V(x) = \|x(t) - c(\alpha)\mathbf{1}_2\|_2$, which satisfies $V(x) > 0$ for any $x \ne c(\alpha)\mathbf{1}_2$. Following the proof of Theorem 4.7 in [13], one can show that there always exists a set $\mathcal{U} \subset \{x \in \mathbb{R}^2 : \|x - c(\alpha)\mathbf{1}_2\| \le \delta\}$ for arbitrary small δ such that $\dot{V}(x) > 0$ for $x \in \mathcal{U}$. Since δ can be chosen such that $\delta < \pi$, one can derive that there exists a set $\mathcal{U}_1 \subset \{\theta \in \mathbb{T}^3 : \operatorname{dist}(\theta, \mathcal{M}_1') \le \delta\}$ such that $\frac{d\operatorname{dist}(\theta, \mathcal{M}_1')}{dt} > 0$ for $\theta \in \mathcal{U}_1$, which proves that the manifold $\mathcal{M}_1$ is unstable. Following similar arguments, one can prove (b) since both eigenvalues of $J(x)$ are positive. ■

We next show that if $u = 0$ the solution $\theta(t)$ converges to $\mathcal{M}_1$ for any initial condition $\theta(0) \in \mathcal{M} - (\mathcal{M}_1 \cup \mathcal{M}_1')$. Observe that $\theta(0) \in \mathcal{M} - (\mathcal{M}_1 \cup \mathcal{M}_1')$ implies: 1) $x_1(t) = x_2(t)$ for all $t \ge 0$ (i.e., invariance of $\mathcal{M}$); 2) x_i belongs to either $(c(\alpha), c(\alpha) + \pi)$ or $(c(\alpha) + \pi, c(\alpha) + 2\pi)$. From (29a), we have $\dot{x}_i = -2A\sin(x_i + \alpha) - A\sin(x_i - \alpha)$ for $i = 1, 2$. It can be computed that $\dot{x}_i < 0$ if $x_i \in (c(\alpha), c(\alpha) + \pi)$, which implies that $x_i(t)$ converges to $c(\alpha)$. Similarly, if $x_i \in (c(\alpha) + \pi, c(\alpha) + 2\pi)$, we have $\dot{x}_i > 0$, which implies that $x_i(t)$ converges to $c(\alpha) + 2\pi$. Since $c(\alpha)$ and $c(\alpha) + 2\pi$ are the same point on $\mathbb{S}^1$ and the equilibrium $x = c(\alpha)\mathbf{1}_2$ of (29) corresponds to $\mathcal{M}_1$ of (22), one can deduce that $\theta(t)$ converges to $\mathcal{M}_1$ for any $\theta(0) \in \mathcal{M} - (\mathcal{M}_1 \cup \mathcal{M}_1')$.

C. Proposition 4

Proposition 4: Let $\eta \in \mathbb{S}^1$ be defined in (24), then the time derivatives of z and ϕ that are defined in (23) are expressed by Eq. (25).

Proof: Let $z' = \theta_0 - \frac{\theta_1 + \theta_2}{2}$, then $\theta_0 - \theta_1 = z' - \phi$, and $\theta_0 - \theta_2 = z' + \phi$. From (22), the time derivatives of ϕ and z' are

$$\begin{aligned}\dot{\phi} &= \frac{1}{2}A\left(\sin(z' - \phi - \alpha) - \sin(z' + \phi - \alpha)\right), \\ \dot{z}' &= u - A(\sin(z' - \phi + \alpha) + \sin(z' + \phi + \alpha)) \\ &\quad - \frac{1}{2}A\left(\sin(z' - \phi - \alpha) + \sin(z' + \phi - \alpha)\right).\end{aligned}$$

Using the trigonometric identity $\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$, we have

$$\begin{aligned}\dot{\phi} &= -A\sin\phi\cos(z' - \alpha), \\ \dot{z}' &= u - 2A\cos\phi\sin\left(z' + \alpha\right) - A\cos\phi\sin(z' - \alpha) \\ &= u - A\cos\phi\left(3\cos\alpha\sin z' + \sin\alpha\cos z'\right).\end{aligned}$$

Since $D = \sqrt{9\cos^2\alpha + \sin^2\alpha}$, one can observe that $3\cos\alpha\sin z' + \sin\alpha\cos z' = D\left(\frac{3\cos\alpha}{D}\sin z' + \frac{\sin\alpha}{D}\cos z'\right) = D\sin(z' + \eta)$ (recall that η satisfies $\cos\eta = 3\cos\alpha/D$ and $\sin\eta = \sin\alpha/D$). Note that $\eta \to 0$ as $\alpha \to 0$, and $\eta \to \frac{\pi}{2}$ as $\alpha \to \frac{\pi}{2}$. Let $z = z' + \eta$, then $\dot{z} = -A\sin\phi\cos(z - \eta - \alpha)$, $\dot{\phi} = u - DA\cos\phi\sin z$, which completes the proof. ■

D. Computation of the Integral $\mathrm{g}(\phi)$

For the simplicity of notation, let $\beta = \alpha + \eta$, and then $g(\phi) = \int_0^{2\pi}\frac{\cos(z - \beta)}{1 - \frac{1}{u}a\sin z}dz$. Since $\cos(z - \beta) = \cos z\cos\beta + \sin z\sin\beta$, it holds that

$$g(\phi) = \cos\beta\underbrace{\int_0^{2\pi}\frac{\cos z}{1 - \frac{1}{u}a\sin z}dz}_{g_1(\phi)} + \sin\beta\underbrace{\int_0^{2\pi}\frac{\sin z}{1 - \frac{1}{u}a\sin z}dz}_{g_2(\phi)}.$$

For $g_1(\phi)$, it follows that $g_1(\phi) = \left[-\frac{u}{a}\ln(u - a\sin z)\right]_0^{2\pi} = 0$. Next, we compute $g_2(\phi)$. Its integral can be described as $us(\phi)$, where

$$s(\phi) = \frac{\sin\phi}{u - a\sin\phi} = \frac{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{u - 2a\sin\frac{\phi}{2}\cos\frac{\phi}{2}}.$$

According to [41], the indefinite integral of $s(\phi)$ is given by

$$\int s(\phi)d\phi = -\frac{\phi}{a} - \frac{1}{a}\frac{2u}{\sqrt{u^2-a^2}}\arctan\left(\frac{a - u\tan\frac{\phi}{2}}{\sqrt{u^2-a^2}}\right).$$

It follows that

$$\begin{aligned} g_2(\phi) &= \lim_{\phi'\to\pi^-}\int_0^{\phi'} us(\phi)d\phi + \lim_{\phi'\to\pi^+}\int_{\phi^+}^{2\pi} us(\phi)d\phi \\ &= -\frac{2u\pi}{a} - \frac{1}{a}\frac{2u^2}{\sqrt{u^2-a^2}}\left(-\frac{\pi}{2} - \arctan\frac{a}{\sqrt{u^2-a^2}}+\right) \\ &\quad - \frac{1}{a}\frac{2u^2}{\sqrt{u^2-a^2}}\left(\arctan\frac{a}{\sqrt{u^2-a^2}} - \frac{\pi}{2}\right) \\ &= \frac{2u\pi}{a\sqrt{u^2-a^2}}\left(u - \sqrt{u^2-a^2}\right). \end{aligned}$$

Combining all computations leads to

$$\begin{aligned} g(\phi) &= \cos\beta g_1(\phi) + \sin\beta g_2(\phi) = -\frac{u\sin\beta}{a}\int_0^{2\pi}(1 - s_1(\phi))d\phi \\ &= \frac{2u\pi\sin\beta}{a\sqrt{u^2-a^2}}\left(u - \sqrt{u^2-a^2}\right). \end{aligned} \tag{33}$$

References

[1] V. Vorotnikov, *Partial Stability and Control*. Springer Science & Business Media, 1997.

[2] W. M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. Princeton, NJ, USA: Princeton University Press, 2011.

[3] V. Sinitsyn, "Stability of the solution of a problem of inertial navigation," *Certain Problems of the Dynamics of Mechanical Systems*, pp. 46–50, 1991.

[4] J. Willems, "A partial stability approach to the problem of transient power system stability," *International Journal of Control*, vol. 19, no. 1, pp. 1–14, 1974.

[5] L. V. Gambuzza, A. Cardillo, A. Fiasconaro, L. Fortuna, J. Gómez-Gardenes, and M. Frasca, "Analysis of remote synchronization in complex networks," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 23, no. 4, p. 043103, 2013.

[6] Y. Qin, Y. Kawano, and M. Cao, "Stability of remote synchronization in star networks of Kuramoto oscillators," in *57th IEEE Conference on Decision and Control*, 2018, pp. 5209–5214.

[7] I. V. Miroshnik, "Attractors and partial stability of nonlinear dynamical systems," *Automatica*, vol. 40, no. 3, pp. 473–480, 2004.

[8] E. J. Hancock and D. J. Hill, "Restricted partial stability and synchronization," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 61, no. 11, pp. 3235–3244, 2014.

[9] C. Kuehn, *Multiple Time Scale Dynamics*. Berlin: Springer, 2015.

[10] K. Al-Naimee, F. Marino, and et. al., "Chaotic spiking and incomplete homoclinic scenarios in semiconductor lasers with optoelectronic feedback," *New Journal of Physics*, vol. 11, no. 7, p. 073022, 2009.

[11] V. Petrov, S. K. Scott, and K. Showalter, "Mixed-mode oscillations in chemical systems," *The Journal of Chemical Physics*, vol. 97, no. 9, pp. 6191–6198, 1992.

[12] V. Chellaboina and W. M. Haddad, "A unification between partial stability and stability theory for time-varying systems," *IEEE Control Systems Magazine*, vol. 22, no. 6, pp. 66–75, 2002.

[13] H. K. Khalil, *Nonlinear Systems,*. Upper Saddle River, NJ, USA: Prentice Hall, 2002.

[14] D. Aeyels and J. Peuteman, "On exponential stability of nonlinear time-varying differential equations," *Automatica*, vol. 35, no. 6, pp. 1091–1100, 1999.

[15] J. Tsinias and A. Stamati, "Averaging criteria for asymptotic stability of time-varying systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 6, pp. 1654–1659, 2013.

[16] F. Varela, J.-P. Lachaux, E. Rodriguez, and J. Martinerie, "The brainweb: Phase synchronization and large-scale integration," *Nature Reviews Neuroscience*, vol. 2, no. 4, p. 229, 2001.

[17] Y. Qin, Y. Kawano, O. Portoles, and M. Cao, "Partial phase cohesiveness in networks of networks of Kuramoto oscillators," *IEEE Transactions on Automatic Control*, doi:10.1109/TAC.2021.3062005, 2021.

[18] T. Menara, G. Baggio, D. Bassett, and F. Pasqualetti, "Stability conditions for cluster synchronization in networks of heterogeneous Kuramoto oscillators," *IEEE Transactions on Control of Network Systems*, 2019.

[19] S. Jafarpour and F. Bullo, "Synchronization of Kuramoto oscillators via cutset projections," *IEEE Transactions on Automatic Control*, vol. 64, no. 7, pp. 2830–2844, 2018.

[20] V. Nicosia, M. Valencia, M. Chavez, A. Díaz-Guilera, and V. Latora, "Remote synchronization reveals network symmetries and functional modules," *Physical Review Letters*, vol. 110, no. 17, p. 174102, 2013.

[21] Y. Qin, M. Cao, B. D. O. Anderson, D. S. Bassett, and F. Pasqualetti, "Mediated remote synchronization of Kuramoto-sakaguchi oscillators: The number of mediators matters," *IEEE Control Systems Letters*, vol. 5, no. 3, pp. 767–772, 2020.

[22] R. Vicente, L. L. Gollo, C. R. Mirasso, I. Fischer, and G. Pipa, "Dynamical relaying can yield zero time lag neuronal synchrony despite long conduction delays," *Proceedings of the National Academy of Sciences*, vol. 105, no. 44, pp. 17 157–17 162, 2008.

[23] I. Fischer, R. Vicente, J. M. Buldú, M. Peil, C. R. Mirasso, M. Torrent, and J. García-Ojalvo, "Zero-lag long-range synchronization via dynamical relaying," *Physical Review Letters*, vol. 97, no. 12, p. 123902, 2006.

[24] A. Wagemakers, J. M. Buldú, and M. A. Sanjuán, "Isochronous synchronization in mutually coupled chaotic circuits," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 17, no. 2, p. 023128, 2007.

[25] R. Banerjee, D. Ghosh, E. Padmanaban, R. Ramaswamy, L. Pecora, and S. K. Dana, "Enhancing synchrony in chaotic oscillators by dynamic relaying," *Physical Review E*, vol. 85, no. 2, p. 027201, 2012.

[26] L. V. Gambuzza, M. Frasca, L. Fortuna, and S. Boccaletti, "Inhomogeneity induces relay synchronization in complex networks," *Physical Review E*, vol. 93, no. 4, p. 042203, 2016.

[27] H. Sakaguchi and Y. Kuramoto, "A soluble active rotater model showing phase transitions via mutual entertainment," *Progress of Theoretical Physics*, vol. 76, no. 3, pp. 576–581, 1986.

[28] F. C. Hoppensteadt and E. M. Izhikevich, *Weakly Connected Neural Networks*. New York, USA: Springer Science & Business Media, 2012.

[29] M. J. Panaggio and D. M. Abrams, "Chimera states: Coexistence of coherence and incoherence in networks of coupled oscillators," *Nonlinearity*, vol. 28, no. 3, p. R67, 2015.

[30] F. Dörfler and F. Bullo, "Synchronization in complex networks of phase oscillators: A survey," *Automatica*, vol. 50, no. 6, pp. 1539–1564, 2014.

[31] M. L. Adams, *Rotating Machinery Vibration: From Analysis to Troubleshooting*. New York, NY, USA: CRC Press, 2009.

[32] J. R. Wertz, *Spacecraft Attitude Determination and Control*. Amsterdam, the Netherlands: Reidel, 1978.

[33] A. R. Teel, J. Peuteman, and D. Aeyels, "Semi-global practical asymptotic stability and averaging," *Systems & Control Letters*, vol. 37, no. 5, pp. 329–334, 1999.

[34] F. W. Wilson, "Smoothing derivatives of functions and applications," *Transactions of the American Mathematical Society*, vol. 139, pp. 413–428, 1969.

[35] T. Yoshizawa, *Stability theory by Liapunov's second method*. Math. Soc. Japan, 1966.

[36] Y. Lin, E. D. Sontag, and Y. Wang, "A smooth converse Lyapunov theorem for robust stability," *SIAM Journal on Control and Optimization*, vol. 34, no. 1, pp. 124–160, 1996.

[37] C. M. Kellett and A. R. Teel, "Weak converse Lyapunov theorems and control-Lyapunov functions," *SIAM Journal on Control and Optimization*, vol. 42, no. 6, pp. 1934–1959, 2004.

[38] C. M. Kellett, "Classical converse theorems in Lyapunov's second method," *arXiv preprint arXiv:1502.04809*, 2015.

[39] Y. Qin, Y. Kawano, B. Anderson, and M. Cao, "Partial exponential stability analysis of slow-fast systems via periodic averaging," *arXiv preprint arXiv:1910.07465*, 2019.

[40] S.-Y. Ha, D. Ko, and Y. Zhang, "Emergence of phase-locking in the Kuramoto model for identical oscillators with frustration," *SIAM Journal on Applied Dynamical Systems*, vol. 17, no. 1, pp. 581–625, 2018.

[41] I. N. Bronshtein and K. A. Semendyayev, *Handbook of Mathematics*. New York: Springer-Verlag, 1997.