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Gijs Heuts
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Simplicial and Dendroidal Homotopy Theory

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Dedicated to the memory of
Michael Boardman and Rainer Vogt

Preface

The goal of this book is to give an exposition of the theory of dendroidal sets and dendroidal spaces, and its relation to topological operads and infinite loop spaces. In the first part we have tried to give a complete treatment of the elementary theory of dendroidal sets, emphasizing the analogies with the theory of simplicial sets. For this reason, the reader less familiar with simplicial sets can also use this text as an introduction to this theory by focusing on Chapters 2 and 5. The reader already acquainted with simplicial sets can skip from Chapter 1 straight to Chapters 3, 4, and 6 to learn about dendroidal sets. To facilitate these different ways of using this text, there will be some measure of repetition of arguments between the simplicial and dendroidal sections.

Dendroidal sets form an extension of the theory of simplicial sets in a very precise sense and this extension leads to several new features. The main one is that while simplicial sets are indexed by linear orders that do not have automorphisms, dendroidal sets are indexed by trees and these have many automorphisms. Another feature is that the Cartesian product on simplicial sets extends to a tensor product on dendroidal sets, the study of which involves a careful analysis of shuffles of trees. These two aspects explain why the theory of dendroidal sets involves quite a bit of combinatorics. The reader will find good illustrations of this phenomenon in the proofs of Proposition 4.26 and Lemma 6.24, for example.

While in the first part homotopical properties of simplicial and of dendroidal sets only occur at a relatively elementary level, many readers will realize that much of the exposition forms a preparation for the study of Quillen model category structures on the categories of simplicial and dendroidal sets. In Part II of this book we prove the existence of several of these structures: amongst others, we establish the Kan–Quillen model structure for simplicial sets (describing the homotopy theory of spaces), the Joyal model structure for simplicial sets (describing the homotopy theory of ∞-categories), and the operadic model structure for dendroidal sets (describing the homotopy theory of ∞-operads). In Part III we develop the homotopy theory of dendroidal spaces. We begin with two general chapters on Reedy model structures and the theory of left Bousfield localization. Then we apply this general theory to describe several model structures on the category of dendroidal spaces and their relation to corresponding homotopy theories of dendroidal sets. This specializes to the equivalence between the Joyal model structure and Rezk's theory of complete Segal spaces in the simplicial case. Then we explore the relation between the homotopy theories of dendroidal sets and spaces on the one hand and the homotopy theories of topological or simplicial operads and of algebras for such operads on the other.

Our aim in this book is to give a reasonably self-contained exposition and to clearly exhibit the parallels and differences between the simplicial and dendroidal theories. For this reason, we also present some material (such as the fundamentals of simplicial sets) that is already covered in earlier sources. Our exposition has been tailored to our needs and, we feel, in several cases provides a more streamlined presentation than is available elsewhere. At the end of each chapter we include a short section with historical comments, indicating some of the main origins of the material we present and providing pointers to the relevant literature.

This book develops the fundamentals of the theory of dendroidal sets and spaces. We have not attempted to cover all of the variations and applications of that theory occurring in the literature, but will point the reader to some further developments and applications in the epilogue.

Our own understanding of the theory owes much to discussions and collaborations with many colleagues, among whom we would in particular like to mention Clemens Berger, Thomas Blom, Pedro Boavida, Hongyi Chu, Denis-Charles Cisinski, Javier Gutierrez, Rune Haugseng, Vladimir Hinich, Eric Hoffbeck, Jacob Lurie, Thomas Nikolaus, and the second author's (former) PhD students Miguel Barata, Matija Basic, Giovanni Caviglia, Peter James, Andor Lukács, Joost Nuiten, and Ittay Weiss.

We started writing this book in 2015 and we would like to thank the many institutions that supported us during our work. A first draft of many chapters of this book was written during our stay at the thematic program 'Homotopy Harnessing Higher structures' at the Isaac Newton Institute in Cambridge in 2018. We thank the Institute for hosting us and providing excellent working conditions. Furthermore, we thank our various host institutions: University of Copenhagen, Harvard University, Radboud University, University of Sheffield, and Utrecht University. We acknowledge the support of the NWO through the project 'Lie algebras and periodic spaces in homotopy theory' (with project number 016-VENI-192-186) and Moerdijk's Spinoza prize (with project number SPI 61-638), as well as the support of the European Research Council through the grant 'Chromatic homotopy theory of spaces' (grant number 950048).

It will be clear to the reader that we owe much to the works of Michael Boardman and Rainer Vogt. Sadly both of them passed away during the writing of this book, which we dedicate to their memory.

Introduction

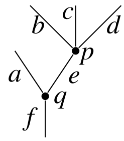
A simplicial set is a system of sets indexed by the natural numbers, or rather the linear orders $0 \leq 1 \leq \cdots \leq n$ for all $n \geq 0$, and maps between these corresponding to monotone maps between linear orders. A simplicial set can be viewed as a way to encode the construction of a topological space, the so-called geometric realization of the simplicial set. With this realization in mind it is possible to develop much if not all of the homotopy theory of topological spaces using simplicial sets as models for spaces, as was convincingly shown already in the 1950s and 1960s by the work of Kan, Curtis, Moore, May, and many others. Good expositions of this early development are the books by May [111], Gabriel and Zisman [61], and Lamotke [101], while the book by Goerss and Jardine [69] provides a more modern perspective. The simplicial theory is now widely used in mathematics and plays a central role not only in topology but also in many aspects of modern algebraic geometry, as illustrated by Illusie [89], Friedlander [59], Artin and Mazur [6], and more recently in the motivic homotopy theory of schemes [119].

There is a close relation between the theory of simplicial sets and that of categories. Any ('small') category defines a simplicial set, known as the nerve of the category. In this way one can view a category as giving rise to a topological space, called the classifying space of the category, simply by considering the geometric realization of its nerve. This viewpoint leads to many useful invariants of categories borrowed from topology, such as homotopy and cohomology groups, and has shown to be extremely useful, for example in the development of higher K-theory initiated by Quillen [126].

From the point of view of category theory, however, the construction of the classifying space is quite a big step, because it forgets the direction of the arrows in the category. As a consequence, many categories have homotopy equivalent realizations; for example, any adjoint pair of functors gives a homotopy equivalence of classifying spaces. The direction of arrows is still present in the simplicial set encoding the nerve of a category. This leads to the question whether there is a more refined 'homotopy theory' for simplicial sets, retaining more of the categorical properties for nerves of categories; a homotopy theory for which an equivalence between two categories corresponds to a new kind of homotopy equivalence between simplicial sets. Joyal and Lurie have shown that it is indeed possible to develop such a more refined homotopy theory of simplicial sets, which goes under the name of the theory of quasi-categories or ∞-categories.

Operads can be viewed as a generalization of categories. While in a category the arrows run from a single source to a single target, in an operad an arrow – or an operation, as one usually says in this context – runs from a finite sequence of sources (or 'inputs') to a single target (or 'output'). Thus, while the structure of composing arrows in a category can be described by linear orders

encoding the structure of composition of an operad requires rooted trees instead of linear orders:



Here the first picture is to be interpreted as the category with objects $0, \ldots, n$ (represented by the edges) and a single morphism $i \longrightarrow j$ whenever $i \leq j$ (with the generating morphisms $i \longrightarrow i + 1$ indicated by the vertices in the picture). Note that this way of using edges and vertices is dual to the more standard conventions for picturing categories and linear orders. This switch is made in anticipation of the kind of pictures we will draw for operads. Indeed, the second picture represents an operad with objects (or *colours*) $a, \ldots, f$ and generating operations $p : (b, c, d) \longrightarrow e$ and $q : (a, e) \longrightarrow f$.

Operads initially arose in an attempt to understand the structure of iterated loop spaces in topology and the higher associativity laws involved in the composition of loops, or the composition of 'loops of loops' in double loop spaces, etc. More explicitly, an operad describes a specific kind of algebraic structure in terms of all the operations involved in that structure, and all the equations that hold between all possible compositions of these operations. The algebraic structure of an n-fold loop space can be described in a very efficient and elegant way as the structure of an algebra over a specific operad, known as the operad of little n-cubes. This method of studying iterated loop spaces goes back to Stasheff [135], Boardman and Vogt [21], and May [112], and forms the origin of the theory of operads. Later it became apparent that operads play an important role in many other parts of mathematics. For example, the moduli spaces of surfaces with boundary form an operad playing an important role in topological field theory [67]. Operads occur in geometric group theory in the analysis of the homology of free groups using ‘outer space’ [44]. They play an essential role in the deformation theory of associative algebras and the precise structure of the Hochschild complex controlling such deformations [100], and in developing a general theory of Koszul duality for algebras of different kinds [68, 65, 104].

As the theory of operads forms an obvious extension – from a single input to multiple inputs – of the theory of categories, the theory of dendroidal sets arose in the search for a similar extension of the theory of simplicial sets, thus completing the square below.

$$\begin{tikzcd}
\text{categories} \arrow[r, "\text{more inputs}"] \arrow[d, "\text{nerve}"'] & \text{operads} \arrow[d] \\
\text{simplicial sets} \arrow[r] & ?
\end{tikzcd}$$

The goal of this book is to show that such a theory indeed exists and to describe some of its main developments. We have organized the exposition into several parts, reflecting the relation of the theory to simplicial sets, operads, and homotopy theory mentioned above.

In Part I, our goal is to explain in detail the basic features of the category of dendroidal sets, emphasizing its relation to simplicial sets, operads and algebras, but saving more advanced homotopy theoretic aspects for later. Chapter 1 reviews the main notions in the theory of operads together with some examples. In Chapter 2 we give an introduction to the theory of simplicial sets, assuming only some familiarity with basic categorical language (categories, functors, limits and colimits, adjunctions, etc.). Then Chapter 3 introduces dendroidal sets and explains how many aspects of the theory of simplicial sets extend to dendroidal sets. Nerves of operads are among the main examples of dendroidal sets, just like nerves of categories give important examples of simplicial sets. The main additional feature at this stage is that due to the existence of non-trivial automorphisms of trees, not every dendroidal set has a well-behaved skeletal filtration in the way that any simplicial set has. This motivates the introduction of the property of normality of a dendroidal set. This property is defined in terms of the groups of automorphisms of trees, which are required to act freely on a normal dendroidal set. For example, the nerve of an operad is a normal dendroidal set precisely if the symmetric groups act freely on the operad (one says that the operad is Σ-*free*). Normal dendroidal sets are much better behaved than arbitrary ones and in particular they do have a good notion of skeletal filtration. A useful analogy to keep in mind here is that among general dendroidal sets the normal ones play a role similar to that of CW-complexes among topological spaces.

In Chapters 5 and 6 we discuss fibrations of simplicial sets and of dendroidal sets. We develop the simplicial theory first and give a detailed treatment of the notion of Kan fibration which is central to the classical homotopy theory of simplicial sets, as well as the weaker notions of inner, left and right fibrations which feature in the ‘categorical’ homotopy theory of simplicial sets. Chapter 6 is a mirror image of Chapter

5, in which we treat similar classes of fibrations for the category of dendroidal sets. While some general categorical features of the simplicial and dendroidal theories are very similar, the two theories differ considerably in detail because of combinatorial aspects arising from the automorphisms of trees, as well as from the fact that the role of the Cartesian product is taken over by a more subtle tensor product of dendroidal sets. In order to secure a solid understanding of how to work with these tensor products we discuss these first, in Chapter 4. The organization and choice of material in Chapters 5 and 6 is motivated to a large extent by the need to prepare the ground for a discussion of the homotopy theory of simplicial sets and dendroidal sets in terms of Quillen model category structures in Part II.

This second part begins with Chapter 7, giving a self-contained introduction to the theory of model categories. Chapter 8 then establishes the two most important examples of model structures on the category of simplicial sets, namely the Joyal (or categorical) model structure describing the homotopy theory of ∞-categories, and the Kan–Quillen model structure describing the homotopy theory of Kan complexes. The latter is one of the original examples of a model structure and much of its relevance derives from the fact that it is, in an appropriate sense, equivalent to a certain standard model structure on the category of topological spaces. Our construction of these model structures is somewhat different from those usually found in the literature and stresses the elementary nature of the arguments. In Chapter 9 we generalize the techniques of Chapter 8 to establish the operadic model structure on the category of dendroidal sets, which describes the homotopy theory of ∞-operads. We then study two variants, namely the covariant and the Picard model structure. In a later part of this book we will relate the first to the homotopy theory of algebras for a given ∞-operad and the second to the homotopy theory of infinite loop spaces.

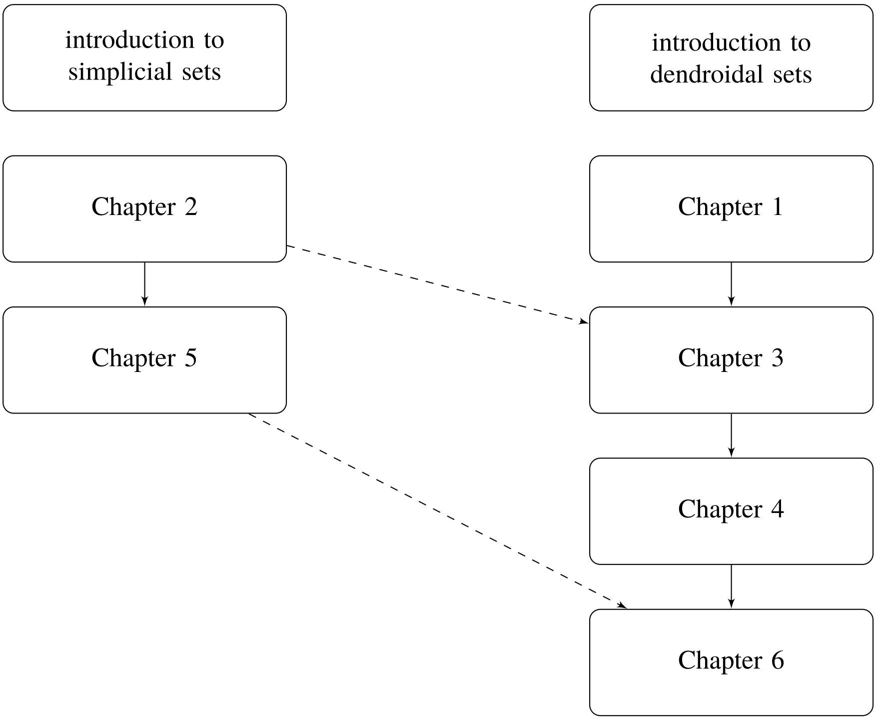
In Part III of this book we develop the homotopy theory of simplicial and dendroidal *spaces*, rather than sets. It starts with two chapters of a rather general nature. Chapter 10 treats the homotopy theory of diagrams of spaces on a given small category, paying particular attention to Reedy model structures. It includes some applications to the theory of homotopy colimits. Chapter 11 develops a version of left Bousfield localization appropriate for our purposes; one of the key ingredients is a general notion of 'mapping space' between objects of a model category, which relies on the construction of Reedy model structures of the preceding chapter. In Chapter 12 we develop the notion of dendroidal Segal spaces and completion; this extends Rezk's theory of (simplicial) Segal spaces and provides yet another model for the homotopy theory of ∞-operads. Chapter 13 adapts the notion of left fibration to the context of dendroidal spaces and proves that these provide a model for the homotopy theory of algebras over an operad. In the concluding Chapter 14, we establish an equivalence of homotopy theories between dendroidal sets (or spaces) and simplicial (or topological) operads.

We stress once more that this book may also be used as a self-contained introduction to the homotopy theory of simplicial sets. We have written Chapter 2 in such a way that it can form an independent first introduction to the theory, followed by Chapter 5 on fibrations and Chapter 8 in which we give an independent and self-contained treatment of the homotopy theory of simplicial sets before extending it to the dendroidal context. Chapter 12 on dendroidal spaces also contains a separate Section 12.7, indicating how our results specialize to give Rezk's theory of (complete) Segal spaces.

How to Use This Book

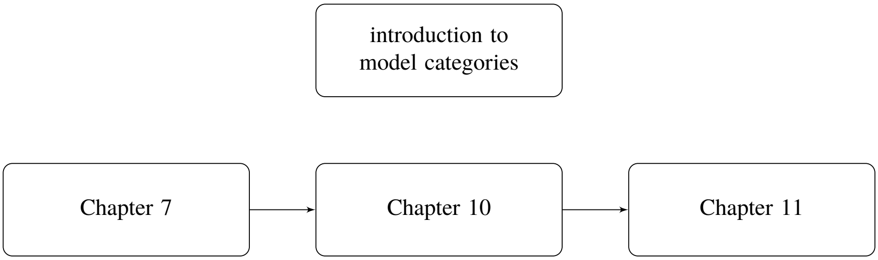
This book provides a comprehensive introduction to the theory of dendroidal sets and dendroidal spaces, but can be read in different ways, for which we outline several paths below. In particular, we stress that this book can also be used as a self-contained introduction to the theory of simplicial sets, including the Kan–Quillen model structure (describing 'classical' homotopy theory) and the Joyal model structure (describing the homotopy theory of ∞-categories).

The first part of this book treats the *basic theory* of simplicial and dendroidal sets and could be split as follows:

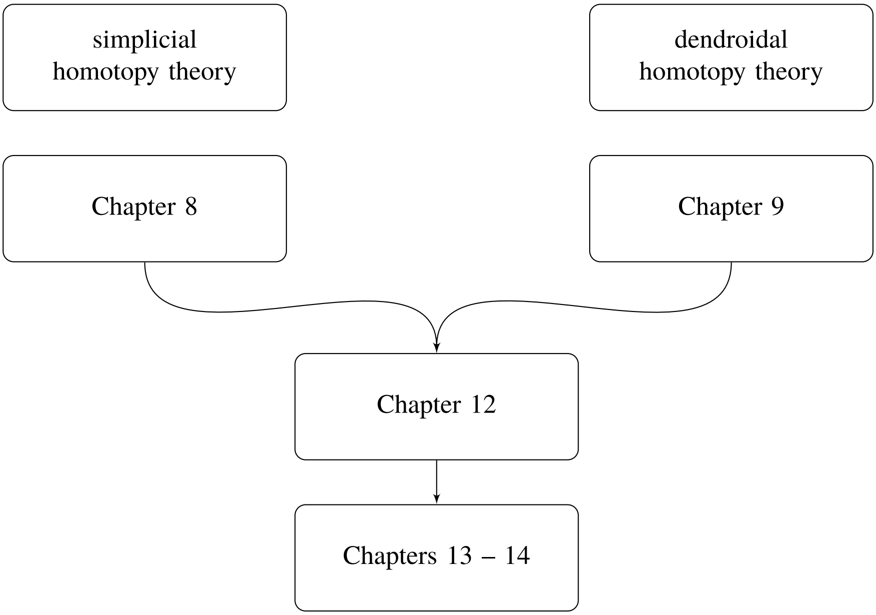


Our exposition of the basic definitions and constructions of simplicial sets in Chapter 2 serves as an introduction to the corresponding material for dendroidal sets in Chapter 3. Similarly, the treatment of various Kan conditions for simplicial sets in Chapter 5, which will be familiar to many readers, can serve as an introduction to the similar but more complicated material for dendroidal sets in Chapter 6, as we have indicated by the dashed arrows above. The reader will observe that some of the results in Chapter 6 make use of those in Chapter 5.

The further parts of the book develop the *homotopy theory* of simplicial and dendroidal sets (and spaces). We use the formalism of Quillen model categories, developed in the following chapters:



The self-contained Chapter 7 contains basic definitions, constructions, and examples. Chapter 10 describes the homotopy theory of diagrams of spaces. Finally, Chapter 11 treats the theory of (co)simplicial resolutions and left Bousfield localization. The reader familiar with this material may skip these chapters and refer back to them as needed. On the other hand, these three chapters can also be used as a self-contained introduction to model categories, possibly supplemented by taking examples in the context of simplicial sets or topological spaces from Chapter 8.



The core of this book is the development of the homotopy theory of simplicial and dendroidal objects. The reader who wishes to learn about the Joyal model structure, but is familiar with the basics of simplicial sets and model categories, can go straight to Chapter 8 and refer back to Chapter 5 as needed. The homotopy theory of dendroidal sets is developed in Chapter 9. Chapter 12 develops the homotopy theory of dendroidal spaces. Chapter 13 explains how dendroidal spaces may be used to model algebras for operads and ∞-operads. In the concluding Chapter 14 we finally explain how dendroidal objects are models for operads.

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Part I
The Elementary Theory of Simplicial and Dendroidal Sets

Chapter 1
Operads

The theory of operads is a convenient framework to define various types of algebraic structures in many different contexts. In this first chapter we will define the notion of an operad as well as that of an algebra for an operad. We present several well-known examples. In particular we will describe the example from which the theory originated, namely that of the little n-cubes operad whose algebras are n-fold loop spaces, and a variant of it, namely the Fulton–MacPherson operad built from compactifications of configuration spaces. We present several constructions which will play an important role in this book. Among these are the construction of the free operad generated by a family of operations, the tensor product of two operads, and the Boardman–Vogt resolution of a given operad, which describes the structure of an algebra-up-to-homotopy for that operad. We would like to emphasize that these constructions, as well as the notational conventions regarding trees, will reoccur in many of the later chapters. For example, already in Chapter 3 the definition of a dendroidal set uses the construction of free operads and morphisms between them, and much later the Boardman–Vogt resolution will be the key tool to relate the homotopy theory of operads to that of dendroidal sets. Many readers will already be familiar with much of this material and they may wish to just glance over these points. They will observe that for us the term operad will always mean coloured symmetric operad.

1.1 Operads

Algebraic structures are sets or spaces equipped with specific operations. For example, a monoid is a set M equipped with a multiplication $\mu\colon M \times M \to M$ and a unit element e in M which can be viewed as an 'operation with zero inputs' (or *nullary* operation) $1 = M^0 \to M$. These operations are of course required to satisfy certain identities. There are many operations which can be defined in terms of these two, such as the multiplication of n elements in a specific order σ, which is a map

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$$M^n \longrightarrow M\colon (x_1, \dots, x_n) \mapsto \mu(x_{\sigma(1)}, \mu(x_{\sigma(2)}, \mu(\dots, \mu(x_{\sigma(n-1)}, x_{\sigma(n)}) \dots))).$$

If M has a topology, one would of course require these operations to be continuous. A central theme in topology and geometry is the behaviour of such algebraic structures under deformations of the space M. It turns out that a simple algebraic structure like that of a monoid will then ‘explode’ into a structure where one still has a multiplication, but one which is only associative up to a (specified) homotopy. Furthermore, the various associativity equations one might write down (when multiplying, say, four elements) are related by further homotopies, and so forth. One speaks of a multiplication which is associative up to *coherent homotopy*. Such coherences are expressed by infinitely many further operations and identities between them. The notion of an operad was initially conceived in order to organize such structures in a tractable way, by equipping the multitude of operations themselves with a geometric structure, and has found many applications since.

Definition 1.1 An *operad* $\mathbf{P}$ consists of a set C of *colours* and for each $n \geq 0$ and each sequence $c_1, \dots, c_n, c$ of colours a set

$$\mathbf{P}(c_1, \dots, c_n; c)$$

of *operations*, thought of as taking n inputs of colours $c_1, \dots, c_n$ respectively to an output of colour c. Moreover, there are three kinds of structure maps:

– for each colour c a *unit* $1_c \in \mathbf{P}(c; c)$,
– for each permutation $\sigma \in \Sigma_n$ a map

$$\sigma^* : \mathbf{P}(c_1, \dots, c_n; c) \longrightarrow \mathbf{P}(c_{\sigma(1)}, \dots, c_{\sigma(n)}; c),$$

usually written $\sigma^* p = p \circ \sigma$,
– for any sequence of colours $c_1, \dots, c_n, c$ and any n-tuple of sequences $d^i_1, \dots, d^i_{k_i}$ for $i = 1, \dots, n$, a *composition* of operations

$$\gamma : \mathbf{P}(c_1, \dots, c_n; c) \times \prod_{i=1}^{n} \mathbf{P}(d^i_1, \dots, d^i_{k_i}; c_i) \longrightarrow \mathbf{P}(d^1_1, \dots, d^1_{k_1}, d^2_1, \dots, d^n_{k_n}; c),$$

usually written $\gamma(p, q_1, \dots, q_n) = p \circ (q_1, \dots, q_n)$ or even just $p(q_1, \dots, q_n)$.

Furthermore, these structure maps have to satisfy a number of axioms, which express that composition is unital and associative, that the permutation operations σ^* give a right action of the symmetric groups and that this action is compatible with the compositions γ. In detail:

– for an operation $p \in \mathbf{P}(c_1, \dots, c_n; c)$, we should have $\gamma(1_c, p) = p$ and $\gamma(p, 1_{c_1}, \dots, 1_{c_n}) = p$,
– for $p, q_1, \dots, q_n$ as above and a further sequence of operations $r^i_1, \dots, r^i_{k_i}$, where r^i_j has output d^i_j, we should have

$$(p(q_1, \dots, q_n))(r^1_1, \dots, r^n_{k_n}) = p(q_1(r^1_1, \dots, r^1_{k_1}), \dots, q_n(r^n_1, \dots, r^n_{k_n}))$$

– for $\sigma, \tau \in \Sigma_n$ and associated operations σ^*, τ^* as above, we should have $(\sigma\tau)^* = \tau^*\sigma^*$,
– for $p, q_1, \ldots, q_n$ as before and $\sigma \in \Sigma_n$, $\tau_i \in \Sigma_{k_i}$, we should have

$$\sigma^* p(\tau^*_{\sigma(1)} q_{\sigma(1)}, \ldots, \tau^*_{\sigma(n)} q_{\sigma(n)}) = (\sigma \circ (\tau_1, \ldots, \tau_n))^*(p(q_1, \ldots, q_n)),$$

where $\sigma \circ (\tau_1, \ldots, \tau_n)$ is the element of the symmetric group on $\Sigma_{i=1}^n k_i$ letters formed by letting τ_i permute the letters within the ith block of length k_i and letting σ permute the n different blocks. Another way of phrasing this condition is by the following two identities:

$$\begin{aligned} \sigma^* p(q_{\sigma(1)}, \ldots, q_{\sigma(n)}) &= p(q_1, \ldots, q_n), \\ p(\tau_1^* q_1, \ldots, \tau_n^* q_n) &= (\tau_1, \ldots, \tau_n)^*(p(q_1, \ldots, q_n)). \end{aligned}$$

Here $(\tau_1, \ldots, \tau_n)$ corresponds to an element of $\Sigma_{k_1+\cdots+k_n}$ via the obvious inclusion $\Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \subseteq \Sigma_{k_1+\cdots+k_n}$.

Remark 1.2 Let $\mathbf{P}$ be an operad, $p \in \mathbf{P}(c_1, \ldots, c_n; c)$ and $q \in \mathbf{P}(d_1, \ldots, d_k; c_i)$ for some $1 \leq i \leq n$. It will be convenient to have the following notation:

$$p \circ_i q := p(1_{c_1}, \ldots, q, \ldots, 1_{c_n}).$$

Thus, $p \circ_i q$ is the composition of p with q in the ith variable. If $c_1, \ldots, c_n$ are all distinct colours of $\mathbf{P}$, we will sometimes also write $p \circ_{c_i} q$. It is possible to rephrase the definition of operad in terms of the operations $\circ_i$, instead of using the 'total composition' of Definition 1.1, as we will explain in Section 1.4.

Example 1.3 (a) As said, one should think of the elements of $\mathbf{P}(c_1, \ldots, c_n; c)$ as operations, taking inputs of types $c_1, \ldots, c_n$ respectively to an output of type c. Thus, a typical example of an operad can be given by taking a family of sets $\{X_c\}_{c \in C}$ indexed by the set of colours C and setting $\mathbf{P}(c_1, \ldots, c_n; c)$ to be the set of functions

$$p : X_{c_1} \times \cdots \times X_{c_n} \longrightarrow X_c.$$

The action by the symmetric groups is given by simply permuting the variables:

$$(p \circ \sigma)(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = p(x_1, \ldots, x_n).$$

Composition in $\mathbf{P}$ is defined as composition of functions.

(b) If the sets X_c have more structure, one can define an operad by defining the set of operations $\mathbf{P}(c_1, \ldots, c_n; c)$ to be the set of functions $X_{c_1} \times \cdots \times X_{c_n} \to X_c$ respecting that structure, provided this condition is compatible with composition and permutation of variables. For example, the X_c could be topological spaces and the operations continuous functions; alternatively, the X_c could be vector spaces and the operations multilinear maps. We are sure the reader can make this list as long as they like.

(c) A more 'universal' way of describing the type of example listed above is as follows: if $\mathbf{C}$ is any symmetric monoidal category and C is a set of objects of $\mathbf{C}$, then one obtains an operad with this set of colours by defining $\mathbf{P}(c_1, \ldots, c_n; c)$ to be the set of morphisms $c_1 \otimes \cdots \otimes c_n \to c$ in $\mathbf{C}$. We will sometimes write $\mathbf{C}^{\otimes}$ for the operad obtained in this way, when C is the set of *all* objects of $\mathbf{C}$ (granted $\mathbf{C}$ is small).

Remark 1.4 (Terminology and notation) (a) If the set C of colours is a singleton $\{c\}$ there is no need to specify colours in our notation and we simply write $\mathbf{P}(n)$ for $\mathbf{P}(c_1, \ldots, c_n; c)$ (where the c_i all coincide with c). Composition is then given by maps

$$\mathbf{P}(n) \times \prod_{i=1}^{n} \mathbf{P}(k_i) \longrightarrow \mathbf{P}(k_1 + \cdots + k_n)$$

and each $\mathbf{P}(n)$ carries a right Σ_n-action. We sometimes call such an operad *uncoloured* or *classical* (the latter because the original definition of operad concerned this uncoloured case). The operads of Definition 1.1 will sometimes be called *coloured operads* for emphasis.

(b) Consider an operad $\mathbf{P}$ which has only unary operations, i.e., where $\mathbf{P}(c_1, \ldots, c_n; c)$ is always empty unless $n = 1$. The definition of operad then comes down to that of a category with C as set of objects and $\mathbf{P}(c; d)$ the set of morphisms from c to d.

(c) Combining (a) and (b), one sees that an uncoloured operad with only unary operations is simply a monoid, i.e. a set with a unital and associative multiplication. Thus there is a diagram of inclusions

$$\begin{tikzcd}
\text{monoids} \arrow[r, tail] \arrow[d, tail] & \text{categories} \arrow[d, tail] \\
\text{uncoloured operads} \arrow[r, tail] & \text{operads,}
\end{tikzcd}$$

with the understanding that we only consider *small* categories here, i.e. those whose objects form a set.

(d) The definition of operad allows for $\mathbf{P}(c_1, \ldots, c_n; c)$ where $n = 0$, which we write as $\mathbf{P}(-; c)$. We will refer to its elements as *constants* of colour c. An operad is called *unital* if for each c there is precisely one constant of colour c, i.e. if $\mathbf{P}(-; c)$ is a singleton. An operad $\mathbf{P}$ is called *open* if it has no constants at all, i.e. if $\mathbf{P}(-; c)$ is empty for all c. Any operad $\mathbf{P}$ has an *interior* $\mathbf{P}^o$, obtained by simply omitting all constants of $\mathbf{P}$. Similarly, each $\mathbf{P}$ has a *closure* $\mathrm{cl}(\mathbf{P})$, in such a way that the formation of closures is a left adjoint to the inclusion of unital operads into all operads. We will come back to this closure in detail later.

Example 1.5 We list some uncoloured operads occurring frequently in the literature.

(a) The *commutative operad* **Com** is defined by

$$\mathbf{Com}(n) = \{*\},$$

i.e. **Com** has a unique n-ary operation for each n. In particular, it is unital.

(b) The *associative operad* **Ass** is defined by

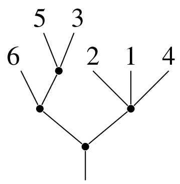
$$\mathbf{Ass}(n) = \Sigma_n,$$

with the following structure maps. The group Σ_n acts on itself on the right in the obvious way; to define composition, we need to specify a map

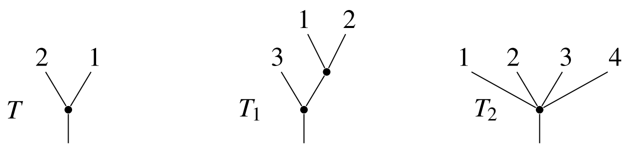
$$\gamma : \Sigma_n \times \Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \longrightarrow \Sigma_{k_1+\cdots+k_n}.$$

If one thinks of the elements $\sigma \in \Sigma_n$ and $\tau_i \in \Sigma_{k_i}$ as permutation matrices, then $\gamma(\sigma, \tau_1, \ldots, \tau_n)$ corresponds to the matrix formed by replacing the 1 in the ith column of σ by the matrix τ_i. As in Definition 1.1, one can think of Σ_n as permuting the n blocks of sizes $k_1, \ldots, k_n$ which partition $k_1 + \ldots + k_n$ and as Σ_{k_i} as permuting the letters within the block corresponding to k_i. Again, **Ass** is a unital operad.

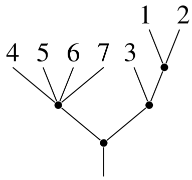
(c) There is an operad $\mathbf{Tree}^{\mathrm{pl}}$ which, roughly speaking, has as its n-ary operations the set of planar rooted trees with n numbered leaves. Our conventions regarding trees will be slightly nonstandard; we will specify them precisely in Section 1.3. For now, here is a typical example of an element of $\mathbf{Tree}^{\mathrm{pl}}(6)$:



The group Σ_n acts on $\mathbf{Tree}^{\mathrm{pl}}(n)$ by permuting the labels on the leaves and the composition is defined by grafting; indeed, $\gamma(T, T_1, \ldots, T_n)$ is given by grafting T_i onto the leaf of T numbered i. The leaves of the resulting grafted tree have to be labelled appropriately: if $T_1, \ldots, T_n$ have $k_1, \ldots, k_n$ leaves respectively, the leaves of $\gamma(T, T_1, \ldots, T_n)$ corresponding to the tree T_i are now labelled $k_1 + \ldots + k_{i-1} + 1$ through $k_1 + \ldots + k_{i-1} + k_i$. For example, the composition of the labelled trees



can be pictured as follows:



One obtains a variation of this example by considering trees without a planar structure, giving an operad **Tree**.

Often, the sets of operations of **P** have more structure. A case of particular interest to us is the following:

Definition 1.6 A *topological operad* is an operad **P** where each set of operations $\mathbf{P}(c_1, \ldots, c_n; c)$ is equipped with a topology and all the structure maps of **P** (i.e. compositions and permutations) are continuous.

Remark 1.7 (a) Notice that we do not include a topology on the set of colours C in this definition.

(b) There are of course many variations on the definition above; one can replace the sets of operations of **P** by abelian groups, vector spaces, manifolds etc. More generally, for a symmetric monoidal category **V**, one can define an *operad in* **V** precisely as in Definition 1.1, but taking the $\mathbf{P}(c_1, \ldots, c_n; c)$ to be objects of **V** (rather than sets) and, in the definition of composition, replacing the product by the tensor product in **V**. With this definition, the operads of Definition 1.1 are *operads in* **Sets** and topological operads are *operads in* **Top**, the latter denoting the category of topological spaces and continuous maps. We will sometimes use this terminology to emphasize the kind of operads we consider. Later in this book we will discuss the case where **V** is the category of *simplicial sets*.

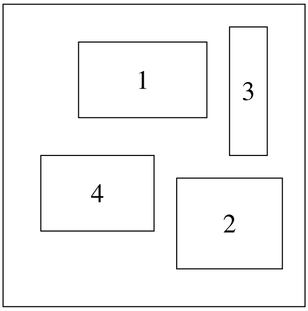
Example 1.8 One of the most classical examples of a topological operad (and, indeed, the motivating example for the original definition of operads) is the *little d-cubes operad* $\mathbf{E}_d$. Intuitively speaking, the space $\mathbf{E}_d(n)$ is the configuration space of n numbered d-dimensional cubes inside the d-dimensional unit cube $[0,1]^d \subset \mathbf{R}^d$. The operadic composition of an operation $p \in \mathbf{E}_d(n)$ with operations $q_1, \ldots, q_n$ is given by substituting the rescaled configuration q_i into the ith cube of p. More precisely, a point of $\mathbf{E}_d(n)$ is an n-tuple of embeddings

$$f_1, \ldots, f_n : [0,1]^d \to [0,1]^d$$

satisfying the following conditions:

(1) Each embedding f_i is *rectilinear*, in the sense that it is itself a product of d affine embeddings $g_1^i, \ldots g_d^i : [0,1] \to [0,1]$ (where affine means of the form $t \mapsto at + b$). Thus, the faces of the cube embedded by f_i are all parallel to the faces of the unit cube in $\mathbf{R}^d$.
(2) The interiors of the cubes embedded by the f_i are mutually disjoint.

As an example, one can picture a typical point of $\mathbf{E}_2(4)$ as follows:



The sets $\mathbf{E}_d(n)$ can be topologized as a subspace of $\mathrm{Map}([0,1]^d \amalg \cdots \amalg [0,1]^d, [0,1]^d)$, where the latter is given the compact-open topology. Composition of operations in $\mathbf{E}_d$ is then formally defined simply by composing embeddings.

Example 1.9 An embedded d-dimensional cube inside $[0,1]^d$ gives a corresponding $d+1$-cube inside $[0,1]^{d+1}$ by adding the identity map as a final coordinate. This gives a morphisms of operads from $\mathbf{E}_d$ into $\mathbf{E}_{d+1}$. With these one may form the following colimit:

$$\mathbf{E}_\infty(n) := \varinjlim_d \mathbf{E}_d(n).$$

The reader can verify that the spaces $\mathbf{E}_\infty(n)$ inherit the structure of an operad in their own right. Moreover, all these spaces turn out to be contractible. Thus, from the point of view of homotopy theory, the operad $\mathbf{E}_\infty$ is much like the commutative operad **Com**, for which each space (or set) of operations is a singleton. More precisely, there is an evident map of operads $\mathbf{E}_\infty \to \mathbf{Com}$ which gives homotopy equivalences $\mathbf{E}_\infty(n) \to \mathbf{Com}(n)$ for each n. However, there is one important difference between these two: the action of the symmetric group Σ_n on $\mathbf{E}_\infty(n)$ is free, whereas its action on $\mathbf{Com}(n)$ is not (at least for $n \geq 2$). Note also that there is a map of operads $\mathbf{E}_1 \to \mathbf{Ass}$; it sends a collection of n numbered subintervals of the unit interval to the permutation $(i_1, \ldots, i_n)$ obtained by reading the labels of the subintervals left to right. This map is also a homotopy equivalence $\mathbf{E}_1(n) \to \mathbf{Ass}(n)$ for every n. Thus one can think of the operads $\mathbf{E}_d$ as a family which 'interpolates' between the associative and commutative operads, at least in a homotopy-theoretic sense.

To conclude this section, we observe that operads form a category **Op** in an evident way: for two operads $\mathbf{P}$ and $\mathbf{Q}$, a morphism $\varphi : \mathbf{P} \to \mathbf{Q}$ is a function $f : C_\mathbf{P} \to C_\mathbf{Q}$ between the corresponding sets of colours and, for each sequence $c_1, \ldots, c_n, c$ of colours of $\mathbf{P}$, a map

$$\varphi_{c_1,\ldots,c_n,c} : \mathbf{P}(c_1, \ldots, c_n; c) \longrightarrow \mathbf{Q}(f(c_1), \ldots, f(c_n); f(c)).$$

These maps should be compatible with Σ_n-actions, compositions and units in the obvious way. For topological operads, one of course requires the maps $\varphi_{c_1,\ldots,c_n,c}$ to be continuous. In general, for operads in $\mathbf{V}$, they should simply be morphisms in $\mathbf{V}$.

1.2 Algebras for Operads

1.2.1 Definitions and Examples

The main role of an operad $\mathbf{P}$ is to define a corresponding 'algebraic' structure on a set or a space. For a given (uncoloured) operad $\mathbf{P}$, a $\mathbf{P}$-algebra is a set A on which the operations $p \in \mathbf{P}(n)$ act as actual functions $A(p) : A^{\times n} \to A$, in a way compatible with the structure of $\mathbf{P}$. The general definition goes as follows:

Definition 1.10 Let $\mathbf{P}$ be an operad. A $\mathbf{P}$-algebra A is a family of sets $\{A_c\}_{c\in C}$ indexed by the colours of $\mathbf{P}$, together with maps

$$\mathbf{P}(c_1, \ldots, c_n; c) \times A_{c_1} \times \cdots \times A_{c_n} \longrightarrow A_c,$$

written $(p, a_1, \ldots, a_n) \mapsto A(p)(a_1, \ldots, a_n)$ or simply $p(a_1, \ldots, a_n)$. These maps should respect the structure of $\mathbf{P}$ in the following sense:

– for each colour c, the unit 1_c acts as an identity:

$$1_c(a) = a \qquad \text{for} \quad a \in X_c,$$

– for a permutation $\sigma \in \Sigma_n$ and $a_i \in A_{c_i}$ for $i = 1, \ldots, n$,

$$(p \circ \sigma)(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = p(a_1, \ldots, a_n),$$

– for a composition $p(q_1, \ldots, q_n)$ of operations as in Definition 1.1 and $a^i_j \in A_{d^i_j}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$,

$$p(q_1, \ldots, q_n)(a^1, \ldots, a^n) = p(q_1(a^1), \ldots, q_n(a^n)),$$

where a^i is short-hand for $a^i_1, \ldots, a^i_{k_i}$.

Remark 1.11 A more concise way of phrasing this definition is as follows. Write $\mathbf{Sets}^{\times}$ for the operad formed out of the symmetric monoidal category $\mathbf{Sets}$ of sets with cartesian product, as described in Remark 1.3(c). Then a $\mathbf{P}$-algebra is simply a morphism of operads $A : \mathbf{P} \to \mathbf{Sets}^{\times}$.

Definition 1.12 A morphism of $\mathbf{P}$-algebras $f : A \to B$ is a family of maps

$$f_c : A_c \longrightarrow B_c, \qquad c \in C,$$

which are compatible with operations of $\mathbf{P}$, meaning

$$f_c(A(p)(a_1, \ldots, a_n)) = B(p)(f_{c_1}(a_1), \ldots, f_{c_n}(a_n))$$

for any $p \in \mathbf{P}(c_1, \ldots, c_n; c)$ and $a_i \in A_{c_i}, a \in A_c$. This notion of morphism defines a category of $\mathbf{P}$-algebras for which we write $\mathrm{Alg}_{\mathbf{P}}$.

Remark 1.13 An analogous definition applies to a topological operad **P**. A **P**-algebra A is now given by a family of spaces $\{A_c\}_{c\in C}$ and continuous maps

$$\mathbf{P}(c_1,\ldots,c_n;c)\times A_{c_1}\times\cdots\times A_{c_n}\longrightarrow A_c$$

satisfying the same equations as before. These algebras form a category in a similar way, where a morphism is now given by a family of continuous maps $A_c\to B_c$. We will again write $\mathrm{Alg}_{\mathbf{P}}$ for this category, or $\mathrm{Alg}_{\mathbf{P}}(\mathbf{Top})$ if we wish to stress that our algebras take values in topological spaces. Further on we will be especially concerned with operads and their algebras in other categories than **Sets** and **Top**, most notably the category **sSets** of simplicial sets.

Example 1.14 We will go through some of the operads listed in the previous section and examine what kind of algebras they encode.

(a) A **Com**-algebra is a set A together with a map $\mu_n : A^{\times n}\to A$ for each $n\geq 0$. One easily verifies that these μ_n are uniquely determined by a commutative and associative multiplication $\mu_2 : A^{\times 2}\to A$ with unit $\mu_0 : *\to A$ (where $*$ denotes a singleton, which is the empty product of sets). If we write $\mu_2(a,b) = ab$ then $\mu_n(a_1,\ldots,a_n)$ is the n-fold product $a_1\cdots a_n$. In other words, the category $\mathrm{Alg}_{\mathbf{Com}}$ is the category of commutative monoids. One can of course view **Com** as a topological operad, in which case **Com**-algebras are topological commutative monoids.

(b) An **Ass**-algebra is given by a set A together with maps

$$\mu_n : \Sigma_n\times A^{\times n}\longrightarrow A.$$

The rule

$$\mu_n(\sigma\tau, a_{\tau(1)},\ldots,a_{\tau(n)}) = \mu_n(\sigma, a_1,\ldots,a_n)$$

shows that μ_n is determined by $\mu_n(1,-,\ldots,-)$ and the rules for units and associativity of composition then show that all the μ_n are completely determined in terms of $\mu_2(1,-,-) : A^{\times 2}\to A$ and $\mu_0 : *\to A$. In this way, one checks that the category of **Ass**-algebras is precisely the category of (associative) monoids.

(c) The next example lies at the heart of the theory of operads and is one of the main motivations for their development. Consider the topological operad $\mathbf{E}_d$ of little d-cubes. Let X be a topological space with basepoint x_0. The *loop space* ΩX of X is the space of basepoint-preserving maps $S^1\to X$, or equivalently the space of maps from the interval $[0,1]$ to X which send the boundary $\partial[0,1]$ to the basepoint x_0. It is equipped with the compact-open topology. This space ΩX has a basepoint itself, given by the constant map with value x_0. The loop space construction can then be iterated to form the *d-fold loop space* $\Omega^d X = \Omega(\Omega^{d-1}X)$. It can be described directly as the space of maps $[0,1]^d\to X$ which map the boundary $\partial([0,1]^d)$ to x_0. This d-fold loop space is an $\mathbf{E}_d$-algebra in a natural way. Indeed, given a configuration $F = (f_1,\ldots,f_n)$ of n little d-cubes corresponding to an element of $\mathbf{E}_d(n)$ and an n-tuple of points $\lambda_1,\ldots,\lambda_n\in\Omega^d X$, interpreted as maps $[0,1]^d\to X$ sending the boundary to the basepoint, one obtains a new such map

$$\lambda = F(\lambda_1,\ldots,\lambda_n) : [0,1]^d\longrightarrow X$$

by letting $\lambda = \lambda_i f_i^{-1}$ on the image of f_i and $\lambda = x_0$ outside of these images. It is a fundamental result in the theory of iterated loop spaces due to Boardman–Vogt and May that this construction can be reversed. If Y is a connected pointed space with the structure of an $\mathbf{E}_d$-algebra, then Y is equivalent to an iterated loop space $\Omega^d X$, in the sense that there exists a pair of morphisms of $\mathbf{E}_d$-algebras

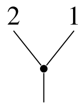
$$Y \longleftarrow Z \longrightarrow \Omega^d X$$

which are both weak homotopy equivalences. This example can be extended to the case $d = \infty$ to yield a corresponding recognition principle for *infinite loop spaces*.

(d) An algebra for the operad $\mathbf{Tree}^{\mathrm{pl}}$ consists of a set A together with for each $n \geq 0$ an operation

$$\mu_n : \Sigma_n \times A^{\times n} \longrightarrow A,$$

corresponding to the tree with n leaves and one internal vertex, usually referred to as the *n-corolla*. For example, for $\tau \in \Sigma_2$ the transposition, $\mu_2(\tau, -, -)$ corresponds to the following planar tree:



As with the operad $\mathbf{Ass}$, one verifies that the symmetry conditions completely determine μ_n in terms of $\mu_n(1, -, \ldots, -)$. Moreover, the action of all other operations of $\mathbf{Tree}^{\mathrm{pl}}$ on A are determined in terms of the μ_n, simply because any tree can be obtained by grafting corollas onto each other. Contrary to $\mathbf{Ass}$, there are no further relations between the μ_n; the operad $\mathbf{Tree}^{\mathrm{pl}}$ is an example of a *free operad*, about which we will say more in Section 1.5. A $\mathbf{Tree}$-algebra A is a set with a similar structure, but now there are maps

$$\mu_n : A^{\times n} \longrightarrow A$$

which should be invariant under the action of Σ_n permuting the coordinates, with no further relations between them.

The previous examples all concerned uncoloured operads. We will now give some examples of algebras for operads with more colours.

Example 1.15 (a) Let $\mathbf{C}$ be a (small) category, viewed as an operad with only unary operations. Then a $\mathbf{C}$-algebra is nothing but a functor $\mathbf{C} \to \mathbf{Sets}$. Similarly, if $\mathbf{C}$ is a category enriched in the category of topological spaces (i.e. a topological operad with only unary operations), a $\mathbf{C}$-algebra is a continuous functor $\mathbf{C} \to \mathbf{Top}$. More explicitly, this is a family of spaces A_c indexed by the objects of $\mathbf{C}$ and continuous maps $\mathbf{C}(c, d) \times A_c \to A_d$ satisfying the obvious functoriality condition.

(b) *(Actions).* Let $\mathbf{Act}$ be the operad with two colours a and m and operations defined as follows:

$$\mathbf{Act}(c_1, \ldots, c_n; d) = \begin{cases} * & \text{if } d = a \text{ and } c_i = a \text{ for all } i \\ * & \text{if } d = m \text{ and } c_i = m \text{ for precisely one } i \\ \varnothing & \text{otherwise} \end{cases}$$

Note that there is an inclusion of the commutative operad into **Act** which sends the unique colour of **Com** to a. An **Act**-algebra consists of two sets A and M, where by the previous remark A has the structure of a commutative monoid and additionally, for $n \geq 1$, there are maps

$$\alpha_n : A^{\times(n-1)} \times M \longrightarrow M.$$

Associativity of composition of the operations of **Act** then show that these maps give an action of the monoid A on the set M. In particular, α_1 is simply the identity and all other α_n's are determined in terms of $\alpha_2 : A \times M \to M$ and the multiplication of A. There is a similar operad whose algebras consist of an *associative* monoid A together with a left action on a set M. We leave it to the reader to spell out its definition.

Remark 1.11 suggests a definition of **P**-algebra in a general symmetric monoidal category **V**, rather than just **Sets**, which we will occasionally use:

Definition 1.16 Let **P** be an operad and **V** a symmetric monoidal category, with associated operad $\mathbf{V}^{\otimes}$ as in Example 1.3(c). Then a **P**-*algebra in* **V** is a morphism of operads $\mathbf{P} \to \mathbf{V}^{\otimes}$.

As before, **P**-algebras in **V** can be organized into a category for which we write $\mathrm{Alg}_{\mathbf{P}}(\mathbf{V})$.

Remark 1.17 Some care is needed if one wants to consider the analogue of Definition 1.16 for **P** a topological operad. In particular, if one wants to think of **P**-algebras in **Top** as morphisms of operads $\mathbf{P} \to \mathbf{Top}^{\times}$, one is implicitly using the existence of a mapping space $\mathrm{Map}(X, Y)$ between two topological spaces X and Y with the universal property that

$$\mathbf{Top}(A \times X, Y) \simeq \mathbf{Top}(A, \mathrm{Map}(X, Y)),$$

naturally in A, X and Y. Unfortunately the category of topological spaces does not admit such a structure; to correct this, one usually works with a slight modification of **Top**, often referred to as a 'convenient category of spaces'. One possibility is to use the category of compactly generated weak Hausdorff spaces.

Example 1.18 Let **C** and **D** be symmetric monoidal categories. Then a $\mathbf{C}^{\otimes}$-algebra in **D**, i.e. a morphism of operads $\mathbf{C}^{\otimes} \to \mathbf{D}^{\otimes}$, is the same thing as a *lax symmetric monoidal functor* $F : \mathbf{C} \to \mathbf{D}$, meaning a functor equipped with natural maps

$$F(c) \otimes F(d) \longrightarrow F(c \otimes d) \qquad \text{and} \qquad I_{\mathbf{D}} \to F(I_{\mathbf{C}}),$$

where $I_{\mathbf{C}}$ and $I_{\mathbf{D}}$ denote the tensor units of **C** and **D** respectively. Moreover, these maps should respect the associativity, unitality and symmetry of the tensor products of both categories.

1.2.2 Free Algebras

Let $\mathbf{P}$ be an operad and let $\mathcal{S} = \{S_c\}_{c \in C}$ be a family of sets indexed by the colours of $\mathbf{P}$. Then $\mathcal{S}$ generates a free $\mathbf{P}$-algebra $\mathrm{Free}_{\mathbf{P}}(\mathcal{S})$ defined as follows: for a colour c, elements of $\mathbf{Free}_{\mathbf{P}}(\mathcal{S})_c$ are equivalence classes of sequences $(p, s_1, \dots, s_n)$ where $n \geq 0$, $p \in \mathbf{P}(c_1, \dots, c_n; c)$ and $s_i \in S_{c_i}$. The equivalence relation on these sequences is generated by

$$(p \circ \sigma, s_{\sigma(1)}, \dots, s_{\sigma(n)}) \sim (p, s_1, \dots, s_n).$$

The family $\{\mathrm{Free}_{\mathbf{P}}(\mathcal{S})_c\}_{c \in C}$ so defined has the structure of a $\mathbf{P}$-algebra, simply using the composition operation of the operad $\mathbf{P}$. Indeed, if we denote the equivalence class of $(p, s_1, \dots, s_n)$ as above by $p \otimes \mathbf{s}$, then for $q \in \mathbf{P}(c_1, \dots, c_k; c)$ and k such equivalence classes $p_i \otimes \mathbf{s}_i$ in $\mathrm{Free}_{\mathbf{P}}(\mathcal{S})_{c_i}$, one has

$$q(p_1 \otimes \mathbf{s}_1, \dots, p_k \otimes \mathbf{s}_k) = q(p_1, \dots, p_k) \otimes (\mathbf{s}_1, \dots, \mathbf{s}_k),$$

where $(\mathbf{s}_1, \dots, \mathbf{s}_k)$ denotes the concatenation of the sequences $\mathbf{s}_1, \dots, \mathbf{s}_k$. This free $\mathbf{P}$-algebra has the usual universal property: given any $\mathbf{P}$-algebra A and a family of maps $\varphi_c : S_c \to A_c$, there is a unique $\mathbf{P}$-algebra map

$$\varphi : \mathrm{Free}_{\mathbf{P}}(\mathcal{S}) \longrightarrow A$$

with the property that $\varphi(1_c \otimes \mathbf{s}) = \varphi_c(s)$ for any $c \in C$ and $s \in S_c$. A more formal way to phrase this property is as follows: write Fam_C for the category of families of sets indexed by C (which is really just the product of categories $\prod_{c \in C} \mathbf{Sets}$). The *forgetful functor*

$$U_{\mathbf{P}} : \mathrm{Alg}_{\mathbf{P}} \longrightarrow \mathrm{Fam}_C$$

assigns to an algebra its underlying collection of sets. The functor $\mathbf{Free}_{\mathbf{P}} : \mathrm{Fam}_C \to \mathrm{Alg}_{\mathbf{P}}$ then provides a left adjoint to $U_{\mathbf{P}}$.

The same construction applies to a topological operad $\mathbf{P}$, where for a family of spaces $\mathcal{S}$ one defines $\mathrm{Free}_{\mathbf{P}}(\mathcal{S})$ as a quotient of

$$\coprod_{n \geq 0} \coprod_{c_1, \dots, c_n, c} \left(\mathbf{P}(c_1, \dots, c_n; c) \times S_{c_1} \times \cdots \times S_{c_n} \right)$$

in the same way as before, now equipping it with the quotient topology.

As a consequence of these constructions, any $\mathbf{P}$-algebra A has a *free resolution*, i.e. it can be written as a coequalizer

$$G \rightrightarrows F \longrightarrow A$$

where F and G are free $\mathbf{P}$-algebras. Indeed, one can take

$$F = \mathrm{Free}_{\mathbf{P}} U_{\mathbf{P}}(A) \qquad \text{and} \qquad G = \mathrm{Free}_{\mathbf{P}} U_{\mathbf{P}} \mathrm{Free}_{\mathbf{P}} U_{\mathbf{P}}(A).$$

The counit of the adjunction between $\mathrm{Free}_{\mathbf{P}}$ and $U_{\mathbf{P}}$ gives a map $F \to A$ and two different maps $G \to F$, which provide the maps in the diagram above.

1.2.3 Change of Operad

To conclude this section we briefly discuss the effect of a map of operads $\varphi : \mathbf{P} \longrightarrow \mathbf{Q}$ on the corresponding categories of algebras. Write C (resp. D) for the set of colours of $\mathbf{P}$ (resp. $\mathbf{Q}$). Obviously, any $\mathbf{Q}$-algebra B pulls back to a $\mathbf{P}$-algebra $\varphi^* B$ described by

$$(\varphi^* B)_c = B_{\varphi(c)}.$$

The operations of $\mathbf{P}$ act on $\varphi^* B$ in the evident way, by

$$\begin{array}{c}
\mathbf{P}(c_1, \ldots, c_n; c) \times B_{\varphi(c_1)} \times \cdots \times B_{\varphi(c_n)} \\
\Big\downarrow{\scriptstyle (\varphi, \mathrm{id}, \ldots, \mathrm{id})} \\
\mathbf{Q}(\varphi(c_1), \ldots, \varphi(c_n); \varphi(c)) \times B_{\varphi(c_1)} \times \cdots \times B_{\varphi(c_n)} \\
\Big\downarrow \\
B_{\varphi(c)}.
\end{array}$$

This defines a functor

$$\varphi^* : \mathrm{Alg}_{\mathbf{Q}} \longrightarrow \mathrm{Alg}_{\mathbf{P}}.$$

It admits a left adjoint

$$\varphi_! : \mathrm{Alg}_{\mathbf{P}} \longrightarrow \mathrm{Alg}_{\mathbf{Q}}$$

which can be conveniently described using free resolutions. Indeed, if

$$\mathrm{Free}_{\mathbf{P}}(\mathcal{T}) \rightrightarrows \mathrm{Free}_{\mathbf{P}}(\mathcal{S}) \longrightarrow A$$

is such a resolution of a $\mathbf{P}$-algebra A, then $\varphi_! A$ is given by a coequalizer

$$\mathrm{Free}_{\mathbf{Q}}(\varphi_! \mathcal{T}) \rightrightarrows \mathrm{Free}_{\mathbf{Q}}(\varphi_! \mathcal{S}) \longrightarrow A,$$

where $\varphi_! \mathcal{S}$ is the collection defined by

$$(\varphi_! \mathcal{S})_d = \coprod_{\varphi(c)=d} S_c$$

and similarly for $\varphi_! \mathcal{T}$.

An alternative, more explicit construction mimics the construction for (left) modules over rings: if $\varphi : R \to S$ is a homomorphism of rings, then the pullback $\varphi^* : \mathrm{Mod}_S \to \mathrm{Mod}_R$ has a left adjoint $\varphi_!$, sending an R-module M to $\varphi_! M = S \otimes_R M$, where S is viewed as an S-R-bimodule. In the same way, for a map of operads

$\varphi : \mathbf{P} \to \mathbf{Q}$ as above, one can define the left adjoint $\varphi_!$ by

$$\varphi_!(A) = \mathbf{Q} \otimes_{\mathbf{P}} A.$$

Here $\mathbf{Q} \otimes_{\mathbf{P}} A$ is a quotient of the set (or space) of tuples $(q, a_1, \ldots, a_n)$ where $q \in \mathbf{Q}(\varphi(c_1), \ldots, \varphi(c_n); d)$ and $a_i \in A_{c_i}$. The quotient is taken by the equivalence relation defined by

$$\big(q(\varphi p_1, \ldots, \varphi p_n), \mathbf{a}_1, \ldots, \mathbf{a}_n\big) \sim \big(q, p_1(\mathbf{a}_1), \ldots, p_n(\mathbf{a}_n)\big).$$

Here $q \in \mathbf{Q}(\varphi(c_1), \ldots, \varphi(c_n); d)$ as before and p_i is an operation in $\mathbf{P}$ with output c_i and inputs $c_i^1, \ldots, c_i^{n_i}$, while each $\mathbf{a}_i$ is a sequence $a_i^1, \ldots, a_i^{n_i}$ with $a_i^j \in A_{c_i^j}$. (It is of course possible to reformulate this in terms of bimodules over operads and their tensor products, but we will not elaborate on this.)

Remark 1.19 So far we have worked almost exclusively in the context of sets or spaces. However, as we have emphasized before, everything we do can be carried over to a general symmetric monoidal category $\mathbf{V}$, provided it has enough colimits to carry out the constructions described above.

1.3 Trees

As should already be evident from Examples 1.5 and 1.14, trees play a fundamental role in the theory of operads. This section serves to fix our definitions and terminology and discuss the free operad associated with a tree.

Definition 1.20 A *tree* T consists of a finite set V of *vertices*, a nonempty finite set E of *edges*, a distinguished element $r \in E$ called the *root*, together with the following data:

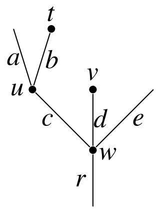
(1) A function $I \colon E - \{r\} \to V$, which we think of as assigning to an edge e the vertex $I(e)$ of which it is an *input*.
(2) A function $O \colon V \to E$, assigning to each vertex v its *output edge* $O(v)$.

For each edge e other than the root r, we obtain a sequence of edges starting at e by repeatedly applying $O \circ I$. We demand that this sequence ends in the root r after finitely many steps, for an arbitrary starting edge e.

The edges in the complement of the image of O are called the *leaves* of the tree T. The vertices in the complement of the image of I are called *stumps*, or *nullary vertices*. An *outer edge* is an edge that is either the root or a leaf. An *inner edge* is any other edge of T, i.e., an edge in the image of O that is not the root. Such an edge is both an output edge and an input edge of some other vertex.

When writing T for a tree, we will usually write $E(T)$ and $V(T)$ for its sets of edges and vertices, respectively. The smallest possible tree consists of a single edge and no vertices; this tree will be denoted by η. The next smallest tree consists of a single edge and a single vertex, pictured as:

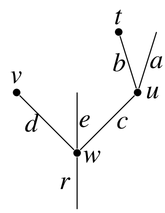
The following is a typical picture of a tree T with $E = \{a, b, c, d, e, r\}$ and $V = \{t, u, v, w\}$:



Here $I(a) = u$, $O(u) = c$, etc. This tree has inner edges b, c and d and outer edges a, e and r. We will always picture trees with the root at the bottom, so that in the picture above T has root r and leaves a and e. The *valence* of a vertex v is the number of its input edges, i.e., the cardinality of the set $I^{-1}\{v\}$, so that a stump is a vertex of valence zero. The sets $E(T)$ and $V(T)$ both have a natural partial ordering, where $e < f$ for edges e and f (or $v < w$ for vertices v and w) if the directed path from e (resp. v) to the root of T passes through f (resp. through w). Thus, any leaf edge of T is minimal in this partial ordering on $E(T)$, whereas the root edge is the unique maximal element. A similar comment applies to vertices.

Definition 1.21 If T is a tree, then a *planar structure* on T is a linear ordering on the set of input edges of every vertex of T.

It is unfortunate that any picture of a tree T automatically endows it with a planar structure, but the reader should note that such a structure is not part of the data of T itself. Thus, the following is a picture of the same tree as above, but corresponding to a different planar structure:



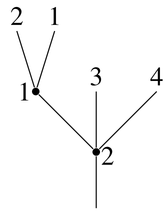
If T is a tree, then we reserve the term *subtree* for a smaller tree S contained in T that satisfies the following condition: if a vertex v of T is contained in S, then so are all input edges of v. Thus a subtree of T is uniquely specified by listing its vertices; conversely, a collection of vertices of T defines a subtree only if the graph consisting of all those vertices together with the edges attached to them is connected.

A tree T defines an operad $\Omega(T)$ in the following way. The colours of $\Omega(T)$ are the edges of T. An operation $p \in \Omega(T)(c_1, \ldots c_n; c)$ is a subtree of T with leaves $c_1, \ldots, c_n$ and root c. Note that for any sequence of colours $c_1, \ldots, c_n, c$ there is at most one such subtree; therefore, the sets of operations of $\Omega(T)$ are either empty or singletons. As an example, consider the tree T pictured above. It has an operation $w \in \Omega(T)(c, d, e; r)$ and an operation $v \in \Omega(T)(-; d)$. The composition $w \circ_d v \in \Omega(T)(c, e; r)$ is the operation corresponding to the subtree of T with leaves c, e and root r. Also, the operation $\tau^*(w \circ_d v) \in \Omega(T)(e, c; r)$ corresponds to precisely the same subtree, the only difference being that we have listed its leaves in a different order. The operad $\Omega(T)$ is an example of a free operad (see Section 1.5). Indeed, its operations are freely generated by the vertices of T, in the sense that any map of operads $\varphi : \Omega(T) \to \mathbf{P}$ is uniquely determined by its effect on colours and, for every vertex v of T with inputs $c_1, \ldots, c_n$ and output c, the operation $\varphi(v) \in \mathbf{P}(\varphi(c_1), \ldots, \varphi(c_n); \varphi(c))$. Conversely, any assignment of a colour of $\mathbf{P}$ to each edge of T and a compatible assignment of an operation of $\mathbf{P}$ to each vertex of T extends uniquely to a map of operads φ.

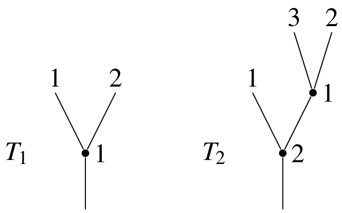
We already used trees in Examples 1.5 and 1.14. Let us give yet another example of an operad defined using trees, which exploits the idea that trees parametrize the operations of an operad and the ways in which they can be composed:

Example 1.22 There is an operad $\mathbf{O}$ whose algebras are uncoloured operads. Its set of colours is the set of nonnegative integers. An operation $T \in \mathbf{O}(k_1, \ldots, k_n; k)$ is represented by a planar tree with k leaves and n vertices, where the leaves are numbered $1, \ldots, k$ and the vertices are numbered $1, \ldots, n$ in such a way that the vertex numbered i has k_i incoming edges. Note that these edges are ordered by the planar structure on T. More precisely, an operation is an isomorphism class of such labelled trees, where the isomorphisms of trees involved are required to preserve planar structures and labellings.

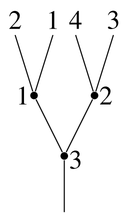
For example,



depicts an element $T \in \mathbf{O}(2, 3; 4)$. The symmetric groups act by permuting the labels on the vertices. The composition of operations of $\mathbf{O}$ is defined as follows: for (isomorphism classes of) trees $T \in \mathbf{O}(k_1, \ldots, k_n; k)$ and $T_i \in \mathbf{O}(l^i_1, \ldots, l^i_{n_i}; k_i)$, the composition $\gamma(T, T_1, \ldots, T_n)$ is the tree obtained by replacing the vertex labelled i in T by the tree T_i. This tree T_i has k_i numbered leaves, which we identify with the incoming edges of the vertex labelled i in T, the leaf j of T_i being identified with the jth incoming edge. Furthermore, the vertices of T_i, which were labelled 1 through k_i, are now relabelled $k_1 + \cdots + k_{i-1} + 1$ through $k_1 + \cdots + k_i$. For example, take T as pictured above and T_1, T_2 as follows:



Then $\gamma(T, T_1, T_2)$ can be pictured as follows:



This example admits a straightforward variation for operads coloured by some set C. To be precise, there exists an operad $\mathbf{O}_C$ for which algebras are precisely C-coloured operads. To obtain $\mathbf{O}_C$ from the setup just described one should introduce an additional labelling of the edges of trees by elements of C. We leave the details to the interested reader for now, although we will return to this operad in Example 13.32.

1.4 Alternative Definitions for Operads

In this short section we review some alternative ways of defining operads. First of all, we discuss a 'coordinate-free' definition where the basic operations are not indexed by natural numbers n and permutations, but rather by finite sets A and bijections between them. Then we will discuss how the axioms for compositions $p \circ (q_1, \dots, q_n)$ of an n-ary operation p with operations $q_1, \dots, q_n$ can alternatively be phrased in terms of the 'partial compositions' $p \circ_i q$ of Remark 1.2, composing p with q in the ith variable. The material of this section will for instance be useful in our discussion of free operads in Section 1.5 and the Fulton–MacPherson operad in Section 1.8.

For simplicity, we begin with the case of operads whose set of colours C is a singleton $\{c\}$. As in Remark 1.4(a), we then abbreviate the set of operations $\mathbf{P}(c, \dots, c; c)$, with n input copies of c, by $\mathbf{P}(n)$. Our first aim is to describe a notion of operad with sets of operations of the form $\mathbf{P}(A)$, for every finite set A. This will reproduce an operad in our previous sense by setting

$$\mathbf{P}(n) := \mathbf{P}(\{1, \dots, n\}).$$

To begin, write $\mathbf{Fin}^{\cong}$ for the category of finite sets and isomorphisms between them. Then a *collection* is a functor $K\colon (\mathbf{Fin}^{\cong})^{\mathrm{op}} \to \mathbf{Sets}$. Thus, a collection defines for every finite set A a corresponding set $K(A)$, and for each bijection $f\colon A \to B$ a function

$$f^*\colon K(B) \to K(A).$$

In particular, any collection K gives a sequence of sets

$$K(n) := K(\{1, \dots, n\})$$

and each $K(n)$ carries a right action of the symmetric group

$$\mathrm{Aut}(\{1, \dots, n\}) = \Sigma_n.$$

Now a *coordinate-free uncoloured operad* is a collection $\mathbf{P}$ together with the following extra structure maps:

- for each singleton $A = \{a\}$ there is a *unit* $1_a \in \mathbf{P}(\{a\})$,
- for any map of finite sets $\varphi\colon B \to A$, there is a *composition*

$$\mathbf{P}(A) \times \prod_{a \in A} \mathbf{P}(B_a) \to \mathbf{P}(B)\colon (p, (q_a)_{a \in A}) \mapsto p \circ (q_a)_{a \in A},$$

  where B_a denotes the inverse image $\varphi^{-1}(a)$.

The axioms to be satisfied by these structure maps are as follows:

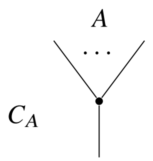
- the elements $1_a \in \mathbf{P}(\{a\})$ act as two-sided units for the composition of operations, in the same sense as before,

– the composition is associative, in the sense that for maps $C \to B \to A$ and operations $p \in P(A)$, $q_a \in B_a$ for a in A, and $r_b \in C_b$ for $a \in A$ and $b \in B_a$, we have
$$(p \circ (q_a)_{a \in A}) \circ (r_b)_{b \in B_a} = p \circ (q_a \circ (r_b)_{b \in B_a})_{a \in A},$$
– for $p \in P(A)$, $q_a \in B_a$ for a in A, and each commutative square of finite sets
$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"] & A \end{tikzcd}$$
in which the horizontal arrows are bijections, we should have
$$g^* p \circ (f_c^* q_{g(c)})_{c \in C} = f^*(p \circ (q_a)_{a \in A}),$$
where $f_c \colon D_c \to B_a$ is the restriction of f.

Each coordinate-free uncoloured operad $\mathbf{P}$ in particular defines an uncoloured operad as in the previous section, essentially by restricting to the finite sets $\{1, \ldots, n\}$ for $n \geq 0$. This restriction induces an equivalence of categories between the two different versions of the theory, simply because the full subcategory of $\mathbf{Fin}^{\cong}$ spanned by the sets $\{1, \ldots, n\}$ is a skeleton, i.e., contains one object in each isomorphism class.

We will now explain how to modify the definitions above for the case of operads with a set of colours C. Although this can easily be done in the same language of finite sets (now equipped with a function to C) and isomorphisms between them, it is convenient to shift perspective slightly and use the language of trees.

A tree with just one vertex is called a *corolla*. Such a tree is determined up to isomorphism by the number of its leaves. We write C_A for a corolla with set of leaves A:



Fix a set C of colours. A C-*coloured corolla* is a corolla C_A equipped with a map from its set of edges $E(C_A)$ to C. In other words, it is a corolla whose edges (including the root) are coloured by C. Write $\mathbf{Cor}_C$ for the category of C-coloured corollas, with maps between these being those isomorphisms of corollas $C_A \to C_B$ (i.e., bijections $A \to B$) compatible with the specified labellings. Then the category of C-coloured collections is defined to be that of functors $\mathbf{Cor}_C^{\mathrm{op}} \to \mathbf{Sets}$. We will write $\mathbf{Coll}_C$ for this category. Observe that if $C = \{c\}$, then the category of C-coloured collections is clearly equivalent to the category of collections (i.e., functors $(\mathbf{Fin}^{\cong})^{\mathrm{op}} \to \mathbf{Sets}$) we considered above.

With this terminology in place, we define a *coordinate-free operad* $\mathbf{P}$ with set of colours C to be a C-coloured collection, thought of as a functor $\mathbf{Cor}_C^{\mathrm{op}} \to \mathbf{Sets}$, equipped with the following additional structure:

- for each corolla C_1 with a single leaf, with both its leaf and its root labelled by the same element $c \in C$, there is a *unit* $1_c \in \mathbf{P}(C_1)$,
- for each C-coloured corolla $C_A \in \mathbf{Cor}_C$ and further C-coloured corollas $\{C_{B_a}\}_{a\in A}$, with the root of C_{B_a} labelled by the same colour as the corresponding leaf a of C_A, there is a *composition*

$$\mathbf{P}(C_A) \times \prod_{a\in A} \mathbf{P}(C_{B_a}) \to \mathbf{P}\big(C_{\amalg B_a}\big) : (p, (q_a)_{a\in A}) \mapsto p \circ (q_a)_{a\in A},$$

  where $C_{\amalg B_a}$ denotes the C-coloured corolla with set of leaves $\coprod_{a\in A} B_a$, labellings of its leaves induced by those of the corollas C_{B_a}, and labelling of its root the same as that of C_A.

As before, this structure has to satisfy the evident three axioms expressing unitality, associativity, and equivariance of these composition maps. Note that in the language of corollas and trees, the composition maps can be pictured as 'grafting' the corollas C_{B_a} onto the leaves of C_A and contracting the inner edges of the resulting tree to obtain a new corolla $C_{\amalg B_a}$. As before, let us observe that each coordinate-free operad $\mathbf{P}$ in particular gives an operad in the sense defined in Section 1.1. Indeed, for a tuple of colours $(c_1, \ldots, c_n, c)$, one considers the corolla C_n with leaves $1, \ldots, n$, labels the leaf i by the colour c_i, and the root by c. Then setting $\mathbf{P}(c_1, \ldots, c_n; c)$ to be the set associated by $\mathbf{P}$ to this C-coloured corolla gives an operad. This process implements an equivalence of categories between coordinate-free C-coloured operads and (ordinary) C-coloured operads as defined previously. We will use the notation $\mathbf{Op}_C$ for the category of coordinate-free C-coloured operads.

To conclude this section, we make some further remarks about the partial compositions of Remark 1.2. Suppose $\mathbf{P}$ is a coordinate-free operad with set of colours C and consider two C-coloured corollas C_A and C_B such that one of the leaves ℓ of C_A is labelled by the same colour c as the root of C_B. Then the composition maps of $\mathbf{P}$ in particular induce a 'composition along ℓ' map

$$- \circ_\ell - : \mathbf{P}(C_A) \times \mathbf{P}(C_B) \to \mathbf{P}(C_{A\circ_\ell B}).$$

Here $A \circ_\ell B$ is the set $B \amalg A - \{\ell\}$. This is precisely the set of leaves of the corolla $C_{A\circ_\ell B}$ obtained by grafting C_B onto the leaf ℓ of C_A and then contracting the resulting inner edge ℓ to obtain a new corolla. As in Remark 1.2, the map $- \circ_\ell -$ can be constructed by inserting identity operations at all the other leaves of the corolla C_A. Conversely, the composition maps

$$\mathbf{P}(C_A) \times \prod_{a\in A} \mathbf{P}(C_{B_a}) \to \mathbf{P}\big(C_{\amalg B_a}\big)$$

of $\mathbf{P}$ can be reconstructed as an iterated application of the compositions $- \circ_\ell -$, one for each leaf ℓ of C_A. The axioms that these composition maps have to satisfy 'restrict' to axioms for the partial compositions $\circ_\ell$, and imposing these would give an equivalent way of defining operads.

1.5 Free Operads

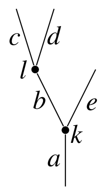
Some of the operads we have encountered (such as $\Omega(T)$ and $\mathbf{Tree}^{\mathrm{pl}}$) are *free* in a precise sense. In this section we will explain this notion of freeness and give a precise description of free operads in general. This will be important in Section 1.7, where we discuss the W-construction, and in many places later on in this book. Throughout this section it will be convenient to use the coordinate-free definition of operads, as explained in the preceding section.

Let C be a fixed set of colours. Recall the notation $\mathbf{Coll}_C$ for the category of C-coloured collections and $\mathbf{Op}_C$ for that of (coordinate-free) C-coloured operads. Any such operad $\mathbf{P}$ has an underlying collection $U(\mathbf{P})$ and this defines a 'forgetful functor'

$$U\colon \mathbf{Op}_C \to \mathbf{Coll}_C.$$

We will now give an explicit construction of the operad $F(K)$ freely generated by a C-coloured collection K. This F will be a functor left adjoint to U. This means that $F(K)$ will be an operad having the familiar universal property; there is a map of collections $\eta\colon K \longrightarrow UF(K)$ and any map of collections $\varphi\colon K \longrightarrow U(\mathbf{P})$ can uniquely be extended to a map of operads $\hat{\varphi}\colon F(K) \to \mathbf{P}$, such that $U(\hat{\varphi}) \circ \eta = \varphi$.

To construct $F(K)$ we will use trees with certain labellings coming from the collection K. For a C-coloured corolla C_A, the elements of $F(K)(C_A)$ are equivalence classes of trees whose set of leaves is A and with labellings as follows. Each edge of T is labelled by an element of C (i.e., it is coloured), with the condition that the labellings of the root and leaves of T agree with those of the corolla C_A. Then every vertex v of T defines a coloured corolla C_v, namely by considering the input edges and output edge of v, and such a vertex gets labelled by an element k_v of $K(C_v)$. Here is a picture of such a tree, representing an element of $F(K)(C_3)$ with C_3 a 3-corolla with leaves labelled by the colours c, d, e and root labelled by a.



In this example k and l are elements of the collection K corresponding to the 2-corollas with colourings of their edges as indicated in the picture. The equivalence relation we impose on such labelled trees is related to the given symmetries of the collection K. If T and T' are two such labelled trees, they represent the same operation in $F(K)(C_A)$ if there exists an isomorphism of trees

$$\varphi\colon T \longrightarrow T'$$

with the following properties:

- φ is the identity on the set of leaves A,
- φ preserves the colourings of the edges,
- for v a vertex of T and $v' = \varphi(v)$ the corresponding vertex of T', we have

$$k_v = \bar{\varphi}^*(k_{v'}),$$

where $\bar{\varphi}$ is the isomorphism of corollas $C_v \cong C_{v'}$ induced by φ.

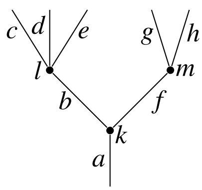
Such an isomorphism $\varphi\colon T \to T'$, if it exists, is unique.

Observe that there is a natural notion of composition of such equivalence classes of labelled trees, simply by grafting. Indeed, if T is a labelled tree with one of its leaves coloured e and S another such tree with root coloured e, then one forms the composition $T \circ_e S$ by simply grafting S onto the leaf of T labelled e and carrying along all the labels. This grafting operation is compatible with the equivalence relation described above. Furthermore, the tree

with unique edge labelled c acts as a unit 1_c for this operation of grafting. These operations give $F(K)$ all the structure required of an operad.

It is not hard to verify that $F(K)$ has the claimed universal property. First of all, the map of collections $\eta\colon K \to UF(K)$ is defined by sending $k \in K(C_A)$, for some coloured corolla C_A, to the element of $F(K)(C_A)$ represented by C_A itself, with its unique vertex labelled by k.

For an operad $\mathbf{P}$ and a map of collections $\varphi\colon K \to U(\mathbf{P})$, one constructs a map of operads $\hat{\varphi}\colon F(K) \to \mathbf{P}$ by sending a tree T with vertices v labelled by elements k_v of K to the composition of the operations $\varphi(k_v)$ of $\mathbf{P}$. More precisely, one combines these operations k_v using the partial composition operators $- \circ_e -$ of the previous section, with e ranging over the inner edges of the tree T. For example, an operation in the operad $F(K)$ represented by the tree



is sent by $\hat{\varphi}$ to the operation $(\varphi(k) \circ_b \varphi(l)) \circ_f \varphi(m) = (\varphi(k) \circ_f \varphi(m)) \circ_b \varphi(l)$ of $\mathbf{P}$. The general procedure for constructing $\hat{\varphi}$ should be clear from this example.

Example 1.23 (a) Let C be a singleton $\{c\}$. Recall that in this case a collection can equivalently be regarded as a functor from $(\mathbf{Fin}^{\cong})^{\mathrm{op}}$ to **Sets**. Now take K to be the terminal collection with $K(A) = *$ for every finite set A. Then $F(K)$ is precisely the operad **Tree** of example 1.5(c). The identification of this operad as a free operad also makes the identification of its algebras as in Example 1.14(d) an easy task.

(b) If $K(A) = \mathrm{Aut}(A)$ for each A, then $F(K)$ is the operad $\mathbf{Tree}^{\mathrm{pl}}$ of planar trees, as in Example 1.5(c).

(c) Let T be a tree. As we claimed before, the operad $\Omega(T)$ of Section 1.3 is a free operad. Indeed, let C be the set of edges of T and K the C-coloured collection for which an element of $K(C_A)$, with C_A a C-coloured corolla, is an embedding $C_A \to T$ compatible with colourings. Note that $K(C_A)$ is either empty or a singleton, the latter happening precisely when there exists a vertex of T for which the labellings of its input and output edges agree with those of C_A. Then $F(K)$ is (isomorphic to) the operad $\Omega(T)$. (Note that the collection K we just used can be interpreted as the functor on $\mathbf{Cor}_C^{\mathrm{op}}$, the opposite of the category of C-coloured corollas and embeddings, 'represented' by the tree T.)

(d) Consider a set C and a C-coloured collection K which has only unary operations, i.e. where $K(C_A)$ is empty unless the corolla C_A has precisely one leaf. Then we may picture K as a directed graph, where the elements of C are vertices and the elements of K are arrows. For example, the $\{a, b, c, d\}$-coloured collection

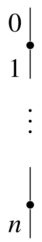
$$\begin{aligned} K(a; b) &= \{f\}, \qquad & K(b; c) &= \{g, h\} \\ K(a; d) &= \{i, j\}, \qquad & K(d; c) &= \{k\}, \end{aligned}$$

is represented by

$$\begin{tikzcd} & b \arrow[dr, shift left, "g"] \arrow[dr, shift right, "h"'] & \\ a \arrow[ur, "f"] \arrow[dr, shift left, "i"] \arrow[dr, shift right, "j"'] & & c \\ & d \arrow[ur, "k"'] & \end{tikzcd}$$

The free operad $F(K)$ is then simply the free category on this directed graph.

(e) There is an overlap between the examples of kinds (c) and (d) in the case where K is a linear tree, i.e., one that has only vertices of valence one. The tree pictured on the left then corresponds to a directed graph as pictured on the right



$$\begin{array}{c} 0 \\ \downarrow \\ 1 \\ \downarrow \\ \vdots \\ \downarrow \\ n \end{array}$$

and interpreted as in (c) and (d) respectively these correspond to the same free category, namely the finite linear order $[n]$ with objects $0, \ldots, n$. We will try to help the reader not to confuse these two notations.

1.6 The Tensor Product of Operads

Let $\mathbf{P}$ and $\mathbf{Q}$ be operads (either of sets or of topological spaces). The category of $\mathbf{P}$-algebras $\mathrm{Alg}_{\mathbf{P}}$ (in either sets or spaces) admits a symmetric monoidal structure, simply by forming Cartesian products. Indeed, if A and B are $\mathbf{P}$-algebras one defines a new $\mathbf{P}$-algebra $A \times B$ by considering the sets (or spaces) $A_c \times B_c$ indexed by colours c of $\mathbf{P}$ and taking the products of structure maps for A and B. Therefore, as in Definition 1.16, it makes sense to speak of $\mathbf{Q}$-algebras in the category $\mathrm{Alg}_{\mathbf{P}}$ and of course $\mathbf{P}$-algebras in the category $\mathrm{Alg}_{\mathbf{Q}}$. These form categories $\mathrm{Alg}_{\mathbf{Q}}(\mathrm{Alg}_{\mathbf{P}})$ and $\mathrm{Alg}_{\mathbf{P}}(\mathrm{Alg}_{\mathbf{Q}})$ respectively. In this section we will define an operad $\mathbf{P} \otimes \mathbf{Q}$, the *Boardman–Vogt tensor product* of $\mathbf{P}$ and $\mathbf{Q}$, which has the property that there are equivalences of categories

$$\mathrm{Alg}_{\mathbf{Q}}(\mathrm{Alg}_{\mathbf{P}}) \simeq \mathrm{Alg}_{\mathbf{P}\otimes\mathbf{Q}} \simeq \mathrm{Alg}_{\mathbf{P}}(\mathrm{Alg}_{\mathbf{Q}}).$$

In the case when $\mathbf{P}$ and $\mathbf{Q}$ both have a single colour, a $\mathbf{P} \otimes \mathbf{Q}$-algebra is a set or space which has both the structure of a $\mathbf{P}$-algebra and a $\mathbf{Q}$-algebra, in such a way that these two structures distribute over one another. In this section we make precise what is meant by this; see also Example 1.24.

We define $\mathbf{P} \otimes \mathbf{Q}$ in terms of generators and relations. The set of colours of this operad is the product of the sets of colours of $\mathbf{P}$ and $\mathbf{Q}$; if c and d are colours of $\mathbf{P}$ and $\mathbf{Q}$ respectively, we write $c \otimes d$ for the corresponding colour of $\mathbf{P} \otimes \mathbf{Q}$. The operad $\mathbf{P} \otimes \mathbf{Q}$ is generated by the following two kinds of operation:

(a) For each operation $p \in \mathbf{P}(c_1, \ldots, c_n; c)$ and colour d of $\mathbf{Q}$, there is an operation

$$p \otimes d \in (\mathbf{P} \otimes \mathbf{Q})(c_1 \otimes d, \ldots, c_n \otimes d; c \otimes d).$$

(b) For each colour c of $\mathbf{P}$ and operation $q \in \mathbf{Q}(d_1, \ldots, d_m; d)$, there is an operation

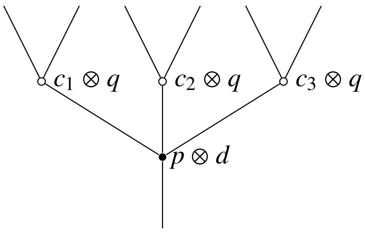
$$c \otimes q \in (\mathbf{P} \otimes \mathbf{Q})(c \otimes d_1, \ldots, c \otimes d_m; c \otimes d).$$

The relations to be satisfied by these generators are the following:

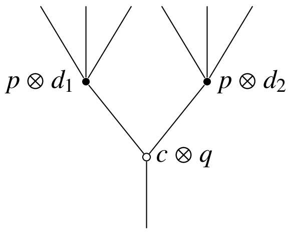
(1) $(p \otimes d)(p_1 \otimes d, \dots, p_n \otimes d) = p(p_1, \dots, p_n) \otimes d$,
(2) $(c \otimes q)(c \otimes q_1, \dots, c \otimes q_m) = c \otimes q(q_1, \dots, q_n)$,
(3) $\sigma^*(p \otimes d) = (\sigma^* p) \otimes d$ for $\sigma \in \Sigma_n$,
(4) $\sigma^*(c \otimes q) = c \otimes (\sigma^* q)$ for $\sigma \in \Sigma_m$,
(5) $(p \otimes d)(c_1 \otimes q, \dots, c_n \otimes q) = \sigma^*_{n,m}\big((c \otimes q)(p \otimes d_1, \dots, p \otimes d_m)\big)$.

Note that relations (1) and (3) express precisely the condition that for any colour d of $\mathbf{Q}$ there is a map of operads $- \otimes d : \mathbf{P} \to \mathbf{P} \otimes \mathbf{Q}$. Likewise, (2) and (4) give maps of operads $c \otimes - : \mathbf{Q} \to \mathbf{P} \otimes \mathbf{Q}$ for colours c of $\mathbf{P}$. Relation (5) is the most interesting; we will refer to it as the *Boardman–Vogt interchange relation*. The permutation $\sigma_{n,m}$ is the appropriate element of Σ_{nm} that makes sense of formula (5), i.e. the element relating the sequences $(c_1 \otimes d_1, \dots, c_1 \otimes d_m, \dots, c_n \otimes d_1, \dots, c_n \otimes d_m)$ and $(c_1 \otimes d_1, \dots, c_n \otimes d_1, \dots, c_1 \otimes d_m, \dots, c_n \otimes d_m)$.

Let us give a graphical interpretation of a small example of the interchange relation. If $p \in \mathbf{P}(c_1, c_2, c_3; c)$ and $q \in \mathbf{Q}(d_1, d_2; d)$ are operations, we can represent the composition $(p \otimes d)(c_1 \otimes q, c_2 \otimes q, c_3 \otimes q)$ by the following picture:

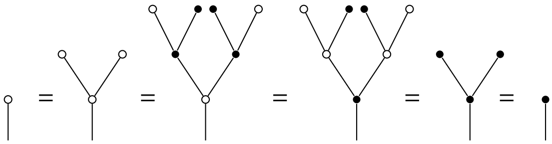


The interchange relation says that this operation of $\mathbf{P} \otimes \mathbf{Q}$ can also be represented by applying $\sigma^*_{2,3}$ to the following picture:

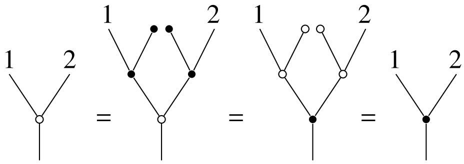


The definition of $\mathbf{P} \otimes \mathbf{Q}$ applies to operads of sets as well as to operads of spaces. In the second case, one has to topologize the spaces of operations of $\mathbf{P} \otimes \mathbf{Q}$ as quotients of appropriate products of spaces of operations of $\mathbf{P}$ and $\mathbf{Q}$.

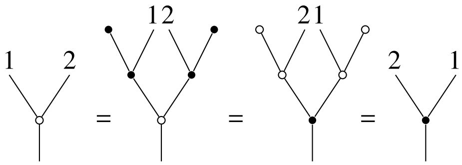
Example 1.24 (a) The tensor product $\mathbf{Ass} \otimes \mathbf{Ass}$ is naturally isomorphic to $\mathbf{Com}$, essentially by the classical Eckmann–Hilton argument. We give a graphical presentation of this argument. An n-ary operation of $\mathbf{Ass}$ can be pictured as an n-corolla with a labelling of its leaves by the numbers $1, \dots, n$. As in the pictures above we use black vertices for operations coming from the first factor and white vertices for operations coming from the second. If we do not explicitly label the leaves of a vertex by numbers, this means the numbering agrees with the planar order of the pictured tree. Recall that **Ass** is generated as an operad by a binary operation μ_2 and a nullary operation μ_0. First, we show that **Ass** $\otimes$ **Ass** has only one operation of arity zero:



Here the middle equality uses the interchange relation; all other identities use the relation that multiplying by the unit is the identity in **Ass**. A priori, there are two generating binary operations for **Ass** $\otimes$ **Ass**; to see that these are the same, observe the chain of equalities



For the middle equality we used the interchange relation and the fact that the two nullary operations agree. To see that this binary operation of **Ass** $\otimes$ **Ass** is commutative, observe that



It follows that **Ass** $\otimes$ **Ass** is a quotient of **Ass** in which the operation μ_2 is forced to be commutative; therefore **Ass** $\otimes$ **Ass** must be the commutative operad, since that is the only such quotient.

(b) Recall that any double loop space $\Omega^2 X$ is naturally an $\mathbf{E}_2$-algebra. However, it also gives rise to an $\mathbf{E}_1 \otimes \mathbf{E}_1$-algebra: indeed, thinking of points of $\Omega^2 X$ as maps $[0,1]^{\times 2} \to X$ which send the boundary to the basepoint, there is a 'vertical' and a 'horizontal' composition law, making $\Omega^2 X$ into an $\mathbf{E}_1$-algebra in two different ways. Furthermore, these two compositions distribute over one another in the appropriate way, so that the interchange relation is respected. In fact, there is a map of operads $\mathbf{E}_1 \otimes \mathbf{E}_1 \to \mathbf{E}_2$, which Dunn proved to be a homotopy equivalence. We will have more to say about tensor products of $\mathbf{E}_d$-operads later, but for now the reader should contrast this example with the previous one. The operads $\mathbf{E}_1$ and **Ass** are homotopy equivalent, but the tensor products $\mathbf{E}_1^{\otimes 2}$ and $\mathbf{Ass}^{\otimes 2}$ behave very differently.

1.7 The Boardman–Vogt Resolution of an Operad

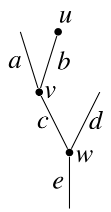
For an operad **P**, we will present a 'resolution' by a map of operads

$$\varepsilon : W(\mathbf{P}) \longrightarrow \mathbf{P}.$$

The new operad $W(\mathbf{P})$ has the same set of colours as **P** and the map ε is the identity on colours. Thus, the induced functor

$$\varepsilon^* \colon \mathrm{Alg}_{\mathbf{P}} \longrightarrow \mathrm{Alg}_{W(\mathbf{P})}$$

gives any **P**-algebra $A = \{A_c\}_{c \in C}$ the structure of a $W(\mathbf{P})$-algebra. The idea is that $W(\mathbf{P})$-algebras are '**P**-algebras up to coherent homotopy'. This operad $W(\mathbf{P})$ will be a topological operad, also when **P** itself is discrete. To illustrate what we mean let us first consider a small example. Let T be the following tree:



Since the operad $\Omega(T)$ is freely generated by the vertices of T, as explained in Example 1.23(c), a topological $\Omega(T)$-algebra A is determined by a collection of spaces indexed by the edges of this tree together with continuous maps

$$\begin{aligned} A(u)&\colon * \longrightarrow A(b), \\ A(v)&\colon A(a) \times A(b) \longrightarrow A(c), \\ A(w)&\colon A(c) \times A(d) \longrightarrow A(e). \end{aligned}$$

Here $*$ denotes the singleton space. Recall that the operad $\Omega(T)$ has operations corresponding to any subtree of T; the corresponding multiplication maps for A are given by appropriate compositions of the three maps above. For example, we have

$$A(w \circ_c v) = A(w) \circ (A(v) \times \mathrm{id}_{A(d)}) : A(a) \times A(b) \times A(d) \longrightarrow A(e).$$

By contrast, a $W(\Omega(T))$-algebra B will be given by a similar collection of spaces (in particular, still indexed on the edges of T), but now equipped with maps

$$\begin{aligned}
B(u)\colon * &\longrightarrow B(b),\\
B(v)\colon B(a)\times B(b) &\longrightarrow B(c),\\
B(w)\colon B(c)\times B(d) &\longrightarrow B(e),\\
B(v\circ_b u)\colon B(a)\times[0,1] &\longrightarrow B(c),\\
B(w\circ_c v)\colon B(a)\times B(b)\times B(d)\times[0,1] &\longrightarrow B(e),\\
B(w\circ_c v\circ_b u)\colon B(a)\times B(d)\times[0,1]^{\times 2} &\longrightarrow B(e),
\end{aligned}$$

satisfying the following conditions:

$$\begin{aligned}
B(v\circ_b u)|_{B(a)\times\{0\}} &= B(v)\circ(\mathrm{id}_{B(a)}\times B(u)),\\
B(w\circ_c v)|_{B(a)\times B(b)\times B(d)\times\{0\}} &= B(w)\circ(B(v)\times\mathrm{id}_{B(d)})\\
B(w\circ_c v\circ_b u)|_{B(a)\times B(d)\times\{0\}\times[0,1]} &= B(w\circ_c v)\circ(\mathrm{id}_{B(a)}\times B(u)),\\
B(w\circ_c v\circ_b u)|_{B(a)\times B(d)\times[0,1]\times\{0\}} &= B(w)\circ(B(v\circ_b u)\times\mathrm{id}_{B(c)}).
\end{aligned}$$

In the algebra A, the multiplication map corresponding to the subtree with leaves a, b, d and root e was simply given by composing the operations $A(v)$ and $A(w)$. For the algebra B the situation is different. There is an operation

$$B(w\circ_c v)|_{B(a)\times B(b)\times B(d)\times\{1\}}\colon B(a)\times B(b)\times B(d)\longrightarrow B(e)$$

which is now homotopic, rather than equal, to the map

$$B(w)\circ(B(v)\times\mathrm{id}_{B(d)}).$$

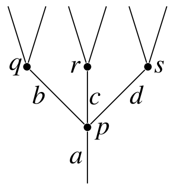
It is in this sense that B can be thought of as a $\Omega(T)$-algebra 'up to homotopy'.

We now explain the W-construction in general. Fix an operad $\mathbf{P}$ with set of colours C. It can be an operad in sets or in topological spaces. As discussed in Section 1.5, we can consider the underlying collection $U(\mathbf{P})$ of $\mathbf{P}$ and the free operad $FU(\mathbf{P})$ which, by its universal property, admits a canonical map $FU(\mathbf{P})\to\mathbf{P}$. The resolution $W(\mathbf{P})$ will sit in between these by a factorization

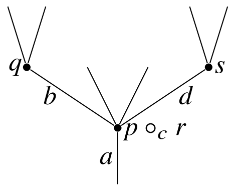
$$FU(\mathbf{P})\to W(\mathbf{P})\xrightarrow{\varepsilon}\mathbf{P}.$$

For its construction, recall that elements in $FU(\mathbf{P})(c_1,\dots,c_n;c)$ are represented by trees T with leaves labelled by the set $\{c_1,\dots,c_n\}$ and root by c. The edges of T are labelled by colours in C and the vertices of T are labelled by operations of $\mathbf{P}$. Two such labelled trees represent the same point in $FU(\mathbf{P})(c_1,\dots,c_n;c)$ if they are related by an isomorphism of trees respecting colours and operations in the way explained in Section 1.5.

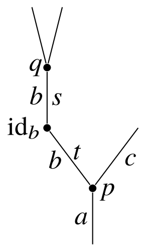
For $W(\mathbf{P})$ we will add a further labelling and more relations. The elements of $W(\mathbf{P})(c_1,\dots,c_n;c)$ are represented by labelled trees as for $FU(\mathbf{P})$, where in addition the edges are all given a length $t\in[0,1]\subseteq\mathbf{R}$. This assignment of lengths must be such that the external edges of T, i.e. the leaves and the root, all have length 1. Two such labelled trees with lengths represent the same operation if they are related by an isomorphism as for $FU(\mathbf{P})$, where the isomorphism must of course respect lengths of edges. However, there are two more relations, one concerning edges of length zero and one related to vertices labelled by unit operations. If an edge has length zero, then the operation in $W(\mathbf{P})$ represented by that labelled tree is to be identified with the one represented by the smaller labelled tree obtained by *contracting* the edge of length zero and labelling the newly arising vertex by the appropriate composition in $\mathbf{P}$. Let us illustrate this relation by an example. Consider an operation of $W(\mathbf{P})$ represented by a labelled tree T, pictured as follows (with only a small part of the labellings indicated):



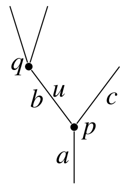
Here p, q, r and s are labels corresponding to operations of $\mathbf{P}$, whereas a, b, c and d are colours of $\mathbf{P}$. If the length assigned to the inner edge c is zero, then the operation of $W(\mathbf{P})$ represented by the above picture is to be identified with the operation represented by the following smaller labelled tree:



The other relation identifies a labelled tree where a vertex v is labelled by a unit operation of $\mathbf{P}$ with the smaller labelled tree obtained by 'erasing' that vertex and giving the newly arising edge as its length the maximum of the lengths of the two edges connected to v. In a picture, consider a labelled tree (again with only part of the labels actually indicated) as follows:

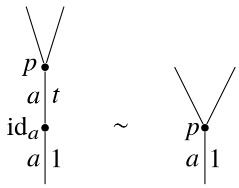


Here t and s indicate the lengths assigned to the relevant edges. The operation represented by this labelled tree is to be identified with



Here the new length u is the maximum of s and t.

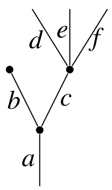
Notice that this identification is compatible with the requirement that external edges have length 1, a compatibility which becomes relevant if the vertex labelled by a unit is attached to an external edge, e.g. near the root as in the following:



To summarize, operations of $W(\mathbf{P})$ are represented by trees whose edges have a colour and a length and whose vertices are labelled by operations of $\mathbf{P}$. The equivalence relation on such representatives is generated by identifications of three kinds:

(i) one related to isomorphisms of trees, respecting the labellings,
(ii) one related to edges of length zero,
(iii) one related to vertices labelled by a unit.

For a fixed tree with labelled edges, the set of operations of $W(\mathbf{P})$ this tree can represent (by adding the necessary lengths and labellings of vertices) has an evident topology by regarding it as the product of spaces of relevant operations of $\mathbf{P}$ and the cube $[0, 1]^{\mathrm{in}(T)}$, where $\mathrm{in}(T)$ is the set of inner edges of T. For example, for the tree with labelled edges



this is the space

$$\mathbf{P}(b, c; a) \times \mathbf{P}(-; b) \times \mathbf{P}(d, e, f; c) \times [0, 1]^{\times 2},$$

the edges coloured b and c being the only internal ones. The spaces

$$W(\mathbf{P})(c_1, \dots, c_n; c)$$

are then topologized as quotients of the disjoint union of spaces associated to trees as above.

The composition operations of the operad $W(\mathbf{P})$ are again defined by grafting, just like for free operads. This grafting

$$W(\mathbf{P})(c_1, \ldots, c_n; c) \times \prod_{i=1}^{n} W(\mathbf{P})(d_1^i, \ldots, d_{k_i}^i; c_i) \longrightarrow W(\mathbf{P})(d_1^1, \ldots, d_{k_n}^n; c)$$

sends operations represented by trees $T, S_1, \ldots, S_n$ to the operation represented by identifying the ith leaf of T with the root of S_i and carrying along all the labels. Note that this ith leaf has the same colour as the root of S_i, as well as the same length (namely 1). Thus, one can recognize operations in $W(\mathbf{P})$ that arise as a composition of operations represented by subtrees by looking for edges of length 1.

The reader will easily be able to check that this composition operation by grafting is well-defined on equivalence classes, i.e. respects the relations (i)–(iii). Moreover, the composition operation is easily seen to be continuous. The maps

$$FU(\mathbf{P}) \xrightarrow{u} W(\mathbf{P}) \xrightarrow{\varepsilon} \mathbf{P}$$

alluded to at the start of this section are defined as follows: the map u assigns to every edge length 1 and the map ε forgets lengths and composes all the operations which label the vertices of a tree. Equivalently, ε changes all the lengths of internal edges to zero.

For now, we list a few more examples of operads of the form $W(\mathbf{P})$ and their algebras other than the one already described earlier in this section.

Example 1.25 (a) Let $\mathbf{C}$ be a category with set of objects C and view it as an operad with unary operations only. Its algebras are then diagrams of sets or spaces indexed by $\mathbf{C}$, or in other words functors from $\mathbf{C}$ to the category of sets or that of spaces. The operad $W(\mathbf{C})$ has the same colours and again only unary operations. So it is a category with the same objects as $\mathbf{C}$ which is enriched in topological spaces; in other words, $W(\mathbf{C})$ is a topological category. For two objects a and b, the space of morphisms $W(\mathbf{C})(a, b)$ can be described as the space of equivalence classes of strings of the form

$$(a = c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots c_{n-1} \xrightarrow{f_n} c_n = b, t_1, \ldots, t_{n-1}) \qquad (*)$$

where the f_i are arrows in $\mathbf{C}$ and the t_i are lengths in the unit interval $[0, 1]$. Intuitively, one can think of the t_i as ‘waiting times’ associated to the objects $c_1, \ldots, c_{n-1}$ respectively, the external objects $a = c_0$ and $b = c_n$ necessarily having waiting times 1. There are two identifications to be made: if a waiting time t_i is zero, the string $(*)$ is to be identified with the one where one composes f_{i+1} and f_i:

$$(a = c_0 \xrightarrow{f_1} \cdots c_{i-1} \xrightarrow{f_{i+1}f_i} c_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} c_n = b, t_1, \ldots, \widehat{t_i}, \ldots, t_{n-1}).$$

Also, if f_j is an identity arrow, then $(*)$ is identified with the string where f_j is deleted and one takes the maximum $t_{j-1} \vee t_j$ of the relevant waiting times:

$$(a = c_0 \xrightarrow{f_1} \cdots c_{j-1} = c_j \xrightarrow{f_{j+1}} \cdots \xrightarrow{f_n} c_n = b, t_1, \ldots, t_{j-1} \vee t_j, \ldots, t_{n-1}).$$

Composition in this category is defined by concatenation of strings, inserting waiting time 1 at the connecting object. Explicitly, the composition of $(*)$ and

$$(b = d_0 \xrightarrow{g_1} \cdots \xrightarrow{g_m} d_m = e, s_1, \ldots, s_{m-1})$$

is

$$(a = c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} c_n = b = d_0 \xrightarrow{g_1} \cdots \xrightarrow{g_m} d_m = e, t_1, \ldots, t_{n-1}, 1, s_1, \ldots, s_{m-1}),$$

where the 1 is to be thought of as the waiting time corresponding to the object b.

Let us now inspect the corresponding notion of $W(\mathbf{C})$-algebra. It is given by a space A_c for each object c and maps corresponding to the points $f \in W(\mathbf{C})(a, b)$ for all a, b. If f is represented by a string of length 1, i.e. just a morphism $a \to b$ in $\mathbf{C}$, there are no waiting times and we have a map

$$A(f) : A_a \longrightarrow A_b.$$

Next for a morphism of $W(\mathbf{C})$ represented by a string of length 2,

$$(a = c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2, t),$$

the algebra structure of A gives a map we denote by

$$A_t(f_2, f_1) : A_a \longrightarrow A_b.$$

These maps fit together in the following way: if one of the f_i is the identity, one has

$$A_t(\mathrm{id}, f) = A(f) = A_t(f, \mathrm{id}).$$

Moreover,

$$\begin{aligned} A_0(f_2, f_1) &= A(f_2 f_1) \\ A_1(f_2, f_1) &= A(f_2) \circ A(f_1). \end{aligned}$$

Thus, the $A_t(f_2, f_1)$ provide a homotopy between $A(f_2 f_1)$ and the composition $A(f_2) \circ A(f_1)$. This shows that A is not a functor on $\mathbf{C}$ itself: it respects identities, but it respects composition only up to homotopy. At the next level, for a string of the form

$$(a = c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} c_3 = b, t_1, t_2)$$

representing a morphism in $W(\mathbf{C})$, the algebra structure of A gives a map

$$A_{t_1,t_2}(f_3, f_2, f_1) : A_a \longrightarrow A_b.$$

In other words, we have a parametrized family of continuous maps

$$A_{-,-}(f_3, f_2, f_1) : [0,1]^{\times 2} \times A_a \longrightarrow A_b.$$

The identifications of strings made to form $W(\mathbf{C})$ now mean that for one of the f_i equal to an identity arrow one has

$$\begin{aligned} A_{t_1,t_2}(f_3, f_2, \mathrm{id}) &= A_{t_2}(f_3, f_2), \\ A_{t_1,t_2}(f_3, \mathrm{id}, f_1) &= A_{t_1 \vee t_2}(f_3, f_1), \\ A_{t_1,t_2}(\mathrm{id}, f_2, f_1) &= A_{t_1}(f_2, f_1) \end{aligned}$$

and that when setting lengths to either 0 or 1 one has

$$\begin{aligned} A_{0,t_2}(f_3, f_2, f_1) &= A_{t_2}(f_3, f_2 f_1), \\ A_{1,t_2}(f_3, f_2, f_1) &= A_{t_2}(f_3, f_2) \circ A(f_1), \\ A_{t_1,0}(f_3, f_2, f_1) &= A_{t_1}(f_3 f_2, f_1), \\ A_{t_1,1}(f_3, f_2, f_1) &= A(f_3) \circ A_{t_1}(f_2, f_1) \end{aligned}$$

which give ‘higher coherence’ conditions on the homotopies $A_t(f_2, f_1)$ described above. This system of homotopies can be pictured schematically as follows:

$$\begin{tikzcd} A(f_3 f_2 f_1) \arrow[r] \arrow[d] & A(f_3)A(f_2 f_1) \arrow[d] \\ A(f_3 f_2)A(f_1) \arrow[r] & A(f_3)A(f_2)A(f_1). \end{tikzcd}$$

The algebra structure of A also gives even higher coherence conditions on these homotopies by examining its action on strings of arbitrary length n. The entire structure is called a *homotopy-coherent diagram* on the category $\mathbf{C}$.

(b) The example above takes a slightly simpler form if the category $\mathbf{C}$ is free on a directed graph. For later use we will make this explicit for the category $[n]$, depicted as

$$0 \to 1 \to \cdots \to n.$$

In this case, a morphism $f : i \longrightarrow j$ in $W([n])$ is again an equivalence class of strings

$$i = i_0 \to i_1 \to \cdots \to i_k = j$$

together with waiting times $t_1, \ldots, t_{k-1}$. But in $[n]$ there is at most one morphism between any two objects, so an arrow $i_0 \to i_1$ can only be the composition of arrows $i_0 \to i_0 + 1 \to \cdots \to i_1 - 1 \to i_1$, so we may as well represent f by the longer string

$$i = i_0 \to i_0 + 1 \to \cdots \to i_1 \to i_1 + 1 \to \cdots \to i_k - 1 \to i_k = j$$

and insert waiting time 0 on any of the intermediate objects we had to add. In this way any morphism f of $W([n])$ admits a unique representative by such a string of maximal length. The waiting times thus give identifications of morphism spaces

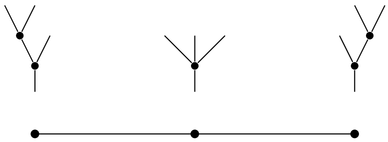
$$W([n])(i, j) = \prod_{i<l<j} [0, 1] \simeq [0, 1]^{\times(j-i-1)}$$

for all $0 \leq i \leq j \leq n$ (with the convention that $[0, 1]^{-1} = *$). Composition is given by the map

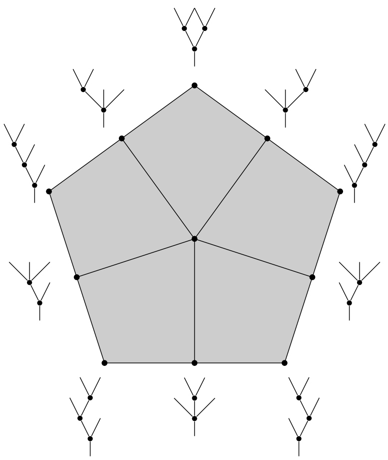
$$\prod_{i<l<j} [0, 1] \times \prod_{j<l<k} [0, 1] \longrightarrow \prod_{i<l<k} [0, 1]$$

which inserts 1 in the jth entry.

(c) Define an operad $\mathbf{Ass}^-$ by setting $\mathbf{Ass}^-(n) = \mathbf{Ass}(n)$ for all $n \geq 1$ and $\mathbf{Ass}(0) = \varnothing$, where all the relevant structure maps are defined as those for the associative operad $\mathbf{Ass}$. We refer to $\mathbf{Ass}^-$ as the *nonunital associative operad*. The operad $W(\mathbf{Ass}^-)$ is closely related to the operad of *Stasheff polytopes*. For example, the space of 3-ary operations $W(\mathbf{Ass}^-)(3)$ consists of two copies of $[0, 1]$ glued together at one endpoint and can be pictured as follows:



The intervals correspond to the lengths assigned to the inner edges of the two trees on the outside. When these lengths are zero, the two resulting operations are identified in $W(\mathbf{Ass}^-)(3)$, which is illustrated by the tree in the middle where the inner edges have been contracted. More interestingly, the following is an illustration of $W(\mathbf{Ass}^-)(4)$, which arises from gluing five squares:



The centre node corresponds to the tree with one vertex and four leaves, which arises from each of the other trees in the picture by contracting all inner edges.

1.8 Configuration Spaces and the Fulton–MacPherson Operad

In this section we will present another important example of an operad, constructed from configuration spaces of points in Euclidean space. In the following section we will then explain how it is related to the operad $\mathbf{E}_d$ of little cubes. These two sections mainly serve to illustrate some of the basic concepts introduced in this chapter, such as free operads and the Boardman–Vogt resolution, and describe an example that is central in applications of operads to be found in the literature. However, the material of these sections is not necessary to study the theory developed in this book, so the reader may decide to skip these sections or revisit them later.

The configuration space of n distinct points in $\mathbb{R}^d$ and variants of it occur frequently in the literature, for example when modelling moving systems of particles, or moduli spaces of points in algebraic geometry. This space is an open subset of $\mathbb{R}^{dn}$ and is not compact. For example, as one lets points in a configuration converge to each other, there exists no 'limiting configuration': once points collide, the result is a configuration with strictly fewer points. One can enlarge the space of configurations of n points by the so-called *Fulton–MacPherson compactification*. Roughly speaking, this is a systematic way of adding 'collisions', but remembering the way in which points came together: at a collision, one assigns an 'infinitesimal' configuration of the colliding points. These points themselves could also have arisen from a collision, which is then also remembered, etc. As we will make precise below, these Fulton–MacPherson spaces consist of such nested collisions; moreover, they are compact and contain the original configuration space as a dense open subset. The surprising fact is that these spaces of nested collisions have the structure of an operad, very similar to the structure of a free operad discussed earlier.

For a fixed dimension d, there is another way of modifying the configuration spaces of n points in $\mathbb{R}^d$ (for varying n) into an operad, namely by considering configurations of little cubes instead. This yields the little d-cubes operad we introduced in Example 1.8. We will see in the next section that there is in fact a (weak) homotopy equivalence of operads between the Fulton–MacPherson operad and $\mathbf{E}_d$.

In this section it will be convenient to work with the coordinate-free versions of the definitions of operad and collection, as in Section 1.4. All of these will be uncoloured. Let us fix a Euclidean space $\mathbb{R}^d$, with $d \geq 0$. For a finite set A, the space of A-configurations $\mathrm{Conf}(A, \mathbb{R}^d)$ is defined to be the space of injective maps

$$x\colon A \to \mathbb{R}^d,$$

topologized as a subspace of $(\mathbb{R}^d)^A$. A point x of $\mathrm{Conf}(A, \mathbb{R}^d)$ will also be written as a family $(x_a)_{a \in A}$ of points satisfying $x_a \neq x_{a'}$ whenever $a \neq a'$. Notice that $\mathrm{Conf}(A, \mathbb{R}^d)$ is a contravariant functor with respect to injective maps $A \to B$ of finite sets. In particular, the symmetric group $\mathrm{Aut}(A)$ of permutations of A acts on $\mathrm{Conf}(A, \mathbb{R}^d)$ from the right.

It will be convenient to identify two A-configurations if one can be obtained from the other by translation and dilation. The group $G_d = \mathbb{R}_{>0} \ltimes \mathbb{R}^d$ acts on a point $x \in \mathrm{Conf}(A, \mathbb{R}^d)$ by

$$((t, v) \cdot x)_a = v + t \cdot x_a.$$

and we will write

$$\mathcal{C}_d(A) = \mathrm{Conf}(A, \mathbb{R}^d)/G_d$$

for the quotient space. Observe that the projection map $\pi\colon \mathrm{Conf}(A, \mathbb{R}^d) \to \mathcal{C}_d(A)$ is a G_d-principal bundle with contractible fibre, hence a homotopy equivalence.

From now on, we assume that the cardinality of A is at least 2. Let us call a configuration $x\colon A \to \mathbb{R}^d$ *normal* if its barycentre is the origin and its diameter is 1, meaning

$$\sum_a x_a = 0, \qquad \max_{a,a'} |x_a - x_{a'}| = 1.$$

Write $N_d(A) \subseteq \mathrm{Conf}(A, \mathbb{R}^d)$ for the subspace of normal configurations. Any configuration x can be translated to have its barycentre at the origin and then dilated to have diameter 1. More precisely, any G_d-orbit of $\mathrm{Conf}(A, \mathbb{R}^d)$ contains a *unique* normal configuration. This observation provides a section s of the bundle map π as in the following diagram:

$$\begin{tikzcd}
\mathrm{Conf}(A, \mathbb{R}^d) \arrow[r, "\nu"] \arrow[d, "\pi"] & N_d(A). \\
\mathcal{C}_d(A) \arrow[ur, "s"] &
\end{tikzcd}$$

We will refer to $\nu = s\pi$ as the *normalization* of configurations. Note that via s we may identify the spaces $\mathcal{C}_d(A)$ and $N_d(A)$. This allows us to take the closure of the space $\mathcal{C}_d(A)$, by which we mean the closure $\overline{N_d(A)}$ of $N_d(A)$ inside $(\mathbb{R}^d)^A$. Since $N_d(A)$ is bounded, this closure is compact. Its points are limits of normal configurations, where 'collisions' can occur. A point of $\overline{N_d(A)}$ is a function $x\colon A \to \mathbb{R}^d$ which is not necessarily injective, but still has barycentre at the origin and diameter 1. Notice that as a functor of A, the space $\overline{N_d(A)}$ is now contravariantly functorial with respect to arbitrary maps of finite sets (rather than just injections); for any $f\colon A \to B$ one obtains a map $\overline{N_d(B)} \to \overline{N_d(A)}$ by precomposing a map $y\colon B \to \mathbb{R}^d$ with f and then normalizing the resulting 'singular' configuration.

We will write $\mathcal{C}_d$ for the collection (in the sense of Section 1.4) formed by the spaces $\mathcal{C}_d(A)$ where the set A is of cardinality at least 2; for smaller A the collection $\mathcal{C}_d$ assigns the empty set. This convention might seem somewhat unnatural, but it is imposed for the following reasons. First of all, further on in this section it will be important that we work with operads without nullary operations. Second of all, for a singleton $A = \{a\}$ the space $\mathcal{C}_d(\{a\})$ consists of a single point, so that omitting these from our collection does not constitute any loss of information. If we were to keep them, then in our consideration of the free operad $F\mathcal{C}_d$ below we would have to quotient out by the relation that the unique element of the space $\mathcal{C}_d(\{a\})$ acts as a unit. It is more efficient to simply exclude this space to begin with, as we are doing.

The collection $\mathcal{C}_d$ does not form an operad, because there is no sensible way to substitute one configuration into another. However, for configurations $x \in \mathcal{C}_d(A)$ and $y \in \mathcal{C}_d(B)$, one can define a 'substitution'

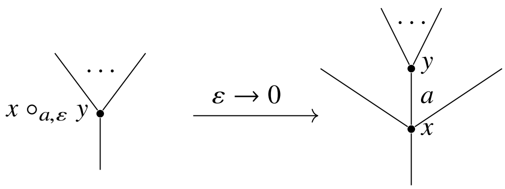
$$x \circ_a y \in \overline{N_d(A \circ_a B)}$$

for any element a of A, which simply forgets about y and takes the value x_a for every element of B. On the other hand, if we shrink y sufficiently and move it to have barycentre x_a, we obtain an actual configuration consisting of the points $x_{a'}$ for $a' \neq a$ and a shrunken copy of y centred at x_a. Formally, for sufficiently small $\varepsilon > 0$, we define a point $x \circ_{a,\varepsilon} y$ in $\mathcal{C}_d(A \circ_a B)$ represented by

$$(x \circ_{a,\varepsilon} y)_i = \begin{cases} x_i & \text{if } i \in A - \{a\} \\ x_a + \varepsilon y_i & \text{if } i \in B. \end{cases}$$

Then the 'collided configuration' $x \circ_a y$ is the limit of the configurations $x \circ_{a,\varepsilon} y$ as $\varepsilon \to 0$. Our goal is now to suitably enlarge the spaces $\mathcal{C}_d(A)$ in such a way that the limits of configurations $x \circ_{a,\varepsilon} y$ exist in them and can be used to give the resulting collection the structure of an operad. The combinatorics of trees is precisely the right tool to describe such 'configurations inside larger configurations' (cf. Remark 1.28 below).

To achieve this goal, consider the free operad $F\mathcal{C}_d$ on the collection $\mathcal{C}_d$. We will retopologize this operad in such a way that the pair $(x, y) \in F\mathcal{C}_d(A \circ_a B)$ corresponding to the grafted tree $C_A \circ_a C_B$, with two vertices corresponding to C_A and C_B and labels x and y respectively, is the limit of the configurations $x \circ_{a,\varepsilon} y$, viewed as elements of $F\mathcal{C}_d(A \circ_a B)$ via the embedding $\mathcal{C}_d \to F\mathcal{C}_d$:



To this end, let us consider the free operad $F\mathcal{C}_d$ in more detail, using the description given in Section 1.5. For a fixed set B, an element of $F\mathcal{C}_d(B)$ is an equivalence class of pairs (T, x), where B is the set of leaves of T and x is a labelling of the set of vertices by elements in the collection $\mathcal{C}_d$. To be precise, x assigns to any vertex v of T a point x_v of the space $\mathcal{C}_d(\text{in}(v))$, with $\text{in}(v)$ denoting the set of input edges of v.

Two pairs (T, x) and (T', x') are equivalent if there exists an isomorphism $\alpha\colon T \longrightarrow T'$ of trees such that $x_v = \alpha^*(x_{\alpha(v)})$ for each vertex v of T. The topology on $F\mathcal{C}_d(B)$ is induced by the topology of the configuration spaces. In detail, $F\mathcal{C}_d(B)$ falls apart as a disjoint sum

$$F\mathcal{C}_d(B) = \coprod_{[T]} F\mathcal{C}_d(B)_T$$

over isomorphism classes of trees (with set of leaves B), where $F\mathcal{C}_d(B)_T$ is the space of equivalence classes of pairs (T, x). Since there is at most one isomorphism between two such trees T and T' with leaves B, we can fix a representative tree T in an isomorphism class $[T]$ and identify $F\mathcal{C}_d(B)_T$ as the product

$$\prod_{v \in V(T)} \mathcal{C}_d(\mathrm{in}(v))$$

of configuration spaces indexed by the vertices of T. It is important to note that the trees T featuring here have no nullary or unary vertices, since by definition the sets $\mathcal{C}_d(A)$ are nonempty only when the cardinality of A is at least 2. Thus, all the trees T that come up in the remainder of this section will have the property that each vertex has at least two input edges.

On our way to changing the topology of $F\mathcal{C}_d$, consider for each finite set B of cardinality ≥ 2 the product space

$$\mathbf{P}(B) = \prod_{A \subseteq B} \overline{N_d(A)},$$

where the product ranges over subsets A of cardinality ≥ 2 as well. This is a compact space, as each $\overline{N_d(A)}$ is itself compact. Using the substitution of configurations discussed above, these spaces $\mathbf{P}(B)$ for all B together form an operad (if one formally adds units for all singletons $B = \{b\}$): for sets B and D and an element $b \in B$, the $\circ_b$-composition

$$- \circ_b -\colon \mathbf{P}(B) \times \mathbf{P}(D) \longrightarrow \mathbf{P}(B \circ_b D)$$

is defined for two points $x \in \mathbf{P}(B)$ and $y \in \mathbf{P}(D)$ and a subset $A \subseteq B \circ_b D$ by the function

$$x_A \circ_b y_A\colon A \longrightarrow \mathbb{R}^d$$

sending $i \in A$ to $x_A(i)$ if $i \in B - \{b\}$ and to $y_A(i)$ if $i \in D$ (or rather the normalization of this function, so as to get an element of $\overline{N_d(A)}$). Now consider the map

$$\varphi_B\colon F\mathcal{C}_d(B) \longrightarrow \mathbf{P}(B)$$

defined as follows. For an element of $F\mathcal{C}_d(B)$ represented by a pair (T, x), we will describe the component $\varphi_B(T, x)_A$ at a subset $A \subseteq B$. Write v_A for the minimal vertex of T (in the partial ordering on the set of vertices $V(T)$) such that for each $a \in A$, the path from the leaf a of T to its root passes through v_A. In other words, v_A is the highest possible vertex in the tree T such that all of the elements of A occur above the vertex v_A. Write

$$p_A : A \longrightarrow \mathrm{in}(v_A)$$

for the map assigning to $a \in A$ the input edge of v_A at which the path from a down to the root arrives at v_A. The configuration $x_{v_A} \in \mathcal{C}_d(\mathrm{in}_{v_A})$ then yields an element $p_A^* x_{v_A} \in \overline{N_d(A)}$ and we define

$$\varphi_B(T, x)_A := p_A^* x_{v_A}.$$

It is straightforward to check that $\varphi_B(T, x)_A$ only depends on the equivalence class of (T, x), so that we indeed have a well-defined map φ_B. Moreover, these maps are clearly continuous and define a map of operads $\varphi : F\mathcal{C}_d \longrightarrow \mathbf{P}$. Its image thus defines a suboperad of $\mathbf{P}$, which we denote by $\mathbf{FM}_d$ and call the *Fulton–MacPherson operad*:

$$\begin{tikzcd} F\mathcal{C}_d \arrow[rr, "\varphi"] \arrow[dr] & & \mathbf{P}. \\ & \mathbf{FM}_d \arrow[ur] & \end{tikzcd}$$

In fact, the maps $\varphi_B : F\mathcal{C}_d(B) \longrightarrow \mathbf{P}(B)$ are injective, so that $\mathbf{FM}_d$ really 'is' the free operad $F\mathcal{C}_d$, but topologized as a suboperad of $\mathbf{P}$:

Proposition 1.26 *For each finite set B (with at least two elements), the map $\varphi_B : F\mathcal{C}_d(B) \longrightarrow \mathbf{P}(B)$ is injective. Its image $\mathbf{FM}_d(B)$ is closed inside $\mathbf{P}(B)$ and hence compact.*

Proof Consider the subspace $X(B)$ of

$$\mathbf{P}(B) = \prod_{A \subseteq B} \overline{N_d(A)}$$

consisting of those points x which satisfy the following condition: for any subsets $A' \subseteq A \subseteq B$, either $x_{A'}$ is the restriction of x_A to A' (normalized so as to get a point in $\overline{N_d(A)}$), or this restriction is constant. The set $X(B) \subseteq \mathbf{P}(B)$ is clearly closed. Moreover, it is easy to check that the image of φ_B is contained in $X(B)$. To check that φ_B is injective and its image is *exactly* $X(B)$, we construct an explicit inverse

$$\psi_B : X(B) \longrightarrow F\mathcal{C}_d(B)$$

as follows. Suppose $\xi \in X(B)$, with components ξ_A for subsets $A \subseteq B$. We will define $\psi_B(\xi)$ by an inductive procedure. First consider $\xi_B : B \longrightarrow \mathbb{R}^d$. The fibres of this map determine a partition of B. We begin building a tree T by taking a corolla with vertex called v_B (which will serve as the root corolla) with one input edge for each point p in the image of ξ_B. This image defines a configuration x_{v_B}, which will be the label of the vertex v_B. Next, for those points p for which the fibre $B_p := \xi_B^{-1}(p)$ has more than one element, attach a vertex v_{B_p} on top of the edge p and an input edge of v_{B_p} for each element q in the image of the map ξ_{B_p}. We label the vertex v_{B_p}

by the configuration that is the image of ξ_{B_p}. Since each of the maps $\xi_A\colon A \to \mathbb{R}^d$ has an image of diameter 1, these maps are not constant, so the cardinality of the fibres of ξ_A is strictly smaller than that of A. Thus the process we are describing can be continued and will eventually terminate to produce a tree T. The set of leaves of this tree is exactly B and for a vertex v of T, its incoming edges are labelled by points in $\mathbb{R}^d$ which form a configuration that is the label x_v of the vertex v. The value ψ_B is now defined to be (the equivalence class of) the pair (T, x). We leave it to the reader to check that ψ_B thus defined is (as a map of sets) an inverse for the map $\varphi_B\colon F\mathcal{C}_d(B) \to X(B)$. This proves that φ_B defines a bijection and $X(B) = \mathbf{FM}_d(B)$ as desired. □

As a consequence of this proposition, the space $\mathbf{FM}_d(B)$ is a union

$$\mathbf{FM}_d(B) = \bigcup_T \mathbf{FM}_d(T),$$

where T ranges over isomorphism classes of trees T with B as set of leaves and $\mathbf{FM}_d(T) := \varphi(F\mathcal{C}_d(B)_T)$. The subspaces $\mathbf{FM}_d(T)$ are disjoint, but as a topological space $\mathbf{FM}_d(B)$ is *not* the coproduct of the spaces $\mathbf{FM}_d(T)$. (See Corollary 1.30 for a more precise statement.) From the construction in the proof above, one reads off that a point $\xi \in \mathbf{FM}_d(B)$ belongs to $\mathbf{FM}_d(T)$ if and only if it has the following property, for any two subsets $A' \subseteq A \subseteq B$:

$$\begin{aligned}
&\xi_A|_{A'} = \xi_{A'} \text{ if and only if } v_A = v_{A'},\\
&\xi_A|_{A'} \text{ is constant if and only if } v_A \neq v_{A'}.
\end{aligned}$$

(Recall that v_A and $v_{A'}$ are the highest vertices below A and A' respectively. Also, we only consider subsets A and A' of cardinality at least 2.) A point $\xi \in \mathbf{FM}_d(B)$ potentially contains a lot of redundant information. In fact if ξ belongs to $\mathbf{FM}_d(T)$ then the function $\xi_A\colon A \to \mathbb{R}^d$ factors through $p_A\colon A \to \mathrm{in}(v_A)$, hence is the restriction to A of ξ_{A_v} where A_v is the set of all leaves above v_A (which is the maximal subset $A' \subseteq B$ with $v_{A'} = v_A$). In particular, ξ is completely determined by the coordinates ξ_{A_v} with A_v ranging over such 'maximal sets' associated to vertices of T. Moreover, if $a, a' \in A_v$ lie over different input edges of v, then $v_{\{a,a'\}} = v$ so $\xi_A|_{\{a,a'\}}$ is not constant. This means that the factorization of ξ_{A_v} through p_A is injective, so ξ_{A_v} defines a point in $\mathcal{C}_d(\mathrm{in}(v))$. So for the tree T, we have a factorization

$$\begin{tikzcd}
F\mathcal{C}_d(B)_T \arrow[r, "\varphi"] \arrow[dr] & \mathbf{FM}_d(T) \arrow[d] \\
& \prod_{v\in V(T)} \mathcal{C}_d(\mathrm{in}(v)),
\end{tikzcd}$$

where the slanted map is a homeomorphism and the other two are bijections, hence also homeomorphisms. In other words, the map φ restricts to a homeomorphism for each tree T. As a consequence, the operad $\mathbf{FM}_d$ inherits the following property from the free operad:

Corollary 1.27 *For a tree $S \circ_e T$ obtained by grafting a tree T onto the leaf e in a tree S, the canonical map*

$$\mathbf{FM}_d(S \circ_e T) \to \mathbf{FM}_d(S) \times \mathbf{FM}_d(T)$$

is a homeomorphism.

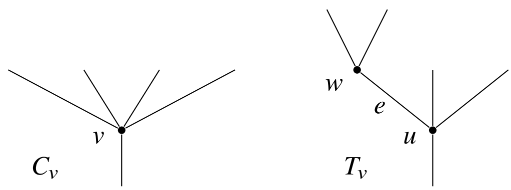
To be clear, the map of the corollary is the one that 'decomposes' a labelling of $S \circ_e T$ into the labellings of the subtrees S and T.

Remark 1.28 One can think of the space $\mathbf{FM}_d(T)$ as that of nested sequences of configurations, indexed by the vertices of T. Any vertex v determines a configuration of the set $\mathrm{in}(v)$ in $\mathbb{R}^d$ and for an input edge e of v with a vertex w attached to the top of e, there is a further configuration of the set $\mathrm{in}(w)$ which we might picture as an 'infinitesimal configuration' around $e \in \mathbb{R}^d$. Thus the union $\mathbf{FM}_d(B) = \bigcup_T \mathbf{FM}_d(T)$ is, as advertised at the beginning of this section, an enlargement of the configuration space $\mathcal{C}_d(B)$ in which one can take limits of configurations; collisions between points are recorded by infinitesimal configurations around the point of collision, as features explicitly in the proof of the following proposition.

As we already described, the space $\mathbf{FM}_d(B)$ is a union of disjoint subspaces $\mathbf{FM}_d(T)$, but not topologically a disjoint union of these spaces. The following result expresses how the distinct pieces (the *strata*) $\mathbf{FM}_d(T)$ are glued together:

Proposition 1.29 *Let S and T be two trees, each with B as set of leaves. Then $\mathbf{FM}_d(S) \subseteq \overline{\mathbf{FM}_d(T)}$ if and only if T can be obtained (up to isomorphism) by contracting inner edges in S.*

Proof First we prove the 'if' direction. Reasoning by induction, it suffices to show this implication in the case where S is obtained from T by contracting a single edge. This means that there is a vertex v in S for which the corolla C_v at this vertex is 'blown up' to a subtree $T_v \subseteq T$ with two vertices u and w, as in the following:



Since $\mathbf{FM}_d(S)$ is a product over the vertices in S of configuration spaces, and similarly for T, it now suffices to prove that $\mathbf{FM}_d(T_v) \subseteq \overline{\mathbf{FM}_d(C_v)}$. But a point (x, y) in $\mathbf{FM}_d(T_v) = \mathcal{C}_d(\mathrm{in}(v)) \times \mathcal{C}_d(\mathrm{in}(w))$ is the limit in $\mathbf{FM}_d(B)$ of the points $x \circ_{e,\varepsilon} y$ in $\mathbf{FM}_d(C_v)$, as described in the beginning of this section.

We now deal with the 'only if' direction. Suppose $\mathbf{FM}_d(S) \subseteq \overline{\mathbf{FM}_d(T)}$. Consider a point $\zeta \in \mathbf{FM}_d(S)$ and write it as a limit of points $\xi \in \mathbf{FM}_d(T)$. Then for any subsets $A' \subseteq A \subseteq B$, if the functions $\xi_A \colon A \to \mathbb{R}^d$ are constant on A', the same is true for ζ_A. Thus, the nested partitions $\{A_v\}$ of B given by the tree T (described above Corollary 1.27) form a refinement of the similar partitions given by the tree S. But this means precisely that S can be obtained from T by contracting inner edges. □

Corollary 1.30 *The embedding $\varphi\colon \mathcal{C}_d(B) \to \mathbf{FM}_d(B)$, identifying $\mathcal{C}_d(B)$ with the stratum $\mathbf{FM}(C_B)$ corresponding to the corolla C_B with leaves B, has dense image.*

Thus, the space $\mathbf{FM}_d(B)$ is a *compactification* of the configuration space $\mathcal{C}_d(B)$ (meaning a compact space containing $\mathcal{C}_d(B)$ as a dense subspace), and for this reason $\mathbf{FM}_d(B)$ is often referred to as the *Fulton–MacPherson compactification* of the configuration space $\mathcal{C}_d(B)$.

1.9 Configuration Spaces and the Operad of Little Cubes

The Fulton–MacPherson operad $\mathbf{FM}_d$ introduced in the preceding section is closely related to the operad $\mathbf{E}_d$ of little d-cubes from Example 1.8. Indeed, a point of the space $\mathbf{E}_d(n)$ is a configuration of n little d-cubes inside the unit cube; assigning to each little cube its barycentre constitutes a homotopy equivalence from the space $\mathbf{E}_d(n)$ to the configuration space of n points in the interior of the unit d-cube, which in turn is homeomorphic to the configuration space of n points in $\mathbb{R}^d$. However, this assignment is not a map of operads in any reasonable sense. We already saw in the previous section that the configuration spaces themselves do not quite form an operad; rather, one should pass to the Fulton–MacPherson compactifications. In this section we will make the relation between the operads $\mathbf{FM}_d$ and $\mathbf{E}_d$ precise by exhibiting a zigzag of maps

$$\mathbf{FM}_d \leftarrow \mathbf{P} \rightarrow \mathbf{E}_d$$

each of which is a homotopy equivalence of operads, in the sense that $\mathbf{P}(n) \to \mathbf{FM}_d(n)$ and $\mathbf{P}(n) \to \mathbf{E}_d(n)$ are homotopy equivalences of spaces for each n.

The operad $\mathbf{P}$ we will use is essentially the Boardman–Vogt W-resolution of $\mathbf{E}_d$. To be precise, in this section we view $\mathbf{E}_d$ as an operad by forgetting $\mathbf{E}_d(0)$ and replacing the contractible space $\mathbf{E}_d(1)$ by just a point, as for the Fulton–MacPherson operad. For the Boardman–Vogt resolution $W(\mathbf{E}_d)$ and a finite set B, elements of $W(\mathbf{E}_d)(B)$ are represented by tuples (T, p, t), where T is a tree with B as its set of leaves, p assigns an element $p(v) \in \mathbf{E}_d(\mathrm{in}(v))$ to each vertex v of T, and t assigns a length $t(e) \in [0, 1]$ to each inner edge e of T. Moreover, we may assume that each vertex of T has at least two incoming edges (as for $\mathbf{FM}_d$) by virtue of the 'unit relation' imposed on the W-construction. If T has an edge e with length $t(e) = 0$, then such a tuple is identified with a tuple $(\partial_e T, p', t')$, where $\partial_e T$ is obtained from T by contracting e, the assignment t' is the restriction of t to the inner edges of $\partial_e T$, and p' is obtained from p by composition in the operad $\mathbf{E}_d$. The composition in the operad $W(\mathbf{E}_d)$ is defined by grafting of trees, assigning length 1 to the edge along which grafting takes place. As we have seen before, there is a morphism of operads $\varepsilon\colon \mathbf{W}(\mathbf{E}_d) \to \mathbf{E}_d$ forgetting lengths of edges and composing all operations in a tree;

this map is easily seen to be a homotopy equivalence via the linear homotopy that 'contracts lengths to zero'. If we forget the topology on the operad $W(\mathbf{E}_d)$, then it is the free operad on the collection of sets $W^\circ(\mathbf{E}_d)(B)$ consisting of points represented by tuples (T, p, t) where each edge e has length $t(e) < 1$. Moreover, extending the analogy with the Fulton–MacPherson operad, $W(\mathbf{E}_d)(B)$ decomposes into strata $W(\mathbf{E}_d)(B)_S$ consisting of those points (T, p, t) for which the tree $\partial_{<1}T$ obtained by contracting all edges of length < 1 is isomorphic to S (by an isomorphism respecting the leaves B).

Now for each finite set B, consider the map

$$\varphi = \varphi_B : W^\circ(\mathbf{E}_d)(B) \to \mathbf{E}_d(B)$$

assigning to a tuple (T, p, t) the composition of the elements $p(v)$ of the little d-cubes operad, *but* rescaled by the lengths of the edges: explicitly, it is the composition of the elements $(1 - t(e_v))p(v)$ where e_v is the edge immediately below v. In this definition, we take the length of the root edge to be zero, so do not rescale the operation at the root of T. Notice that more generally, if an edge e_v has length zero then no rescaling takes place, so that φ is well-defined on equivalence classes of tuples. If we compose this map φ_B with the map

$$c: \mathbf{E}_d(B) \to \mathcal{C}_d(B)$$

taking the centres of the cubes involved, we obtain a map

$$c\varphi_B : W^\circ(\mathbf{E}_d)(B) \to \mathcal{C}_d(B) \subseteq \mathbf{FM}_d(B).$$

These maps $c\varphi_B$ are obviously natural with respect to bijections between finite sets. So, as $W(\mathbf{E}_d)$ is free as an operad (ignoring the topology for a moment) over the collection of spaces $W^\circ(\mathbf{E}_d)(B)$, we obtain a map of operads (in **Sets**)

$$\overline{c\varphi}: W(\mathbf{E}_d) \to \mathbf{FM}_d.$$

It is not difficult to check that this is in fact a map of *topological* operads, i.e., each $\overline{c\varphi}_B$ is continuous. (The reason is that if the length of an edge e_v below a vertex e converges to 1, then by the rescaling factor $1 - t(e_v)$, the configuration of the centres of the cubes in $p(v)$ converges to a single point, exactly as in the topology of the Fulton–MacPherson operad.) Notice also that the map $\overline{c\varphi}$ maps the stratum $W(\mathbf{E}_d)(B)_S$ for a tree S exactly to the stratum $\mathbf{FM}_d(S)$ of $\mathbf{FM}_d(B)$.

We claim that the map $\overline{c\varphi}$ gives a homotopy equivalence of spaces $W(\mathbf{E}_d)(B) \to \mathbf{FM}_d(B)$ for each finite set B. To see this, consider the diagram

$$\begin{tikzcd} & W(\mathbf{E}_d) \arrow[r, "\overline{c\varphi}"] \arrow[dl, "\varepsilon"'] & \mathbf{FM}_d \\ \mathbf{E}_d \arrow[r, "\eta"] & W^\circ(\mathbf{E}_d) \arrow[r, "c\varphi"] \arrow[u] & \mathcal{C}_d. \arrow[u] \end{tikzcd}$$

In this diagram $\overline{c\varphi}$ and ε are maps of operads, but the other arrows are merely maps of collections. The map η identifies $\mathbf{E}_d(B)$ with the part of $W^\circ(\mathbf{E}_d)(B)$ given by the corolla with B as its set of leaves. The maps in the triangle on the left are homotopy equivalences by 'contracting lengths of inner edges to zero', as before. The composition $c\varphi\eta$ simply takes centres of cubes and is a homotopy equivalence as well. Thus it suffices to show that the inclusion of the 'configuration space' $\mathcal{C}_d(B)$ into its compactification $\mathbf{FM}_d(B)$ is a homotopy equivalence for each finite set B. Briefly said, this is the case because $\mathbf{FM}_d(B)$ is a manifold with corners and $\mathcal{C}_d(B)$ is precisely its interior. Let us take the remainder of this section to explain this point in some more detail.

Recall that the space $\mathcal{C}_d(B)$ itself is a smooth manifold; indeed, it is the quotient of $\mathrm{Conf}(B, \mathbb{R}^d)$ by a free and proper action of the group $\mathbb{R}_{>0} \times \mathbb{R}^d$ of translations and dilations, while $\mathrm{Conf}(B, \mathbb{R}^d)$ itself is simply on open submanifold of $(\mathbb{R}^d)^B$. Now recall that a manifold with corners of dimension N is covered by charts of the form $(0, \varepsilon)^n \times [0, \varepsilon)^{N-n}$ for $\varepsilon > 0$ and $0 \leq n \leq N$. To exhibit such charts for $\mathbf{FM}_d(B)$, consider a point $\xi \in \mathbf{FM}_d(B)$ lying in a stratum $\mathbf{FM}_d(T)$ corresponding to a tree T with set of leaves B. Then the union of the strata $\mathbf{FM}_d(S)$, where S ranges over trees obtained from T by contracting inner edges, is an open set in $\mathbf{FM}_d(B)$ containing the point ξ (cf. Proposition 1.29). We will describe a chart around ξ within this open set, using the identifications

$$\mathbf{FM}_d(S) = \prod_{w \in V(S)} \mathcal{C}_d(\mathrm{in}(w)).$$

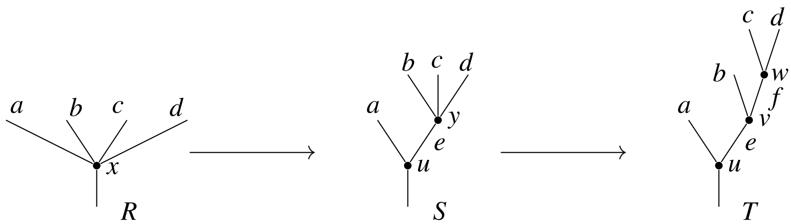
So let us view ξ as a point in $\prod_{v \in V(T)} \mathcal{C}_d(\mathrm{in}(v))$ and let W be an open neighbourhood of ξ in this product space. Choose W small enough so that there is an $\varepsilon > 0$ such that all the configurations $\zeta(v)$ for points $\zeta \in W$ have mesh at least 2ε, i.e., the points in the configuration $\zeta(v)$ are at least a distance 2ε apart. This implies that if in such a configuration we replace one or more points x by a configuration centred at x with diameter $< \varepsilon$, the result is still a configuration of *distinct* points.

Now write $I(T)$ for the set of inner edges of T and define a map

$$\psi\colon W \times [0, \varepsilon)^{I(T)} \to \mathbf{FM}_d(B)$$

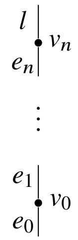
as follows. For a point (ζ, t) with $\zeta \in W$ and t a sequence of lengths $t(e)$ assigned to inner edges of T, let S_t be the tree obtained by contracting all edges in T of length > 0. Then $\psi(\zeta, t)$ will land in the stratum $\mathbf{FM}_d(S_t)$, so is given by a family of configurations $\psi(\zeta, t)(w)$ indexed by the vertices w of S_t. Such a vertex w arises as the contraction of a subtree $T_w \subseteq T$ all of whose inner edges e have length $0 < t(e) < \varepsilon$. The configuration $\psi(\zeta, t)(w)$ is obtained by iteratively substituting the configurations $\zeta(v)$ (of diameter 1) rescaled by the lengths $t(e)$. Before giving the formula, let us consider a small example to illustrate the idea.

Example 1.31 Consider the trees



and a point $\zeta \in \mathbf{FM}_d(T)$ given by configurations $\zeta(u) \in \mathcal{C}(\{a, e\})$, $\zeta(v) \in \mathcal{C}(\{b, f\})$, and $\zeta(w) \in \mathcal{C}(\{c, d\})$. Let $(t(e), t(f))$ be an assignment of lengths in $[0, \varepsilon)$ to the inner edges of T. If $t(e) = 0 < t(f)$, then $\psi(\zeta, t)$ is a point in the stratum for the tree S. It is given by the old configuration $\zeta(u)$ at the vertex u and the new three-point configuration $\psi(\zeta, t)(y)$ at the vertex w, obtained from the configuration $\zeta(v)$ by replacing the point $\zeta(v)(f)$ by the tiny configuration $t(f) \cdot \zeta(w)$ now centred around $\zeta(v)(f)$. If $t(e)$ and $t(f)$ are both strictly positive, then $\psi(\zeta, t)$ is a four-point configuration in the stratum $\mathbf{FM}_d(R) = \mathcal{C}_d(\{a, b, c, d\})$. It is defined by additionally replacing the point $\zeta(u)(e)$ by the configuration $t(e) \cdot \psi(\zeta, t)(y)$.

In order to give a general formula for $\psi(\zeta, t)(w)$, consider the incoming edges of w in the tree S_t. Since w was obtained by contracting the subtree T_w of T, these incoming edges are exactly the leaves of T_w. For such a leaf l, the point $\psi(\zeta, t)(w)(l)$ in $\mathbb{R}^d$ is the point in the configuration obtained by rescaled substitution along the path from l down to the root of T_w. If we depict this path as



then $\psi(\zeta, t)(w)(l)$ is the point

$$\zeta(v_0)(e_1) + t(e_1)\zeta(v_1)(e_2) + t(e_1)t(e_2)\zeta(v_2)(e_3) + \cdots + t(e_1)\cdots t(e_n)\zeta(v_n)(l).$$

This completes the definition of the map

$$\psi : W \times [0, \varepsilon)^{I(T)} \to \mathbf{FM}_d(B).$$

It is not difficult to check that this map is a homeomorphism onto its image. For example, to see that it is injective, notice that we can recover the rescaling factors $t(e)$ from $\psi(\zeta, t)$. Indeed, $\psi(\zeta, t)$ lies in a stratum S, already telling us which of the $t(e)$ are nonzero. The values of these nonzero $t(e)$ can then be read off from the diameters of the substituted rescaled configurations. For instance, in the first example above where $t(e) = 0 < t(f)$, the configuration $\psi(\zeta, t)(y)$ of three points b, c, and d is normalized with diameter 1, but its subconfiguration of just the points labelled c and d has exactly diameter $t(f)$.

Historical Notes

The theory of operads first demonstrated its importance through the characterization of iterated loop spaces by means of the little cubes operads, by Boardman–Vogt [21] and May [112]. The latter reference contains the first occurrence of the definition of an operad as we gave it. Boardman and Vogt formulated their results in terms of the closely related notion of a 'prop', and also described what we have called the 'Boardman–Vogt resolution' and the tensor product of operads in these terms. While operads describe general algebraic structures in terms of operations with a finite number of inputs and one output, a prop does the same thing for a finite number of inputs and any finite number of outputs. These props arose in the work on higher homotopies of Adams, MacLane and others in the 1960s [107]. A precursor of the characterization of iterated loop spaces in terms of operads is Stasheff's characterization [135] of loop spaces in terms of 'Stasheff polytopes', closely related to the operad of little 1-cubes (i.e., little intervals). The compactification of the configuration space in terms of trees discussed in the last section of this chapter goes back to Fulton–MacPherson [60] and Axelrod–Singer [7]. The Fulton–MacPherson operad was first introduced by Getzler–Jones [65]. Our description of the compactification and its operad structure is based on Kontsevitch–Soibelman [100], Sinha's work [134] and on the PhD thesis of Dean Barber [9]. A further useful reference is Salvatore's work [132].

The force of the notion of operad is that it makes sense in any symmetric monoidal category, so that operads and their algebras can easily be transported along functors between such categories, and can be dualized to obtain 'cooperads'. Nonetheless, after having demonstrated their importance in topology, it took a while before operads and cooperads were explicitly used in other, more general categories. Decisive steps here were taken by Getzler–Jones [65] and Ginzburg–Kapranov [68], who discussed bar-cobar and Koszul duality for (co)operads and studied the operad structure on moduli spaces of curves. The role of operads in deformation theory and quantization was emphasized by Kontsevich [100]. Nowadays, operads play a crucial role in many parts of mathematics. For more details on the history of the theory of operads, the reader is referred to the books by Markl, Shnider, and Stasheff [110] and the book of Loday and Vallette [104], which gives a very comprehensive treatment of operads in the context of homological algebra.

Chapter 2
Simplicial Sets

Simplicial sets form a very convenient tool to study the homotopy theory of topological spaces. In this chapter we will present an introduction to the theory of simplicial sets. We assume some basic acquaintance with the language of category theory, but no prior knowledge of simplicial sets on the side of the reader. We present the basic definitions and constructions, including the geometric realization of a simplicial set, the nerve of a category, and the description of the product of two simplicial sets in terms of shuffles. The category of simplicial sets is an example of a category of presheaves, and we also take the opportunity to discuss Kan extensions and several other constructions for presheaves that will be used again later in this book. The chapter ends with some examples of other types of simplicial objects, such as bisimplicial sets and simplicial operads. The material in this chapter is quite classical, and different presentations each having their own virtues can be found in the books already mentioned in the introduction. Our particular way of selecting and presenting the material was mainly motivated by the need to prepare the ground for the extension of the theory to that of dendroidal sets in the next chapter.

2.1 The Simplex Category $\mathbf{\Delta}$

In this section we recall the definition of the category $\mathbf{\Delta}$ of finite linear orders, which lies at the basis of the theory of simplicial sets. In fact there are two equivalent definitions of this category, a skeletal and a non-skeletal one. The skeletal category $\mathbf{\Delta}$ has as its objects the natural numbers, which are denoted $[n]$ (for $n \geq 0$) and are thought of as linear orders

$$[n] = \{0 \leq 1 \leq 2 \leq \cdots \leq n\}$$

or equivalently as free categories

$$[n] = \big(0 \to 1 \to 2 \to \cdots \to n\big).$$

G. Heuts, I. Moerdijk, *Simplicial and Dendroidal Homotopy Theory*,
Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern
Surveys in Mathematics 75, https://doi.org/10.1007/978-3-031-10447-3_2

The morphisms $\alpha : [n] \to [m]$ in $\mathbf{\Delta}$ are the non-decreasing functions or, from the second perspective, functors $[n] \to [m]$.

Sometimes it is convenient to consider a larger version of $\mathbf{\Delta}$, whose objects are finite non-empty linearly ordered sets and whose morphisms are non-decreasing functions between them. The difference between the two versions will not matter very much, but we will usually stick to the skeletal one described above for notational convenience.

There are some morphisms in $\mathbf{\Delta}$ for which we introduce additional notation. First, there is for each $0 \leq i \leq n$ the injective monotone function

$$\delta_i : [n-1] \longrightarrow [n]$$

which skips the value i. Also, for each $0 \leq j \leq n-1$ there is the surjective function

$$\sigma_j : [n] \longrightarrow [n-1]$$

which hits the value j twice and every other value once; in other words, it is given by $\sigma_j(k) = k$ for $k \leq j$ and $\sigma(k) = k - 1$ for $k > j$. These morphisms are called the *elementary faces* and *elementary degeneracies* respectively.

Note that any injective function $[m] \to [n]$ can be written as a composition of elementary face maps (although not necessarily uniquely). Also, any surjective function factors as a composition of elementary degeneracies. Since any morphism $[m] \to [n]$ factors as a surjection $[m] \to [k]$ followed by an injection $[k] \to [n]$, this shows that the elementary faces and degeneracies generate all the morphisms of $\mathbf{\Delta}$. One easily figures out the relations satisfied by these generating maps. For example, if $0 \leq i < j \leq n$ then the composition

$$[n-2] \xrightarrow{\delta_i} [n-1] \xrightarrow{\delta_j} [n]$$

is the injective map skipping i and j in its image, as is

$$[n-2] \xrightarrow{\delta_{j-1}} [n-1] \xrightarrow{\delta_i} [n].$$

In other words, we have the relation

(1) $\delta_j \delta_i = \delta_i \delta_{j-1}$ for $i < j$.

The other relations are as follows:

(2) $\sigma_i \sigma_j = \sigma_{j-1} \sigma_i$ for $i < j$.

(3) $\sigma_i \delta_j = \begin{cases} \delta_{j-1} \sigma_i & \text{if } i < j-1 \\ \mathrm{id} & \text{if } i = j-1 \text{ or } i = j \\ \delta_j \sigma_{i-1} & \text{if } i > j. \end{cases}$

These relations are called the *cosimplicial identities*. As a consequence, a functor F from $\mathbf{\Delta}$ into any other category $\mathbf{C}$ can be specified by giving the values $F([n])$ for all $n \geq 0$ together with the maps $F(\delta_i)$ and $F(\sigma_j)$ corresponding to the elementary faces and degeneracies, provided that these maps satisfy the cosimplicial identities.

The category $\mathbf{\Delta}$ has very few limits and colimits, but there are some which we wish to single out. Suppose we have inclusions $f : [k] \to [n]$ and $g : [k] \to [m]$ where f is an 'initial segment' and g is a 'terminal segment', i.e. they satisfy

$$f(i) = i \qquad \text{and} \qquad g(i) = i + m - k.$$

Then the pushout square

$$\begin{tikzcd} {[k]} \arrow[r, "f"] \arrow[d, "g"'] & {[n]} \arrow[d] \\ {[m]} \arrow[r] & {[n] \cup_{[k]} [m]} \end{tikzcd}$$

exists in $\mathbf{\Delta}$; indeed, the bottom right corner is the linear order $[m + n - k]$. The simplest example of this is the pushout square

$$\begin{tikzcd} {[0]} \arrow[r, "0"] \arrow[d, "m"'] & {[n]} \arrow[d] \\ {[m]} \arrow[r] & {[m + n].} \end{tikzcd}$$

Iterating this type of pushout we can write $[n]$ as the colimit of a diagram involving only $[0]$'s and $[1]$'s:

$$[n] = [1] \cup_{[0]} [1] \cup_{[0]} \cdots \cup_{[0]} [1].$$

Here there are n copies of $[1]$ and each $[0]$ includes as the vertex 1 of the copy of $[1]$ on its left and the vertex 0 of the copy of $[1]$ to its right.

An example of a different kind is the pushout of two surjections

$$[k] \xleftarrow{p} [n] \xrightarrow{q} [l]$$

between linear orders. One can think of $[k]$ as obtained from $[n]$ by collapsing certain segments to points and similarly for $[l]$. When one collapses both families of (possibly overlapping) segments to points, one obtains a further quotient $[m]$ which is the pushout of p and q. For example, for $0 \leq i < j < n$,

$$\begin{tikzcd} {[n]} \arrow[r, "\sigma_j"] \arrow[d, "\sigma_i"'] & {[n-1]} \arrow[d, "\sigma_i"] \\ {[n-1]} \arrow[r, "\sigma_{j-1}"'] & {[n-2]} \end{tikzcd}$$

is such a pushout. These pushouts in $\mathbf{\Delta}$ have a special property, expressed by the following proposition:

Proposition 2.1 (i) *In the square above with $i < j$ there exist sections $\alpha\colon [n-1] \to [n]$ of σ_i and $\beta\colon [n-2] \to [n-1]$ of σ_i, which are compatible in the sense that $\sigma_j \alpha = \beta \sigma_{j-1}$.*

(ii) *Consider a commutative square*

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "r"] \\ C \arrow[r, "s"'] & D \end{tikzcd}$$

in a category **C**, *with p a split epimorphism (i.e., admitting a section) and q an epimorphism. If there exist compatible sections α of p and β of r (in the sense described in (i)), then the square is a pushout. In fact it is an* absolute pushout, *meaning any functor from* $\mathbf{\Delta}$ *to another category sends the square to a pushout square.*

(iii) *Let $[k] \xleftarrow{p} [n] \xrightarrow{q} [l]$ be surjections in* $\mathbf{\Delta}$. *The pushout*

$$\begin{tikzcd} {[n]} \arrow[r, "q"] \arrow[d, "p"'] & {[l]} \arrow[d, "r"] \\ {[k]} \arrow[r, "s"'] & {[m]} \end{tikzcd}$$

exists in $\mathbf{\Delta}$ *and is an absolute pushout.*

Proof (i) Define $\alpha = \delta_i\colon [n-1] \to [n]$ and $\beta = \delta_i\colon [n-2] \to [n-1]$. The equation $\sigma_j \delta_i = \delta_i \sigma_{j-1}$ is one of the cosimplicial identities discussed above.

(ii) If X is an object of **C** and $f : B \to X$, $g : C \to X$ are maps such that $fq = gp$, then one defines a corresponding map $h : D \to X$ by $h := f\beta$. We should check that $hs = g$ and $hr = f$. The first equality is clear from $f\beta s = fq\alpha = gp\alpha = g$. For the second equality, it suffices to prove $hrq = fq$ because q is epi. The left-hand side equals $hrq = f\beta sp = fq\alpha p = gp\alpha p = gp$, which equals fq by assumption. To see that our choice of extension $h : D \to X$ is uniquely determined by (f, g), one observes that r is an epimorphism. The conclusion that the square is an absolute pushout follows from the fact that our proof only uses structures (split epis, commutative diagrams) that are preserved by any functor.

(iii) The surjections p and q can both be factored as compositions of elementary degeneracies, so that the conclusion follows by repeatedly applying (i) and (ii). □

Finally, let us record the following two existence results:

Proposition 2.2 (i) *If $f\colon [m] \to [n]$ is a monomorphism, then the pullback of any morphism $g\colon [k] \to [n]$ along f exists, provided that the image of g intersects the image of f nontrivially.*

(ii) *If $f\colon [m] \to [n]$ is an epimorphism, then the pushout of any morphism $g\colon [m] \to [k]$ along f exists.*

Proof (i) is clear by restricting g to the preimage of $f([m])$. For (ii), it suffices to treat the case where f is an elementary degeneracy $\sigma_i : [m] \to [m-1]$. Then the pushout of g is the map which collapses the interval $[g(i), g(i+1)]$ to a single point.□

2.2 Simplicial Sets and Geometric Realization

Let $\mathcal{E}$ be a category. The reader should keep in mind the examples where $\mathcal{E}$ is the category of sets, of topological spaces, or of groups. A *simplicial object* in $\mathcal{E}$ is a functor

$$X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{E}.$$

With natural transformations between such functors as morphisms, one obtains a category of simplicial objects in $\mathcal{E}$, which we denote by $\mathbf{s}\mathcal{E}$. One generally refers to a simplicial object in **Sets** as a *simplicial set*, and similarly for simplicial spaces, simplicial groups, simplicial schemes etc. We will soon see plenty of examples of such simplicial objects.

In more detail, a simplicial object X in $\mathcal{E}$ is given by a sequence of objects $X_n := X([n])$ in $\mathcal{E}$ ($n \geq 0$), together with maps $\alpha^* : X_n \to X_m$ for morphisms $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$. These maps should be functorial, in the sense that

$$\begin{aligned} &\mathrm{id}^* = \mathrm{id} : X_n \to X_n, \\ &(\alpha\beta)^* = \beta^*\alpha^* : X_n \to X_k \quad \text{for} \quad [k] \xrightarrow{\beta} [m] \xrightarrow{\alpha} [n]. \end{aligned}$$

A morphism f between two such simplicial objects X and Y is then a sequence of morphisms $f : X_n \to Y_n$ in $\mathcal{E}$ compatible with all the α^*, in the sense that

$$f_m\alpha^* = \alpha^* g_n$$

for $\alpha : [m] \to [n]$. When $\mathcal{E} = \mathbf{Sets}$, we will often refer to the elements of the set X_n as the *n-simplices* of X.

By our description of the morphisms in $\mathbf{\Delta}$ in the previous section, one may equivalently describe a simplicial object by specifying the operations α^* only when α is an elementary face or degeneracy. These are usually denoted

$$\begin{aligned} d_i &= (\delta_i)^* : X_n \to X_{n-1} \qquad i = 0, \ldots, n, \\ s_j &= (\sigma_j)^* : X_{n-1} \to X_n \qquad i = 0, \ldots, n-1. \end{aligned}$$

These maps are called the *face maps* and *degeneracy maps* of the simplicial object X. To distinguish them from the corresponding elementary face and degeneracy maps in the category $\mathbf{\Delta}$, the latter are in the literature sometimes referred to as cofaces and codegeneracies. The functoriality requirement on the α^* is equivalent to the requirement that the d_i and s_j satisfy the following *simplicial identities* (dual to the cosimplicial identities of the previous section):

(i) $d_i d_j = d_{j-1} d_i$ for $i < j$
(ii) $s_j s_i = s_i s_{j-1}$ for $i < j$
(iii) $d_j s_i = \begin{cases} s_i d_{j-1} & \text{if } i < j-1 \\ \text{id} & \text{if } i = j-1 \text{ or } i = j \\ s_{i-1} d_j & \text{if } i > j. \end{cases}$

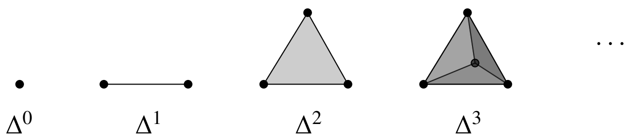
Similarly, a collection of maps $f : X_n \to Y_n$ determines a morphism of simplicial objects if and only if it is compatible with the face and degeneracy maps, as in

$$f_{n-1} d_i = d_i f_n \text{ for } n \geq 0, \quad i = 0, \dots, n,$$
$$f_n s_j = s_j f_{n-1} \text{ for } n \geq 0, \quad j = 0, \dots, n-1.$$

For the remainder of this section we will focus on the category **sSets** of simplicial sets. The main motivation for the concept of a simplicial set is to give a combinatorial procedure for building a topological space, as we will recall below, although the uses of simplicial sets and simplicial objects are now much more widespread.

Consider for each $n \geq 0$ the *standard topological n-simplex*

$$\Delta^n := \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_0 + \cdots + t_n = 1,\ t_i \geq 0 \quad \forall i\}.$$



This standard simplex has $n+1$ vertices $v_0, \dots, v_n$, where

$$v_i = (0, \dots, 0, 1, 0, \dots, 0)$$

with the 1 in the ith entry. Thus, any function of sets $f : \{0, \dots, m\} \to \{0, \dots, n\}$ defines an affine map

$$f_* : \Delta^m \to \Delta^n$$

which is uniquely determined by the requirement $f(v_i) = v_{f(i)}$. In particular, this makes the family of standard simplices into a functor

$$\Delta^\bullet : \boldsymbol{\Delta} \longrightarrow \mathbf{Top}.$$

We write Δ^α as α_* as for f above. Explicitly, for $\alpha : [m] \to [n]$,

$$\alpha_*(t_0, \dots, t_m) = (s_0, \dots, s_n) \quad \text{with} \quad s_i = \sum_{\alpha(j)=i} t_j.$$

In particular, for an elementary face map $\delta_i : [n-1] \to [n]$, the map

$$(\delta_i)_* : \Delta^{n-1} \to \Delta^n$$

embeds Δ^{n-1} as the face opposite the vertex v_i. More generally, for an injective map $\alpha : [m] \to [n]$, the corresponding map α_* embeds the m-simplex Δ^m as a face of Δ^n of possibly high codimension. Also, for the elementary degeneracy $\sigma_j : [n] \to [n-1]$, the map

$$(\sigma_j)_* : \Delta^n \to \Delta^{n-1}$$

collapses Δ^n onto Δ^{n-1} by a projection parallel to the line connecting v_j and v_{j+1}.

We will use these topological n-simplices to define for each simplicial set X its *geometric realization* $|X|$. This is a topological space defined as a quotient of the large disjoint sum of simplices

$$\coprod_{n \geq 0} X_n \times \Delta^n = \coprod_{n \geq 0} \coprod_{x \in X_n} \Delta^n,$$

the points of which we denote by

$$(x, t) \qquad \text{for} \quad x \in X_n, \quad t \in \Delta^n.$$

This quotient is formed by making the identification

$$(x, \alpha_* t) \sim (\alpha^* x, t)$$

for each morphism $\alpha\colon [m] \to [n]$ of $\mathbf{\Delta}$ and each $x \in X_n$, $t \in \Delta^m$. We write $x \otimes t$ for the equivalence class of a pair $(x, t) \in X_n \times \Delta^n$. This notation comes from the idea that X is a 'right module' over $\mathbf{\Delta}$ and $\Delta^\bullet$ is a 'left module', where left and right correspond to co- and contravariant functoriality respectively. There is a sense in which $|X|$ can be interpreted as a 'tensor product' $X \otimes_{\mathbf{\Delta}} \Delta^\bullet$ of such modules, but we will not elaborate on it here.

A map $f : X \to Y$ between simplicial sets induces an obvious continuous map

$$|f| : |X| \to |Y| : x \otimes t \mapsto f(x) \otimes t,$$

where we have suppressed the subscript n on f in the expression $f(x)$ for $x \in X_n$. This assignment makes geometric realization into a functor

$$|\cdot| : \mathbf{sSets} \longrightarrow \mathbf{Top}.$$

For a simplicial set, every n-simplex $x \in X_n$ defines a map

$$\hat{x} : \Delta^n \to |X| : t \mapsto x \otimes t.$$

The images of all these maps evidently cover all of $|X|$ and we will examine more closely how they overlap in the next section. For now, observe that by the equivalence relation imposed to form the geometric realization, these maps respect the simplicial structure of X, in the sense that for any $\alpha : [m] \to [n]$ and $y \in X_m$ such that $\alpha^* x = y$, the diagram

$$\begin{tikzcd} \Delta^m \arrow[rr, "\hat{y}"] \arrow[dr, "\alpha_*"'] & & {|X|} \\ & \Delta^n \arrow[ur, "\hat{x}"'] & \end{tikzcd}$$

commutes.

If $C^\bullet : \mathbf{\Delta} \to \mathcal{E}$ is any functor, each object E of $\mathcal{E}$ defines a simplicial set $\mathrm{Sing}_{C^\bullet}(E)$ by the formula

$$\mathrm{Sing}_{C^\bullet}(E)_n = \mathcal{E}(C^n, E),$$

where for $\alpha : [m] \to [n]$ the map α^* is defined by precomposition with C^α, the image of α under the functor $C^\bullet$. This general way of constructing simplicial sets applies in particular to the standard simplices $\Delta^\bullet : \mathbf{\Delta} \to \mathbf{Top}$, so that any topological space T defines a simplicial set $\mathrm{Sing}_{\Delta^\bullet}(T)$. It is usually more briefly denoted $\mathrm{Sing}(T)$ and called the *singular complex* of T.

In this way we obtain a functor

$$\mathrm{Sing} : \mathbf{Top} \longrightarrow \mathbf{sSets}$$

which bears a special relation to geometric realization. Indeed, a continuous map $\varphi : |X| \to T$ of topological spaces is given by a family of continuous maps

$$\varphi \circ \hat{x} : \Delta^n \to |X| \to T \qquad x \in X_n$$

for which each diagram of the form

$$\begin{tikzcd} \Delta^m \arrow[rr, "\varphi\circ\hat{y}"] \arrow[dr, "\alpha_*"'] & & T \\ & \Delta^n \arrow[ur, "\varphi\circ\hat{x}"'] & \end{tikzcd}$$

commutes. Thus, φ defines for each n a map of sets

$$\varphi_n : X_n \to \mathrm{Sing}(T)_n : x \mapsto \varphi \circ \hat{x}$$

which group together into a map of simplicial sets by the compatibility described above. In fact, this gives a natural bijective correspondence

$$\mathbf{Top}(|X|, T) \simeq \mathbf{sSets}(X, \mathrm{Sing}(T)),$$

so that the singular complex functor is right adjoint to geometric realization:

$$|\cdot| : \mathbf{sSets} \rightleftarrows \mathbf{Top} : \mathrm{Sing}.$$

2.3 The Geometric Realization as a Cell Complex

In this section we will examine the cellular structure of the geometric realization of a simplicial set X. Recall that we refer to the elements of X_n as the n-simplices of X. An n-simplex $x \in X_n$ is called *degenerate* if it lies in the image of one of the degeneracy operators $s_i : X_{n-1} \to X_n$ for $0 \le i \le n-1$. Equivalently, x is degenerate if there exists a surjection $\alpha : [n] \to [m]$ and $y \in X_m$ such that $x = \alpha^* y$. In fact, by choosing a further surjection in case y itself is degenerate, it is clearly possible to arrange that $x = \beta^* z$ for a surjection $\beta : [n] \to [k]$ and z a *non-degenerate* k-simplex of X. Furthermore, given $x \in X_n$, this choice of (β, z) with z non-degenerate is unique. Indeed, if (γ, w) was another such pair with $\gamma^* w = x$ and w non-degenerate, one forms the following pushout:

$$\begin{tikzcd} {[n]} \arrow[r, "\beta"] \arrow[d, "\gamma"'] & {[k]} \arrow[d] \\ {[l]} \arrow[r] & {[j].} \end{tikzcd}$$

It is an absolute pushout by Proposition 2.1 and therefore the resulting square

$$\begin{tikzcd} X_n & X_k \arrow[l, "\beta^*"'] \\ X_l \arrow[u, "\gamma^*"] & X_j \arrow[l] \arrow[u] \end{tikzcd}$$

is a pullback. Thus there is an element $v \in X_j$ whose image is z (resp. w) in X_k (resp. X_l). By the assumption that w and z are non-degenerate, this can only happen if the maps $[k] \to [j]$ and $[l] \to [j]$ are identities. It follows that $\beta = \gamma$ and $z = w$. The reader should also note that any 0-simplex is non-degenerate.

Every point of $|X|$ can be represented in the form $y \otimes s$ with y a non-degenerate simplex of X. Indeed, for any $x \otimes t \in |X|$, choose y non-degenerate and $\alpha : [n] \to [m]$ so that $x = \alpha^* y$. Then $\alpha_* : \Delta^n \to \Delta^m$ is surjective, so that there is an $s \in \Delta^m$ with $\alpha_* s = t$, and $x \otimes t = y \otimes s$. This section serves to explain the much more precise statement formulated in the theorem below. Recall that every n-simplex $x \in X_n$ determines a map $\hat{x} : \Delta^n \to |X|$.

Theorem 2.3 *Let X be a simplicial set. Its geometric realization $|X|$ naturally has the structure of a CW complex with precisely one closed n-cell $\hat{x} : \Delta^n \to |X|$ for every non-degenerate n-simplex $x \in X_n$.*

We begin by describing the CW structure of the theorem in more detail. Recall that we write $x \otimes t$ for the point of $|X|$ determined by a pair $(x, t) \in X_k \times \Delta^k$. Denote by $|X|^{(n)}$ the subspace of $|X|$ consisting of points which can be represented as $x \otimes t$ for some $(x, t) \in X_k \times \Delta^k$ with $k \leq n$. This describes a filtration of X,

$$|X|^{(0)} \subseteq |X|^{(1)} \subseteq |X|^{(2)} \subseteq \cdots, \quad \bigcup_n |X|^{(n)} = |X|,$$

and $|X|$ has the weak topology with respect to these subspaces. Indeed, the latter is clear from the definition of $|X|$ as a quotient of $\coprod_n X_n \times \Delta^n$. This filtration will serve as the skeletal filtration for the CW structure of $|X|$.

First, we claim that the space $|X|^{(0)}$ is discrete and is in fact given by $X_0 \times \Delta^0$, so that the elements of X_0 will serve as the 0-cells of $|X|$. Indeed, it is clear that the evident map $X_0 \times \Delta^0 \longrightarrow |X|^{(0)}$ is surjective. To see that is a bijection, we should argue that no two distinct 0-simplices of X are identified in $|X|$. If $x, y \in X_0$, then $x \otimes 1$ and $y \otimes 1$ represent the same point of $|X|$ only if there exists $z \otimes t$ with $z \in X_n$ and $t \in \Delta^n$, together with morphisms $\alpha, \beta : [0] \longrightarrow [n]$ such that $\alpha^* z = x$, $\beta^* z = y$ and $\alpha_* 1 = \beta_* 1 = t$. The last condition immediately implies that $\alpha = \beta$, from which it follows that $x = y$. This establishes our claim.

We should show that for $n \geq 1$ the space $|X|^{(n)}$ can be obtained from $|X|^{(n-1)}$ by attaching an n-cell for each non-degenerate n-simplex if X. More precisely, consider the square

$$\begin{tikzcd} \coprod_{x \in \mathrm{nd}(X_n)} \partial\Delta^n \arrow[r] \arrow[d] & {|X|^{(n-1)}} \arrow[d] \\ \coprod_{x \in \mathrm{nd}(X_n)} \Delta^n \arrow[r, "\{\hat{x}\}"] & {|X|^{(n)},} \end{tikzcd}$$

where $\mathrm{nd}(X_n)$ denotes the subset of X_n consisting of non-degenerate n-simplices. The conclusion of the theorem is clear if we can show that this square is a pushout. Note that it is a pullback: indeed, a point $x \otimes t$ of $|X|^{(n)}$ with $x \in \mathrm{nd}(X_n)$ and $t \in \Delta^n$ is contained in $|X|^{(n-1)}$ if and only if t is contained in the boundary of Δ^n. (This conclusion would not hold if we replaced the collection of non-degenerate n-simplices by the collection of all n-simplices.)

To see that the square is a pushout, we should argue that if $x \otimes t = y \otimes s$ for points (x, t) and (y, s) of $\coprod_{x \in \mathrm{nd}(X_n)} \Delta^n$, then either $(x, t) = (y, s)$ or both (x, t) and (y, s) are contained in $\coprod_{x \in \mathrm{nd}(X_n)} \partial\Delta^n$. This will follow from:

Proposition 2.4 *Let $\xi \in |X|$. Choose $x \in X_n$ and $t \in \Delta^n$ with $\xi = x \otimes t$ and with n as small as possible. Then x is non-degenerate and if $n \geq 1$ then t is contained in the interior of Δ^n. Also, the pair (x, t) representing ξ with x non-degenerate and t in the interior of Δ^n is unique.*

Indeed, to conclude the theorem from this, suppose $x \otimes t = y \otimes s$, still with $x, y \in X_n$ non-degenerate as above. If both s and t are in the interior of Δ^n, then the proposition implies $(x, t) = (y, s)$. If one of them, say t, is on the boundary of Δ^n, then we can write $x \otimes t = z \otimes r$ for z of smaller dimension k and r in the interior of Δ^k, uniquely. But then s must also be on the boundary of Δ^n; if it were in the interior this would contradict the uniqueness of representatives expressed by the proposition.

Proof (of Proposition 2.4) First we show x is non-degenerate. If it were not then there would exist a nontrivial surjection $\alpha : [n] \to [m]$ and $y \in X_m$ with $\alpha^* y = x$. But then $x \otimes t = \alpha^* y \otimes t = y \otimes \alpha_* t$, contradicting the minimality of n. It is also straightforward to see that t is in the interior of Δ^n (assuming $n \geq 1$); indeed, if it were on the boundary $\partial\Delta[n]$ then there would exist a nontrivial injection $\beta : [k] \to [n]$ such that t is in the image of $\beta_* : \Delta^k \to \Delta^n$, so we may write $t = \beta_* s$. In that case $x \otimes t = x \otimes \beta_* s = \beta^* x \otimes s$, which again contradicts the assumption that n is minimal.

It remains to argue that the representative pair (x, t) of the proposition is unique. So suppose $\xi = x \otimes t = y \otimes s$ where both x and y are non-degenerate and s, t are interior points of Δ^n. By the equivalence relation involved in the definition of $\otimes$, this means that there is a zigzag in $\mathbf{\Delta}$ of the form

$$\begin{tikzcd}[column sep=small]
{[n]} & & {[m_1]} & & {[m_2]} & \cdots & & {[n]} \\
& {[k_1]} \arrow[ul, "\alpha_1"] \arrow[ur, "\beta_1"'] & & {[k_2]} \arrow[ul, "\alpha_2"] \arrow[ur, "\beta_2"'] & \cdots & & {[k_N]} \arrow[ul, "\alpha_N"] \arrow[ur, "\beta_N"'] &
\end{tikzcd}$$

and elements $(a_i, u_i) \in X_{k_i} \times \Delta^{k_i}$, $(b_i, v_i) \in X_{m_i} \times \Delta^{m_i}$ for which

$$\alpha_i^* b_{i-1} = a_i, \quad (\alpha_i)_* u_i = v_{i-1},$$
$$\beta_i^* b_i = a_i, \quad (\beta_i)_* u_i = v_i.$$

Here we have written $(x, t) = (b_0, v_0)$ and $(y, s) = (b_N, v_N)$. Since $t = v_0$ and $s = v_N$ are interior points, the maps α_1 and β_N must be surjective. We now reason by induction on the length of the zig-zag. If $N = 1$, then the pushout of the surjections $[n] \leftarrow [k_1] \to [n]$ exists in $\mathbf{\Delta}$ and is absolute, so that X turns it into a pullback

$$\begin{tikzcd}
X_l \arrow[r, "\gamma^*"] \arrow[d] & X_n \arrow[d, "\alpha_1^*"] \\
X_n \arrow[r, "\beta_1^*"] & X_k
\end{tikzcd}$$

for the appropriate value of $l \leq n$. But then there is a $z \in X_l$ with $\gamma^* z = x$, meaning that l must equal n (otherwise we reach a contradiction with the minimality of n) and $\alpha_1 = \beta_1 = \mathrm{id}$. Thus $(x, t) = (y, s)$. If $N > 1$, factor β_1 as

$$[k_1] \xrightarrow{\varepsilon} [m_1'] \xrightarrow{\delta} [m_1]$$

with δ a monomorphism and ε surjective. Then one can apply the same argument as above to the pushout of the degeneracies

$$[n] \xleftarrow{\alpha_1} [k_1] \xrightarrow{\varepsilon} [m_1']$$

and the elements $(x, t) \in X_n \times \Delta^n$ and $(\delta^* b_1, \varepsilon_* u_1) \in X_{m'_1} \times \Delta^{m'_1}$ to conclude that ε must be the identity. Hence $\beta_1 = \delta$ is a monomorphism. But then the pullback of β_1 and α_2 exists in $\mathbf{\Delta}$ (cf. Proposition 2.2(i)) and such pullbacks are easily checked to be preserved by the functor $\Delta^\bullet$:

$$\begin{tikzcd} {[k'_1]} \arrow[r, "\theta"] \arrow[d, "\eta"] & {[k_2]} \arrow[d, "\alpha_2"] \\ {[k_1]} \arrow[r, "\beta_1"] & {[m_1],} \end{tikzcd} \qquad \begin{tikzcd} \Delta^{k'_1} \arrow[r, "\theta_*"] \arrow[d, "\eta_*"] & \Delta^{k_2} \arrow[d, "(\alpha_2)_*"] \\ \Delta^{k_1} \arrow[r, "(\beta_1)_*"] & \Delta^{m_1}. \end{tikzcd}$$

So we can shorten the zigzag by replacing the first two spans by the single span

$$[n] \xleftarrow{\alpha_1\eta} [k'_1] \xrightarrow{\beta_2\theta} [m_2]$$

and using the element $(c, w) \in X_{k'_1} \times \Delta^{k'_1}$, with $c = \eta^* a_1 = \theta^* a_2$ and w the unique point in $\Delta^{k'_1}$ satisfying $\eta_* w = u_1$ and $\theta_* w = u_2$. This completes the inductive step. □

The filtration of the geometric realization $|X|$ by subspaces $|X|^{(n)}$ has a counterpart in the theory of simplicial sets, called the *skeletal filtration* of X. It is a fundamental tool when one proves properties of X 'simplex by simplex'. In fact, our filtration of $|X|$ above simply arises as the geometric realization of the skeletal filtration of the simplicial set X.

We define $\mathrm{sk}_n X$ to be the simplicial subset of X generated by its simplices of dimension at most n. In other words, it is the smallest subobject $\mathrm{sk}_n X \subseteq X$ which contains every simplex $x \in X_k$ for $k \leq n$. Clearly $\cup_n \mathrm{sk}_n X = X$. The crucial property is that $\mathrm{sk}_n X$ can be built from $\mathrm{sk}_{n-1} X$ by 'cell attachments' as follows:

Proposition 2.5 *The evident square*

$$\begin{tikzcd} \coprod_{x \in \mathrm{nd}(X_n)} \partial\Delta[n] \arrow[r] \arrow[d] & \mathrm{sk}_{n-1} X \arrow[d] \\ \coprod_{x \in \mathrm{nd}(X_n)} \Delta[n] \arrow[r] & \mathrm{sk}_n X, \end{tikzcd}$$

is a pushout. As before, the coproduct is over the set $\mathrm{nd}(X_n)$ *of non-degenerate n-simplices of X.*

Proof For the length of this proof we write P for the pushout in the square above and $p : P \to \mathrm{sk}_n X$ for the evident map. We should demonstrate that P is an isomorphism of simplicial sets.

To see that p is surjective, consider an n-simplex $x \in X_n$. Then we can write $x = \alpha^* y$ for some degeneracy $\alpha : [n] \to [m]$ and a unique non-degenerate simplex $y \in X_m$. If $m < n$, then y (and hence also x) is already contained in $\mathrm{sk}_{n-1} X$ and hence in the image of p. If $m = n$ then α is the identity and x is non-degenerate, so that x occurs in the coproduct in the lower left corner of the square of the proposition. Again, x is therefore in the image of p.

It remains to argue that p is injective. There are two things to check:

(a) If $x \in \mathrm{nd}(X_n)$, then the pullback of the corresponding span

$$\Delta[n] \xrightarrow{x} \mathrm{sk}_n X \leftarrow \mathrm{sk}_{n-1} X$$

is precisely the boundary $\partial\Delta[n]$.

(b) For two distinct non-degenerate simplices $x, y \in \mathrm{nd}(X_n)$, consider the pullback square

$$\begin{tikzcd} Q \arrow[r, "v"] \arrow[d, "w"] & \Delta[n] \arrow[d, "x"] \\ \Delta[n] \arrow[r, "y"] & \mathrm{sk}_n X. \end{tikzcd}$$

Then v and w factor through the boundary inclusion $\partial\Delta[n] \subseteq \Delta[n]$.

Indeed, (a) and (b) express the idea that all identifications to be made when adding non-degenerate n-simplices to $\mathrm{sk}_{n-1}X$ concern only the boundary of those simplices.

Proof of (a): Say $[k] \xrightarrow{\alpha} [n]$ is a map so that $\alpha^* x$ is contained in $\mathrm{sk}_{n-1}X$. We should show that α is not surjective, so that it factors through $\partial\Delta[n]$. We reason by contradiction; suppose α *is* surjective. By definition of the $(n-1)$-skeleton we can write $\alpha^* x = \beta^* y$ for some non-degenerate m-simplex $y \in X_m$ (with $m < n$) and a surjective map $\beta : [k] \to [l]$. Form the absolute pushout square

$$\begin{tikzcd} {[k]} \arrow[r, "\alpha"] \arrow[d, "\beta"] & {[n]} \arrow[d, "\delta"] \\ {[m]} \arrow[r, "\gamma"] & {[l].} \end{tikzcd}$$

Since X turns it into a pullback, there exists a $z \in X_l$ with $\gamma^* z = y$ and $\delta^* z = x$. Now, δ is surjective and x non-degenerate, so we must have that δ is the identity. But then $k = m$, contradicting the fact that $m < n \leq k$.

Proof of (b): Consider maps $\alpha, \beta : [k] \to [n]$ so that $\alpha^* x = \beta^* y$. We should show that both α and β are *not* surjective. Factor these maps as

$$[k] \xrightarrow{\alpha_-} [m_1] \xrightarrow{\alpha_+} [n], \quad [k] \xrightarrow{\beta_-} [m_2] \xrightarrow{\beta_+} [n],$$

with α_-, β_- surjective and α_+, β_+ injective. Now form the absolute pushout

$$\begin{tikzcd} {[k]} \arrow[r, "\alpha_-"] \arrow[d, "\beta_-"] & {[m_1]} \arrow[d, "\delta"] \\ {[m_2]} \arrow[r, "\gamma"] & {[l].} \end{tikzcd}$$

As before it follows that there is a $z \in X_l$ with $(\gamma\alpha_-)^* z = x$ and $(\delta\beta_-)^* z = y$. The non-degeneracy of x and y then implies that all the surjections in the above square are in fact identities. We conclude that $\alpha = \alpha_+$ and $\beta = \beta_+$ are both injective. It remains to argue that neither can be the identity. But if one of them was, then $[k] = [n]$ and clearly both of them are identities. It follows that $x = y$, contradicting our assumption. □

2.4 Simplicial Sets as a Category of Presheaves

For a small category $\mathbf{C}$, a functor

$$X : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$$

is called a *presheaf* (of sets) on $\mathbf{C}$. Together with the natural transformations between them, these presheaves form a category which we denote by

$$\mathrm{PSh}(\mathbf{C}).$$

(Other common notations are $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ and $\widehat{\mathbf{C}}$.) Thus, the category **sSets** of simplicial sets is the category $\mathrm{PSh}(\mathbf{\Delta})$ of presheaves on $\mathbf{\Delta}$ and as such enjoys the general properties of such categories of presheaves. In this section we review several of those properties which will be relevant to us.

First some notation: for a presheaf X as above and a morphism $\alpha : c \to d$ in $\mathbf{C}$, its value under X is denoted

$$\alpha^* : X(d) \to X(c).$$

If $f : X \to Y$ is a morphism between presheaves, consisting of a natural family of morphisms $f_c : X(c) \to Y(c)$ for c ranging through the objects of $\mathbf{C}$, we often abbreviate f_c by f again if no confusion can arise.

Representable presheaves. Each object $c \in \mathbf{C}$ determines a so-called representable presheaf $\mathbf{y}(c)$, defined on objects by

$$\mathbf{y}(c)(d) = \mathbf{C}(d, c)$$

and with the evident action of morphisms in $\mathbf{C}$ by precomposition. It can also be denoted $\mathbf{C}(-, c)$. This construction is also functorial in c and determines a functor

$$\mathbf{y} : \mathbf{C} \to \mathrm{PSh}(\mathbf{C})$$

called the *Yoneda embedding*. The basic *Yoneda lemma* states that for any presheaf X there is a natural bijective correspondence between morphisms of presheaves $f : \mathbf{y}(c) \to X$ and elements $x \in X(c)$. This correspondence is given by $x = f(\mathrm{id}_c)$ and $f(\alpha) = \alpha^* x$. We write $\overline{x}$ for this morphism f corresponding to x.

Standard simplices. For the special case of simplicial sets, the representable presheaf $y([n])$ is denoted by $\Delta[n]$ and referred to as the (simplicial) standard n-simplex. It mirrors the topological n-simplex Δ^n in the sense that

$$|\Delta[n]| = \Delta^n,$$

as one easily checks. The Yoneda lemma gives a correspondence between n-simplices $x \in X_n$ and maps $\overline{x} : \Delta[n] \to X$ and the geometric realization of the latter is precisely the map we denoted by $\hat{x} : \Delta^n \to |X|$ in previous sections.

Limits and colimits. Each presheaf category $\mathrm{PSh}(\mathbf{C})$ has all small limits and colimits and these are all computed 'pointwise'. To be precise, if

$$X : I \to \mathrm{PSh}(\mathbf{C}) : i \mapsto X_i$$

is a diagram of presheaves indexed by a small category I, then

$$(\varinjlim_I X_i)(c) \simeq \varinjlim_I X_i(c),$$

the colimit on the left being computed in $\mathrm{PSh}(\mathbf{C})$, the one on the right in **Sets**. The same applies to limits. To give a simple example, the product of simplicial sets X and Y is constructed as

$$(X \times Y)_n = X_n \times Y_n,$$

with simplicial operators (e.g. faces and degeneracies) defined componentwise, as $d_i(x, y) = (d_i x, d_i y)$, etc.

A similar observation applies to epimorphisms, monomorphisms and images: a map $f : X \to Y$ between presheaves is epi (resp. mono) if and only if each of its components $f : X(c) \to Y(c)$ is. For a general $f : X \to Y$, its image $f(X) \subseteq Y$ is constructed as $f(X)(c) = f(X(c))$ for each object c of $\mathbf{C}$. A monomorphism $A \to Y$ for which each component $A(c) \to Y(c)$ is the inclusion of a subset is referred to as a *subpresheaf* of Y. Sometimes we will also use this terminology to refer to an isomorphism class of monos $A \to Y$, secretly identifying them with their common image.

Colimits of representables. Every presheaf X on a category $\mathbf{C}$ is canonically isomorphic to a colimit of representable presheaves. To see this, one first constructs the *category of elements* of X, variously denoted $\mathrm{El}(X)$, $\int_{\mathbf{C}} X$ or $\mathbf{C}/X$ in the literature. We will use the latter notation. The objects of $\mathbf{C}/X$ are pairs (c, x) with $c \in \mathbf{C}$ and $x \in X(c)$. A morphism $(c, x) \to (d, y)$ is a morphism $\alpha : c \to d$ with the property that $\alpha^* y = x$. There is an evident projection

$$\pi_X : \mathbf{C}/X \to \mathbf{C} : (c, x) \mapsto c$$

and an isomorphism

$$\theta_X : \varinjlim_{\mathbf{C}/X} \mathbf{y} \circ \pi_X \to X.$$

This natural transformation θ_X is induced by the morphisms

$$\overline{x} : \mathbf{y}(c) \to X$$

for (c, x) ranging over the objects of $\mathbf{C}/X$.

Kan extension. The category $\mathrm{PSh}(\mathbf{C})$ is the *free* category with all small colimits generated by $\mathbf{C}$. What this means is that for any category $\mathcal{E}$ with all small colimits, any functor $F : \mathbf{C} \to \mathcal{E}$ extends (uniquely up to natural isomorphism) to a functor $F_! : \mathrm{PSh}(\mathbf{C}) \to \mathcal{E}$, such that $F_!$ preserves all small colimits. To make sense of the word 'extends' here, one should regard $\mathbf{C}$ as a subcategory of $\mathrm{PSh}(\mathbf{C})$ via the Yoneda embedding. In other words, there is a natural isomorphism $F_! \circ \mathbf{y} \simeq F$. Another common notation for $F_!$ is $\mathrm{Lan}_{\mathbf{y}} F$, indicating that it is the *left Kan extension* of F along $\mathbf{y}$. The functor $F_!$ can be constructed explicitly by writing every presheaf as a colimit of representables:

$$F_!(X) = \varinjlim_{\mathbf{C}/X} F \circ \pi_X.$$

More informally, one might also write

$$F_!(X) = \varinjlim_{c \in \mathbf{C}, x \in F(c)} F(c).$$

To check that $F_!$ is indeed a functor, one observes that the construction of the category of elements is itself functorial in X. To see that $F_!$ extends F, one observes that the category of elements $\mathbf{C}/\mathbf{y}(c)$ is isomorphic to the slice category $\mathbf{C}/c$. The latter has a terminal object, namely id_c, so that the colimit over this category may be computed by evaluation at this object.

The functor $F_!$ just constructed admits a right adjoint F^*. Indeed, for $E \in \mathcal{E}$ we simply define the presheaf F^*E by

$$F^*E(c) = \mathcal{E}(F(c), E).$$

Functoriality of F^* is clear; to see it is indeed right adjoint, consider a presheaf $X \in \mathrm{PSh}(\mathbf{C})$ and observe the sequence of natural isomorphisms

$$\begin{aligned}
\mathcal{E}(F_!X, E) &\simeq \varprojlim_{\mathbf{C}/X} \mathcal{E}(F \circ \pi_X, E) \\
&\simeq \varprojlim_{\mathbf{C}/X} F^*E \circ \pi_X \\
&\simeq \varprojlim_{\mathbf{C}/X} \mathrm{PSh}(\mathbf{C})(\mathbf{y} \circ \pi_X, F^*E) \\
&\simeq \mathrm{PSh}(\mathbf{C})(X, F^*E).
\end{aligned}$$

Here we have applied the Yoneda lemma to go from the second to the third line.

Geometric realization. The category **Top** of topological spaces has all small colimits. Therefore the left Kan extension explained above applies to the functor of standard topological simplices

$$\Delta^\bullet : \mathbf{\Delta} \to \mathbf{Top}.$$

The resulting functor from **sSets** to **Top** is precisely the geometric realization discussed in previous sections. Indeed, geometric realization preserves colimits and the composition $|\cdot| \circ \mathbf{y}$ is (isomorphic to) the functor $\Delta^\bullet$. Therefore geometric realization is the left Kan extension of $\Delta^\bullet$ to the category of simplicial sets. In this specific example, the right adjoint discussed in the previous paragraph yields the singular complex functor

$$\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSets}.$$

The nerve of a category. An important construction which is analogous to the adjoint pair $|\cdot|$ and Sing is the following. Consider the category **Cat** of small categories. It contains the categories of partially ordered and linearly ordered sets as full subcategories and in particular there is a fully faithful functor

$$\iota \colon \mathbf{\Delta} \to \mathbf{Cat}$$

sending an object $[n]$ to the corresponding linear order $(0 \to 1 \to \cdots \to n)$. The left Kan extension of i defines a functor which is usually denoted τ in the literature,

$$\tau = \iota_! \colon \mathbf{sSets} \to \mathbf{Cat}.$$

Following the general pattern explained above, this functor τ has a right adjoint called the *nerve functor* and usually written

$$N \colon \mathbf{Cat} \to \mathbf{sSets}.$$

Spelling out the general formula for the right adjoint in this specific case, we see that for a small category $\mathbf{C}$, its nerve can be described as follows: the set of 0-simplices $(N\mathbf{C})_0$ is the set of objects of $\mathbf{C}$ and the set of n-simplices $(N\mathbf{C})_n$ is the set of strings of n composable morphisms

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n.$$

The simplicial operators $d_i : (N\mathbf{C})_n \to (N\mathbf{C})_{n-1}$ and $s_j : (N\mathbf{C})_{n-1} \to (N\mathbf{C})_n$ can somewhat cryptically be described by 'd_i deletes c_i' and 's_j inserts the identity $c_j = c_j$'. To be more precise, for $0 < i < n$ we have:

$$\begin{aligned}
d_0(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n) &= c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n, \\
d_n(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n) &= c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_{n-1}, \\
d_i(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n) &= c_0 \xrightarrow{f_1} \cdots c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \cdots \xrightarrow{f_n} c_n, \\
s_j(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} c_{n-1}) &= c_0 \xrightarrow{f_1} \cdots \to c_j \xrightarrow{\mathrm{id}_{c_j}} c_j \cdots \xrightarrow{f_n} c_{n-1}.
\end{aligned}$$

One easily checks that the nerve functor $N\colon \mathbf{Cat} \longrightarrow \mathbf{sSets}$ is fully faithful. In other words, a functor $\mathbf{C} \longrightarrow \mathbf{D}$ is essentially the same thing as a morphism of simplicial sets $N\mathbf{C} \longrightarrow N\mathbf{D}$. An equivalent statement is that the counit $\tau N\mathbf{C} \longrightarrow \mathbf{C}$ is an isomorphism. This can also easily be checked using the following explicit description of τ.

For a simplicial set X, the category τX has X_0 as its set of objects. Any 1-simplex $f \in X_1$ defines a morphism $x \longrightarrow y$ in τX, with $x = d_1 f$ and $y = d_0 f$. These morphisms generate all morphisms in τX, in the sense that any arrow $x \longrightarrow y$ in τX can be represented by a finite string of 'composable' 1-simplices $(f_1, \ldots, f_k)$, i.e. these 1-simplices satisfy $d_1 f_1 = x$, $d_0 f_k = y$ and $d_0 f_i = d_1 f_{i+1}$. The relations satisfied by these generators are of two kinds: each degenerate 1-simplex is identified with an identity morphism in τX and each 2-simplex $\xi \in X_2$ describes a composition relation, namely

$$d_1\xi = d_0\xi \circ d_2\xi.$$

More graphically, the 2-simplex ξ imposes that the following be a commutative diagram in τX:

$$\begin{tikzcd}
& y \arrow[dr, "d_0\xi"] & \\
x \arrow[ur, "d_2\xi"] \arrow[rr, "d_1\xi"'] & & z.
\end{tikzcd}$$

One may wonder why this explicit description indeed describes τ. This can be proved by showing directly that the functor we just described is left adjoint to the nerve functor.

For later use, we note that the functor $\tau\colon \mathbf{Cat} \longrightarrow \mathbf{sSets}$ preserves products. Indeed, for representable simplicial sets $\Delta[n]$ and $\Delta[m]$, this follows from the chain of natural isomorphisms

$$\tau(\Delta[n] \times \Delta[m]) \cong \tau N(\iota[n] \times \iota[m]) \cong \iota[n] \times \iota[m].$$

General simplicial sets are colimits of representables and the assertion follows since the functors involved preserve colimits in each variable separately.

The classifying space. Composing the nerve functor with geometric realization, one recovers the well-known and important construction of a space out of a category $\mathbf{C}$, namely its *classifying space*, denoted

$$B\mathbf{C} := |N\mathbf{C}|.$$

This classifying space functor is particularly useful in relating the (co)homology of categories to the (co)homology of spaces; in the case where $\mathbf{C}$ is a group G (i.e., $\mathbf{C}$ has a single object and all its morphisms are isomorphisms), then the (co)homology of G as defined by homological algebra coincides with the singular (co)homology of its classifying space BG. This is related to the fact that the geometric realization and singular complex functors are homotopy inverse to each other in an appropriate sense, a fact we will discuss extensively in the second part of this book.

Internal hom or exponential. Any presheaf category $\mathrm{PSh}(\mathbf{C})$ is *cartesian closed*, meaning that for any object $X \in \mathrm{PSh}(\mathbf{C})$, the product functor

$$- \times X : \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{C})$$

has a right adjoint. This right adjoint is referred to as the *internal hom* or *exponential* and accordingly denoted $\mathbf{hom}(X, -)$ or $(-)^X$ respectively. The construction of this adjoint can actually be viewed as another example of an adjoint pair $F_!$ and F^* obtained by Kan extension as discussed above. Indeed, the functor $- \times X$ preserves colimits, because the same is true in the category of sets and colimits of presheaves are computed objectwise. So $- \times X$ is the left Kan extension of its restriction to representables,

$$F : \mathbf{C} \to \mathrm{PSh}(\mathbf{C}) : c \mapsto y(c) \times X.$$

Therefore the right adjoint F^* exists and gives the exponential alluded to above. The adjointness of these functors is the usual exponential relation

$$\mathrm{PSh}(\mathbf{C})(Z \times X, Y) \simeq \mathrm{PSh}(\mathbf{C})(Z, Y^X).$$

For the special case of simplicial sets, we thus have the formula

$$(Y^X)_n = \mathbf{sSets}(\Delta[n] \times X, Y).$$

Some of the functors we have discussed in this section behave well with respect to exponentials. For left adjoints this is rarely the case, but right adjoints are generally better. Explicitly, consider an adjoint pair $\varphi_! : \mathbf{D} \leftrightarrows \mathbf{E} : \varphi^*$ between categories with finite products and exponentials. Then the exponential law gives, for objects $X, Y \in \mathbf{E}$, a canonical map

$$\gamma : \varphi^*(Y^X) \to (\varphi^* Y)^{\varphi^* X}.$$

Indeed, we have natural maps

$$\varphi_!(Z \times \varphi^* X) \to \varphi_!(Z) \times \varphi_!(\varphi^* X) \to \varphi_!(Z) \times X.$$

The first one derives from the universal property of the product, the second uses the counit of the adjoint pair $(\varphi_!, \varphi^*)$. Write p for the composition of these two maps. Then we can form the sequence of natural maps

$$\begin{aligned} \mathbf{D}(Z, \varphi^*(Y^X)) &\simeq \mathbf{E}(\varphi_! Z \times X, Y) \\ &\xrightarrow{p^*} \mathbf{E}(\varphi_!(Z \times \varphi^* X), Y) \\ &\simeq \mathbf{D}(Z \times \varphi^* X, \varphi^* Y) \\ &\simeq \mathbf{D}(Z, (\varphi^* Y)^{\varphi^* X}). \end{aligned}$$

Applying this to the case $Z = \varphi^*(Y^X)$ and its identity map this gives the promised comparison map γ. At the same time, we conclude that γ is an isomorphism for all $X, Y \in \mathbf{E}$ if and only if p is an isomorphism for all $X \in \mathbf{E}$ and $Z \in \mathbf{D}$. For example, this applies to the adjoint pair $\tau : \mathbf{sSets} \leftrightarrows \mathbf{Cat} : N$. Indeed, it is straightforward to verify that τ commutes with products. We already noted that the counit $\tau N\mathbf{C} \to \mathbf{C}$ is an isomorphism, so that the map $p : \tau(X \times N\mathbf{C}) \to \tau X \times \tau N\mathbf{C} \to \tau X \times \mathbf{C}$ is an isomorphism for any simplicial set X and small category $\mathbf{C}$. Thus, the nerve functor preserves exponentials.

The case of the adjoint pair $|\cdot|$ and Sing is different; a similar argument to the above would apply if **Top** had exponentials, if geometric realizations would preserve products and if Sing was fully faithful. However, all three of these statements are in general false. The first two can be corrected by replacing the category of spaces by 'a convenient category of spaces', such as the category of compactly generated weak Hausdorff spaces. Still, not much can be done about the third: for a topological space X, the map $|\mathrm{Sing}(X)| \to X$ is generally not a homeomorphism. It is, however, a weak homotopy equivalence, as we shall discuss in Section 8.6. We will come back to the relation between geometric realization and products in the next section.

Dependence on $\mathbf{C}$. We include a general remark on how the presheaf category $\mathrm{PSh}(\mathbf{C})$ depends on $\mathbf{C}$. Consider a functor $\varphi : \mathbf{C} \to \mathbf{D}$ between small categories. It induces an obvious restriction functor

$$\varphi^* : \mathrm{PSh}(\mathbf{D}) \to \mathrm{PSh}(\mathbf{C}), \quad \varphi^* Y(c) = Y(\varphi c).$$

Since colimits in presheaf categories are computed pointwise, the functor φ^* preserves colimits. Therefore it is the left Kan extension of its restriction to representables and by the same logic as before, it must admit a right adjoint for which we write

$$\varphi_* : \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{D}), \quad \varphi_* X(d) = \mathrm{PSh}(\mathbf{C})(\varphi^*(y(d)), X).$$

But $\varphi : \mathbf{C} \to \mathbf{D}$ also induces an obvious functor

$$y \circ \varphi : \mathbf{C} \to \mathbf{D} \to \mathrm{PSh}(\mathbf{D})$$

resulting in another pair of adjoint functors, which we should for now denote by $(y \circ \varphi)_!$ and $(y \circ \varphi)^*$, in accordance with our earlier discussion of Kan extensions. But

$$(y \circ \varphi)^* : \mathrm{PSh}(\mathbf{D}) \to \mathrm{PSh}(\mathbf{C})$$

is given by the formula

$$(y \circ \varphi)^* Y(c) = \mathrm{PSh}(\mathbf{D})((y \circ \varphi)(c), Y) \simeq Y(\varphi(c)),$$

the latter by the Yoneda lemma. In other words, $(y \circ \varphi)^* Y$ is (isomorphic to) the presheaf $\varphi^* Y$ consider before, so we need not distinguish between φ^* and $(y \circ \varphi)^*$. Similarly, we will abbreviate the notation $(y \circ \varphi)_!$ to $\varphi_!$. Up to natural isomorphism it is the unique functor

$$\varphi_! : \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{D})$$

which is left adjoint to the restriction functor φ^*. Also, it is up to natural isomorphism the unique functor preserving small colimits and agreeing with φ on representables, in the precise sense that there is an isomorphism $\varphi_! y(c) \simeq y(\varphi c)$, natural in c.

As an example, consider the inclusion

$$\iota : \mathbf{\Delta}_{\leq n} \to \mathbf{\Delta}$$

of the full subcategory $\mathbf{\Delta}_{\leq n}$ of $\mathbf{\Delta}$ on the objects $[k]$ for $k \leq n$. It gives rise to three functors

$$\begin{tikzcd} \mathrm{PSh}(\mathbf{\Delta}_{\leq n}) \arrow[r, bend left, "\iota_!"] \arrow[r, bend right, "\iota_*"'] & \mathbf{sSets}, \arrow[l, "\iota^*"'] \end{tikzcd}$$

with $\iota_!$ and ι_* the left and right adjoint of ι^* respectively. For a simplicial set X, the counit of the first and unit of the second adjunction give rise to maps

$$\iota_!\iota^* X \to X \to \iota^*\iota_* X.$$

The simplicial set $\iota_!\iota^* X$ is precisely the n-skeleton $\mathrm{sk}_n X$ discussed at the end of Section 2.3. Dually, the simplicial set $\iota^*\iota_* X$ is called the *n-coskeleton* of X and usually denoted $\mathrm{cosk}_n X$.

Constant presheaves. There is an evident notion of *constant presheaf* on a category $\mathbf{C}$. Indeed, the constant presheaf F with value a set S is the functor which satisfies $F(c) = S$ for every object c of $\mathbf{C}$ and which sends every morphism of $\mathbf{C}$ to the identity map of S. With the notation of the previous paragraph, one can consider the functor $\varphi : \mathbf{C} \to 1$, where 1 denotes the trivial category with one object and only the identity morphism. Then under the obvious isomorphism $\mathrm{PSh}(1) \cong \mathbf{Sets}$, the constant presheaf with value S is precisely $\varphi^* S$. The left adjoint $\varphi_!$ (resp. the right adjoint φ_*) is now the functor which takes the colimit (resp. the limit) of a presheaf F over the category $\mathbf{C}^{\mathrm{op}}$.

In the context of simplicial sets we introduce some terminology and notation for this situation. We will say that a simplicial set X is *discrete* if it is constant as a presheaf on $\mathbf{\Delta}^{\mathrm{op}}$. The reason for this terminology is the relation to topology; for a space Y with the discrete topology, the singular complex $\mathrm{Sing}(Y)$ is a discrete simplicial set. The functor which assigns to a set S the corresponding discrete simplicial set admits a left adjoint (called $\varphi_!$ in the previous paragraph) for which we will write

$$\pi_0 : \mathbf{sSets} \to \mathbf{Sets}.$$

As the notation suggests, we will refer to $\pi_0 X$ as the set of *connected components* of X. Again the reason is the analogy with topology. To be precise, $\pi_0 X$ is exactly the set of connected components of the geometric realization $|X|$. Indeed, the inclusion

$$\mathrm{dis} : \mathbf{Sets} \to \mathbf{CW}$$

which equips a set with the discrete topology (thought of as a CW-complex) admits a left adjoint (also denoted π_0), sending a CW-complex to its set of connected components. The composition of right adjoints Sing $\circ$ dis sends a set to the corresponding discrete simplicial set. Hence the composition of left adjoints $\pi_0 \circ |\cdot|$ agrees with the functor π_0 we defined above.

The interested reader may wish to verify that to compute $\pi_0 X$ in practice, one can simply take the coequalizer

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \longrightarrow \pi_0 X,$$

rather than the colimit of the entire diagram $X_\bullet$ on $\mathbf{\Delta}^{\mathrm{op}}$.

2.5 Products of Simplicial Sets and Shuffle Maps

The goal of this section is twofold: first, we discuss the product $X \times Y$ of two simplicial sets X and Y, in particular in the case where X and Y are representable. The reason for the latter is that for simplicial sets, as for any cartesian closed category, the product as a functor

$$X \times - \quad \text{or} \quad - \times Y : \mathbf{sSets} \to \mathbf{sSets}$$

admits a right adjoint and hence preserves colimits. In other words, the product preserves colimits in each variable separately. Since every simplicial set is canonically a colimit of representables, many properties of the product can be deduced from those of the product of two standard simplices. We will discuss these products and their description in terms of *shuffle maps* in some detail, since an analysis of shuffle maps and generalizations thereof will return at various places in this book. In the second part of this section we examine the behaviour of the geometric realization functor with respect to products and, more generally, finite limits.

Consider a p-simplex σ of the simplicial set $\Delta[n] \times \Delta[m]$, the binary product of two standard simplices. It corresponds to a pair of maps

$$\sigma_1 : \Delta[p] \to \Delta[n], \quad \sigma_2 : \Delta[p] \to \Delta[m]$$

or more simply a morphism (still denoted by the same symbol)

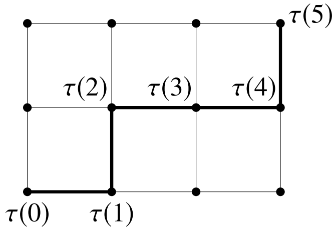
$$\sigma : [p] \to [n] \times [m]$$

in the category of partially ordered sets. We will call the simplex σ *non-degenerate* if this morphism is injective. Moreover, such a simplex σ is a face of another non-degenerate simplex τ precisely if the map of partially ordered sets σ can be extended to an injective map τ as follows:

$$\begin{tikzcd}
{[p]} \arrow[rr, "\sigma"] \arrow[dr] & & {[n] \times [m].} \\
& {[q]} \arrow[ur, "\tau"'] &
\end{tikzcd}$$

Let us say such an injective map $\tau : [q] \to [n] \times [m]$ is *maximal* if it cannot be factored further in this way. Such maximal simplices τ correspond precisely to the injective maps $[n+m] \to [n] \times [m]$. Any non-degenerate simplex σ of the product $\Delta[n] \times \Delta[m]$ is clearly a face of some maximal non-degenerate simplex τ, although this τ need not be unique.

An injective map $\tau : [n+m] \to [n] \times [m]$ of partially ordered sets can be pictured as a staircase. The following is an example with $(n, m) = (3, 2)$:

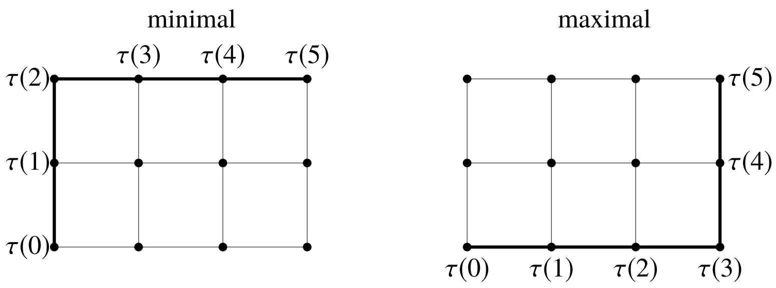


Indeed, the values of τ trace out a path through the rectangle starting at $\tau(0) = (0, 0)$ and ending at $\tau(n+m) = (n, m)$. We will refer to such a maximal injective map τ as a *shuffle* of $[n]$ and $[m]$. The reason for this terminology is that such a shuffle is uniquely described by specifying the 'steps' in this staircase. Indeed, observe that the staircase consists of $n+m$ edges, of which n are horizontal and m are vertical. Those vertical edges are specified by a strictly increasing map $v_\tau : \{1, \ldots, m\} \to \{1, \ldots, n+m\}$. Equivalently, one can specify the horizontal edges by a strictly increasing map $h_\tau : \{1, \ldots, n\} \to \{1, \ldots, n+m\}$, whose image is the complement of the previous map. In this sense, the staircase above corresponds to a 'shuffle' of the linearly ordered sets $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$. Observe that there are $\binom{n+m}{n}$ such shuffles.

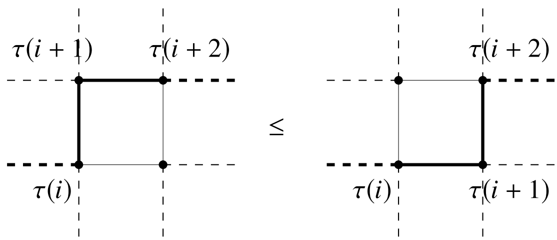
For later use, we note that there is a natural partial ordering on these shuffles. Indeed, for shuffles τ_1 and τ_2 with associated maps

$$v_{\tau_1}, v_{\tau_2} : \{1, \ldots, m\} \to \{1, \ldots, n+m\}$$

as above, one sets $\tau_1 \leq \tau_2$ if $v_{\tau_1}(i) \leq v_{\tau_2}(i)$ for each $1 \leq i \leq m$. This partial order has a minimal and a maximal element. These are pictured below:



Also, the following illustrates a typical relation between two shuffles:



The conclusion of our discussion is that one can write

$$\Delta[n] \times \Delta[m] = \bigcup_{\tau} \Delta[n+m],$$

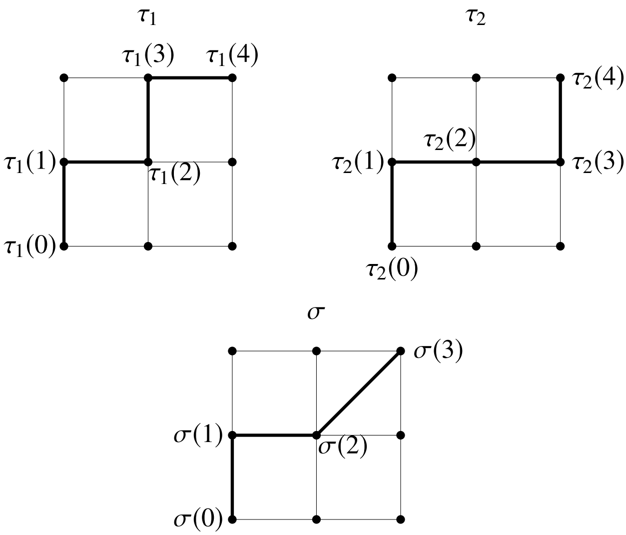
where the union is over all monomorphisms

$$\Delta[n+m] \longrightarrow \Delta[n] \times \Delta[m]$$

corresponding to shuffles τ. These simplices overlap in a way which is easily expressed in terms of shuffles: for two shuffles τ_1 and τ_2, the pullback

$$\begin{tikzcd}
\Delta[k] \arrow[r] \arrow[d] & \Delta[n+m] \arrow[d, "\tau_2"] \\
\Delta[n+m] \arrow[r, "\tau_1"'] & \Delta[n] \times \Delta[m]
\end{tikzcd}$$

corresponds to the map of partially ordered sets $\sigma : [k] \rightarrow [n] \times [m]$ enumerating the common values of τ_1 and τ_2. It always satisfies $\sigma(0) = (0, 0)$ and $\sigma(k) = (n, m)$. A typical example is as follows:



The remainder of this section will concern the behaviour of the geometric realization functor with respect to products (and more generally finite limits) of simplicial sets. First, a remark on the kind of topological spaces we consider is in order. As we have alluded to before, the category of topological spaces is *not* cartesian closed; in particular, products of topological spaces do not in general behave well with respect to colimits, in contrast to the case of simplicial sets. To remedy this, one works in a 'convenient category of spaces'. For us this will be the category of compactly generated weak Hausdorff spaces, which includes all CW complexes and *is* cartesian closed. The product of two such spaces X and Y agrees with the usual product of topological spaces in the case that both are compact. For general compactly generated weak Hausdorff spaces X and Y, one retopologizes the product $X \times Y$ with the *compactly generated topology*, for which a subset A is open precisely if its intersection with every compact subset $K \subset X \times Y$ is open in K. From now **Top** will always refer to this category of compactly generated weak Hausdorff spaces. Geometric realization obviously takes values in these compactly generated weak Hausdorff spaces. Hence with this new interpretation of **Top**, geometric realization is still left adjoint to the functor Sing. We will prove the following result:

Proposition 2.6 *The geometric realization functor*

$$|\cdot| : \mathbf{sSets} \to \mathbf{Top}$$

preserves finite limits.

To prove that a functor preserves finite limits it suffices to show it preserves finite products and equalizers. The fact that geometric realization preserves equalizers is rather easy to show (see Lemma 2.7) and clearly it preserves the empty product, i.e. the terminal object, because $|\Delta[0]| \simeq \Delta^0$. Lemma 2.8 will show that it preserves binary products of simplices. From this it follows that $|X \times Y| \simeq |X| \times |Y|$ for general simplicial sets X and Y; indeed, one expresses X and Y as colimits of simplices, uses that the left adjoint functor geometric realization preserves colimits and finally the fact that the product in (our new interpretation of) **Top** preserves colimits in each variable separately. This last step is why using a convenient category of spaces is necessary.

Lemma 2.7 *The geometric realization functor preserves equalizers.*

Proof If X is a simplicial set and $E \subseteq X$ a simplicial subset, then $|E|$ is a subcomplex of $|X|$, considered as a CW complex as in Theorem 2.3. In particular, the topology of $|E|$ is the subspace topology inherited from $|X|$. Thus it suffices to show that if

$$E \longrightarrow X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

is an equalizer of simplicial sets, then the resulting diagram

$$|E| \longrightarrow |X| \underset{|g|}{\overset{|f|}{\rightrightarrows}} |Y|$$

is an equalizer of sets. It suffices to show that if $x \otimes t$ is a point of $|X|$ such that $f(x) \otimes t = g(x) \otimes t$ in $|Y|$, then $f(x) = g(x)$. We may assume that $x \otimes t$ is in the form described in Proposition 2.4, so that x is a non-degenerate n-simplex and t is in the interior of Δ^n when $n \geq 1$. As explained at the beginning of Section 2.3, there is a unique non-degenerate k-simplex y of Y and a surjection $\alpha : [n] \to [k]$ such that $f(x) = \alpha^* y$. Similarly, there is a unique non-degenerate l-simplex z and a surjection $\beta : [n] \to [l]$ with $g(x) = \beta^* z$. Then we have

$$f(x) \otimes t = y \otimes \alpha_* t = z \otimes \beta_* t = g(x) \otimes t$$

and by the uniqueness of representatives in Proposition 2.4 this implies $y = z$ (in particular $k = l$) and $\alpha_* t = \beta_* t$. But one easily checks that if the two surjective maps $\alpha_*, \beta_* : \Delta^n \to \Delta^k$ agree on an interior point t, then $\alpha = \beta$. It follows that $f(x) = \alpha^* y = \beta^* z = g(x)$, as was to be shown. □

Lemma 2.8 *The natural map*

$$|\Delta[n] \times \Delta[m]| \to |\Delta[n]| \times |\Delta[m]|$$

is a homeomorphism.

Proof We write the points of $\Delta^n = |\Delta[n]|$ as convex linear combinations of its vertices:

$$t_0 v_0 + \cdots + t_n v_n, \quad \sum_i t_i = 1, \quad t_i \geq 0.$$

Similarly we denote the points of Δ^m by

$$s_0 w_0 + \cdots + s_m w_m.$$

Then $\Delta^n \times \Delta^m$ is the space of all such pairs

$$(t_0 v_0 + \cdots + t_n v_n, s_0 w_0 + \cdots + s_m w_m).$$

On the other hand, each shuffle map $\tau : [n+m] \to [n] \times [m]$ defines an embedding

$$|\tau| : \Delta^{n+m} \to \Delta^n \times \Delta^m$$

sending $(r_0, \ldots, r_{n+m})$ to the convex combination

$$\sum_i r_i (v_{\tau_1(i)}, v_{\tau_2(i)})$$

where τ_1 and τ_2 are the components of τ. Now $|\Delta[n] \times \Delta[m]|$ is the colimit of these embeddings, glued together along their intersections as discussed above. It suffices to check that the resulting map

$$T : \bigcup_\tau \Delta^{n+m} \to \Delta^n \times \Delta^m$$

is a bijection, because the spaces involved are compact Hausdorff.

To see that T is injective, let τ and σ be two shuffle maps and suppose that

$$|\tau|(r_0, \ldots, r_{n+m}) = |\sigma|(r'_0, \ldots, r'_{n+m}),$$

so that

$$r_i(v_{\tau_1(i)}, w_{\tau_2(i)}) = r'_i(v_{\sigma_1(i)}, w_{\sigma_2(i)})$$

for each $i = 0, \ldots, n+m$. Since the vertices $(v_i, w_j) \in \Delta^n \times \Delta^m$ are linearly independent this can only happen if $r_i = r'_i$ for all i and $\tau(i) = \sigma(i)$ whenever $r_i \neq 0$. Let $\rho : [k] \to [n+m]$ enumerate those i for which $r_i \neq 0$. Then $\tau \circ \rho = \sigma \circ \rho$ and

$$|\tau|(r_0, \ldots, r_{n+m}) = |\tau \circ \rho|(r_{\rho(0)}, \ldots, r_{\rho(k)}) = |\sigma|(r'_0, \ldots, r'_{n+m}).$$

Therefore the points $\tau \otimes (r_0, \ldots, r_{n+m})$ and $\sigma \otimes (r'_0, \ldots, r'_{n+m})$ are already identified in $|\Delta[n] \times \Delta[m]|$, proving injectivity.

To prove surjectivity of T, we should check that every point of $\Delta^n \times \Delta^m$ lies in an $n+m$-simplex spanned by the vertices $(v_{\tau_1(i)}, w_{\tau_2(i)})$ enumerated by a shuffle map $\tau : [n+m] \to [n] \times [m]$. We work by induction on $n+m$, noting that the cases $n+m \leq 1$ are trivial. Consider a point

$$(x, y) = (t_0 v_0 + \cdots + t_n v_n, s_0 w_0 + \cdots + s_m w_m)$$

of $\Delta^n \times \Delta^m$. If $t_n \leq s_m$, we can write it as

$$t_n(v_n, w_m) + (t_0 v_0 + \cdots + t_{n-1} v_{n-1}, s_0 w_0 + \cdots + (s_m - t_n) w_m) = t_n(v_n, w_m) + (x', y').$$

In this case we set $\tau(n+m) = (n,m)$ and $\tau(n+m-1) = (n-1,m)$. Note that if $t_n = s_m = 1$ then $(x,y) = (v_n, w_m)$, which is clearly in the image of φ. Therefore assume $t_n < 1$. The point

$$(x'', y'') := \frac{1}{1-t_n}(x', y')$$

lies in the product of simplices $\Delta^{n-1} \times \Delta^m$ spanned by the vertices (v_i, w_j) for $i = 0, \dots, n-1$ and $j = 0, \dots, m$. By the inductive hypothesis there is a shuffle map $\tau' : [n+m-1] \to [n-1] \times [m]$ such that the image of the map $|\tau'|$ contains (x', y'). In other words, there exist coefficients r_i with $\sum_i r_i = 1$ such that

$$(x'', y'') = \sum_{i=0}^{n+m-1} r_i (v_{\tau_1'(i)}, w_{\tau_2'(i)}).$$

Extend the definition of τ by $\tau(i) = \tau'(i)$ for $i \le n+m-1$. Then

$$\begin{aligned}(x,y) &= t_n(v_n, w_m) + (1-t_n) \sum_{i=0}^{n+m-1} r_i(v_{\tau_1'(i)}, w_{\tau_2'(i)}) \\ &= \sum_{i=0}^{n+m} r_i'(v_{\tau_1(i)}, w_{\tau_2(i)})\end{aligned}$$

with $r_i' = (1-t_n)r_i$ for $i < n+m$ and $r_{n+m} = t_n$, proving that (x,y) is in the image of φ. If $t_n > s_m$ then one sets $\tau(n+m-1) = (n, m-1)$ and proceeds similarly. □

We conclude this section with a well-known consequence of the fact that geometric realization preserves products.

Corollary 2.9 *Consider functors $F, G : \mathbf{C} \to \mathbf{D}$ between small categories and a natural transformation $\nu : F \to G$ between them. Then ν induces a homotopy*

$$|N\nu| : \Delta^1 \times B\mathbf{C} \to B\mathbf{D}$$

between the corresponding maps of classifying spaces.

Proof The natural transformation ν can be seen as functor

$$\nu : (0 \to 1) \times \mathbf{C} \to \mathbf{D},$$

which upon applying the nerve functor (which preserves products, being a right adjoint) gives

$$N\nu : \Delta[1] \times N\mathbf{C} \to N\mathbf{D}.$$

Applying geometric realization and using that it preserves products gives a map

$$|N\nu| : \Delta^1 \times B\mathbf{C} \to B\mathbf{D}.$$ □

In particular, considering the unit and counit of an adjoint pair of functors gives the following:

Corollary 2.10 *An adjoint pair of functors between categories* $\mathbf{C}$ *and* $\mathbf{D}$ *induces a homotopy equivalence between the classifying spaces* $B\mathbf{C}$ *and* $B\mathbf{D}$*.*

2.6 Simplicial Spaces and Bisimplicial Sets

As mentioned at the start of Section 2.2, one can define simplicial objects in any category $\mathcal{E}$ as functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{E}$. The cases where $\mathcal{E}$ is the category of spaces or of simplicial sets itself frequently occur in the literature and we will need some general facts and constructions for these.

2.6.1 Simplicial Spaces

A *simplicial space* is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$. In other words, X is a simplicial set equipped with a topology on each X_n for which all the face maps $d_i : X_n \to X_{n-1}$ and degeneracy maps $s_j : X_{n-1} \to X_n$ are continuous. With natural transformations between them, they form a category for which we write **sTop**.

For example, if X is a simplicial set and T a topological space, one can define a simplicial space $X \otimes T$ by setting

$$(X \otimes T)_n = X_n \times T,$$

giving X_n the discrete topology and $X_n \times T$ the product topology, while defining the face and degeneracy maps in the obvious way.

A second example, perhaps one of the most important, is the following. If G is a topological group, its nerve NG is naturally a simplicial space. Recall from Section 2.4 that as a simplicial set it is defined by $(NG)_n = G^n$, which we can now equip with the product topology. Its face and degeneracy maps are

$$d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0, \\ (g_1, \ldots, g_{i+1}g_i, \ldots, g_n) & \text{if } 0 < i < n, \\ (g_0, \ldots, g_{n-1}) & \text{if } i = n, \end{cases}$$

and

$$s_j(g_1, \ldots, g_{n-1}) = (g_1, \ldots, g_j, 1, g_{j+1}, \ldots, g_{n-1}).$$

For a simplicial space X one can define its *geometric realization* $|X|$ by exactly the same formula as for simplicial sets, again describing it as a quotient space of

$$\coprod_{n \geq 0} X_n \times \Delta^n,$$

except that now each X_n is a topological space (rather than just a set). In this way one obtains a functor

$$| \cdot | : \mathbf{sTop} \to \mathbf{Top},$$

which again preserves colimits. In the first example above, for a simplicial set X and a topological space T, one has

$$|X \otimes T| \simeq |X| \times T.$$

For a topological group G as in our second example one defines

$$BG := |N(G)|,$$

the *classifying space* of G.

The homotopical properties of this realization functor have been widely discussed in the literature and we will address some aspects of this in Chapter 8.

2.6.2 Bisimplicial Sets

A *bisimplicial set* is a simplicial object in the category of simplicial sets, i.e. a functor $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{sSets}$ or equivalently a functor

$$X : (\mathbf{\Delta} \times \mathbf{\Delta})^{\mathrm{op}} \longrightarrow \mathbf{Sets}.$$

Thus, such an X is given by a collection of sets $X_{p,q}$ for $p, q \geq 0$ together with 'horizontal' and 'vertical' face and degeneracy maps:

$$\begin{tikzcd} X_{p,q} \arrow[r, shift left, "d_i^h"] \arrow[d, shift right, "d_i^v"'] & X_{p-1,q} \arrow[l, shift left, "s_j^h"] \\ X_{p,q-1} . \arrow[u, shift right, "s_j^v"'] & \end{tikzcd}$$

These horizontal and vertical operations should satisfy the simplicial identities, while 'horizontal' and 'vertical' commute with one another. For instance,

$$d_i^h d_j^h = d_{j-1}^h d_i^h \quad (i < j), \quad d_i^h d_j^v = d_j^v d_i^h, \quad \text{etc.}$$

Bisimplicial sets and natural transformations between them form a category denoted **bisSets**. Applying the general facts about presheaf categories from Section 2.4, we find that the diagonal functor

$$\delta : \mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$$

induces a triple of adjoint functors

$$\begin{tikzcd} \mathbf{sSets} \arrow[r, bend left, "\delta_!"] \arrow[r, bend right, "\delta_*"'] & \mathbf{bisSets}. \arrow[l, "\delta^*"'] \end{tikzcd}$$

The functor δ^* is usually referred to as the diagonal and for a bisimplicial set X it gives

$$(\delta^* X)_n = X_{n,n}.$$

The functor $\delta_!$ is essentially uniquely determined by the fact that it preserves colimits and sends representables to representables according to δ. If we write

$$\mathbf{\Delta}[p,q] = (\mathbf{\Delta} \times \mathbf{\Delta})(-, ([p],[q]))$$

for the bisimplicial set represented by $([p],[q]) \in \mathbf{\Delta} \times \mathbf{\Delta}$, then

$$\delta_! \Delta[n] = \Delta[n,n].$$

Any two simplicial sets X and Y define a bisimplicial set $X \boxtimes Y$, their *external product*, by

$$(X \boxtimes Y)_{p,q} := X_p \times Y_q.$$

For example, the representable bisimplicial set $\Delta[p,q] = \Delta[p] \boxtimes \Delta[q]$ is such an external product. Notice that δ^* maps the external product to the ordinary product, i.e.

$$\delta^*(X \boxtimes Y) = X \times Y.$$

Any bisimplicial set X gives rise to two simplicial spaces, obtained by geometric realization in the horizontal and vertical direction respectively. The horizontal one

$$|X|_q^{(h)} = |X_{\bullet,q}|$$

is obtained by taking the geometric realization of the qth row for each fixed q and the vertical one

$$|X|_p^{(v)} = |X_{p,\bullet}|$$

is similarly obtained by realizing the columns. One can next realize these simplicial spaces to obtain spaces

$$||X|_\bullet^{(h)}| \quad \text{and} \quad ||X|_\bullet^{(v)}|.$$

It is a simple and very useful observation that these two spaces are naturally homeomorphic and furthermore coincide with a third way of 'realizing' X, namely by taking the geometric realization of the diagonal. We record this result as follows:

Proposition 2.11 *There are natural homeomorphisms*

$$||X|_\bullet^{(h)}| \simeq |\delta^* X| \simeq ||X|_\bullet^{(v)}|.$$

Proof Observe that the three functors involved are colimit preserving functors **bisSets** $\to$ **Top**, so it suffices to show that they are naturally isomorphic on representables, i.e. that there are natural homeomorphisms

$$||\Delta[p,q]|_\bullet^{(h)}| \simeq |\delta^*\Delta[p,q]| \simeq ||\Delta[p,q]|_\bullet^{(v)}|.$$

Recall that $\Delta[p,q] = \Delta[p] \boxtimes \Delta[q]$. But for general simplicial sets Y and Z one has natural identifications

$$|Y \boxtimes Z|^{(v)} = Y \otimes |Z| \quad \text{and} \quad |Y \boxtimes Z|^{(h)} = |Y| \otimes Z$$

and so by the observation at the beginning of this section also

$$||Y \boxtimes Z|^{(v)}| = |Y| \times |Z| \quad \text{and} \quad ||Y \boxtimes Z|^{(h)}| = |Y| \times |Z|.$$

Finally, we have

$$|\delta^*(Y \boxtimes Z)| = |Y \times Z| \simeq |Y| \times |Z|$$

since geometric realization preserves products. We conclude by setting $Y = \Delta[p]$ and $Z = \Delta[q]$. $\square$

For a bisimplicial set X we will sometimes write $||X||$ for $|\delta^* X|$ and refer to it as *the* geometric realization of X. The observations above are useful when studying the classifying spaces of *simplicial groups*. Such a simplicial group is of course a functor $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Gp}$ or equivalently a simplicial set G with a group structure on each G_n such that the face and degeneracy maps of G are group homomorphisms between them. Taking the nerve of each G_n (where one regards a group as a category with one object) yields a bisimplicial set,

$$NG_{p,q} = N(G_p)_q = \underbrace{G_p \times \cdots \times G_p}_{q \text{ times}}$$

and its *classifying space* is the geometric realization,

$$BG = ||NG||.$$

In particular, by the proposition above, BG coincides with the classifying space of the topological group $|G|$. (Note that $|G|$ indeed inherits a group structure from G, because geometric realization preserves products.)

2.7 Simplicial Categories and Simplicial Operads

We encountered some examples of topological categories and topological operads in Chapter 1. These come up naturally when describing algebraic structures in homotopy theory (e.g. the $\mathbf{E}_n$-operads for n-fold loop spaces) and when one wishes to describe structures 'up to coherent homotopy' (by means of the W-resolution, for example). In this section we replace topological spaces by simplicial sets and discuss the resulting notions.

2.7.1 Internal Versus Enriched Categories and Operads

For a category $\mathcal{E}$ with pullbacks there is a notion of *internal category in* $\mathcal{E}$ or a *category object in* $\mathcal{E}$. Such an internal category $\mathbf{C}$ is given by two objects $\mathrm{ob}(\mathbf{C})$ and $\mathrm{ar}(\mathbf{C})$, the 'object of objects' of $\mathbf{C}$ and the 'object of arrows' of $\mathbf{C}$, with structure maps for domain, codomain, identity morphisms and composition in $\mathbf{C}$, together making up a diagram in $\mathcal{E}$ of the form

$$\mathrm{ar}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} \mathrm{ar}(\mathbf{C}) \longrightarrow \mathrm{ar}(\mathbf{C}) \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \mathrm{ob}(\mathbf{C})$$

satisfying the usual equations for a category. A morphism $f : \mathbf{C} \to \mathbf{D}$ between two such internal categories, also called an *internal functor*, consists of two morphisms $f_{\mathrm{ob}} : \mathrm{ob}(\mathbf{C}) \to \mathrm{ob}(\mathbf{D})$ and $f_{\mathrm{ar}} : \mathrm{ar}(\mathbf{C}) \to \mathrm{ar}(\mathbf{D})$ satisfying the usual equations. In this way we obtain a category of internal categories in $\mathcal{E}$.

Each such internal category $\mathbf{C}$ gives rise to a simplicial object $N\mathbf{C}$ in $\mathcal{E}$, its nerve, by means of pullbacks in $\mathcal{E}$. One sets

$$N\mathbf{C}_n := \underbrace{\mathrm{ar}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} \cdots \times_{\mathrm{ob}(\mathbf{C})} \mathrm{ar}(\mathbf{C})}_{n \text{ times}}$$

for $n \geq 1$ and

$$N\mathbf{C}_0 := \mathrm{ob}(\mathbf{C}).$$

A closely related notion is that of a category *enriched in* $\mathcal{E}$, or briefly an $\mathcal{E}$-*category*. An $\mathcal{E}$-category $\mathbf{C}$ consists of a collection of objects $\mathrm{ob}(\mathbf{C})$ and for any two objects x, y of $\mathbf{C}$ an object $\mathbf{C}(x, y)$ of $\mathcal{E}$, the 'object of arrows' from x to y. Furthermore, there are structure maps

$$\mathbf{C}(y, z) \times \mathbf{C}(x, y) \to \mathbf{C}(x, z)$$

for composition and $1 \to \mathbf{C}(x, x)$ for identities, where 1 denotes the terminal object of $\mathcal{E}$. A morphism between such $\mathcal{E}$-categories $f : \mathbf{C} \to \mathbf{D}$ consists of a map $f : \mathrm{ob}(\mathbf{C}) \to \mathrm{ob}(\mathbf{D})$ and for each $x, y \in \mathrm{ob}(\mathbf{C})$ a morphism $\mathbf{C}(x, y) \to \mathbf{D}(f(x), f(y))$ of $\mathcal{E}$, compatible with composition and identities. In this way one obtains a category of $\mathcal{E}$-categories.

As we have just described it the notion of enriched category is more restrictive than that of internal category: if $\mathcal{E}$ has coproducts which distribute over products, one can view an enriched category $\mathbf{C}$ as an internal category in which the object of objects is a coproduct of copies of the terminal object indexed by the set $\mathrm{ob}(\mathbf{C})$. On the other hand, the definition of enriched category only uses products, not pullbacks, and in fact makes sense if $\mathcal{E}$ is any monoidal category: in the definition one simply replaces the product and terminal object by the tensor and unit of the monoidal structure.

Exactly the same dichotomy applies to operads. For $\mathcal{E}$ with pullbacks, there is a notion of internal operad $\mathbf{P}$ in $\mathcal{E}$, given by an object of colours C and for each $n \geq 0$ an object of operations $\mathbf{P}_n$. The structure maps are $\mathbf{P}_n \to C^n \times C$ (for domain and codomain), an action of the symmetric group Σ_n on $\mathbf{P}_n$, and maps for unit and composition

$$\begin{aligned} \mathbf{P}_n \times_{C^n} (\mathbf{P}_{k_1} \times \cdots \times \mathbf{P}_{k_n}) &\to \mathbf{P}_{k_1+\cdots+k_n}, \\ C &\to \mathbf{P}_1, \end{aligned}$$

all satisfying equations we leave for the reader to spell out.

On the other hand, an operad $\mathbf{P}$ *enriched in* $\mathcal{E}$ (or more simply an $\mathcal{E}$*-operad*) consists of a *set* of colours C and for each sequence of colours $(c_1, \ldots, c_n; c)$ an object $\mathbf{P}(c_1, \ldots, c_n; c)$ of $\mathcal{E}$, all equipped with structure maps for composition, symmetries and units, much like the definition of operad we gave in Chapter 1.

Thus, when one speaks of topological or simplicial operads or categories, it is a priori not clear whether one is referring to the internal or the enriched notion. In this book *we will always mean the enriched notion*, unless explicitly stated otherwise, which is also fairly standard in the literature. This is consistent with our use of the phrases *topological category* and *topological operad* in Chapter 1. Note that the ambiguity disappears when the object of objects is the terminal object of $\mathcal{E}$, as is the case for topological or simplicial groups, monoids, or operads with a single colour.

Remark 2.12 Another valid interpretation of the phrase 'simplicial category' would have it be a simplicial object in **Cat**. We leave it to the reader to verify that this is essentially the same thing as a category internal to simplicial sets.

2.7.2 Simplicial Categories

According to the convention above, a simplicial category $\mathbf{C}$ is given by a set of objects $\mathrm{ob}(\mathbf{C})$ and for any two $x, y \in \mathrm{ob}(\mathbf{C})$ a simplicial set $\mathbf{C}(x, y)$ of arrows from x to y. Furthermore, there are identity elements $\mathrm{id}_x \in \mathbf{C}(x, x)_0$ and maps of simplicial sets $\mathbf{C}(y, z) \times \mathbf{C}(x, y) \to \mathbf{C}(x, z)$ for composition. The equations for these structure maps come down to the requirement that for each fixed n, one has a category $\mathbf{C}_n$, all having *the same* set of objects $\mathrm{ob}(\mathbf{C})$, with the face and degeneracy operators between the various $\mathbf{C}_n$ being the identity on objects. With morphisms of simplicial categories as defined above we obtain a category which we denote by **sCat**.

The nerve of a simplicial category gives a bisimplicial set $N\mathbf{C}$ with

$$N\mathbf{C}_{p,q} = N(\mathbf{C}_p)_q = \coprod_{x_0,\ldots,x_q} \mathbf{C}_p(x_0, x_1) \times \cdots \times \mathbf{C}_p(x_{q-1}, x_q)$$

for $q > 0$ and

$$N\mathbf{C}_{p,0} = \mathrm{ob}(\mathbf{C}_p) = \mathrm{ob}(\mathbf{C}),$$

i.e. this bisimplicial set is *constant* (or *discrete*) in its bottom row. Another relevant construction is the following: one can take the geometric realizations $|\mathbf{C}(x, y)|$ of all the objects of arrows to obtain a topological category $|\mathbf{C}|$. (The reason that this is still naturally a category is, again, that geometric realization preserves products.) We can define the *classifying space* of $\mathbf{C}$ in the evident way and express it in the following two ways, cf. Proposition 2.11:

$$B\mathbf{C} = ||N\mathbf{C}|| \simeq |N(|\mathbf{C}|)|.$$

2.7.3 Boardman–Vogt Resolution

For a simplicial category $\mathbf{C}$ one can construct its *Boardman–Vogt resolution* $W\mathbf{C}$ exactly as in the topological case, now using the representable 1-simplex $\Delta[1]$ instead of the topological unit interval $[0, 1]$. Indeed, the construction essentially only uses the elements 0 and 1 of $[0, 1]$ together with the supremum operation $\vee : [0, 1] \times [0, 1] \to [0, 1]$. These are now replaced by the vertices $0, 1 : \Delta[0] \to \Delta[1]$ (which correspond to ∂_1 and ∂_0 respectively) and the map

$$\vee : \Delta[1] \times \Delta[1] \to \Delta[1]$$

corresponding to the map of partially ordered sets $\vee : [1] \times [1] \to [1]$ taking the supremum of a pair. Since this is really all the structure we need, we will discuss the W-resolution slightly more generally, with respect to an arbitrary *interval object*. For us, this will be a simplicial set I together with maps

$$+ : \Delta[0] \to I \quad \text{and} \quad - : \Delta[0] \to I$$

and an associative operation $\vee : I \times I \to I$ for which $-$ is a unit and $+$ is absorbing, in the sense that $x \vee + = + = + \vee x$. One could define such intervals in more general monoidal categories as well (and carry out much of what we do in this section), but we will remain in the relatively explicit setting of simplicial sets. The most important examples of intervals we have in mind are the following:

Example 2.13 (i) The representable $\Delta[1]$, with $- = 0$ and $+ = 1$ and $\vee$ being the supremum, as described above.

(ii) The opposite of the previous example, with again $I = \Delta[1]$ but now $- = 1$ and $+ = 0$, with $\vee$ taking the infimum rather than supremum.

(iii) The nerve J of the 'free isomorphism'. To be precise, write $\mathbf{J}$ for the category with objects $-$ and $+$ together with morphisms $t : - \to +$ and $s : + \to -$ satisfying the relations $ts = \mathrm{id}_+$ and $st = \mathrm{id}_-$. Thus, $\mathbf{J}$ consists of two objects and an isomorphism between them. One then defines $J = N\mathbf{J}$. The geometric realization of J is homeomorphic to the infinite-dimensional sphere. Indeed, J has two nondegenerate simplices of dimension n for every $n \geq 0$, given by the alternating sequences $sts\cdots$ and $tst\cdots$ of length n. Thus, $|J|$ has a CW structure with precisely two cells in each dimension. It is therefore clear that $|J|^{(0)} = S^0$ and more generally (by induction) one sees that $|J|^{(n)}$ is the n-sphere S^n with its 'equatorial' cell structure, i.e. the one consisting of two closed n-cells glued along their common boundary S^{n-1}. One makes J into an interval object in the same way as for $\Delta[1]$, setting $- = 0$ and $+ = 1$ and taking the supremum operation. This makes the evident inclusion $\Delta[1] \to J$ into a morphism of intervals.

Let us now describe the simplicial category $W\mathbf{C}$ in detail. Its objects are the same as those of $\mathbf{C}$. For any two objects x, y of $\mathbf{C}$ we construct $W\mathbf{C}(x, y)$ by induction as a filtered simplicial set,

$$W\mathbf{C}(x, y)^{(0)} \subseteq W\mathbf{C}(x, y)^{(1)} \subseteq \cdots, \quad \bigcup_n W\mathbf{C}(x, y)^{(n)} = W\mathbf{C}(x, y).$$

We start with $W\mathbf{C}(x, y)^{(0)} := \mathbf{C}(x, y)$. At each stage, $W\mathbf{C}(x, y)^{(n)}$ will come equipped with a map

$$\coprod_{x_1,\ldots,x_n} I^n \times \mathbf{C}(x, x_1) \times \cdots \times \mathbf{C}(x_n, y) \xrightarrow{\xi^{(n)}} W\mathbf{C}(x, y)^{(n)}$$

which is the identity for $n = 0$. One constructs $W\mathbf{C}(x, y)^{(n)}$ and the map $\xi^{(n)}$ from $W\mathbf{C}(x, y)^{(n-1)}$ and $\xi^{(n-1)}$ by means of the pushout square of simplicial sets

$$\begin{tikzcd} A \arrow[r, "\zeta"] \arrow[d] & W\mathbf{C}(x, y)^{(n-1)} \arrow[d] \\ \coprod_{x_1,\ldots,x_n} I^n \times \mathbf{C}(x, x_1) \times \cdots \times \mathbf{C}(x_n, y) \arrow[r, "\xi^{(n)}"'] & W\mathbf{C}(x, y)^{(n)}. \end{tikzcd}$$

Here A is the simplicial subset of the bottom left corner consisting of those elements which have $-$ in one of the n entries of I^n or an identity arrow of $\mathbf{C}$ in one of the slots $\mathbf{C}(x_i, x_{i+1})$. More explicitly, A is the coproduct of simplicial subsets of two types; the first are of the form

$$I^{i-1} \times \Delta[0] \times I^{n-i} \times \mathbf{C}(x, x_1) \times \cdots \times \mathbf{C}(x_n, y),$$

which are included by means of the map $- : \Delta[0] \to I$ and which map to $W\mathbf{C}(x, y)^{(n-1)}$ by using the composition

$$\mathbf{C}(x_{i-1}, x_i) \times \mathbf{C}(x_i, x_{i+1}) \to \mathbf{C}(x_{i-1}, x_{i+1})$$

and the evident identification

$$I^{i-1} \times \Delta[0] \times I^{n-i} \simeq I^{n-1},$$

and then composing these with $\xi^{(n-1)}$. The subsets of the second type are of the form

$$I^n \times \mathbf{C}(x, x_1) \times \cdots \times \{\mathrm{id}_{x_i}\} \times \mathbf{C}(x_{i+1}, x_{i+2}) \times \cdots \times \mathbf{C}(x_n, y),$$

which map to

$$I^{n-1} \times \mathbf{C}(x, x_1) \times \cdots \times \mathbf{C}\widehat{(x_i, x_{i+1})} \times \cdots \times \mathbf{C}(x_n, y)$$

by using the map

$$I^n \simeq I^{i-1} \times I^2 \times I^{n-i-1} \xrightarrow{\mathrm{id} \times \vee \times \mathrm{id}} I^{i-1} \times I \times I^{n-i-1} \simeq I^{n-1}$$

and then to $W\mathbf{C}(x, y)^{(n-1)}$ by composing with $\xi^{(n-1)}$ again.

This construction mimics the identifications made in the topological case and gives a well-defined simplicial category $W\mathbf{C}$. The composition in $W\mathbf{C}$ is uniquely determined by the fact that each diagram of the form

$$\begin{tikzcd}
(I^n \times \prod_{i=0}^{n} \mathbf{C}(x_i, x_{i+1})) \times (I^m \times \prod_{j=0}^{m} \mathbf{C}(y_j, y_{j+1})) \arrow[r] \arrow[d] & W\mathbf{C}(x, y)^{(n)} \times W\mathbf{C}(y, z)^{(m)} \arrow[d] \\
I^n \times I^m \times \prod_{k=0}^{n+m+1} \mathbf{C}(z_k, z_{k+1}) \arrow[d, "u \times \mathrm{id}"'] & W\mathbf{C}(x, z)^{(n+m+1)} \\
I^{n+m+1} \times \prod_{k=0}^{n+m+1} \mathbf{C}(z_k, z_{k+1}) \arrow[ur, "\xi^{(n+m+1)}"'] &
\end{tikzcd}$$

commutes, where $x_0 = x$, $x_{n+1} = y = y_0$, $y_{m+1} = z$. Also, the sequence of z_k's is the sequence $x_0, \ldots, x_{n+1} = y_0, \ldots, y_{m+1}$ and

$$u : I^n \times I^m \simeq I^n \times \Delta[0] \times I^m \to I^n \times I \times I^m \simeq I^{n+m+1}$$

inserts $+$ in the $(n+1)$st coordinate.

2.7.4 Homotopy-Coherent Nerve

As for the topological Boardman–Vogt resolution, the case where $\mathbf{C}$ is the free category

$$[n] = (0 \to 1 \to \cdots \to n)$$

(regarded as a discrete simplicial category) is much easier to describe. In this section we take $I = \Delta[1]$ with $- = 0$ and $+ = 1$, as in Example 2.13(i). Explicitly, $W[n]$ has the same objects as $[n]$ and

$$W[n](i, j) = \Delta[1]^{j-i-1}$$

for $i \leq j$, with the convention that $\Delta[1]^{-1} = \Delta[0]$. (As before, we heuristically think of an arrow $i \longrightarrow j$ in $W[n]$ as a sequence of waiting times on the objects $i+1, \ldots, j-1$ in a virtual composition $i \to i+1 \to \cdots \to j-1 \to j$, so there are $j-i-1$ such waiting times.) Composition

$$W[n](i, j) \times W[n](j, k) \longrightarrow W[n](i, k)$$

is given by the map which inserts 1 in the appropriate slot, as in the following diagram:

$$\begin{tikzcd}
\Delta[1]^{j-i-1} \times \Delta[1]^{k-j-1} \arrow[r, equal] \arrow[d] & \Delta[1]^{j-i-1} \times \Delta[0] \times \Delta[1]^{k-j-1} \arrow[d, "\mathrm{id}\times\partial_0\times\mathrm{id}"] \\
\Delta[1]^{k-i-1} \arrow[r, equal] & \Delta[1]^{j-i-1} \times \Delta[1] \times \Delta[1]^{k-j-1}.
\end{tikzcd}$$

This construction gives a functor

$$W : \mathbf{\Delta} \longrightarrow \mathbf{sCat}.$$

By the general procedure of left Kan extension it induces an adjoint pair

$$w_! : \mathbf{sSets} \rightleftarrows \mathbf{sCat} : w^*.$$

Explicitly, for a simplicial category $\mathbf{C}$, the simplicial set $w^*\mathbf{C}$ is defined by

$$w^*\mathbf{C}_n = \mathrm{Hom}(W[n], \mathbf{C}),$$

where Hom is the set of functors of simplicial categories. This simplicial set $w^*\mathbf{C}$ is called the *homotopy-coherent nerve* of $\mathbf{C}$. If $\mathbf{C}$ happens to be a discrete simplicial category, it agrees with the usual nerve $N\mathbf{C}$. The left adjoint $w_!$ is uniquely determined (up to isomorphism) by the fact that it preserves colimits and agrees with W on representables, i.e.

$$w_!(\Delta[n]) = W[n].$$

In the second part of this book we will explain in what sense this adjoint pair gives an equivalence of homotopy theories between simplicial sets and simplicial categories.

Remark 2.14 The adjoint pair $(w_!, w^*)$ is commonly denoted $(\mathfrak{C}, N)$ in the literature, but we will not use this notation.

2.7.5 Simplicial Operads

Our discussion of simplicial categories has a parallel for simplicial operads. A simplicial operad $\mathbf{P}$ is given by a set of colours C and for each sequence $c_1, \ldots, c_n, c$ of colours a simplicial set $\mathbf{P}(c_1, \ldots, c_n; c)$, thought of as the simplicial set of operations from $c_1, \ldots, c_n$ to c. Furthermore, there are composition maps and symmetric group actions, as well as units $q \in \mathbf{P}(c,c)_0$ for each colour c, similar to the cases of operads in **Sets** and **Top** discussed in the first chapter. In particular, for each simplicial degree q, there is an operad $\mathbf{P}_q$ in **Sets** with the same set of colours C and the sets $\mathbf{P}(c_1, \ldots, c_n; c)_q$ as operations. Moreover, the simplicial face and degeneracy operators give morphisms of operads $\mathbf{P}_q \to \mathbf{P}_{q-1}$ and $\mathbf{P}_{q-1} \to \mathbf{P}_q$ respectively. A morphism of simplicial operads $\mathbf{P} \to \mathbf{Q}$ consists of a function $\varphi : C \to D$ between the respective sets of colours of $\mathbf{P}$ and $\mathbf{Q}$ and a family of morphisms of simplicial sets (all denoted φ again)

$$\varphi : \mathbf{P}(c_1, \ldots, c_n; c) \to \mathbf{Q}(\varphi(c_1), \ldots, \varphi(c_n); \varphi(c)),$$

one such for each sequence of colours $c_1, \ldots, c_n, c$ of $\mathbf{P}$. These morphisms are required to be compatible with composition, symmetries and units. A different way of describing such a morphism is as a collection $\varphi_q : \mathbf{P}_q \to \mathbf{Q}_q$ of morphisms of operads in **Sets**, natural in q with respect to face and degeneracy operators. In this way one obtains a category of simplicial operads, which we denote by **sOp**.

Since geometric realization preserves products, each such simplicial operad $\mathbf{P}$ yields a topological operad with the same set of colours C and spaces of operations $|\mathbf{P}(c_1, \ldots, c_n; c)|$. In fact, this defines a functor

$$|\cdot| : \mathbf{sOp} \to \mathbf{Op}_{\mathbf{Top}}$$

between the categories of simplicial and topological operads. When compared to the discussion of simplicial categories above, the parallel seems to stop here, since we do not have a nerve functor for operads at our disposal which parallels the nerve functor for categories. This gap will be filled in the next chapter and is in fact a major theme of this book.

Finally, we note that any simplicial operad $\mathbf{P}$ of course 'contains' a simplicial category $j^*\mathbf{P}$ with the set C as its set of objects and the simplicial sets $\mathbf{P}(c; d)$ as morphisms from c to d. Conversely, any simplicial category $\mathbf{C}$ can be regarded as a simplicial operad (which we denote $j_!\mathbf{C}$) with only unary operations, i.e. one for which the simplicial sets $j_!\mathbf{C}(c_1, \ldots, c_n; c)$ are empty unless $n = 1$. This procedure is easily seen to define an adjoint pair

$$j_! : \mathbf{sCat} \rightleftarrows \mathbf{sOp} : j^*.$$

2.7.6 The Barratt–Eccles Operad

In this section we discuss an important example of a simplicial operad, the so-called *Barratt–Eccles operad*. As before we write Σ_n for the symmetric group on n letters, for $n \geq 0$. Every group G (or in fact every monoid) gives rise to a category usually denoted EG, whose objects are the elements of G and where an arrow $g \to h$ is an element $k \in G$ with $g = hk$. Of course for groups this element k is unique, namely $k = h^{-1}g$. Alternatively, viewing G as a category with one object $*$ and the elements of G as morphisms, EG is simply the slice category $G/*$. Then the identity of the object $*$ is a terminal object of EG, so that its classifying space $|N(EG)|$ is contractible. Moreover G acts on EG from the left in the obvious way, using multiplication of elements of G. Clearly this action is free.

The simplicial sets $N(E\Sigma_n)$ fit together to form an operad with one colour, as we will now explain. We will define the operadic composition maps by first specifying group homomorphisms

$$\gamma : \Sigma_n \times \Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \to \Sigma_k$$

for each $n, k_1, \ldots, k_n \geq 0$ and $k = k_1 + \cdots + k_n$. One way to do this is to view the elements $\sigma \in \Sigma_n$ as permutation matrices and defining $\gamma(\sigma, \tau_1, \ldots, \tau_n)$ by replacing the 1 in the ith column (and $\sigma(i)$th row) of σ by the permutation matrix τ_i. Another way is to combine the embedding

$$\text{block} : \Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \to \Sigma_k,$$

given by letting Σ_{k_i} act on the ith 'block' of length k_i in the set $\{1, \ldots, k\}$, with the homomorphism $\Sigma_n \to \Sigma_k$ permuting the n blocks. If we write

$$f : \{1, \ldots, k\} \to \{1, \ldots, n\}$$

for the map sending every element of the ith block to the number i, we denote this homomorphism by $f^* : \Sigma_n \to \Sigma_k$. Then the map γ can be described as a product in the group Σ_k:

$$\begin{aligned}\gamma(\sigma, \tau_1, \ldots, \tau_n) &= f^*\sigma \cdot \text{block}(\tau_1, \ldots, \tau_n) \\ &= \text{block}(\tau_1, \ldots, \tau_n) \cdot f^*\sigma.\end{aligned}$$

Now the maps $N(E\gamma)$ give the desired operadic composition. The action of Σ_n on $E\Sigma_n$ is as described above and one easily verifies that these are compatible with the operadic composition in the desired way. In this way we obtain a simplicial and a topological operad $\mathrm{BE}_{\mathbf{\Delta}}$ and $\mathrm{BE}_{\mathbf{Top}}$ with

$$\begin{aligned}\mathrm{BE}_{\mathbf{\Delta}}(n) &= N(E\Sigma_n) \\ \mathrm{BE}_{\mathbf{Top}}(n) &= |N(E\Sigma_n)|\end{aligned}$$

which are called the (simplicial and topological) Barratt–Eccles operad. As we already observed, it is an operad whose spaces are contractible and have a free Σ_n-action. In this respect it resembles the little disks operad $\mathbf{E}_\infty$ of Example 1.9.

2.7.7 The Simplicial Boardman–Vogt Resolution of an Operad

For a simplicial operad $\mathbf{P}$, one can mimic the topological Boardman–Vogt resolution for topological operads (as in Section 1.7) and construct a simplicial operad $W\mathbf{P}$ equipped with a map to $\mathbf{P}$ which is the identity on colours. The modification to the simplicial case is done completely analogously to the W-construction for simplicial categories of Section 2.7.3, replacing the topological unit interval by the simplex $\Delta[1]$. In particular, for a sequence of colours $c_1, \ldots, c_n, c$ the simplicial set $W\mathbf{P}(c_1, \ldots, c_n; c)$ is a quotient of a coproduct of simplicial sets $W\mathbf{P}^{(T)}$ indexed over planar trees T with numbered leaves and edges labelled by colours of $\mathbf{P}$, so that the leaves of T are labelled by $c_1, \ldots, c_n$ (not necessarily in that order) and the root by c. For such a labelled tree T, the simplicial set $W\mathbf{P}^{(T)}$ is the product

$$\prod_{v \in V(T)} \mathbf{P}(v) \times \prod_{e \in \mathrm{in}(T)} \Delta[1],$$

where v ranges over the vertices of $\mathbf{P}$ and e over the inner edges, while $\mathbf{P}(v) = \mathbf{P}(e_1, \ldots, e_n; e)$ with $e_1, \ldots, e_n$ the incoming edges of v and e its outgoing edge.

We will return to this simplicial W-construction many times in this book. For now, we note that if $\mathbf{P}$ has only unary operations (i.e. is a simplicial category), this W-construction agrees with the one given in Section 2.7.3. Recall that in that case the description of $W[n]$ was much simpler than the general case. The same is true for the free operads $\Omega(T)$ corresponding to trees, see Section 1.3. Recall that the colours of $\Omega(T)$ are the edges of T and its operations are generated by the vertices of T. The Boardman–Vogt resolution $W\Omega(T)$ has the edges of T as colours again. For a sequence $e_1, \ldots, e_n, e$ of such edges, the simplicial set $W\Omega(T)(e_1, \ldots, e_n; e)$ is empty unless there exists a subtree S of T whose leaves are $e_1, \ldots, e_n$ and whose root is e. In this case

$$W\Omega(T)(e_1, \ldots, e_n; e) = \prod_{s \in \mathrm{in}(S)} \Delta[1],$$

the product now ranging over the inner edges s of S. Operadic composition in $W\Omega(T)$ is defined by grafting subtrees, now inserting length 1 for the edges along which the grafting takes place, exactly as in the topological case.

One would hope that this construction gives rise to a ‘homotopy-coherent nerve’ for simplicial operads. This is indeed the case, once we have found a suitable context for nerves of operads in the next chapter.

Historical Notes

Simplicial sets were introduced as a tool to describe the homotopical properties of topological spaces in a combinatorial way. The first definition of simplicial sets (then called complete semi-simplicial complexes) appears in a 1950 paper of Eilenberg–Zilber [54]. Much of the basic theory was developed soon after; Moore's lectures in the Séminaire Henri Cartan of 1954–1955, his overview paper [118], and the early works of Kan [96, 97, 98] are classic references. These include the fundamentals of homotopy theory (such as homotopy groups and fibrations) from the simplicial point of view; much of this material will appear in Chapters 5 and 7 of this book. Standard textbook references on simplicial sets were written in the 1960s by Gabriel–Zisman [61], Lamotke [101], and May [111]. Another very useful survey of work from this time is given by Curtis [47].

All of the above references focus on the application of simplicial sets to the homotopy theory of topological spaces. The shift in focus to general simplicial objects and their applications came a bit later. The application of simplicial techniques to problems of algebra is perhaps most famously promoted in the works of Quillen [123, 126] on the (co)homology of commutative rings and on higher algebraic K-theory. Simplicial categories, as defined in this chapter, rose to importance in the work of Dwyer–Kan [51, 50] from the 1980s. The more recent textbook on simplicial homotopy theory by Goerss–Jardine [69] takes some of these developments into account.

In this chapter we have also introduced the homotopy-coherent nerve, which produces a simplicial set out of a simplicial category. This construction first arose in the work of Vogt [139] on homotopy-coherent diagrams of spaces, building on his earlier work on homotopy-coherent algebraic structures with Boardman [21]. Cordier [45] systematically studied the simplicial version of the homotopy-coherent nerve and made the connection to the work of Dwyer–Kan on simplicial categories.

The Barratt–Eccles operad we described just above originates in [10].

Chapter 3
Dendroidal Sets

In this chapter we will introduce the category of dendroidal sets, which is the main object of study of this book. The definition of a dendroidal set mirrors that of a simplicial set, except that the category $\mathbf{\Delta}$ is replaced by a larger category $\mathbf{\Omega}$ of trees. The morphisms in the latter category are the operad morphisms between the free operads generated by these trees. The category of dendroidal sets contains the categories of simplicial sets and of coloured operads (in **Sets**) as full subcategories, and the Boardman-Vogt resolution discussed in Chapter 1 can be used to construct for each coloured operad (in topological spaces or in simplicial sets) a homotopy-coherent nerve which has the structure of a dendroidal set. The category of dendroidal sets is a category of presheaves, just like that of simplicial sets, and many constructions can be lifted from the simplicial context to the dendroidal one. At the same time there are several important differences, some of which are caused by the fact that unlike the category $\mathbf{\Delta}$, the category $\mathbf{\Omega}$ of trees contains non-trivial automorphisms. This leads to the introduction of normality and normalization of dendroidal sets, not present in the simplicial theory, and a more refined notion of skeletal filtration.

3.1 Trees

We already defined the trees we will work with in Section 1.3. In this section we fix some more terminology. The reader is advised to quickly glance over it and refer back to it when necessary.

Recall that all our trees are finite, rooted, and are allowed to have *inner* edges, connecting two vertices, and *outer* edges, only attached to a vertex at one end. One of these outer edges is the root and all others are leaves. A typical picture of a tree was already provided in Section 1.3. For a tree T, we will write $E(T)$ for its set of edges and $V(T)$ for its set of vertices. These vertices come in two kinds, namely *external vertices* which only have one inner edge attached to them, and *internal vertices*. Note that removing an external vertex and all the outer edges attached to it yields a new tree; a similar procedure is not available for internal vertices. We will often refer to

G. Heuts, I. Moerdijk, *Simplicial and Dendroidal Homotopy Theory*,
Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern
Surveys in Mathematics 75, https://doi.org/10.1007/978-3-031-10447-3_3

this removing of an external vertex as *pruning*. An external vertex occurring at the top of T (i.e. minimal in the partial order on $V(T)$) will sometimes be called a *leaf vertex*.

Both $V(T)$ and $E(T)$ come with a natural partial ordering. We will mostly use the one on $E(T)$, defined by $e \leq f$ if the path from e to the root of T passes through f. The root itself is of course the maximal element. We call two edges e and f *incomparable* if they are not related in this partial order and denote this by $e \perp f$. The edges on a path from a given edge e to the root are linearly ordered; from this one easily deduces that

$$e \perp f, \quad e' \leq e, \quad f' \leq f \quad \Rightarrow \quad e' \perp f'.$$

The partial ordering on $E(T)$ defines for any vertex v an *outgoing edge* $\mathrm{out}(v)$ and a set $\mathrm{in}(v)$ of *incoming edges*. The number of incoming edges is the *valence* of v and sometimes denoted $|v|$. A vertex of valence zero is a *nullary vertex* or a *stump*.

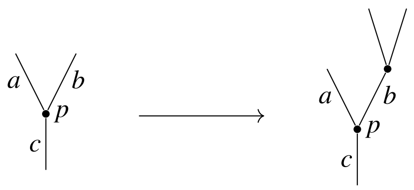
There is one rather special tree having no vertices at all, which can be pictured as follows:

It has a unique edge which is both a leaf and a root. We often write η for this tree.

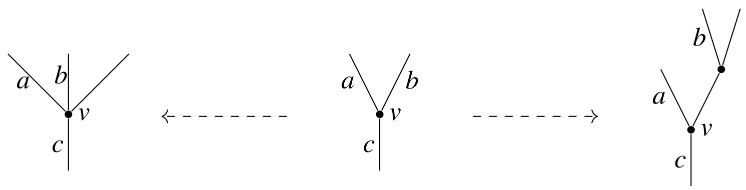
A tree is called *open* if it has no nullary vertices; on the other hand, it is called *closed* if it has no leaves. For example, the tree η is open but not closed and generally a tree cannot be both open and closed. A tree is called *linear* if all its vertices have valence one. The linear tree with $n + 1$ edges running from the leaf 0 to the root n will be denoted $[n]$:

The reason for this notation, if not yet clear, will become apparent in this chapter. Note that by our convention the tree obtained from $[n]$ by putting a stump on top will *not* be referred to as a linear tree.

An *embedding* $\varphi : S \longrightarrow T$ of trees consists of injective maps $\varphi_E : E(S) \longrightarrow E(T)$ and $\varphi_V : V(S) \longrightarrow V(T)$ such that for any vertex v of S, the map φ_E gives a bijection between $\mathrm{in}(v)$ and $\mathrm{in}(\varphi v)$ and $\varphi(\mathrm{out}(v)) = \mathrm{out}(\varphi v)$. It is an easy exercise to see that such an embedding will automatically respect the partial orderings on sets of edges and vertices. A typical example of an embedding looks as follows:



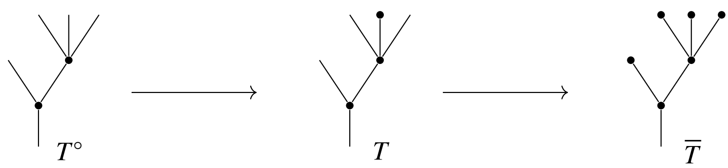
However, there are no embeddings of trees as indicated below:



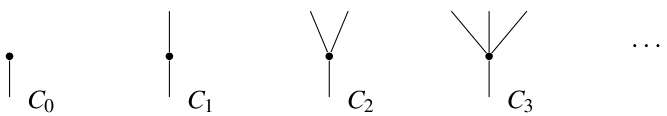
An embedding which is bijective (on edges as well as on vertices) is an *isomorphism* of trees. If $\varphi : S \longrightarrow T$ is an embedding and the map φ is just given by inclusion of subsets $E(S) \subseteq E(T)$ and $V(S) \subseteq V(T)$, then we call S a *subtree* of T. In other words then, an embedding $S \longrightarrow T$ is the same thing as an isomorphism of S onto a subtree of T. Note that if S is a subtree of T, then S can be obtained from T by successively pruning away external vertices (and the outer edges attached to them) from T. Note that any edge e of T determines an embedding denoted

$$e : \eta \rightarrow T.$$

Any tree T embeds into its *closure* $\overline{T}$ defined by putting a nullary vertex on top of each leaf of T. Conversely, any tree T has an embedded *interior* $T^\circ \longrightarrow T$ obtained by pruning away its nullary vertices.



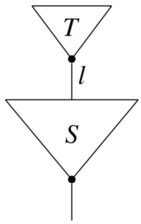
A *corolla* is a tree with precisely one vertex. We will write C_n for the corolla with root 0 and leaves $1, \ldots, n$.



If S and T are trees and l is a leaf of S, one can *graft* T on top of S by identifying l with the root of T. The resulting tree will be denoted

$$S \circ_l T$$

and can be depicted as below.



Similarly, if $l_1, \ldots, l_n$ are leaves of S and $T_1, \ldots, T_n$ are trees, one can graft each T_i onto l_i to obtain a tree

$$S \circ_{l_1,\ldots,l_n} (T_1, \ldots, T_n).$$

One can inductively build any tree by grafting together corollas. Indeed, any tree T is either isomorphic to η or isomorphic to one of the form

$$C_n \circ_{1,\ldots,n} (T_1, \ldots, T_n)$$

for strictly smaller trees $T_1, \ldots, T_n$. This description will sometimes be helpful when using induction on the size of trees. For example, the automorphism group of a tree can be described as an iterated semidirect product of symmetric groups as follows. First observe that $\mathrm{Aut}(C_n) \cong \Sigma_n$. Then for T as above, partition the set of trees $T_1, \ldots, T_n$ so that T_i and T_j are in the same equivalence class if and only if they are isomorphic. Say the sizes of the equivalence classes are $n_1, \ldots, n_k$ with $n_1 + \cdots + n_k = n$ and choose a representative $T^{(i)}$ for each isomorphism class, $i = 0, \ldots, k$. Then

$$\mathrm{Aut}(T) \cong (\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}) \ltimes (\mathrm{Aut}(T^{(1)})^{n_1} \times \cdots \times \mathrm{Aut}(T^{(k)})^{n_k}).$$

We already discussed planar structures on trees in Section 1.3. It is worth noting that the evident action of $\mathrm{Aut}(T)$ on the set of planar structures of T is free. However, it is generally *not* transitive. Indeed, consider the two pictures below, which indicate two different planar structures (now we *do* use the planar structure induced by our pictures!) on the same tree T which cannot be related by an automorphism of T.

We end this section with some observations on the poset of edges $E(T)$ of a tree T.

Lemma 3.1 *Let T be a tree. Then T is uniquely determined (up to isomorphism) by the partially ordered set $E(T)$ and the subset of leaves $L(T) \subseteq E(T)$.*

Proof We will show how to reconstruct T from $E(T)$ with its partial order and the subset $L(T)$. First of all, the set of vertices $V(T)$ is in bijection with the set $E(T) - L(T)$, by assigning to each vertex its outgoing edge. We will now inductively reconstruct T from $E(T)$. Start with the root r, which is the unique maximal element of $E(T)$. If $r \in L(T)$ then $T \cong \eta$. Otherwise, attach a vertex v_r to the top of r; as incoming edges of this vertex, take the elements $e < r$ of $E(T)$ such that there does not exist any $f \in E(T)$ with $e < f < r$. Next, for every input edge e of v_r which is not contained in $L(T)$, attach a vertex v_e to the top of e. Give it the incoming edges $e' \in E(T)$ with $e' < e$ so that there are no f' with $e' < f' < e$. Continuing in this way one ends up with T itself. □

Conversely, we can characterize which posets arise as sets of the form $E(T)$, for some tree T.

Lemma 3.2 *Let E be a finite poset with a unique maximal element r and assume that E satisfies the following property:*

$(*)$ *For every $e \in E$, the poset $E_{e\leq} := \{x \in E \mid e \leq x\}$ is linearly ordered.*

Let $L \subseteq E$ be a subset consisting of minimal elements. Then there exists a tree T with $E(T) = E$ and $L(T) = L$. This T is unique up to isomorphism by the previous lemma.

Proof One constructs T as in the proof of the previous lemma. To see that the process described there yields a tree, we need to check that every edge e with $e \neq r$ is the input edge of a uniquely defined vertex v. This is guaranteed by assumption $(*)$, since there is a unique element f with the property that $e < f$ and there exists no f' with $e < f' < f$. To see that $E(T) = E$, one simply notes that any $e \in E$ is connected to r by the finite chain of elements $E_{e\leq}$, so that such an e is added to T at some stage of the process. Finally, it is clear that $L = L(T)$ by construction. □

Note that the previous two lemmas make it possible to give an alternative definition of tree, simply in terms of a pair (E, L) as above.

3.2 The Category Ω of Trees

In Section 1.3 we discussed how any tree generates an operad $\Omega(T)$ in **Sets**. The colours of $\Omega(T)$ are the edges of T. Given a sequence of colours $e_1, \ldots, e_n, e$, the set of operations $\Omega(T)(e_1, \ldots, e_n; e)$ is either empty or a singleton. There is an operation in $\Omega(T)(e_1, \ldots, e_n; e)$ if and only if there exists a subtree of T with leaves $e_1, \ldots, e_n$ and root e. In particular any vertex v of T gives, for any ordering of its input edges $\mathrm{in}(v) = \{e_1, \ldots, e_n\}$, an operation $v \in \Omega(T)(e_1, \ldots, e_n; \mathrm{out}(v))$. The operad $\Omega(T)$ is free and any planar structure on T fixes a set of generating operations corresponding to the vertices of T.

Definition 3.3 The category $\mathbf{\Omega}$ is the category whose objects are trees and whose morphisms $S \to T$ are maps of operads $\Omega(S) \to \Omega(T)$.

Thus, by definition, $\mathbf{\Omega}$ is a full subcategory of the category of operads in **Sets**. The linear trees $[n]$ form a full subcategory of $\mathbf{\Omega}$. For such a tree $[n]$, the corresponding operad $\Omega[n]$ is precisely the category

$$0 \to 1 \to \cdots \to n,$$

so that we may identify this full subcategory with $\mathbf{\Delta}$. We will denote the resulting inclusion by

$$i : \mathbf{\Delta} \to \mathbf{\Omega}.$$

This inclusion is compatible with the inclusion of **Cat** into **Op**, in the sense that the square below commutes.

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r] \arrow[d, "i"'] & \mathbf{Cat} \arrow[d, "j"] \\ \mathbf{\Omega} \arrow[r] & \mathbf{Op} \end{tikzcd}$$

Let us consider the morphisms in $\mathbf{\Omega}$ more closely. A morphism $\varphi : S \to T$ sends edges to edges and vertices v to subtrees $\varphi(v)$ of T, by definition of the operads $\Omega(S)$ and $\Omega(T)$. If $e_1, \dots e_n$ are the input edges of v and e the output, then $\varphi(e_1), \dots, \varphi(e_n)$ are the leaves of $\varphi(v)$, whereas $\varphi(e)$ is the root. Since the operad $\Omega(T)$ has at most one operation from a given sequence of colours to another colour, it is clear that the morphism φ is uniquely determined by its effect on colours. This map on edges $\varphi_E : E(S) \to E(T)$ is a map of posets, i.e.

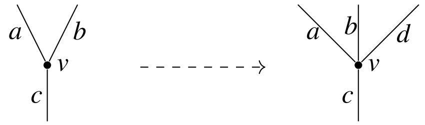
$$e \leq f \quad \Rightarrow \quad \varphi(e) \leq \varphi(f),$$

and preserves independence of edges,

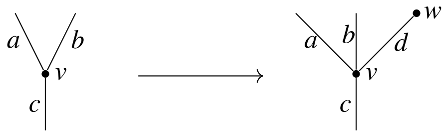
$$e \perp f \quad \Rightarrow \quad \varphi(e) \perp \varphi(f).$$

To see the latter fact, observe that the paths from e and f to the root of S first meet in a vertex v. This vertex has two leaves e' and f' such that $e \leq e'$ and $f \leq f'$. These two leaves are distinct and therefore incomparable; moreover, the edges $\varphi(e')$ and $\varphi(f')$ are distinct leaves of the subtree $\varphi(v)$ and therefore also incomparable. Since $\varphi(e) \leq \varphi(e')$ and $\varphi(f) \leq \varphi(f')$ it follows that $\varphi(e)$ and $\varphi(f)$ are incomparable as well.

Each embedding of trees $S \to T$ (as defined in the previous section) defines a morphism in $\mathbf{\Omega}$. In fact, this morphism sends generators to generators, or in other words it sends vertices to subtrees with one vertex. We already noted that there is no embedding as follows:



In fact, there is no such morphism in $\mathbf{\Omega}$. Indeed, the binary operation in $\Omega(S)(a, b; c)$ would have to map to a binary operation in $\Omega(T)$, but there are none. On the other hand, there *is* a map between the following trees:



It sends the unique operation in $\Omega(S)(a, b; c)$ to the composition $v \circ_d w$. This example shows that taking the interior of a tree *is not a functor*. The situation is better for the closure of a tree. Let us first observe the following:

Lemma 3.4 *Let T be a closed tree. Let $e_1, \ldots, e_n, e$ be edges in T for which $e_i \leq e$ and $e_i \perp e_j$ for any distinct i and j. Then there is a unique subtree of T with leaves $e_1, \ldots, e_n$ and root e.*

Proof Uniqueness is clear, since subtrees of any tree T (closed or not) are always determined by their leaves and root. For existence, one prunes away everything above the edges $e_1, \ldots, e_n$. To be more precise, consider the subset X of $E(T)$ consisting of the edges f for which there exists an e_i such that $e_i \leq f \leq e$. Also, let Y be the subset of $V(T)$ consisting of those vertices v for which the path from v to the root of T contains only edges of X. Then it is easily verified that the pair (X, Y) defines the necessary subtree. □

Proposition 3.5 *Let S and T be trees and let $\varphi : E(S) \to E(T)$ be a map of posets which preserves the incomparability relation $\perp$ on edges. If T is closed, then φ is the underlying map on edges of a unique morphism $S \to T$ in $\mathbf{\Omega}$.*

Proof For a vertex v of S, the input edges and output edge of v satisfy the conditions of the previous lemma, so we can define $\varphi(v)$ to be the resulting unique subtree. □

Corollary 3.6 *Any morphism $\varphi : S \to T$ in $\mathbf{\Omega}$ extends uniquely to a morphism between closures $\overline{\varphi} : \overline{S} \to \overline{T}$.*

We write $\overline{\mathbf{\Omega}}$ for the full subcategory of $\mathbf{\Omega}$ on the closed trees. It is worth noting that the combination of Proposition 3.5 and Lemma 3.2 gives an alternative way of describing the category of closed trees. Indeed, since $L(T) = \varnothing$ for a closed tree T, one concludes that $\overline{\mathbf{\Omega}}$ is equivalent to the category whose objects are finite posets satisfying the assumptions of 3.2 and whose morphisms are the maps of posets which respect the incomparability relation.

Corollary 3.6 shows that the operation of taking the closure of a tree defines a functor which is left adjoint to the inclusion:

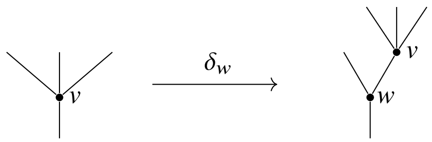
$$\mathbf{\Omega} \overset{\overline{()}}{\rightleftarrows} \overline{\mathbf{\Omega}}.$$

Recall from Corollary 2.10 that this implies in particular that the classifying spaces of $\mathbf{\Omega}$ and $\overline{\mathbf{\Omega}}$ are homotopy equivalent. We use this observation to prove the following:

Proposition 3.7 *The classifying space $B\mathbf{\Omega}$ is contractible.*

Proof By the adjunction above it suffices to prove this for $B\overline{\mathbf{\Omega}}$. Consider the object $\overline{\eta} = C_0$ and the corresponding constant functor denoted $\overline{\eta} : \overline{\mathbf{\Omega}} \to \overline{\mathbf{\Omega}}$. It suffices to relate the identity functor to $\overline{\eta}$ by a zig-zag of natural transformations. For a closed tree T, let T^+ denote the tree obtained from T by adjoining a unary vertex to the root. In other words, if r is the root of T, then $T^+ = C_1 \circ_r T$. Write e_T for the newly arising root edge of T^+. The procedure assigning T^+ to T is in fact a functor; indeed, any map of closed trees φ induces an evident map $E(S^+) \to E(T^+)$ which satisfies the condition of Proposition 3.5. Clearly there is an embedding $i_T : T \to T^+$. Moreover, since T is closed, the map $e_T : \eta \to T^+$ extends to a map $\overline{e_T} : \overline{\eta} \to T^+$ by Corollary 3.6. Both i_T and e_T are parts of natural transformations, completing the proof. $\square$

Remark 3.8 The use of closed trees in the previous proof is essential. Although the construction of T^+ from T makes sense for any tree, it cannot generally be made functorial in a way that renders the map $e_T : \eta \to T^+$ to the new root natural in T. For example, while there is a map δ_w as pictured below, there is no root-preserving map between the trees S^+ and T^+ at all.



S T

3.3 Faces and Degeneracies in $\mathbf{\Omega}$

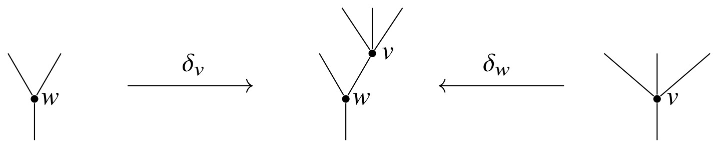
In Chapter 2 we explained in what sense the category $\mathbf{\Delta}$ is generated by the face maps $\partial_i : [n-1] \to [n]$ and the degeneracy maps $\sigma_j : [n] \to [n-1]$. We will now discuss a similar set of generating morphisms for the category $\mathbf{\Omega}$. There is one important difference however, in that the automorphism groups of objects of $\mathbf{\Delta}$ are all trivial while in $\mathbf{\Omega}$ they need not be.

3.3.1 Outer Faces

For a tree T and an external vertex v of T, write $\partial_v T$ for the tree obtained by pruning away v and all the outer edges attached to it. This vertex v can be any leaf vertex or it can be the vertex attached to the root of T, provided it is only attached to one inner vertex of T. We write

$$\partial_v T \xrightarrow{\delta_v} T$$

for the inclusion of this tree and refer to it as an *outer face*. If v is a leaf vertex we will sometimes specifically speak of a *leaf face* and similarly for a *root face*. Below is an example of a leaf face δ_v and a root face δ_w.



There is one case which deserves special mention. For most trees T the notions of leaf vertex and root vertex are distinct and the preceding discussion suffices. However, for corollas (i.e. trees with one vertex) this is not the case. There are $n + 1$ different maps

$$\eta \to C_n$$

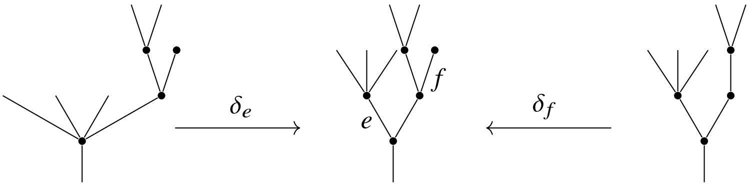
corresponding to the inclusion of each of the n leaves and the root. All of these are by definition outer faces as well.

3.3.2 Inner Faces

For a tree T and an inner edge e, let $\partial_e T$ denote the tree obtained by deleting the edge e and identifying the vertices v and w at either end of e. We will usually say $\partial_e T$ has been obtained from T by *contracting* e. This gives an inclusion of trees denoted

$$\partial_e T \xrightarrow{\delta_e} T$$

which sends the new vertex obtained by contraction to the subtree of T obtained by grafting the two corollas with vertices v and w, i.e. the smallest subtree of T containing these vertices. In terms of the associated operads, it sends the new vertex in $\partial_e T$ to the composition in $\Omega(T)$ of the operations correspondingly given by v and w along e. A morphism of this kind will be referred to as an *inner face*.

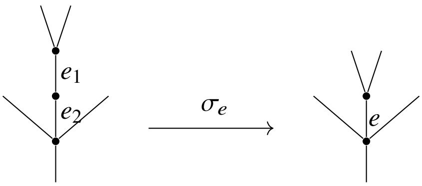


3.3.3 Degeneracies

Let e be an edge in a tree T. Consider the tree $\sigma_e T$ obtained from T by splitting this edge into two edges e_1 and e_2 by placing a unary vertex v in the middle of e. Then there is a morphism

$$\sigma_e T \xrightarrow{\sigma_e} T$$

in $\mathbf{\Omega}$ which sends both edges e_1 and e_2 to e and the new unary vertex v to the subtree $\eta \xrightarrow{e} T$ of T. In operadic terms, it sends the new vertex v to the identity operation of the colour e in the operad $\Omega(T)$. A morphism of this kind is called a *degeneracy*.



Notice that face maps are injective on edges and increase the number of vertices by one, while degeneracy maps are surjective on edges and decrease the number of vertices as well as the number of edges by one. It need not be true that a face map increases the number of edges, as witnessed by the inclusion $\eta \to \overline{\eta}$. We will sometimes refer to the face and degeneracy maps discussed above as *elementary* faces and degeneracies and use the general terms face (resp. degeneracy) also for compositions of elementary faces (resp. elementary degeneracies). We will say these general face maps are of 'higher codimension' if it is necessary to make the distinction with elementary face maps. Similar terminology applies to degeneracy maps. Note that a composition of elementary outer faces is precisely the same as the embedding of a subtree.

3.3.4 Codendroidal Identities

Analogous to the cosimplicial identities, there are equations governing the various ways in which elementary faces and degeneracies compose. These equations are the *codendroidal identities*, dual to the dendroidal identities to be introduced in the next section. For completeness' sake we provide a list below, although we will rarely make use of it in this form. This list is necessarily more complicated than that of the cosimplicial identities. In the list below T is a tree, e and f are edges of T and v and w are vertices.

(1) $\delta_v\delta_w = \delta_w\delta_v$ if v and w are distinct external vertices.

This equation makes sense, because w is still an external vertex of $\partial_v T$ and v an external vertex of $\partial_w T$. It is to be read as stating the commutativity of the diagram in $\mathbf{\Omega}$ pictured below.

$$\begin{tikzcd}
 & \partial_w(\partial_v T) = \partial_v(\partial_w T) \arrow[dl, "\delta_w"'] \arrow[dr, "\delta_v"] & \\
\partial_v T \arrow[dr, "\delta_v"'] & & \partial_w T \arrow[dl, "\delta_w"] \\
 & T &
\end{tikzcd}$$

Similar remarks apply to the remaining identities below.

(2) $\delta_e\delta_f = \delta_f\delta_e$ if e and f are distinct inner edges.
(3) $\delta_e\delta_v = \delta_v\delta_e$ if e is an inner edge and v an external vertex not attached to e.
(4) $\delta_e\delta_x = \delta_v\delta_w$ if v is an external vertex and e is an inner edge connecting v to another vertex w, provided w is an external vertex of the tree $\partial_v T$. Here x is the new vertex of $\partial_e T$ arising as the composition of v and w along e.
(5) $\sigma_e\sigma_f = \sigma_f\sigma_e$ if e and f are distinct edges.
(6) $\sigma_e\sigma_{e_1} = \sigma_e\sigma_{e_2}$ if e_1 and e_2 are the two new edges in the tree $\sigma_e T$.
(7) $\sigma_e\delta_{e_i} = \mathrm{id}$ for $i = 1, 2$ if e_i is an internal edge.
(8) $\sigma_e\delta_x = \mathrm{id}$ if e is an outer edge and x is the new vertex connecting e_1 and e_2.
(9) $\sigma_e\delta_f = \delta_f\sigma_e$ if e and f are distinct and f is an inner edge.
(10) $\sigma_e\delta_v = \delta_v\sigma_e$ if v is an external vertex and e is not an outer edge attached to v.

3.3.5 Factorization of Morphisms Between Trees

Recall that we use the term face map to mean a composition of elementary face maps as described above. Similarly a degeneracy map is a composition of elementary degeneracies.

Proposition 3.9 *Any morphism $\varphi : S \to T$ in $\mathbf{\Omega}$ can be factored as*

$$S \xrightarrow{\sigma} S' \xrightarrow{\alpha} T' \xrightarrow{\delta} T,$$

where σ is a degeneracy, α an isomorphism and δ a face map. This factorization is unique up to unique isomorphism, in the sense that for any other factorization $(\sigma', \alpha', \delta')$ there is a commutative diagram

$$\begin{tikzcd} & S' \arrow[r, "\alpha"] \arrow[dd, "\simeq"'] & T' \arrow[dd, "\simeq"'] \arrow[dr, "\delta"] & \\ S \arrow[ur, "\sigma"] \arrow[dr, "\sigma'"'] & & & T \\ & S'' \arrow[r, "\alpha'"'] & T'' \arrow[ur, "\delta'"'] & \end{tikzcd}$$

in which the vertical maps are isomorphisms, and moreover the choice of vertical isomorphisms is unique.

Proof First consider the effect of φ on edges and factor it as a surjection followed by an injection:

$$E(S) \xrightarrow{\varphi_s} A \xrightarrow{\varphi_i} E(T).$$

Here one takes A to the quotient of $E(S)$ by the equivalence relation $\sim$ identifying two elements whenever they have the same image in $E(T)$. If $e \sim e'$ then e and e' must be related in $E(S)$; indeed, any map of trees preserves independence of edges. Let us assume $e \leq e'$. Then φ must send all the edges on the path from e down to e' to the same element of $E(S)$. It follows that every vertex on this path must be unary. Hence there is a degeneracy $S \longrightarrow S_{e,e'}$ collapsing the segment from e to e' in S to a single edge in $S_{e,e'}$. Applying this reasoning to every equivalence class of edges gives a degeneracy $\sigma : S \longrightarrow S'$ such that $E(S') = A$. Moreover, φ factors as $i \circ \sigma$ for some injective map $S' \longrightarrow T$. This map i may be factored as

$$S' \xrightarrow{\alpha} T' \xrightarrow{\delta} T$$

where α is an isomorphism onto the image of i. The map δ is an injective map of trees which on the level of edges is simply an inclusion of subsets $E(T') \subseteq E(T)$. Any such map is a composition of elementary face maps, as we shall prove in detail in the next proposition.

The uniqueness of this factorization up to isomorphism is straightforward to establish. Indeed, it is clear on the level of edges, and one uses that maps of trees are uniquely determined by their effect on edges. $\square$

Proposition 3.10 *Any face map $\varphi : S \longrightarrow T$ in $\mathbf{\Omega}$ can be factored as*

$$S \xrightarrow{\delta} S' \xrightarrow{\varepsilon} T,$$

in the following two ways:

(1) *δ is a composition of outer faces and ε a composition of inner faces.*
(2) *δ is a composition of inner faces and ε a composition of outer faces. In this case the factorization is unique up to unique isomorphism.*

Proof We start with the factorization of type (2). Write $l_1, \ldots, l_n$ for the leaves of S and r for its root. Then take S' to be the maximal subtree of T with leaves $\varphi(l_1), \ldots, \varphi(l_n)$ and root $\varphi(r)$. Clearly φ factors into injective maps $\delta : S \longrightarrow S'$ and

$\varepsilon : S' \longrightarrow T$. The map ε is a composition of elementary outer face maps, where each such outer face adds a single vertex to S'. We should argue that the map δ is a composition of elementary inner faces. Recall that every face map sends a vertex v of S to a subtree of T. If δ sends every vertex of S to a corolla, then δ is the inclusion of a subtree. But S and S' have the same leaves and root, so δ is the identity. Otherwise there is a vertex v of S such that the subtree $\varphi(v)$ of T has an inner edge, say e. Then φ factors as

$$S \xrightarrow{\delta'} \partial_e S' \xrightarrow{\delta_e} S',$$

where the map δ' still preserves the leaves and root. We complete the proof by induction on the number of inner edges in the trees $\varphi(v)$, for $v \in V(S)$. For uniqueness we argue as follows: if (δ', ε') is another factorization as in (2), then ε' has to be the inclusion of the subtree of T with leaves $\varphi(l_1), \ldots, \varphi(l_n)$ and root $\varphi(r)$. Such a subtree is unique (namely S'), so that $\varepsilon' = \varepsilon$. Now the map $\delta' : S \longrightarrow S'$ is uniquely determined by the induced map on edges $E(S) \longrightarrow E(S')$. But this map has to coincide with the map induced by δ, so that $\delta' = \delta$.

For (1), it suffices to argue that a composition

$$\partial_e(\partial_v T) \xrightarrow{\delta_e} \partial_v T \xrightarrow{\delta_v} T$$

of an inner face δ_e followed by an outer face δ_v can also be factored as an outer face followed by an inner face. Note that the inner edge e of $\partial_v T$ is also an inner edge of T, which moreover is *not* connected to v (otherwise it could never be inner in $\partial_v T$). But then we simply have $\partial_v \partial_e = \partial_e \partial_v$ by the codendroidal identity (3) of the previous subsection. □

Definition 3.11 A morphism $\varphi : S \longrightarrow T$ in $\mathbf{\Omega}$ is *positive* if it is injective on edges. A morphism is *negative* if it is surjective on edges and on vertices.

By *surjective on vertices* we mean that any vertex of T is contained in a subtree $\varphi(v)$ for some vertex v of S. In fact, the surjectivity of φ on edges implies that any such $\varphi(v)$ is simply a corolla. The classes of positive and of negative maps are clearly closed under composition, so that they define subcategories $\mathbf{\Omega}^+$ and $\mathbf{\Omega}^-$ respectively. It is easily verified that their intersection is precisely the class of isomorphisms in $\mathbf{\Omega}$. Note that Proposition 3.9 implies that every positive morphism is the composition of an isomorphism followed by a face map, whereas every negative morphism is the composition of a degeneracy map followed by an isomorphism. Also, the proposition implies:

Corollary 3.12 *Every morphism $S \longrightarrow T$ in $\mathbf{\Omega}$ factors as the composition of a negative morphism $S \longrightarrow R$ followed by a positive morphism $R \longrightarrow T$. Moreover, this factorization is unique up to unique isomorphism.*

Finally, the following observation will be useful later on:

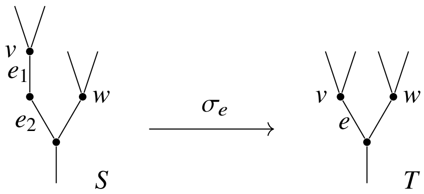
Proposition 3.13 *Let $\sigma : S \to T$ be a degeneracy and α an automorphism of S. Then there is at most one automorphism β of T making the following square commute:*

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & S \arrow[d, "\sigma"] \\ T \arrow[r, "\beta"] & T. \end{tikzcd}$$

Conversely, given an automorphism β of T, there is at most one automorphism α of S making the square commute.

Proof The first statement is clear from the fact that σ is an epimorphism. For the second statement we use that the map α is completely determined by its effect on edges, like any map in $\mathbf{\Omega}$. By commutativity of the diagram, it must send the fibre $\sigma^{-1}\{d\}$ to $\sigma^{-1}\{\beta(d)\}$, for any edge d of T. This fibre is a linear tree, which does not have nonidentity automorphisms. It follows that the effect of α on edges is completely determined by β. □

Remark 3.14 The phrase 'at most' in the previous proposition is crucial. It is *not* true that an automorphism α of S always induces an automorphism of T, or conversely that an automorphism of T induces one of S. As an example, consider the following trees S and T:



Then T has an automorphism interchanging the vertices v and w, but S does not.

3.3.6 Some Limits and Colimits in $\mathbf{\Omega}$

As in Section 2.1 we will now establish the existence of certain pushouts and pullbacks in $\mathbf{\Omega}$.

Lemma 3.15 *Let $S \leftarrow T \rightarrow R$ be two negative morphisms in $\mathbf{\Omega}$. Then their pushout exists in $\mathbf{\Omega}$ and is an absolute pushout.*

Proof As in the proof of Proposition 2.1 it suffices to show this for elementary degeneracy maps $\sigma_e : T \to S$ (collapsing two neighbouring edges e_1 and e_2 to a single edge e, deleting the vertex v in between) and $\sigma_f : T \to R$ (collapsing f_1 and f_2 to f, deleting a vertex w). The evident candidate for a pushout square is

$$\begin{tikzcd} T \arrow[r, "\sigma_e"] \arrow[d, "\sigma_f"] & S \arrow[d, "\sigma'_f"] \\ R \arrow[r, "\sigma'_e"] & Q, \end{tikzcd}$$

where Q is obtained from T by deleting both v and w and performing the collapses involving e and f. Again as in Proposition 2.1, it suffices to give sections α of σ_f and α' of σ'_f which are compatible in the sense that $\sigma_e\alpha = \alpha'\sigma'_e$. When $v = w$, one can take any section α of σ_f (there are two choices, depending on whether $\alpha(f)$ is f_1 or f_2) and take α' to be the identity section of $\sigma'_f = \mathrm{id}$. If v and w are different and not connected by a single edge, then one can take α to be any of the two available sections and α' to be the section with $\alpha'(f) = \alpha(f)$.

The remaining case is where v and w are connected by a single edge. If v is directly above w, so that $e_2 = f_1$, one takes $\alpha(f) = f_1$ and $\alpha'(f) = f_1$. If v is directly below w, so that $f_2 = e_1$, one takes $\alpha(f) = \alpha'(f) = f_2$. □

Lemma 3.16 *Let $\sigma : S \to T$ be a negative morphism in $\mathbf{\Omega}$. Then the pushout of any morphism along σ exists in $\mathbf{\Omega}$.*

Proof Let $\tau : S \to R$ be a morphism. The negative morphism σ collapses several linear subtrees of S; the images of those subtrees in R are also linear and we define $R \to \sigma_! R$ to be the degeneracy morphism collapsing those linear subtrees of R. Then $S \to \sigma_! R$ factors uniquely through a morphism $T \to \sigma_! R$ and it is easily verified that the resulting morphism is the pushout of τ along σ. □

Lemma 3.17 *Let $\partial : S \to T$ be a positive morphism and $\sigma : R \to T$ a negative morphism. Then their pullback exists in $\mathbf{\Omega}$.*

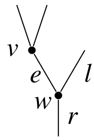
Proof The morphism σ collapses several linear subtrees R_i onto single edges i of T. For every edge i in the image of ∂, replace the edge $\partial^{-1}i$ in S by the linear subtree R_i. It is easily verified that the resulting tree gives the pullback $S \times_T R$. □

We proved in Section 2.1 that in $\mathbf{\Delta}$ all pullbacks along face maps exist. In the previous lemma we only established the analogous statement in $\mathbf{\Omega}$ for the pullback of a negative morphism along a face map. The case of a positive map is potentially problematic, as the example after the following lemma shows:

Lemma 3.18 *Let $\partial : S \to T$ and $\varepsilon : R \to T$ be two elementary face maps, for a tree T with at least two vertices. Then their pullback exists in $\mathbf{\Omega}$, except in the case of $\partial_e T \to T$ and $\partial_v T \to T$, where v is an external vertex, e an inner edge attached to v and the other vertex attached to e is* not *unary.*

Proof If we are not in the exceptional case of the lemma, we can use the codendroidal identities established earlier to describe the pullback of two elementary faces. For example, if ∂ and ε are outer faces corresponding to distinct outer vertices v and w, one uses identity (1) to see that the pullback is the tree obtained by removing both v and w from T. One uses (2) for two inner faces, and (3) and (4) for the combination of an outer face and an inner face; note that in (4), where e is the inner edge attached to v, our assumptions guarantee that the other vertex w attached to e is an external vertex of $\partial_v T$. □

Example 3.19 We illustrate the problem arising in the exceptional case of the lemma. Let T be the following tree:



Then the 'intersection' between $\partial_e T$ and $\partial_v T$ consists of the two edges r and l; it is disconnected and can no longer be described as a single tree.

3.4 Dendroidal Sets

The definition of a *dendroidal set* is analogous to that of a simplicial set, except that the category $\mathbf{\Delta}$ is replaced by the category $\mathbf{\Omega}$ of trees described in the previous two sections. Thus, a dendroidal set is a functor

$$X : \mathbf{\Omega}^{\mathrm{op}} \longrightarrow \mathbf{Sets}.$$

With natural transformations as morphisms between them, dendroidal sets form a category which we denote **dSets**. More explicitly, a dendroidal set is given by a family of sets X_T, one for each $T \in \mathbf{\Omega}$, together with a map

$$\alpha^* : X_T \longrightarrow X_S$$

for every morphism $\alpha : S \longrightarrow T$ in $\mathbf{\Omega}$. These maps should be functorial in the sense that $(\beta\alpha)^* = \alpha^*\beta^*$ for morphisms $R \xrightarrow{\alpha} S \xrightarrow{\beta} T$ in $\mathbf{\Omega}$ and $\mathrm{id}_T^* = \mathrm{id}_{X_T}$. The elements of X_T are called *dendrices* of X of shape T (or simply T-dendrices of X).

In particular, each set X_T carries an action of the group $\mathrm{Aut}(T)$ and the dendroidal set X is completely determined by the collection of all these $\mathrm{Aut}(T)$-sets together with the elementary face and degeneracy operators of the following three kinds:

$$\begin{aligned} d_e &= \partial_e^* : X_T \longrightarrow X_{\partial_e T} && \textit{inner face}, \text{ for each inner edge } e \text{ in } T, \\ d_v &= \partial_v^* : X_T \longrightarrow X_{\partial_v T} && \textit{outer face}, \text{ for each external vertex } v \text{ of } T, \\ s_e &= \sigma_e^* : X_T \longrightarrow X_{\sigma_e T} && \textit{degeneracy}, \text{ for each edge } e \text{ of } T. \end{aligned}$$

These operators satisfy the *dendroidal identities*, dual to the identities of the previous section:

(1) $d_w d_v = d_v d_w$ if v and w are distinct outer vertices.
(2) $d_f d_e = d_e d_f$ if e and f are distinct inner edges.

(3) $d_v d_e = d_e d_v$ if e is an inner edge and v an external vertex not attached to e.
(4) $d_x d_e = d_w d_v$ if v is an external vertex and e is an inner edge connecting v to another vertex w, provided w is an external vertex of the tree $\partial_v T$. Here x is the new vertex of $\partial_e T$ arising as the composition of v and w along e.
(5) $s_f s_e = s_e s_f$ if e and f are distinct edges.
(6) $s_{e_1} s_e = s_{e_2} s_e$ if e_1 and e_2 are the two new edges in the tree $\sigma_e T$.
(7) $d_{e_i} s_e = \mathrm{id}$ for $i = 1, 2$ if e_i is an internal edge.
(8) $d_x s_e = \mathrm{id}$ if e is an outer edge and x is the new vertex connecting e_1 and e_2.
(9) $d_f s_e = s_e d_f$ if e and f are distinct and f is an inner edge.
(10) $d_v s_e = s_e d_v$ if v is an external vertex and e is not an external edge attached to v.

Moreover, the face and degeneracy operators are equivariant with respect to isomorphisms of trees, in the sense that if $\alpha : S \to T$ is an isomorphism and e an inner edge of T, then the square

$$\begin{tikzcd} X(T) \arrow[r, "\alpha^*"] \arrow[d, "d_e"'] & X(S) \arrow[d, "d_{\alpha(e)}"] \\ X(\partial_e T) \arrow[r, "(\partial_e \alpha)^*"'] & X(\partial_{\alpha(e)} S) \end{tikzcd}$$

commutes. Of course a similar statement applies to outer faces and degeneracies.

The following is a list of first examples of dendroidal sets.

Example 3.20 (a) As for any presheaf category, any object T of $\mathbf{\Omega}$ gives rise to a representable presheaf which we denote by $\Omega[T]$. Thus

$$\Omega[T]_S = \mathrm{Hom}_{\mathbf{\Omega}}(S, T).$$

(b) One can form limits and colimits of diagrams of dendroidal sets by calculating them 'pointwise', as in any presheaf category. For instance, one can form the product $X \times Y$ of dendroidal sets X and Y as

$$(X \times Y)_T = X_T \times Y_T,$$

and similarly for pullbacks, coproducts, pushouts, images, unions, etc.

(c) As a particular case, each face $\delta_x : \partial_x T \to T$, where x is an inner edge or outer vertex, defines a monomorphism of dendroidal sets

$$\Omega[\partial_x T] \to \Omega[T].$$

We will write $\partial_e \Omega[T] \subseteq \Omega[T]$ for the image of this morphism and call it the *face of T opposite e*. The union of all these (for all inner and outer faces) defines the *boundary* $\partial\Omega[T]$ of $\Omega[T]$. Note that these faces and the boundary can be explicitly described as follows:

$$\begin{aligned}
(\partial_e\Omega[T])_S &= \{\alpha : S \to T \mid \text{the edge } e \text{ is not contained in the image of } \alpha\},\\
(\partial_v\Omega[T])_S &= \{\alpha : S \to T \mid \text{no subtree } \alpha(w) \text{ contains the vertex } v\},\\
(\partial\Omega[T])_S &= \{\alpha : S \to T \mid \alpha \text{ factors through a proper inclusion } R \to T\}.
\end{aligned}$$

(d) For a tree T consider the set $\mathrm{Pl}(T)$ of planar structures on T. If $\alpha : S \to T$, then it induces a map

$$\alpha^* : \mathrm{Pl}(T) \to \mathrm{Pl}(S).$$

Indeed, for v a vertex of S, the morphism α gives an bijection between $\mathrm{in}(v)$ and the set of leaves of $\alpha(v)$. The latter is linearly ordered by the planar structure of T, so that the former inherits a linear ordering. This construction is easily seen to be functorial in α, so that planar structures define a dendroidal set Pl.

(e) The category $\mathbf{\Omega}$ has been defined as a full subcategory of the category of operads in **Sets**. Thus any such operad $\mathbf{P}$ defines a dendroidal set $N\mathbf{P}$, its *dendroidal nerve*, by

$$N\mathbf{P}_T = \mathbf{Op}(\Omega(T), \mathbf{P}).$$

Note that this definition is completely analogous to that of the nerve of a category. Let us take a closer look at the dendroidal set $N\mathbf{P}$. For the tree η consisting of a single edge, the set $N\mathbf{P}_\eta$ is the set of colours of $\mathbf{P}$. With C_n the n-corolla and $(c_1, \ldots, c_n, c)$ a sequence of colours of $\mathbf{P}$, there is a pullback square

$$\begin{tikzcd}
\mathbf{P}(c_1, \ldots, c_n; c) \arrow[r] \arrow[d] & N\mathbf{P}_{C_n} \arrow[d] \\
* \arrow[r, "\mathbf{c}"'] & \prod_{i=0}^n N\mathbf{P}_\eta .
\end{tikzcd}$$

Here the vertical map on the right has as its ith component the map $i^* : N\mathbf{P}_{C_n} \to N\mathbf{P}_\eta$ induced by the edge inclusion $i : \eta \to C_n$, while $\mathbf{c}$ denotes the sequence $(c_1, \ldots, c_n, c)$. More generally, for a tree T a dendrex in $N\mathbf{P}_T$ can be pictured as follows. It consists of an assignment c of colours of $\mathbf{P}$ to the edges of T and an equivalence class of pairs (p, f), where p is a planar structure on T and f assigns an operation $f(v) \in \mathbf{P}(c(e_1), \ldots, c(e_n); c(e))$ to each vertex v of T, where e denotes the output edge of v and $e_1, \ldots, e_n$ enumerates the input edges of v in the order provided by the planar structure p. Two such pairs (p, f) and (q, g) are equivalent if for each vertex v, the permutation σ relating the planar structures p and q as in

$$\begin{tikzcd}
\{1, \ldots, n\} \arrow[r, "p_v"] \arrow[d, "\sigma"'] & \mathrm{in}(v) \\
\{1, \ldots, n\} \arrow[ur, "q_v"'] &
\end{tikzcd}$$

satisfies

$$f(v) = \sigma^* g(v).$$

A little more informally, a dendrex in $N\mathbf{P}_T$ is an assignment of colours of $\mathbf{P}$ to the edges of T and a compatible assignment of an operation in $\mathbf{P}$ to each vertex of T. As explained above, this is only a precise statement if one has a planar structure on T in mind.

(f) We list a few special cases of the construction of the nerve of an operad. First of all, representables are nerves: for the operad $\Omega(T)$ generated by a tree T, we have

$$N\Omega(T) = \Omega[T].$$

Secondly, for the commutative operad **Com** and any tree T, the set $N\mathbf{Com}_T$ is a singleton. Thus the dendroidal set $N\mathbf{Com}$ is a terminal object of the category **dSets**. We already encountered the nerve of the associative operad in example (d) above; indeed, there is an isomorphism

$$N\mathbf{Ass} \simeq \mathrm{Pl}.$$

Finally, every symmetric monoidal category $\mathbf{V}$ can be viewed as a coloured operad $\mathbf{V}^{\otimes}$ and hence gives rise to a (large) dendroidal set $N\mathbf{V}^{\otimes}$.

(g) Every simplicial set M gives a dendroidal set $i_!M$ by 'extension by zero',

$$(i_!M)_T = \begin{cases} M_n & \text{if } T \simeq [n], \\ \varnothing & \text{otherwise.} \end{cases}$$

This is indeed a well-defined dendroidal set, because if $\alpha : S \to T$ is a morphism in $\mathbf{\Omega}$ and T is a linear tree, then S must be linear as well. As the notation suggests, the functor

$$i_! : \mathbf{sSets} \to \mathbf{dSets}$$

is the left Kan extension of the functor

$$i : \mathbf{\Delta} \to \mathbf{dSets}$$

which sends $[n]$ to the representable $\Omega[n]$. Its right adjoint $i^* : \mathbf{dSets} \to \mathbf{sSets}$ gives for every dendroidal set X its *underlying simplicial set* i^*X. For a symmetric monoidal $\mathbf{V}$ as in the previous example, the simplicial set $i^*N\mathbf{V}^{\otimes}$ is precisely the usual nerve $N\mathbf{V}$ of the category $\mathbf{V}$. Note that the dendroidal nerve $N\mathbf{V}^{\otimes}$ also records the monoidal structure of $\mathbf{V}$.

(h) Let $\mathbf{P}$ be an operad and let A be a $\mathbf{P}$-algebra. In particular, A defines a set A_c for each colour c of $\mathbf{P}$. The *nerve of* A, written

$$N(\mathbf{P}, A),$$

is the dendroidal set defined as follows. A dendrex of $N(\mathbf{P}, A)_T$ is a dendrex $\xi \in N\mathbf{P}_T$ together with an assignment of an element $a_e \in A_{\xi(e)}$ for every edge e of T, satisfying a compatibility which we spell out below. The dendrex ξ gives a planar structure on T and for every vertex v of T an operation $f(v) \in \mathbf{P}(e_1, \ldots, e_n; e)$, with $e_1, \ldots, e_n$ the ordered set of inputs of v. The assignment of elements of the algebra A should satisfy the equation

$$f(v)(a_{e_1}, \ldots, a_{e_n}) = a_e.$$

This description of $N(\mathbf{P}, A)_T$ is evidently functorial in T and hence defines a dendroidal set. Note that it would suffice to specify just the elements a_l, for l ranging through the leaves of T, since any other a_e is then uniquely determined by ξ because of the compatibility specified above.

For example, if $\mathbf{P}$ is **Com** and A is a commutative monoid, an element of $N(\mathbf{P}, A)_T$ is an assignment of elements a_e for e ranging through the edges of T, with the property that for any vertex v of T the element assigned to $\mathrm{out}(v)$ is the product of the elements assigned to the input edges of v. Similarly, if $\mathbf{P}$ is **Ass** and A is an associative monoid, an element of $N(\mathbf{P}, A)_T$ is a pair (p, a) where p is a planar structure on T and a is an assignment as before, now taking the order of the multiplication at each vertex as prescribed by p into account.

We should also point out a more abstract way of describing the nerve of an algebra. For a tree T, the set of edges of T defines an $\Omega(T)$-algebra $E(T)$. In fact it is the terminal $\Omega(T)$-algebra, because $E(T)_e$ consists of just the edge e. But it is also the free algebra generated by the leaves of T. An element of $N(\mathbf{P}, A)_T$ is a pair (ξ, a) where $\xi : \Omega(T) \longrightarrow \mathbf{P}$ is a map of operads and $a : E(T) \longrightarrow \xi^* A$ is a map of $\Omega(T)$-algebras.

(i) As a final example in this section, recall the Boardman–Vogt resolution $W\mathbf{P}$ of a topological operad $\mathbf{P}$ as discussed in Section 1.7. When applied to each of the discrete operads $\Omega(T)$ associated to objects T in $\mathbf{\Omega}$, we obtain a functor

$$w : \mathbf{\Omega} \longrightarrow \mathbf{Op}_{\mathbf{Top}}, T \mapsto W(\Omega(T)).$$

Thus we can define for each topological operad $\mathbf{P}$ its *homotopy-coherent nerve* $w^*\mathbf{P}$, which is the dendroidal set given by

$$w^*\mathbf{P}_T = \mathbf{Op}_{\mathbf{Top}}(w(T), \mathbf{P}).$$

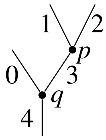
In exactly the same way one can use the simplicial Boardman–Vogt resolution

$$w : \mathbf{\Omega} \longrightarrow \mathbf{sOp}, T \mapsto W_{\mathbf{\Delta}}(\Omega(T))$$

to define a homotopy-coherent nerve $w^*\mathbf{P}$ for any simplicial operad $\mathbf{P}$. Let us make the description of the homotopy-coherent nerve a little more explicit in the topological case by means of an example. For a tree T with one internal edge, as depicted below, a point x in $w^*\mathbf{P}_T$ consists of an assignment of a colour x_i of $\mathbf{P}$ to each of the edges i of T, as well as three operations

$$x_p \in \mathbf{P}(x_1, x_2; x_3), \quad x_q \in \mathbf{P}(x_0, x_3; x_4), \quad x_r \in \mathbf{P}(x_0, x_1, x_2; x_4)$$

of $\mathbf{P}$, together with a path between $x_q \circ_{x_3} x_p$ and x_r in the topological space $\mathbf{P}(x_0, x_1, x_2; x_4)$.

$T:$ 

More generally, for an arbitrary tree T, a point $x \in w^*\mathbf{P}_T$ can be described in terms of a chosen planar structure on T (which we are also implicitly using above) as follows. It is given by a collection of maps

$$x_S : [0,1]^{I(S)} \to \mathbf{P}(\ell(S), \mathrm{root}(S)),$$

one for every subtree S of T, where $I(S)$ is the set of inner edges of S, $\ell(S)$ is the sequence of leaves of S (in the order prescribed by the planar structure of T) and $\mathrm{root}(S)$ is the root of S. These maps should be compatible in the following sense. If S is a subtree and e is an inner edge of S, then 'cutting along e' results in two subtrees S/e and e/S of S, the first having e as its root and the second having e as a leaf. Then on the face of $[0,1]^{I(S)}$ having 1 at coordinate e, the map x_S should take as value the appropriate composition of $x_{S/e}$ and $x_{e/S}$. In a diagram:

$$\begin{tikzcd}
{[0,1]}^{I(S)-\{e\}} \arrow[r, equal] \arrow[dd, "1_e"] & {[0,1]}^{I(e/S)} \times {[0,1]}^{I(S/e)} \arrow[d, "x_{e/S} \times x_{S/e}"] \\
& \mathbf{P}(\ell(e/S), \mathrm{root}(S)) \times \mathbf{P}(\ell(S/e), e) \arrow[d, "\circ_e"] \\
{[0,1]}^{I(S)} \arrow[r, "x_S"] & \mathbf{P}(\ell(S), \mathrm{root}(S)).
\end{tikzcd}$$

We end this section by recording an observation analogous to the one at the beginning of Section 2.3. Let X be a dendroidal set. A dendrex $x \in X_T$ is called *degenerate* if there is a degeneracy map $\sigma : T \to S$ which is not an isomorphism and a dendrex $y \in X_S$ with $\sigma^* y = x$. Of course a dendrex is *non-degenerate* if such a pair (σ, y) does not exist.

Proposition 3.21 *For any dendrex $x \in X_T$ there exists a degeneracy $\sigma : T \to S$ and a non-degenerate dendrex $y \in X_S$ with $\sigma^* y = x$. Furthermore the pair (σ, y) is unique up to isomorphism, in the sense that for any other such pair (σ', y') there is an isomorphism $\alpha : S \to S'$ with $\sigma' = \alpha\sigma$ and $y = \alpha^* z$.*

Proof The existence of (σ, y) is clear. Indeed, if x is degenerate one starts with a degeneracy $\tau : T \to R$ and $z \in X_R$ such that $\tau^* z = x$. If z is non-degenerate we are done; if it is not one simply repeats the process for z. Clearly one arrives at a non-degenerate dendrex after finitely many steps. For uniqueness of (σ, y), suppose that (σ', y') is another such pair as in the proposition. Then one forms the pushout

$$\begin{tikzcd} T \arrow[r, "\sigma"] \arrow[d, "\sigma'"'] & S \arrow[d, "\tau'"] \\ S' \arrow[r, "\tau"] & R, \end{tikzcd}$$

which exists and is an absolute pushout by Lemma 3.15. Therefore the map

$$X_R \to X_S \times_{X_T} X_{S'}$$

is a bijection. Thus there is an element z in X_R whose image is y (resp. y') in X_S (resp. in $X_{S'}$). By the assumption that y and y' are non-degenerate, the maps τ and τ' must both be isomorphisms. The desired isomorphism α is then $\tau^{-1}\tau'$. $\square$

3.5 Categories Related to Dendroidal Sets

In this section we exploit the fact that **dSets** is a presheaf category to construct adjunctions between it and other categories using the method of Kan extension. These categories include those of simplicial sets, operads and Γ-spaces, as well as variations on the category of dendroidal sets. Recall from Section 2.4 that if $f\colon \mathbf{C} \to \mathbf{D}$ is a functor between small categories, restriction along f defines a functor

$$f^* : \mathbf{Sets}^{\mathbf{D}^{\mathrm{op}}} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} : X \mapsto X \circ f$$

which admits a left adjoint $f_!$ and a right adjoint f_*. These are described by the formulas

$$f_! X(d) = \varinjlim_{d \to f(c)} X(c),$$
$$f_* X(d) = \varprojlim_{f(c) \to d} X(c),$$

where the colimit and limit are taken over the slice categories d/f and f/d respectively. The left adjoint $f_!$ is determined up to natural isomorphism by the fact that it coincides with f on representables (after identifying $\mathbf{C}$ and $\mathbf{D}$ with their images under the Yoneda embedding) and preserves colimits. Let us observe the following elementary properties:

Lemma 3.22 *Let $f\colon \mathbf{C} \to \mathbf{D}$ be a functor between small categories, inducing functors $f_!, f^*, f_*$ as above.*

(i) *If f is fully faithful, then so is $f_!$.*
(ii) *If f has a left adjoint $g\colon \mathbf{D} \to \mathbf{C}$, then $g_!$ is also left adjoint to $f_!$. Equivalently, g^* is naturally isomorphic to $f_!$ and g_* to f^*.*

Proof (i) It suffices to show that for each presheaf X on $\mathbf{C}$, the unit $\eta_X : X \longrightarrow f^* f_! X$ is an isomorphism. Since both f^* and $f_!$ preserve colimits and η is natural, it suffices to check this when X is representable. Indeed, any other presheaf is canonically a colimit of representables. But if $X = \mathbf{C}(-, c)$, we have

$$\begin{aligned}(f^* f_! X)(c') &= (f^* \mathbf{D}(-, f(c)))(c') \\ &= \mathbf{D}(f(c'), f(c)),\end{aligned}$$

and $\eta_X(c')$ is the map

$$\mathbf{C}(c', c) \longrightarrow \mathbf{D}(f(c'), f(c)).$$

This map is a bijection if f is fully faithful, so that η_X is indeed an isomorphism.

(ii) Since adjoints are unique up to natural isomorphism, the various assertions in part (ii) are all equivalent. If g is left adjoint to f, then for a representable $\mathbf{C}(-, c)$ there are natural isomorphisms

$$\begin{aligned}g^* \mathbf{C}(-, c) &= \mathbf{C}(g-, c) \\ &\cong \mathbf{D}(-, f(c)) \\ &= f_! \mathbf{C}(-, c).\end{aligned}$$

Since $f_!$ and g^* both preserve colimits, this shows that $f_! \cong g^*$. □

3.5.1 Dendroidal Sets and Operads

Recall that every operad in **Sets** has a *nerve* $N\mathbf{P}$ defined by

$$N\mathbf{P}_T = \mathbf{Op}(\Omega(T), \mathbf{P}).$$

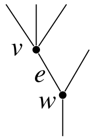
This construction defines an adjoint pair (left adjoint on the left)

$$\tau : \mathbf{dSets} \rightleftarrows \mathbf{Op} : N.$$

Up to natural isomorphism, the left adjoint τ is the unique colimit-preserving functor which coincides with Ω on representables, i.e.

$$\tau(\Omega[T]) = \Omega(T).$$

More explicitly, for a dendroidal set X, the operad $\tau(X)$ has as its colours the elements of the set X_η, while the operations are generated under composition by elements of X_{C_n} for $n \geq 0$. These are subject to the following two types of relations: a degenerate element of X_{C_1} corresponding to a colour $c \in X_\eta$ gives the identity morphism id_c, whereas the operation corresponding to an inner face $\partial_e T$ of a tree T with two vertices (as pictured below) is identified with the composition of the two operations corresponding to the vertices v and w of T.



There is a simplified description of $\tau(X)$ for dendroidal sets satisfying the *inner Kan condition*, to be discussed in Chapter 6. Using our description of $\tau(X)$ (or the simplified one of Chapter 6) it follows easily that the counit $\tau N\mathbf{P} \to \mathbf{P}$ is an isomorphism for any operad $\mathbf{P}$. In other words, the nerve functor N is fully faithful, as it is for simplicial sets and categories.

We observed before that the functor τ, when applied to simplicial sets, preserves products. The same is true for the functor $\tau\colon \mathbf{dSets} \to \mathbf{Op}$ under consideration here. Indeed, for representables $\Omega[S]$ and $\Omega[T]$ we have

$$\tau(\Omega[S] \times \Omega[T]) \cong \tau N(\Omega(S) \times \Omega(T)) \cong \Omega(S) \times \Omega(T).$$

The case of general dendroidal sets X and Y now follows from the case of representables, because the expressions $\tau(X \times Y)$ and $\tau(X) \times \tau(Y)$ both commute with colimits in each variable separately.

3.5.2 Dendroidal Sets and Simplicial Sets

The inclusion $i : \mathbf{\Delta} \to \mathbf{\Omega}$ defines three adjoint functors

$$\begin{tikzcd} \mathbf{sSets} \arrow[r, bend left, "i_!"] \arrow[r, bend right, "i_*"'] & \mathbf{dSets} \arrow[l, "i^*"'] \end{tikzcd}.$$

Thus, each dendroidal set X has an underlying simplicial set i^*X defined by

$$(i^*X)_n = X_{[n]}.$$

Conversely, a simplicial set M gives rise to a dendroidal set $i_!M$ by 'extension by zero', as already mentioned in Example 3.20(g) above. Since $i : \mathbf{\Delta} \to \mathbf{\Omega}$ is fully faithful, so is $i_!$ by part (i) of Lemma 3.22. Thus, we can identify the category of simplicial sets with a full subcategory of the category of dendroidal sets. In fact, it is also a slice category: indeed, the equivalence

$$\mathbf{\Delta} \simeq \mathbf{\Omega}/\eta$$

also implies the equivalence of categories

$$\mathbf{sSets} \simeq \mathbf{dSets}/\Omega[\eta].$$

The commutative square of functors on the left gives rise to the commutative square of adjoint pairs on the right (meaning that the square of left adjoints commutes up to natural isomorphism, as does the square of right adjoints):

$$
\begin{tikzcd}
\mathbf{\Delta} \arrow[r] \arrow[d, "i"'] & \mathbf{Cat} \arrow[d, "j"] \\
\mathbf{\Omega} \arrow[r] & \mathbf{Op},
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbf{sSets} \arrow[r, shift left, "\tau"] \arrow[d, shift right, "i_!"'] & \mathbf{Cat} \arrow[l, shift left, "N"] \arrow[d, shift right, "j_!"'] \\
\mathbf{dSets} \arrow[r, shift left, "\tau"] \arrow[u, shift right, "i^*"'] & \mathbf{Op}. \arrow[l, shift left, "N"] \arrow[u, shift right, "j^*"']
\end{tikzcd}
$$

In the diagram on the right we follow our standard convention of writing left adjoints to the left or on top of their right adjoints. The commutativity of the square translates into natural isomorphisms of functors

$$j_!\tau \simeq \tau i_! \qquad \text{and} \qquad Nj^* \simeq i^*N.$$

Note that since $i_!$, $j_!$ and both functors labelled N are fully faithful, we also have the following natural isomorphisms:

$$Nj_! \simeq i_!N \qquad \text{and} \qquad \tau i^* \simeq j^*\tau.$$

3.5.3 Dendroidal Sets and Simplicial Operads

The category of simplicial operads has all small colimits. Therefore the simplicial Boardman–Vogt resolution $W_{\mathbf{\Delta}}$ induces an adjoint pair

$$w_! : \mathbf{dSets} \rightleftarrows \mathbf{sOp} : w^*$$

for which

$$w_!\Omega[T] = W_{\mathbf{\Delta}}(\Omega(T)).$$

In other words, the homotopy-coherent nerve functor w^* introduced in the previous section has a left adjoint. A similar procedure applies to the category of topological (rather than simplicial) operads. One of the main goals of the second part of this book will be to show that this adjoint pair induces an equivalence of 'homotopy categories' in an appropriate sense.

3.5.4 Open Dendroidal Sets

The inclusion $u : \mathbf{\Omega}^\circ \to \mathbf{\Omega}$ of the category of open trees into the category of all trees induces adjoint functors

$$\begin{tikzcd} \mathbf{odSets} \arrow[r, bend left, "u_!"] \arrow[r, bend right, "u_*"'] & \mathbf{dSets}, \arrow[l, "u^*"'] \end{tikzcd}$$

where we have written **odSets** for the category of presheaves on $\mathbf{\Omega}^\circ$, which we will refer to as the category of *open dendroidal sets*. The functors involved are very easy to describe. Any dendroidal set X restricts to a presheaf u^*X on open trees, whereas any open dendroidal set Y can be 'extended by zero' to form $u_!Y$:

$$(u_!Y)_T = \begin{cases} Y_T & \text{if } T \text{ is open,} \\ \varnothing & \text{otherwise.} \end{cases}$$

This is indeed a presheaf, because if $S \to T$ is any morphism in $\mathbf{\Omega}$ then S must be open if T is. The functor $u_!$ is a fully faithful embedding because u is (cf. Lemma 3.22) and we will usually identify **odSets** with the corresponding full subcategory of **dSets**.

The category of open dendroidal sets can also be described as a slice category. Indeed, write $1 = N\mathbf{Com}$ for the terminal dendroidal set and define $O := u_!u^*1$. It can be described by

$$O_T = \begin{cases} * & \text{if } T \text{ is open,} \\ \varnothing & \text{otherwise.} \end{cases}$$

It is then clear that a general dendroidal set X is open (i.e., is in the essential image of $u_!$) if and only if it admits a (necessarily unique) map $X \to O$. Hence $u_!$ gives an equivalence of categories

$$\mathbf{odSets} \xrightarrow{\simeq} \mathbf{dSets}/O.$$

An operad $\mathbf{P}$ is called *open* if $\mathbf{P}(\varnothing, c)$ is empty for each colour c. The nerve of an open operad is an open dendroidal set.

3.5.5 Closed Dendroidal Sets

The adjoint pair

$$\mathrm{cl} : \mathbf{\Omega} \rightleftarrows \overline{\mathbf{\Omega}} : \mathrm{incl}$$

given by the inclusion of closed trees into $\mathbf{\Omega}$ defines, according to Lemma 3.22(ii), adjoint functors

$$\mathrm{cl}_! \dashv \mathrm{cl}^* = \mathrm{incl}_! \dashv \mathrm{cl}_* = \mathrm{incl}^* \dashv \mathrm{incl}_*.$$

Here $f \dashv g$ is shorthand for 'f is left adjoint to g'. We write **cdSets** for the category of closed dendroidal sets, i.e., the category of presheaves on $\overline{\mathbf{\Omega}}$. The functor

$$\mathrm{incl}_! : \mathbf{cdSets} \to \mathbf{dSets}$$

is fully faithful and we will usually identify **cdSets** with the corresponding full subcategory of **dSets**. There is a useful way to characterize the dendroidal sets in the essential image of $\mathrm{incl}_!$. Indeed, a dendroidal set X is isomorphic to one of the form $\mathrm{incl}_! Y$ if and only if for every tree T, the restriction map

$$\mathbf{dSets}(\Omega[\overline{T}], X) \to \mathbf{dSets}(\Omega[T], X)$$

is a bijection. If $X = \mathrm{incl}_! Y$, then one uses that $\mathrm{incl}_! = \mathrm{cl}^*$. Conversely, for X satisfying the condition above, one checks that the counit $\mathrm{incl}_!\mathrm{incl}^* X \to X$ is an isomorphism. To do this, observe that

$$\begin{aligned}\mathbf{dSets}(\Omega[T], \mathrm{incl}_!\mathrm{incl}^* X) &= \mathbf{dSets}(\Omega[T], \mathrm{cl}^*\mathrm{incl}^* X)\\ &\cong \mathbf{dSets}(\mathrm{incl}_!\mathrm{cl}_!\Omega[T], X)\end{aligned}$$

and $\mathrm{incl}_!\mathrm{cl}_!\Omega[T] = \Omega[\overline{T}]$.

An operad **P** is called *unital* if $\mathbf{P}(\varnothing; c)$ is a one-point set for every colour c of **P**. In other words, **P** has a unique constant for each colour. The terminology ‘unital’ comes from operads like **Ass** and **Com**, where the constant in $\mathbf{P}(0)$ gives the unit in each associative or commutative algebra. The nerve of a unital operad is a closed dendroidal set.

3.5.6 Uncoloured Dendroidal Sets

A dendroidal set X is called *uncoloured* if X_η is a one-point set. The category of uncoloured dendroidal sets will be denoted **udSets**. The nerve of an operad with a single colour is an uncoloured dendroidal set.

A *pointed* dendroidal set is a pair (X, x_0) with X a dendroidal set and $x_0 \in X_\eta$ a chosen basepoint. These pointed dendroidal sets and maps preserving the basepoint form a category which we denote by $\mathbf{dSets}_*$. For later use we construct some functors relating this category to that of uncoloured dendroidal sets. There are evident ‘forgetful’ functors

$$\mathbf{udSets} \to \mathbf{dSets}_* \to \mathbf{dSets},$$

the first of which is fully faithful, while the second is only faithful. This second functor admits a left adjoint

$$\mathbf{dSets} \to \mathbf{dSets}_* \colon X \mapsto X_+ := X \amalg \eta,$$

which ‘freely adds a basepoint’. The forgetful functor from $\mathbf{dSets}_* \to \mathbf{dSets}$ does *not* admit a right adjoint though, simply because it does not preserve coproducts. The functor $\mathbf{udSets} \to \mathbf{dSets}_*$, which we will denote r^*, does admit a right adjoint

$$r_* \colon \mathbf{dSets}_* \to \mathbf{udSets}$$

which is given by 'restriction to the basepoint': for a pointed dendroidal set (X, x_0) and a tree T, one defines $r_*(X, x_0)_T \subseteq X(T)$ as the subset of dendrices all of whose edges are x_0. More formally, there is a pullback square

$$\begin{tikzcd} r_*(X, x_0)_T \arrow[r] \arrow[d] & X_T \arrow[d] \\ * \arrow[r, "\overline{x_0}"] & \prod_{e \in E(T)} X_\eta, \end{tikzcd}$$

where $\overline{x_0}$ takes the value x_0 in every coordinate. The inclusion **udSets** $\to$ **dSets**$_*$ also admits a left adjoint

$$r_! \colon \mathbf{dSets}_* \to \mathbf{udSets},$$

defined by simply collapsing all the elements of X_η to the basepoint. More formally, each $x \in X_\eta$ defines a map $x \colon \eta \to X$ of dendroidal sets, and $r_!(X, x_0)$ is the pushout

$$\begin{tikzcd} \coprod_{x \in X_\eta} \eta \arrow[r] \arrow[d] & X \arrow[d] \\ \eta \arrow[r] & r_!(X, x_0) \end{tikzcd}$$

We will sometimes call $r_!(X, x_0)$ the *reduction* of (X, x_0). The x_0 may be omitted from the notation, since the reduction is independent of the choice of basepoint.

3.5.7 Dendroidal Sets and Γ-Sets

An important tool in the theory of infinite loop spaces is Segal's category $\mathbf{\Gamma}$. We shall recall the definition of this category and construct several functors relating dendroidal sets to presheaves on $\mathbf{\Gamma}$. In later parts of this book we will return to the relation between dendroidal sets and the theory of infinite loop spaces in depth.

We write $\mathbf{F}_{\text{part}}$ for the category of which the objects are finite sets and the morphisms are *partial maps*. A partial map from A to B is by definition a pair (U, f), with $U \subseteq A$ and $f : U \to B$ an ordinary map of sets. Given another such partial map (W, g) from B to C, the composition of the two is given by $(f^{-1}W, g \circ f|_{f^{-1}W})$. The category $\mathbf{\Gamma}$ is the opposite category of $\mathbf{F}_{\text{part}}$.

We will now define functors

$$\lambda : \mathbf{\Omega} \to \mathbf{\Gamma} \qquad \text{and} \qquad V : \mathbf{\Omega} \to \mathbf{\Gamma}.$$

Here λ sends a tree T to its set of leaves. For a morphism of trees $\varphi : S \to T$ one defines $\lambda(\varphi)$ as follows. Consider an element $\ell \in \lambda(T)$, i.e. a leaf of T, and the path from ℓ to the root. If this path meets an edge $\varphi(\ell')$ with ℓ' some leaf of S (which is unique if it exists) then $\lambda(\varphi)(\ell) = \ell'$. If that path does not meet any such edge then $\lambda(\varphi)$ is not defined on ℓ. As before, the functor λ induces adjoint functors

$$\begin{tikzcd} \mathbf{dSets} \arrow[r, bend left, "\lambda_!"] \arrow[r, bend right, "\lambda_*"'] & \mathrm{PSh}(\mathbf{\Gamma}). \arrow[l, "\lambda^*"'] \end{tikzcd}$$

The functor V sends a tree T to its set of vertices $V(T)$. Consider a map $\varphi\colon S \to T$ and a vertex $v \in V(T)$. If v occurs in a subtree $\varphi(w)$ of T, with w some vertex of S, then we set $V(\varphi)(v) = w$. If v does not occur in such a subtree, then $V(\varphi)$ is not defined on v. Again there are induced functors $V_!$, V^*, and V_*. Of course there is also a functor $E\colon \mathbf{\Omega} \to \mathbf{Sets}$ sending a tree T to its set of edges $E(T)$, inducing analogous adjoint pairs, but we will not have any use for these.

Let us also briefly consider the restriction of the functor V to the subcategory $\mathbf{\Delta} \subset \mathbf{\Omega}$. For an object $[n] \in \mathbf{\Delta}$, interpreted as a linear tree with $n + 1$ edges, its set of vertices $V([n])$ can be identified with the linear order

$$1 < 2 < \cdots < n.$$

Write $\mathbf{\Gamma}_{\mathrm{ord}}$ for the opposite of the category of *linearly ordered* finite sets and partial maps $(U, f)\colon A \to B$ with the property that U is a convex subset of A. Then the restricted functor $V\colon \mathbf{\Delta} \to \mathbf{\Gamma}$ factors through this category $\mathbf{\Gamma}_{\mathrm{ord}}$ in an evident way. This category admits a perhaps more familiar interpretation in terms of *intervals*, as follows. Write **Int** for the category whose objects are the linearly ordered finite sets

$$\mathbf{n} = (-\infty < 1 < \cdots < n < \infty)$$

for $n \geq 0$, and whose maps are the monotone maps f between these preserving the endpoints $-\infty$ and ∞. Any such map $f\colon \mathbf{n} \to \mathbf{m}$ defines a partial map $V([n]) \to V([m])$ by restricting it to the preimage of $V([m])$ in $V([n])$, which is always convex. This assignment defines an equivalence of categories

$$\mathbf{Int}^{\mathrm{op}} \xrightarrow{\simeq} \mathbf{\Gamma}_{\mathrm{ord}}.$$

Thus we may interpret the restriction of V to $\mathbf{\Delta}$ as a functor $V\colon \mathbf{\Delta} \to \mathrm{Int}^{\mathrm{op}}$. The reader can verify that this functor is in fact an isomorphism of categories, thus giving a duality between linearly ordered finite sets and intervals. Note that the vertices of the linear tree corresponding to $[n]$ form precisely the set of 'cuts' of the linearly ordered set $[n]$ into two pieces; it is this interpretation of V one often finds in the literature.

3.6 Normal Dendroidal Sets and Skeletal Filtration

For a dendroidal set X and a tree T in $\mathbf{\Omega}$, the group $\mathrm{Aut}(T)$ acts on the set X_T. The dendroidal set X is called *normal* if this action is free for any tree T. More generally, a monomorphism $u\colon X \to Y$ of dendroidal sets is called *normal* if for each tree T the group $\mathrm{Aut}(T)$ acts freely on the complement $Y_T - u(X_T)$ of the image of u.

Thus, X is normal precisely if the inclusion of the empty dendroidal set $\varnothing \to X$ is a normal monomorphism. In Part II we will see that normality plays an important role in the homotopy theory of dendroidal sets, comparable to that of CW complexes in the homotopy theory of topological spaces. In this section we will show that normal dendroidal sets have a well-behaved skeletal filtration analogous to that of simplicial sets. But before that, we start with some examples.

Example 3.23 (a) Any representable dendroidal set $\Omega[T]$ is normal. To see this, suppose $f: S \to T$ is a morphism in $\mathbf{\Omega}$ and α is an automorphism of S such that $f\alpha = f$. Factor f as $\delta\beta\sigma$, where $\sigma : S \to S'$ is a degeneracy, β is an isomorphism, and δ is a face. Since $\delta\beta$ is a monomorphism in δ, we find that $\sigma\alpha = \sigma$. By Proposition 3.13 it follows that α is the identity.

(b) The dendroidal set of planar structures $\mathrm{Pl} \cong N\mathbf{Ass}$ is normal.

(c) The terminal dendroidal set $N\mathbf{Com}$ is *not* normal.

(d) Example (b) is a special case of the following general fact. Let $\mathbf{P}$ be an operad with the property that for each colour c, the symmetric group Σ_n acts freely on the set

$$\mathbf{P}_n(-;c) := \coprod_{c_1,\ldots,c_n} \mathbf{P}(c_1,\ldots,c_n;c)$$

of all n-ary operations into c. (An operad with this property is called Σ-*free*.) Then its nerve $N\mathbf{P}$ is a normal dendroidal set. To see this, first note that by assumption the automorphisms of a corolla C_n act freely on $N\mathbf{P}_{C_n}$. For a general tree T we argue by induction on its size. Suppose α is an automorphism of T and that $\alpha^*\xi = \xi$ for some dendrex $\xi \in N\mathbf{P}_T$. Then α restricts to an automorphism α_r of the root corolla C_{v_r}, which is the corolla consisting of the root r of T, the root vertex v_r, and its incoming edges $e_1,\ldots,e_n$. Write $j : C_{v_r} \to T$ for the inclusion of this corolla in T. Then $\alpha_r^*(j^*\xi) = j^*\xi$, so that α_r must be the identity. It follows that α restricts to an automorphism α_i of each of the subtrees T/e_i obtained by chopping off the root corolla. The restriction of ξ to T/e_i is fixed by α_i, so that (by induction) α_i must be the identity. It follows that α is the identity as well.

(e) The same reasoning as in the previous example shows that for a Σ-free topological or simplicial operad $\mathbf{P}$, its homotopy-coherent nerve $w^*\mathbf{P}$ is normal.

The following simple observation will sometimes be useful:

Proposition 3.24 *If X is a dendroidal set such that* $\mathrm{Aut}(T)$ *acts freely on the set* $\mathrm{nd}(X_T)$ *of non-degenerate T-dendrices for every T, then X is normal. More generally, if $u : X \to Y$ is a monomorphism and* $\mathrm{Aut}(T)$ *acts freely on the non-degenerate T-dendrices in $Y_T - u(X_T)$, then u is normal.*

Proof Implicit in the formulation of the proposition is that a T-dendrex x is non-degenerate if and only if α^*x is, for any automorphism α of T, so that the action of $\mathrm{Aut}(T)$ restricts to give an action on non-degenerate T-dendrices. This fact follows easily from Proposition 3.13. We prove the second claim of the proposition, since the first is a special case. Suppose $\mathrm{Aut}(T)$ acts freely on non-degenerate elements of $Y_T - u(X_T)$ for all T. Let $x \in Y_T - u(X_T)$ be an arbitrary element and write

$x = \sigma^* y$ for some degeneracy $\sigma : T \longrightarrow S$ and a non-degenerate $y \in Y_S - u(X_S)$ (cf. Proposition 3.21). If $\alpha \in \mathrm{Aut}(T)$ fixes x, let β be the unique automorphism of S such that $\beta\sigma = \sigma\alpha$. Then

$$\sigma^*(\beta^* y) = \alpha^*(\sigma^* y) = \alpha^* x = x = \sigma^*(y).$$

Since σ^* is a monomorphism (σ has a section), we conclude that $\beta^* y = y$. Since y is non-degenerate, we have $\beta = \mathrm{id}$ by assumption. Therefore α is the identity as well, since the homomorphism $\sigma_! : \mathrm{Aut}(T) \longrightarrow \mathrm{Aut}(S)$ is injective. □

We will now discuss skeletal filtrations of dendroidal sets. For a dendroidal set X, we define a dendroidal subset

$$\mathrm{sk}_n X \subseteq X$$

by declaring that a dendrex $x \in X_T$ belongs to $(\mathrm{sk}_n X)_T$ if there exists a tree S with at most n vertices and an element $y \in X_S$ together with a morphism $\alpha : T \longrightarrow S$ such that $\alpha^* y = x$. Notice that if $\beta : T' \longrightarrow T$ is any morphism in $\mathbf{\Omega}$ and $x \in (\mathrm{sk}_n X)_T$, then $\beta^* x \in (\mathrm{sk}_n X)_{T'}$. So $\mathrm{sk}_n X$ is a well-defined dendroidal set and the inclusion $\mathrm{sk}_n X \longrightarrow X$ is a morphism of dendroidal sets. In this way one obtains a filtration

$$\mathrm{sk}_0 X \subseteq \mathrm{sk}_1 X \subseteq \mathrm{sk}_2 X \subseteq \cdots$$

with

$$\bigcup_{n=0}^{\infty} \mathrm{sk}_n X = X.$$

One refers to $\mathrm{sk}_n X$ as the *n-skeleton of* X and to the above filtration as the *skeletal filtration* of X. Note that

$$\mathrm{sk}_0 X = \coprod_{x \in X_\eta} \Omega[\eta].$$

In contrast to the case of simplicial sets this skeletal filtration is not necessarily of much use in proving properties of a general dendroidal set X, but it turns out to be very useful when dealing with *normal* dendroidal sets. The reason is that in this case the n-skeleton of X is obtained from the $(n-1)$-skeleton by 'cell attachments', as expressed by the following:

Proposition 3.25 *Let X be a normal dendroidal set. Then for each $n > 0$, the dendroidal set* $\mathrm{sk}_n X$ *can be obtained from* $\mathrm{sk}_{n-1} X$ *by means of a pushout square*

$$\begin{tikzcd}
\coprod_{(T,x)} \partial\Omega[T] \arrow[r, "f"] \arrow[d] & \mathrm{sk}_{n-1} X \arrow[d] \\
\coprod_{(T,x)} \Omega[T] \arrow[r, "g"] & \mathrm{sk}_n X.
\end{tikzcd}$$

Here the coproduct is over pairs (T, x) with T a tree with precisely n vertices and $x \in X_T$ a non-degenerate dendrex, where we take one such pair in each isomorphism class. By definition, two pairs (T, x) and (S, y) are isomorphic if there is an isomorphism $\alpha : T \longrightarrow S$ with $\alpha^* y = x$. The map g sends the summand $\Omega[T]$ indexed by (T, x) to $\mathrm{sk}_n X$ by means of the map $\Omega[T] \to X$ corresponding to x. The map f is the restriction of g.

Proof Let us temporarily write P for the pushout in the square and $p : P \longrightarrow \mathrm{sk}_n X \subseteq X$ for the evident map. We will show that this map is an isomorphism.

Let T be a tree and let $x \in (\mathrm{sk}_n X)_T$. Then $x = \alpha^* y$ for some $\alpha : T \longrightarrow S$ with S at most n vertices and y a non-degenerate dendrex in X_S. If S has fewer than n vertices, x is contained in $\mathrm{sk}_{n-1} X$. If S has precisely n vertices, then there is an isomorphism $\beta : S \longrightarrow S'$ and $y' \in X_{S'}$ with $\beta^* y' = y$ such that the summand (S', y') occurs in the indexing set of the coproduct. So x must lie in the image of g. We conclude that $p : P \longrightarrow \mathrm{sk}_n X$ is surjective.

We will now prove that if X is normal, the map p is also injective. Since $\mathrm{sk}_{n-1} X \longrightarrow \mathrm{sk}_n X$ is a monomorphism there are two things to check:

(a) If (T, x) occurs as a summand on the left and $u : S \longrightarrow T$ is a morphism in $\mathbf{\Omega}$ with $u^* x \in (\mathrm{sk}_{n-1} X)_S$, then u factors through a face of T (so that u is an element of $\partial\Omega[T]_S$).

(b) If (T, x) and (S, y) both occur as summands on the left and $T \xleftarrow{u} R \xrightarrow{v} S$ are morphisms in $\mathbf{\Omega}$ such that $u^* x = v^* y$ in X_R, then either $(T, x) = (S, y)$ and $u = v$ or u factors through a face of T and v through a face of S.

Loosely speaking, item (b) expresses the idea that any two elements identified by the map g are already identified by f.

Proof of (a): Take (T, x) and u as in the statement of (a). Then $u^* x = v^* y$ for some morphism $v : S \longrightarrow R$ and $y \in X_R$ non-degenerate, with R having fewer than n vertices. Suppose to the contrary that u does *not* factor through a face of T. Then u must necessarily be a negative map, i.e. a composition of a degeneracy and an isomorphism. Factor v as $S \xrightarrow{v_-} S' \xrightarrow{v_+} R$ with v_- negative and v_+ positive. Then consider the diagram

$$\begin{tikzcd} & S \arrow[r, "u"] \arrow[dl, "v"'] \arrow[d, "v_-"] & T \arrow[d] \\ R & S' \arrow[l, "v_+"] \arrow[r] & V, \end{tikzcd}$$

in which the square is a pushout (which exists by Lemma 3.15). Since x is non-degenerate, the map $T \longrightarrow V$ must be an isomorphism. Then V has n vertices, but at the same time the bottom line of the diagram shows it has fewer vertices than R (since v_+ is injective), which is a contradiction.

Proof of (b): Choose factorizations of u and v into a negative morphism followed by a positive morphism, say $R \xrightarrow{u_-} R' \xrightarrow{u_+} T$ and $R \xrightarrow{v_-} R'' \xrightarrow{v_+} S$. Form the pushout of u_- and v_- to obtain a diagram

$$\begin{tikzcd} & & T \\ & R \arrow[r, "u_-"] \arrow[d, "v_-"'] & R' \arrow[u, "u_+"] \arrow[d, "p"] \\ S & R'' \arrow[l, "v_+"] \arrow[r, "q"'] & Q. \end{tikzcd}$$

Since the pushout is absolute the resulting map

$$X_Q \longrightarrow X_{R'} \times_{X_R} X_{R''}$$

is bijective. Hence there is an element $z \in X_Q$ with $p^*z = u_+^*x$ and $q^*z = v_+^*y$. But x and y are both non-degenerate, so that if u_+ is an isomorphism then so is p; similarly if v_+ is an isomorphism then so is q. Now, if both u_+ and v_+ are *not* isomorphisms then u and v factor through faces and we are done. So suppose one of them is an isomorphism, say u_+, so that p is an isomorphism as well as we just said. Then T, R' and Q all have the same number of vertices, namely n. The bottom row of the diagram gives inequalities

$$n = |V(Q)| \leq |V(R'')| \leq |V(S)|.$$

But $|V(S)| = n$ so that in fact these are both equalities. It follows that q and v_+ are isomorphisms as well. The composition $\alpha = v_+q^{-1}pu_+^{-1}$ is an isomorphism satisfying $x = \alpha^*y$. Therefore $(T, x) = (S, y)$, since in the coproduct we picked only one pair in each isomorphism class. Since X is assumed to be normal, the isomorphism α must be the identity. Now observe that

$$\begin{aligned} u &= u_+u_- \\ &= \alpha u_+u_- \\ &= v_+q^{-1}pu_- \\ &= v_+q^{-1}qv_- \\ &= v, \end{aligned}$$

finishing the proof. □

In exactly the same way one can prove a relative version of the above statement about skeletal filtrations. To state it, let $A \to X$ be a normal monomorphism of dendroidal sets. For ease of notation let us identify A with its image in X. Let

$$\mathrm{sk}_n(X, A) = A \cup \mathrm{sk}_n X,$$

giving a filtration

$$A =: \mathrm{sk}_{-1}(X, A) \subseteq \mathrm{sk}_0(X, A) \subseteq \mathrm{sk}_1(X, A) \subseteq \cdots$$

with union X. The argument of the previous proposition then also gives the following more general statement:

Proposition 3.26 *Let $A \to X$ be a monomorphism of dendroidal sets. If it is a normal monomorphism, then for each $n \geq 0$ there is a pushout square*

$$\begin{tikzcd}
\coprod_{(T,x)} \partial\Omega[T] \arrow[r, "f"] \arrow[d] & \mathrm{sk}_{n-1}(X, A) \arrow[d] \\
\coprod_{(T,x)} \Omega[T] \arrow[r, "g"] & \mathrm{sk}_n(X, A).
\end{tikzcd}$$

The coproduct ranges over isomorphism classes of pairs (T, x) where T has n vertices and $x \in X_T$ is a non-degenerate dendrex not contained in A_T.

Finally, there is also a variant of the skeletal filtration for closed dendroidal sets which we will sometimes use. For a closed tree T, let us define the *closed boundary* $\partial^{\mathrm{cl}}\Omega[T]$ of T to be the union of all faces S of T which are themselves closed trees. We define the *closed n-skeleton* $\mathrm{sk}_n^{\mathrm{cl}}X$ to be the dendroidal subset of X generated by all dendrices $x \in X_T$ for T a *closed* tree with at most n vertices. A straightforward adaptation of the proof of Proposition 3.27 can be used for the following:

Proposition 3.27 *Let X be a closed normal dendroidal set. Then for each $n > 0$, the closed dendroidal set $\mathrm{sk}_n^{\mathrm{cl}}X$ can be obtained from $\mathrm{sk}_{n-1}^{\mathrm{cl}}X$ by means of a pushout square*

$$\begin{tikzcd}
\coprod_{(T,x)} \partial^{\mathrm{cl}}\Omega[T] \arrow[r, "f"] \arrow[d] & \mathrm{sk}_{n-1}^{\mathrm{cl}}X \arrow[d] \\
\coprod_{(T,x)} \Omega[T] \arrow[r, "g"] & \mathrm{sk}_n^{\mathrm{cl}}X.
\end{tikzcd}$$

Here the coproduct is over pairs (T, x) with T a closed tree with precisely n vertices and $x \in X_T$ a non-degenerate dendrex, where we take one such pair in each isomorphism class. The map g sends the summand $\Omega[T]$ indexed by (T, x) to $\mathrm{sk}_n^{\mathrm{cl}}X$ by means of the map $\Omega[T] \to X$ corresponding to x. The map f is the restriction of g.

3.7 Normal Monomorphisms and Normalization

The previous section indicates that normal monomorphisms play an important role in the theory of dendroidal sets, analogous to relative CW complexes in the homotopy theory of topological spaces. Here we will discuss several elementary properties of the class of normal monomorphisms. In particular, we show it is a *saturated* class of morphisms (see Definition 3.30). Saturated classes naturally occur when studying *lifting properties* of morphisms and will feature heavily in Part II of this book. In this section we present Quillen's small object argument, which allows us to factor every morphism into a normal monomorphism followed by a *trivial fibration*. In particular, we discuss normalizations of a dendroidal set X.

Proposition 3.28 (i) *If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are normal monomorphisms, then so is their composition gf.*

(ii) *If*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

is a pullback in which $C \to D$ is a normal mono, then $A \to B$ is a normal mono as well.

(iii) *If $A \to B$ is a morphism of dendroidal sets and B is normal, then A is normal as well.*

(iv) *If*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

is a pushout in which $A \to B$ is a normal mono, then $C \to D$ is a normal mono as well.

(v) *If $A \xrightarrow{u} B$ is a retract of $C \xrightarrow{v} D$, i.e. if there exists a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "u"'] & C \arrow[r, "r"] \arrow[d, "v"'] & A \arrow[d, "u"'] \\ B \arrow[r, "j"] & D \arrow[r, "s"] & B \end{tikzcd}$$

in which $ri = \mathrm{id}_A$ and $sj = \mathrm{id}_B$, then u is a normal mono whenever v is.

(vi) *If $\{A_i \to B_i\}_{i \in I}$ is a family of normal monomorphisms, then their coproduct $\amalg_i A_i \to \amalg_i B_i$ is a normal mono as well.*

(vii) *If $A_0 \to A_1 \to A_2 \to \cdots$ is a sequence with colimit A_∞ for which each $A_i \to A_{i+1}$ is a normal mono, then each $A_i \to A_\infty$ is a normal mono as well.*

Remark 3.29 (a) A property similar to (vii) holds for longer 'continuous' sequences indexed by a limit ordinal λ. Explicitly, if $\{A_\zeta \xrightarrow{f_{\zeta\xi}} A_\xi \mid \zeta < \xi < \lambda\}$ is a family of normal monos such that $f_{\eta\xi} \circ f_{\zeta\eta} = f_{\zeta\xi}$ for all $\zeta < \eta < \xi$, and $A_\mu = \varinjlim_{\xi<\mu} A_\xi$ for μ a limit ordinal, then each map $A_\zeta \to \varinjlim_{\xi<\lambda} A_\xi$ of the colimit cone is a normal mono.

(b) Property (vi), which is easily proved directly, also follows formally from properties (iv) and (vii), if necessary using its generalization just described in (a). Indeed, if for instance $\{A_n \to B_n\}$ is a countable family of normal monos, then

$\amalg_n A_n \to \amalg_n B_n$ is the map $C_0 \to C_\infty$ defined by the sequence $C_0 \to C_1 \to C_2 \to \cdots$ with

$$C_i = \coprod_{k<i} B_k \amalg \coprod_{k \geq i} A_k.$$

The map $C_i \to C_{i+1}$ is a pushout of the map $A_i \to B_i$.

Proof (of Proposition 3.28) All parts of this proposition can be proved by elementary verification, using the fact that limits and colimits of dendroidal sets are computed 'pointwise'.

(i) Let us identify A and B with their images in C. Then for any tree T in $\mathbf{\Omega}$, the complement $C_T - A_T$ is the disjoint union of $C_T - B_T$ and $B_T - A_T$. So if $\mathrm{Aut}(T)$ acts freely on the latter two, it acts freely on $C_T - A_T$ as well.

(ii) For any tree T, the map $B_T \to D_T$ restricts to a map $B_T - A_T \to D_T - C_T$ because the square is a pullback. By assumption $\mathrm{Aut}(T)$ acts freely on the latter, so it must act freely on $B_T - A_T$ as well.

(iii) Consider the pullback square

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \varnothing \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

and apply (ii).

(iv) For any tree T we have a pushout of sets

$$\begin{tikzcd} A_T \arrow[r] \arrow[d] & C_T \arrow[d] \\ B_T \arrow[r] & D_T. \end{tikzcd}$$

If we identify A_T with its image in B_T we can rewrite this as

$$\begin{tikzcd} A_T \arrow[r] \arrow[d] & C_T \arrow[d] \\ A_T \amalg (B_T - A_T) \arrow[r] & C_T \amalg (B_T - A_T). \end{tikzcd}$$

In particular, $B_T \to D_T$ induces an isomorphism

$$B_T - A_T \to D_T - C_T,$$

from which the assertion is clear.

(v) The complement of the image of $A \xrightarrow{u} B$ maps to the complement of the image of $C \xrightarrow{v} D$. Indeed, if $x \in B_T$ and $j(x) = v(y)$ for some $y \in C_T$, then $x = sj(x) = sv(y) = ur(y)$, so that x is in the image of u. The assertion follows, because $B_T - A_T$ maps to a free $\mathrm{Aut}(T)$-set and is therefore free.

(vi) This is clear from the fact that the complement of the image of $(\amalg_n A_n)_T \to (\amalg_n B_n)_T$ is the disjoint union of the complements of the images of $(A_n)_T \to (B_n)_T$.

(vii) For any tree T, the complement of the image of $(A_n)_T \to (A_\infty)_T$ is isomorphic to the disjoint union of the complements of the images of each of the $(A_n)_T \to (A_{n+1})_T$, from which the assertion is clear. □

Definition 3.30 Let $\mathbf{C}$ be a category with all small colimits. A class of morphisms $\mathcal{A}$ in $\mathbf{C}$ is *weakly saturated* if it contains all isomorphisms and is closed under pushouts, composition and transfinite composition (as in (vii) above and its 'longer' version of the subsequent remark). The class $\mathcal{A}$ is *saturated* if moreover it is closed under retracts.

Thus, the previous proposition implies that the normal monomorphisms form a saturated class of morphisms in dendroidal sets. Moreover, remark (b) shows that any saturated class is closed under coproducts. The arguments of the previous section show that the normal monomorphisms are 'generated' by boundary inclusions of trees in the following sense:

Theorem 3.31 *The class of normal monomorphisms is the smallest weakly saturated class containing the boundary inclusion $\partial\Omega[T] \to \Omega[T]$ for every tree T in $\mathbf{\Omega}$.*

Proof The class of normal monos is saturated, so in particular weakly saturated. Moreover, each boundary inclusion $\partial\Omega[T] \to \Omega[T]$ is in particular a normal monomorphism; indeed, observe that $\Omega[T]$ is normal (cf. Example 3.23(a)) and apply Proposition 3.28(iii). Therefore the class described in the theorem is contained in the class of normal monomorphisms. To see that these classes are equal one applies Proposition 3.26, which shows that any normal monomorphism can be written as a transfinite composition of pushouts of boundary inclusions of trees. □

As mentioned at the start of this section, saturated classes arise when studying lifting properties. Perhaps the most familiar examples of such occur in the definitions of Serre fibrations and cofibrations in classical homotopy theory.

Definition 3.32 Let $\mathbf{C}$ be a category and $\mathcal{F}$ a class of morphisms in $\mathbf{C}$. We say that a morphism $i\colon A \to B$ has the *left lifting property* with respect to $\mathcal{F}$ if, for every $f\colon X \to Y$ in $\mathcal{F}$ and any commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "v"'] & Y, \end{tikzcd}$$

there exists a morphism $g\colon B \to X$ such that $gi = u$ and $fg = v$. (More briefly, there exists a *lift* in the square.) We denote the class of morphisms having the left lifting property with respect to $\mathcal{F}$ by ${}^{\perp}\mathcal{F}$.

Dually, for a class of morphisms $\mathcal{C}$, we say a morphism $f\colon X \to Y$ of $\mathbf{C}$ has the *right lifting property* with respect to $\mathcal{C}$ if for any $i\colon A \to B$ in $\mathcal{C}$ and any square as above there exists a lift. The class of morphisms having the right lifting property with respect to $\mathcal{C}$ is denoted $\mathcal{C}^{\perp}$. If $\mathbf{C}$ has a terminal object 1, we say that an object X has the *extension property* with respect to $\mathcal{C}$ if the morphism $X \to 1$ has the right lifting property with respect to $\mathcal{C}$.

It is clear from the definition that for any class of morphisms $\mathcal{C}$ there is an inclusion $\mathcal{C} \subseteq {}^{\perp}(\mathcal{C}^{\perp})$.

Lemma 3.33 *If $\mathcal{F}$ is any class of morphisms of $\mathbf{C}$, then the class ${}^{\perp}\mathcal{F}$ is saturated.*

Proof Closure of ${}^{\perp}\mathcal{F}$ under pushouts and (transfinite) compositions follows from the universal property of colimits. Suppose

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[r, "r"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "j"] & D \arrow[r, "s"] & B \end{tikzcd}$$

is a retract diagram and g has the left lifting property with respect to $\mathcal{F}$. Suppose $h : X \to Y$ is in $\mathcal{F}$ and we have a commutative square as on the right in the following diagram:

$$\begin{tikzcd} C \arrow[r, "r"] \arrow[d, "g"'] & A \arrow[r] \arrow[d, "f"] & X \arrow[d, "h"] \\ D \arrow[r, "s"] & B \arrow[r] & Y. \end{tikzcd}$$

We can find a lift $l : D \to X$ in the rectangle by our assumption on g. Then $lj : B \to X$ is a lift in the square, so that $f \in {}^{\perp}\mathcal{F}$. □

Definition 3.34 A morphism of dendroidal sets is a *trivial fibration* if it has the right lifting property with respect to all normal monomorphisms.

For now the terminology ‘trivial fibration’ is merely suggestive. It is inspired by homotopy theory: a map of topological spaces having the right lifting property with respect to all relative CW complexes is a Serre fibration which is also a weak equivalence. We will come back to the homotopy-theoretic properties of trivial fibrations of dendroidal sets in Part II. One can think of the following result and the subsequent corollary as the dendroidal analogue of CW approximation:

Proposition 3.35 *Every morphism of dendroidal sets $f : X \to Y$ admits a factorization $X \xrightarrow{i} Z \xrightarrow{p} Y$ with i a normal monomorphism and p a trivial fibration.*

If X is a dendroidal set, one applies this proposition to the map $\varnothing \to X$ to obtain:

Corollary 3.36 *For any dendroidal set X there exists a trivial fibration $X' \to X$ with X' a normal dendroidal set.*

We will refer to such a trivial fibration as a *normalization* of X. Such normalizations need not be unique, but we will see later that they are unique ‘up to homotopy’. Before proving the proposition, we note the following useful lemma:

Lemma 3.37 *A map $f : X \to Y$ of dendroidal sets is a trivial fibration if and only if it has the right lifting property with respect to all boundary inclusions of trees $\partial\Omega[T] \to \Omega[T]$.*

Proof Since boundary inclusions are normal monomorphisms, any trivial fibration has the right lifting property with respect to them. Conversely, suppose f has the right lifting property with respect to all boundary inclusions. Then Theorem 3.31 and Lemma 3.33 imply that f has the right lifting property with respect to all normal monomorphisms. □

Proof (of Proposition 3.35) The proof follows the classical 'small object argument'. We explain it here in elementary form, but see also Remark 3.38 below. We will first construct factorizations $X \xrightarrow{i_k} Z_k \xrightarrow{p_k} Y$ by an inductive procedure. Set $Z_0 = X$, $i_0 = \mathrm{id}_X$ and $p_0 = f$. If Z_k, i_k and p_k have been defined, we construct Z_{k+1}, i_{k+1} and p_{k+1} as follows. Consider the set $\mathcal{S}_k$ of commutative squares S of dendroidal sets of the form

$$\begin{tikzcd} \partial\Omega[T] \arrow[r, "\alpha_S"] \arrow[d] & Z_k \arrow[d, "p_k"] \\ \Omega[T] \arrow[r] & Y. \end{tikzcd}$$

Define Z_{k+1} by the pushout square

$$\begin{tikzcd} \coprod_{S\in\mathcal{S}_k} \partial\Omega[T] \arrow[r, "\amalg_S \alpha_S"] \arrow[d] & Z_k \arrow[d] \\ \coprod_{S\in\mathcal{S}_k} \Omega[T] \arrow[r] & Z_{k+1}. \end{tikzcd}$$

Define i_{k+1} to be the composition of the vertical map on the right with i_k. Also, by the universal property of pushouts we find a map $p_{k+1} : Z_{k+1} \to Y$. We take the colimit over k to obtain a factorization

$$X \xrightarrow{i_\infty} Z_\infty \xrightarrow{p_\infty} Y$$

which we claim to have the desired properties. Since each i_k is a normal monomorphism, the colimit i_∞ is a normal monomorphism as well by Proposition 3.28(vii). We wish to show that p_∞ is a trivial fibration. We will demonstrate how to solve a lifting problem of the following form:

$$\begin{tikzcd} \partial\Omega[T] \arrow[r, "u"] \arrow[d] & Z_\infty \arrow[d, "p_\infty"] \\ \Omega[T] \arrow[r] & X. \end{tikzcd}$$

This suffices by Lemma 3.37. We claim that u must factor as $\partial\Omega[T] \xrightarrow{u_k} Z_k \to Z_\infty$ for some k. Indeed, since $\partial\Omega[T]$ is a finite union $\cup_j \Omega[S_j]$ of representable dendroidal sets, we have

$$\begin{aligned}
\mathrm{Hom}(\partial\Omega[T], \cup_k Z_k) &= \cap_j \mathrm{Hom}(\Omega[S_j], \cup_k Z_k) \\
&\cong \cap_j \cup_k (Z_k)_{S_j} \\
&= \cup_k \cap_j (Z_k)_{S_j} \\
&= \cup_k \mathrm{Hom}(\partial\Omega[T], Z_k).
\end{aligned}$$

Here we used that finite intersections commute with infinite unions of sets. The square

$$\begin{tikzcd}
\partial\Omega[T] \arrow[r, "u"] \arrow[d] & Z_k \arrow[d, "p_k"] \\
\Omega[T] \arrow[r] & Y
\end{tikzcd}$$

is contained in $\mathcal{S}_k$, so by construction there is a commutative square

$$\begin{tikzcd}
\partial\Omega[T] \arrow[r, "u"] \arrow[d] & Z_k \arrow[d] \\
\Omega[T] \arrow[r, "v_k"] & Z_{k+1}.
\end{tikzcd}$$

The composition $\Omega[T] \xrightarrow{v_k} Z_{k+1} \to Z_\infty$ provides a solution to the lifting problem above, completing the proof. □

Remark 3.38 The small object argument is a general procedure for producing factorizations as above. For a set $\mathcal{A}$ of morphisms, it will give a factorization of a given morphism $f : X \to Y$ into morphisms $i : X \to Z$ and $p : Z \to Y$, where p has the right lifting property with respect to morphisms in $\mathcal{A}$ and i is a (transfinite) composition of pushouts of morphisms in $\mathcal{A}$. In the argument above, $Z = Z_\infty$ is constructed as a colimit indexed by the natural numbers. The factorization thus obtained had the required properties because any map from $\partial\Omega[T]$ to the colimit factored through a finite stage, using that $\partial\Omega[T]$ itself is finite in the appropriate sense. In general, the object Z will be constructed as the colimit of a sequence indexed over a limit ordinal λ, where λ is chosen large enough so that any map from the domain of a morphism in $\mathcal{A}$ factors through some stage of the colimit. In this book we will only need the countable case, as exemplified in the preceding proof.

We defined trivial fibrations of dendroidal sets to be those maps which have the right lifting property with respect to normal monomorphisms. Dually, the class of trivial fibrations determines the normal monomorphisms in the following way:

Lemma 3.39 *Suppose $f : A \to B$ is a morphism of dendroidal sets which has the left lifting property with respect to trivial fibrations. Then f is a normal monomorphism.*

Proof Use Proposition 3.35 to factor f as $A \xrightarrow{i} A' \xrightarrow{p} B$ with i a normal monomorphism and p a trivial fibration. By the assumption on f there exists a lift as indicated by the dashed arrow in the following square:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & A' \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dashed] & B. \end{tikzcd}$$

This lift exhibits f as a retract of i, so that f is a normal monomorphism as well. □

Historical Notes

The observation that trees are a relevant organizing tool to describe homotopy coherent algebraic structures goes back to the work of Boardman–Vogt [21]. In fact, it seems that early on in the development of the theory it was already recognized that trees provide a convenient way to label the cells of Stasheff's associahedra, as well as the cells of certain decompositions of configuration spaces, hinting at the Fulton–MacPherson operad we described in Chapter 1. Ginzburg–Kapranov [68] use trees very explicitly, also in their description of free operads. Trees feature prominently in many of the subsequent works on bar constructions of operads, e.g., the papers of Fresse [58] for algebraic operads and Ching [36] for topological operads. In particular, Ching already proposes to study presheaves on a certain category of trees; he refers to these as *arboreal objects*. All of the aforementioned authors use trees to organize the ways in which different operations can be composed, as well as the contraction of inner edges of a tree to represent composition of operations. In this chapter we have also taken external face maps and degeneracies between trees into account to construct the category $\mathbf{\Omega}$ of trees. This viewpoint, and the fact that $\mathbf{\Omega}$ generalizes the simplex category $\mathbf{\Delta}$, first appears in [116]. An alternative approach to the category $\mathbf{\Omega}$ is presented in [99].

Chapter 4
Tensor Products of Dendroidal Sets

In the first chapter we described the Boardman–Vogt tensor product of two operads. This operadic tensor product induces a related tensor product on dendroidal sets by the method of Kan extension. This new dendroidal tensor product agrees with the Cartesian product when restricted to simplicial sets, and reproduces the Boardman–Vogt tensor product when dendroidal sets are realized as operads by the functor τ. In this chapter we will show how the dendroidal tensor product can be explicitly described in terms of a notion of *shuffles of trees*. This notion extends that of shuffle of linear orders featuring in the product of simplicial sets and many related notions, such as Eilenberg–Zilber maps of chain complexes. To prepare for an analysis of the homotopical properties of the dendroidal tensor product in Chapters 6 and 9, we will carefully analyze the behaviour of the tensor product with respect to normal monomorphisms. While the tensor product of operads provides the category of operads with the structure of a symmetric monoidal category, the extension to dendroidal sets is symmetric but no longer associative up to isomorphism. To describe the more subtle associativity properties of this tensor product, we equip the category of dendroidal sets with an 'unbiased' tensor product of n variables. This notion will be useful in Part II, where we demonstrate that the tensor product of dendroidal sets *is* associative up to homotopy in many case, for example when the dendroidal sets involved are either all open or all closed, or when some factors are simplicial.

4.1 Elementary Properties and Shuffles of Trees

Recall from Section 1.6 that for operads $\mathbf{P}$ and $\mathbf{Q}$ of sets (or of topological spaces), their tensor product $\mathbf{P} \otimes \mathbf{Q}$ is an operad with colours $C \times D$ and can be defined in terms of generators and relations. The generators are

$$p \otimes d \in (\mathbf{P} \otimes \mathbf{Q})(c_1 \otimes d, \ldots, c_n \otimes d; c \otimes d)$$

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for $d \in D$ and $p \in \mathbf{P}(c_1, \ldots, c_n; c)$, and

$$c \otimes q \in (\mathbf{P} \otimes \mathbf{Q})(c \otimes d_1, \ldots, c \otimes d_m; c \otimes d)$$

for $c \in C$ and $q \in \mathbf{Q}(d_1, \ldots, d_m; d)$. There are relations expressing the fact that for $c \in C$ and $d \in D$ the assignments

$$- \otimes d : \mathbf{P} \to \mathbf{P} \otimes \mathbf{Q} \qquad \text{and} \qquad c \otimes - : \mathbf{Q} \to \mathbf{P} \otimes \mathbf{Q}$$

are maps of operads. The crucial relation is the Boardman–Vogt interchange, which states that

$$(p \otimes d)(c_1 \otimes q, \ldots, c_n \otimes q) = \sigma^*_{n,m}(c \otimes q)(p \otimes d_1, \ldots, p \otimes d_m)$$

for an appropriate permutation $\sigma_{n,m}$.

In particular, we can use this construction to define a functor

$$\text{ten} : \mathbf{\Omega} \times \mathbf{\Omega} \to \mathbf{dSets}$$

by assigning to a pair of trees (S, T) the corresponding free operads $\Omega(S)$ and $\Omega(T)$, forming their tensor product $\Omega(S) \otimes \Omega(T)$ and finally taking the dendroidal nerve:

$$\text{ten}(S, T) = N(\Omega(S) \otimes \Omega(T)).$$

By the method of Kan extension discussed in Section 2.4, this functor determines a functor (again denoted as tensor product)

$$\otimes : \mathbf{dSets} \times \mathbf{dSets} \to \mathbf{dSets},$$

uniquely determined up to natural isomorphism by the requirement that it agrees with ten on representables and preserves colimits in each variable separately. Explicitly,

$$\begin{aligned} \Omega[S] \otimes \Omega[T] &= N(\Omega(S) \otimes \Omega(T)), \\ \varinjlim_i (X \otimes Y_i) &\simeq X \otimes (\varinjlim_i Y_i), \\ \varinjlim_i (X_i \otimes Y) &\simeq (\varinjlim_i X_i) \otimes Y. \end{aligned}$$

Remark 4.1 Although these three natural isomorphisms characterize the tensor product of dendroidal sets up to natural isomorphism, this is strictly speaking not directly a special case of the method of Kan extension introduced in Section 2.4. Rather, it is a multi-variable extension of that method, which we leave the reader to formalize.

The category of dendroidal sets, being a presheaf category, is Cartesian closed and therefore has inner hom objects adjoint to the cartesian product as discussed in Section 2.4. In exactly the same way, but more importantly for us, there exist inner hom objects adjoint to the tensor product we have just defined, characterized by natural isomorphisms

$$\mathbf{dSets}(X \otimes Y, Z) \simeq \mathbf{dSets}(X, \mathbf{Hom}(Y, Z))$$

for dendroidal sets X, Y and Z. In later parts of this book we will make extensive use of the underlying simplicial sets of these hom objects, for which we introduce the notation

$$\mathbf{hom}(Y, Z) := i^*\mathbf{Hom}(Y, Z).$$

Proposition 4.2 *The tensor product of dendroidal sets satisfies the following properties:*

(i) *For dendroidal sets X and Y there is a natural isomorphism $X \otimes Y \simeq Y \otimes X$, i.e., the tensor product is* symmetric.
(ii) *The functor $\tau \colon \mathbf{dSets} \to \mathbf{Op}$ (cf. Section 3.5) commutes with tensor products, i.e., there is a natural isomorphism of operads $\tau(X \otimes Y) \simeq \tau(X) \otimes \tau(Y)$.*
(iii) *The functor $i_! \colon \mathbf{sSets} \to \mathbf{dSets}$ sends products to tensor products, i.e., there is a natural isomorphism $i_!(M \times N) \simeq i_!M \otimes i_!N$ for simplicial sets M and N.*

Proof Since the functors $\otimes$, τ and $i_!$ are all compatible with colimits it suffices to establish these natural isomorphisms for representables. Then (i) follows from the symmetry of the Boardman–Vogt tensor product of operads. For (ii), recall from Section 3.5 that the nerve functor N is fully faithful, so that the counit $\tau N \to \mathrm{id}$ is a natural isomorphism. It follows that

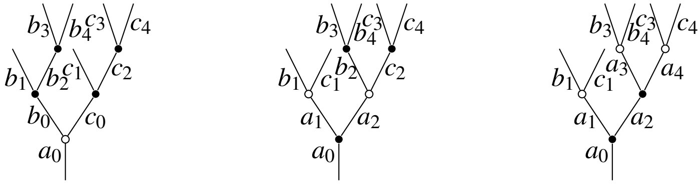
$$\tau(\Omega[S] \otimes \Omega[T]) = \tau N(\Omega(S) \otimes \Omega(T)) \simeq \Omega(S) \otimes \Omega(T) = (\tau N\Omega(S)) \otimes (\tau N\Omega(T)).$$

Finally, (iii) follows from the fact that for categories $\mathbf{C}$ and $\mathbf{D}$ the tensor product $\mathbf{C} \otimes \mathbf{D}$ coincides with the Cartesian product $\mathbf{C} \times \mathbf{D}$, as one easily checks. □

Remark 4.3 We warn the reader that the tensor product of dendroidal sets is not associative up to coherent isomorphism, so it is not part of a symmetric monoidal structure on the category **dSets**. However, in Section 4.4 we will introduce natural maps expressing associativity which are not necessarily isomorphisms, but which we will show to be equivalences in a homotopical sense in Part II.

Example 4.4 Let us examine the tensor product of representables $\Omega[S]$ and $\Omega[T]$ more closely. From the description of the operad $\Omega(S) \otimes \Omega(T)$ in terms of generators and relations and the fact that $\Omega(S)$ and $\Omega(T)$ are free operads, we see that any operation of $\Omega(S) \otimes \Omega(T)$ can be obtained as a composition of operations $p \otimes d$ and $c \otimes q$ where p and q are themselves generating operations of the operads $\Omega(S)$ and $\Omega(T)$, i.e., correspond to vertices in the trees S and T (together with an ordering of the leaves of those vertices). To give a small example, consider the following trees:

Let us fix the planar structures on S and T induced by this picture, giving a choice of generators for the operads $\Omega(S)$ and $\Omega(T)$, namely $v \in \Omega(S)(b, c; a)$ and similarly for p and q. Then $\Omega(S) \otimes \Omega(T)$ is an operad with colours (a, i), (b, i) and (c, i) for $i = 0, \ldots, 4$. The specified generators make it easy to describe the operations in $\Omega(S) \otimes \Omega(T)$. Indeed, each such operation will arise from a subtree of one of the following three trees (where we write a_1 instead of $(a, 1)$ etc.):



A B C

Any subtree of A, B or C, together with an ordering of its leaves, determines an operation of $\Omega(S) \otimes \Omega(T)$ by composing the operations corresponding to the vertices of the subtree. Each such vertex corresponds to a pair (p, t) (with p a generating operation of $\Omega(S)$ and t an edge of T) or (s, q) (with s an edge of S and q a generating operation of $\Omega(T)$). The trees A, B and C represent 'maximal composable words' in the generating operations of $\Omega(S)$ and $\Omega(T)$. Note that the root of each is labelled by $(a, 0)$ (i.e., the roots of S and T) and the leaves by the product of the sets of leaves of S and T. Recall that the sets of edges $E(S)$ and $E(T)$ are partially ordered, with the root as maximal element. Each of the sets $E(A)$, $E(B)$, $E(C)$ is a subset of $E(S) \times E(T)$ and the partial order on each of them, given by the tree structures of respectively A, B and C, is precisely the one inherited from $E(S) \times E(T)$.

The representation of an operation of $\Omega(S) \otimes \Omega(T)$ as a subtree of A, B or C need not be unique. The Boardman–Vogt interchange relation describes which identifications to make. For example, the operation defined by the subtree of A with leaves b_1, b_2, c_1, c_2 and root a_0 describes the same operation as the one with leaves b_1, c_1, b_2, c_2 and root a_0 inside the tree B (modulo a permutation of the inputs). In fact, one can view the tree B as obtained from A by interchanging the operations $b \otimes p$ and $c \otimes p$ with $v \otimes 0$, i.e., by *shuffling up* the white vertex $v \otimes 0$ through the two black vertices $b \otimes p$ and $c \otimes p$. Similarly, the tree C is obtained from B by shuffling up the white vertex $v \otimes 2$. On the other hand, while certain operations like the one above occur in more than one tree, there are also operations which only occur in one, such as the operation with inputs b_1, b_2, c_0 and output a_0 in A.

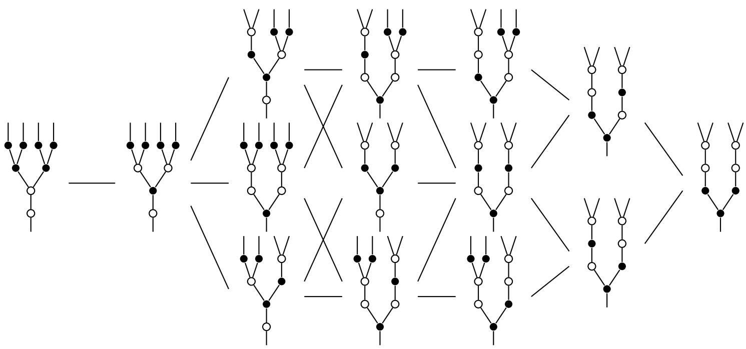
What all this means for the dendroidal set $\Omega[S] \otimes \Omega[T] = N(\Omega(S) \otimes \Omega(T))$ is the following. Like any dendroidal set it is a colimit of representables. In this case it is the union of the dendroidal sets represented by the trees A, B and C, glued together by the Boardman–Vogt interchange relations. These can be described as follows. The tree $\partial_{b_0}\partial_{c_0}A$ obtained by contracting the inner edges a_0 and b_0 is isomorphic to $\partial_{a_1}\partial_{a_2}B$, and similarly $\partial_{b_2}\partial_{c_2}B$ is isomorphic to $\partial_{a_3}\partial_{a_4}C$. Moreover, contracting *all* inner edges in each of A, B and C results in a corolla $C_A = \partial_{b_2}\partial_{c_2}\partial_{b_0}\partial_{c_0}A$, and similarly for C_B and C_C. These three are isomorphic. Together this gives the following diagram in the category $\mathbf{\Omega}$:

$$
\begin{tikzcd}
 & \partial_{b_0}\partial_{c_0}A \arrow[dr] \arrow[dd, "\wr|"] & \\
C_A \arrow[ur] \arrow[dd, "\wr|"] & & A \\
 & \partial_{a_1}\partial_{a_2}B \arrow[dr] & \\
C_B \arrow[ur] \arrow[dr] \arrow[dd, "\wr|"] & & B \\
 & \partial_{b_2}\partial_{c_2}B \arrow[ur] \arrow[dd, "\wr|"] & \\
C_C \arrow[dr] & & C. \\
 & \partial_{a_3}\partial_{a_4}C \arrow[ur] &
\end{tikzcd}
$$

The dendroidal set $\Omega[S]\otimes\Omega[T]$ is the colimit of this diagram, viewed as a diagram of representable dendroidal sets. It is a *union* of $\Omega[A]$, $\Omega[B]$ and $\Omega[C]$, in the sense that $\Omega[A] \to \Omega[S]\otimes\Omega[T]$ is a monomorphism, as are the maps from $\Omega[B]$ and $\Omega[C]$, and these three maps are jointly surjective.

Example 4.5 As in the previous example, a tensor product $\Omega[S] \otimes \Omega[T]$ of two representable dendroidal sets can be explicitly described as the union of representables, glued together along certain isomorphic inner faces. The number of these representables (A, B and C in the previous example) however can increase very rapidly as the trees S and T increase in size. For example, consider the two trees

with their planar structure as pictured. Then the set of representables making up their tensor product $\Omega[S] \otimes \Omega[T]$ can be listed as follows:



Notice that there is a natural partial order on this list, starting with copies of T on top of S and ending with copies of S on top of T. This partial order is generated by imposing the relation $A < B$ when B is obtained from A by shuffling up a white vertex coming from the tree S. This relation is indicated by the lines in the picture above.

We will now formalize the description of the tensor product $\Omega[S] \otimes \Omega[T]$ of two representable dendroidal sets as a union of representables by means of the notion of a *shuffle* of the trees S and T. Before giving the formal definition, let us recall the classical notion of a shuffle of two numbers $p, q \geq 0$, which we discussed in the context of products of simplicial sets in Chapter 2. As explained there, a shuffle is an order-preserving injection

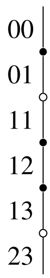
$$\{1, \ldots, p\} \to \{1, \ldots, p+q\}$$

or equivalently an order-preserving injection

$$\{1, \ldots, q\} \to \{1, \ldots, p+q\}$$

whose image is the complement of the previous one. If one thinks of the linear orders $[p]$ and $[q]$ as representing linear trees, with p white vertices and edges labelled $0, \ldots, p$ for $[p]$, and similarly q black vertices and edges $0, \ldots, q$ for $[q]$, a shuffle can also be represented by a tree with black and white vertices and edges labelled by pairs of numbers. For $p = 2$ and $q = 3$, the relevant trees are

and the picture



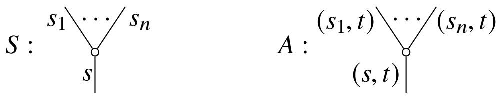
represents the shuffle $\{1, 2\} \to \{1, 2, 3, 4, 5\}$ mapping 1 to 2 and 2 to 5.

The following definition of shuffle admits a more economical reformulation (see Proposition 4.8 below), but the elaborate one we give here is perhaps the most intuitive.

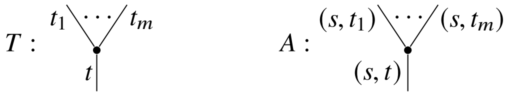
Definition 4.6 A *shuffle* of two trees S and T is a tree A together with a labelling of its edges by pairs (s,t), where s and t are edges of S and T respectively, satisfying the following conditions:

(1) The root of A is labelled by the pair (r_S, r_T) of root edges of S and T.
(2) The set of labels of the leaves of A is the cartesian products of the sets of leaves of S and T.
(3) If (s,t) is the label of an edge in A, then

– either (s,t) is a leaf of A and by (2) s and t are leaves of S and T respectively,
– or the vertex above (s,t) has input edges labelled $(s_1,t),\ldots,(s_n,t)$, where $s_1,\ldots,s_n$ are the input edges of the vertex above s in S (here s is not a leaf of S),



– or the vertex above (s,t) has input edges labelled $(s,t_1),\ldots,(s,t_m)$, where $t_1,\ldots,t_m$ are the input edges of the vertex above t in T (and so t is not a leaf of T).



(4) If (s,t) is not a leaf of A and the vertex above it is a stump (i.e., has no input edges), then s is either a leaf or has a stump above it in S *and* t is either a leaf or has a stump above it in T.

Remark 4.7 (a) A choice of planar structures on S and T induces a planar structure on a shuffle A.

(b) The set of edges $E(A)$ of a shuffle A is a subset of the cartesian product $E(S)\times E(T)$. The partial order on $E(A)$ induced by the tree structure of A coincides with the one induced by the product of the partial orders on $E(S)$ and $E(T)$.

(c) To explain condition (4), note that if the vertex above (s,t) in A is a stump, it must correspond to a stump in S or in T or in both. For two trees

the tree on the left below satisfies conditions (1)–(3), but it is not ‘maximal’ in the sense that it is a face (contract b_2 and c_2) of the labelled tree on the right, which *is* a shuffle.

(d) For linear trees defined by linear orders $[p]$ and $[q]$, a shuffle in the sense of Definition 4.6 is exactly the same as a shuffle as in the discussion immediately preceding the definition.

If A is a shuffle, we observed that the labelling of its edges allows us to identify the partially ordered set $E(A)$ with a subset of $E(S) \times E(T)$. Note that for a tree T, the minimal elements in the partially ordered set $E(T)$ are precisely the leaves and the edges immediately below a stump. The following characterizes the shuffles of two trees:

Proposition 4.8 *For two trees S and T, their shuffles correspond precisely to subsets $E(A) \subseteq E(S) \times E(T)$ satisfying the following:*

(i) *the partially ordered set $E(A)$ satisfies the conditions of Lemma 3.2,*
(ii) *the maximal element of $E(A)$ coincides with the maximal element (r_S, r_T) of $E(S) \times E(T)$ and the subset of minimal elements of $E(A)$ is precisely the product of the subsets of minimal elements of $E(S)$ and $E(T)$,*
(iii) *the linear order given by the path in the tree A from a minimal element (s, t) to the unique maximal element (r_S, r_T) is a 'classical' shuffle of the two linear orders given by the paths from s to r_S in S and from t to r_T in T.*

Proof Clearly the set of edges of a shuffle A satisfies (i)–(iii). Conversely, suppose we are given $E(A) \subseteq E(S) \times E(T)$ satisfying (i)–(iii) and we wish to construct the corresponding shuffle. Let $L(A) \subseteq E(A)$ be the subset consisting of pairs (s, t) where s is a leaf of S and t is a leaf of T. Lemma 3.2 gives an essentially unique tree A with edges $E(A)$ and leaves $L(A)$. Conditions (1), (2) and (4) of Definition 4.6 are satisfied by construction. For condition (3) we use that A is a tree. Indeed, suppose (s, t) is an element of $E(A)$ which is not minimal. Pick one of the input edges (s', t') of the vertex v immediately above (s, t) in A. Then condition (iii) forces that either $s = s'$ or $t = t'$. Assume $t = t'$ (the other case is of course treated analogously). Then s' immediately precedes s in the partial order on $E(S)$, again by condition (iii). In other words, s' is an input edge of the vertex w above s in S. We need to show that the set of input edges of v is precisely the set of edges of the form (s_i, t), with s_i ranging over the input edges of w. There are two issues to address:

– If there were another input edge of v of the form (s, t'), with t' immediately preceding t in $E(T)$, then by condition (iii) the tree $E(A)$ has an edge (s', t') occurring somewhere in the branch above (s, t'). But this edge must also occur in the branch above (s', t), which contradicts the fact that A is a tree. Therefore *all* input edges of v are of the form (s', t) with s' an input edge of w.
– For every input edge s_i of w, there is a corresponding input edge (s_i, t) of v. Indeed, pick a path in S going from some minimal edge $\bar{s}$ to s which passes through s_i. Similarly, pick a path in T from some minimal edge $\bar{t}$ to t. Then condition (iii)

forces the existence of a path in A from the minimal element $(\bar{s}, \bar{t})$ to (s, t) which passes through the edge (s_i, t). In particular, this edge exists in A and moreover it is clear that it must be an input edge of v. □

Remark 4.9 To be precise, the subsets $E(A)$ considered in Proposition 4.8 correspond to isomorphism classes of shuffles. When we speak of 'the' shuffles of S and T, we will from now on always mean the ones arising from subsets $E(A) \subseteq E(S) \times E(T)$ as above.

We will say an operad $\mathbf{P}$ is *thin* if for any sequence $c_1, \ldots, c_n, c$ of colours of $\mathbf{P}$, there is at most one operation $p \in \mathbf{P}(c_1, \ldots, c_n; c)$. The free operad $\Omega(T)$ generated by a tree is thin. Moreover, a tensor product $\Omega[S] \otimes \Omega[T]$ of such is also thin. Indeed, any two ways of expressing an operation in $\Omega[S] \otimes \Omega[T]$ as a composition of generating operations can be related by applications of the Boardman–Vogt interchange. This observation will be useful in Proposition 4.10 to express a tensor product $\Omega[S] \otimes \Omega[T]$ as a union of shuffles.

In any category of (set-valued) presheaves, an epimorphism $Y \to X$ is the coequalizer of its pullback projections:

$$Y \times_X Y \rightrightarrows Y \longrightarrow X.$$

Thus, the following proposition gives an explicit description of the way in which a tensor product $\Omega[S] \otimes \Omega[T]$ of two representable dendroidal sets is a colimit of representables.

Proposition 4.10 *Let S and T be objects of $\mathbf{\Omega}$.*

(a) *The evident map*

$$\coprod_A \Omega[A] \to \Omega[S] \otimes \Omega[T]$$

from the coproduct indexed by the shuffles of S and T is an epimorphism.

(b) *The inclusion of each shuffle*

$$\Omega[A] \to \Omega[S] \otimes \Omega[T]$$

is a monomorphism.

(c) *For finitely many shuffles $A_1, \ldots, A_n$, the inclusion of their intersection*

$$\Omega[A_1] \cap \cdots \cap \Omega[A_n] \to \Omega[S] \otimes \Omega[T]$$

in the tensor product is isomorphic to the map

$$\Omega[B_i] \to \Omega[S] \otimes \Omega[T],$$

where B_i is the inner face of A_i obtained by contracting all inner edges in A_i which do not occur in each of the A_j for $j \neq i$. In particular, $\Omega[B_i]$ is isomorphic to $\Omega[B_j]$ by a unique isomorphism over $\Omega[S] \otimes \Omega[T]$.

Proof (a) Consider an R-dendrex ξ of $\Omega[S] \otimes \Omega[T] = N(\Omega(S) \otimes \Omega(T))$, i.e., an element $\xi \in (\Omega[S] \otimes \Omega[T])_R$. By definition of the nerve of an operad, ξ is given by a labelling of the edges of R by pairs of edges (s, t) of S and T and of the vertices of R by operations of the operad $\Omega(S) \otimes \Omega(T)$. By writing each of those operations as a composition of generating operations, we can factor ξ as

$$\begin{tikzcd} \Omega[R] \arrow[rr, "m"] \arrow[dr, "\xi"'] & & \Omega[R'] \arrow[dl, "\xi'"] \\ & \Omega[S] \otimes \Omega[T] & \end{tikzcd}$$

where $m : R \to R'$ is a composition of inner faces and ξ' sends vertices of R' to generating operations of $\Omega[S] \otimes \Omega[T]$. But then ξ' is itself an outer face of a shuffle of S and T, i.e., there is a further factorization

$$\begin{tikzcd} \Omega[R'] \arrow[rr, "m'"] \arrow[dr, "\xi'"'] & & \Omega[A] \arrow[dl] \\ & \Omega[S] \otimes \Omega[T] & \end{tikzcd}$$

where m' is an outer face map.

(b) Let A be a shuffle and write i_A for the inclusion of $\Omega[A]$ into the tensor product. To check that i_A is a monomorphism, suppose φ and ψ are maps in $\mathbf{\Omega}$ giving a commutative diagram

$$\Omega[R] \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \Omega[A] \xrightarrow{i_A} \Omega[S] \otimes \Omega[T].$$

Then for each edge e of R, the corresponding map $\Omega[\eta] \xrightarrow{e} \Omega[R]$ satisfies $i_A \varphi e = i_A \psi e$. But i_A is injective on edges, so $\varphi e = \psi e$. Consequently φ and ψ determine the same map on edges and it follows easily that $\varphi = \psi$, since any operation in $\Omega[S] \otimes \Omega[T]$ is uniquely determined by its output edge and the collection of its input edges (cf. the discussion about thin operads preceding the proposition).

(c) We only consider the case $n = 2$, the rest is similar. Let A_1 and A_2 be shuffles of S and T and let B_1 be the face of A_1 obtained by contracting all edges not occurring in A_2. Note that these edges are indeed inner, because A_1 and A_2 have the same root and leaves. Similarly define the face B_2 of A_2. Then B_1 and B_2 have the same edges and consequently correspond to the same subset of $E(S) \times E(T)$. But then B_1 and B_2 are isomorphic as objects of $\mathbf{\Omega}$ and there is a commutative diagram

$$\begin{tikzcd}[column sep=small]
& \Omega[B_1] \simeq \Omega[B_2] \arrow[dl] \arrow[dr] & \\
\Omega[A_1] \arrow[dr, "i_{A_1}"'] & & \Omega[A_2] \arrow[dl, "i_{A_2}"] \\
& \Omega[S] \otimes \Omega[T]. &
\end{tikzcd}$$

To check that $\Omega[B_1] \simeq \Omega[B_2]$ is the pullback of i_{A_1} and i_{A_2}, consider maps

$$\Omega[A_1] \xleftarrow{\psi_1} \Omega[R] \xrightarrow{\psi_2} \Omega[A_2]$$

for which $i_{A_1}\psi_1 = i_{A_2}\psi_2$. Then ψ_1 sends all edges of R to edges which also occur in A_2, so that it factors through $\Omega[B_1]$. Similarly ψ_2 factors through $\Omega[B_2]$. From this the pullback property is clear. □

Corollary 4.11 *For any two trees S and T, the dendroidal set $\Omega[S] \otimes \Omega[T]$ is normal.*

Proof For a tree R, any dendrex $\xi \in (\Omega[S] \otimes \Omega[T])_R$ factors through some shuffle A, i.e., there is a commutative diagram

$$\begin{tikzcd}
\Omega[R] \arrow[r] \arrow[dr, "\xi"'] & \Omega[A] \arrow[d, "i_A"] \\
& \Omega[S] \otimes \Omega[T].
\end{tikzcd}$$

Since $\Omega[A]$ is normal and i_A is mono, it is clear that the automorphisms of R act freely on ξ. □

Remark 4.12 It follows that $\Omega(S) \otimes \Omega(T)$ is a Σ-free operad. The more general statement that a tensor product of Σ-free operads is again Σ-free is false, as the example **Ass** $\otimes$ **Ass** = **Com** shows (cf. Example 1.24(a)).

Another obvious consequence of Proposition 4.10 is the following.

Proposition 4.13 *The tensor product of two open dendroidal sets is open and the tensor product of two closed dendroidal sets is closed.*

Proof If S and T are open trees, then any shuffle of S and T is open as well, as is each intersection of such shuffles. Thus $\Omega[S] \otimes \Omega[T]$ is a colimit of open dendroidal sets and thus itself open. The general statement follows by writing an arbitrary open dendroidal set as a colimit of representables given by open trees. The proof for closed dendroidal sets proceeds in exactly the same way. □

Remark 4.14 Another way to phrase the previous proposition is to say that the category **odSets** carries a tensor product for which the inclusion functor

$$u_! : \mathbf{odSets} \to \mathbf{dSets}$$

is equipped with an isomorphism $u_!(X \otimes Y) \cong u_!(X) \otimes u_!(Y)$. A similar remark applies to the inclusion **cdSets** $\to$ **dSets** of closed dendroidal sets. Moreover, its left adjoint

$$\mathrm{cl}_! \colon \mathbf{dSets} \to \mathbf{cdSets}$$

commutes with tensor products, i.e., for dendroidal sets X and Y there is a natural isomorphism

$$\mathrm{cl}_!(X \otimes Y) \cong \mathrm{cl}_! X \otimes \mathrm{cl}_! Y,$$

as one easily checks on representables.

4.2 The Tensor Product of a Simplicial and a Dendroidal Set

Restricting one of the factors of the tensor product to be a simplicial (rather than dendroidal) set defines a functor

$$\mathbf{sSets} \times \mathbf{dSets} \to \mathbf{dSets} \colon (A, X) \mapsto i_! A \otimes X$$

of which we will make ample use throughout this book. Since it will play such an important role, we include this short section to highlight an alternative simple description of this functor. Throughout this section we abbreviate $i_! A \otimes X$ simply by $A \otimes X$.

Recall that the set of edges $E(T)$ of a tree T carries a natural partial order, in which the root is the largest element, and that any morphism $S \to T$ in particular gives a map of posets $E(S) \to E(T)$. Therefore, for each $n \geq 0$ we can define a dendroidal set $\mathcal{E}[n]$ by

$$\mathcal{E}[n]_T = \mathrm{Hom}(E(T), [n]),$$

where on the right-hand side we consider maps of posets only. Thus, an element of $\mathcal{E}[n]_T$ is simply a labelling of the edges of T by numbers $0, \ldots, n$ which is monotone along each branch of T. Notice that $\mathcal{E}[n]$ is in fact the nerve of an operad; its colours are the numbers $0, \ldots, n$ and there is a unique operation $(i_1, \ldots, i_k) \to j$ if and only if each of the indices $i_1, \ldots, i_k$ is less than or equal to j. This construction is evidently functorial in $[n]$, so in fact we obtain for each simplicial set M a dendroidal set

$$\mathcal{E}[M] = \varinjlim_{[n] \to M} \mathcal{E}[n].$$

In other words,

$$\mathcal{E} \colon \mathbf{sSets} \to \mathbf{dSets}$$

is the left Kan extension of the functor

$$\mathbf{\Delta} \to \mathbf{dSets} \colon [n] \mapsto \mathcal{E}[n].$$

Remark 4.15 For a simplicial set M there is a natural isomorphism $i^*\mathcal{E}[M] \to M$, as one easily verifies. Nonetheless, the functor $\mathcal{E}$ should not be confused with one of the functors $i_!, i_* \colon \mathbf{sSets} \to \mathbf{dSets}$. For example, $i_!\Delta[2]_T$ is empty for $T = C_2$, and

$$i_*\Delta[2]_T = \mathbf{sSets}(i^*C_2, \Delta[2])$$

is the set of all maps from the discrete three-element set of edges of C_2 to $[2]$, whereas $\mathcal{E}[2]_T$ only contains the order-preserving maps.

Lemma 4.16 *For a simplicial set M and a dendroidal set X, there is a natural isomorphism*

$$M \otimes X \cong \mathcal{E}[M] \times X.$$

Proof Since both sides preserve colimits in each variable separately, it suffices to construct a natural isomorphism

$$\alpha \colon \Delta[n] \otimes \Omega[T] \to \mathcal{E}[n] \times \Omega[T].$$

Both sides are nerves of operads, so such an isomorphism is the same as one of operads

$$\alpha \colon [n] \otimes \Omega(T) \to \tau\mathcal{E}[n] \times \Omega(T)$$

where $[n]$ is the category $(0 \to 1 \to \cdots \to n)$ viewed as an operad with unary operations only. The operad $\tau\mathcal{E}[n]$ can be described as the operad associated to the symmetric monoidal category $[n]$ with as tensor product the operation of taking the maximum. The two operads above have the same colours and we define α to be the identity on those. The generating operations of the operad $[n] \otimes \Omega(T)$ are of two kinds. First, there are unary operations $(i, e) \to (j, e)$ for $i \leq j$. Then there are operations

$$((i, e_1), \ldots, (i, e_k)) \to (i, e)$$

for $(e_1, \ldots, e_k) \to e$ corresponding to a vertex of T. Clearly, both of these define operations in $\mathcal{E}[n] \times \Omega(T)$. The Boardman–Vogt interchange relation is obviously respected, so α becomes a map of operads. It is automatically faithful, because $[n] \otimes \Omega(T)$ is a thin operad (i.e., there is at most one operation from any tuple of colours to another colour). It is also full, because any operation of $\mathcal{E}[n] \times \Omega(T)$ is clearly the image of a composition of generating operations in $[n] \otimes \Omega(T)$. $\square$

The product $M \times N$ of simplicial sets of course admits projections onto each of its two factors M and N. For the tensor product $X \otimes Y$ of dendroidal sets, however, no such projections generally exist. Nonetheless, in the case of a simplicial set M and dendroidal set X, one can use the unique map $M \to \Delta[0]$ to form a projection

$$M \otimes X \to \Delta[0] \otimes X \cong X$$

onto the 'dendroidal factor'. This provides the vertical maps in the following:

Corollary 4.17 *For any simplicial set M and any map of dendroidal sets $X \to Y$, the square*

$$\begin{tikzcd} M \otimes X \arrow[r] \arrow[d] & M \otimes Y \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is a pullback.

4.3 Tensor Products and Normal Monomorphisms

In much of what follows it will be important to understand how normal monomorphisms between dendroidal sets behave under tensor products. In particular, we will be interested in whether the tensor product with a fixed (normal) dendroidal set X maps a normal mono $B \to Y$ to a normal mono $X \otimes B \to X \otimes Y$. More generally, we would like to know for which normal monos $f : A \to X$ and $g : B \to Y$ the induced map

$$A \otimes Y \amalg_{A \otimes B} X \otimes B \to X \otimes Y$$

is again a normal mono. This map is usually referred to as the *pushout-product* of f and g. In the context of simplicial sets, with cartesian product playing the role of the tensor, it is easily seen that the pushout-product of two monomorphisms is again a monomorphism. The following minimal counterexample shows that the situation for dendroidal sets is more complicated.

Example 4.18 Consider the tree $\bar{\eta} = C_0$ which consists of one edge and a nullary vertex attached to it and write $i : \eta \to \bar{\eta}$ for the inclusion. Note that $\bar{\eta} \otimes \bar{\eta} \cong \bar{\eta}$, since the Boardman–Vogt interchange relation identifies the nullary operations coming from each of the factors. The pushout-product of $\Omega[i]$ with itself is (isomorphic to) the map

$$\Omega[\bar{\eta}] \amalg_{\Omega[\eta]} \Omega[\bar{\eta}] \to \Omega[\bar{\eta}],$$

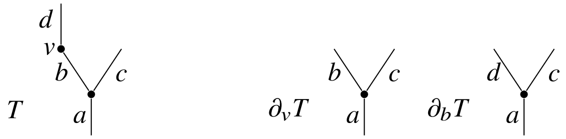
which is *not* a monomorphism. Indeed, the left-hand side has two dendrices of shape $\bar{\eta}$, whereas the right-hand side has only one.

The presence of nullary vertices also gives rise to issues of the following kind:

Example 4.19 For any tree T the tensor product $\Omega[\bar{\eta}] \otimes \Omega[T]$ consists of just one shuffle and is simply given by the dendroidal set $\Omega[\bar{T}]$ represented by the closure of T. It follows that tensoring with $\Omega[\bar{\eta}]$ is (isomorphic to) the closure functor

$$\mathrm{cl}_! : \mathbf{dSets} \to \mathbf{cdSets}.$$

Indeed, both functors commute with colimits and agree on representables. This closure functor does *not* preserve normal monomorphisms. For example, let T be the tree on the left



and write V for the subobject of $\Omega[T]$ which is the union of the representables $\Omega[\partial_v T]$ and $\Omega[\partial_b T]$. Their intersection consists of two copies of $\Omega[\eta]$, for which we write

$$\Omega[\eta_a] \amalg \Omega[\eta_c]$$

to indicate the relevant edges in the notation. The inclusion $V \longrightarrow \Omega[T]$ is a normal mono. However, $\Omega[\bar{\eta}] \otimes V$ is the pushout

$$\Omega[\overline{\partial_v T}] \amalg_{\Omega[\bar{\eta}_a] \amalg \Omega[\bar{\eta}_c]} \Omega[\overline{\partial_b T}]$$

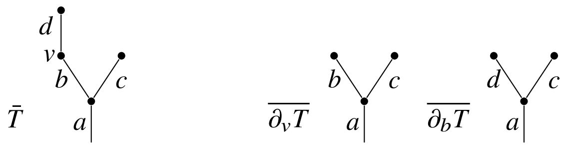
and the 'inclusion'

$$\Omega[\bar{\eta}] \otimes V \longrightarrow \Omega[\bar{\eta}] \otimes \Omega[T] \cong \Omega[\bar{T}]$$

is *not* a monomorphism. Indeed, the intersection

$$\Omega[\overline{\partial_v T}] \cap \Omega[\overline{\partial_b T}]$$

inside $\Omega[\bar{\eta}] \otimes V$ contains two copies of $\Omega[\partial_b(\overline{\partial_v T})] = \Omega[\partial_d(\overline{\partial_b T})]$, once inside $\Omega[\overline{\partial_v T}]$ and once inside $\Omega[\overline{\partial_b T}]$:



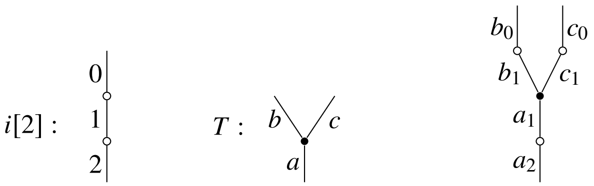
Note that only their edges a and c are identified in the pushout $\Omega[\bar{\eta}] \otimes V$.

An essential feature of this example is that $\partial_v T$ and $\partial_b T$ are two faces of T which have a *disconnected* intersection, as is typical for the intersection of a leaf face corresponding to a leaf vertex v and the inner face corresponding to the outgoing edge of v. This feature cannot occur for the intersection of two inner faces. It also cannot occur for intersections of faces of a *linear* tree, explaining why this phenomenon does not occur in the theory of simplicial sets. As we see from the example above, the dendroidal theory will be more complicated. We will have to analyze intersections of various kinds of faces of trees in great detail in the proofs of Lemma 4.24.

As already becomes clear from the previous example, the notation $\Omega[T]$ for a representable dendroidal set can make the notation rather cluttered.

Notation: *From now on, if no confusion can arise, we will often simply write T for the dendroidal set represented by a tree T.*

In this section we will investigate several special cases in which the pushout-product of normal monos is again a normal mono. Let us begin with the case in which one of the factors is a morphism of simplicial sets (or rather its image in the category **dSets** under the inclusion $i_!$). Consider a linear tree $i[n]$ and a tree T. A shuffle of $i[n]$ and T is easily visualized: it is a tree of the same shape as T, where on each edge of T one has inserted a number of unary vertices corresponding to vertices of $i[n]$. For example, here is a shuffle of $i[2]$ and a binary corolla T:



There is a (necessarily unique) operation $(k_1, e_1), \ldots, (k_n, e_n) \to (k, e)$ in $i[n] \otimes T$ if and only if there is an operation $e_1, \ldots, e_n \to e$ in T and $k_1, \ldots, k_n \leq k$. For the statement of the next lemma, note that for faces $\partial_k i[n]$ of $i[n]$ and $\partial_x T$ of T, the operad maps $\Omega(\partial_k i[n]) \otimes \Omega(T) \to \Omega(i[n]) \otimes \Omega(T)$ and $\Omega(i[n]) \otimes \Omega(\partial_x T) \to \Omega(i[n]) \otimes \Omega(T)$ are maps between *thin* operads which are injective on colours and therefore monomorphisms. Consequently, the nerves of these maps are monos as well.

Lemma 4.20 *Let T be an arbitrary tree and $i[n]$ a linear tree. Consider faces $\partial_x T$ and $\partial_y T$ of T and numbers $0 \leq p, q \leq n$. Then the following monomorphisms between dendroidal sets are isomorphisms:*

(i) $i[n] \otimes (\partial_x T \cap \partial_y T) \to (i[n] \otimes \partial_x T) \cap (i[n] \otimes \partial_y T)$,
(ii) $(\partial_p i[n] \cap \partial_q i[n]) \otimes T \to (\partial_p i[n] \otimes T) \cap (\partial_q i[n] \otimes T)$,
(iii) $\partial_p i[n] \otimes \partial_x T \to (\partial_p i[n] \otimes T) \cap (i[n] \otimes \partial_x T)$.

Proof Notice that the arrows above can all be interpreted as inclusions between subobjects of $i[n] \otimes T$. We need to show they are surjective. An operation $(k_1, e_1), \ldots, (k_n, e_n) \to (k, e)$ of $i[n] \otimes T$ belongs to $\partial_p i[n] \otimes T$ if and only if none of the $k_1, \ldots, k_n, k$ equals p. From this it readily follows that (ii) and (iii) are surjective. For (i), suppose $(k_1, e_1), \ldots, (k_n, e_n) \to (k, e)$ is an operation of $i[n] \otimes T$ which occurs both in $i[n] \otimes \partial_x T$ and in $i[n] \otimes \partial_y T$. Then $k_1, \ldots, k_n \leq k$, while $e_1, \ldots, e_n \to e$ is an operation in $\Omega(T)$ which belongs to both $\Omega(\partial_x T)$ and $\Omega(\partial_y T)$. This operation is represented by a subtree $T(e_1, \ldots, e_n; e)$ of T with leaves $e_1, \ldots, e_n$ and root e. If $\partial_x T$ is a leaf face of T removing a leaf vertex v (and its leaves, if it has any), then this v cannot be a leaf vertex of $T(e_1, \ldots, e_n; e)$. Indeed, the latter would not be a subtree of $\partial_x T$ anymore. Similarly, if $\partial_x T$ is the root face of T, then e cannot be the root edge of T. If $\partial_x T$ is an inner face, then the corresponding inner edge cannot be any of the edges $e_1, \ldots, e_n, e$. The same analysis applies to $\partial_y T$ and one concludes that there is a subtree of the intersection $\partial_x T \cap \partial_y T$ with leaves $e_1, \ldots, e_n$ and root e (even though this intersection might not be connected!). Therefore the operation $(k_1, e_1), \ldots, (k_n, e_n) \to (k, e)$ also occurs in $i[n] \otimes (\partial_x T \cap \partial_y T)$. □

From this lemma one concludes the pushout-product property if one of the factors is simplicial by a standard induction:

Proposition 4.21 *Let $M \xrightarrow{f} N$ be a monomorphism of simplicial sets and let $X \xrightarrow{g} Y$ be a normal monomorphism of dendroidal sets. Then the pushout-product*

$$i_! M \otimes Y \amalg_{i_! M \otimes X} i_! N \otimes X \to i_! N \otimes Y$$

is again a normal monomorphism.

Remark 4.22 An efficient proof of the proposition can be given by using Lemma 4.16 from the previous section. However, we include the proof below as a warm-up for the more complicated cases in the remainder of this section.

Proof First consider the special case where $M \to N$ is a boundary inclusion $\partial\Delta[n] \to \Delta[n]$ and $X \to Y$ is a boundary inclusion $\partial\Omega[T] \to \Omega[T]$. Since $i_!\Delta[n] \otimes \Omega[T]$ is normal, it suffices to check that the map

$$i_!\partial\Delta[n] \otimes \Omega[T] \amalg_{i_!\partial\Delta[n]\otimes\partial\Omega[T]} i_!\Delta[n] \otimes \partial\Omega[T] \to i_!\Delta[n] \otimes \Omega[T]$$

is a monomorphism. The two maps

$$i_!\partial\Delta[n] \otimes \Omega[T] \to i_!\Delta[n] \otimes \Omega[T] \qquad \text{and} \qquad i_!\Delta[n] \otimes \partial\Omega[T] \to i_!\Delta[n] \otimes \Omega[T]$$

are monos. Indeed, Lemma 4.20(i) guarantees that $i_!\Delta[n] \otimes -$ preserves intersections of faces of T, whereas item (ii) shows that $- \otimes \Omega[T]$ preserves intersections of faces of $[n]$. Regarding $i_!\partial\Delta[n] \otimes \Omega[T]$ and $i_!\Delta[n] \otimes \partial\Omega[T]$ as subobjects of $i_!\Delta[n] \otimes \Omega[T]$, we can now use Lemma 4.20 again (also including (iii)) to conclude that the intersection of these subobjects is precisely $i_!\partial\Delta[n] \otimes \partial\Omega[T]$. This completes the proof in this special case. For the case of general normal monos $M \to N$ and $X \to Y$, recall that every normal monomorphism of dendroidal sets (or monomorphism of simplicial sets) can be obtained as a transfinite composition of pushouts of boundary inclusions. Therefore it suffices to argue that the class of (f, g) for which the pushout-product of f and g is a normal monomorphism is closed under pushouts and transfinite composition in each variable. This is a standard argument, which we record here in general form as Lemma 4.23 below for future reference. $\square$

In the following lemma, we take $\mathbf{C}$ to be a category which admits pushouts and transfinite compositions and which comes equipped with a functor

$$- \otimes - : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

which preserves pushouts and transfinite compositions in each variable.

Lemma 4.23 *Let $\mathcal{A}$ be a class of morphisms in* $\mathbf{C}$ *which is closed under pushouts and transfinite composition and let $f : X \to Y$ be a fixed morphism. Write $\mathcal{B}$ for the class of morphisms $i : A \to B$ for which the pushout-product*

$$A \otimes Y \amalg_{A \otimes X} B \otimes X \to B \otimes Y$$

of i and f is contained in $\mathcal{A}$. Then $\mathcal{B}$ is also closed under pushouts and transfinite composition.

Proof Suppose $i \in \mathcal{B}$ and we have a pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & C \arrow[d, "j"] \\ B \arrow[r] & D. \end{tikzcd}$$

We claim that the square

$$\begin{tikzcd} A \otimes Y \amalg_{A \otimes X} B \otimes X \arrow[r] \arrow[d] & C \otimes Y \amalg_{C \otimes X} D \otimes X \arrow[d] \\ B \otimes Y \arrow[r] & D \otimes Y. \end{tikzcd}$$

is also a pushout. The vertical map on the left is in $\mathcal{A}$ by assumption, so that the map on the right is in $\mathcal{A}$ as well. To establish our claim, note that the span

$$\begin{tikzcd} A \otimes Y \amalg_{A \otimes X} B \otimes X \arrow[r] \arrow[d] & C \otimes Y \amalg_{C \otimes X} D \otimes X \\ B \otimes Y & \end{tikzcd}$$

is the pushout of the following diagram of spans:

$$\begin{tikzcd} (B \otimes X \leftarrow A \otimes X \to C \otimes X) \arrow[r] \arrow[d] & (B \otimes X \leftarrow B \otimes X \to D \otimes X) \\ (B \otimes Y \leftarrow A \otimes Y \to C \otimes Y). & \end{tikzcd}$$

Taking the pushout of each of these three spans (and using that $\otimes$ commutes with pushouts in each of its variables) gives the diagram

$$D \otimes Y \longleftarrow D \otimes X \longrightarrow D \otimes X$$

whose pushout is indeed $D \otimes Y$. Let us now treat the case of a countable composition of morphisms

$$A_0 \xrightarrow{i_1} A_1 \xrightarrow{i_2} A_2 \to \cdots \to A_\infty.$$

Finite compositions will be a special case, whereas transfinite compositions for larger ordinals are treated analogously. We assume that each i_k is contained in $\mathcal{B}$. Consider the following diagram:

$$\begin{tikzcd} A_0 \otimes Y \amalg_{A_0 \otimes X} A_{k-1} \otimes X \arrow[r] \arrow[d] & A_0 \otimes Y \amalg_{A_0 \otimes X} A_k \otimes X \arrow[d] \\ A_{k-1} \otimes Y \arrow[r] & A_{k-1} \otimes Y \amalg_{A_{k-1} \otimes X} A_k \otimes X. \end{tikzcd}$$

It is a pushout by a similar argument as above. Indeed, the span formed by all but the lower right corner of the diagram is itself a pushout of the following diagram of spans:

$$\begin{tikzcd} (A_{k-1} \otimes X \leftarrow A_0 \otimes X \rightarrow A_0 \otimes X) \arrow[r] \arrow[d] & (A_{k-1} \otimes X \leftarrow A_{k-1} \otimes X \rightarrow A_k \otimes X) \\ (A_{k-1} \otimes Y \leftarrow A_0 \otimes Y \rightarrow A_0 \otimes Y). & \end{tikzcd}$$

Taking the pushout of each of these gives the diagram

$$A_{k-1} \otimes Y \longleftarrow A_{k-1} \otimes X \longrightarrow A_k \otimes X,$$

whose pushout is indeed the lower right corner in the square above. By induction on the length of compositions we may assume that the left vertical map in that square is in $\mathcal{A}$, so that the vertical map on the right is in $\mathcal{A}$ as well. Taking the colimit over k (and using that $\otimes$ commutes with this particular colimit in both of its variables), we conclude that the map

$$A_0 \otimes Y \amalg_{A_0 \otimes X} A_\infty \otimes X \rightarrow A_\infty \otimes Y \amalg_{A_\infty \otimes X} A_\infty \otimes X = A_\infty \otimes Y$$

is in $\mathcal{A}$ as well. The case of a transfinite composition indexed by a larger ordinal is treated in the same way. □

We now turn to the pushout-product in the case where both factors are dendroidal (rather than one of them being simplicial). Like before, observe that any map of the form $\partial_x S \otimes T \rightarrow S \otimes T$ is a monomorphism. Indeed, it is the nerve of a map between thin operads which is injective on colours. We will be considering intersections $\partial_y T \cap \partial_z T$ of faces of a tree T. Recall that such an intersection is always representable, *except* in the following cases:

(1) Suppose v is a leaf vertex of T connected to an inner edge e and suppose the bottom vertex w of e has at least two incoming edges (so at least one besides e). Then the intersection $\partial_v T \cap \partial_e T$ is a disjoint union of representables, namely of the tree obtained from T by chopping off w and everything above it and each of the maximal subtrees of T whose root is an incoming edge of w other than e.

(2) Suppose v is the root vertex of T and that v has precisely one inner edge e attached to it, so that the root face $\partial_v T$ is defined. Write w for the vertex immediately above v, i.e., the vertex which has e as its outgoing edge. Then $\partial_v T \cap \partial_e T$ is the disjoint union of the maximal subtrees of T with root edges the incoming edges of w. Thus, if w has more than one incoming edge, this is not a representable dendroidal set.

(3) If T is a corolla, so that its faces are simply the inclusions of edges $\eta \to T$, then the intersection of any two distinct faces of T is empty.

Lemma 4.24 *Let S and T be trees.*

(i) *Let e be an inner edge of S and $\partial_y T$ any face of T. Then the following map is an isomorphism:*

$$\partial_e S \otimes \partial_y T \to (\partial_e S \otimes T) \cap (S \otimes \partial_y T).$$

(ii) *The statement of (i) remains true if $\partial_e S$ is replaced with the root face of S (which exists only if the root of S has precisely one inner edge attached to it) or with a leaf face $\partial_v S$ of S corresponding to a leaf vertex v which has at least one leaf attached to it, i.e., v is* not *a stump.*

(iii) *If $\partial_y T$, $\partial_z T$ are faces of T for which the intersection $\partial_y T \cap \partial_z T$ is* not *of the exceptional kind (1) described above, then the following map is an isomorphism:*

$$S \otimes (\partial_y T \cap \partial_z T) \to (S \otimes \partial_y T) \cap (S \otimes \partial_z T).$$

(iv) *If S is an open tree and $\partial_y T$, $\partial_z T$ are any two faces of T, then the map of (iii) is an isomorphism.*

Proof For (i), note first that this map is a monomorphism, simply because the maps

$$\partial_e S \otimes \partial_y T \to \partial_e S \otimes T \qquad \text{and} \qquad \partial_e S \otimes \partial_y T \to S \otimes \partial_y T$$

are monomorphisms. Let F be a face of a shuffle A of $S \otimes \partial_y T$ and suppose $F \to S \otimes T$ also factors through $\partial_e S \otimes T$. This means that F cannot contain any edges of the form (e, t) (for some edge t of T). So if we contract all edges in A of this form, we obtain a face of G of A which still contains F. Moreover, this face is a shuffle of $\partial_e S \otimes \partial_y T$, so that

$$(\partial_e S \otimes T) \cap (S \otimes \partial_y T) \subseteq \partial_e S \otimes \partial_y T.$$

For (ii) one uses the same argument as above to see that the stated map is a monomorphism. To establish surjectivity, let F be a face of a shuffle A of $S \otimes \partial_y T$ and suppose $F \to S \otimes T$ also factors through $\partial_x S \otimes T$, for $\partial_x S$ a face of S as in (ii). First consider the case where $\partial_x S$ is the root face of S and write e for the unique inner edge attached to the root vertex of S. Then the shuffle A will have one or several edges of the form (e, t). Consider the collection of lowest such edges in A (lowest meaning closest to the root), say $(e, t_1), \ldots, (e, t_n)$. Write T_i for the maximal subtree of $\partial_y T$ with root edge t_i and similarly write A_i for the maximal subtree of A with root edge (e, t_i). Since F does not contain any edge of the form (r, t), with r the root edge of S, this F must factor through one of the A_i. But each A_i is a shuffle of $\partial_x S \otimes T_i$ and therefore a face (possibly of high codimension) of a shuffle of $\partial_x S \otimes \partial_y T$, showing that F factors through $\partial_x S \otimes \partial_y T$. Next, consider the case of a leaf face $\partial_v S$ as described in (ii). Write e for the inner edge attached to the leaf vertex v and $l_1, \ldots, l_n$ for its leaves. Now list the highest occurrences of edges of the form (e, t) in A, say $(e, t_1), \ldots, (e, t_n)$. Again write T_i for the maximal subtree of $\partial_y T$ with root edge t_i and A_i for the maximal subtree of A with root edge (e, t_i). Form a face A' of A (possibly of high codimension) by contracting all edges of A_i (if T_i is a closed tree) or chopping off A_i (if T_i is not closed), for each i. Since F does not contain any edges of the form (l_i, t) and v is not a nullary vertex, F must factor through A'. But A' is a shuffle of the tensor product $\partial_v S \otimes T'$, where T' is the tree obtained from $\partial_y T$ by contracting all of T_i (if T_i closed) or chopping off T_i (if T_i is not closed), for each i. In particular, A' is contained in $\partial_v S \otimes T$.

We now consider (iii). Again, the nontrivial part is to establish surjectivity. There are several cases to check, of which the easiest is the one where $\partial_y T$ and $\partial_z T$ are inner faces, contracting edges e and f respectively. Let F be a face of a shuffle A of $S \otimes \partial_f T$ and suppose F is also contained in $S \otimes \partial_e T$. Form a face A' of A by contracting all edges of the form (s, e), with s ranging over the edges of S. Then F must be a face of A'. But A' is a shuffle of $S \otimes \partial_e \partial_f T$, which gives the desired conclusion in the case of two inner faces. Now consider the case where $\partial_y T = \partial_v T$ is a leaf face deleting a leaf vertex v and $\partial_z T$ is any other face (inner or outer) of T such that the intersection $\partial_v T \cap \partial_z T$ is not of the exceptional kind (1). So z may correspond to an inner edge of T, or the root vertex, or another leaf vertex. Label the leaves of v by $l_1, \ldots, l_n$ and its outgoing edge by k. As before, consider a face F of a shuffle A of $S \otimes \partial_z T$ and suppose F is contained in $S \otimes \partial_v T$. If z is not the outgoing edge k of the leaf vertex v, then we reason as follows. Consider edges of the form (s, k) in A and list the collection of highest such edges, say $(s_1, k), \ldots, (s_m, k)$. Then the edges of A above these are all of the form (s, l_j) and therefore do not occur in F. So F is also a face of the tree A' obtained from A by replacing the part above (s_i, k) by $S_i \otimes k$, where $S_i \subseteq S$ is the maximal subtree of S with root edge s_i. This A' is a shuffle of $S \otimes \partial_v \partial_z T$, finishing this case. We should still address the case $z = k$. This works the same way, only now considering the edge k' of T immediately below k and using the highest occurrences of edges of the form $s \otimes k'$ in A. To finish the proof of (iii), one still has to deal with the case where $\partial_y T$ is the root face of T and $\partial_z T$ is any other face, as well as the case where T is a corolla. The latter is trivial, because the intersection of distinct faces of a corolla is empty. In the case where T is not a corolla and $\partial_y T$ is the root face of T, the proof is analogous to the case of a leaf face as above, where the role of k is now played by the (unique) inner edge attached to the root vertex of T and one considers the lowest occurrences of edges of the form (s, k) in A.

Finally, we prove (iv). In almost all cases this is subsumed by (iii); the only case left to deal with is where the intersection $\partial_y T \cap \partial_z T$ is of the exceptional kind (1), so y is a leaf vertex v and z is the outgoing inner edge e of v. Write F for a face of a shuffle A of $S \otimes \partial_v T$ and suppose F is also contained in $S \otimes \partial_e T$. The shuffle A will have one or several edges of the form (s, e), for s ranging through the edges of S. List the collection of lowest occurrences of such edges by $(s_1, e), \ldots, (s_n, e)$.

Since S is assumed to be an open tree, the maximal subtree S_i of A with root edge (s_i, e) is open (for every i). The intersection of $S \otimes \partial_e T$ with each of the subtrees S_i is empty and it follows that the intersection of $S \otimes \partial_e T$ with A breaks up into several connected components. To describe these, write w for the bottom vertex of e, then r for the outgoing edge of w and $f_1, \ldots, f_m$ for the incoming edges of w other than e. Then the maximal subtrees of A with root edges of the form (s_i, f_j) form all but one of these connected components; the remaining one is the tree obtained from A by chopping off everything above the edges (s_i, r) for $1 \leq i \leq n$. Now F must factor through one of these connected components and each of the components is clearly contained in a shuffle of S with a corresponding connected component of the intersection $\partial_v T \cap \partial_e T$. □

Remark 4.25 The assumptions in parts (ii) and (iii) of the previous lemma are necessary: indeed, the statement of (ii) fails for $S = T = C_0$ (cf. Example 4.18), whereas (iii) fails for $S = C_0$ and the tree T of Example 4.19 with faces $\partial_b T$ and $\partial_v T$.

From the previous lemma we deduce the pushout-product property of normal monomorphisms for two important classes of dendroidal sets:

Proposition 4.26 *Let $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$ be normal monomorphisms between dendroidal sets and consider the pushout-product*

$$A \otimes D \amalg_{A \otimes C} B \otimes C \to B \otimes D.$$

(i) *If A, B, C, and D are open dendroidal sets, then the pushout-product is a normal monomorphism.*
(ii) *If A, B, C, and D are closed dendroidal sets, then the pushout-product is a normal monomorphism.*

Proof For case (i), observe that every normal monomorphism between open dendroidal sets is a transfinite composition of pushouts of boundary inclusions of *open* trees. Therefore, the same inductive argument as in the proof of 4.21 shows that it suffices to treat the case where the maps f and g are boundary inclusions, say $\partial S \to S$ and $\partial T \to T$, with S and T open trees. Lemma 4.24(iv) shows that $S \otimes -$ and $- \otimes T$ preserve intersections between faces of open trees, so that the maps $S \otimes \partial T \to S \otimes T$ and $\partial S \otimes T \to S \otimes T$ are monomorphisms. Using items (i) and (ii) of Lemma 4.24 now shows that the intersection of $S \otimes \partial T$ and $\partial S \otimes T$ inside $S \otimes T$ is precisely $\partial S \otimes \partial T$. We conclude that the pushout-product of f and g is a monomorphism. Since $S \otimes T$ is normal, it is also a normal monomorphism.

In case (ii) the argument is similar, now using the observation that any monomorphism between closed dendroidal sets is a transfinite composition of pushouts of the modified boundary inclusions $\partial^{\mathrm{cl}} T \to T$ for closed trees T. One replaces the use of Lemma 4.24(iv) above by item (iii) of that same lemma. The faces from which $\partial^{\mathrm{cl}} T$ is built do not include leaf faces of T (since these do not give closed trees), so the problematic case in item (iii) never arises. □

4.4 Unbiased Tensor Products

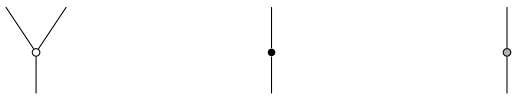
In this section we study the associativity properties of the tensor product of dendroidal sets, which turn out to be somewhat subtle. They will not feature very prominently in the rest of this book, except for a few isolated occurrences. Therefore most readers might wish to skip this material on first reading and refer back to it as needed.

The tensor product of two dendroidal sets has been defined in terms of the nerve of the Boardman–Vogt tensor product of operads. Indeed, for representable dendroidal sets $\Omega[S]$ and $\Omega[T]$ one has

$$\Omega[S] \otimes \Omega[T] = N(\Omega(S) \otimes \Omega(T)).$$

This definition is extended to arbitrary dendroidal sets in the essentially unique way for which the tensor product commutes with colimits in each variable separately. The Boardman–Vogt tensor product of operads is symmetric and associative up to coherent natural isomorphism and makes the category of operads into a symmetric monoidal category. It follows that the tensor product of dendroidal sets is symmetric as well, as already observed, but since the nerve functor does not commute with colimits one cannot conclude the same for associativity. In fact, the tensor product of dendroidal sets is not associative up to isomorphism, as the following example shows.

Example 4.27 Consider the corolla C_2 and two copies of the linear tree $i[1]$, pictured as follows:

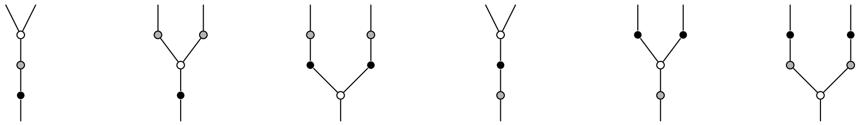


Then $i[1] \otimes i[1]$ is the union of two representables, given by the shuffles

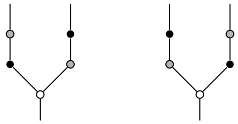
whose intersection is their common inner face (just like the product of simplicial sets $\Delta[1] \times \Delta[1]$ decomposes into two 2-simplices glued along their unique inner faces). Thus $C_2 \otimes (i[1] \otimes i[1])$ is a pushout

$$C_2 \otimes i[2] \amalg_{C_2 \otimes i[1]} C_2 \otimes i[2]$$

and may be depicted as the union of the following six shuffles:



It does not contain the shuffles



which occur in $(C_2 \otimes i[1]) \otimes i[1]$.

More generally, for three trees R, S and T, the tensor product $R \otimes (S \otimes T)$ is the union of all shuffles of $R \otimes A$ where A ranges over shuffles in $S \otimes T$, whereas $(R \otimes S) \otimes T$ is a similar union of shuffles of $B \otimes T$ with B ranging over all shuffles of $R \otimes S$. In general these are different, as the example above shows. Note that this difference does not occur when all three factors are linear trees; indeed, the product of simplicial sets is of course associative.

Although in practice we will rarely consider tensor products of more than three factors, one can define an n-fold tensor product of dendroidal sets as a functor

$$\otimes_n : \mathbf{dSets} \times \cdots \times \mathbf{dSets} \to \mathbf{dSets}$$

which is uniquely determined up to isomorphism by the requirement that it preserves colimits in each of its n variables separately and is given on representable dendroidal sets by the nerve of the Boardman–Vogt tensor product:

$$\otimes_n(\Omega[T_1], \ldots, \Omega[T_n]) = N(\Omega(T_1) \otimes \cdots \otimes \Omega(T_n)).$$

For general dendroidal sets $X_1, \ldots, X_n$ we will usually write

$$\otimes_n(X_1, \ldots, X_n) = X_1 \otimes \cdots \otimes X_n.$$

Although the tensor product of dendroidal sets is not associative up to isomorphism, there are still 'associator maps' of the kind

$$\alpha : X \otimes (Y \otimes Z) \to X \otimes Y \otimes Z.$$

Indeed, the functors $\mathbf{dSets}^{\times 3} \to \mathbf{dSets}$ represented by these two expressions both commute with colimits in each variable separately, so it suffices to define α for representable dendroidal sets. By adjunction, a map

$$\alpha : \Omega[R] \otimes N(\Omega(S) \otimes \Omega(T)) \to N(\Omega(R) \otimes \Omega(S) \otimes \Omega(T))$$

corresponds to a map of operads

$$\widehat{\alpha} : \tau(\Omega[R] \otimes N(\Omega(S) \otimes \Omega(T))) \to \Omega(R) \otimes \Omega(S) \otimes \Omega(T).$$

There is a canonical choice for such a map, even an isomorphism, because the functor τ commutes with tensor products (cf. Proposition 4.2) and is left inverse (up to isomorphism) to the nerve functor N (cf. Section 3.5).

More generally, for $0 \leq i < j \leq n$ there are maps α (or $\alpha_{i,j}$ if it is useful to be more explicit),

$$\alpha_{i,j} : X_1 \otimes \cdots \otimes X_{i-1} \otimes (X_i \otimes \cdots \otimes X_j) \otimes X_{j+1} \otimes \cdots \otimes X_n \to X_1 \otimes \cdots \otimes X_n$$

which are natural in each of the n variables. These natural transformations α are defined exactly as in the simple case above. The coherence isomorphisms for the tensor product of operads then immediately imply that these maps α, although no longer isomorphisms, satisfy similar coherence relations. In particular, the dendroidal set η still acts as a unit. There are relations expressing symmetry and associativity of the maps α, which are straightforward to describe but somewhat tedious for n large. For a threefold tensor product, one such relation expressing the compatibility of the maps α with symmetry is expressed by the following diagram:

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\alpha"] \arrow[d, "\cong"] & X \otimes Y \otimes Z \arrow[d, "\cong"] \\
(Y \otimes X) \otimes Z \arrow[r, "\alpha"] & Y \otimes X \otimes Z.
\end{tikzcd}$$

An n-fold tensor product of representables $T_1 \otimes \cdots \otimes T_n$ can again be described in terms of shuffles. To state this precisely we use shuffles of an n-tuple of numbers $p_1, \ldots, p_n \geq 0$, which are permutations

$$\sigma : p_1 + \cdots + p_n \to p_1 + \cdots + p_n$$

which restrict to monotone maps on each segment $p_i + 1, \ldots, p_{i+1}$ (for $i = 0, \ldots, n$ with the convention $p_0 = 0$). Notice that by removing one such segment and its image, such a shuffle σ of $p_1, \ldots, p_n$ 'projects' to a shuffle of $p_1, \ldots, \widehat{p_i}, \ldots, p_n$, with the hat denoting omission. A shuffle of $p_1, \ldots, p_n$ can also be viewed as a directed path from $(0, \ldots, 0)$ to $(p_1, \ldots p_n)$ through the lattice

$$\{0, \ldots, p_1\} \times \cdots \times \{0, \ldots, p_n\} \subset \mathbb{R}^n$$

by unit steps along the coordinate axes.

Having said all this, we define a shuffle of trees $T_1, \ldots, T_n$ as a tree A whose edges are labelled by n-tuples of edges $(t_1, \ldots, t_n)$ where t_i is an edge in T_i, and for which the partial order on these edges induced by the tree structure of A agrees with the one given by the product of the partial orders on the edges of each of the trees T_i. Moreover, the following conditions (cf. Proposition 4.8) should be satisfied, where we think of the poset of edges $E(A)$ as a subset of $E(T_1) \times \cdots \times E(T_n)$.

(i) The maximal element of the poset of edges $E(A)$ is the maximal element $(r_{T_1}, \dots, r_{T_n})$ of $E(T_1) \times \cdots \times E(T_n)$. The subset of minimal elements of $E(A)$ is precisely the product of the subsets of minimal elements of each of the $E(T_i)$.
(ii) For a tuple $(t_1, \dots, t_n)$ of minimal elements, the labelling on the branch in A down from $(t_1, \dots, t_n)$ to the root of A defines a shuffle of the n linear orders given by the branches in T_i down from t_i, for $i = 1, \dots, n$.

One then has the following analogue of Proposition 4.10, which is proved in exactly the same way.

Proposition 4.28 *Let $T_1, \dots, T_n$ be trees. The dendroidal set $\Omega[T_1] \otimes \cdots \otimes \Omega[T_n]$ is the colimit of representables $\Omega[A]$ indexed by all the shuffles A of those trees. Moreover, each*

$$\Omega[A] \to \Omega[T_1] \otimes \cdots \otimes \Omega[T_n]$$

is a monomorphism and each intersection $\Omega[A_1] \cap \cdots \cap \Omega[A_k]$ as a subobject of $\Omega[T_1] \otimes \cdots \otimes \Omega[T_n]$ is represented by the tree $B_i \subseteq A_i$ obtained by contracting all inner edges which do no occur in each of the other A_j. In particular, these B_i are all isomorphic (for $i = 1, \dots, k$).

The following is proved in the same way as Corollary 4.11.

Corollary 4.29 *An n-fold tensor product of trees $\Omega[T_1] \otimes \cdots \otimes \Omega[T_n]$ is normal.*

It is easy to describe the associators α in terms of shuffles. We leave the proof of the following as an exercise to the reader. A simple instance of it is illustrated in Example 4.27 above.

Proposition 4.30 *Each associator*

$$\alpha : R_1 \otimes \cdots \otimes R_k \otimes (S_1 \otimes \cdots \otimes S_l) \otimes T_1 \otimes \cdots \otimes T_m \to R_1 \otimes \cdots \otimes T_m$$

is a normal monomorphism, for trees $R_1, \dots, R_k$, etc. Its domain is the union of those shuffles A of the $k + l + m$ trees involved with the following property: for a sequence of minimal elements $s_1 \in S_1, \dots, s_l \in S_l$, the shuffles of the linear orders given by branches from minimal elements $(r_1, \dots, r_k, s_1, \dots, s_l, t_1, \dots, t_m)$ in $E(A)$ down to the root of A all project to the same shuffle of the l linear orders of the branches down from s_i in S_i.

Corollary 4.31 *The associativity map α in the previous proposition is an isomorphism whenever $R_1, \dots, R_k$ and $T_1, \dots, T_m$ are all linear trees.*

Proof This follows because linear trees have a unique minimal edge. □

Corollary 4.32 *Let $X_1, \dots, X_n$ be normal dendroidal sets. Then each associativity map*

$$\alpha\colon X_1 \otimes \cdots \otimes X_{i-1} \otimes (X_i \otimes \cdots \otimes X_j) \otimes X_{j+1} \otimes \cdots \otimes X_n \to X_1 \otimes \cdots \otimes X_n$$

is a normal monomorphism. It is an isomorphism if $X_1, \dots, X_{i-1}$ and $X_{j+1}, \dots, X_n$ are simplicial sets (viewed as dendroidal sets via the embedding $i_!$).

Proof This follows from the previous two statements by skeletal induction, as before. ◻

Example 4.33 A minimal example to keep in mind is 4.27. There we described $C_2 \otimes (i[1] \otimes i[1])$ as a union of six shuffles. It is a subobject of $(C_2 \otimes i[1]) \otimes i[1] = C_2 \otimes i[1] \otimes i[1]$, which is a union of eight shuffles. The inclusion is precisely an example of one of the associators α. We will see in Proposition 6.32 that these associator maps, while not isomorphisms, are still weak equivalences in an appropriate sense.

To conclude this section, let us briefly consider the pushout-product property for n-fold tensor products. For a tensor product of three factors, this property can be stated as follows. Consider three maps $A \to X$, $B \to Y$ and $C \to Z$. Then the various tensor products can be organized in a cube, where we write ABC for $A \otimes B \otimes C$, etc.:

$$\begin{tikzcd}
ABC \arrow[rr] \arrow[dr] \arrow[dd] & & ABZ \arrow[dr] \arrow[dd] & \\
& AYC \arrow[rr] \arrow[dd] & & AYZ \arrow[dd] \\
XBC \arrow[rr] \arrow[dr] & & XBZ \arrow[dr] & \\
& XYC \arrow[rr] & & XYZ.
\end{tikzcd}$$

Omitting the last vertex XYZ yields a 'punctured cube', of which one can take the colimit. The pushout-product property then states that the map from this colimit to the last vertex XYZ is a normal monomorphism whenever each of the three maps $A \to X$, $B \to Y$ and $C \to Z$ is. This property will hold in three cases, namely where two of the three factors are simplicial, when all dendroidal sets involved are open, or when all of them are closed. These last two cases can be proved in a way analogous to the case of binary products, but the first one is actually a consequence of the binary case and the associativity isomorphism of Corollary 4.31.

To state the general case of a tensor product of n factors, consider the n-fold product $\{0,1\}^n$, viewed as a category with a unique arrow $(i_1, \ldots, i_n) \to (j_1, \ldots, j_n)$ if and only if $i_k \leq j_k$ for each k. Denote the full subcategory on all objects except the terminal one by $\{0,1\}^n_-$. Then an n-tuple of maps $X_k^0 \to X_k^1$ between dendroidal sets defines a functor

$$X : \{0,1\}^n \to \mathbf{dSets} : (i_1, \ldots, i_n) \mapsto X_1^{i_1} \otimes \cdots \otimes X_n^{i_n}.$$

Write X_- for the restriction of this functor to the category $\{0,1\}^n_-$. Then the n-fold pushout-product property concerns the map

$$\varinjlim X_- \to X_1^1 \otimes \cdots \otimes X_n^1.$$

The domain of this map can also suggestively be written as

$$\bigcup_{k=1}^{n} X_1^1 \otimes \cdots \otimes X_{k-1}^1 \otimes X_k^0 \otimes X_{k+1}^1 \otimes \cdots \otimes X_n^1.$$

The following is then the general analogue of the results on pushout-products of the previous section. The proof is analogous, we will omit it here.

Proposition 4.34 *Let $X_k^0 \to X_k^1$ be normal monomorphisms between dendroidal sets, for $k = 1, \ldots, n$. Then the pushout-product map*

$$\varinjlim X_- \to X_1^1 \otimes \cdots \otimes X_n^1$$

is again a normal monomorphism in each of the following three cases:

(i) *All but one of the X_k^1 are simplicial.*
(ii) *All X_k^1 are open (and hence all X_k^0 are as well).*
(iii) *All X_k^i are closed.*

Historical Notes

The construction of a tensor product of dendroidal sets from the Boardman–Vogt tensor product of operads is already contained in the original papers [116, 117]. However, several aspects were at first overlooked. A correct description of the behaviour of the tensor product with respect to normal monomorphisms first appears in [43]. This aspect is discussed systematically in [80], where the weak form of associativity that the tensor product of dendroidal sets satisfies is also made explicit.

Chapter 5
Kan Conditions for Simplicial Sets

In this chapter we will define several important classes of simplicial sets and of maps between simplicial sets by extension and lifting conditions. These include the classical notions of Kan complexes and Kan fibrations, as well as the notion of inner Kan complex (or ∞-category). Kan complexes and Kan fibrations play a central role in homotopy theory and will therefore be of fundamental importance in Part II of this book. These concepts extend to the theory dendroidal sets, as we will explain in the next chapter. However, the necessary combinatorics in that case will be more involved. The current chapter serves as an introduction to and blueprint for these more general results, but is more accessible to the uninitiated reader. Those readers familiar with the classical notions of fibrations between simplicial sets and the basics of the theory of ∞-categories might wish to only glance over this chapter and proceed to the next.

5.1 Kan Complexes and ∞-Categories

Important examples of simplicial sets are obtained by applying the singular complex functor (adjoint to geometric realization)

$$\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSets}$$

to a topological space or by applying the nerve functor (adjoint to τ)

$$N : \mathbf{Cat} \to \mathbf{sSets}$$

to a small category. The simplicial sets thus obtained have special 'extension' properties, which we will make explicit in this section. To this end we define the *horn* $\Lambda^k[n]$, for any $n > 0$ and $0 \leq k \leq n$. It is the subobject of the n-simplex $\Delta[n]$ given by the union of all the faces of $\Delta[n]$ containing the vertex k (equivalently, the union of all the faces except the one given by $\partial_k : \Delta[n-1] \to \Delta[n]$). It is a simplicial subset of the boundary $\partial\Delta[n]$, which is the union of all faces of $\Delta[n]$. These simplicial sets

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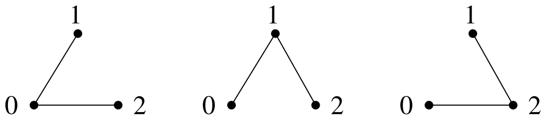
are described by

$$\begin{aligned}\Lambda^k[n]_p &= \{\alpha : [p] \to [n] \mid \exists i \neq k : i \notin \mathrm{im}(\alpha)\},\\ \partial\Delta[n]_p &= \{\alpha : [p] \to [n] \mid \exists i : i \notin \mathrm{im}(\alpha)\}.\end{aligned}$$



Λ^2_0 Λ^2_1 Λ^2_2

Definition 5.1 A simplicial set X is a *Kan complex* if any map $\Lambda^k[n] \to X$ admits an extension to a map $\Delta[n] \to X$, for any $n > 0$ and $0 \leq k \leq n$. In a diagram:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & X. \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

A simplicial set X is called an ∞*-category* or an *inner Kan complex* if it satisfies this condition only for $0 < k < n$.

The horns $\Lambda^k[n]$ for $0 < k < n$ are called the *inner horns* of the n-simplex. The extensions which the preceding definition asks for need not be unique. If they are, one calls X a *strict* (inner) Kan complex. Later in this section we will explain that an inner Kan complex is a kind of 'weak' category, thus providing some justification for the use of the term ∞-category.

Example 5.2 (a) If T is a topological space, then its singular complex $\mathrm{Sing}(T)$ is a Kan complex. Indeed, by adjunction, an extension problem on the left corresponds to one on the right:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & \mathrm{Sing}(T) \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} \mid \Lambda^k[n] \mid \arrow[r] \arrow[d] & X. \\ \mid \Delta[n] \mid \arrow[ur, dashed] & \end{tikzcd}$$

A dashed arrow making the diagram on the right commute always exists, since the inclusion $|\Lambda^k[n]| \subseteq |\Delta[n]| = \Delta^n$ admits a retraction (even a deformation retraction).

(b) If $\mathbf{C}$ is a small category, its nerve $N\mathbf{C}$ is a strict inner Kan complex. Using the left adjoint $\tau : \mathbf{sSets} \to \mathbf{Cat}$, this follows as in the previous example because for $0 < k < n$ the functor τ maps the inclusion $\Lambda^k[n] \to \Delta[n]$ to an *isomorphism* of categories. To see this, note first that for any simplicial set X the generators and relations of $\tau(X)$ are contained in $\mathrm{sk}_2 X$. Since $\Lambda^k[n]$ contains all the 2-simplices of $\Delta[n]$ if $n \geq 4$, it is clear that $\tau(\Lambda^k[n]) \to \tau(\Delta[n])$ is an isomorphism in those cases (just as $\tau(\partial\Delta[n]) \to \tau(\Delta[n])$ is for $n \geq 3$). Thus, we only need to check the cases $n = 2$, $k = 1$ and $n = 3$, $k = 1$ or $k = 2$. For the first one, simply observe that

$$\tau(\Delta[2]) = [2] = (0 \to 1 \to 2)$$

is the pushout in **Cat** of $0 \to 1$ and $1 \to 2$ along their common object 1, which is by definition $\tau(\Lambda^1[2])$. For the second case, note that $\tau(\Lambda^1[3])$ is the category generated by arrows $\alpha_{ji} : i \to j$ for $0 \leq i < j \leq 3$ subject to the relations $\alpha_{21}\alpha_{10} = \alpha_{20}$, $\alpha_{32}\alpha_{21} = \alpha_{31}$, and $\alpha_{31}\alpha_{10} = \alpha_{30}$. It follows that the relation $\alpha_{32}\alpha_{20} = \alpha_{30}$ also holds and that this category is simply isomorphic to [3]. The case $\tau(\Lambda^2[3])$ is of course similar.

In fact, the property described in (b) above characterizes the nerves of categories among simplicial sets:

Proposition 5.3 *For a simplicial set X, the following are equivalent:*

(i) *X is a strict inner Kan complex.*
(ii) *For each $n \geq 2$, the map*

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

induced by the morphisms $[1] \xrightarrow{(i-1,i)} [n]$ for $i = 1, \ldots, n$ is an isomorphism.
(iii) *X is isomorphic to $N\mathbf{C}$ for some small category $\mathbf{C}$.*
(iv) *The unit $X \to N\tau(X)$ is an isomorphism.*

Proof We will postpone the proof of the implication (i) ⇒ (ii) to Section 5.4 (see Remark 5.33), but all the others are elementary. Suppose X satisfies (ii). Then for any pair of 1-simplices $a, b \in X_1$ with $d_0 a = d_1 b$, depicted as

$$x_0 \xrightarrow{a} x_1 \xrightarrow{b} x_2$$

there is a *unique* 2-simplex e with $d_2 e = a$ and $d_0 e = b$,

$$\begin{tikzcd} x_0 \arrow[r, "a"] \arrow[dr, dashed] & x_1 \arrow[d, "b"] \\ & x_2. \end{tikzcd}$$

We define $b \circ a = d_1 e$. It is readily checked that this makes X_1 into the set of arrows of a category $\mathbf{C}$ with X_0 as its set of objects. The associativity of composition follows by applying property (ii) for $n = 3$, where it shows that given 1-simplices

$$x_0 \xrightarrow{a} x_1 \xrightarrow{b} x_2 \xrightarrow{c} x_3,$$

there is a *unique* 3-simplex f with edges a, b, c, and the various compositions formed out of them. Consider for this category $\mathbf{C}$ the diagram

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & N\mathbf{C}_n \arrow[d, equal] \\
X_1 \times_{X_0} \cdots \times_{X_0} X_1 \arrow[r] & \mathrm{ar}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} \cdots \times_{\mathrm{ob}(\mathbf{C})} \mathrm{ar}(\mathbf{C}).
\end{tikzcd}$$

The bottom map is an isomorphism for any $n \geq 1$ by definition of $\mathbf{C}$ and the left-hand map is an isomorphism by hypothesis. Hence the top map is an isomorphism as well, showing that $X \cong N\mathbf{C}$. (The category constructed here is actually $\tau(X)$, as will be clear from what follows.) Now assume X satisfies (iii), i.e., $X \cong N\mathbf{C}$. Since the nerve functor is fully faithful, the counit $\tau N\mathbf{C} \xrightarrow{\varepsilon} \mathbf{C}$ is an isomorphism. But then the triangle identity

$$\begin{tikzcd}
N\mathbf{C} \arrow[r] \arrow[dr, equal] & N\tau N\mathbf{C} \arrow[d, "N\varepsilon"] \\
& N\mathbf{C}
\end{tikzcd}$$

shows that the unit $N\mathbf{C} \to N\tau N\mathbf{C}$ is an isomorphism, so that the same is true of $X \to N\tau X$. Finally, the implication (iv) ⇒ (i) is Example 5.2(b) above. □

The previous proposition suggests thinking of an ∞-category as some kind of 'weak' category, where compositions exist but are not necessarily unique. They are unique 'up to homotopy' however, in the following sense. Let $x_0 \xrightarrow{a} x_1$ and $x_1 \xrightarrow{b} x_2$ be two 1-simplices in an ∞-category X, compatible in the sense that $d_0 a = d_1 b$ as indicated. Then a 2-simplex e as follows defines *some* choice of composition $\gamma = d_1 e$:

$$\begin{tikzcd}
x_0 \arrow[r, "a"] \arrow[dr, "\gamma"'] & x_1 \arrow[d, "b"] \\
& x_2.
\end{tikzcd}$$

If f is another such 2-simplex with $d_0 f = b$ and $d_2 f = a$, defining another composition $\delta = d_1 f$, then γ and δ fit into a 2-simplex h with a degenerate face $d_0 h = s_1 x_2$,

$$\begin{tikzcd}
x_0 \arrow[r, "\delta"] \arrow[dr, "\gamma"'] & x_2 \arrow[d, equal] \\
& x_2.
\end{tikzcd}$$

Such a 2-simplex can be obtained by first 'filling' the horn $\Lambda^1[3] \to \Delta[3]$ depicted as

$$\begin{tikzcd}
& x_0 \arrow[d, "a"] \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\
& x_1 \arrow[dl, "b"] \arrow[dr, "b"'] & \\
x_2 \arrow[rr, equal, "s_0 b"'] & & x_2
\end{tikzcd}$$

and taking the face of the resulting 3-simplex opposite x_1.

We will return to this homotopy relation at the end of this section. But first let us make the following observation. The nerve of a category is a strict inner Kan complex as we have seen, but it is rarely a Kan complex.

Proposition 5.4 *Let* **C** *be a small category. Then* $N\mathbf{C}$ *is a Kan complex if and only if* **C** *is a groupoid.*

Proof Suppose $N\mathbf{C}$ is a Kan complex and let $f : c \to d$ be a morphism in **C**. Then filling the horns $\Lambda^i[2] \to N\mathbf{C}$ for $i = 0, 2$ depicted as

$$
\begin{tikzcd}
c \arrow[r, "f"] \arrow[dr, equal] & d \arrow[d, dashed] \\
& c
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
d \arrow[r, dashed] \arrow[dr, equal] & c \arrow[d] \\
& d
\end{tikzcd}
$$

shows that f has both a left and a right inverse in **C**, hence is an isomorphism. Since f was arbitrary, **C** is a groupoid. The fact that $N\mathbf{C}$ is also a strict Kan complex follows easily from the uniqueness of inverses. Now suppose **C** is a groupoid. To see that $N\mathbf{C}$ is a Kan complex, consider a horn filling problem

$$
\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & N\mathbf{C}. \\
\Delta[n] \arrow[ur, dashed] &
\end{tikzcd}
$$

We already know a unique filling exists for $0 < k < n$. Moreover, for $n \geq 4$ there is nothing to prove since τ maps $\Lambda^k[n] \to \Delta[n]$ to an isomorphism, as observed before. Moreover, for $n = 2$, $k = 0, 2$, inverses in **C** provide fillings (see the diagrams above), while for $n = 1$ identity morphisms provide the relevant fillings. Thus, the only remaining cases are $n = 3$, $k = 0$ and $k = 3$. For the first of these, this means that given a diagram

$$
\begin{tikzcd}
& x_0 \arrow[d, "a"] \arrow[ddl] \arrow[ddr] & \\
& x_1 \arrow[dl, "b"] \arrow[dr] & \\
x_2 \arrow[rr, "c"'] & & x_3
\end{tikzcd}
$$

in **C** in which all faces commute except possibly the bottom one, in fact the bottom one commutes as well. This is indeed the case if a is an isomorphism (in fact, being an epimorphism suffices). The case $k = 3$ is similar. □

In this book another important source of examples of ∞-categories is the homotopy-coherent nerve construction. It provides further motivation for thinking of ∞-categories as a generalization of the concept of category.

Proposition 5.5 *Suppose* $\mathbf{C}$ *is a simplicial category for which* $\mathbf{C}(x, y)$ *is a Kan complex, for every two objects* $x, y \in \mathbf{C}$*. Then its homotopy-coherent nerve* $w^*\mathbf{C}$ *is an* ∞*-category.*

We postpone the proof of this proposition to Section 6.1, where we prove a more general version for simplicial operads and dendroidal sets.

We conclude this section with a discussion of the category $\tau(X)$ generated by an ∞-category X. We will sometimes also refer to $\tau(X)$ as the *homotopy category* of X. For a general simplicial set X, the category $\tau(X)$ is hard to control, since it is only defined by generators and relations. But if X is an inner Kan complex the category $\tau(X)$ is much easier to describe. Indeed, if X has fillers for inner horns, we construct a category $\tau_1(X)$ as follows. The objects of $\tau_1(X)$ are the vertices of X (like for $\tau(X)$). The arrows $x \to y$ in $\tau_1(X)$ are equivalence classes of 1-simplices f with $d_0 f = y$ and $d_1 f = x$, where two such 1-simplices f and f' are declared to be equivalent if there exists a 2-simplex $e \in X_2$ with $d_2 e = f$, $d_1 e = f'$ and $d_0 e = s_0 y$, the degenerate 1-simplex on the vertex y:

$$\begin{tikzcd} & y \arrow[dr, equal] & \\ x \arrow[ur, "f"] \arrow[rr, "f'"] & & y. \end{tikzcd}$$

Writing $[f]$ for the equivalence class of f, the composition of two arrows $[f] : x \to y$ and $[g] : y \to z$ is defined to be $[h] : x \to z$, where $h = d_1 b$ for any 2-simplex b with $d_2 b = f$ and $d_0 b = g$. One can think of this 2-simplex b as a 'witness' to the fact that z is a composition of the arrows g and f.

Lemma 5.6 (i) *The relation just defined is an equivalence relation.*

(ii) *Two 1-simplices* f *and* f' *as above are equivalent if and only if there exists a 2-simplex* $b \in X_2$ *with* $d_1 b = f$*,* $d_0 b = f'$*, and* $d_2 b = s_0 x$*:*

$$\begin{tikzcd} & x \arrow[dr, "f'"] & \\ x \arrow[ur, equal] \arrow[rr, "f"] & & y. \end{tikzcd}$$

(iii) *The composition described above is well-defined on equivalence classes.*

(iv) *The evident functor* $\tau_1(X) \to \tau(X)$ *is an isomorphism of categories (and we will no longer distinguish them in notation).*

Proof The proofs are all elementary horn filling diagrams, which nicely illustrate the use of the inner Kan condition.

(i) Reflexivity of the relation is witnessed by the degenerate 2-simplex $b = s_1 f$. To see that the given relation is transitive, suppose we are given 2-simplices a and b with common inner face $d_1 a = d_1 b = f'$ as pictured below:

$$\begin{tikzcd}
 & y \arrow[dr, equal] & \\
x \arrow[ur, "f"] \arrow[rr, "f'"] \arrow[dr, "f''"'] & & y \\
 & y. \arrow[ur, equal] &
\end{tikzcd}$$

Together with a degenerate simplex on $\partial_0\Delta[3]$, these define a map $\Lambda^2[3] \to X$. Then picking an extension $\alpha : \Delta[3] \to X$ and considering the face $d_2\alpha$ shows that $f \sim f''$. Finally, the relation is symmetric: given a 2-simplex e witnessing the relation $f \sim f'$, define a map $h : \Lambda^1[3] \to X$ which is e on the face opposite the third vertex and degenerate opposite the vertices 0 and 2 (as in the picture below). Then the face d_1k of an extension $k : \Delta[3] \to X$ witnesses the symmetry (for clarity we label the vertices by the objects of $[3]$ rather than x and y):

$$\begin{tikzcd}
 & 0 \arrow[d, "f"] \arrow[ddl, "f'"'] \arrow[ddr, "f"] & \\
 & 1 \arrow[dl, equal] \arrow[dr, equal] & \\
2 \arrow[rr, equal] & & 3
\end{tikzcd}$$

(ii) Suppose we are given a 2-simplex e witnessing $f \sim f'$ as above. Then construct a map $\Lambda^2[3] \to X$

$$\begin{tikzcd}
 & 0 \arrow[d, equal] \arrow[ddl, "f"'] \arrow[ddr, "f"] & \\
 & 1 \arrow[dl, "f"] \arrow[dr, "f'"'] & \\
2 \arrow[rr, equal] & & 3
\end{tikzcd}$$

which is e on the bottom face $\partial_0\Delta[3]$, degenerate s_0f on the face $\partial_3\Delta[3]$ and degenerate s_1f on the front face $\partial_1\Delta[3]$. Then the face d_2k of a filler $k : \Delta[3] \to X$ shows that f and f' are related as in the statement of (ii). The converse direction can be proved by a very similar argument.

(iii) Suppose $g \sim g' : y \to z$ are two equivalent 1-simplices (with the equivalence witnessed by a 2-simplex a) and $f : x \to y$ is another 1-simplex. If b and c are 2-simplices 'witnessing' compositions $g \circ f$ and $g' \circ f$ as in

$$\begin{tikzcd}
& 0 \arrow[d, "f"] \arrow[ddl, "g\circ f"'] \arrow[ddr, "g'\circ f"] & \\
& 1 \arrow[dl, "g"] \arrow[dr, "g'"'] & \\
2 \arrow[rr, equal] & & 3
\end{tikzcd}$$

then a, b, and c together define a map $\Lambda^1[3] \to X$, and the d_1 face of an extension shows $g \circ f \sim g' \circ f$, as in the picture. A similar tetrahedron, now using (ii) of the lemma, applies to show that composition is well-defined on equivalence classes in the other variable. Moreover, this proves that composition does not depend on the choice of a 2-simplex (b and c in the diagram above). It is now clear that degenerate 1-simplices act as units for the composition. Finally, to see that composition is associative, one fills a horn of the form

$$\begin{tikzcd}
& 0 \arrow[d, "f"] \arrow[ddl, "gf"'] \arrow[ddr, "h(gf)"] & \\
& 1 \arrow[dl, "g"] \arrow[dr, "hg"'] & \\
2 \arrow[rr, "h"'] & & 3
\end{tikzcd}$$

by taking a 2-simplex for a composition gf on the face $d_3\Delta[3]$, a 2-simplex for a composition hg on the face $d_0\Delta[3]$, and finally a 2-simplex for a composition of the 1-simplices h and gf on the face $d_1\Delta[3]$. Together this defines a map $\Lambda^2[3] \to \Delta[3]$, and d_2 of an extension gives a 2-simplex equating $[h(gf)]$ with the composition of $[hg]$ and $[f]$.

(iv) Clearly the relations defining $\tau_1(X)$ are contained in those defining $\tau(X)$, so there is a quotient map $\tau_1(X) \to \tau(X)$ which is the identity on objects (the vertices of X) and on generators for the arrows (the 1-simplices of X). On the other hand, now that we know $\tau_1(X)$ is a well-defined category, the universal property of $\tau(X)$ gives a functor $\tau(X) \to \tau_1(X)$ satisfying the same properties, so that these must be each other's inverse. □

5.2 Fibrations Between Simplicial Sets

The extension conditions defining Kan complexes and ∞-categories can be generalized to morphisms between simplicial sets. This generalization will also help us to better understand the properties of (inner) Kan complexes themselves, as we will see. First we refine our terminology for horns.

Definition 5.7 An inclusion $\Lambda^k[n] \to \Delta[n]$ of simplicial sets is called a *horn* for $0 \leq k \leq n$, an *inner horn* for $0 < k < n$, a *left horn* for $0 \leq k < n$, and a *right horn* for $0 < k \leq n$.

Definition 5.8 A morphism $Y \to X$ between simplicial sets is said to be

- a *Kan fibration* if it has the right lifting property with respect to all horn inclusions,
- a *left fibration* if it has the right lifting property with respect to all left horn inclusions,
- a *right fibration* if it has the right lifting property with respect to all right horn inclusions,
- an *inner fibration* if it has the right lifting property with respect to all inner horn inclusions.

We record the following properties of these fibrations, which are immediate from their definitions.

Lemma 5.9 *A composition of Kan fibrations, a pullback of a Kan fibration, or a retract of a Kan fibration is again a Kan fibration. The analogous statements hold for left, right, and inner fibrations.*

Example 5.10 (a) A simplicial set X is a Kan complex if and only if the unique map $X \to \Delta[0]$ is a Kan fibration, and an ∞-category if and only if that map is an inner fibration.

(b) We will see later that if the map $X \to \Delta[0]$ is a left or right fibration, it is automatically a Kan fibration (more generally, this is true for any map $X \to K$ with K a Kan complex). This explains why left or right horns did not feature explicitly in the previous section.

Example 5.11 For a map $T \to S$ between topological spaces, the map $\mathrm{Sing}(T) \to \mathrm{Sing}(S)$ is a Kan fibration if and only if $T \to S$ is a Serre fibration. Indeed, by adjunction there is a correspondence between lifting problems

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & \mathrm{Sing}(T) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathrm{Sing}(S) \end{tikzcd} \qquad \begin{tikzcd} \mid \Lambda^k[n] \mid \arrow[r] \arrow[d] & T \arrow[d] \\ \mid \Delta[n] \mid \arrow[r] \arrow[ur, dashed] & S. \end{tikzcd}$$

and the right-hand one essentially defines Serre fibrations (see Section 7.2). We will come back to the relation between Kan and Serre fibrations in much more detail in Chapter 8, also investigating the effect of geometric realization on a Kan fibration.

Example 5.12 Recall that the nerve $N\mathbf{C}$ of a small category $\mathbf{C}$ is a strict inner Kan complex, and a Kan complex if and only if $\mathbf{C}$ is a groupoid. One can ask for a functor $p : \mathbf{D} \to \mathbf{C}$ when the corresponding map of simplicial sets $N\mathbf{D} \to N\mathbf{C}$ is a fibration of some sort. It follows easily from the uniqueness of inner horn fillings that $N\mathbf{D} \to N\mathbf{C}$ is always an inner fibration.

The property of $N\mathbf{D} \to N\mathbf{C}$ being a right fibration is related to the classical notion of a *fibred category*, which is defined in terms of *cartesian arrows*. Let us fix an arrow $f : c' \to c$ in $\mathbf{C}$. An arrow $g : d' \to d$ in $\mathbf{D}$ with $p(g) = f$ is said to be a *cartesian lift* of f if for any arrow $h : d'' \to d$ in $\mathbf{D}$ and any factorization $p(h) = f \circ f'$ in $\mathbf{C}$, there is a unique arrow $g' : d'' \to d'$ in $\mathbf{D}$ with $g \circ g' = h$ and $p(g') = f$. In a picture:

$$
\begin{tikzcd}
d'' \arrow[d, dashed, "g'"'] \arrow[dr, "h"] & \\
d' \arrow[r, "g"] & d
\end{tikzcd}
\qquad
\begin{tikzcd}
c'' \arrow[d, "f'"'] \arrow[dr, "p(h)"] & \\
c' \arrow[r, "f"] & c.
\end{tikzcd}
$$

The functor $p : \mathbf{D} \to \mathbf{C}$ is called a *fibred category* if for any arrow $f : c' \to c$ in $\mathbf{C}$ and any object d of $\mathbf{D}$ with $p(d) = c$, a cartesian lift g of f exists. The *fibre* $p^{-1}(c)$ of p over an object c is by definition the subcategory of $\mathbf{D}$ given by all arrows mapped by p to the identity of c. If each fibre is a groupoid, then $p : \mathbf{D} \to \mathbf{C}$ is called a *category fibred in groupoids*. In a category fibred in groupoids any arrow of $\mathbf{D}$ is in fact cartesian, as one easily verifies by noting that it is related to a cartesian arrow via an isomorphism. Conversely, the reader may like to check that a fibred category in which every arrow of $\mathbf{D}$ is cartesian is fibred in groupoids.

We claim that for a functor $p : \mathbf{D} \to \mathbf{C}$, the map $N\mathbf{D} \to N\mathbf{C}$ is a right fibration if and only if p is such a category fibred in groupoids. Indeed, assume $N\mathbf{D} \to N\mathbf{C}$ is a right fibration. Then for any object c in $\mathbf{C}$ so is its pullback

$$
\begin{tikzcd}
N(p^{-1}c) \arrow[r] \arrow[d] & N\mathbf{D} \arrow[d] \\
\Delta[0] \arrow[r, "c"] & N\mathbf{C}.
\end{tikzcd}
$$

As in the proof of Proposition 5.4, using the right lifting property with respect to horn inclusions $\Lambda^2[2] \to \Delta[2]$, it follows that every arrow in $p^{-1}(c)$ has a right inverse, which implies that $p^{-1}(c)$ is a groupoid. (To see this, note that each such right inverse has itself a right inverse, hence is an isomorphism.) Next, let us consider a lifting problem

$$
\begin{tikzcd}
\Lambda^n[n] \arrow[r] \arrow[d] & N\mathbf{D} \arrow[d] \\
\Delta[n] \arrow[r] & N\mathbf{C}.
\end{tikzcd}
$$

Since we are dealing with nerves of categories, this problem is only interesting for $n \leq 3$ as noted before. For $n = 1$, to solve such a lifting problem one should provide, given an arrow $f : c' \to c$ in $\mathbf{C}$ and a lift $d \in \mathbf{D}$ of c, an arrow $g : d' \to d$ in $\mathbf{D}$ lifting f. For $n = 2$ one is given a commutative triangle in $\mathbf{C}$

$$\begin{tikzcd} c'' \arrow[d] \arrow[dr] & \\ c' \arrow[r] & c \end{tikzcd}$$

and a partial lift to a diagram in $\mathbf{D}$ (without the dashed arrow)

$$\begin{tikzcd} d'' \arrow[d, dashed, "f"] \arrow[dr, "h"] & \\ d' \arrow[r, "g"'] & d. \end{tikzcd}$$

Solving the lifting problem amounts to providing the dashed arrow so that the resulting triangle commutes and projects to the given diagram in $\mathbf{C}$ under p. To conclude that the arrows of $\mathbf{D}$ are cartesian, we still have to settle the uniqueness of the dashed arrow f above. Suppose we have another candidate f' for this factorization. Then we consider the map $\Lambda^3[3] \to N\mathbf{D}$ depicted by

$$\begin{tikzcd} & d'' \arrow[ddl, "f"'] \arrow[d, "f'"'] \arrow[ddr, "h"] & \\ & d' \arrow[dl, equal] \arrow[dr, "g"] & \\ d' \arrow[rr, "g"'] & & d. \end{tikzcd}$$

A lift in

$$\begin{tikzcd} \Lambda^3[3] \arrow[r] \arrow[d] & N\mathbf{D} \arrow[d] \\ \Delta[3] \arrow[r] \arrow[ur, dashed] & N\mathbf{C} \end{tikzcd}$$

proves that the left triangle in the previous diagram commutes, giving the desired uniqueness. This concludes the proof that $p : \mathbf{D} \to \mathbf{C}$ is a category fibred in groupoids.

Conversely, if $p : \mathbf{D} \to \mathbf{C}$ is a category fibred in groupoids, we should argue that Np is a right fibration. Indeed, we already observed it is an inner fibration, so we only need to check the right lifting property with respect to horn inclusions $\Lambda^n[n] \to \Delta[n]$ for $1 \leq n \leq 3$. For $n = 1, 2$ this is clear from the discussion above. For $n = 3$, we should check that in a diagram in $\mathbf{D}$ of shape

$$\begin{tikzcd}
 & 0 \arrow[d] \arrow[ddl] \arrow[ddr] & \\
 & 1 \arrow[dl] \arrow[dr] & \\
2 \arrow[rr] & & 3.
\end{tikzcd}$$

with the property that all faces except the one opposite the vertex 3 commute, that last one has to commute as well. This property follows from the uniqueness condition in the definition of a fibred category.

Properties like the one discussed for right fibrations in the last example of course have dual versions for left fibrations. More generally, if X is an arbitrary simplicial set, one can define its *opposite* as the composition

$$\begin{tikzcd}
\mathbf{\Delta}^{\mathrm{op}} \arrow[r, "\mathrm{rev}^{\mathrm{op}}"] \arrow[dr, "X^{\mathrm{op}}"'] & \mathbf{\Delta}^{\mathrm{op}} \arrow[d, "X"] \\
 & \mathbf{Sets},
\end{tikzcd}$$

where $\mathrm{rev} : \mathbf{\Delta} \to \mathbf{\Delta}$ is the functor which reverses a linear order. When we view linear orders as categories, then rev is the functor sending a category to its opposite. For $\mathbf{\Delta}$, this functor is the identity on objects, but acts on an arrow $\alpha : [n] \to [m]$ by

$$\mathrm{rev}(\alpha)(i) = m - \alpha(n - i) \qquad \text{for} \quad i = 0, \ldots, n.$$

If X is the nerve of a category, the notion of opposite simplicial set corresponds to the usual opposite of a category, i.e.,

$$(N\mathbf{C})^{\mathrm{op}} = N(\mathbf{C}^{\mathrm{op}}).$$

For an ∞-category X the opposite X^{op} is again an ∞-category, which we refer to as the *opposite ∞-category*. More generally, $(-)^{\mathrm{op}}$ preserves inner fibrations, and turns left fibrations into right fibrations (and vice versa). Thus, many properties of left fibrations automatically transfer to properties of right fibrations by applying them to opposites.

We conclude this section with a different, much more restrictive notion of fibration.

Definition 5.13 A morphism $Y \to X$ between simplicial sets is called a *trivial fibration* if it has the right lifting property with respect to the boundary inclusion $\partial\Delta[n] \to \Delta[n]$ for each $n \geq 0$. A simplicial set Y is called an *acyclic Kan complex* if $Y \to \Delta[0]$ is a trivial fibration.

We will have ample occasion to discuss the properties of trivial fibrations later in this book. For now, we limit ourselves to stating the relation to Kan fibrations and considering examples analogous to the ones above.

Lemma 5.14 *A trivial fibration is also a Kan fibration.*

In particular, this lemma implies that if $Y \to \Delta[0]$ is a trivial fibration, then Y is a Kan complex, justifying the terminology 'acyclic Kan complex' in the definition above.

Proof The lemma follows from the fact that any monomorphism between simplicial sets, in particular any horn inclusion $\Lambda^k[n] \to \Delta[n]$, can be written as a composition of pushouts of boundary inclusions of simplices. (In fact, the reader is invited to check that a composition of two such morphisms suffices for a horn.) □

Example 5.15 (a) Let $f : T \to S$ be a Serre fibration between topological spaces. Then $\mathrm{Sing}(T) \to \mathrm{Sing}(S)$ is a Kan fibration as we observed. It is a trivial fibration if and only if $T \to S$ has the left lifting property with respect to any boundary inclusion $\partial\Delta^n \to \Delta^n$ of a topological simplex. This condition is equivalent to the condition that f induces an isomorphism $\pi_0 T \to \pi_0 S$, as well as isomorphisms of homotopy groups $\pi_n(T, t_0) \to \pi_n(T, f(t_0))$ for any $n \geq 0$ and any choice of basepoint $t_0 \in T$. In other words, $\mathrm{Sing}(f)$ is a trivial fibration if and only if f is a Serre fibration which is in addition a weak homotopy equivalence.

(b) Let $\mathbf{C}$ be a category. Then $N\mathbf{C}$ is an acyclic Kan complex if and only if $\mathbf{C}$ is a non-empty *contractible* groupoid, i.e., a groupoid with the property that there is a unique morphism between any two objects. A functor $\mathbf{D} \to \mathbf{C}$ defines a trivial fibration $N\mathbf{D} \to N\mathbf{C}$ if and only if it is a category fibred in such non-empty contractible groupoids. We leave it to the reader to verify that such a functor is precisely an equivalence of categories which is also surjective on objects.

5.3 Saturated Classes and Anodyne Morphisms

If $Y \to X$ is a fibration of some kind (Kan, inner, left, right) then the defining right lifting property of $Y \to X$ with respect to certain horns implies that it has the right lifting property with respect to many more morphisms. We will state this using saturated classes, which we already introduced in Section 3.7. Recall that a class of morphisms is *saturated* if it is closed under pushouts, transfinite composition and retracts. If $\mathcal{F}$ is a class of morphisms, then the class of morphisms $\mathcal{A}$ having the left lifting property with respect to $\mathcal{F}$ is saturated (Lemma 3.33). Of course, this result has dual version for classes of morphisms defined by having the right lifting property with respect to a given collection of morphisms. For the various classes of fibrations considered before, this was essentially already noted in Lemma 5.9.

Definition 5.16 An *anodyne morphism* of simplicial sets is a morphism having the left lifting property with respect to all Kan fibrations. Similarly, an *inner*, *left*, or *right anodyne morphism* is a morphism having the left lifting property with respect to inner, left, or right fibrations, respectively.

Thus the class of inner anodyne morphisms in particular contains all inner horn inclusions $\Lambda^k[n] \to \Delta[n]$, $0 < k < n$, and similarly for the other kinds of anodynes. Lemma 3.33 implies the following:

Lemma 5.17 *The class of (inner, left, right) anodyne morphisms is saturated.*

Showing that a given morphism is anodyne is often not accomplished by checking the defining left lifting property, but rather by building it from simple kinds of anodynes, such as horn inclusions. This is similar to how we described a general monomorphism of simplicial sets as a composition of pushouts of boundary inclusions of simplices, using skeletal filtration.

Definition 5.18 Let $\mathcal{J}$ be a class of morphisms. Then a morphism f is $\mathcal{J}$-*cellular* if it is a transfinite composition of pushouts of elements of $\mathcal{J}$.

Note that if $\mathcal{J}$ is the set of horn inclusions $\Lambda^k[n] \to \Delta[n]$, then an $\mathcal{J}$-cellular map is in particular an anodyne map. A similar statement applies to maps which are cellular with respect to inner, left, or right horns and inner, left, or right anodynes respectively. Using the small object argument (as described in Remark 3.38) we immediately have the following analogue of a result in Section 3.7.

Lemma 5.19 *Let $\mathcal{J}$ be the set of all horn inclusions $\Lambda^k[n] \to \Delta[n]$. Then any morphism $f : X \to Y$ of simplicial sets can be factored as*

$$X \xrightarrow{i} Z \xrightarrow{p} Y,$$

with i an $\mathcal{J}$-cellular map and p a Kan fibration. The analogous statement is true for inner, left, or right horns and inner, left, or right fibrations respectively.

Lemma 5.20 *A map $f : A \to B$ of simplicial sets is anodyne if and only it is a retract of an $\mathcal{J}$-cellular map, with $\mathcal{J}$ the set of horn inclusions $\Lambda^k[n] \to \Delta[n]$. Similarly it is inner, left, or right anodyne if and only if it is a retract of a map which is cellular with respect to inner, left, or right horn inclusions respectively.*

Proof We repeat the argument of Lemma 3.39, using Lemma 5.19 to obtain the relevant factorizations. If f is anodyne, factor it into an $\mathcal{J}$-cellular map followed by a Kan fibration:

$$A \xrightarrow{i} X \xrightarrow{p} B.$$

Then one picks a lift as indicated by the dashed arrow in the following square, which exists by the assumption on f:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dashed] & B. \end{tikzcd}$$

It follows that f is a retract of i. Conversely, any retract of an $\mathcal{J}$-cellular map is anodyne because the class of anodynes is saturated. $\square$

We can now characterize the (inner, left, or right) anodyne maps in the following way:

Corollary 5.21 *The class of (inner, left, or right) anodyne maps is the smallest saturated class containing the (inner, left, or right) horn inclusions.*

Proof The smallest saturated class containing a given collection of morphisms $\mathcal{J}$ must at least contain all retracts of $\mathcal{J}$-cellular maps, so the statement is immediate from the preceding lemma. □

For this reason, we will sometimes refer to the class of (inner, left, or right) anodynes as the *saturation* of the class of (inner, left, or right) horn inclusions. The reader should note that the arguments above have little to do with the specifics of horn inclusions, or even simplicial sets. Indeed, they apply to a general collection of morphisms $\mathcal{J}$ as long as it 'admits the small object argument', meaning that a statement like Lemma 5.19 applies.

In the remainder of this section we give some examples of anodyne morphisms of various kinds. For the first example, let us write $(0 \leftrightarrow 1)$ for the groupoid of two objects and one isomorphism between them, and

$$J = N(0 \leftrightarrow 1)$$

for its nerve. The arrow $0 \to 1$ in this groupoid defines a morphism of simplicial sets $\Delta[1] \to J$.

Proposition 5.22 *The morphism $\Delta[1] \to J$ is both left and right anodyne.*

Proof A non-degenerate n-simplex of J i s a sequence of non-identity morphisms like

$$0 \to 1 \to 0 \to \cdots \to 0 \to 1.$$

There are four types of such, depending on whether the sequence starts or ends with 0 or with 1. Any such simplex is obviously a face of one that starts with 0. So if we let $B^{(n)} \subseteq J$ be the simplicial subset generated by the non-degenerate k-simplices starting with 0, for $k \leq n$, then J is the colimit of the sequence of inclusions

$$\Delta[1] = B^{(1)} \subseteq B^{(2)} \subseteq B^{(3)} \subseteq \cdots .$$

Furthermore, $B^{(n-1)} \subseteq B^{(n)}$ fits into a pushout of the form

$$\begin{tikzcd}
\Lambda^0[n] \arrow[r] \arrow[d] & B^{(n-1)} \arrow[d] \\
\Delta[n] \arrow[r] & B^{(n)}.
\end{tikzcd}$$

Indeed, for the non-degenerate n-simplex starting with 0 (i.e., $(0 \to 1 \to \cdots \to 0)$ for n even and $(0 \to 1 \to \cdots \to 1)$ for n odd), all its inner faces are degeneracies of shorter sequences starting with a 0, as is its last face. Moreover, its initial face $(1 \to 0 \to \cdots \to 0)$ or $(1 \to 0 \to \cdots \to 1)$ cannot belong to $B^{(n-1)}$. This proves that $\Delta[1] \to J$ is left anodyne.

One can of course prove in the same way that $\Delta[1] \to J$ is right anodyne, now adjoining sequences starting with 1. Alternatively, one uses that $(-)^{\mathrm{op}}$ turns left anodyne maps into right anodynes and considers the square

$$\begin{tikzcd} \Delta[1]^{\mathrm{op}} \arrow[r] \arrow[d, "\wr"] & J^{\mathrm{op}} \arrow[d, "\wr"] \\ \Delta[1] \arrow[r] & J. \end{tikzcd}$$

□

For our next example, as well as for other instances of anodyne maps considered later, the following lemma will be very useful.

Lemma 5.23 *Let $E \subset \{0, \ldots, n\}$ be a non-empty subset and let $\Lambda^E[n] \subseteq \Delta[n]$ be the union of all the faces $\partial_i \Delta[n]$ for i* not *in E; or, to put it more positively, the union of those faces containing all the vertices in E. Then*

(i) $\Lambda^E[n] \to \Delta[n]$ *is an anodyne morphism,*
(ii) $\Lambda^E[n] \to \Delta[n]$ *is inner anodyne if $E \subseteq \{1, \ldots, n-1\}$,*
(iii) $\Lambda^E[n] \to \Delta[n]$ *is left anodyne if $E \subseteq \{0, \ldots, n-1\}$,*
(iv) $\Lambda^E[n] \to \Delta[n]$ *is right anodyne if $E \subseteq \{1, \ldots, n\}$.*

Proof Let us prove case (ii). Note that $n \geq 2$ in that case. If $n = 2$ and $E = \{1\}$ the lemma is tautologically true, since the inclusion in question is $\Lambda^1[2] \to \Delta[2]$. We proceed by induction on n as well as on the number of elements in E. For larger E, write $E = E_0 \cup \{e\}$, so that we have a pushout

$$\begin{tikzcd} \partial_e \Delta[n] \cap \Lambda^E[n] \arrow[r] \arrow[d] & \partial_e \Delta[n] \arrow[d] \\ \Lambda^E[n] \arrow[r] & \Lambda^{E_0}[n]. \end{tikzcd}$$

Now notice that $\partial_e \Delta[n] \cap \Lambda^E[n] \to \partial_e \Delta[n]$ is the inclusion of the union of all the images of the maps $\alpha : \Delta[k] \to \Delta[n]$ where α misses e as well as some element not in E. In other words, there are isomorphisms

$$\begin{tikzcd} \partial_e \Delta[n] \cap \Lambda^E[n] \arrow[r, "\cong"] \arrow[d] & \Lambda^{E_0}[n-1] \arrow[d] \\ \partial_e \Delta[n] \arrow[r, "\cong"] & \Delta[n-1], \end{tikzcd}$$

where we identify E_0 with its preimage under $\partial_e : [n-1] \to [n]$. Thus, the top map in the previous square is inner anodyne by the induction hypothesis, and therefore so is the bottom map. Since inner anodyne maps are closed under composition, we conclude that $\Lambda^E[n] \to \Lambda^{E_0}[n] \to \Delta[n]$ is inner anodyne.

The proofs of the other cases are similar. For instance, in case (iii) one observes that in the argument above, if $e \neq n$ then $\partial_e(n-1) = n$. And for case (iv) one notes that if $e \neq 0$ then $\partial_e(0) = 0$. □

We will need the following to recognize the kinds of inclusions occurring in the previous lemma:

Lemma 5.24 *Consider a simplicial subset $A \subseteq \Delta[n]$. Suppose A is not equal to $\Delta[n]$ or $\partial\Delta[n]$ and satisfies the following condition: if $F_1, \ldots, F_k$ are codimension 1 faces of $\Delta[n]$ not contained in A, then also*

$$F_1 \cap \cdots \cap F_k \nsubseteq A.$$

Then A is equal to $\Lambda^E[n]$ for some nonempty subset $E \subseteq \{0, \ldots, n\}$.

Proof Consider a non-degenerate m-simplex $\xi : \Delta[m] \to A$. It suffices to show that there exists a face $\partial_i\Delta[n]$ so that $\partial_i\Delta[n] \subseteq A$ and ξ is contained in $\partial_i\Delta[n]$. Indeed, it would then follow that A is a union of codimension one faces of $\Delta[n]$ and since we assumed A is not $\partial\Delta[n]$, it must be of the form $\Lambda^E[n]$. So let us write $i_0, \ldots, i_{n-m-1}$ for the vertices of $\Delta[n]$ which are not in the image of the inclusion

$$\Delta[m] \xrightarrow{\xi} A \to \Delta[n].$$

Then the image of ξ is the intersection

$$\partial_{i_0}\Delta[n] \cap \cdots \cap \partial_{i_{n-m-1}}\Delta[n].$$

One of these faces has to be contained in A, because otherwise our hypothesis would be violated. □

For our next and last example in this section, consider a small category $\mathbf{C}$ and two objects a and b in $\mathbf{C}$. Suppose we wish to freely adjoin a new arrow $f : a \to b$ to get a new category $\mathbf{C}[f]$. This category is the pushout

$$\begin{tikzcd} \{0,1\} \arrow[r] \arrow[d] & \mathbf{C} \arrow[d] \\ (0 \to 1) \arrow[r] & \mathbf{C}[f] \end{tikzcd}$$

in the category **Cat** of small categories. It has the same objects as $\mathbf{C}$, but potentially many more arrows: the arrows in $\mathbf{C}[f]$ are all possible compositions of old arrows of $\mathbf{C}$ and the new arrow f (including compositions of f with itself if $a = b$). On the other hand, the analogous pushout in simplicial sets

$$\begin{array}{ccc} \partial\Delta[1] & \longrightarrow & N\mathbf{C} \\ \downarrow & & \downarrow \\ \Delta[1] & \longrightarrow & (N\mathbf{C})[f] \end{array}$$

is simple to understand: besides the simplices in $N\mathbf{C}$ it contains just one new non-degenerate 1-simplex, namely f. Of course this difference is to be expected, because there is no reason why the right adjoint nerve functor should preserve pushouts. However, it does preserve this pushout (and similar ones) 'up to inner anodyne morphisms':

Proposition 5.25 *The canonical morphism* $(N\mathbf{C})[f] \to N(\mathbf{C}[f])$ *is inner anodyne.*

Proof Morphisms in $\mathbf{C}[f]$ are composites of the form

$$g_1 f g_2 f g_3 f \cdots f g_n,$$

with all the g_i morphisms in $\mathbf{C}$. Let us call a morphism *elementary* if it is a morphism in $\mathbf{C}$, or just f. Let us call an n-simplex

$$c_0 \to c_1 \to \cdots \to c_n$$

in $N(\mathbf{C}[f])$ *elementary* if all its constituent arrows $c_i \to c_{i+1}$ are elementary. Then any simplex in $N(\mathbf{C}[f])$ is a face of an elementary simplex, because the morphisms of $\mathbf{C}[f]$ are generated by elementary morphisms. Each non-degenerate elementary simplex has a certain number of occurrences of f which we call the *height* of the simplex. Let us write

$$A^{(k)} \subseteq N(\mathbf{C}[f])$$

for the simplicial subset generated by $N(\mathbf{C})[f]$ together with the non-degenerate elementary simplices of height at most k. Thus $A^{(0)} = (N\mathbf{C})[f]$ and $N(\mathbf{C}[f]) = \cup_k A^{(k)}$, so it suffices to prove that each

$$A^{(k)} \to A^{(k+1)}$$

is inner anodyne. Write

$$A^{(k)} = A_0^{(k+1)} \subseteq A_1^{(k+1)} \subseteq \cdots$$

where $A_n^{(k+1)} \subseteq N(\mathbf{C}[f])$ is generated by $A^{(k)}$ together with all non-degenerate elementary simplices of dimension n and height $k+1$. Thus

$$\bigcup_n A_n^{(k+1)} = A^{(k+1)} \qquad \text{and} \qquad A_n^{(k+1)} = A^{(k)} \quad \text{for} \quad n \leq k.$$

We claim that for each $n > k$ the map $A_{n-1}^{(k+1)} \to A_n^{(k+1)}$ is inner anodyne. To see this, list the non-degenerate elementary n-simplices ξ of height exactly $k+1$. For a given one

$$\xi = (c_0 \xrightarrow{g_1} c_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} c_n)$$

with $k + 1$ of the g_i equal to f, its outer faces $d_0\xi$ and $d_n\xi$ are still elementary, so belong to $A^{(k+1)}_{n-1}$. Also, inner faces $d_i\xi$ for which neither g_i nor g_{i+1} is f are elementary, so belong to $A^{(k+1)}_{n-1}$ as well. However, an inner face $d_i\xi$ for which one or both of g_i and g_{i+1} equal f is not elementary, and it cannot be a face of a non-degenerate elementary n-simplex of height $k + 1$ other than ξ itself, nor a face of a simplex in $A^{(k)}$ of course, because the $k + 1$ occurrences of f are still there. The same reasoning applies to intersections of these particular kinds of inner faces $d_i\xi$, showing that they are not contained in $A^{(k+1)}_{n-1}$. Thus Lemma 5.24 applies to show that the following square is a pullback as well as a pushout,

$$\begin{tikzcd}
\coprod_\xi \Lambda^{E_\xi}[n] \arrow[r] \arrow[d] & A^{(k+1)}_{n-1} \arrow[d] \\
\coprod_\xi \Delta[n] \arrow[r] & A^{(k+1)}_{n},
\end{tikzcd}$$

in which the disjoint union ranges over all the non-degenerate elementary n-simplices ξ of height $k+1$. Here we define for each such ξ the set $E_\xi \subseteq \{1, \ldots, n-1\}$ to be the set of those i for which at least one of g_i, g_{i+1} is f. This shows that $A^{(k+1)}_{n-1} \to A^{(k+1)}_{n}$ is inner anodyne, and hence so is the infinite composition

$$A^{(k+1)}_0 = A^{(k)} \to A^{(k+1)} = \bigcup_n A^{(k+1)}_n.$$

This completes the proof. $\square$

5.4 Products, Joins, and Spines of Simplices

In this section we will investigate the behaviour of anodyne morphisms (possibly inner, left, or right) with respect to the formation of products and joins of simplices, and analyze the spine of a simplex from the point of view of inner anodyne morphisms. The latter analysis will also help to complete the proof of Proposition 5.3.

We begin with products. Recall from Section 2.5 that a product $\Delta[n] \times \Delta[m]$ of two simplices is itself a union of $(n + m)$-simplices

$$\Delta[n] \times \Delta[m] = \bigcup_\tau \Delta[n + m]$$

where τ ranges over the (n, m)-shuffles. These shuffles are injective maps of linear orders $\tau : [n + m] \to [n] \times [m]$, which one can think of as 'staircases' on an n-by-m rectangular grid. By considering the horizontal and vertical steps, each shuffle corresponds to a pair of strictly increasing injective maps

$$\{1, \ldots, n\} \xrightarrow{\alpha} \{1, \ldots, n+m\} \xleftarrow{\beta} \{1, \ldots, m\}$$

with complementary images. This second perspective connects well with our discussion of shuffles of trees in Section 4.1. Indeed, as explained there, one can think of $[n]$ (resp. $[m]$) as a linear tree with n white vertices (resp. m black vertices), and a shuffle corresponds to a linear tree with $n+m$ vertices of which n are coloured white and m are coloured black. One can equip the set of shuffles with a linear order given by the lexicographic order on the image of α. The minimal object in this linear order is the shuffle starting with all white vertices and ending with all black vertices, and vice versa for the maximal object. If a shuffle σ is obtained from another shuffle τ by shuffling up a black vertex through a white vertex, then $\tau < \sigma$ in this linear order. Throughout this section we will mostly use this last perspective (trees with black and white vertices) in our proofs. It facilitates the discussion here and hopefully makes the step to dendroidal sets in the next chapter transparent.

Combinatorial lemmas like the following (and their analogues for dendroidal sets) will play a major role in this book.

Lemma 5.26 *For $0 < k < n$ and any m, the inclusion*

$$(\Lambda^k[n] \times \Delta[m]) \cup (\Delta[n] \times \partial\Delta[m]) \to \Delta[n] \times \Delta[m]$$

is inner anodyne. Moreover, it is left anodyne for $k = 0$ and right anodyne for $k = n$.

Proof Let us write A for the domain of the inclusion in the statement of the lemma. We will build $\Delta[n] \times \Delta[m]$ out of A by adjoining the shuffles of the product in the linear order described before the lemma. Consider a shuffle τ and let $B \subseteq \Delta[n] \times \Delta[m]$ be the union of A and all the shuffles – or more precisely, the corresponding copies of $\Delta[n+m] \subseteq \Delta[n] \times \Delta[m]$ – occurring before τ. As explained, we think of τ as a linear tree with edges $\{0, \ldots, n+m\}$ and vertices coloured black or white (white corresponding to the first factor $\Delta[n]$, black to the second $\Delta[m]$). Consider a face $\partial_i\tau$, for $0 \leq i \leq n+m$. If $i = 0$, then this face corresponds to chopping off the first vertex of the linear tree corresponding to τ. If this vertex is white then $\partial_0\tau$ factors through $\partial_0\Delta[n] \times \Delta[m]$, whereas it factors through $\Delta[n] \times \partial_0\Delta[m]$ if this vertex is black. Similarly, $\partial_{n+m}\tau$ factors through $\partial_n\Delta[n] \times \Delta[m]$ or $\Delta[n] \times \partial_m\Delta[m]$. If $\partial_i\tau$ is an inner face, there are various cases to consider:

(i) If i corresponds to an inner edge connecting two white vertices, then $\partial_i\tau$ is contained in $\partial_j\Delta[n] \times \Delta[m]$ for some $0 < j < n$.
(ii) If i corresponds to an inner edge connecting two black vertices, then $\partial_i\tau$ is contained in $\Delta[n] \times \partial_j\Delta[m]$ for some $0 < j < m$.
(iii) If i corresponds to an inner edge connecting a black vertex to a white vertex below it, then $\partial_i\tau$ is also a face of the shuffle τ' obtained from τ by swapping these two vertices (i.e., shuffling up the white vertex). But $\tau' < \tau$ in the linear order on shuffles, so $\partial_i\tau$ is already contained in B.
(iv) If i corresponds to an inner edge connecting a white vertex to a black vertex below it, then $\partial_i\tau$ is not contained in B. Indeed, it cannot be contained in A and also cannot be contained in an earlier shuffle.

This analysis shows that the faces of τ not yet contained in B are the ones described in (i) in the case where $j = k$, as well as the faces described in (iv). Write E for the collection of these missing faces. Note that all of these are inner faces of τ, at least in the first case $0 < k < n$ of the lemma. A similar analysis applies to show that each intersection of elements of E is also missing, so that we may apply Lemmas 5.23 and 5.24 to conclude the existence of a pushout square

$$\begin{tikzcd}
\Lambda^E[n+m] \arrow[r] \arrow[d] & B \arrow[d] \\
\Delta[n+m] \arrow[r] & B \cup \tau
\end{tikzcd}$$

in which the vertical map on the left is inner anodyne. Therefore the map on the right is inner anodyne as well. Once we have dealt with all the shuffles τ in this way, we have written

$$A \to \bigcup_{\tau} \Delta[n+m] = \Delta[n] \times \Delta[m]$$

as a composition of inner anodyne maps, which proves the first part of the lemma.

Applying the same analysis in the case $k = 0$, we see that the collection E of missing faces again consists of inner faces and possibly the face $\partial_0 \tau$ (only in case the shuffle τ starts with a white vertex). In that case one concludes that $B \to B \cup \tau$ is left anodyne. When $k = n$ one concludes it is right anodyne. □

Remark 5.27 For later use, we note that the preceding proof gives slightly more. In proving that the map of the lemma is left anodyne for $k = 0$, we used certain pushouts along left horn inclusions $\Lambda^0[n+m] \to \Delta[n+m]$. It will be important that the 'initial edge' $\Delta[0, 1]$ of this simplex is mapped to the edge $\Delta[0, 1] \times \{0\}$ of the product $\Delta[n] \times \Delta[m]$. In particular, it is degenerate in the second factor, but not in the first. A similar (but dual) comment applies to the case $k = n$.

Corollary 5.28 *For an inner anodyne map $i : A \to B$ and a monomorphism $j : M \to N$, the pushout-product*

$$A \times N \cup_{A \times M} B \times M \to B \times N$$

is again inner anodyne. The corresponding statements for left and right anodynes hold as well.

Proof The previous lemma settles the basic case of pushout-products of inner, left, or right horn inclusions with boundary inclusions, whereas Lemma 4.23 guarantees that the class of pairs (i, j) for which this pushout-product property holds is saturated in both variables. □

Next, we turn to the discussion of joins of simplices and prove a similar (but easier) lemma for these. The *join* of two simplices $\Delta[n]$ and $\Delta[m]$ is denoted $\Delta[n] \star \Delta[m]$ and defined by

$$\Delta[n] \star \Delta[m] = \Delta[n+m+1],$$

where $[n+m+1]$ is to be thought of as the two linear orders $[n]$ and $[m]$ put next to one another as

$$0 < 1 < \cdots < n < 0' < 1' < \cdots < m'.$$

In this way, $\Delta[n] \star \Delta[m]$ comes equipped with obvious inclusions

$$\Delta[n] \longrightarrow \Delta[n+m+1] \longleftarrow \Delta[m],$$

which are moreover natural in n and m. The join operation can be extended to arbitrary simplicial sets A and B to give a diagram

$$A \longrightarrow A \star B \longleftarrow B$$

natural in A and B. This operation is essentially uniquely determined by the property that it behaves as described above on simplices and that $A \star (-)$ and $(-) \star B$ preserve colimits, when viewed as functors from the category **sSets** to the slice categories $A/$**sSets** and $B/$**sSets** respectively. For readers wishing to see a more explicit construction of the join of two simplicial sets, we will return to it in Remark 5.31. For now, let us observe that

$$\partial_i \Delta[n] \star \Delta[m] = \partial_i \Delta[n+m+1], \qquad 0 \le i \le n$$

as simplicial subsets of $\Delta[n+m+1]$ and similarly

$$\Delta[n] \star \partial_j \Delta[m] = \partial_{n+1+j} \Delta[n+m+1], \qquad 0 \le j \le m.$$

Thus

$$\partial \Delta[n] \star \Delta[m] = \bigcup_{i \le n} \partial_i \Delta[n+m+1]$$

and

$$\Delta[n] \star \partial \Delta[m] = \bigcup_{n < j} \partial_j \Delta[n+m+1],$$

so that

$$\partial \Delta[n] \star \Delta[m] \cup \Delta[n] \star \partial \Delta[m] = \partial \Delta[n+m+1].$$

If we take out a face on one side of the join, we observe that for $0 \le k \le n$ and $0 \le l \le m$

$$\Lambda^k[n] \star \Delta[m] \cup \Delta[n] \star \partial \Delta[m] = \Lambda^k[n+m+1],$$
$$\Delta[n] \star \Lambda^l[m] \cup \partial \Delta[n] \star \Delta[m] = \Lambda^{n+1+l}[n+m+1].$$

In particular, we conclude the following:

Lemma 5.29 *For any $n, m \geq 0$ and $0 \leq k \leq n$, the inclusion*

$$\Lambda^k[n] \star \Delta[m] \cup \Delta[n] \star \partial\Delta[m] \longrightarrow \Delta[n] \star \Delta[m]$$

is always left anodyne and inner anodyne for $0 < k$. Similarly, for $0 \leq l \leq m$ the inclusion

$$\Delta[n] \star \Lambda^l[m] \cup \partial\Delta[n] \star \Delta[m] \longrightarrow \Delta[n] \star \Delta[m]$$

is always right anodyne and inner anodyne for $l < m$.

As with Corollary 5.28 one immediately concludes the following:

Corollary 5.30 *For an anodyne map $i \colon A \longrightarrow B$ and a monomorphism $j \colon M \longrightarrow N$, the map*

$$A \star N \cup_{A \star M} B \star M \longrightarrow B \star N$$

is left anodyne, and inner anodyne whenever i is right anodyne. Similarly,

$$M \star B \cup_{M \star A} N \star A \longrightarrow N \star B$$

is always right anodyne, and inner anodyne whenever i is left anodyne.

Remark 5.31 For two simplicial sets A and B it is not difficult to describe $A \star B$ explicitly. Indeed, one has the formula

$$(A \star B)_k = \coprod_{p+q=k-1,\ p,q \geq -1} A_p \times B_q,$$

where A_{-1} and B_{-1} are interpreted as one-point sets. In other words,

$$(A \star B)_k = A_k \amalg B_k \amalg \coprod_{p+q=k-1,\ p,q \geq 0} A_p \times B_q.$$

The simplicial structure of $A \star B$ is defined as follows. An element $\xi = (\xi_1, \xi_2) \in (A \star B)_k$ in particular determines p and q with $p + q = k - 1$, giving an identification

$$\Delta^p \star \Delta^q = \Delta^k.$$

A map $\alpha : [l] \longrightarrow [k]$ then defines (by considering the preimages of Δ^p and Δ^q) a compatible decomposition $\Delta^{p'} \star \Delta^{q'} = \Delta^l$ and a commutative diagram

$$\begin{tikzcd}
\Delta^l \arrow[r, "\alpha"] \arrow[d, equal] & \Delta^k \arrow[d, equal] \\
\Delta^{p'} \star \Delta^{q'} \arrow[r, "\alpha_1 \star \alpha_2"] & \Delta^p \star \Delta^q.
\end{tikzcd}$$

The simplex $\alpha^*\xi$ is then the element $(\alpha_1^*\xi_1, \alpha_2^*\xi_2)$.

As a third and last topic in this section, we will now discuss spines of simplices. In order to define these, let us introduce the following notation. For the representable simplicial set $\Delta[n]$ and numbers $0 \leq i_0 < \cdots < i_p \leq n$, we write

$$\Delta[i_0, \ldots, i_p] \subseteq \Delta[n]$$

for the p-dimensional face spanned by the vertices $i_0, \ldots, i_p$. In other words, it is the image of the corresponding map $\Delta[p] \to \Delta[n]$. The *spine* of $\Delta[n]$ is the union of the successive edges

$$\mathrm{Sp}[n] = \Delta[0,1] \cup \Delta[1,2] \cup \cdots \cup \Delta[n-1,n].$$

In other words, it is the pushout of n copies of $\Delta[1]$:

$$\mathrm{Sp}[n] \cong \Delta[1] \cup_{\Delta[0]} \cdots \cup_{\Delta[0]} \Delta[1].$$

We also define for each $0 < k < n$ the *grafting* of $\Delta[k]$ and $\Delta[n-k]$ as

$$\mathrm{Gr}^k[n] = \Delta[0, \ldots, k] \cup \Delta[k, \ldots, n] \subseteq \Delta[n].$$

In other words, it is the pushout

$$\mathrm{Gr}^k[n] = \Delta[k] \cup_{\Delta[0]} \Delta[n-k],$$

where $\Delta[0]$ includes as the final vertex in $\Delta[k]$ and the initial vertex in $\Delta[n-k]$. Note that for $n = 2$ we have

$$\Lambda^1[2] = \mathrm{Sp}[2] = \mathrm{Gr}^1[2].$$

For $n = 3$, here is a picture of the spine of a 3-simplex:

$$\begin{tikzcd} & 0 \arrow[d] & & & & 0 \arrow[d] \arrow[ddl] \arrow[ddr] & \\ & 1 \arrow[dl] & & \longrightarrow & & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 & & 2 \arrow[rr] & & 3. \end{tikzcd}$$

Lemma 5.32 *For any n and any $0 < k < n$, the inclusions*

$$\mathrm{Sp}[n] \to \mathrm{Gr}^k[n] \to \Delta[n]$$

are inner anodyne.

Proof For $n = 2$ this is clear as pointed out above, since the first two simplicial sets are equal to $\Lambda^1[2]$. We proceed by induction on n. Notice that there is a pushout square

$$\begin{tikzcd} \mathrm{Sp}[k] \amalg \mathrm{Sp}[n-k] \arrow[r] \arrow[d] & \mathrm{Sp}[n] \arrow[d] \\ \Delta[k] \amalg \Delta[n-k] \arrow[r] & \mathrm{Gr}^k[n]. \end{tikzcd}$$

The vertical map on the left is inner anodyne by the inductive hypothesis, so the map on the right is inner anodyne as well. It remains to prove that each $\mathrm{Gr}^k[n] \to \Delta[n]$ is inner anodyne. To this end form a sequence

$$\mathrm{Gr}^k[n] = A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = \Delta[n].$$

Here the simplicial set A_p is the union of all p-dimensional faces of $\Delta[n]$ which contain k as a vertex, together with $\mathrm{Gr}^k[n]$:

$$A_p = \mathrm{Gr}^k[n] \cup \bigcup \Delta[i_0, \ldots, i_{q-1}, k, i_{q+1}, \ldots, i_p].$$

The union is over all $0 \leq q \leq p$ and all sequences $0 \leq i_0 < \ldots < i_p \leq n$ with $i_q = k$. (Of course the simplices with $i_0 = k$ or $i_p = k$ are redundant, since they are already contained in $\mathrm{Gr}^k[n]$.) We claim that $A_{p-1} \to A_p$ is inner anodyne. Indeed, if we adjoin to A_p the simplices $\Delta[i_0, \ldots, i_{q-1}, k, i_{q+1}, \ldots, i_p]$ in the union above one by one, each time we form a pushout along a map (isomorphic to)

$$\Lambda^q[p] \to \Delta[p],$$

which is an inner horn inclusion. □

Remark 5.33 For a given simplicial set X, consider the class of monomorphisms $A \to B$ with the property that any map $A \to X$ extends *uniquely* to B, i.e., the class of monomorphisms $A \to B$ for which

$$\mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$$

is an isomorphism. This class is obviously saturated. So if it contains the inner horn inclusions $\Lambda^k[n] \to \Delta[n]$, $0 < k < n$, then it also contains the spine inclusions $\mathrm{Sp}[n] \to \Delta[n]$. This observation proves the implication (i) ⇒ (ii) of Proposition 5.3, which we left open at the time.

It is not true that the saturated class generated by the spine inclusions is the class of inner anodyne morphisms. However, this can be fixed by enlarging this class slightly:

Proposition 5.34 *Let $\mathcal{A}$ be a saturated class of monomorphisms between simplicial sets which contains all spine inclusions and satisfies the following additional closure property: if $i : A \to B$ and $j : B \to C$ are monomorphisms such that i and ji are in $\mathcal{A}$, then j is in $\mathcal{A}$ as well. Then the class $\mathcal{A}$ contains all inner anodynes.*

Proof Since $\mathcal{A}$ is saturated it suffices to show it contains all inner horn inclusions $\Lambda^k[n] \to \Delta[n]$, for $0 < k < n$. It will be convenient to prove the slightly more general claim that each inclusion $\Lambda^E[n] \to \Delta[n]$ is in $\mathcal{A}$, for E a nonempty subset of $\{1, \ldots, n-1\}$. Consider the inclusions

$$\mathrm{Sp}[n] \xrightarrow{i} \Lambda^E[n] \xrightarrow{j} \Delta[n].$$

Then ji is in $\mathcal{A}$ by assumption, so it suffices to show that i is in $\mathcal{A}$. Factor i as

$$\mathrm{Sp}[n] \xrightarrow{i_1} \Delta[n-1,n] \cup \partial_n\Delta[n] \xrightarrow{i_2} \partial_0\Delta[n] \cup \partial_n\Delta[n] \xrightarrow{i_3} \Lambda^E[n].$$

We will show that all three of these maps are in $\mathcal{A}$ by induction on n. For $n = 2$ there is nothing to prove, because they are all identities. For larger n, note that i_1 is a pushout of $\mathrm{Sp}[n-1] \to \Delta[n-1] \cong \partial_n\Delta[n]$ and therefore in $\mathcal{A}$. Also, i_2 is a pushout of

$$\Delta[n-1,n] \cup \Delta[1,\ldots,n-1] \to \Delta[1,\ldots,n] = \partial_0\Delta[n].$$

This is a map of the form i_2 again, but for lower n, and therefore in $\mathcal{A}$ by the inductive hypothesis. Finally, we prove i_3 is in $\mathcal{A}$ by a further induction on the size of the complement of E. If E is $\{1,\ldots,n-1\}$ then i_3 is an identity and there is nothing to prove. Otherwise, pick an element $i \in \{1,\ldots,n-1\}$ which is *not* contained in E and write $E' = E \cup \{i\}$. The map

$$\partial_0\Delta[n] \cup \partial_n\Delta[n] \to \Lambda^{E'}[n]$$

is in $\mathcal{A}$ by the inductive hypothesis. Consider the pushout square

$$\begin{tikzcd}
\Lambda^E\partial_i\Delta[n] \arrow[r] \arrow[d] & \Lambda^{E'}[n] \arrow[d] \\
\partial_i\Delta[n] \arrow[r] & \Lambda^E[n].
\end{tikzcd}$$

Here the upper left-hand corner denotes the union of all faces of the $(n-1)$-simplex $\partial_i\Delta[n]$ corresponding to elements of $\{0,\ldots,\widehat{i},\ldots,n\}$ *not* contained in E. Thus, the left vertical map is (isomorphic to) an inner horn inclusion, so that the right vertical map is in $\mathcal{A}$. It follows that

$$\partial_0\Delta[n] \cup \partial_n\Delta[n] \xrightarrow{i_3} \Lambda^E[n]$$

is in $\mathcal{A}$ as well. □

5.5 Fibrations Between Mapping Spaces

The classes of fibrations of various kinds (Kan, inner, etc.) enjoy closure properties dual to those of a saturated class of morphisms, cf. Lemma 5.9. For a different kind of closure property, dual to the pushout-product property of (inner, left, right) anodynes, one considers a fibration $f : Y \to X$ and a map $u : A \to B$. Then composition with f and restriction along u together induce a morphism

$$Y^B \to X^B \times_{X^A} Y^A$$

and one can ask whether this is a fibration again. This will involve checking the right lifting property with respect to certain morphisms $C \to D$. In this context we observe the following correspondence.

Lemma 5.35 *Consider morphisms $A \to B$, $C \to D$ and $Y \to X$. Then there is a bijective correspondence between the following three types of commutative squares and a diagonal filling (as dashed arrow) exists in one if and only if it exists in the other two:*

$$\begin{tikzcd} (B \times C) \cup_{A \times C} (A \times D) \arrow[r] \arrow[d] & Y \arrow[d] \\ B \times D \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r] \arrow[d] & Y^B \arrow[d] \\ D \arrow[r] \arrow[ur, dashed] & X^B \times_{X^A} Y^A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & Y^D \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & X^D \times_{X^C} Y^C. \end{tikzcd}$$

Proof This follows easily from exponential correspondences such as the one between morphisms $D \to Y^B$ and morphisms $B \times D \to Y$, together with the universal properties of the pullbacks and pushouts involved. □

Of course the argument just given applies equally to a hom-tensor adjunction, with tensor product not necessarily equal to cartesian product. We can use this lemma to construct many fibrations from a given one. We summarize this in the following:

Theorem 5.36 *Consider two morphisms $f : Y \to X$ and $i : A \to B$ and the induced map*

$$p : Y^B \to X^B \times_{X^A} Y^A.$$

Suppose $Y \to X$ is a fibration of one of the five types considered (Kan, inner, left, right, or trivial).

(i) *If i is a monomorphism, then p is again a fibration of the same type.*
(ii) *If i is an anodyne morphism of the type corresponding to the fibration f, then p is a trivial fibration. (Thus, for example, this applies if i is inner anodyne and f is an inner fibration. The case of f a trivial fibration is already covered by (i).)*

Proof The statements are all proved in the same formal way; we just prove (i) and (ii) for an inner fibration $f : Y \to X$. For (i), we should check that p has the right lifting property with respect to inner anodyne morphisms $j : C \to D$. By Lemma 5.35, this is the case precisely if f has the right lifting property with respect to the corresponding morphisms

$$B \times C \cup_{A \times C} A \times D \to B \times D.$$

But these are also inner anodyne by Corollary 5.21, so that the conclusion follows from the fact that f itself is an inner fibration. For (ii) one uses the same argument, now checking the right lifting property of p with respect to arbitrary monomorphisms j. The pushout-product of j and i is again inner anodyne, this time because j is a monomorphism and i is inner anodyne. □

We list several special cases of Theorem 5.36.

Corollary 5.37 *Suppose $f : Y \to X$ is a fibration of one of the five types considered and B is a simplicial set. Then*

$$p : Y^B \to X^B$$

is a fibration of the same type as f.

Proof This is the case $A = \varnothing$. □

Corollary 5.38 *Suppose Y is an ∞-category (resp. a Kan complex) and B a simplicial set. Then Y^B is also an ∞-category (resp. a Kan complex).*

Corollary 5.39 *Suppose Y is an ∞-category (resp. a Kan complex) and $i : A \to B$ a map of simplicial sets.*

(i) *If i is a monomorphism, then $Y^B \to Y^A$ is an inner fibration (resp. a Kan fibration).*
(ii) *If i is an inner anodyne (resp. an anodyne), then $Y^B \to Y^A$ is a trivial fibration.*

Proof This is the case $X = \Delta[0]$. □

We will now run through roughly the same pattern, replacing the product $A \times B$ of simplicial sets by the join $A \star B$. While the functor

$$A \times - : \mathbf{sSets} \to \mathbf{sSets}$$

has a right adjoint

$$(-)^A : \mathbf{sSets} \to \mathbf{sSets},$$

the situation for the join is a bit more subtle. Indeed, $A \star (-)$ only preserves colimits as a functor

$$A \star (-) : \mathbf{sSets} \to A/\mathbf{sSets},$$

so its right adjoint is a functor

$$A/\mathbf{sSets} \to \mathbf{sSets},$$

whose value at an object $A \xrightarrow{\alpha} X$ will be denoted $X_{\alpha/}$ or simply $X_{A/}$ when the map α is left implicit. We call $X_{A/}$ the *slice of X under A* (or under α if necessary). By definition, the n-simplices of $X_{A/}$ correspond to maps $A \star \Delta[n] \to X$ under A. We can do the same for the other variable of the join, to obtain a right adjoint to the functor

$$(-) \star A : \mathbf{sSets} \to A/\mathbf{sSets},$$

whose value at $A \xrightarrow{\alpha} X$ is $X_{/A}$, the slice of X *over* A. Note that there are natural projection maps

$$X_{A/} \to X \leftarrow X_{/A}.$$

Indeed, since an n-simplex of $X_{A/}$ corresponds to a map $A \star \Delta[n] \to X$, it can be restricted along the inclusion $\Delta[n] \to A \star \Delta[n]$ to give an n-simplex of X. The map $X_{/A} \to X$ is defined similarly. In fact, the construction $X_{A/}$ (and also $X_{/A}$) is contravariantly functorial in A. For a map $i : A \to B$, one uses precomposition with $A \star \Delta[n] \to B \star \Delta[n]$ to obtain an n-simplex of $X_{A/}$ from an n-simplex of $X_{B/}$ and similarly for $X_{/A}$ and $X_{/B}$.

Remark 5.40 The terminology *slice* refers to the construction of slice categories. Indeed, if X is the nerve of a category $\mathbf{C}$ and c is a vertex of X (i.e., an object of $\mathbf{C}$), then

$$X_{/c} = N(\mathbf{C}/c),$$

with $\mathbf{C}/c$ denoting the usual slice category of $\mathbf{C}$ over c. More generally, for a diagram $F : \mathbf{D} \to \mathbf{C}$ and $A = N\mathbf{D}$, the simplicial set $X_{/A}$ is precisely the nerve of the category of cones in $\mathbf{C}$ over F.

We are now ready to state the analogue of Theorem 5.36 for slices. We replace the pushout-product map

$$A \times D \cup_{A \times C} B \times C \to B \times D$$

by the map

$$A \star D \cup_{A \star C} B \star C \to B \star D$$

and the morphism

$$Y^B \to X^B \times_{X^A} Y^A$$

between mapping spaces by the morphism between slices

$$Y_{B/} \to X_{B/} \times_{X_{A/}} Y_{A/}.$$

Remember that our notation is a bit misleading here, because the slice $Y_{B/}$ depends on a morphism $B \to Y$ which is implicit in the notation. Given such a morphism $B \to Y$ and morphisms $A \to B$ and $Y \to X$, one obtains morphisms $B \to X$, $A \to X$ and $A \to Y$ by composition, so that the other slices in the formula above also make sense. This convention is implicit in the statement of the following theorem.

Theorem 5.41 *Let $f : Y \to X$ be a morphism of simplicial sets and let $i : A \to B$ be a monomorphism over Y. Consider the map*

$$p : Y_{B/} \to X_{B/} \times_{X_{A/}} Y_{A/}.$$

(i) *If f is an inner fibration, then p is a left fibration.*
(ii) *If f is a right fibration then p is a Kan fibration.*

(iii) *If f is an inner fibration and i is right anodyne, then p is a trivial fibration.*
(iv) *If f is a left fibration and i is anodyne, then p is a trivial fibration.*

Proof The proofs of these statements all proceed in the same way, using Corollary 5.30. For (i), we should solve lifting problems of the form

$$\begin{tikzcd} C \arrow[r] \arrow[d, "j"'] & Y_{B/} \arrow[d] \\ D \arrow[r] \arrow[ur, dashed] & X_{B/} \times_{X_{A/}} Y_{A/} \end{tikzcd}$$

where j is left anodyne. But these correspond to lifting problems

$$\begin{tikzcd} B \star C \cup_{A \star C} A \star D \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ B \star D \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

and the map on the left is inner anodyne by Corollary 5.30, so that a lift exists by the assumption that f is an inner fibration. For (ii) we let j be a general anodyne. Then the map on the left is right anodyne, so a lift exists if f is a right fibration. For (iii) and (iv) one uses the remaining cases of Corollary 5.30 in the evident way. □

Of course Theorem 5.41 has a dual version for the map

$$Y_{/B} \to X_{/B} \times_{X_{/A}} Y_{/A},$$

simply by swapping the terms 'left' and 'right' everywhere. For ease of reference in the next section as well as later in this book we list a few special cases. As just noted, all these statements will have evident duals, which we will not state explicitly.

Corollary 5.42 *Let $f : Y \to X$ be a morphism of simplicial sets and let B be a simplicial set over Y. If f is an inner fibration, a right fibration, a left fibration, or a Kan fibration, then $Y_{B/} \to X_{B/}$ is an inner fibration, a right fibration, a left fibration, or a Kan fibration respectively.*

Proof Note that $Y_{\varnothing/} = Y$ and consider the diagram

$$\begin{tikzcd} Y_{B/} \arrow[r] \arrow[dr] & X_{B/} \times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & X_{B/} \arrow[r] & X. \end{tikzcd}$$

The first horizontal map is a fibration of the relevant kind by items (i) and (ii) of Theorem 5.41. Both vertical maps are such fibrations as well: the one on the right by assumption and the other because it is a pullback. Since the classes of fibrations of the relevant types are closed under composition, the conclusion for the slanted map follows. □

Corollary 5.43 *Let Y be a simplicial set and let $i : A \to B$ be a monomorphism over Y.*

(i) *If Y is an ∞-category, then $Y_{B/} \to Y_{A/}$ is a left fibration. Moreover it is a trivial fibration if i is right anodyne.*
(ii) *If Y is a Kan complex, then $Y_{B/} \to Y_{A/}$ is a Kan fibration. Moreover it is a trivial fibration if i is anodyne.*

Proof This follows from Theorem 5.41 by taking $X = \Delta[0]$. □

Corollary 5.44 *Let $B \to Y$ be a morphism of simplicial sets. If Y is an ∞-category (or a Kan complex), then $Y_{B/}$ is also an ∞-category (or a Kan complex respectively).*

5.6 Equivalences in ∞-Categories

In the first section of this chapter we gave an explicit description of the homotopy category $\tau(X)$ of an ∞-category X and we noticed that if X is a Kan complex, then every arrow in $\tau(X)$ is an isomorphism. In this section we will investigate the role of isomorphisms in $\tau(X)$ more systematically. Amongst other things we will prove the converse statement to the above, namely that any ∞-category X for which $\tau(X)$ is a groupoid is a Kan complex (cf. Corollary 5.51 below). We begin with a few definitions.

Definition 5.45 Let X be an ∞-category. A 1-simplex $\alpha : \Delta[1] \to X$ is called an *equivalence* if the corresponding arrow $\tau(\alpha)$ in $\tau(X)$ is an isomorphism.

Definition 5.46 (i) A functor $f : \mathbf{D} \to \mathbf{C}$ between small categories is called an *isofibration* if for any isomorphism $\alpha : c \to c'$ in $\mathbf{C}$ and any $d \in \mathbf{D}$ with $f(d) = c$ there exists an isomorphism $\beta : d \to d'$ in $\mathbf{D}$ with $f(\beta) = \alpha$. In other words, if every commutative square of small categories

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathbf{D} \arrow[d] \\ (0 \leftrightarrow 1) \arrow[r] \arrow[ur, dashed] & \mathbf{C} \end{tikzcd}$$

admits a diagonal lift.

(ii) A morphism $f : Y \to X$ between ∞-categories is said to have *path lifting for equivalences* if for every commutative square of simplicial sets

$$\begin{tikzcd} \Delta[0] \arrow[r] \arrow[d, "0"'] & Y \arrow[d] \\ \Delta[1] \arrow[r, "\alpha"] \arrow[ur, dashed] & X \end{tikzcd}$$

in which α is an equivalence, a diagonal lift as indicated exists and is moreover an equivalence in Y.

(iii) A morphism $Y \to X$ between ∞-categories is said to have *J-path lifting* if every commutative square

$$\begin{tikzcd} \Delta[0] \arrow[r] \arrow[d, "0"'] & Y \arrow[d] \\ J \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

admits a diagonal lift. (Recall that J is the nerve of the groupoid $(0 \leftrightarrow 1)$ featuring in part (i).)

These three notions are closely related, as we will see.

Lemma 5.47 *Let $f : Y \to X$ be an inner fibration between ∞-categories. Then f has path lifting with respect to equivalences if and only if $\tau(f) : \tau(Y) \to \tau(X)$ is an isofibration.*

Proof The only if direction is clear. For the converse, consider a commutative square as in part (ii) of Definition 5.46. Then $\tau(\alpha)$ is an isomorphism in $\tau(X)$, which lifts to an isomorphism $\tau(\beta) : y_0 \to y_1$ in $\tau(Y)$ by assumption. Thus $f(\beta)$ is equivalent to α, as witnessed by a 2-simplex a in X_2 pictured as follows:

$$\begin{tikzcd} & 1 \arrow[dr, equal] & \\ 0 \arrow[ur, "f(\beta)"] \arrow[rr, "\alpha"] & & 2. \end{tikzcd}$$

Since f is an inner fibration, we can find a lift in

$$\begin{tikzcd} \Lambda^1[2] \arrow[r, "\widehat{\beta}"] \arrow[d] & Y \arrow[d] \\ \Delta[2] \arrow[r, "a"'] \arrow[ur, dashed, "b"] & X \end{tikzcd}$$

where $\widehat{\beta}$ is β on $\partial_2\Delta[2]$ and degenerate on $\partial_0\Delta[2]$. A diagonal b as indicated will produce a 1-simplex $\beta' = d_1 b$ defining the same arrow $\tau(\beta') = \tau(\beta)$ in $\tau(Y)$ and with $f(\beta') = \alpha$. $\square$

A functor $f : \mathbf{D} \to \mathbf{C}$ is *conservative* if a morphism α in $\mathbf{D}$ is an isomorphism whenever $f(\alpha)$ is an isomorphism in $\mathbf{C}$.

Lemma 5.48 *Let $f : Y \to X$ be a left or right fibration between ∞-categories. Then $\tau(f)$ is a conservative isofibration. In particular, if $Y \to \Delta[0]$ is a left or right fibration then $\tau(Y)$ is a groupoid.*

Proof Suppose f is a left fibration. We will first check that any 1-simplex β of Y for which $f(\beta)$ is an equivalence is itself an equivalence in Y, i.e., that $\tau(f)$ is conservative. Let $a \in X_2$ be a witness of the fact that $\tau(f(\beta))$ has a left inverse in $\tau(X)$, as in

$$\begin{tikzcd} & f(d_0\beta) \arrow[dr, "\gamma"] & \\ f(d_1\beta) \arrow[ur, "f(\beta)"] \arrow[rr, equal] & & f(d_1\beta). \end{tikzcd}$$

Now lift in a diagram of the form

$$\begin{tikzcd} \Lambda^0[2] \arrow[r] \arrow[d] & Y \arrow[d] \\ \Delta[2] \arrow[r, "a"] \arrow[ur, dashed] & X \end{tikzcd}$$

to find a 2-simplex b in Y of the form

$$\begin{tikzcd} & d_0\beta \arrow[dr] & \\ d_1\beta \arrow[ur, "\beta"] \arrow[rr, equal] & & d_1\beta. \end{tikzcd}$$

This shows that $\tau(\beta)$ itself has a left inverse $\tau(d_0b)$ in $\tau(Y)$. The same argument applied to this left inverse $\tau(d_0b)$ shows that $\tau(d_0b)$ itself also has a left inverse in $\tau(Y)$, so that d_0b is indeed an equivalence. But then β is an equivalence as well, which proves that $\tau(f)$ is conservative. To see that it is an isofibration, use the right lifting property of f with respect to the horn inclusion $\Lambda^0[1] \to \Delta[1]$ (which is simply the inclusion of the vertex 0) to see that for any 1-simplex α of X and vertex y_0 of Y with $f(y_0) = d_1\alpha$, there exists a 1-simplex β of Y with $f(\alpha) = \beta$. Thus, any morphism φ of $\tau(X)$ can be lifted to $\tau(Y)$ once we have specified a lift of the domain of φ. If φ is an isomorphism, then this lift is an isomorphism as well, since we already proved that $\tau(f)$ is conservative. □

Using these two lemmas and the results of the previous sections we can now easily deduce the following theorem, which states that inner fibrations between ∞-categories additionally have the right lifting property with respect to certain 'exceptional' horn inclusions. It is the main technical result of this section.

Theorem 5.49 *Let $f : Y \to X$ be an inner fibration between ∞-categories. Then a diagonal lift exists in any commutative square (with $n \geq 2$)*

$$\begin{tikzcd} \Lambda^n[n] \arrow[r, "\beta"] \arrow[d] & Y \arrow[d] \\ \Delta[n] \arrow[r, "\alpha"] \arrow[ur, dashed] & X \end{tikzcd}$$

with the property that the edge $\beta(\Delta[n-1,n])$ in Y is an equivalence. Moreover, a similar statement holds for $\Lambda^0[n] \to \Delta[n]$ under the condition that $\beta(\Delta[0,1])$ is an equivalence in Y.

Proof Note that the statement for $\Lambda^0[n] \to \Delta[n]$ follows from the first statement for $\Lambda^n[n] \to \Delta[n]$ by passing to opposite ∞-categories. To prove the first, write

$$\Delta[n] = \Delta[n-2] \star \Delta[1].$$

Then $\Lambda^n[n] \to \Delta[n]$ is

$$\partial\Delta[n-2] \star \Delta[1] \cup \Delta[n-2] \star \Lambda^1[1] \to \Delta[n-2] \star \Delta[1],$$

and by adjunction the lifting problem of the theorem translates into

$$\begin{tikzcd}
\Lambda^1[1] \arrow[r, "\widehat{\beta}"] \arrow[d] & Y_{\Delta[n-2]/} \arrow[d] \\
\Delta[1] \arrow[r, "\widehat{\alpha}"'] \arrow[ur, dashed] & Y_{\partial\Delta[n-2]/} \times_{X_{\partial\Delta[n-2]/}} X_{\Delta[n-2]/}.
\end{tikzcd}$$

The right-hand map is a left fibration between ∞-categories by Theorem 5.41 and Corollary 5.43. So by Lemmas 5.47 and 5.48, it suffices to prove that $\widehat{\alpha}$ is an equivalence. To this end, consider the projection π from the codomain of $\widehat{\alpha}$ to Y. This projection is the composition of the pullback of a left fibration with the left fibration π' as in the diagram below (see Corollary 5.43):

$$\begin{tikzcd}
Y_{\partial\Delta[n-2]/} \times_{X_{\partial\Delta[n-2]/}} X_{\Delta[n-2]/} \arrow[r] \arrow[d] & X_{\Delta[n-2]/} \arrow[d] \\
Y_{\partial\Delta[n-2]/} \arrow[r] \arrow[d, "\pi'"] & X_{\partial\Delta[n-2]/} \\
Y. &
\end{tikzcd}$$

Therefore π is itself a left fibration. The image of $\widehat{\alpha}$ under this projection is precisely $\beta(\Delta[n-1,n])$, which is assumed to be an equivalence. Since left fibrations are conservative in the sense of Lemma 5.48, this shows that $\widehat{\alpha}$ is itself an equivalence, which completes the proof of the theorem. □

Theorem 5.49 has a number of very useful consequences.

Corollary 5.50 *If $f: Y \to X$ is an inner fibration and X is a Kan complex, then the following are equivalent:*

(i) *f is a Kan fibration,*
(ii) *$\tau(f)$ is a conservative isofibration,*
(iii) *f is a left fibration,*

(iv) *f is a right fibration.*

Proof We prove (i) ⇒ (iii) ⇒ (ii) ⇒ (i). The equivalence to (iv) then follows since f is a Kan fibration if and only if $f^{\mathrm{op}} : Y^{\mathrm{op}} \to X^{\mathrm{op}}$ is. Now (i) ⇒ (iii) is true by definition, whereas (iii) ⇒ (ii) follows from Lemma 5.48. Now assume (ii). Since every edge of X is an equivalence and $\tau(f)$ is conservative, the same is true of the edges of Y. Theorem 5.49 now implies that f has the right lifting property with respect to all horn inclusions $\Lambda^k[n] \to \Delta[n]$ for $n \geq 2$. For the case $n = 1$ (which corresponds to 'path lifting') one simply uses Lemma 5.47. □

The following is the special case $X = \Delta[0]$:

Corollary 5.51 *Let Y be an ∞-category. Then the following are equivalent:*

(i) *Y is a Kan complex,*
(ii) *$\tau(Y)$ is a groupoid,*
(iii) *$Y \to \Delta[0]$ is a left fibration,*
(iv) *$Y \to \Delta[0]$ is a right fibration.*

Corollary 5.52 (i) *Let $Y \to X$ be an inner fibration between ∞-categories. Then a commutative square*

$$\begin{tikzcd} \Delta[1] \arrow[r, "b"] \arrow[d] & Y \arrow[d] \\ J \arrow[r, "a"] \arrow[ur, dashed] & X \end{tikzcd}$$

admits a diagonal filling if and only if b is an equivalence in Y.

(ii) *In particular, a morphism $b\colon \Delta[1] \to Y$ into an ∞-category extends to J if and only if b is an equivalence.*

Proof Part (ii) is the special case where $X = \Delta[0]$. For part (i), note that the condition is clearly necessary, because every 1-simplex in J is an equivalence. Conversely, recall that $\Delta[1] \to J$ is left anodyne (Proposition 5.22), and in fact a composition of pushouts of left horns $\Lambda^0[n] \to \Delta[n]$ for which the $(0, 1)$-edge of $\Lambda^0[n]$ is $\Delta[1] \subseteq J$, i.e., for which

$$\begin{tikzcd} \Delta[0,1] \arrow[r] \arrow[d] & J \\ \Lambda^0[n] \arrow[r] & \Delta[n] \arrow[u] \end{tikzcd}$$

commutes. One can now apply Theorem 5.49 to each step of this composition to get the desired lift. □

For an ∞-category X, let $k(X)$ be the simplicial subset of X whose n-simplices are the maps $\xi : \Delta[n] \to X$ with the property that every edge of ξ is an equivalence, i.e., for each $0 \leq i \leq j \leq n$, the edge

$$\xi|_{\Delta[i,j]} : \Delta[i, j] \cong \Delta[1] \to X$$

is an equivalence. In particular, $k(X)$ has the same vertices as X. The following property now follows immediately from Corollary 5.51.

Corollary 5.53 *The simplicial subset $k(X)$ is a Kan complex. Moreover, it is the largest Kan complex contained in X, in the sense that any other Kan complex contained in X is in fact contained in $k(X)$.*

Another way of stating this corollary is to say that k is a functor which is right adjoint to the inclusion of Kan complexes into ∞-categories. Our final corollary of Theorem 5.49 concerns the behaviour of this functor with respect to Kan fibrations.

Corollary 5.54 *Let $f\colon Y \to X$ be an inner fibration between ∞-categories. Then the following properties are equivalent:*

(i) *$k(f)\colon k(Y) \to k(X)$ is a Kan fibration,*
(ii) *$\tau(f)\colon \tau(Y) \to \tau(X)$ is an isofibration,*
(iii) *f has J-path lifting,*
(iv) *f has path lifting with respect to equivalences.*

Proof The equivalence between (ii) and (iv) is Lemma 5.47, whereas (ii) $\Rightarrow$ (i) follows from Corollary 5.50 applied to $k(f)$. We will prove (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv). For (i) $\Rightarrow$ (iii) we need to check that a diagonal lift exists in any commutative square

$$\begin{tikzcd} \Delta[0] \arrow[r] \arrow[d] & Y \arrow[d] \\ J \arrow[r] \arrow[ur, dashed] & X. \end{tikzcd}$$

But $J \to X$ factors through $k(X)$, since every 1-simplex in J is an equivalence, and $\Delta[0] \to J$ is anodyne (Proposition 5.22), so that a lift $J \to k(Y)$ exists. For (iii) $\Rightarrow$ (iv), one considers a square

$$\begin{tikzcd} \Delta[0] \arrow[r] \arrow[d] & Y \arrow[d] \\ \Delta[1] \arrow[r, "\alpha"] & X \end{tikzcd}$$

where α is an equivalence in X. We can extend α to $\tilde{\alpha} : J \to X$ by Corollary 5.52 and then lift as in

$$\begin{tikzcd} \Delta[0] \arrow[r] \arrow[d] & Y \arrow[d] \\ J \arrow[r, "\tilde{\alpha}"'] \arrow[ur, dashed, "\tilde{\beta}"] & X. \end{tikzcd}$$

Then $\tilde{\beta}$ restricts to the required lift $\beta : \Delta[1] \to Y$ of α. □

We will conclude this section by investigating equivalences in mapping spaces X^A, where X is an ∞-category and A is an arbitrary simplicial set. Recall that X^A is also an ∞-category (cf. Corollary 5.38). The vertices of X^A are the maps $A \to X$ and the edges in X^A are homotopies

$$h : \Delta[1] \times A \to X$$

parametrized by the 1-simplex. So a priori, these homotopies are *directed*. The edge h is an equivalence in X^A if and only if this homotopy can be extended to J. The main result of the remainder of this section will be that such a homotopy h is an equivalence if and only if it is a 'pointwise' equivalence, i.e., if for each vertex $a \in A_0$, the edge

$$h(-, a) : \Delta[1] \to X$$

is an equivalence. To discuss this point in more detail we introduce the following notation. For a simplicial set and its set of vertices A_0, viewed as a (discrete) simplicial subset $A_0 \subseteq A$, note that

$$X^{A_0} = \prod_{a \in A_0} X$$

and hence

$$k(X^{A_0}) = \prod_{a \in A_0} k(X),$$

and define $k(A, X)$ via the pullback square

$$\begin{tikzcd} k(A,X) \arrow[r] \arrow[d] & X^A \arrow[d] \\ k(X^{A_0}) \arrow[r] & X^{A_0}. \end{tikzcd}$$

Thus, $k(A, X)$ consists of those simplices in X^A whose edges are pointwise equivalences. Clearly

$$k(X^A) \subseteq k(A, X)$$

and we will prove that this inclusion is in fact an equality:

Theorem 5.55 (a) *Let X be an ∞-category and let $A \to B$ be any monomorphism between simplicial sets. Then $k(B, X) \to k(A, X)$ is a Kan fibration.*
(b) *In particular, each $k(A, X)$ is a Kan complex, and hence $k(A, X) = k(X^A)$.*

Proof Part (b) follows from part (a) applied to the monomorphism $\varnothing \to A$ and the fact that $k(X^A)$ is the *largest* Kan complex contained in X^A. To prove (a), we should show that a diagonal lift exists in any commutative square of the following form:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & k(B,X) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & k(A,X). \end{tikzcd}$$

By adjunction this is equivalent to a lifting problem of the form

$$\begin{tikzcd} \Lambda^k[n] \times B \cup_{\Lambda^k[n]\times A} \Delta[n] \times A \arrow[r, "\gamma"] \arrow[d] & X \\ \Delta[n] \times B \arrow[ur, dashed] & \end{tikzcd}$$

where γ maps each 'horizontal edge' in $\Delta[n] \times \Delta[m]$ (automatically in the domain of γ since $n > 0$) to an equivalence in X. If $0 < k < n$ then the map on the left is inner anodyne by Corollary 5.28 and the assumption that X is an ∞-category. For $k = 0$, the map on the left lies in the saturation of the class of monomorphisms of the form

$$\Lambda^k[n] \times \Delta[m] \cup_{\Lambda^k[n]\times\partial\Delta[m]} \Delta[n] \times \partial\Delta[m] \to \Delta[n] \times \Delta[m].$$

Lemma 5.26 and Remark 5.27 in turn show that this map is in the saturation of the class of inner horn inclusions and left horns whose initial edge is 'horizontal', i.e., degenerate in the second factor. Since such an edge is mapped to an equivalence by γ, we find a lift by Theorem 5.49 in case $n + m \geq 2$. The only case not yet covered is $n = 1$, $m = 0$, but then the map under consideration is $\{0\} \to \Delta[1]$. The desired extension exists simply by picking a degenerate simplex of X. A dual argument applies for $k = n$ (or one replaces X by X^{op}). □

Almost exactly the same argument as in the preceding proof applies to the relative case of an inner fibration $f : Y \to X$ between ∞-categories, rather than a single ∞-category. The only modification to be made is in the case $n = 1$, $m = 0$ at the end of the proof. In order for a lift to exist, one now has to assume that f has path lifting with respect to equivalences, or equivalently that f has J-path lifting (Corollary 5.54).

Theorem 5.56 *Let $f : Y \to X$ be an inner Kan fibration between ∞-categories which has J-path lifting. Then for any monomorphism $A \to B$, the map*

$$k(B, Y) \to k(B, X) \times_{k(A,X)} k(A, Y)$$

is a Kan fibration between Kan complexes.

For later use we mention the following application of Theorem 5.55. First we make an obvious definition: for $A \subseteq B$, two maps

$$\begin{tikzcd} B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \end{tikzcd}$$

which agree on A are said to be *J-homotopic relative to A* if there exists a homotopy $h : J \times B \to X$ making the following diagram commute:

$$\begin{tikzcd} J \times A \cup \partial J \times B \arrow[r] \arrow[d] & X. \\ J \times B \arrow[ur, "h"] & \end{tikzcd}$$

The horizontal map consists of the constant homotopy $J \times A \to A \to X$ between $f|_A$ and $g|_A$ and (f, g) on $\partial J \times B = B \amalg B$. The vertical map is the evident inclusion. This definition in particular applies to the case where $A \subseteq B$ is $\partial\Delta[1] \subseteq \Delta[1]$, so that f and g represent two arrows in $\tau(X)$ with common domain and codomain.

Corollary 5.57 *Let X be an ∞-category and let $f, g \in X_1$ be two edges with common domain and codomain. Then f and g represent the same arrow in $\tau(X)$ if and only if they are J-homotopic relative to their endpoints.*

Proof Suppose f and g represent the same arrow in $\tau(X)$. Write $d_1 f = x = d_1 g$ and $d_0 f = y = d_0 g$, so f and g represent arrows $x \to y$. If $f \sim g$ via a 2-simplex of X

$$\begin{tikzcd} & x \arrow[dr, equal] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y, \end{tikzcd}$$

we can extend this to a map $j : \Delta[1] \times \Delta[1] \to X$ which sends $\Delta[1] \times \partial\Delta[1]$ to degeneracies, as in

$$\begin{tikzcd} y \arrow[r, equal] & y \\ x \arrow[u, "f"] \arrow[r, equal] \arrow[ur, "g"] & x. \arrow[u, "g"'] \end{tikzcd}$$

Here the left 2-simplex is the one above and the right 2-simplex is the degenerate simplex $s_0 g$. Thus, j gives a map $\Delta[1] \to k(\Delta[1], X)$. In the commutative square

$$\begin{tikzcd} \Delta[1] \arrow[r, "j"] \arrow[d] & k(\Delta[1], X) \arrow[d] \\ J \arrow[r, "(\bar{x}, \bar{y})"] & k(\partial\Delta[1], X) = k(X) \times k(X) \end{tikzcd}$$

with constant homotopies $\bar{x}, \bar{y} : J \to k(X)$ on the bottom, Theorem 5.49 gives a diagonal lift h, which corresponds to the required homotopy $J \times \Delta[1] \to X$.

Conversely, if f and g are J-homotopic relative to their endpoints, then applying τ to a homotopy between them immediately shows that $[f] = [g]$ in $\tau(X)$. □

5.7 Minimal ∞-Categories and Minimal Kan Complexes

Consider an ∞-category X. In the previous section we defined two maps $f, g : B \to X$ which agree on a simplicial subset $A \subseteq B$ to be J-homotopic relative to A if there exists a homotopy $h : J \times B \to X$ making the diagram

$$\begin{tikzcd} J \times A \cup \partial J \times B \arrow[r] \arrow[d] & X. \\ J \times B \arrow[ur, "h"'] & \end{tikzcd}$$

commute. Here the top horizontal map is the 'constant homotopy' $J \times A \xrightarrow{\pi_2} A \xrightarrow{f} X$ on the first term and (f, g) on the second. One can also express the existence of such a homotopy h as the existence of a solution to the following lifting problem:

$$\begin{tikzcd} \partial J \arrow[r] \arrow[d] & X^B \arrow[d] \\ J \arrow[r] \arrow[ur, dashed] & X^A, \end{tikzcd}$$

where the bottom map is degenerate with image the vertex $f|_A = g|_A$. Here the vertical map on the right is an inner fibration by Corollary 5.39. The horizontal maps in this square factor through the maximal Kan complexes $k(X^B)$ and $k(X^A)$, respectively, as would the dashed lift if it exists. Therefore finding a lift is equivalent to finding a lift h' in the diagram below:

$$\begin{tikzcd} \partial\Delta[1] \arrow[r] \arrow[d] & k(X^B) \arrow[d] \\ \Delta[1] \arrow[r] \arrow[ur, dashed, "h'"] & k(X^A). \end{tikzcd}$$

Indeed, Corollary 5.52 implies that such an h' would extend from $\Delta[1]$ to J to give the desired lift in the previous square. Observe that the map $k(X^B) \to k(X^A)$ is a Kan fibration.

Notice that if X is itself a Kan complex, then so are X^B and X^A, while $X^B \to X^A$ is a Kan fibration. So in this case f and g are J-homotopic relative to A if and only if they are $\Delta[1]$-homotopic relative to A, in the sense that there is a lift in the diagram

$$\begin{tikzcd} \partial\Delta[1] \arrow[r] \arrow[d] & X^B \arrow[d] \\ \Delta[1] \arrow[r] \arrow[ur, dashed] & X^A. \end{tikzcd}$$

Also notice that for a general ∞-category X and $A \subseteq B$ as above, the relation on maps $B \to X$ of being J-homotopic relative to A is an equivalence relation. Indeed, the diagram with h' above shows that this relation coincides with the relation of lying in the same connected component of the fibre of the Kan fibration $k(X^B) \to k(X^A)$ over the vertex $f|_A = g|_A$.

We will now apply these considerations to the case where $A \subseteq B$ is the inclusion $\partial\Delta[n] \subseteq \Delta[n]$ of the boundary of an n-simplex.

Definition 5.58 (i) Two n-simplices x and y in an ∞-category X are said to be *J-equivalent* if, when viewed as maps $\Delta[n] \to X$, they agree on $\partial\Delta[n]$ and are J-homotopic relative to $\partial\Delta[n]$.

(ii) An ∞-category X is *minimal* if any two J-equivalent simplices are equal.

Remark 5.59 By what we said above, replacing J by $\Delta[1]$ gives the same notion of equivalence in the case when X is a Kan complex. A Kan complex which is minimal as an ∞-category will of course be referred to as a *minimal Kan complex*.

Example 5.60 Let $\mathbf{C}$ be a small category. Then its nerve $N\mathbf{C}$ is a minimal ∞-category if and only if any two isomorphic objects of $\mathcal{C}$ are equal; in other words, if and only if $\mathbf{C}$ is skeletal. For a general small category $\mathbf{D}$, one can construct a full subcategory $\mathbf{C}$ which is skeletal by picking precisely one object from each isomorphism class in $\mathbf{D}$. The inclusion $\mathbf{C} \to \mathbf{D}$ is an equivalence of categories.

Our goal in this section is to prove a similar result for ∞-categories, namely that every ∞-category contains an equivalent minimal subcategory. But first, let us discuss some elementary properties of minimal ∞-categories.

Proposition 5.61 (a) *If X and Y are minimal ∞-categories, then so is their product $X \times Y$. More generally, if an ∞-category X is the limit of a diagram of minimal ∞-categories, then X is itself minimal.*

(b) *If $X_0 \to X_1 \to \cdots$ is a diagram of minimal ∞-categories, then $\varinjlim_i X_i$ is again a minimal ∞-category. The same applies more generally to filtered colimits of minimal ∞-categories.*

(c) *If X is a minimal ∞-category, then $k(X)$ is a minimal Kan complex.*

(d) *A subcategory of a minimal ∞-category is again minimal.*

In item (d), a *subcategory* of an ∞-category X is a simplicial subset $Y \subseteq X$ which is itself an ∞-category. We omit the easy proof of the proposition. The following expresses the key property of minimal ∞-categories:

Proposition 5.62 *Any J-homotopy equivalence between minimal ∞-categories is an isomorphism.*

Remark 5.63 This proposition in particular implies that if X is a minimal ∞-category, it cannot contain a smaller ∞-category Y such that the inclusion $i : Y \to X$ is a J-homotopy equivalence. Indeed, a J-homotopy inverse $r : X \to Y$ would yield a J-homotopy equivalence $ir : X \to X$, which has to be an isomorphism since X is minimal. This implies that the inclusion i is surjective, hence the identity. This is one explanation of the use of the word 'minimal'.

Let us return to the statement of Proposition 5.62. If $g : Y \to X$ is a J-homotopy inverse to a given J-homotopy equivalence $f : X \to Y$, then $gf : X \to X$ is an endomorphism which is J-homotopic to the identity. Thus Proposition 5.62 is essentially equivalent to the following statement, which we will prove:

Proposition 5.64 *Let X be a minimal ∞-category. Then any endomorphism of X which is J-homotopic to the identity is an isomorphism.*

In the proof of this proposition, as well as in the proof of Theorem 5.67, we will use the following elementary lemma:

Lemma 5.65 *Consider an inclusion of simplicial sets $A \subseteq B$ and an ∞-category X. Suppose $h, k : J \times B \to X$ are two J-homotopies such that the restrictions of h and k to $\{0\} \times B \cup_{\{0\}\times A} J \times A$ agree. Then the maps*

$$h_1, k_1 : B \cong \{1\} \times B \to X$$

are J-homotopic relative to A.

Proof The assumption can be translated into a commutative square

$$\begin{tikzcd} \Lambda^0[2] \arrow[r, "h \cup k"] \arrow[d] & k(X^B) \arrow[d] \\ \Delta[2] \arrow[r, "l"] & k(X^A). \end{tikzcd}$$

Here $h \cup k$ denotes the map which restricts to (the adjoint of) the homotopy h on $\Delta[0, 1]$ and to k on $\Delta[0, 2]$. Also, l denotes the composition of the degeneracy $\sigma_1 : \Delta[2] \to \Delta[1]$ (collapsing the edge $1 \to 2$) and the map $\Delta[1] \subseteq J \to k(X^A)$ corresponding to $h = k : J \times A \to X$. Since $k(X^B) \to k(X^A)$ is a Kan fibration, there exists a lift $\Delta[2] \to k(X^B)$ in the square. Its restriction along $\partial_0 : \Delta[1] \to \Delta[2]$ (i.e. the edge $\Delta[1, 2]$) extends to a map $J \to k(X^B)$ by Corollary 5.52(ii), which gives the required homotopy between h_1 and k_1. □

Remark 5.66 Note that in the statement of Lemma 5.65 we are *not* assuming that the homotopies h and k are constant on $J \times A$. Nonetheless, its conclusion involves a homotopy relative to A.

Proof (of Proposition 5.64) Consider an endomorphism $\varphi : X \to X$ for which there exists a J-homotopy from φ to the identity:

$$h : J \times X \to X, \quad h_0 = \varphi, \quad h_1 = \mathrm{id}_X.$$

We shall prove by induction on n that φ restricts to an automorphism of the n-skeleton $\mathrm{sk}_n X$ for every n. Taking the colimit over n then gives the desired conclusion. For $n = 0$ the claim is clear, because by minimality X has only one vertex in each J-connected component (i.e., in each connected component of the maximal Kan complex $k(X)$). Suppose we have proved that φ restricts to an automorphism of $\mathrm{sk}_{n-1} X$. Let us show that the restriction of φ to the n-skeleton of X gives an isomorphism $\mathrm{sk}_n X \to \mathrm{sk}_n X$.

To see that φ is injective on $\mathrm{sk}_n X$, take two n-simplices $x, y \in X_n$ and suppose that $\varphi x = \varphi y$. Then $\partial x = \partial y : \partial\Delta[n] \to X$, because φ is injective on $\mathrm{sk}_{n-1} X$. The J-homotopies

$$J \times \Delta[n] \xrightarrow{\mathrm{id}\times x} J \times X \xrightarrow{h} X$$

from $\varphi(x)$ to x and

$$J \times \Delta[n] \xrightarrow{\mathrm{id}\times y} J \times X \xrightarrow{h} X$$

from $\varphi(y)$ to y agree on $J \times \partial\Delta[n]$ (since $\partial x = \partial y$) and on $\{0\} \times \Delta[n]$ (where they both equal $\varphi(x) = \varphi(y)$). Thus Lemma 5.65 implies that x and y are J-homotopic relative to $\partial\Delta[n]$. Minimality of X then implies $x = y$.

To complete the proof let us show that $\varphi : \mathrm{sk}_n X \longrightarrow \mathrm{sk}_n X$ is surjective. Take an n-simplex $z \in X_n$. Since φ is an isomorphism on the $(n-1)$-skeleton we can write $\partial z = \varphi u$ for a unique $u : \partial\Delta[n] \longrightarrow X$. Now consider the homotopy $h_u : \Delta[1] \longrightarrow k(X^{\partial\Delta[n]})$ from ∂_z to u defined by the map

$$\Delta[1] \times \partial\Delta[n] \subseteq J \times \partial\Delta[n] \xrightarrow{h\circ(\mathrm{id}\times u)} X.$$

The map $k(X^{\Delta[n]}) \longrightarrow k(X^{\partial\Delta[n]})$ is a Kan fibration by Theorem 5.55, so there exists a lift in the following diagram:

$$\begin{tikzcd}
\Delta[0] \arrow[r, "z"] \arrow[d, "\partial_1"'] & k(X^{\Delta[n]}) \arrow[d] \\
\Delta[1] \arrow[r, "h_u"'] \arrow[ur, dashed, "g"] & k(X^{\partial\Delta[n]}).
\end{tikzcd}$$

This lift extends to a J-homotopy, again denoted

$$g : J \longrightarrow k(X^{\Delta[n]})$$

from $g_0 = z$ to another n-simplex $y := g_1$. Then $\partial y = u$, while g and

$$h_y : J \times \Delta[n] \xrightarrow{\mathrm{id}\times y} J \times X \xrightarrow{h} X$$

define two J-homotopies which agree on $J \times \partial\Delta[n]$ and on $\{1\} \times \Delta[n]$. By Lemma 5.65 (or rather its 'opposite' version) we conclude that $g_0 = z$ and $(h_y)_0 = \varphi(y)$ are J-equivalent. Since X is minimal, we conclude that $z = \varphi(y)$, completing the proof. □

The final result of this section establishes the existence of minimal ∞-categories:

Theorem 5.67 *Let X be an ∞-category. Then X contains a minimal ∞-category M as a strong J-deformation retract. Explicitly, there are maps*

$$M \xrightarrow{i} X \xrightarrow{r} M, \quad h : J \times X \longrightarrow X$$

with $ri = \mathrm{id}_M$ and h a J-homotopy relative to M from ir to id_X. Moreover, the retraction r is a trivial fibration.

Remark 5.68 Since a retract of a Kan complex is again a Kan complex, the theorem also implies that any Kan complex contains a minimal Kan complex as a deformation retract.

Proof (of Proposition 5.67) The proof of the first part of the theorem consists of an explicit construction of M. The second part will then follow relatively easily. We will build $M \subseteq X$ by inductively defining its skeleta $\mathrm{sk}_n M =: M^{(n)}$, together with retractions $r^{(n)}$ and homotopies $h^{(n)}$, while i will just be the inclusion:

$$M^{(n)} \xrightarrow{i^{(n)}} X^{(n)} \xrightarrow{r^{(n)}} M^{(n)}, \quad h^{(n)} : J \times X^{(n)} \to X.$$

For $n = 0$, start by choosing a single vertex in each connected component of $k(X)$ and let $M^{(0)} = M_0$ be the set of these chosen vertices (thought of as a discrete simplicial set). Then for each vertex $x \in X_0$ there is a unique vertex $r(x) \in M_0$ for which there is a 'path' $h_x : J \to X$ from $r(x)$ to x. This defines

$$M^{(0)} \xrightarrow{i^{(0)}} X^{(0)} \xrightarrow{r^{(0)}} M^{(0)}, \quad h^{(0)} : J \times X^{(0)} \to X,$$

which will satisfy the necessary equations for a J-deformation retract, provided we choose h_x to be degenerate if $x \in M_0$ (so that $r(x) = x$).

Suppose now that for $n > 0$ we have defined $M^{(n-1)}$, $r^{(n-1)}$, and $h^{(n-1)}$. We then define $M^{(n)}$ to be the simplicial subset of X generated by $M^{(n-1)}$ together with a chosen collection of n-simplices: we consider the collection of all the simplices $x \in X_n$ whose boundary ∂x lies in $M^{(n-1)} \subseteq X^{(n-1)}$ and which are *not* fibrewise homotopic to a degenerate simplex, and choose precisely *one* such x in each J-equivalence class (in the sense of Definition 5.58). Thus, $M^{(n)}$ fits into a pushout square

$$\begin{tikzcd}
\coprod_x \partial\Delta[n] \arrow[r] \arrow[d] & M^{(n-1)} \arrow[d] \\
\coprod_x \Delta[n] \arrow[r] & M^{(n)}
\end{tikzcd}$$

where the coproduct is over the set of chosen n-simplices. Next we define $r^{(n)}$ and $h^{(n)}$, extending $r^{(n-1)}$ and $h^{(n-1)}$. On simplices of $M^{(n)}$, of course we have to define $r^{(n)}$ to be the identity and $h^{(n)}$ to be the constant homotopy. For a non-degenerate n-simplex $x \in X_n$ which is *not* contained in $M^{(n)}$ we proceed as follows. The map $h^{(n-1)}$ defines a homotopy

$$J \times \partial\Delta[n] \xrightarrow{\mathrm{id}\times\partial x} J \times X^{(n-1)} \xrightarrow{h^{(n-1)}} X.$$

(Notice that if ∂x happened to be contained in $M^{(n-1)}$ then this homotopy is constant.) Taking adjoints and restricting along the inclusion $\Delta[1] \subseteq J$ defines the bottom horizontal map in the following square:

$$\begin{tikzcd}
\Delta[0] \arrow[r, "x"] \arrow[d, "\partial_0"'] & k(X^{\Delta[n]}) \arrow[d] \\
\Delta[1] \arrow[r] \arrow[ur, dashed, "f"] & k(X^{\partial\Delta[n]}).
\end{tikzcd}$$

The right vertical map is a Kan fibration and the left vertical map is anodyne, so we may pick a lift f. Thus f defines a homotopy to x from another n-simplex whose boundary lies in $M^{(n-1)}$, which we denote by $s(x)$. By the definition of $M^{(n)}$, there exists a *unique* n-simplex of $M^{(n)}$ which is J-equivalent to $s(x)$ and we define $r^{(n)}(x)$ to be this simplex. The J-equivalence between $s(x)$ and $r^{(n)}(x)$ means that there is a homotopy

$$g : \Delta[1] \to k(X^{\Delta[n]})$$

relative to $\partial\Delta[n]$ such that $g_0 = r^{(n)}(x)$ and $g_1 = s(x)$. Now f and g together define the top horizontal map in the following square:

$$\begin{tikzcd}
\Lambda^1[2] \arrow[r, "g\cup f"] \arrow[d] & k(X^{\Delta[n]}) \arrow[d] \\
\Delta[2] \arrow[r] \arrow[ur, dashed, "j"] & k(X^{\partial\Delta[n]}).
\end{tikzcd}$$

The bottom horizontal arrow is the composition $\Delta[2] \xrightarrow{s_0} \Delta[1] \to k(X^{\partial\Delta[n]})$ where the second map is defined by the evaluation of $h^{(n-1)}$ on ∂x as above. Again the left vertical map is anodyne, so a lift j as indicated exists. Restricting j along the inner face $\Delta[1] \xrightarrow{\partial_1} \Delta[2]$ then defines a homotopy

$$\Delta[1] \to k(X^{\Delta[n]})$$

from $r^{(n)}(x)$ to x, compatible with $h^{(n-1)}$ on the boundary ∂x. This homotopy can be extended along the left anodyne map $\Delta[1] \to J$ to finally define a map

$$h_x^{(n)} : J \times \Delta[n] \to X.$$

Doing this for every non-degenerate n-simplex of X defines $h^{(n)} : J \times X^{(n)} \to X$.

It remains to show that the retraction $r : X \to M$ is a trivial fibration. To this end, consider a lifting problem

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r, "u"] \arrow[d, "j"'] & X \arrow[d, "r"] \\
\Delta[n] \arrow[r, "v"] \arrow[ur, dashed] & M.
\end{tikzcd}$$

Taking the map $iv : \Delta[n] \to X$ as the dashed diagonal would make the bottom square commute, since $riv = v$. However, the top triangle would only commute up to the homotopy h from the first part of the theorem: it gives a homotopy

$$h_u : J \to X^{\partial\Delta[n]}$$

from $ivj = iru$ to u. As before, we can use that the right vertical map in the square below is a Kan fibration to find a lift:

$$\begin{tikzcd} \Delta[0] \arrow[r, "iv"] \arrow[d, "\partial_1"'] & k(X^{\Delta[n]}) \arrow[d] \\ J \arrow[r, "h_u"] \arrow[ur, dashed, "f"] & k(X^{\partial\Delta[n]}). \end{tikzcd}$$

This map f is a J-homotopy from $f_0 = iv$ to a map f_1 which agrees with h_u on the boundary $\partial\Delta[n]$. Since $h_{f_1} : J \to X^{\Delta[n]}$ is a J-homotopy from irf_1 to f_1 agreeing with h_u on the boundary, Lemma 5.65 implies that iv and irf_1 are J-equivalent n-simplices of X. Applying the retraction r then gives J-equivalent simplices $riv = v$ and $rirf_1 = rf_1$. By minimality of M we conclude that $v = rf_1$. Hence f_1 can be used as the diagonal filling in the lifting problem above. □

5.8 Minimal Fibrations Between ∞-Categories

Under suitable conditions, the notions and results of the previous section can be extended to the 'relative' case where X is not simply an inner Kan complex, but rather an inner fibration $X \to S$ over a 'base' S. A property of ∞-categories playing a crucial role in the proofs in the 'absolute' case of the previous section is the fact that for an ∞-category X and inclusion of simplicial sets $A \to B$, the map $X^B \to X^A$ is an inner fibration with J-path lifting, and hence the induced map $k(X^B) \to k(X^A)$ is a Kan fibration (Theorem 5.55). In this section we will need the relative version of this fact, as expressed by Theorem 5.56. We will refer to an inner fibration with J-path lifting more briefly as a *J-fibration*. Thus, for ∞-categories X and S, a map $X \to S$ is a J-fibration if and only if it is an inner fibration for which $k(X) \to k(S)$ is a Kan fibration (Corollary 5.54). Also if $X \to S$ is a J-fibration, then so is $X^B \to X^A \times_{S^A} S^B$ for any monomorphism $A \to B$.

For a J-fibration $p : X \to S$ and a monomorphism $i : A \to B$, we call two maps $f, g : B \to X$ over S (i.e., $pf = pg$) *fibrewise J-homotopic relative to A* if there is a homotopy $h : J \times B \to X$ from f to g which restricts to the constant homotopy $J \times A \to A \to X$ on A and composes to the constant homotopy $J \times B \to B \to S$ to S. More concisely, h is a map making the following diagram commute:

$$\begin{tikzcd} \partial J \arrow[r, "(f,g)"] \arrow[d] & X^B \arrow[d] \\ J \arrow[r] \arrow[ur, "h"] & X^A \times_{S^A} S^B. \end{tikzcd}$$

Here the bottom horizontal map is the constant map with value $(f|_A, pf) = (g|_A, pg)$. Thus, the existence of such a fibrewise homotopy h is equivalent to f and g, as vertices of X^B, being in the same connected component of the relevant fibre of the Kan fibration

$$k(X^B) \to k(X^A \times_{S^A} S^B) = k(X^A) \times_{k(S^A)} k(S^B).$$

Note that if $p : X \to S$ is itself a Kan fibration, then so is $X^B \to X^A \times_{S^A} S^B$, and this last fact does not require X and S to be ∞-categories. Thus in this case two maps are fibrewise J-homotopic if and only if they are fibrewise $\Delta[1]$-homotopic. This makes the notion of minimal fibration we consider in this section slightly more versatile in the case of Kan fibrations, since it does not require the base S to be an ∞-category.

For the following we again consider the special case where $A \subseteq B$ is the boundary inclusion $\partial\Delta[n] \subseteq \Delta[n]$:

Definition 5.69 Let $p : X \to S$ be a J-fibration between ∞-categories or a Kan fibration between general simplicial sets.

(a) Two n-simplices $x, y \in X_n$ are *fibrewise J-equivalent* if, when viewed as maps $\Delta[n] \to X$, these simplices are fibrewise J-homotopic relative to $\partial\Delta[n]$. (In particular, $px = py$ and x and y agree on $\partial\Delta[n]$.)
(b) The map $p : X \to S$ is a *minimal J-fibration* (or a *minimal Kan fibration* if p is a Kan fibration) if any two fibrewise J-equivalent simplices are equal.

Notice that if $S = \Delta[0]$ then these definitions agree with the absolute versions of the previous section. We will now state the following two basic results. Their proofs are straightforward adaptations of the proofs given in the previous section and will be omitted. In any case, in Section 6.8 we will state a more general result for dendroidal sets and include a detailed proof. The remainder of this section will focus on some properties which cannot be formulated in the absolute case.

Proposition 5.70 *Let $p : X \to S$ and $q : Y \to S$ be minimal J-fibrations between ∞-categories. Then any fibrewise J-homotopy equivalence $\varphi : X \to Y$ over S is an isomorphism. The same is true for minimal Kan fibrations between general simplicial sets.*

Here we have used the evident terminology, where φ is a fibrewise J-homotopy equivalence over S if there exists another map $\psi : Y \to X$ over S such that each of the compositions $\varphi\psi$ and $\psi\varphi$ is fibrewise J-homotopic to the identity.

Theorem 5.71 *Let $p : X \to S$ be a J-fibration between ∞-categories. Then there exists a minimal J-fibration $q : M \to S$ which is a fibrewise J-deformation retract of $X \to S$. Moreover, the retraction is a trivial fibration. The same applies to a Kan fibration between arbitrary simplicial sets.*

To be completely explicit, the theorem provides maps

$$M \xrightarrow{i} X \xrightarrow{r} M \quad \text{and} \quad h : J \times X \to X$$

such that $h_0 = ir$, $h_1 = \mathrm{id}_X$, $p \circ h = p \circ \pi_2 : J \times X \to S$, and $h \circ (\mathrm{id}_J \times i) = i \circ \pi_2 : J \times M \to X$.

Now let us turn to some closure properties of the class of minimal J-fibrations between ∞-categories. The same properties hold for minimal Kan fibrations between arbitrary simplicial sets and we will not state these separately.

Proposition 5.72 (a) *Consider a pullback square of ∞-categories*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "q"] & X \arrow[d, "p"] \\ T \arrow[r] & S. \end{tikzcd}$$

If p is a minimal J-fibration, then so is q. In particular, the fibres of a minimal J-fibration are minimal ∞-categories.

(b) *Let*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[dl] & \cdots \arrow[dll] \\ S & & \end{tikzcd}$$

be a sequence of minimal J-fibrations over an ∞-category S. Then $\varinjlim_i X_i \to S$ is also a minimal J-fibration.

(c) *If $\{X_i \to S\}_{i\in I}$ is a collection of minimal J-fibrations over an ∞-category S, then $\amalg_i X_i \to S$ is again a minimal J-fibration.*

The proofs of these are elementary and left to the reader. The following expresses a kind of 'homotopy invariance' for minimal J-fibrations, somewhat analogous to the homotopy invariance of fibre bundles over paracompact topological spaces.

Proposition 5.73 *Let $p : X \to S$ be a minimal J-fibration between ∞-categories. If $f, g : T \to S$ are two J-homotopic maps from an ∞-category T, then the pullbacks f^*p and g^*p are isomorphic minimal J-fibrations over T.*

Proof (of Proposition 5.73) Write Y_f (resp. Y_g) for the pullback of X along f (resp. g) and write $h : J \times T \to S$ for a J-homotopy from f to g. By Proposition 5.70 it suffices to prove that Y_f and Y_g are J-homotopy equivalent over T. In fact we will prove that Y_f (and similarly Y_g) is J-homotopy equivalent over T to the pullback Y_J of X along h. For this, consider the pullback square

$$\begin{tikzcd} Y_f \arrow[r] \arrow[d] & Y_J \arrow[d, "h^*p"] \\ T \arrow[r, "0\times\mathrm{id}_T"] & J \times T. \end{tikzcd}$$

The bottom arrow is part of an obvious J-deformation retract. Therefore the top arrow is as well by the following lemma. □

Lemma 5.74 *Consider a pullback square*

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d] & Y \arrow[d, "p"] \\ A \arrow[r, "i"] & X \end{tikzcd}$$

in which the right vertical map is a J-fibration between ∞-categories. If i is part of a J-deformation retract of X onto A, then likewise j is part of a J-deformation retract.

Proof Write the J-deformation retract of X onto A as

$$A \xrightarrow{i} X \xrightarrow{r} A, \quad h : J \times X \to X$$

with $ri = \mathrm{id}_A$, $h_0 = \mathrm{id}_X$, $h_1 = ir$, and $h \circ (\mathrm{id} \times i) = i\pi_2 : J \times A \to X$. Then $k := h \circ (\mathrm{id} \times p)$ is a J-homotopy from p to irp. Let l be a lift in the following square:

$$\begin{tikzcd} J \times B \cup \{0\} \times Y \arrow[r, "j\pi_2 \cup \pi_2"] \arrow[d] & Y \arrow[d, "p"] \\ J \times Y \arrow[r, "k"] \arrow[ur, dashed, "l"] & X. \end{tikzcd}$$

Such a lift exists because in the adjoint lifting problem the map $Y^Y \to Y^B \times_{X^B} X^Y$ is again a J-fibration. Now l_1 has the property that $pl_1 = k_1 = irp$, so that l_1 factors uniquely as js for some map $s : Y \to B$. Then $sj = \mathrm{id}_B$ because $jsj = l_1 j = j$ and moreover $js = l_1$ is J-homotopic (via l of course) to $l_0 = \mathrm{id}_Y$. This proves the lemma. □

Corollary 5.75 *A minimal J-fibration $p \colon X \to S$ for which S is J-contractible is trivial, in the sense that it is isomorphic to one of the form $M \times S \xrightarrow{\pi_2} S$ for a minimal ∞-category M.*

Proof In the statement, J-contractible means that S admits a J-deformation retract onto $\Delta[0]$. The corollary follows from Proposition 5.73 by using that the identity map of S is J-homotopic to a constant map. □

As for the earlier results, if $p \colon X \to S$ is a Kan fibration one may replace J by $\Delta[1]$ and the condition that X and S are ∞-categories is unnecessary in Proposition 5.73, as well as in the lemma used in its proof. In particular, we find the following variant of the previous corollary:

Corollary 5.76 *A minimal Kan fibration $p \colon X \to S$ between simplicial sets, for which S is $\Delta[1]$-contractible, is trivial.*

In particular the corollary applies when S is a standard simplex $\Delta[n]$. This fact is often expressed by saying that a minimal Kan fibration $p \colon X \to S$ (with S arbitrary) is *locally trivial*. This is yet another indication that the theory of minimal fibrations resembles that of topological fibre bundles.

Historical Notes

The extension (or lifting) condition that defines a Kan complex (or Kan fibration) first appeared in Kan's paper [95], although in the context of cubical sets. The switch to simplicial sets and the development of homotopy theory in this language was made soon after that [118, 96, 97, 98]. The notion of an inner Kan complex (also called a *weak* Kan complex) was first singled out by Boardman–Vogt [21]. Joyal studied the notion of inner Kan complex (under the name *quasi-category*) in [92] and proved many of the basic results contained in the present chapter, among which the crucial Theorem 5.49. Lurie's book [105] uses the terminology ∞-*category* like we do here and is the most comprehensive reference. It develops the analogue of a substantial portion of category theory in this more sophisticated context.

Chapter 6
Kan Conditions for Dendroidal Sets

In this chapter we study extension conditions for dendroidal sets and lifting conditions for maps of dendroidal sets which parallel the conditions for simplicial sets of the previous chapter. The structure of this chapter follows the same plan to stress the analogy. The proofs of various pushout-product properties, which we develop in Section 6.3, become more technical in the case of dendroidal sets.

6.1 Dendroidal Kan Complexes and ∞-Operads

We begin by discussing the dendroidal analogues of the various Kan conditions of the previous chapter. Suppose T is a tree and $\partial_x T$ is an elementary face of it. Then we define the *horn* $\Lambda^x[T]$ to be the subobject of the representable dendroidal set $\Omega[T]$ which is the union of all faces of T *except* $\partial_x T$. If $\partial_x T$ is an inner face of T, contracting some inner edge x, we will call the corresponding horn $\Lambda^x[T]$ an *inner horn*. It will also be convenient to have terminology for *leaf horns*. These are horns of one of the following two types:

(1) a horn $\Lambda^v[T]$ for T a tree with at least two vertices and v a leaf vertex of T,
(2) for a corolla C_n, the inclusion of its leaves

$$\coprod_{i=1,\ldots,n} \eta \to \Omega[C_n].$$

At the opposite end, we define a *root horn* to be a horn of one of the following types:

(1) a horn $\Lambda^r[T]$ for T a tree with root vertex r, where r has precisely one inner edge attached to it (so that the root face of T exists),
(2) for a corolla C_n, the inclusion of its root and all but one of its leaves (say the kth one):

$$\coprod_{i=0,\ldots,\hat{k},\ldots,n} \eta \to \Omega[C_n].$$

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Definition 6.1 A dendroidal set X is a *dendroidal Kan complex* if any map from a horn $\Lambda^x[T] \to X$ admits an extension to a map $\Omega[T] \to X$. In a diagram:

$$\begin{tikzcd} \Lambda^x[T] \arrow[r] \arrow[d] & X. \\ \Omega[T] \arrow[ur, dashed] & \end{tikzcd}$$

A dendroidal set X is an ∞*-operad* or a *dendroidal inner Kan complex* if it satisfies this condition for every *inner* horn inclusion. It is a *dendroidal left Kan complex* if it satisfies the extension condition for every inner or leaf horn. Finally, it is a *dendroidal right Kan complex* if it satisfies the extension condition for every inner or root horn. We will say X is a *strict* Kan complex if the necessary extensions are unique and similarly for strict inner, left, or right Kan complexes.

As with ∞-categories, one should think of ∞-operads roughly as 'weak' operads where composition is only defined up to homotopy. We will discuss these in much more detail later. The notion of a dendroidal right Kan complex will play a far less prominent role for us than that of a dendroidal left Kan complex. In fact there is an asymmetry here which was not present for simplicial sets. In the simplicial case our definitions and results for left horns and left fibrations had evident duals for right horns and right fibrations, simply by using the automorphism of the category $\mathbf{\Delta}$ which reverses the order of a simplex. This automorphism does *not* extend to an automorphism of the category of trees $\mathbf{\Omega}$, so that there is no natural notion of the 'opposite' of a dendroidal set.

The reader should note that Definition 6.1 generalizes our definitions for simplicial sets in the following sense: for a horn inclusion of simplicial sets $\Lambda^k[n] \to \Delta[n]$, the corresponding map

$$i_!\Lambda^k[n] \to i_!\Delta[n]$$

is a horn inclusion of the dendroidal set $i_!\Delta[n]$ represented by the linear tree $i[n]$. Moreover, $i_!$ sends inner horns to inner horns, left horns of the form $\Lambda^0[n]$ to leaf horns, and right horns of the form $\Lambda^n[n]$ to root horns. This simple observation has the following consequences:

Lemma 6.2 *If X is a dendroidal (left or right) Kan complex, then i^*X is a Kan complex. If X is an ∞-operad, then i^*X is an ∞-category. Also, if M is a simplicial set which is an inner Kan complex, then $i_!M$ is a dendroidal inner Kan complex. This last conclusion does not extend to dendroidal (left or right) Kan complexes.*

Proof The first two sentences follow immediately from the remark preceding the lemma and the adjunction between the functors $i_!$ and i^*, combined with the fact that if $i^*X \to \Delta[0]$ is a left or right fibration of simplicial sets then it is automatically a Kan fibration (cf. Corollary 5.51). For the third sentence, consider an extension problem

$$\begin{tikzcd}
\Lambda^e[T] \arrow[r] \arrow[d] & i_!M \\
\Omega[T] \arrow[ur, dashed] &
\end{tikzcd}$$

with $\Lambda^e[T]$ an inner horn of T. The horn $\Lambda^e[T]$ in particular contains every corolla C_v corresponding to a vertex v of T. Because any dendroidal set admitting a map to $\eta = i_!\Delta[0]$ 'is' a simplicial set (in the sense that is in the image of $i_!$) each such corolla must be linear, i.e., each vertex v has precisely one input edge. It follows that the entire tree T is linear, so it is of the form $i[n]$. Because $i_!$ is a fully faithful functor, this lifting problem corresponds to one in the category of simplicial sets, where it admits a solution by assumption. This line of reasoning fails for leaf horns (and for root horns by similar reasoning). Indeed, consider a corolla C_n with $n \neq 1$ and a lifting problem

$$\begin{tikzcd}
\coprod_{i=1,\ldots,n} \eta \arrow[r] \arrow[d] & i_!M \\
\Omega[C_n] \arrow[ur, dashed] &
\end{tikzcd}$$

where the vertical map on the left is the inclusion of the leaves of C_n. The horizontal map simply picks out a collection of n vertices of M; in particular there exist such maps (as long as M is non-empty). However, a map as indicated by the dashed arrow cannot exist because C_n is not a linear tree. □

We now list some basic examples of ∞-operads and dendroidal (left) Kan complexes. We will then spend most of the remainder of this section checking that these examples indeed satisfy the necessary extension conditions.

Example 6.3 (a) Let $\mathbf{P}$ be an operad in **Sets**. Then its nerve $N\mathbf{P}$ is a strict dendroidal inner Kan complex (or strict ∞-operad). We will prove this below in Proposition 6.4. Recall that an operad $\mathbf{P}$ always defines an 'underlying category' $j^*\mathbf{P}$, simply by discarding all non-unary operations of $\mathbf{P}$, and the simplicial set $i^*N\mathbf{P}$ coincides with the strict inner Kan complex $Nj^*\mathbf{P}$.

(b) Let $\mathbf{C}$ be a symmetric monoidal groupoid. One can associate to it the operad $\mathbf{C}^\otimes$ with

$$\mathbf{C}^\otimes(c_1, \ldots, c_n; d) = \mathbf{C}(c_1 \otimes \cdots \otimes c_n; d).$$

Then the nerve $N\mathbf{C}^\otimes$ is a dendroidal left Kan complex, not necessarily strict. We will come back to this example and related ones in Section 9.6, but the special case of a discrete groupoid features in Example 6.10(e). Of course the underlying simplicial set $i^*N\mathbf{C}^\otimes$, being simply the nerve of the groupoid $\mathbf{C}$, is a Kan complex which is strict as an inner Kan complex (i.e., all fillers for *inner* horns are unique).

(c) A *Picard groupoid* $\mathbf{C}$ is a symmetric monoidal groupoid in which every object is invertible with respect to the tensor product, i.e., for every $X \in \mathbf{C}$ there exists an $X^\vee \in \mathbf{C}$ such that $X \otimes X^\vee$ is isomorphic to the unit $\mathbf{1}$ of the symmetric monoidal structure. The dendroidal set $N\mathbf{C}^\otimes$ is a dendroidal left Kan complex by (b), but it is in fact a dendroidal Kan complex. This implication can be reversed; if $\mathbf{C}$ is a symmetric monoidal groupoid for which $N\mathbf{C}^{\otimes}$ is a dendroidal Kan complex, then $\mathbf{C}$ is a Picard groupoid. We will come back to this example in Section 9.7, as well as in Example 6.10(f) in the case of a discrete groupoid.

(d) Let $\mathbf{P}$ be a simplicial operad with the property that for any two colours x, y of $\mathbf{C}$, the simplicial set $\mathbf{P}(x, y)$ is a Kan complex. Then the homotopy-coherent nerve $w^*\mathbf{P}$ is an ∞-operad, as we will demonstrate in Proposition 6.5 below.

The following is the generalization of Proposition 5.3 to dendroidal sets. Its formulation uses the notion of the *spine* of a tree T, which generalizes the corresponding notion for simplices we discussed in the previous chapter. Roughly speaking, the spine $\mathrm{Sp}[T]$ of T is the union of all its corollas. More precisely, every vertex v of T defines a subtree C_v of T with one vertex (namely v), the output edge of v as its root, and the input edges of v as its leaves. With this notation, $\mathrm{Sp}[T]$ is the following subobject of $\Omega[T]$:

$$\mathrm{Sp}[T] := \bigcup_{v\in V(T)} \Omega[C_v].$$

Proposition 6.4 *For a dendroidal set X, the following are equivalent:*

(i) *X is a strict inner Kan complex.*
(ii) *For each tree T, the map*

$$X_T = \mathbf{dSets}(\Omega[T], X) \to \mathbf{dSets}(\mathrm{Sp}[T], X)$$

is an isomorphism.
(iii) *X is isomorphic to $N\mathbf{P}$ for some operad $\mathbf{P}$ in* **Sets**.
(iv) *The unit $X \to N\tau(X)$ is an isomorphism.*

Proof We postpone the proof of the equivalence between (i) and (ii) to a more systematic discussion of the relation between inner horns and spines in Section 6.3 (see Remark 6.40 specifically), although the motivated reader can take this as a useful exercise in the combinatorics of horn inclusions of trees. We will prove the other fairly elementary equivalences now. The implication (iv) $\Rightarrow$ (ii) is more or less direct from the definition of the nerve functor and the fact that the operads $\Omega(T)$ are free. Indeed, suppose X satisfies (iv). To prove X satisfies (ii) it suffices to check that for any operad $\mathbf{P}$ and a tree T, the map

$$\mathbf{dSets}(\Omega[T], N\mathbf{P}) \to \mathbf{dSets}(\mathrm{Sp}[T], N\mathbf{P})$$

is an isomorphism. This is easily verified by recalling that a T-dendrex of $N\mathbf{P}$ is essentially described by a labelling of the edges of T by colours of $\mathbf{P}$ and of each vertex of T by an operation of $\mathbf{P}$ with the appropriate in- and outputs (see Example 3.20(e) for a precise discussion). An alternative way to phrase this argument is by saying that

$$\tau(\mathrm{Sp}[T]) \to \tau(\Omega[T]) = \Omega(T)$$

is an isomorphism of operads, which is yet another way to say that the operad $\Omega(T)$ is free. Now suppose X satisfies (ii). Then define an operad $\mathbf{P}$ with X_η as its set of colours as follows. For $x_1, \ldots, x_n, y \in X_\eta$, an operation

$$p \in \mathbf{P}(x_1, \ldots, x_n; y)$$

is a corolla $\xi_p \in X_{C_n}$ whose leaves are precisely $x_1, \ldots, x_n$ and whose root is y. For another such operation

$$q \in \mathbf{P}(y_1, \ldots, y_m; z)$$

with $y_i = y$, one defines the composition $q \circ_i p$ as the operation corresponding to the restriction to the inner face of the map

$$\Omega[C_m \circ_i C_n] \xrightarrow{\xi_q \circ_{y_i} \xi_p} X,$$

with $C_m \circ_i C_n$ the tree obtained by grafting C_n onto the ith leaf of C_m. That such a map exists and is uniquely determined by ξ_p and ξ_q follows from the assumption that X satisfies (ii). To see that this composition defines an operad one has to check the various axioms; for associativity of composition one uses trees obtained by grafting three corollas together, the others are straightforward. Consider the commutative square

$$\begin{tikzcd} X_T \arrow[r] \arrow[d] & N\mathbf{P}_T \arrow[d] \\ \mathbf{dSets}(\mathrm{Sp}[T], X) \arrow[r] & \mathbf{dSets}(\mathrm{Sp}[T], N\mathbf{P}). \end{tikzcd}$$

The right-hand vertical map is an isomorphism, as we verified above when proving (iv) ⇒ (ii). The bottom horizontal map is an isomorphism by construction and the left-hand map is an isomorphism by hypothesis. Therefore the top horizontal map is an isomorphism, showing that $X \cong N\mathbf{P}$. The implication (iii) ⇒ (iv) works exactly as in the proof of Proposition 5.3. □

An important source of ∞-operads is the homotopy-coherent nerve construction. In fact, in Part II of this book we will demonstrate that in an appropriate sense this construction gives an equivalence of homotopy theories between simplicial operads and ∞-operads. The first result in that direction is the following:

Proposition 6.5 *Let* $\mathbf{P}$ *be a simplicial operad such that for every tuple of colours* $(x_1, \ldots, x_n, y)$ *of* $\mathbf{P}$ *the simplicial set* $\mathbf{P}(x_1, \ldots, x_n; y)$ *is a Kan complex. Then its homotopy-coherent nerve* $w^*\mathbf{P}$ *is an ∞-operad.*

Proof We should check that for an inner horn inclusion $\Lambda^e[T] \to \Omega[T]$ of a tree T, there exists a lift in any square of the following form:

$$\begin{tikzcd} w_!\Lambda^e[T] \arrow[r] \arrow[d] & \mathbf{P}. \\ w_!\Omega[T] \arrow[ur, dashed] & \end{tikzcd}$$

We will use the explicit description of the functor $w_!$ given in Example 3.20(i) and Section 3.5.3, as well as the notations S/e and e/S used in Example 3.20(i). In particular, recall that for a tuple $(c_1, \ldots, c_n, d)$ of edges of T, there is an equality of simplicial sets

$$(w_!\Omega[T])(c_1, \dots, c_n; d) = \Delta[1]^{I(S)},$$

where $I(S)$ is the set of inner edges of the maximal subtree S of T with leaves $c_1, \dots, c_n$ and root d (if such S exists, otherwise this simplicial set is empty). The morphism of operads $w_!\Lambda^e[T] \to w_!\Omega[T]$ induces an isomorphism of simplicial sets

$$(w_!\Lambda^e[T])(c_1, \dots, c_n; d) \cong (w_!\Omega[T])(c_1, \dots, c_n; d)$$

for any tuple of colours $(c_1, \dots, c_n, d)$, *except* when $c_1, \dots, c_n$ is precisely the collection of leaves of T and d is the root of T. Indeed, in all other cases, the relevant maximal subtree S (if it exists) with leaves $c_1, \dots, c_n$ and root d factors through some outer face of T and is therefore already contained in $\Lambda^e[T]$. In the exceptional case, the map above can be identified with the inclusion

$$V \subseteq \Delta[1]^{I(T)},$$

where V is the union of the subcubes of the following two types:

(a) The subcubes

$$\Delta[1]^{I(T)-\{f\}} \xrightarrow{0_f} \Delta[1]^{I(T)}$$

where f ranges over the set $I(T) - \{e\}$, and the inclusion gives the edge f the coordinate 0. Each such cube is contributed by the inner face $\partial_f T$ of T.

(b) For each inner edge f of T, the subcube

$$\Delta[1]^{I(T)-\{f\}} \xrightarrow{1_f} \Delta[1]^{I(T)}.$$

This subcube arises from the 'composition along f' map

$$\Delta[1]^{I(T)-\{f\}} = \Delta[1]^{I(T/f)} \times \Delta[1]^{I(f/T)} \to \Delta[1]^{I(T)},$$

where both factors arise from trees (namely T/f and f/T) which are contained in outer faces of T and hence in $\Lambda^e[T]$.

To provide a solution to our original lifting problem, it suffices to show that a lift exists in the following (with $c_1, \dots, c_n$ the leaves and d the root of T):

$$\begin{tikzcd} V \arrow[r] \arrow[d] & \mathbf{P}(c_1, \dots, c_n; d). \\ \Delta[1]^{I(T)} \arrow[ur, dashed] & \end{tikzcd}$$

The simplicial set on the right is a Kan complex by assumption, so it suffices to show that the left vertical map is anodyne. To do this, write

$$\partial(\Delta[1]^{I(T)-\{e\}}) := \bigcup_{f \in I(T)-\{e\}} \partial\Delta[1] \times \Delta[1]^{I(T)-\{e,f\}}$$

for the 'boundary' of the cube $\Delta[1]^{I(T)-\{e\}}$ and note that the inclusion $V \to \Delta[1]^{I(T)}$ can be identified with the pushout-product map

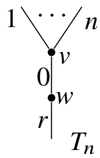
$$\{1\} \times \Delta[1]^{I(T)-\{e\}} \cup \Delta[1] \times \partial(\Delta[1]^{I(T)-\{e\}}) \to \Delta[1]^{I(T)}.$$

This map is right anodyne by Corollary 5.28. □

Remark 6.6 For the reader familiar with the homotopy theory of simplicial sets, the preceding proof can also be concluded by observing that the inclusion $V \to \Delta[1]^{I(T)}$ is a weak homotopy equivalence of simplicial sets. Indeed, the domain and codomain are easily seen to be weakly contractible. We will return to arguments of this kind in Part II of this book, once we study the homotopy theory of simplicial sets more seriously.

We conclude this section with a description of $\tau(X)$ in the case where X satisfies the dendroidal inner Kan condition. We will sometimes refer to $\tau(X)$ as the *homotopy operad* of the ∞-operad X. The following discussion parallels the one at the end of Section 5.1.

For an ∞-operad X we will define an operad $\tau_1(X)$ and prove it is naturally isomorphic to $\tau(X)$. First we need the notions of *root homotopy* and *leaf homotopy*. Say $\xi, \zeta \in X_{C_n}$ are two n-corollas of X. Write T_n for the grafted tree $C_1 \circ C_n$, pictured as follows:

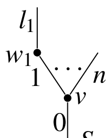


T_n

Observe the evident identifications $C_n \cong \partial_w T_n$ (chopping off the root), $C_n \cong \partial_0 T_n$ (contracting the edge 0), and $C_1 \cong \partial_v T$ (chopping off v and its leaves). We say ξ and ζ are *root homotopic* if there exists an element $\psi \in X_{T_n}$ such that:

- The dendrex $d_v\psi \in X_{C_1}$ is degenerate.
- Under the identifications described above, $d_w\psi = \xi$ and $d_0\psi = \zeta$.

More precisely, we say ψ is a root homotopy from ξ to ζ and write $\xi \sim_r \zeta$ to indicate the relation of root homotopy. In case $n = 1$, so that T is linear, this notion reduces to the homotopy relation we used in Section 5.1. Also, write $S_{n,i}$ for the grafted tree $C_n \circ_i C_1$. The following is an illustration of $S_{n,1}$:



$S_{n,1}$

There are identifications $C_n \cong \partial_{w_i} S_{n,i}$ (chopping off the leaf vertex w_i connected to i and its leaf) and $C_n \cong \partial_i S_{n,i}$ (contracting all the input edge i of v). Also, the vertex w_i determines a corolla isomorphic to C_1. We say ξ and ζ are *leaf homotopic* if there exists i and $\varphi \in X_{S_{n,i}}$ such that:

- The restriction of φ to the unary corolla with vertex w_i is a degenerate corolla of X.
- Under the identifications described above, $d_i\varphi = \xi$ and $d_{w_i}\varphi = \zeta$.

In this case φ is said to define a leaf homotopy from ξ to ζ, indicated by the notation $\xi \sim_l \zeta$.

We are now ready to define the operad $\tau_1(X)$. The set of colours of $\tau_1(X)$ is X_η. The set of operations

$$\tau_1(X)(x_1, \ldots, x_n; y)$$

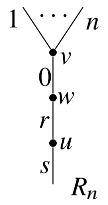
is a quotient of the set of pairs (ξ, p) with $\xi \in X_{C_n}$ an n-corolla of X and p a planar structure of C_n (i.e. a linear ordering of its leaves), such that the sequence of leaves of ξ in the chosen ordering is precisely $(x_1, \ldots, x_n)$ and its root is y. The quotient is determined by the equivalence relation that has $(\xi, p) \sim (\zeta, q)$ precisely if there is an automorphism σ of C_n (i.e. a permutation of its n leaves) such that $\sigma^*p = q$, the permutation σ leaves the sequence $(x_1, \ldots, x_n)$ invariant, and $\sigma^*\xi$ is root homotopic to ζ. Composition in $\tau_1(X)$ is defined using the fact that X satisfies the inner Kan condition. Indeed, for two operations $a \in \tau_1(X)(x_1, \ldots, x_n; y_i)$ and $b \in \tau_1(X)(y_1, \ldots, y_m; z)$ with $1 \le i \le m$, represented by pairs (ξ, p) and (ζ, q) respectively, the composite operation $b \circ_{y_i} a$ is represented as follows. Consider the grafted tree $T = C_m \circ_i C_n$ and pick a dendrex $\theta \in X_T$ such that T restricts to ξ on C_n and to ζ on C_m. Such a θ exists since X is inner Kan. Clearly p and q define a planar structure t on T and $b \circ_{y_i} a$ is represented by the pair $(d_i\theta, t)$, with $d_i\theta$ the inner face of θ. That our definitions give a well-defined operad $\tau_1(X)$ which is moreover isomorphic to $\tau(X)$ is guaranteed by:

Lemma 6.7 *Assume X is a dendroidal inner Kan complex.*

(i) *The relation of root homotopy is an equivalence relation.*
(ii) *Two corollas of X are root homotopic if and only if they are leaf homotopic.*
(iii) *The composition of operations in $\tau_1(X)$ described above is well-defined on equivalence classes and gives $\tau_1(X)$ the structure of an operad.*
(iv) *The evident morphism $\tau_1(X) \to \tau(X)$ is an isomorphism.*

Proof This lemma is a generalization of Lemma 5.6 to dendroidal sets; the reader might find it instructive to compare the proofs.

(i) Reflexivity $\xi \sim_r \xi$ follows from considering dendrices of shape T_n which are degenerate at the root; more precisely, for $\xi \in C_n$ and 0 denoting the root of C_n, the degenerate dendrex $s_0\xi$ shows that ξ is root homotopic to itself. To see that the relation of root homotopy is transitive, suppose we have n-corollas ξ_1, ξ_2, and ξ_3 of X, and dendrices $\psi, \varphi \in X_{T_n}$ expressing root homotopies $\xi_1 \sim_r \xi_2$ and $\xi_2 \sim_r \xi_3$ respectively. Consider the tree R_n obtained by grafting an n-corolla onto a linear tree $i[2]$ with two vertices:



Define a map f from the inner horn $\Lambda^r[R_n]$ to X as follows:

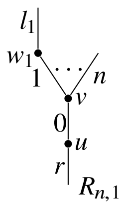
- f sends the outer face $\partial_v R_n$ chopping off v and its leaves to the degenerate dendrex of shape $i[2]$ with colour that of the root edge of ξ_1 (which is the same as that of ξ_2 and ξ_3).
- f sends the inner face $\partial_0 R_n$ to φ.
- f sends the root face $\partial_u R_n$ to ψ.

Since X satisfies the inner Kan condition, f can be extended to a map $g : \Omega[R_n] \to X$. The restriction of g to the face $\partial_r R_n$ now defines a root homotopy from ξ_1 to ξ_3. It remains to prove symmetry of the root homotopy relation. So suppose $\psi \in X_{T_n}$ defines a root homotopy from ξ to ζ. With R_n as above, define a map f from the inner horn $\Lambda^0[R_n]$ to X as follows:

- f sends the outer face $\partial_v R_n$ to the degenerate dendrex of shape $i[2]$ with colour that of the root edge of ξ.
- f sends the inner face $\partial_r R_n$ to the degenerate dendrex $s_0\xi$.
- f sends the root face $\partial_u R_n$ to φ.

As before, f can be extended to a map $g : \Omega[R_n] \to X$. Restricting g to the face $\partial_0 R_n$ defines a root homotopy from ζ to ξ, as desired.

(ii) Suppose $\varphi \in X_{T_n}$ is a dendrex giving a root homotopy $\xi \sim_r \zeta$ between two corollas $\xi, \zeta \in X_{C_n}$. Consider a tree $R_{n,1}$ as follows:



In words, $R_{n,1}$ is obtained from the corolla C_n by grafting a 1-corolla onto its root and another one onto the leaf labelled 1. Note the evident identifications $\partial_{w_1} R_{n,1} \cong T_n$ and $\partial_u R_{n,1} \cong S_{n,1}$. Define a map f from the inner horn $\Lambda^0[R_{n,1}]$ to X as follows:

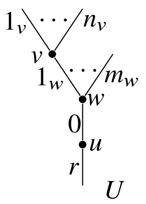
- f sends the outer face $\partial_{w_1} R_{n,1}$ to φ.
- f sends the inner face $\partial_1 R_{n,1}$ to the degenerate dendrex $s_0\xi$.
- f sends the root face $\partial_u R_{n,1}$ to the degenerate dendrex $s_1\xi$.

Extend f to a map $g : \Omega[R_{n,1}] \to X$. Then the restriction of g to $\partial_0 R_{n,1}$ provides a leaf homotopy $\xi \sim_l \zeta$. Conversely, to see that leaf homotopy implies root homotopy one uses an entirely analogous argument, now exploiting the existence of a filler for an inner horn of the kind $\Lambda^i[R_{n,i}]$.

(iii) Suppose we have elements

$$a \in \tau_1(X)(x_1, \dots, x_n; y_i) \text{ and } b \in \tau_1(X)(y_1, \dots, y_m; z).$$

Say (ξ, p) is a representative of a and (ζ, q) and (ζ', q') are two representatives of b, so that there exists a dendrex $\varphi \in X_{T_m}$ giving a root homotopy $\zeta \sim_r \zeta'$. As in the definition of composition before the lemma, suppose we have picked pairs (θ, t) and (θ', t'), with θ and θ' dendrices of X of the shape $C_m \circ_i C_n$, so that the inner faces of θ and θ' represent the compositions of (ζ, q) and (ζ', q') with (ξ, p) respectively. To show that these two represent the same element, consider the grafted tree $U = C_1 \circ C_m \circ_i C_n$ (illustrated here for $i = 1$):



We will tacitly make the obvious identifications of the kind $\partial_v U \cong T_m$. Define a map f from the inner horn $\Lambda[U]$ to X as follows:

- f sends the leaf face $\partial_v U$ to φ.
- f sends the inner face $\partial_0 U$ to θ'.
- f sends the root face $\partial_u U$ to θ.

Pick an extension of f to a map $g : \Omega[U] \to X$. Then the restriction of g to $\partial_{1_w} U$ defines a root homotopy showing that (θ, t) and (θ', t') represent the same operation of $\tau_1(X)$. To see that composition is independent of the chosen representative of a one uses a similar argument, now exploiting the existence of a leaf homotopy between any two representatives (ξ, p) and (ξ', p'), which is guaranteed by (ii).

(iv) This is now formal and follows by an argument of the same kind used in the proof of Lemma 5.6. □

6.2 Fibrations and Anodyne Morphisms Between Dendroidal Sets

The extension conditions of the previous paragraph all have relative versions, defining corresponding notions of fibrations between dendroidal sets. As with simplicial sets, these various kinds of fibrations will play a central role in what follows.

Definition 6.8 A morphism $Y \to X$ between dendroidal sets is said to be

- a *dendroidal Kan fibration* if it has the right lifting property with respect to all horn inclusions of trees,
- a *dendroidal inner fibration* if it has the right lifting property with respect to all inner horn inclusions of trees,
- a *dendroidal left fibration* if it has the right lifting property with respect to all leaf and inner horn inclusions of trees,
- a *dendroidal right fibration* if it has the right lifting property with respect to all root and inner horn inclusions of trees.

We will omit the adjective 'dendroidal' when no confusion can arise.

As already remarked before, the notion of dendroidal left fibration will be much more important for us than that of dendroidal right fibration. For the record we mention the following obvious fact:

Lemma 6.9 *Each of the classes of fibrations just defined is closed under composition, pullback, and retracts.*

Example 6.10 (a) A dendroidal set X is an ∞-operad precisely if the unique map to the terminal dendroidal set $X \to 1$ is an inner fibration. Here we use the standard convention of writing 1 for the terminal object. Recall from Example 3.20(f) that $1 = N\mathbf{Com}$ for the category of dendroidal sets.

(b) Contrary to the case of simplicial sets, a left fibration of dendroidal sets over the terminal object $X \to 1$ is *not* automatically a Kan fibration, as is witnessed by the difference between parts (b) and (c) of Example 6.3.

(c) For a morphism of operads $f : \mathbf{P} \to \mathbf{Q}$, the dendroidal nerve Nf is an inner fibration. This follows easily from the fact that $N\mathbf{P}$ and $N\mathbf{Q}$ are *strict* inner Kan complexes, so that fillers for inner horns are unique.

(d) Let $\mathbf{P}$ be an operad in **Sets** and A a $\mathbf{P}$-algebra. Recall the nerve $N(\mathbf{P}, A)$ of Example 3.20(h): a T-dendrex of $N(\mathbf{P}, A)$ is a T-dendrex $\xi \in N\mathbf{P}_T$ together with a labelling of each leaf l of T by an element $x_l \in A_{\xi(l)}$. Assigning to such a dendrex the element ξ defines a map of dendroidal sets

$$\pi_A : N(\mathbf{P}, A) \to N\mathbf{P},$$

which is a left fibration. Indeed, it is an inner fibration as a consequence of example (c) above. For a lifting problem

$$\begin{tikzcd} \Lambda^v[T] \arrow[r] \arrow[d] & N(\mathbf{P}, A) \arrow[d, "\pi_A"] \\ \Omega[T] \arrow[r] \arrow[ur, dashed] & N\mathbf{P} \end{tikzcd}$$

involving a leaf horn (so v is a leaf vertex of T) one argues as follows. Since the bottom horizontal arrow already defines the necessary T-dendrex $\xi \in N\mathbf{P}_T$, all that is necessary is to give an appropriate labelling $x_l \in A_{\xi(l)}$ of each leaf l of T. The leaf horn $\Lambda^v[T]$ in particular contains all the leaves of T, so that this labelling is uniquely determined by the top horizontal map.

(e) Consider a commutative monoid A. In other words, A is a **Com**-algebra in **Sets**. Then $N(\mathbf{Com}, A)$ is a dendroidal left Kan complex. This is a special case of (d) above, and simultaneously a special case of Example 6.3(b) by considering A as a *discrete* symmetric monoidal groupoid. Note that a dendrex $\xi \in N(\mathbf{Com}, A)$ is simply a labelling of each leaf l of T by an element $x_l \in A$.

(f) Let A be an abelian group. By the previous example, the dendroidal set $N(\mathbf{Com}, A)$ is a dendroidal left Kan complex. It is in fact a dendroidal Kan complex. This example is also a special case of Example 6.3(c). To see explicitly that $N(\mathbf{Com}, A)$ has the extension property with respect to root horn inclusions, first observe that there is nothing to prove for a root horn of a tree T with at least two vertices; in that case the dendroidal set $\Lambda^r[T]$ already contains all the leaves of T, so that any map $\Lambda^r[T] \to N(\mathbf{Com}, A)$ extends uniquely to $\Omega[T]$. The remaining case is where $T = C_n$ is a corolla and we consider a horn inclusion of the kind

$$\coprod_{i=0,\ldots,\hat{k},\ldots,n} \eta =: R \to \Omega[C_n],$$

missing one of the leaves of C_n. Suppose we have a map $f : R \to N(\mathbf{Com}, A)$, which corresponds to elements $f(i) \in A$ for $i \neq k$. To extend it to $\Omega[C_n]$ we should give an element $f(k) \in A$ such that $f(1) + \cdots + f(n) = f(0)$ in the abelian group A. Clearly the only possibility is

$$f(k) := f(0) - (f(1) + \cdots + \widehat{f(k)} + \cdots + f(n)).$$

Remark 6.11 There is a generalization of Example 6.10(d) involving the notion of an *operad cofibred in groupoids*, a notion which generalizes the fibred categories of Example 5.12. Fibred categories $\mathbf{D} \to \mathbf{C}$ are closely related to pseudofunctors on $\mathbf{C}$, and similarly such operads cofibred in groupoids are closely related to (a weak kind of) algebras. More precisely, for a map of operads $f : \mathbf{P} \to \mathbf{Q}$ the nerve Nf is a left fibration of dendroidal sets if and only f exhibits $\mathbf{P}$ as an operad cofibred in groupoids. Since we will not need the details of this notion we leave them to the reader. We will come back to the connection between left fibrations over a dendroidal set X and 'X-algebras' in Section 14.8. It turns out that left fibrations are an excellent tool to study the homotopy theory of such algebras.

Recall that a *trivial fibration* of dendroidal sets is a map which has the right lifting property with respect to all normal monomorphisms or, equivalently, with respect to all boundary inclusions of trees $\partial\Omega[T] \to \Omega[T]$. Since all of the fibrations of Definition 6.8 are defined in terms of right lifting properties with respect to certain normal monomorphisms, we immediately obtain the following:

Lemma 6.12 *A trivial fibration of dendroidal sets is in particular a dendroidal Kan fibration (and hence also a left, right, and inner fibration).*

Let us also record the following evident generalization of Lemma 6.2:

Lemma 6.13 *If f is a dendroidal inner (or left, right, Kan, trivial) fibration, then $i^* f$ is an inner (resp. left, right, Kan, trivial) fibration of simplicial sets. Conversely, if g is an inner fibration of simplicial sets, then $i_! g$ is an inner fibration of dendroidal sets. However, this conclusion does not extend to left, right, Kan, and trivial fibrations.*

In Section 5.3 we discussed anodyne morphisms of simplicial sets. The general theory of lifting properties and saturated classes carries over without change; we will now collect the corresponding facts for dendroidal sets. After that we discuss several examples of anodynes which will be useful throughout this book.

Definition 6.14 An *anodyne morphism* of dendroidal sets is a morphism having the left lifting property with respect to all dendroidal Kan fibrations. Similarly, an *inner*, *leaf*, or *root anodyne morphism* is a morphism having the left lifting property with respect to inner, left, or right fibrations, respectively.

Lemma 6.15 *The class of anodyne morphisms is saturated, and similarly for inner, leaf, and root anodyne morphisms.*

The small object argument still applies and yields the following:

Lemma 6.16 *Let $\mathfrak{I}$ be the set of all horn inclusions of trees $\Lambda^x[T] \to \Omega[T]$. Then any morphism $f : X \to Y$ of dendroidal sets can be factored as*

$$X \xrightarrow{i} Z \xrightarrow{p} Y,$$

with i an $\mathfrak{I}$-cellular map and p a dendroidal Kan fibration. The analogous statement is true for inner horns and inner fibrations, as well as for leaf and inner horns and left fibrations, and similarly for right fibrations.

Furthermore, the same arguments used in Section 5.3 give:

Lemma 6.17 *A map $f : A \to B$ of dendroidal sets is anodyne if and only it is a retract of an $\mathfrak{I}$-cellular map, with $\mathfrak{I}$ the set of horn inclusions $\Lambda^x[T] \to \Omega[T]$. It is inner anodyne if and only if it is a retract of a map which is cellular with respect to inner horn inclusions, and similarly for leaf and root anodynes.*

Corollary 6.18 *The class of anodyne maps is the smallest saturated class containing the horn inclusions. The analogous statements apply to the classes of inner, leaf, and root anodynes.*

We will now discuss several examples of anodyne morphisms. The proof of the following lemma is straightforward, but we postpone it until after a discussion of spines of trees in Section 6.5 (see Remark 6.43).

Lemma 6.19 *Consider a tree with at least one vertex. Then the inclusion*

$$\coprod_{l \in \mathrm{leaves}(T)} \eta \to \Omega[T]$$

of the leaves of T *is a leaf anodyne morphism. This includes the case where* T *is closed and hence has no leaves, where one takes the domain of the map above to be empty.*

We will often make use of the following kind of generalized inner horn inclusions:

Lemma 6.20 *Let* T *be a tree and let* E *be a non-empty subset of the set* $I(T)$ *of inner edges. Write* $\Lambda^E[T] \subseteq \Omega[T]$ *for the union of all the faces* $\partial_x \Omega[T]$ *for* x not *in* E*. In other words,* $\Lambda^E[T]$ *is the union of those faces of* T *which contain all the inner edges in the set* E*. Then the inclusion* $\Lambda^E[T] \to \Omega[T]$ *is a composition of pushouts of inner horn inclusions. In particular it is an inner anodyne morphism.*

Proof The proof is by induction on the size of E. If it consists of a single element e, then $\Lambda^E[T] = \Lambda^e[T]$ and the inclusion into $\Omega[T]$ is inner anodyne by definition. For larger E, pick an element $e \in E$ and write $E' = E - \{e\}$. It suffices to prove that each of the two inclusions

$$\Lambda^E[T] \subseteq \Lambda^{E'}[T] \subseteq \Omega[T]$$

is a composition of pushouts of inner horn inclusions. For the second map this is the inductive hypothesis. The first map fits into a pushout square

$$\begin{tikzcd}
\Lambda^{E'}[\partial_e T] \arrow[r] \arrow[d] & \Lambda^E[T] \arrow[d] \\
\Omega[\partial_e T] \arrow[r] & \Lambda^{E'}[T].
\end{tikzcd}$$

Note that the notation in the upper left corner makes sense, because the elements of E' are also inner edges of the tree $\partial_e T$. The left vertical map is a composition of pushouts of inner horns by the inductive hypothesis. □

There is an evident variant of the preceding lemma for leaf anodynes:

Lemma 6.21 *Let* T *be a tree and let* L *be a non-empty subset of the set of leaf vertices of* T*. Write* $\Lambda^L[T] \subseteq \Omega[T]$ *for the union of all the faces of* T except *for the leaf faces corresponding to elements of* L*. Then the inclusion* $\Lambda^L[T] \to \Omega[T]$ *is a composition of pushouts of leaf horn inclusions of trees. In particular it is a leaf anodyne morphism.*

Proof The proof proceeds by induction on the size of L, using the same technique as in the previous lemma. Now one uses that if $L = L' \cup \{v\}$, then there is a pushout square

$$\begin{tikzcd}
\Lambda^{L'}[\partial_v T] \arrow[r] \arrow[d] & \Lambda^L[T] \arrow[d] \\
\Omega[\partial_v T] \arrow[r] & \Lambda^{L'}[T].
\end{tikzcd}$$

□

In practice we will recognize the dendroidal sets of the kind $\Lambda^E[T]$ as follows:

Lemma 6.22 *Consider a dendroidal subset $A \subseteq \Omega[T]$. Suppose A is not equal to $\Omega[T]$ or $\partial\Omega[T]$ and satisfies the following condition: if $F_1, \ldots, F_k$ are elementary faces of T not contained in A, then also*

$$F_1 \cap \cdots \cap F_k \not\subseteq A.$$

Then A is equal to $\Lambda^V[T]$ for some nonempty subset of faces V of T.

Proof Consider a nondegenerate dendrex $\xi : \Omega[S] \to A$. It suffices to show that there is some elementary face $\partial_x T$ of T so that $\Omega[\partial_x T] \subseteq A$ and ξ is contained in $\Omega[\partial_x T]$. Indeed, it would then follow that A is the union of elementary faces. Since it is not equal to $\partial\Omega[T]$, it must be a horn of the form $\Lambda^V[T]$. To establish the existence of a suitable face $\partial_x T$ we reason as follows. Consider the collection of all elementary faces $F_1, \ldots, F_n$ which contain the dendrex ξ. Then ξ is contained in the intersection

$$F_1 \cap \cdots \cap F_k$$

and hence our assumption guarantees that at least one of these faces is contained in A. □

We will conclude this section with a generalization of Proposition 5.25 to dendroidal sets. Consider an operad $\mathbf{P}$ in sets and a collection of colours $c_1, \ldots, c_n, d$ of $\mathbf{P}$. We can freely adjoin a new operation f from $(c_1, \ldots, c_n)$ to d to get a new operad $\mathbf{P}[f]$. To be precise, this operad is the pushout

$$\begin{tikzcd}
\coprod_{i=0,\ldots,n} \eta \arrow[r] \arrow[d] & \mathbf{P} \arrow[d] \\
\Omega(C_n) \arrow[r] & \mathbf{P}[f]
\end{tikzcd}$$

in the category **Op** of operads. The vertical arrow on the left is the inclusion of the colours $0, \ldots, n$ of the operad $\Omega(C_n)$, whereas the top horizontal arrow sends 0 to d and i to c_i for $i \geq 1$. This new operad $\mathbf{P}[f]$ has the same colours as $\mathbf{P}$, but many more operations: the operations in $\mathbf{P}[f]$ are all possible compositions of operations of $\mathbf{P}$ with the new operation f (including possible compositions of f with itself). As was the case with simplicial sets, the corresponding pushout

$$\begin{tikzcd}
\partial\Omega[C_n] \arrow[r] \arrow[d] & N\mathbf{P} \arrow[d] \\
\Omega[C_n] \arrow[r] & (N\mathbf{P})[f]
\end{tikzcd}$$

of dendroidal sets is much simpler to understand: the only 'new' non-degenerate dendrex of $(N\mathbf{P})[f]$ compared to $N\mathbf{P}$ is precisely f. As with simplicial sets, the process of adjoining a new operation in this way is not preserved by the nerve functor, but it is often preserved up to an inner anodyne morphism:

Proposition 6.23 *If* $\mathbf{P}$ *is* Σ*-free, then the canonical morphism* $(N\mathbf{P})[f] \to N(\mathbf{P}[f])$ *is inner anodyne.*

Proof We will generalize the proof of Proposition 5.25, which the reader might wish to refer to for comparison. The operations of the operad $\mathbf{P}[f]$ are generated from f and those of $\mathbf{P}$ by forming compositions and imposing the relations dictated by $\mathbf{P}$. We call an operation of $\mathbf{P}[f]$ *elementary* if it is simply f or when it is in the image of the inclusion $\mathbf{P} \subseteq \mathbf{P}[f]$. We call a dendrex $\xi \in N(\mathbf{P}[f])_T$ *elementary* if for every vertex v of T, the corolla $\xi_v \in N(\mathbf{P}[f])_{C_v}$ corresponds to an elementary operation of $\mathbf{P}[f]$. Then any dendrex of $N(\mathbf{P}[f])$ is an inner face (possibly of high codimension) of some elementary dendrex, because the operations of $\mathbf{P}[f]$ are generated by elementary operations under composition. Every non-degenerate elementary dendrex ξ of $N(\mathbf{P}[f])$ features a certain number of occurrences of the operation f, which we call the *height* of ξ. For $k \geq 1$ write

$$A^{(k)} \subseteq N(\mathbf{P}[f])$$

for the dendroidal subset generated by $(N\mathbf{P})[f]$ together with the non-degenerate elementary dendrices of height at most k. Thus $A^{(1)} = (N\mathbf{P})[f]$ and $\cup_k A^{(k)} = N(\mathbf{P}[f])$, so it suffices to prove that

$$A^{(k)} \to A^{(k+1)}$$

is inner anodyne for every k. We will decompose this inclusion further as

$$A^{(k)} = A_0^{(k+1)} \subseteq A_1^{(k+1)} \subseteq A_2^{(k+1)} \subseteq \cdots, \qquad \bigcup_i A_i^{(k+1)} = A^{(k+1)},$$

where $A_i^{(k)}$ is the dendroidal subset of $N(\mathbf{P}[f])$ generated by $A^{(k)}$ together with the non-degenerate elementary dendrices of height $k+1$ corresponding to trees with at most i vertices. We claim that each of the maps

$$A_{i-1}^{(k+1)} \to A_i^{(k+1)}$$

is inner anodyne. For $i \leq k$ this map is the identity and there is nothing to prove. For $i > k$, consider a non-degenerate elementary dendrex ξ of height $k+1$ and 'size' i. We observe the following:

- Any outer face (leaf or root) of ξ is still elementary, but of smaller size, and hence contained in $A_{i-1}^{(k+1)}$.
- Any inner face contracting an inner edge between two vertices which correspond to operations of $\mathbf{P}$ (i.e. do *not* correspond to f) is still elementary and hence contained in $A_{i-1}^{(k+1)}$.
- Any inner face contracting an inner edge between two vertices of which at least one corresponds to f is no longer elementary and cannot be contained in $A_{i-1}^{(k+1)}$. Also, it cannot be an inner face of another non-degenerate dendrex ξ of height $k+1$ and size i, unless it is isomorphic to ξ. Similar statements apply to any intersection of such inner faces.

These observations, together with Lemma 6.22 and the fact that $\mathbf{P}$ is Σ-free, imply that there is a pushout square

$$\begin{tikzcd}
\coprod_{[\xi]} \Lambda^{E_\xi}[T] \arrow[r] \arrow[d] & A_{i-1}^{(k+1)} \arrow[d] \\
\coprod_{[\xi]} \Omega[T] \arrow[r] & A_i^{(k+1)},
\end{tikzcd}$$

where the coproduct is over isomorphism classes of non-degenerate elementary dendrices ξ of height $k+1$ and size i. Here for each such ξ the set E_ξ is the set of inner edges which connect to at least one vertex corresponding to the operation f. The pushout square above shows that $A_{i-1}^{(k+1)} \to A_i^{(k+1)}$ is inner anodyne and hence $A^{(k)} \to A^{(k+1)}$ is inner anodyne as well. $\square$

6.3 Tensor Products and Anodyne Morphisms

We will now investigate the behaviour of various kinds of anodyne morphisms with respect to tensor products. This section is rather technical, but the results we prove are central to the development of the homotopy theory of dendroidal sets.

Recall from Section 4.1 that the tensor product of two representable dendroidal sets $\Omega[S]$ and $\Omega[T]$ can be described as a union of *shuffles*. According to Proposition 4.8 a shuffle A of S and T is completely determined by its set of edges $E(A)$, which is a subset of the product of sets of edges $E(S) \times E(T)$. The subset $E(A)$ satisfies several conditions, of which the crucial one is that the path from a minimal element (s,t) to the maximal element (r_S, r_T) is a shuffle (in the more classical sense) of the two linear orders corresponding to the paths from s to r_S in the tree S and t to r_T in the tree T. Many of the combinatorial proofs that we carry out involve an induction on the shuffles of a tensor product of trees. For this it is useful to equip this set with an ordering. As already explained in Section 4.1 the set of shuffles of S and T can be partially ordered in such a way that the minimal element is the shuffle consisting of copies of S grafted on top of the leaves of T and the maximal element is the shuffle built by grafting copies of T on top of the leaves of S. The partial order is generated by the relations $A < B$ whenever B is a shuffle obtained from A by 'shuffling a vertex of S down through a vertex of T', or equivalently 'shuffling a vertex of T up through a vertex of S'. We will use this partial ordering (or sometimes its opposite) in our proofs.

As we did when discussing tensor products, we will often abbreviate the notation for representable dendroidal sets from $\Omega[T]$ to T in order to avoid excessive cluttering of notation. The following is the analogue of Lemma 5.26:

Lemma 6.24 *Consider trees S and T and let e be an inner edge of S. Suppose that S or T is linear, or that both S and T are open. Then the pushout-product*

$$\Lambda^e[S] \otimes T \cup S \otimes \partial T \to S \otimes T$$

is a composition of pushouts of inner horn inclusions. In particular it is inner anodyne.

The proof of this lemma requires some rather elaborate combinatorics. The reader might wish to refer back to the simpler proof of Lemma 5.26 while working through it, also noting that Lemma 5.26 is a special case of the result we shall prove shortly. The proof will make use of the following notion:

Definition 6.25 A *pruning* of a tree T is a subtree $P \subseteq T$ whose root coincides with that of T. In other words, P is obtained from T by iteratively chopping off leaf corollas.

Proof (of Lemma 6.24) The assumptions on S and T guarantee that the map under consideration is a normal monomorphism, using Propositions 4.21 and 4.26. Let us first treat the case where S and T are open. The other cases are similar and we discuss the necessary modifications at the end of the proof. Write v_e for the top vertex of the edge e in S. In other words, v_e is the unique vertex which has e as its outgoing edge. If R is a shuffle of S and T, it will contain one or several vertices of the form $v_e \otimes t$. Such a vertex has as its outgoing edge $e \otimes t$ and we will call these *special edges*. In other words, special edges are the highest occurrences of edges of the form $e \otimes t$, for t ranging over the edges of T and highest meaning furthest from the root.

We will construct a rather elaborate filtration of the map of the lemma. To begin with, write

$$A_0 := \Lambda^e[S] \otimes T \cup S \otimes \partial T.$$

We will think of A_0 as a subobject of $S \otimes T$, which is justified by the first sentence of this proof. Arbitrarily choose a linear ordering on the set of shuffles of S and T which extends the partial order described before the statement of the lemma. Adjoin these shuffles one by one in the chosen order to obtain a filtration

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots, \qquad \bigcup_i A_i = S \otimes T.$$

Thus, A_i is the union of A_0 with the first i shuffles. We will show that each of the inclusions $A_i \subseteq A_{i+1}$ is a composition of pushouts of inner horn inclusions. Say A_{i+1} is obtained from A_i by adjoining a shuffle R. Define a further filtration

$$A_i =: A_i^0 \subseteq A_i^1 \subseteq A_i^2 \subseteq \cdots, \qquad \bigcup_j A_i^j = A_{i+1}$$

by adjoining the prunings of R one by one, in an order which extends the partial ordering of inclusions of prunings. Consider a step $A_i^j \subseteq A_i^{j+1}$ in this filtration given by adjoining some pruning P. Denote by Σ_P the collection of inner edges of P which are also special edges. Without loss of generality we may assume that Σ_P is non-empty. Indeed, if it were empty then either P contains no special edges at all or they are leaf edges. In both cases P cannot contain any vertex of the form $v_e \otimes t$, which implies P factors through $e/S \otimes T$ (where e/S is the tree obtained from S by chopping off everything above e). But then P is also contained in $\Lambda^e[S] \otimes T$ and the inclusion $A_i^j \subseteq A_i^{j+1}$ is the identity.

Recall that $I(P)$ denotes the set of inner edges of P and write

$$J(P) := I(P) - \Sigma_P.$$

For each subset $H \subseteq J(P)$, write P^H for the tree obtained from P by contracting all edges in the complement $J(P) - H$. In particular, $P^{J(P)} = P$. Pick a linear order on the collection of subsets of $J(P)$ which extends the partial order of inclusion and adjoin the trees P^H to A_i^j in this order to obtain a further filtration

$$A_i^j =: A_i^{j,0} \subseteq A_i^{j,1} \subseteq A_i^{j,2} \subseteq \cdots, \qquad \bigcup_k A_i^{j,k} = A_i^j \cup P = A_i^{j+1}.$$

Finally we will now argue that each inclusion $A_i^{j,k} \subseteq A_i^{j,k+1}$ is a composition of pushouts of inner horn inclusions. The inclusion under consideration is given by adjoining some tree of the form P^H. If the map

$$P^H \longrightarrow S \otimes T$$

factors through $A_i^{j,k}$ then this inclusion is the identity and there is nothing to prove. If it does not, we can say the following:

(1) Any outer face of P^H chopping off a leaf corolla factors through a smaller pruning and is therefore contained in $A_i^{j,k}$ by our induction on the size of prunings.
(2) The root face of P^H, if it exists, is contained in A_0. Indeed, it deletes all occurrences of the root vertex of either S or T in P.
(3) Any inner face of P^H which contracts an inner edge f which is not special is contained in $P^{H'}$ with $H' = H - \{f\}$ and is therefore contained in $A_i^{j,k}$ by our induction on the size of H.
(4) Finally, any inner face Q of P^H contracting a special edge $e \otimes t$ (or multiple such special edges) *cannot* be contained in $A_i^{j,k}$. We will prove this in (a)–(c) below.

It follows from these four observations and Lemma 6.22 that $A_i^{j,k} \subseteq A_i^{j,k+1}$ is a pushout of the map

$$\Lambda^{\Sigma_P}[P^H] \to P^H,$$

which is a composition of pushouts of inner horn inclusions by Lemma 6.20 as desired. It remains to verify the following:

(a) The face Q cannot be contained in $\Lambda^e[S] \otimes T$. If it were, then Q would be contained in $\partial_x S \otimes T$ for some face $\partial_x S$ of S other than the inner face $\partial_e S$. Since Q is obtained from P^H by contracting one or several edges of the form $e \otimes t$, this would imply that P^H itself is contained in $\partial_x S \otimes T$. This is a contradiction, because we assumed that P^H is not contained in A_0.

(b) The face Q cannot be contained in $S \otimes \partial T$. Indeed, suppose Q is contained in $S \otimes \partial_x T$ for some face $\partial_x T$ of T. If $\partial_x T$ is the root face (assuming it exists) then P^H itself must be contained in $S \otimes \partial_x T$, giving a contradiction. If $\partial_x T$ is a leaf face chopping off some leaf vertex w, then Q cannot contain any vertices of the form $s \otimes w$ or vertices arising from such by contracting inner edges in P. But then since S is an open tree, such vertices also cannot occur in P^H. It follows that P^H is also contained in $S \otimes \partial_x T$, which is a contradiction. Finally, consider the case where $\partial_x T$ is an inner face of T contracting some inner edge t. Write w for the top vertex of t. The tree Q cannot contain any edges of the form $s \otimes t$ for s ranging through the edges of S. Hence the only possible occurrences of edges $s \otimes t$ in P^H are special edges (which have been contracted to form Q). It follows that $s = e$. The edge $e \otimes t$ in P has top vertex $v_e \otimes t$. All the edges in P of the form $s \otimes t$ *above* $v_e \otimes t$ must have been contracted to form P^H (since P^H and Q 'look the same' above the special edges). But then P^H factors through an earlier shuffle, in which the vertex w is shuffled down all the way through v_e. In particular P^H is contained in A_i, a contradiction.

(c) By (a) and (b) it follows that Q is not contained in A_0. Also it cannot factor through an earlier shuffle because of how special edges are defined, in particular the fact that their top vertex is of the form $v_e \otimes t$, 'a vertex of S'. Indeed, the tree P^H is not contained in an earlier shuffle by assumption, and contracting special edges in P^H can only create overlap with shuffles in which v_e has been shuffled down. Thus Q is not contained in A_i. Given this, it is clear that Q cannot be contained in $A_i^{j'}$ for $j' \leq j$ because of the size of the pruning P under consideration, and also it cannot be contained in $A_i^{j,k'}$ for $k' \leq k$: if it were, then P^H is evidently contained in $A_i^{j,k'}$ as well, which is a contradiction again.

This completes our proof under the assumption that S and T are open trees. The assumption that T is open was only used to ensure that the map under consideration is a normal monomorphism. It is also a normal monomorphism if S is linear, without further assumptions on T, and the proof above still applies.

For the case where S is a general tree and T is linear some modifications are needed. In this case we define a special edge of a shuffle R to be the *lowest* occurrence of an edge of the form $e \otimes t$. Since T is linear, every shuffle now contains precisely *one* such edge (contrary to the more general case considered above). In other words, if u_e

denotes the *bottom* vertex attached to e, the special edge is the edge of R occurring immediately above the unique vertex of the form $u_e \otimes t$. Also, we reverse the order of the shuffles of S and T: the first one is now the shuffle where one grafts copies of T onto the leaves of S and the last one is the shuffle obtained by grafting S onto the (unique) leaf of T. In this ordering we have $R_1 < R_2$ when R_2 is obtained from R_1 by shuffling down a vertex of T, or equivalently shuffling up a vertex of S. The argument proceeds as in the other cases above, but one replaces item (b) above with the following:

(b') The face Q cannot be contained in $S \otimes \partial T$. Indeed, suppose Q is contained in $S \otimes \partial_x T$ for some face $\partial_x T$. It cannot be the root face of T by the same argument as in (b). Since T is linear, the face $\partial_x T$ is uniquely determined by the unique edge t it misses. Write w for the bottom vertex of t (which exists because t is not the root of T). So Q does not contain any edges of the form $s \otimes t$ (for any edge s of S). But P^H must contain at least one such edge, or it would automatically be contained in $S \otimes \partial_x T$ as well, using the fact that T is linear. It follows that Q is obtained from P^H by contracting the special edge $e \otimes t$. Since there are no other occurrences of t in P^H, in particular not below the special edge $e \otimes t$, it follows that P^H must in fact be contained in an earlier shuffle in which w has been shuffled up all the way through u_e (or u_e has been shuffled down through w). This is a contradiction. □

As before we use the standard arguments on saturated classes to conclude:

Corollary 6.26 *For an inner anodyne map $i : A \to B$ and a normal monomorphism $j : M \to N$ of dendroidal sets, the pushout-product*

$$A \otimes N \cup_{A \otimes M} B \otimes M \to B \otimes N$$

is inner anodyne, provided that all dendroidal sets involved are open or that either B or N is simplicial.

We will also need some variations on the preceding result concerning root and leaf anodynes. Let us state these results and their consequences now; the remainder of this section will consist of their proofs. The following terminology will be useful: a *unary* root horn is a root horn of a tree S whose root vertex is unary, i.e., has only one input edge. The class of *unary root anodyne morphisms* is the smallest saturated class containing the inner horn inclusions and the unary root horn inclusions of trees. These unary root horn inclusions enjoy the following pushout-product property:

Lemma 6.27 *Consider a tree S with unary root vertex and another tree T. Let $K \to S$ be a root horn inclusion (of one of the two kinds described at the start of Section 6.1). Suppose that S or T is linear, or that both S and T are open. Then the pushout-product*

$$K \otimes T \cup S \otimes \partial T \to S \otimes T$$

is a unary root anodyne map.

Remark 6.28 The case of a general root horn is somewhat more complicated. It features in Section 9.7, specifically Lemmas 9.90 and 9.91.

Leaf horn inclusions satisfy an analogous property:

Lemma 6.29 *Consider trees S and T and a leaf horn inclusion $L \to S$ (of one of the two kinds described at the start of Section 6.1). Suppose that S or T is linear, or that both S and T are open. Then the pushout-product*

$$L \otimes T \cup S \otimes \partial T \to S \otimes T$$

is a leaf anodyne map.

The previous two lemmas imply the following for the corresponding saturated classes:

Corollary 6.30 *For a unary root anodyne map or leaf anodyne map $i : A \to B$ and a normal monomorphism $j : M \to N$ of dendroidal sets, the pushout-product*

$$A \otimes N \cup_{A \otimes M} B \otimes M \to B \otimes N$$

is unary root anodyne or leaf anodyne respectively, provided that all dendroidal sets involved are open or that either B or N is simplicial.

Although the statements of Lemmas 6.27 and 6.29 appear 'dual' (at least for the case of unary leaf and root vertices), as in the duality between left and right anodynes of simplicial sets, they do not admit a common proof. As we mentioned before, the involution of the category $\mathbf{\Delta}$ that reverses linear orders does not extend to $\mathbf{\Omega}$ and hence there is no formal way to turn statements about leaf anodynes into statements about root anodynes and vice versa. In fact, there will be some subtle differences in the proofs of these lemmas, although the general approach is of course very similar.

Proof (of Lemma 6.27) Our strategy is similar to the proof of Lemma 6.24. First of all, the map under consideration is a normal monomorphism by Propositions 4.21 and 4.26. As before, we begin with the case where S and T are open and indicate the modifications for the other cases at the end of the proof. We will write r for the root edge of S and v_r for the root vertex, i.e., the unique vertex having r as its outgoing edge.

Again we will consider a filtration

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots, \qquad \bigcup_i A_i = S \otimes T,$$

where $A_0 := K \otimes T \cup S \otimes \partial T$. We think of A_0 as a subobject of $S \otimes T$. Choose a linear order on the set of shuffles of S and T which extends the partial order described before, in which the minimal element is given by grafting copies of S onto the leaves of T. Then A_i is the union of A_0 with the first i shuffles in this linear order. We will show that each of the inclusions $A_i \subseteq A_{i+1}$ is a composition of pushouts of inner horn and root horn inclusions. Say A_{i+1} is obtained from A_i by adjoining a shuffle R. We distinguish two cases:

Case 1. The root vertex of the shuffle R is *not* of the form $v_r \otimes t$ for some edge t of T. In this case we will show that $A_i \subseteq A_{i+1}$ is a composition of pushouts of inner horn inclusions.

Case 2. The root vertex of the shuffle R *is* of the form $v_r \otimes t$. In this case we will show that $A_i \subseteq A_{i+1}$ is a composition of pushouts of unary root horn inclusions.

Case 1. The shuffle R will contain one or several vertices of the form $v_r \otimes t$, none of which are root vertices. We will call the outgoing edges $r \otimes t$ of these vertices the *special edges* of R. Note that these special edges are in particular inner edges, since the $v_r \otimes t$ are not root vertices. Define a further filtration

$$A_i =: A_i^0 \subseteq A_i^1 \subseteq A_i^2 \subseteq \cdots, \qquad \bigcup_j A_i^j = A_{i+1}$$

by adjoining the prunings of R one by one, in an order which extends the partial ordering of inclusion. Say $A_i^j \subseteq A_i^{j+1}$ is given by adjoining some pruning P. Denote by Σ_P the collection of inner edges which are also special edges. Without loss of generality we may assume that Σ_P is non-empty. Indeed, if it were empty then either P contains no special edges at all or they are leaf edges. In both cases P cannot contain any vertex of the form $v_r \otimes t$, which implies P factors through $\eta_r \otimes T$ (where η_r denotes the root edge of T). But then P is also contained in $K \otimes T$ and the inclusion $A_i^j \subseteq A_i^{j+1}$ is the identity.

As in the proof of Lemma 6.24, write $J(P) = I(P) - \Sigma_P$ and for each subset $H \subseteq J(P)$ write P^H for the tree obtained from P by contracting all edges in $J(P) - H$. Pick a linear order on the collection of subsets of $J(P)$ which extends the partial order of inclusion and adjoin the trees P^H to A_i^j in this order to obtain a further filtration

$$A_i^j =: A_i^{j,0} \subseteq A_i^{j,1} \subseteq A_i^{j,2} \subseteq \cdots, \qquad \bigcup_k A_i^{j,k} = A_i^j \cup P = A_i^{j+1}.$$

Finally we will now argue that each inclusion $A_i^{j,k} \subseteq A_i^{j,k+1}$ is a composition of pushouts of inner horn inclusions. The inclusion under consideration is given by adjoining some tree of the form P^H. If the map

$$P^H \longrightarrow S \otimes T$$

factors through $A_i^{j,k}$ then this inclusion is the identity and there is nothing to prove. If it does not, we can say the following:

(1) Any outer face of P^H chopping off a leaf corolla factors through a smaller pruning and is therefore contained in $A_i^{j,k}$ by our induction on the size of prunings.

(2) The root face of P^H, if it exists, is contained in A_0. Indeed, it deletes the root vertex v_T of T (or more precisely, a vertex of the form $s \otimes v_T$) and therefore factors through $S \otimes \partial T \subseteq A_0$.

(3) Any inner face of P^H which contracts an inner edge f which is not special is contained in $P^{H'}$ with $H' = H - \{f\}$ and is therefore contained in $A_i^{j,k}$ by our induction on the size of H.
(4) Finally, any inner face Q of P^H contracting a special edge $r \otimes t$ (or multiple such special edges) *cannot* be contained in $A_i^{j,k}$. This follows by an argument completely analogous to that in the proof of 6.24, using items (a)–(c) there.

It follows from these four observations and Lemma 6.22 that $A_i^{j,k} \subseteq A_i^{j,k+1}$ is a pushout of the map

$$\Lambda^{\Sigma_P}[P^H] \to P^H,$$

which is a composition of pushouts of inner horn inclusions by Lemma 6.20.

Case 2. The root vertex of R is of the form $v_r \otimes t$. Define A_i^j as before, adjoining prunings of R to A_i one by one. Say $A_i^j \subseteq A_i^{j+1}$ is given by adjoining a pruning P. This time we consider all possible subsets $H \subseteq I(P)$ and the corresponding trees P^H obtained by contracting the all edges in $I(P) - H$. Adjoining these to A_i^j in an order extending inclusion of subsets H gives a further filtration

$$A_i^j =: A_i^{j,0} \subseteq A_i^{j,1} \subseteq A_i^{j,2} \subseteq \cdots, \qquad \bigcup_k A_i^{j,k} = A_i^j \cup P = A_i^{j+1}.$$

Consider a map $A_i^{j,k} \subseteq A_i^{j,k+1}$ adjoining some tree of the form P^H. If P^H is already contained in $A_i^{j,k}$ then this map is the identity. Note that this is in particular the case if the (unique) incoming edge $e \otimes t$ of the root vertex $v_r \otimes t$ does not occur in P^H: indeed, then P^H either factors through $\partial_e S \otimes T$ (if $e \otimes t$ connects two vertices coming from S), or the Boardman–Vogt relation implies that P^H factors through an earlier shuffle in which the vertex v_r of S has been shuffled up (if $e \otimes t$ connects $v_r \otimes t$ to a vertex coming from T). Without loss of generality we will thus assume that P^H is not contained in $A_i^{j,k}$ and its root vertex is of the form $v_r \otimes t$, with incoming edge $e \otimes t$. We will argue that $A_i^{j,k} \subseteq A_i^{j,k+1}$ is a pushout of the root horn inclusion of P^H. Indeed:

(1) Any outer face of P^H chopping off a leaf corolla factors through a smaller pruning and is therefore contained in $A_i^{j,k}$.
(2) Any inner face of P^H factors through some $P^{H'}$ for H' a proper subset of H and is therefore contained in $A_i^{j,k}$.
(3) The root face of P^H, chopping off the unary root corolla with vertex $v_r \otimes t$, cannot be contained in $A_i^{j,k}$. Indeed, it cannot factor through $K \otimes T$ or $S \otimes \partial T$ or through an earlier shuffle. Also, it cannot factor through $A_i^{j'}$ for $j' \leq j$ because of the size of the pruning P under consideration, or through $A_i^{j,k'}$ for $k' \leq k$ by the definition of the trees P^H.

It follows that $A_i^{j,k} \subseteq A_i^{j,k+1}$ is a pushout of the unary root horn inclusion

$$\Lambda^{\text{root}}[P^H] \to \Omega[P^H].$$

This completes the proof of the lemma under the assumption that S and T are open trees. In fact, the given proof also covers the case where T is a general tree (we never used that it is open) and S is linear.

We will now deduce the case where S is a general tree (still with unary root vertex of course) and T is linear from what we have already done. First of all, we claim that the root horn inclusion $K \to S$ is a retract of the pushout-product map

$$f : i[1] \otimes K \cup \eta_1 \otimes S \to i[1] \otimes S,$$

where η_1 denotes the root edge of the linear tree $i[1]$. To see this, consider the evident inclusion

$$j : S \cong \eta_0 \otimes S \to i[1] \otimes S.$$

It admits a retraction

$$p : i[1] \otimes S \to S$$

which is characterized by its effect on edges as follows: it sends an edge of colour $k \otimes s$ to the edge s, except when $k = 1$ and s is the incoming edge of the root vertex v_r, in which case $p(k \otimes s) = r$, the root of S. It is easily verified that the map p is well-defined and indeed gives the desired retraction. It now suffices to check that the pushout-product of f with the boundary inclusion of T is unary root anodyne:

$$(i[1] \otimes K \cup \eta_1 \otimes S) \otimes T \cup (i[1] \otimes S) \otimes \partial T \to (i[1] \otimes S) \otimes T.$$

Using Corollary 4.32 and the fact that T is linear, this map can be rewritten (up to isomorphism) as

$$i[1] \otimes (K \otimes T \cup S \otimes \partial T) \cup \eta_1 \otimes (S \otimes T) \to i[1] \otimes (S \otimes T).$$

This is the pushout-product of $\eta_1 \to i[1]$ (which is a unary root horn inclusion) with the map $K \otimes T \cup S \otimes \partial T \to S \otimes T$, which is a normal monomorphism of dendroidal sets. This map is unary root anodyne, because we have already covered the case of a pushout-product of a unary root anodyne between simplicial sets and a normal monomorphism of dendroidal sets. □

Proof (of Lemma 6.29) As before, the map of the lemma is a normal monomorphism by Propositions 4.21 and 4.26. We will write v_ℓ for the leaf vertex of S corresponding to the 'missing face' of the leaf horn $L \subset S$. Also, we write e for the outgoing edge of v_ℓ and $\ell_1, \ldots, \ell_n$ for its incoming edges, which are leaves of S. In this proof we will treat all cases simultaneously: S and T open, or one of them simplicial. Essentially the same argument will work for all of these, apart from a few small distinctions depending on whether the leaf vertex v_ℓ of S has leaves or is a stump (meaning it has no leaves).

As usual, we define a filtration

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots, \qquad \bigcup_i A_i = S \otimes T,$$

with $A_0 := K \otimes T \cup S \otimes \partial T$ and A_i the union of A_0 with the first i shuffles of $S \otimes T$. However, for this proof we *reverse* the ordering on these shuffles, so that the first one is given by grafting copies of T onto the leaves of S. Thus, one progresses through this ordering by shuffling vertices of S *up* through vertices of T. We will show that each of the inclusions $A_i \subseteq A_{i+1}$ is a composition of pushouts of inner horn and leaf horn inclusions. Say A_{i+1} is obtained from A_i by adjoining a shuffle R. We distinguish two cases:

Case 1. At least one of the vertices of the form $v_\ell \otimes t$ occurring in R (with t some edge of T) is *not* a leaf vertex of R (which is only possible if v_ℓ is not a stump of S).

Case 2. All vertices of the form $v_\ell \otimes t$ occurring in R are leaf vertices of R.

Case 1. Define a further filtration

$$A_i =: A_i^0 \subseteq A_i^1 \subseteq A_i^2 \subseteq \cdots, \qquad \bigcup_j A_i^j = A_{i+1}$$

by adjoining the prunings of R one by one, in an order extending the partial order of inclusion of prunings. Consider an inclusion $A_i^j \subseteq A_i^{j+1}$ adjoining some pruning P. Without loss of generality, P still contains one or several vertices of the form $v_\ell \otimes t$. Indeed if it does not, then all corollas of R occurring *above* such vertices have been chopped off as well and P is already contained $S \otimes \partial T \subseteq A_0$, so there is nothing to prove. Now consider the incoming edges $\ell_1 \otimes t, \ldots, \ell_n \otimes t$ of the vertices $v_\ell \otimes t$ still occurring in P. At least some of these have to be inner edges of P; if not, $v_\ell \otimes t$ would be a leaf vertex of P and again P would be contained in $S \otimes \partial T$. We refer to such inner edges as *special edges* of P and write Σ_P for the collection of special edges. As before, we define $J(P) = I(P) - \Sigma_P$ and for any subset $H \subseteq J(P)$ we write P^H for the tree obtained by contracting all inner edges of P contained in the complement $J(P) - H$. Adjoining the P^H in an order extending the partial order of inclusion of subsets H gives a filtration

$$A_i^j =: A_i^{j,0} \subseteq A_i^{j,1} \subseteq A_i^{j,2} \subseteq \cdots, \qquad \bigcup_k A_i^{j,k} = A_i^j \cup P = A_i^{j+1}.$$

Consider an inclusion $A_i^{j,k} \subseteq A_i^{j,k+1}$ adjoining a tree P^H. If that tree is already contained in $A_i^{j,k}$ there is nothing to prove. If it is not, we observe:

(1) Any outer face of P^H chopping off a leaf corolla factors through a smaller pruning and is therefore contained in $A_i^{j,k}$.
(2) The root face of P^H, if it exists, is contained in A_0. Indeed, it deletes the root vertex v_T of T (or more precisely, a vertex of the form $s \otimes v_T$) and therefore factors through $S \otimes \partial T \subseteq A_0$.
(3) Any inner face of P^H which contracts an inner edge f which is not special is contained in $P^{H'}$ with $H' = H - \{f\}$ and is therefore contained in $A_i^{j,k}$ by our induction on the size of H.

(4) Finally, any inner face Q of P^H contracting a special edge $\ell_k \otimes t$ (or multiple such special edges) *cannot* be contained in $A_i^{j,k}$. This follows by an argument completely analogous to that in the proof of 6.24, using items (a)–(c) there. For (b) some care is required; we are no longer assuming S is open. However, the argument only needs that no stumps of S occur above v_ℓ, which is true in the current situation.

As before we conclude that $A_i^{j,k} \subseteq A_i^{j,k+1}$ is a pushout of the map

$$\Lambda^{\Sigma_P}[P^H] \to P^H,$$

which is inner anodyne.

Case 2. Each of the vertices of the form $v_\ell \otimes t$ occurring in R is a leaf vertex. For this case we introduce a variation on the notion of pruning. An *ℓ-pruning* of R is a pruning P of R satisfying the following property: if there is a vertex $v_\ell \otimes t$ of R whose outgoing edge $e \otimes t$ is contained in P, then $v_\ell \otimes t$ itself is contained in P. This time we define a filtration

$$A_i =: A_i^0 \subseteq A_i^1 \subseteq A_i^2 \subseteq \cdots, \qquad \bigcup_j A_i^j = A_{i+1}$$

by adjoining the ℓ-prunings of R one by one in an order compatible with the partial order of inclusion of prunings. Consider an inclusion $A_i^j \subseteq A_i^{j+1}$ adjoining some ℓ-pruning P. Again, we consider subsets $H \subseteq I(P)$ and the corresponding subtrees P^H contracting the inner edges of P not contained in H. Adjoining these in an appropriate order yields $A_i^{j,k}$ as before. Say a step $A_i^{j,k} \subseteq A_i^{j,k+1}$ in this filtration adjoins a tree P^H. If this tree is contained in $A_i^{j,k}$ there is nothing to prove. Note that this is in particular the case if H does not contain any outgoing edges $e \otimes t$ of a leaf vertex $v_\ell \otimes t$. Indeed, if all such edges are contracted, the resulting tree either factors through $\partial_e S \otimes T$ or through an earlier shuffle of $S \otimes T$ in which such a vertex $v_\ell \otimes t$ has been shuffled down. Without loss of generality we will assume that P^H is not contained in $A_i^{j,k}$, so that it contains several edges of the form $e \otimes t$ and leaf vertices $v_\ell \otimes t$ attached to the top of those. Write L for the corresponding collection of leaf corollas. We observe:

(1) Any outer face of P^H chopping off a leaf corolla *not* of the form $v_\ell \otimes t$ factors through a smaller ℓ-pruning and is therefore contained in $A_i^{j,k}$.
(2) The root face of P^H, if it exists, is contained in A_0. Indeed, it deletes the root vertex v_T of T (or more precisely, a vertex of the form $s \otimes v_T$) and therefore factors through $S \otimes \partial T \subseteq A_0$.
(3) Any inner face of P^H factors through some $P^{H'}$ for H' a proper subset of H and is therefore contained in $A_i^{j,k}$.
(4) Any face chopping off one of the leaf corollas in L (or any intersection of such faces) cannot be contained in $A_i^{j,k}$. Indeed, it cannot factor through $K \otimes T$ or $S \otimes \partial T$ or through an earlier shuffle. Also, it cannot factor through $A_i^{j'}$ for $j' \leq j$

because of the size of the ℓ-pruning P under consideration and the fact that chopping off a leaf corolla $v_\ell \otimes t$ yields a tree which is *not* an ℓ-pruning. Finally, it cannot factor through $A_i^{j,k'}$ for $k' \leq k$ by the definition of the trees P^H.

It follows that $A_i^{j,k} \subseteq A_i^{j,k+1}$ is a pushout of the horn inclusion

$$\Lambda^L[P^H] \to \Omega[P^H],$$

which is leaf anodyne by Lemma 6.21. $\square$

Remark 6.31 The following is analogous to Remark 5.27. The proof of Lemma 6.29 gives a little more. If the leaf horn in the statement of the lemma concerns a *unary* leaf vertex v (i.e. a vertex having only one leaf), then the leaf horns used in the proof will also concern unary leaf vertices. Moreover, those leaf vertices will be mapped to corollas of the tensor product $S \otimes T$ which are of the form $v \otimes t$, with t an edge in T. An analogous comment applies to the proof of Lemma 6.27 and a unary root vertex of S.

We end this section with some further discussion of the associator morphisms

$$\alpha\colon X_1 \otimes \cdots \otimes X_{i-1} \otimes (X_i \otimes \cdots \otimes X_j) \otimes X_{j+1} \otimes \cdots \otimes X_n \to X_1 \otimes \cdots \otimes X_n$$

for the tensor product of dendroidal sets discussed in Section 4.4. In Corollary 4.32 we observed that these maps are normal monomorphisms whenever the dendroidal sets involved are normal. Much more is true; these maps are in fact inner anodyne. We will now prove this in the case of corollas, since this is all we will need, but Remark 9.48 gives a statement that applies more generally.

Proposition 6.32 *For corollas $C_{k_1}, \ldots, C_{k_n}$, the associator*

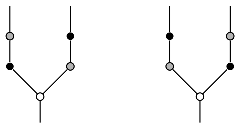
$$\alpha\colon C_{k_1} \otimes \cdots \otimes C_{k_{i-1}} \otimes (C_{k_i} \otimes \cdots \otimes C_{k_j}) \otimes C_{k_{j+1}} \otimes \cdots \otimes C_{k_n} \to C_{k_1} \otimes \cdots \otimes C_{k_n}$$

is inner anodyne.

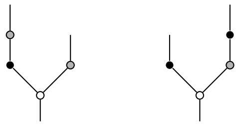
Let us first illustrate the proof in the case of Examples 4.27 and 4.33, which describe the associator

$$C_2 \otimes (C_1 \otimes C_1) \to C_2 \otimes C_1 \otimes C_1.$$

The codomain $C_2 \otimes C_1 \otimes C_1$ has two shuffles R_1 and R_2



which are not already contained in the domain $C_2 \otimes (C_1 \otimes C_1)$. Consider the one on the left and the following two prunings of it:



Write E for the two incoming edges of the 2-corolla at the root. Then each of these two prunings P can be adjoined via a pushout along the inner anodyne $\Lambda^E[P] \to P$. Subsequently the shuffle R_1 itself can be adjoined via a pushout along $\Lambda^E[R_1] \to R_1$. The other shuffle R_2 is treated analogously.

Proof (of Proposition 6.32) Write Σ for the set of shuffles of the n corollas $C_{k_1}, \dots, C_{k_n}$, so that the tensor product $C_{k_1} \otimes \cdots \otimes C_{k_n}$ is the union of the shuffles in Σ. The domain of α

$$C_{k_1} \otimes \cdots \otimes C_{k_{i-1}} \otimes (C_{k_i} \otimes \cdots \otimes C_{k_j}) \otimes C_{k_{j+1}} \otimes \cdots \otimes C_{k_n}$$

is a union of some subset of these shuffles. Write A_0 for this, regarded as a subobject of $C_{k_1} \otimes \cdots \otimes C_{k_n}$. For every shuffle $S \in \Sigma$ we write V_S for the set of vertices of S corresponding to vertices of the corollas $C_{k_1}, \dots, C_{k_{i-1}}, C_{k_{j+1}}, \dots, C_{k_n}$. We give Σ a partial order where $S_1 < S_2$ whenever S_2 is obtained from S_1 by shuffling vertices of V_{S_1} down, i.e., towards the root. Adjoining the shuffles in Σ to A_0 in an order that respects this partial order defines a filtration

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_N = C_{k_1} \otimes \cdots \otimes C_{k_n}$$

where A_i is obtained from A_{i-1} by adjoining the ith shuffle. Consider an inclusion $A_i \subseteq A_{i+1}$, adjoining a shuffle S. Define a *V-pruning* to be a subtree P of S (i.e. a tree obtained by iteratively chopping off leaf corollas and root corollas) satisfying the following condition: if P contains any vertex of S occurring above some vertex $v \in V_S$, then P also contains v. Adjoining such V-prunings in an order compatible with the partial order of inclusion of prunings defines a further filtration

$$A_i =: A_i^0 \subseteq A_i^1 \subseteq \cdots \subseteq \bigcup_j A_i^j = A_{i+1}.$$

Note that any V-pruning in which all the vertices above those of V_S have been removed (i.e. in which the vertices of V_S are leaf vertices) is a face of the shuffle of $C_{k_1} \otimes \cdots \otimes C_{k_n}$ where all the vertices corresponding to $C_{k_i}, \dots, C_{k_j}$ are below those of V_S. Such a shuffle is contained in A_0. Now consider an inclusion $A_i^j \subseteq A_i^{j+1}$ adjoining a V-pruning P. Call an inner edge of P *special* if it is an input edge of a vertex in V_S. Write $\mathcal{E}$ for the collection of special edges. Without loss of generality we may assume it to be non-empty; if it were empty, P would contain no vertices above V_S and is already contained in A_0. As usual (by now), consider trees P^H obtained from P by contracting all inner edges that are not special and not contained in some specified subset H of $I(P) - \mathcal{E}$. Adjoining the P^H in an order that extends the partial order of inclusion of subsets H, we obtain a refinement

$$A_i^j =: A_i^{j,0} \subseteq A_i^{j,1} \subseteq \cdots \subseteq \bigcup_k A_i^{j,k} = A_i^{j+1}.$$

Consider one of the inclusions $A_i^{j,k} \subseteq A_i^{j,k+1}$ adjoining a tree P^H. If P^H is already contained in $A_i^{j,k}$ there is nothing to prove. Otherwise:

(1) Any inner face of P^H contracting a special edge, or some collection of special edges, is not contained in $A_i^{j,k}$. Indeed, P^H is not contained in $A_i^{j,k}$ because the trees on top of the various vertices of V do not all stem from the same shuffle of $C_{k_i} \otimes \cdots \otimes C_{k_j}$. Contracting several special edges does not change this fact.
(2) Any inner face of P^H contracting a non-special edge is contained in $A_i^{j,k}$ by our induction on H.
(3) Any outer face of P^H is contained in A_i^j by our induction on the size of V-prunings.

We conclude that $A_i^{j,k} \subseteq A_i^{j,k+1}$ is a pushout of the inclusion $\Lambda^{\mathcal{E}}[P] \to P$ and hence inner anodyne. □

6.4 Fibrations Between Mapping Spaces of Dendroidal Sets

Recall that in Section 4.1 we introduced, for dendroidal sets A and X, the 'mapping space' $\mathbf{hom}(A, X)$. It is a simplicial set characterized by a natural isomorphism

$$\mathbf{sSets}(\Delta[n], \mathbf{hom}(A, X)) \cong \mathbf{dSets}(A \otimes i_!\Delta[n], X).$$

In other words, it is the underlying simplicial set of the 'internal hom' between A and X in the category of dendroidal sets. We put this term in quotation marks only to remind the reader that the tensor product does not make **dSets** a symmetric monoidal category, so that these mapping spaces do not automatically come equipped with a natural notion of composition. Nonetheless they will be very useful.

The following result parallels Theorem 5.36 for simplicial sets:

Theorem 6.33 *Consider two morphisms of dendroidal sets $f : Y \to X$ and $i : A \to B$, together giving a map of simplicial sets*

$$p : \mathbf{hom}(B, Y) \to \mathbf{hom}(B, X) \times_{\mathbf{hom}(A,X)} \mathbf{hom}(A, Y).$$

Suppose f is a fibration of one of the five types considered (Kan, inner, left, right, or trivial).

(i) *If i is a normal monomorphism, then p is again a fibration of the same type.*
(ii) *If i is an anodyne morphism corresponding to the type of the fibration f, then p is a trivial fibration. Here inner anodynes correspond to inner fibrations, leaf anodynes to left fibrations, etc. The case where f is a trivial fibration is already covered by (i), where i is allowed to be any normal monomorphism.*

Proof The proofs all proceed by using the analogue of Lemma 5.35 for the adjunction between $\mathbf{hom}(-, -)$ and the tensor product, together with the closure properties of various classes of anodynes stated in Corollaries 6.26 and 6.30. For example, suppose f is a left fibration and i a normal monomorphism. To check that p is a left fibration we should argue the existence of a lift in a square of the following kind:

$$\begin{tikzcd} \Lambda^0[n] \arrow[r] \arrow[d] & \mathbf{hom}(B, Y) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathbf{hom}(B, X) \times_{\mathbf{hom}(A,X)} \mathbf{hom}(A, Y). \end{tikzcd}$$

By adjunction this lifting problem corresponds to the following:

$$\begin{tikzcd} B \otimes i_!\Lambda^0[n] \cup_{A \otimes i_!\Lambda^0[n]} A \otimes i_!\Delta[n] \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ B \otimes i_!\Delta[n] \arrow[r] \arrow[ur, dashed] & X. \end{tikzcd}$$

The vertical map on the left is leaf anodyne by Corollary 6.30 and so a lift indeed exists. For part (ii), say for example that i is inner anodyne and f an inner fibration. We should check that a lift existence in the following kind of square, with $0 < k < n$:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & \mathbf{hom}(B, Y) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathbf{hom}(B, X) \times_{\mathbf{hom}(A,X)} \mathbf{hom}(A, Y). \end{tikzcd}$$

This time the conclusion follows from the fact that the map

$$B \otimes i_!\Lambda^k[n] \cup_{A \otimes i_!\Lambda^k[n]} A \otimes i_!\Delta[n] \to B \otimes i_!\Delta[n]$$

is inner anodyne, see Corollary 6.26. □

Remark 6.34 With care it is possible to weaken some of the assumptions in the preceding theorem. For example, in order for p to be a left fibration, it is only necessary that f has the right lifting property with respect to inner horns and leaf horns *corresponding to a unary leaf corolla*. This can be checked using the observation of Remark 6.31. Similar remarks apply to the corollaries that follow.

Corollary 6.35 *Suppose $f : Y \to X$ is a fibration of one of the five types considered in Theorem 6.33 and B is a normal dendroidal set. Then*

$$\mathbf{hom}(B, Y) \to \mathbf{hom}(B, X)$$

is a fibration of simplicial sets of the corresponding type.

Proof Take $A = \varnothing$. □

Another special case of Theorem 6.33 is where X is the terminal dendroidal set and one obtains statements similar to Corollary 5.39. In particular one finds:

Corollary 6.36 *If Y is an ∞-operad and B a normal dendroidal set, then* $\mathbf{hom}(B, Y)$ *is an ∞-category.*

6.5 Spines and Leaves of Trees

We will consider the relation between spine inclusions of trees and inner anodyne morphisms. We will use this relation to complete the proof of Proposition 6.4. We also include a useful characterization of the class of leaf anodyne morphisms.

Recall that for a tree T, its *spine* is the union of all the subtrees of T with precisely one vertex:

$$\mathrm{Sp}[T] = \bigcup_{v \in V(T)} \Omega[C_v].$$

Also, if e is an inner edge of T, one can consider the trees e/T (chopping off everything above e) and T/e (the maximal subtree of T with root e) and express T as the result of grafting these two along the edge e:

$$T = e/T \circ_e T/e$$

Lemma 6.37 *For any tree T with at least two vertices and an inner edge e, the inclusions*

$$\mathrm{Sp}[T] \to \Omega[e/T] \cup_e \Omega[T/e] \to \Omega[T]$$

are inner anodyne.

Proof If T has two vertices (and hence precisely one inner edge e), the first map is an equality and the second is equal to the inner horn inclusion $\Lambda^e[T] \to \Omega[T]$. We proceed by induction on the number of vertices of T. There is a pushout square

$$\begin{tikzcd}
\mathrm{Sp}[e/T] \amalg \mathrm{Sp}[T/e] \arrow[r] \arrow[d] & \mathrm{Sp}[T] \arrow[d] \\
\Omega[e/T] \amalg \Omega[T/e] \arrow[r] & \Omega[e/T] \cup_e \Omega[T/e].
\end{tikzcd}$$

The left vertical map is inner anodyne by induction, so the right vertical map is inner anodyne as well. Hence it suffices to prove that $\Omega[e/T] \cup_e \Omega[T/e] \to \Omega[T]$ is inner anodyne. Form a sequence of inclusions

$$\Omega[e/T] \cup_e \Omega[T/e] = A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_n = \Omega[T],$$

where A_p is the union of $\Omega[e/T] \cup_e \Omega[T/e]$ with all the faces S of T that have at most p vertices and contain e as an inner edge. We claim that $A_{p-1} \subseteq A_p$ is inner anodyne for every p. Indeed, write F_p for the collection of faces being adjoined to form A_p. Then there is a pushout square

$$\begin{tikzcd} \coprod_{S \in F_p} \Lambda^e[S] \arrow[r] \arrow[d] & A_{p-1} \arrow[d] \\ \coprod_{S \in F_p} \Omega[S] \arrow[r] & A_p. \end{tikzcd}$$

□

Remark 6.38 Similarly to Remark 5.33, the preceding lemma leads to a proof of the implication (i) ⇒ (ii) in Proposition 6.4. For a given dendroidal set X, consider the class $\mathcal{C}$ of monomorphisms $A \to B$ with the property that any map $A \to X$ extends *uniquely* to B, i.e., the class of monomorphisms $A \to B$ for which

$$\mathbf{dSets}(B, X) \to \mathbf{dSets}(A, X)$$

is an isomorphism. Clearly $\mathcal{C}$ is a saturated class. If it contains all inner horn inclusions of trees, then it also contains all the spine inclusions by virtue of the preceding lemma.

As with simplicial sets we can also characterize the inner anodynes in terms of the spine inclusions in the following way:

Proposition 6.39 *Let $\mathcal{A}$ be a saturated class of normal monomorphisms between dendroidal sets which contains all spine inclusions of trees and satisfies the following additional closure property: if $i : A \to B$ and $j : B \to C$ are monomorphisms such that i and ji are in $\mathcal{A}$, then j is in $\mathcal{A}$ as well. Then the class $\mathcal{A}$ contains all inner anodynes.*

Proof Since $\mathcal{A}$ is saturated it suffices to show it contains all inner horn inclusions $\Lambda^e[T] \to \Omega[T]$, for T any tree with inner edge e. We will prove the slightly more general claim that each inclusion $\Lambda^E[T] \to \Omega[T]$ is in $\mathcal{A}$, for E a non-empty subset of the set of inner edges $I(T)$ of T. Consider the inclusions

$$\mathrm{Sp}[T] \xrightarrow{i} \Lambda^E[T] \xrightarrow{j} \Omega[T].$$

Then ji is in $\mathcal{A}$ by assumption, so it suffices to show that i is in $\mathcal{A}$. Factor it as

$$\mathrm{Sp}[T] \xrightarrow{i_1} \partial^{\mathrm{ext}}\Omega[T] \xrightarrow{i_2} \Lambda^E[T],$$

where $\partial^{\mathrm{ext}}\Omega[T]$ denotes the union of the *outer* (or *external*) faces of T. We will show that both of these maps are in $\mathcal{A}$ by induction on the number k of vertices of T. For $k = 2$ there is nothing to prove, because both are identities. For larger k, let us consider proper subtrees S of T (so S is obtained from T by iteratively taking leaf and root faces, but not inner faces). Give the set of such subtrees a linear order which extends the partial order of inclusion and adjoin them to $\mathrm{Sp}[T]$ in that order to form a filtration as follows:

$$\mathrm{Sp}[T] = B_1 \subseteq B_2 \subseteq \cdots, \qquad \bigcup_i B_i = \partial^{\mathrm{ext}}\Omega[T].$$

If $B_i \longrightarrow B_{i+1}$ is given by adjoining a subtree S, then it is a pushout of the map

$$\partial^{\mathrm{ext}}\Omega[S] \to \Omega[S].$$

Indeed, all the outer faces of S are contained in B_i by the induction on the size of S, but no inner face of it can be contained in a smaller subtree. We conclude that $B_i \longrightarrow B_{i+1}$ is in $\mathcal{A}$, because it is a pushout of a map of the form i_2 but for a tree smaller than T. Hence $i_1 : \mathrm{Sp}[T] \longrightarrow \partial^{\mathrm{ext}}\Omega[T]$ is in $\mathcal{A}$ as well.

We prove that $i_2 : \partial^{\mathrm{ext}}\Omega[T] \longrightarrow \Lambda^E[T]$ is in $\mathcal{A}$ by an induction on the size of the complement of E. If E is $I(T)$ then i_2 is the identity and there is nothing to prove. Otherwise, pick an element $e \in I(T)$ which is *not* contained in E and write $E' = E \cup \{e\}$. The map

$$\partial^{\mathrm{ext}}\Omega[T] \to \Lambda^{E'}[T]$$

is in $\mathcal{A}$ by the inductive hypothesis. Consider the pushout square

$$\begin{tikzcd}
\Lambda^E[\partial_e T] \arrow[r] \arrow[d] & \Lambda^{E'}[T] \arrow[d] \\
\Omega[\partial_e T] \arrow[r] & \Lambda^E[T].
\end{tikzcd}$$

Here the upper left-hand corner denotes the union of all faces of the tree $\partial_e T$ *except* for the inner faces corresponding to elements of E. Thus the left vertical map is of the form i_2, but for the smaller tree $\partial_e T$. By the inductive hypothesis it is in $\mathcal{A}$, so that the right vertical map is in $\mathcal{A}$ as well. It follows that i_2, which is the composition

$$\partial^{\mathrm{ext}}\Omega[T] \to \Lambda^{E'}[T] \to \Lambda^E[T],$$

is in $\mathcal{A}$. □

Remark 6.40 We can now continue Remark 6.38 and prove the implication (ii) $\Rightarrow$ (i) of Proposition 6.4. Let $\mathcal{C}$ be the same class of monomorphisms as in that remark. Clearly $\mathcal{C}$ has the closure property described in the statement of Proposition 6.39 (simply because isomorphisms are closed under two-out-of-three). So if $\mathcal{C}$ contains the spine inclusions of trees, the proposition implies that it contains all inner anodynes.

We conclude this section with analogous results describing the class of leaf anodyne morphisms. We will denote the inclusion of the leaves of a tree T by $\ell[T] \longrightarrow \Omega[T]$. We promised in Lemma 6.19 that it is a leaf anodyne morphism, as we will prove below (see Remark 6.43). Conversely, the class of leaf anodynes may be characterized in terms of such leaf inclusions as follows:

Proposition 6.41 *Let $\mathcal{A}$ be a saturated class of normal monomorphisms between dendroidal sets which satisfies the following additional closure property: if $i: A \to B$ and $j: B \to C$ are monomorphisms such that i and ji are in $\mathcal{A}$, then j is in $\mathcal{A}$ as well. If $\mathcal{A}$ satisfies either of the following two properties, then it contains all leaf anodyne morphisms:*

(1) *The class $\mathcal{A}$ contains the leaf inclusion $\ell[T] \to \Omega[T]$ of any tree T.*
(2) *The class $\mathcal{A}$ contains the leaf inclusion $\ell[C_n] \to \Omega[C_n]$ of the n-corolla, for any $n \geq 0$, as well as the spine inclusion $\mathrm{Sp}[T] \to T$ of any tree T.*

The proof of this proposition will use the following elementary observation:

Lemma 6.42 *For any tree T, the inclusion of its leaves into its spine*

$$\ell[T] \to \mathrm{Sp}[T]$$

is a composition of pushouts of leaf inclusions of corollas.

Proof If T has one vertex (so T is a corolla) the statement of the lemma is trivial. We proceed by induction on the size of T. A larger tree T can be described as a collection of trees $(T_1, \ldots, T_n)$ grafted onto the leaves of its root corolla C_n, and we have

$$\mathrm{Sp}[T] = \Omega[C_n] \cup (\mathrm{Sp}[T_1] \amalg \cdots \amalg \mathrm{Sp}[T_n]).$$

Consider the composition

$$\ell[T] \xrightarrow{i} \mathrm{Sp}[T_1] \amalg \cdots \amalg \mathrm{Sp}[T_n] \xrightarrow{j} \mathrm{Sp}[T].$$

The map i is a composition of pushouts of leaf inclusions of corollas by the inductive hypothesis. The map j is a pushout of the leaf inclusion of the corolla C_n, finishing the proof. $\square$

Proof (of Proposition 6.41) First we demonstrate that properties (1) and (2) are equivalent. Indeed, consider the diagram

$$\begin{tikzcd}
& \ell[T] \arrow[dl, "i"'] \arrow[dr, "j"] & \\
\mathrm{Sp}[T] \arrow[rr, "k"] & & \Omega[T].
\end{tikzcd}$$

If $\mathcal{A}$ satisfies (1) then it contains the map j by assumption and the map i by the conclusion of Lemma 6.42. By the assumed closure property it must contain k as well. Conversely, if $\mathcal{A}$ satisfies (2) then it contains k by assumption and i by Lemma 6.42. Thus it also contains $j = ki$.

We should prove that $\mathcal{A}$ contains all leaf horn inclusions. In fact, it will be convenient to prove a slightly more general statement. Consider a composition of inclusions

$$\ell[T] \xrightarrow{f_1} V \xrightarrow{f_2} W \xrightarrow{f_3} \Lambda^L[T] \xrightarrow{f_4} \Omega[T].$$

Here V is a union of $\ell[T]$ with a collection of *inner* faces of T and W is the union of V with $D_{\text{root}}T$, the 'modified' root face of T. This is the disjoint union of trees obtained from T by removing the root edge and root vertex; in other words, writing $T = C_n \circ (T_1, \dots, T_n)$ we set $D_{\text{root}}T = \Omega[T_1] \amalg \cdots \amalg \Omega[T_n]$. (In case the root vertex of T is unary then $D_{\text{root}}T = \Omega[\partial_{\text{root}}T]$.) The superscript L denotes a non-empty collection of leaf vertices of T and $\Lambda^L[T]$ is the corresponding leaf horn. We will prove that each of f_1, f_2, f_3, and f_4 belongs to $\mathcal{A}$. The proposition follows by considering f_4 in the special case where L consists of a single element. We work by induction on the size of T, the base case where T is a corolla being trivial: all the maps are identities except f_4, which is in $\mathcal{A}$ by assumption.

To see that f_1 is in $\mathcal{A}$, first consider the case where V consists of only one inner face $\Omega[\partial_e T]$. Then $\ell[T] \to \Omega[\partial_e T]$ is in $\mathcal{A}$ by the inductive hypothesis, since the tree $\partial_e T$ is smaller than T. For larger V we work by induction. Single out a face $\partial_e T$ occurring in the union and write $V = V' \cup \Omega[\partial_e T]$ for a smaller V'. Consider the following square, which is both a pushout and a pullback:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & V' \arrow[d] \\ \Omega[\partial_e T] \arrow[r] & V. \end{tikzcd}$$

Here P is a union of inner faces of $\Omega[\partial_e T]$. Thus the left vertical map is of the form $f_4 f_3 f_2$ for a tree smaller than T (namely $\partial_e T$) and hence in $\mathcal{A}$ by the inductive hypothesis. Since $\mathcal{A}$ is closed under pushouts the right vertical map is in $\mathcal{A}$ as well. We conclude that both of the maps $\ell[T] \to V' \to V$ are in $\mathcal{A}$ and therefore so is their composite f_1.

Now recall that $D_{\text{root}}T = \Omega[T_1] \amalg \cdots \amalg \Omega[T_n]$. To see that f_2 is in $\mathcal{A}$ consider the following square, which again is both a pushout and a pullback:

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & V \arrow[d, "f_2"] \\ D_{\text{root}}T \arrow[r] & W. \end{tikzcd}$$

Here the left vertical map is a disjoint union of maps $Q_i \to \Omega[T_i]$ for $1 \leq i \leq n$. Each Q_i is a union of inner faces of T_i and potentially $D_{\text{root}}T_i$ (if V contains the inner face of T corresponding to the root edge of T_i). In particular, each of these maps is of the form $f_4 f_3$ for the smaller tree T_i and therefore contained in $\mathcal{A}$. It follows that the pushout f_2 in the square is also in $\mathcal{A}$.

To see that f_3 is in $\mathcal{A}$, we work by downward induction on the size of L. If it consists of *all* leaf vertices of T, then f_3 is the identity and there is nothing to prove. For smaller L, pick a leaf vertex v not contained in L and consider the following square, which is easily checked to be a pushout:

$$\begin{tikzcd}
\Lambda^L[\partial_v T] \arrow[r] \arrow[d] & \Lambda^{\{v\}\cup L}[T] \arrow[d] \\
\Omega[\partial_v T] \arrow[r] & \Lambda^L[T].
\end{tikzcd}$$

Here the upper left-hand corner makes sense because all the elements of L are also leaf vertices of the smaller tree $\partial_v T$. The left vertical map is of the form f_4 for the smaller tree $\partial_v T$ and hence in $\mathcal{A}$ by induction. Therefore the right vertical map is also in $\mathcal{A}$, from which it follows that f_3 is in $\mathcal{A}$.

Finally, consider the diagram

$$\begin{tikzcd}
\ell[T] \arrow[r, "f_3 f_2 f_1"] \arrow[dr] & \Lambda^L[T] \arrow[d, "f_4"] \\
& \Omega[T].
\end{tikzcd}$$

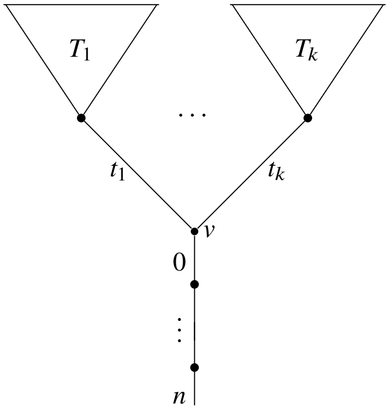
We have proved that the horizontal map is in $\mathcal{A}$, whereas the slanted map is in $\mathcal{A}$ by assumption. By our hypotheses on $\mathcal{A}$ (specifically its 'cancellation property'), the map f_4 is in $\mathcal{A}$ as well. $\square$

Remark 6.43 Note that Lemma 6.19 follows immediately from Lemma 6.42 and the fact that spine inclusions are inner anodyne, so in particular leaf anodyne.

6.6 Joins of Trees

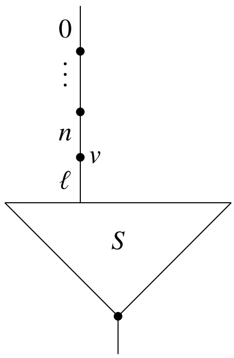
In this section we discuss the behaviour of certain kinds of anodyne morphisms with respect to joins of trees. The construction of the join, which is relatively simple for simplicial sets, becomes a bit more subtle in the dendroidal case. For example, one has to take care to express the correct functoriality of these constructions, which we will do below. The main reason for introducing the join is Corollary 6.46, which we will need in the next section.

We discussed the join of two simplicial sets in Section 5.4. For simplices $\Delta[n]$ and $\Delta[m]$ it is essentially given by putting the linear orders $[n]$ and $[m]$ 'next to each other'. This definition admits a generalization to trees, which we will now explain. In fact, it will be useful to phrase our constructions more generally in terms of *forests*: by a forest we will mean a tuple $(T_1, \ldots, T_k)$ of trees. We explicitly allow the 'empty tuple' () as a forest. For the linear order $[n]$, interpreted as a linear tree $i[n]$, we define the join $(T_1, \ldots, T_k) \star [n]$ to be the tree formed by grafting the root of every tree T_i (label it t_i) onto a single new vertex v and grafting the result on top of $[n]$:



For the empty forest () we define the join () $\star$ $[n]$ to be the tree obtained from $[n]$ by simply putting a vertex v without leaves on top; in other words, () $\star$ $[n]$ is the closure of the linear tree $i[n]$. This construction is evidently functorial in the linear order $[n]$. We will describe the relevant functoriality in the forest variable below.

Another kind of join, which we only mention here for the sake of completeness, is the following. Consider a tree S and a leaf ℓ of S. Then we form the join $[n] \star_\ell S$ by grafting $[n]$ onto a new vertex v and grafting that vertex on top of ℓ:



Remark 6.44 Both constructions of joins above are special cases of a single slightly more general construction. Indeed, consider a tree S, a set of leaves $\{\ell_1, \ldots, \ell_m\}$ of S, and for every ℓ_i a forest F_i. Then one can form the join

$$(F_1, \ldots, F_m) \star_{(\ell_1, \ldots, \ell_m)} S$$

by grafting the forest F_i onto a new vertex v_i and then grafting v_i onto the leaf ℓ_i. The first case above (joining a forest onto a linear tree) is the only construction we will actually use.

When considering joins of trees rather than joins of simplices, some new features arise when considering functoriality in the forest F. For example, consider the tree $T = C_2$ and let e be one of the two leaves of T. Then the inclusion $e \longrightarrow T$ does *not* induce a corresponding map

$$(e) \star [n] \to T \star [n].$$

Indeed, in the first tree the new vertex v is unary, whereas in the second v is a binary vertex. The situation is better if one considers the forest (e, f) consisting of both leaves of T; in that case there *is* an evident face map

$$(e, f) \star [n] \to T \star [n].$$

In fact it is an inner face corresponding to the root edge t of T (which is indeed an inner edge of the join $T \star [n]$). To take these issues into account we will now introduce an appropriate notion of map of forests.

To any forest $F = (T_1, \ldots, T_n)$ we associate a dendroidal set

$$\Omega[F] := \Omega[T_1] \amalg \cdots \amalg \Omega[T_n],$$

with the understanding that $\Omega[F] = \varnothing$ if F is the empty forest. Then we define a *wide* map of forests $F \to G$ to be a map of dendroidal sets $f : \Omega[F] \to \Omega[G]$ satisfying the following condition:

(∗) For every constituent tree S of the forest G, any path in the poset $E(S)$ from a minimal element to the maximal root element r_S meets precisely one edge of the form $f(r_{T_i})$ for some constituent tree T_i of F (with r_{T_i} denoting its root edge as usual).

In particular, the edges $f(r_{T_i})$ are pairwise *independent* edges of the forest G; either they lie in different components of $\Omega[G]$, or when they lie in the same component S then they are independent edges of that tree. This clearly implies that the images of the trees T_i under f are disjoint in $\Omega[G]$. However, note that different trees T_i and T_j may be taken to the same component S of G, but with disjoint images. In particular, the number of constituent trees of G need not be the same as that of F, but will always be less than or equal to it. We write $\mathbf{\Phi}_w$ for the category with objects the forests as defined above, and morphisms the wide maps of forests. In this section we will use the presheaf category $\mathrm{PSh}(\mathbf{\Phi}_w)$ of *wide forest sets*. It comes equipped with a 'realization functor'

$$\omega : \mathrm{PSh}(\mathbf{\Phi}_w) \to \mathbf{dSets}$$

which is the left Kan extension of the functor $\mathbf{\Phi}_w \to \mathbf{dSets} : F \mapsto \Omega[F]$.

One of the crucial examples of a map in $\mathbf{\Phi}_w$ is the following. If T is a tree with at least one vertex, write $\mathbf{D}_r T$ for the 'forest root face' of T. It is the forest given defined by removing the root edge and the root vertex from T. Thus, if $T = C_n \circ (T_1, \ldots, T_n)$, then $\mathbf{D}_r T = (T_1, \ldots, T_n)$. The evident inclusion $\mathbf{D}_r T \to (T)$ is a wide map of forests. In fact, the reader might wish to check that any wide map of forests can be obtained as a composition of maps which (on components) are maps of trees which preserve the root, together with these 'modified root faces' $\mathbf{D}_r T \to (T)$. Note that for a tree T there is an evident map of trees $\mathbf{D}_r T \star [n] \to T \star [n]$. Indeed, it is an inner face map corresponding to the root edge r of T (which is indeed an inner edge of the join $T \star [n]$). More generally, any wide map of forests $F \to G$ defines a map $F \star [n] \to G \star [n]$. This defines a functor

$$\mathbf{\Phi}_w \times \mathbf{\Delta} \to \mathbf{dSets} : (F, [n]) \mapsto F \star [n]$$

and, by Kan extension, a functor

$$\mathrm{PSh}(\mathbf{\Phi}_w) \times \mathbf{sSets} \to \mathbf{dSets} : (Z, K) \mapsto Z \star K,$$

which preserves colimits in each variable separately and agrees with the previous functor on representable objects. The dendroidal set $Z \star K$ receives an evident map from K, as well as from $\omega(Z)$. In particular, for a fixed object $Z \in \mathrm{PSh}(\mathbf{\Phi}_w)$ one obtains a colimit-preserving functor

$$\mathbf{sSets} \to \omega(Z)/\mathbf{dSets} : K \mapsto Z \star K$$

with right adjoint

$$\omega(Z)/\mathbf{dSets} \to \mathbf{sSets} : (\omega(Z) \to X) \mapsto X_{Z/}.$$

A map $Y \to Z$ of wide forest sets induces a map $X_{Z/} \to X_{Y/}$ of simplicial sets.

We define a *face* of a forest F in the expected way, namely as a proper inclusion $G \to F$ in the category $\mathbf{\Phi}_w$. The union of all faces of F defines the *wide boundary* $\partial^w F \in \mathrm{PSh}(\mathbf{\Phi}_w)$. We write $\partial^w \Omega[F]$ for the corresponding subobject of the dendroidal set, which we define to be the image of $\omega(\partial^w F)$ inside $\Omega[F]$.

In the next section we will need to analyze root horns $\Lambda^r[T] \to T$, where r is a unary root vertex. In this case one can always write $T = F \star [1]$ for some forest $F = (T_1, \ldots, T_k)$. Indeed, the root vertex r is the unique vertex of the linear tree $[1]$ and F is all that remains once one deletes that copy of C_1 and the vertex of T attached to the top of it, which again we label v. More generally, let us consider faces of a tree $F \star [n]$. The faces of $F \star [n]$ can be listed as follows:

(i) The faces of T corresponding to the faces of $[n]$; each deletes an edge i and the resulting face is $F \star \partial_i \Delta[n]$. Note that all these faces are inner except for $i = n$, which is the root face.
(ii) For every face $\partial_x T_i$ of a constituent tree T_i of the forest F which is *not* a root face, there is a corresponding face of T which can be written $(T_1, \ldots, \partial_x T_i, \ldots, T_k) \star [n]$. More briefly, if we write $\partial_x F$ for the forest obtained from F by replacing T_i with $\partial_x T_i$, we can rewrite this face of T as $\partial_x F \star [n]$.
(iii) For every tree T_i of F which is not η, replacing T_i by the 'modified root face' $\mathbf{D}_{r_i} T_i$ gives a forest $\mathbf{D}_{r_i} F$ and a face $(\mathbf{D}_{r_i} F) \star [n]$ of $F \star [n]$. Note that this face is an inner face of the tree $F \star [n]$ corresponding to the root edge r_i of T_i.
(iv) If all the constituent trees of F are copies of η (including the case where $F = ()$, so no copies at all), the vertex v is a leaf vertex of T. So then T has a leaf face deleting v and all its leaves (if any).

The union of the faces in (i) is precisely the join $F \star \partial\Delta[n]$. Similarly, one can summarize the faces occurring in (ii)–(iv) much more succinctly; their union forms precisely the join $\partial^w F \star [n]$. Note that if F consists of copies of η as in (iv), then $\partial^w F = \varnothing$ and $\partial^w F \star [n] \cong i_!\Delta[n]$.

If $T = F \star [1]$ is a tree with unary root vertex r, we may use our analysis above to write

$$\Lambda^r[T] = F \star \{1\} \cup \partial^w F \star [1].$$

Indeed, the missing root face $\partial_r T$ is the face $F \star \{0\}$. We have set up our constructions so that they satisfy the following property:

Lemma 6.45 *For any* $0 \leq m \leq n$*, the inclusion*

$$F \star i_!\Lambda^m[n] \cup \partial^w F \star [n] \to F \star [n]$$

is a root anodyne map of dendroidal sets. Moreover, it is inner anodyne if $m < n$*.*

Proof Simply observe that

$$F \star i_!\Lambda^m[n] \cup \partial^w F \star [n] = \Lambda^m(F \star [n]),$$

with $\Lambda^m(F \star [n])$ the union of all faces of the tree $F \star [n]$ which contain the edge m. So the only missing face is an inner face contracting m (when $m < n$) or the root face (if $m = n$). □

In analogy with our terminology for dendroidal sets, let us say that a map of wide forest sets is a *normal monomorphism* if it is a composition of pushouts of wide boundary inclusions $\partial^w F \to F$. Using the arguments for saturated classes that are by now standard, we conclude the following:

Corollary 6.46 *Let* $A \to B$ *be a normal monomorphism of wide forest sets and* $\omega(B) \to Y$ *a map of dendroidal sets. If* $f : Y \to X$ *is an inner fibration of dendroidal sets, then the induced map*

$$Y_{B/} \to X_{B/} \times_{X_{A/}} Y_{A/}$$

is a left fibration of simplicial sets.

Proof Consider a lifting problem of the form

$$\begin{tikzcd} \Lambda^m[n] \arrow[r] \arrow[d] & Y_{B/} \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & X_{B/} \times_{X_{A/}} Y_{A/} \end{tikzcd}$$

with $m < n$. By adjunction it corresponds to a lifting problem

$$\begin{tikzcd} B \star i_!\Lambda^m[n] \cup A \star i_!\Delta[n] \arrow[r] \arrow[d] & Y \arrow[d] \\ B \star i_!\Delta[n] \arrow[r] \arrow[ur, dashed] & X, \end{tikzcd}$$

which admits a solution because the left vertical map is inner anodyne by the preceding lemma. □

The special case where X is terminal gives the following:

Corollary 6.47 *Let $A \to B$ be a normal monomorphism of wide forest sets and $\omega(B) \to Y$ a map of dendroidal sets. If Y is an ∞-operad then the induced map*

$$Y_{B/} \to Y_{A/}$$

is a left fibration of simplicial sets.

Specializing further to the case where $A = \varnothing$ gives:

Corollary 6.48 *If B is a normal wide forest set and $\omega(B) \to Y$ a map into an ∞-operad, then*

$$Y_{B/} \to Y_{\varnothing/} \cong i^*Y$$

is a left fibration. In particular, $Y_{B/}$ is an ∞-category.

There are also versions of the preceding lemmas for a join of the form $[n] \star_\ell S$, which we leave to the determined reader. They will not feature in this book.

6.7 Equivalences in ∞-Operads

The purpose of this section is to give an extension of Theorem 5.49 to dendroidal sets and discuss its consequences. We begin with a definition. Note that for a dendroidal set X, a 1-corolla $\alpha \in X_{C_1}$ can equivalently be thought of as a 1-simplex of the underlying simplicial set i^*X, simply using the fact that $C_1 = i_!\Delta[1]$.

Definition 6.49 Let X be an ∞-operad. A 1-corolla $\alpha \in X_{C_1}$ is an *equivalence* if the corresponding 1-simplex is an equivalence in the ∞-category i^*X in the sense of Definition 5.45.

The result we are after is the following:

Theorem 6.50 *Let $f : Y \to X$ be an inner fibration between ∞-operads and consider a tree T with a unary root vertex v and T itself with at least two vertices. Then a lift exists in any square of the form*

$$\begin{tikzcd} \Lambda^v[T] \arrow[r, "\beta"] \arrow[d] & Y \arrow[d, "f"] \\ T \arrow[r, "\alpha"] \arrow[ur, dashed] & X, \end{tikzcd}$$

provided that β sends the 1-corolla with vertex v to an equivalence in Y.

Proof The tree T may be written as $F \star C_1$, where F is the forest obtained from T by removing the unary root corolla and the vertex attached to the top of it. As in the preceding section we may write $\Lambda^v[T] = F \star \{1\} \cup \partial^w F \star [1]$, where $\{1\}$ corresponds to the root edge of T. The lifting problem of the theorem then corresponds to the following:

$$\begin{tikzcd} \{1\} \arrow[r, "\widehat{\beta}"] \arrow[d] & Y_{F/} \arrow[d] \\ \Delta[1] \arrow[r, "\widehat{\alpha}"] \arrow[ur, dashed] & X_{F/} \times_{X_{\partial^w F/}} Y_{\partial^w F/}. \end{tikzcd}$$

The vertical map on the right is a left fibration between ∞-categories by Corollary 6.46. It suffices to prove that $\widehat{\alpha}$ is an equivalence. Indeed, the existence of the lift then follows from Lemmas 5.47 and 5.48, which in particular guarantee that left fibrations have path lifting with respect to equivalences. To see that $\widehat{\alpha}$ is indeed an equivalence, consider the projection π from the codomain of $\widehat{\alpha}$ to i^*Y. This projection is the composition of the pullback of a left fibration with the left fibration π' as in the diagram below (see Corollary 6.47):

$$\begin{tikzcd} X_{F/} \times_{X_{\partial^w F/}} Y_{\partial^w F/} \arrow[r] \arrow[d] & X_{F/} \arrow[d] \\ Y_{\partial^w F} \arrow[r] \arrow[d, "\pi'"] & X_{\partial^w F/} \\ i^*Y. & \end{tikzcd}$$

Therefore π is itself a left fibration. The image of $\widehat{\alpha}$ under π is precisely the image of the unary root corolla of T under β, which is assumed to be an equivalence. Since left fibrations are conservative in the sense of Lemma 5.48, this shows that $\widehat{\alpha}$ is itself an equivalence, which completes the proof of the theorem. □

We will now use Theorem 6.50 to deduce some results about mapping spaces between ∞-operads analogous to the results of Section 5.6. If X is an ∞-operad and A a normal dendroidal set, we will say two maps $f, g : A \to X$ are J-homotopic if any of the following three equivalent conditions is satisfied:

(1) There is a map $h\colon i_! J \otimes A \to X$ such that $h_0 = f$ and $h_1 = g$.
(2) The maps f and g are equivalent (cf. Definition 5.45) as vertices of the ∞-category $\mathbf{hom}(A, X)$.

(3) The vertices f and g lie in the same connected component of the Kan complex $k(\mathbf{hom}(A, X))$.

We will prove that f and g are J-homotopic if and only if there exists a map

$$h\colon C_1 \otimes A \to X$$

which is a *pointwise* equivalence, i.e., for every colour $a \in A_\eta$ the corresponding 1-corolla $h(-, a)\colon C_1 \to X$ is an equivalence in X.

Extending the notation introduced at the end of Section 5.6, we write $k(A, X)$ for the simplicial subset of $\mathbf{hom}(A, X)$ consisting of the simplices all of whose edges are *pointwise* equivalences. Thus, an n-simplex of $k(A, X)$ is a map

$$\xi : i_!\Delta[n] \otimes A \to X$$

such that for every colour $a \in A_\eta$ the corresponding map

$$\xi(-, a) : \Delta[n] \to i^*X$$

factors through the maximal Kan complex $k(i^*X)$ in i^*X. According to Corollary 6.36 the simplicial set $\mathbf{hom}(A, X)$ is an ∞-category. Clearly any equivalence in this ∞-category is a pointwise equivalence between maps from A to X, so that there is an inclusion of its maximal Kan complex into $k(A, X)$:

$$k(\mathbf{hom}(A, X)) \subseteq k(A, X) \subseteq \mathbf{hom}(A, X).$$

We will prove that the first inclusion is an equality:

Theorem 6.51 (a) *Let X be an ∞-operad and let $A \to B$ be a normal monomorphism between dendroidal sets. Then $k(B, X) \to k(A, X)$ is a Kan fibration.*
(b) *In particular, if A is a normal dendroidal set then $k(A, X)$ is a Kan complex and hence $k(A, X) = k(\mathbf{hom}(A, X))$.*

Proof Part (b) follows by applying (a) to the normal monomorphism $\varnothing \to A$ and using that $k(\mathbf{hom}(A, X))$ contains any other Kan complex in $\mathbf{hom}(A, X)$. To prove part (a) it suffices to show that $k(B, X) \to k(A, X)$ is a *right* fibration. Indeed, as a special case it will follow that $k(A, X) \to k(\varnothing, X) = \Delta[0]$ is a right fibration, hence a Kan fibration by Corollary 5.50. But then $k(B, X) \to k(A, X)$ is a right fibration over a Kan complex, hence itself a Kan fibration by that same corollary.

Thus we should demonstrate that any lifting problem

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & k(B, X) \arrow[d] \\ \Delta[n] \arrow[r] & k(A, X) \end{tikzcd}$$

with $0 < k \leq n$ admits a solution. It is adjoint to a lifting problem of the form

$$\begin{tikzcd} B \otimes i_!\Lambda^k[n] \cup A \otimes i_!\Delta[n] \arrow[r, "\gamma"] \arrow[d] & X, \\ B \otimes i_!\Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

where γ maps each 1-corolla of the form $\{b\} \otimes i_!\Delta[j, j']$ to an equivalence in X. For $0 < k < n$ the vertical map is inner anodyne by Corollary 6.26 and hence a lift exists because X is assumed to be an ∞-operad. For $k = n$, Corollary 6.30 and Remark 6.31 guarantee that the map on the left is a composition of pushouts of inner horn inclusions and root horn inclusions

$$\Lambda^r[T] \to T$$

for which the root vertex of T is unary. Moreover, that unary root vertex is mapped to a 1-corolla of the form $b \otimes i_!\Delta[j, j']$ in $B \otimes i_!\Delta[n]$ and hence to an equivalence in X. If T has at least two vertices then we conclude that a lift exists by applying Theorem 6.50. If $T = C_1$, the horn under consideration is the inclusion of the root edge of C_1 and the lifting problem is trivial: one can simply use a degenerate 1-corolla of X. □

For simplicial sets we also expressed a relative version of the preceding theorem (namely Theorem 5.56). A similar statement holds for dendroidal sets. We will say a map $f : Y \to X$ is a *J-fibration* if it is an inner fibration and the map of underlying simplicial sets i^*f has J-path lifting (so that in particular i^*f is a J-fibration of simplicial sets).

Theorem 6.52 *Let $f: Y \to X$ be a J-fibration between ∞-operads and $A \to B$ a normal monomorphism between dendroidal sets. Then the map*

$$k(B, Y) \to k(B, X) \times_{k(A,X)} k(A, Y)$$

is a Kan fibration between Kan complexes.

6.8 Minimal Fibrations Between ∞-Operads

In this section we will extend the notion of minimal J-fibration to dendroidal sets and generalize some of the results of Section 5.8. The main extra feature to deal with is the presence of nontrivial automorphisms in the category $\mathbf{\Omega}$. We begin by defining several terms in evident analogy with the simplicial case. Throughout this section it will be convenient to abbreviate notation for the dendroidal set $i_!J$ simply to J. The crucial tool to be used many times in this section is Theorem 6.52 above.

Consider a J-fibration $p : X \to S$ between ∞-operads and a normal monomorphism $j : A \to B$ between dendroidal sets. If $f, g : B \to X$ are maps for which $fj = gj$ and $pf = pg$, then we say that f and g are *fibrewise J-homotopic relative to A* if there exists a J-homotopy $h : J \otimes B \to X$ from f to g (so $h_0 = f$ and $h_1 = g$) so that $h \circ (\mathrm{id} \times j)$ is the constant homotopy

$$J \otimes A \xrightarrow{\pi_2} A \xrightarrow{i} X$$

and $p \circ h$ is the constant homotopy

$$J \otimes B \xrightarrow{\pi_2} B \xrightarrow{pf} S.$$

Observe that we are using the existence of a projection map $J \otimes A \xrightarrow{\pi_2} A$ and similarly for B. Generally the tensor product of dendroidal sets does *not* admit natural projection maps onto its factors, but the maps π_2 above exist because J is a simplicial set: indeed, π_2 can be thought of as the composition $J \otimes A \to i_!\Delta[0] \otimes A \cong A$, which we will continue to do throughout this section. By adjunction, it is straightforward to verify that f and g are fibrewise J-homotopic relative to A if and only if they lie in the same connected component of the fibre of the Kan fibration

$$k(B, X) \to k(A, X) \times_{k(A,S)} k(B, S)$$

over the vertex $(fj, pf) = (gj, pg)$.

Definition 6.53 Let $p : X \to S$ be a J-fibration between ∞-operads and let $T \in \mathbf{\Omega}$ be a tree.

(a) Two dendrices $x, y \in X_T$ are *fibrewise J-equivalent* if, when viewed as maps $\Omega[T] \to X$, they are fibrewise J-homotopic relative to the boundary $\partial\Omega[T]$. (In particular, $px = py$ and x and y agree on $\partial\Omega[T]$.)
(b) The map $p : X \to S$ is a *minimal J-fibration* if for any two fibrewise J-equivalent dendrices $x, y \in X_T$, there exists an automorphism α of T so that $\alpha^* x = y$.

Of course there is an alternative to (b) in which one demands x and y to be equal, rather than related by an automorphism. However, this stricter notion turns out to be less useful in practice. In particular, we need the more general notion to have a result such as Theorem 6.56 below. Observe that a minimal J-fibration p between ∞-operads in particular gives a minimal J-fibration i^*p between ∞-categories. As before, the crucial property of minimal fibrations is the following rigidity:

Proposition 6.54 *Let $p : X \to S$ and $q : Y \to S$ be minimal J-fibrations between ∞-operads and suppose both X and Y are normal. Then any fibrewise J-homotopy equivalence $X \to Y$ over S is an isomorphism.*

We will use the following dendroidal analogue of Lemma 5.65, which is proved by the same argument:

Lemma 6.55 *Consider a normal monomorphism of dendroidal sets $A \subseteq B$ and an ∞-category X. Suppose $h, k : J \otimes B \to X$ are two J-homotopies such that the restrictions of h and k to $\{0\} \otimes B \cup_{\{0\}\otimes A} J \otimes A$ agree. Then the maps*

$$h_1, k_1 : B \cong \{1\} \otimes B \to X$$

are J-homotopic relative to A.

Proof (of Proposition 6.54) By the same formal argument used just before Proposition 5.64 it suffices to show that any endomorphism $\varphi : X \to X$ over S (i.e. $p\varphi = \varphi$) which is fibrewise J-homotopic to the identity is in fact an isomorphism. So let $h : J \otimes X \to X$ be a homotopy (over S) from φ to id_X. As in the simplicial case we will use skeletal induction on X (which is why we need normality of X).

The case $n = 0$ is clear: $\mathrm{sk}_0 X$ is a disjoint union of copies of η, and there is a unique one in each J-connected component of every fibre of the J-fibration $i^*X \to i^*S$. In fact this even shows that φ is the identity on the 0-skeleton. Suppose we have proved that φ restricts to an automorphism of $\mathrm{sk}_{n-1}X$. Let us show that φ then also gives an automorphism of $\mathrm{sk}_n X$.

To see that the restriction of φ to $\mathrm{sk}_n X$ is injective, consider a tree T with n vertices and dendrices $x, y \in X_T$ for which $\varphi x = \varphi y$. Then the restriction of x and y to $\partial\Omega[T]$ agree because φ is injective on $\mathrm{sk}_{n-1}X$. The fibrewise J-homotopies

$$J \otimes \Omega[T] \xrightarrow{\mathrm{id}\otimes x} J \otimes X \xrightarrow{h} X$$

from φx to x and

$$J \otimes \Omega[T] \xrightarrow{\mathrm{id}\otimes y} J \otimes X \xrightarrow{h} X$$

from φy to y agree on $J \otimes \partial\Omega[T]$ and on $\{0\} \otimes \Omega[T]$, where they both equal $\varphi x = \varphi y$. Therefore Lemma 6.55 implies that x and y are fibrewise J-homotopic relative to $\partial\Omega[T]$. The minimality of p then implies that there is an automorphism α of T so that $\alpha^* x = y$. Applying φ gives $\alpha^*\varphi x = \varphi y = \varphi x$. Since X is normal, we conclude that α must be the identity and thus $x = y$.

To see that the restriction of φ to $\mathrm{sk}_n X$ is surjective onto $\mathrm{sk}_n X$ take a dendrex $z \in X_T$, again for some tree T with n vertices. By the inductive hypothesis we can write $\partial z = \varphi u$ for a unique map $u : \partial\Omega[T] \to X$. We consider the fibrewise homotopy defined by the map

$$J \otimes \partial\Omega[T] \xrightarrow{h\circ(\mathrm{id}\otimes u)} X$$

and write h_u for its restriction along $i_!\Delta[1] \otimes \partial\Omega[T] \subseteq J \otimes \partial\Omega[T]$. Equivalently, we may think of h_u as a map $\Delta[1] \to k(\partial\Omega[T], X)$. Theorem 6.52 guarantees that the vertical map on the right in the following square is a Kan fibration, so that a lift g exists:

$$\begin{tikzcd}
\Delta[0] \arrow[rr, "z"] \arrow[d, "\partial_1"'] & & k(\Omega[T], X) \arrow[d] \\
\Delta[1] \arrow[r, "h_u"'] \arrow[urr, dashed, "g"] & k(\partial\Omega[T], X) \times_{k(\partial\Omega[T], S)} k(\Omega[T], S). &
\end{tikzcd}$$

This lift gives a fibrewise J-homotopy (again denoted g)

$$J \otimes \Omega[T] \to X$$

from $g_0 = z$ to another dendrex $y := g_1$. Then $\partial y = u$, while g and

$$h_y : J \otimes \Omega[T] \xrightarrow{\mathrm{id}\otimes y} J \otimes X \xrightarrow{h} X$$

define two fibrewise J-homotopies which agree on $J \otimes \partial\Omega[T]$ and $1 \otimes \Omega[T]$. Again applying Lemma 6.55 now shows that $g_0 = z$ and $(h_y)_0 = \varphi y$ are p-equivalent and hence $z = \alpha^*\varphi y = \varphi(\alpha^* y)$ for some automorphism α of T. In particular, z is in the image of φ. □

Theorem 6.56 *Let $p : X \to S$ be a J-fibration between ∞-operads. If X is normal, then there exists a minimal J-fibration $q : M \to S$ which is a fibrewise J-deformation retract of $X \to S$. If S is also normal, then the retraction $r : X \to M$ is a trivial fibration.*

Proof We will start by inductively constructing $M^{(n)}$, which we will be the n-skeleton of M, together with maps

$$M^{(n)} \xrightarrow{i^{(n)}} X^{(n)} \xrightarrow{r^{(n)}} M^{(n)}, \quad h^{(n)} : J \otimes X^{(n)} \to X,$$

so that $r^{(n)}i^{(n)} = \mathrm{id}_{M^{(n)}}$ and $h^{(n)}$ is a homotopy (relative to $M^{(n)}$) from $i^{(n)}r^{(n)}$ to the identity of $X^{(n)} := \mathrm{sk}_n X$ (or rather the composition of those maps with the inclusion $X^{(n)} \subseteq X$). Moreover, the homotopies $h^{(n)}$ are fibrewise over S in the sense that the composition $p \circ h^{(n)}$ is the constant homotopy

$$J \otimes X^{(n)} \xrightarrow{\pi_2} X^{(n)} \to S.$$

To start the induction we define $M^{(0)}$. Consider for every $s \in S_\eta$ the fibre $\eta_s \times_S X =: X_s$, which can be thought of as a simplicial set (it admits a map to η) and is in fact an ∞-category. Choose a single vertex in each connected component of the Kan complex $k(X_s)$ and set $M^{(0)}$ to be the coproduct of η's indexed by the set of these chosen vertices. Then for each $x \in X^{(0)}$ there is a unique vertex $r(x) \in M^{(0)}$ in the same fibre $X_{p(x)}$ and a path $h_x : J \to X$ from $r(x)$ to x so that the composition $p \circ h_x$ is constant with value $p(x)$. This defines the relevant maps

$$M^{(0)} \xrightarrow{i^{(0)}} X^{(0)} \xrightarrow{r^{(0)}} M^{(0)}, \quad h^{(0)} : J \otimes X^{(0)} \to X,$$

provided we arrange h_x to be degenerate if $x \in M^{(0)}$, in which case $r(x) = x$.

Now suppose we have defined $M^{(n-1)}$, $r^{(n-1)}$, and $h^{(n-1)}$. Consider the dendrices $x \in X_T$, for T ranging through trees with n vertices, whose boundary ∂x lies in $M^{(n-1)} \subseteq X^{(n-1)}$ and which are *not* fibrewise homotopic to a degenerate dendrex. We say two such dendrices x and y are *equivalent* if x is fibrewise J-equivalent to $\alpha^* y$, for some automorphism α of T. This defines an equivalence relation, because fibrewise homotopy defines one. Pick exactly one dendrex x in each equivalence class and write the collection of chosen dendrices as D_n. Then define $M^{(n)}$ to be the dendroidal subset generated by $M^{(n-1)}$ together with the elements of D_n. Since X was assumed to be normal, so that it has a good skeletal filtration, it follows that $M^{(n)}$ fits into a pushout square as follows:

$$\begin{tikzcd}
\coprod_{x \in D_n} \partial\Omega[T_x] \arrow[r] \arrow[d] & M^{(n-1)} \arrow[d] \\
\coprod_{x \in D_n} \Omega[T_x] \arrow[r] & M^{(n)}.
\end{tikzcd}$$

Here T_x denotes the shape of the dendrex x. We should now define $r^{(n)}$ and $h^{(n)}$, extending the maps $r^{(n-1)}$ and $h^{(n-1)}$. On simplices of $M^{(n)}$ we have no choice but to define $r^{(n)}$ to be the identity and $h^{(n)}$ the constant homotopy. For a tree T with n vertices and a non-degenerate dendrex $x \in X_T$ which is *not* contained in $M^{(n)}$ we proceed as follows. We have a fibrewise homotopy over S

$$J \otimes \partial\Omega[T] \xrightarrow{\mathrm{id}\otimes\partial x} J \otimes X^{(n-1)} \xrightarrow{h^{(n-1)}} X,$$

which is constant if ∂x happened to be contained in $M^{(n-1)}$. Restricting along $\Delta[1] \subseteq J$ and taking adjoints gives the bottom map in the following square:

$$\begin{tikzcd}
\Delta[0] \arrow[rr, "x"] \arrow[d, "\partial_0"] & & k(\Omega[T], X) \arrow[d] \\
\Delta[1] \arrow[r] \arrow[urr, dashed, "f"] & k(\partial\Omega[T], X) \times_{k(\partial\Omega[T],S)} k(\Omega[T], S). &
\end{tikzcd}$$

A lift f exists because the map on the right is a Kan fibration and the map on the left is anodyne. This map f defines a fibrewise homotopy to x from another dendrex $s(x) \in X_T$, whose boundary lies in $M^{(n-1)}$. The definition of $M^{(n)}$ now assures that there is a *unique* T-dendrex in one of the D_m (for $m \leq n$) which is equivalent to $s(x)$. We write $r^{(n)}(x)$ for this dendrex. By definition there exists a fibrewise homotopy

$$g : \Delta[1] \to k(\Omega[T], X)$$

relative to $\partial\Omega[T]$ such that $g_0 = r^{(n)}(x)$ and $g_1 = s(x)$. As before we 'compose' f and g. More precisely, consider the following square:

$$\begin{tikzcd}
\Lambda^1[2] \arrow[rr, "g\cup f"] \arrow[d] & & k(\Omega[T], X) \arrow[d] \\
\Delta[2] \arrow[r] \arrow[urr, dashed, "j"] & k(\partial\Omega[T], X) \times_{k(\partial\Omega[T],S)} k(\Omega[T], S). &
\end{tikzcd}$$

The bottom arrow is the degenerate 2-simplex

$$\Delta[2] \xrightarrow{s_0} \Delta[1] \to k(\partial\Omega[T], X) \times_{k(\partial\Omega[T],S)} k(\Omega[T], S),$$

where the second map is defined by the evaluation of $h^{(n-1)}$ on ∂x as above. The left vertical map in the square is inner anodyne, so a lift j indeed exists. The restriction of j along the inner face $\Delta[1] \xrightarrow{\partial_1} \Delta[2]$ is a fibrewise homotopy

$$\Delta[1] \to k(\Omega[T], X)$$

from $r^{(n)}(x)$ to x, compatible with $h^{(n-1)}$ on the boundary ∂x. One can of course extend j along the anodyne inclusion $\Delta[1] \subseteq J$ to obtain the desired J-homotopy

$$h_x^{(n)} : J \otimes \Omega[T] \to X.$$

This concludes our constructions. Note that $M \to S$, being a retract of $X \to S$, is also a J-fibration between ∞-categories. Moreover, it is evidently minimal.

It remains to check that if S is normal, then the retraction $r : X \to M$ is a trivial fibration. Consider a lifting problem

$$\begin{tikzcd}
\partial\Omega[T] \arrow[r, "u"] \arrow[d, "j"'] & X \arrow[d, "r"] \\
\Omega[T] \arrow[r, "v"] \arrow[ur, dashed] & M.
\end{tikzcd}$$

Using the map h constructed above, we find a fibrewise homotopy

$$h_u : J \to k(\partial\Omega[T], X) \times_{k(\partial\Omega[T],S)} k(\Omega[T], S)$$

from $ivj = iru$ to u. Pick a lift f in the square below:

$$\begin{tikzcd}
\Delta[0] \arrow[r, "iv"] \arrow[d, "\partial_1"'] & k(\Omega[T], X) \arrow[d] \\
J \arrow[r, "h_u"] \arrow[ur, dashed, "f"] & k(\partial\Omega[T], X) \times_{k(\partial\Omega[T],S)} k(\Omega[T], S).
\end{tikzcd}$$

This map f gives a fibrewise J-homotopy (agreeing with h_u on the boundary $\partial\Omega[T]$) from $f_0 = iv$ to another map f_1. Now

$$h_{f_1} : J \to k(\partial\Omega[T], X) \times_{k(\partial\Omega[T],S)} k(\Omega[T], S)$$

is another fibrewise J-homotopy from irf_1 to f_1, also agreeing with h_u on the boundary. Hence Lemma 6.55 implies that iv and irf_1 are fibrewise J-equivalent. Applying the retraction r gives fibrewise J-equivalent dendrices $riv = v$ and $rirf_1 = rf_1$. Since $M \to X$ is minimal, we conclude that $v = \alpha^*(rf_1)$ for some $\alpha \in \mathrm{Aut}(T)$. If we can argue that α is the identity, it follows that f_1 is a solution to our original lifting problem above. To see this, write q for the fibration $M \to S$ and note that $\alpha^*(qrf_1) = qv = qrf_1$, where the second equality follows from the fact that our homotopies are fibrewise over S. Since S is assumed to be normal, this implies $\alpha = \mathrm{id}$. □

In the case of simplicial sets it was rather evident that the class of minimal J-fibrations is closed under pullbacks. This is not quite the case for dendroidal sets. However, we still have:

Lemma 6.57 *Let $f : S' \to S$ be a map of ∞-operads such that for any dendrex $x \in X_T$, the induced map of stabilizers $\mathrm{Aut}(T)_x \to \mathrm{Aut}(T)_{f(x)}$ is an isomorphism. If $p : X \to S$ is a minimal J-fibration between ∞-operads, with X normal, then the pullback $q : f^*X \to S'$ is again a minimal J-fibration between ∞-operads. In particular, note that the condition on stabilizers is automatically satisfied when S' and S are normal.*

Proof The class of J-fibrations is stable under pullback, so it suffices to check minimality of f^*p. Write $g : f^*X \to X$ for the evident map. Let $x, y \in f^*X$ be two dendrices which are fibrewise J-equivalent over S'. Then $g(x)$ and $g(y)$ are fibrewise J-equivalent over S, so that minimality of p implies that $g(x) = \alpha^*g(y) = g(\alpha^*y)$ for some $\alpha \in \mathrm{Aut}(T)$. Projecting to S, we find that α^* fixes the dendrex $pg(y) = fq(y)$. By the condition of the lemma α^* must also fix $q(y)$, so that $q(x) = q(y) = \alpha^*q(y)$. By the defining pullback square of f^*X, it follows that $x = \alpha^*y$. □

Essentially the same argument used to prove Proposition 5.73 can be used for the following homotopy invariance property:

Proposition 6.58 *Let $p : X \to S$ be a minimal J-fibration between ∞-operads and assume X is normal. If $f, g : S' \to S$ are two J-homotopic maps satisfying the condition of the previous lemma, then the pullbacks f^*p and g^*p are isomorphic minimal J-fibrations over S'.*

Historical Notes

The notion of an inner dendroidal Kan complex (or ∞-operad) was introduced and developed in [116, 117]. These references study the behaviour of the inner Kan condition with respect to tensor products and also prove Theorem 6.50, the analogue of Joyal's theorem in the context of dendroidal sets. Leaf anodynes and left fibrations of dendroidal sets were first considered systematically in [77]. Minimal fibrations of dendroidal sets originate in [114].

Part II
The Homotopy Theory of Simplicial and Dendroidal Sets

Chapter 7
Model Categories

In [123] Quillen proposed an axiomatic framework for homotopy theory through the notion of a *model structure* on a category. Such a structure consists of three distinguished classes of morphisms, called weak equivalences, fibrations, and cofibrations, required to satisfy several axioms reminiscent of the properties of the corresponding notions in the usual homotopy theory of topological spaces. This axiomatization allows one to carry out many of the basic manipulations of homotopy theory in any context where such a model structure is present. One of Quillen's early applications of the theory was to the category of simplicial commutative rings, where it leads to an elegant development of the theory of homology of commutative rings [125]. This is now usually referred to as *André–Quillen homology* or the *cotangent complex*. More modern applications of the theory include the study of *motivic homotopy theory*, which uses model structures on the category of simplicial presheaves on the category of smooth schemes (over some fixed base S) [119].

The notion of model category also allows one to make rigorous sense of the idea that two homotopy theories are 'equivalent'; this applies, for example, to the homotopy theories of topological spaces and of simplicial sets. Any sufficiently homotopy-theoretic statement can thus be carried over from one of these theories to the other without essential change. More interestingly, in [124] Quillen compared the homotopy theory of rational spaces to various 'models' constructed in a completely algebraic fashion; in particular, he exhibited equivalences between the model category of rational spaces and model categories constructed from differential graded Lie algebras, or from cocommutative coalgebras, among others.

In this book we will use model categories to study the homotopy theory of topological and simplicial operads and of dendroidal sets (of various flavours). We will prove various comparison results relating these theories.

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7.1 Axioms for a Model Category

In this section we introduce the definition of a model category and give the first few examples. As we will see, in some cases it is very easy to verify the axioms, but one learns little new from the fact that they hold. In other cases this verification is hard, but the validity of the axioms captures crucial properties of the category at hand.

We follow Quillen's formulation:

Definition 7.1 Let $\mathcal{E}$ be a category. A *model structure* (or more explicitly a *Quillen model structure*) on $\mathcal{E}$ is given by three classes of morphisms in $\mathcal{E}$, called the *fibrations*, the *cofibrations*, and the *weak equivalences*. These are required to satisfy the following axioms:

(M1) The category $\mathcal{E}$ has all small limits and colimits.
(M2) If two out of three morphisms $f : X \to Y$, $g : Y \to Z$, and their composition $gf : X \to Z$ are weak equivalences, then so is the third.
(M3) The classes of fibrations, cofibrations, and weak equivalences are closed under retracts.
(M4) For any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

in which i is a cofibration and f is a fibration, a lift $B \to X$ making both triangles commutes exists as soon as i or f is also a weak equivalence.
(M5) Any morphism $f : X \to Y$ can be factored as a cofibration $i : X \to Z$ followed by a fibration $p : Z \to Y$ in two ways: one in which i is also a weak equivalence, and one in which p is.

A category equipped with a model structure will also be referred to as a *model category* (sometimes *Quillen model category*).

When there is no danger of confusion we will often use the phrase 'Let $\mathcal{E}$ be a model category' and leave the choice of model structure on $\mathcal{E}$ implicit.

Remark 7.2 (a) Quillen's original form of axiom (M1) only demands the existence of finite limits and colimits. However, in all our examples, $\mathcal{E}$ will have all small limits and colimits, so that for us it is convenient to use the stronger version of (M1) given above, as is also standard in the literature.

(b) A map which is both a fibration and a weak equivalence is called a *trivial fibration*. Similarly, a cofibration which is also a weak equivalence is referred to as a *trivial cofibration*. So (M5) states that every map factors as a trivial cofibration followed by a fibration and as a cofibration followed by a trivial fibration. With this terminology, (M4) states that fibrations have the right lifting property with respect to trivial cofibrations, and trivial fibrations have the right lifting property with respect to cofibrations.

(c) In fact, the converse statements also hold. For example, if a map $f : X \to Y$ has the right lifting property with respect to cofibrations, then it must be a trivial fibration. Indeed, factor f as a cofibration $i : X \to Z$ followed by a trivial fibration $p : Z \to Y$. Then by assumption we can find a lift g in

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "f"] \\ Z \arrow[r, "p"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

and this makes f a retract of p:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Z \arrow[r, "g"] \arrow[d, "p"] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y. \end{tikzcd}$$

So f is a trivial fibration by (M2). Thus

(i) a map is a trivial fibration if and only if it has the right lifting property with respect to cofibrations.

In exactly the same way one proves

(ii) a map is a fibration if and only if it has the right lifting property with respect to trivial cofibrations,
(iii) a map is a trivial cofibration if and only if it has the left lifting property with respect to fibrations,
(iv) a map is a cofibration if and only if it has the left lifting property with respect to trivial fibrations.

In particular, the classes of weak equivalences and fibrations together determine the class of cofibrations, and dually the class of fibrations is determined by the classes of weak equivalences and cofibrations. Also, the class of weak equivalences is determined in terms of the classes of fibrations and cofibrations. Indeed, the latter two also determine the classes of *trivial* cofibrations and fibrations by (i)–(iv) above, while a map $f : X \to Y$ is a weak equivalence if and only if it is a composition of a trivial cofibration $i : X \to Z$ with a trivial fibration $p : Y \to Z$. This composition is guaranteed to exist by the factorization axiom (M5), using (M2) to conclude that both constituent maps are weak equivalences. Summarizing, any two of the three classes defining a model structure determine the third.

(d) The characterization of the classes of cofibrations and of trivial cofibrations given in (iii) and (iv) above in particular show that these classes are saturated (in the sense of Definition 3.30). The classes of fibrations and of trivial fibrations satisfy dual properties. Isomorphisms are automatically in each of the three classes.

Notation 7.3 *It will often be convenient to decorate arrows denoting morphisms belonging to one or more of the three classes in a model structure:*

$$\begin{array}{ll} A \xrightarrow{\sim} B & \textit{for a weak equivalence,} \\ A \rightarrowtail B & \textit{for a cofibration,} \\ A \overset{\sim}{\rightarrowtail} B & \textit{for a trivial cofibration,} \\ X \twoheadrightarrow Y & \textit{for a fibration,} \\ X \overset{\sim}{\twoheadrightarrow} Y & \textit{for a trivial fibration.} \end{array}$$

Definition 7.4 Let $\mathcal{E}$ be a model category. An object X of $\mathcal{E}$ is called *fibrant* if the unique map $X \to 1$ into the terminal object 1 of $\mathcal{E}$ (which exists by (M1)) is a fibration. Dually, X is called *cofibrant* if the unique map $0 \to X$ from the initial object to X is a cofibration.

In practice one often needs to replace objects by weakly equivalent fibrant and/or cofibrant objects. Let us introduce the relevant terminology:

Definition 7.5 Let $\mathcal{E}$ be a model category and X an object of $\mathcal{E}$. A *fibrant replacement* of X is a weak equivalence $X \xrightarrow{\sim} X_f$ where X_f is fibrant. Dually, a *cofibrant replacement* is a weak equivalence $X_c \xrightarrow{\sim} X$ with X_c cofibrant.

Of course, for an arbitrary object X one can always factor the unique map $X \to 1$ as

$$X \overset{\sim}{\rightarrowtail} X_f \twoheadrightarrow 1$$

to find a fibrant replacement X_f of X. Dually, one factors $0 \to X$ as

$$0 \rightarrowtail X_c \overset{\sim}{\twoheadrightarrow} X$$

to obtain a cofibrant replacement. Notice that these replacements have the additional property that the relevant weak equivalences are in fact a trivial cofibration and trivial fibration, respectively.

We end this section by listing some very elementary examples of model categories. The reader less familiar with Quillen's axioms should do the useful exercise of verifying that the axioms hold in each of them.

First examples. (a) Consider the category **Cat** of small categories. Call a morphism (i.e., a functor) $f : \mathbf{C} \to \mathbf{D}$ a *weak equivalence* if it is an equivalence of categories, a *cofibration* if it is injective on objects, and a *fibration* if it is an isofibration. Recall that this means that for any object c of $\mathbf{C}$ and any isomorphism $\alpha : f(c) \to d$ in $\mathbf{D}$, there exists an isomorphism β in $\mathbf{C}$ with domain c satisfying $f(\beta) = \alpha$ (cf. Definition 5.46). It is not difficult to verify that the axioms (M1–5) hold for these classes of maps. This model structure on **Cat** if usually referred to as the *naive* or *folk* model structure. Recall that a functor $\mathbf{C} \to \mathbf{D}$ is a weak equivalence if and only if it is essentially surjective and fully faithful. One can check that the trivial fibrations are precisely the functors which are both fully faithful and surjective on objects. Also, any category is fibrant as well as cofibrant. We notice for later use that if

$$\mathbf{C}_0 \leftarrow \mathbf{C}_1 \leftarrow \mathbf{C}_2 \leftarrow \cdots \leftarrow \mathbf{C}_\xi \leftarrow \mathbf{C}_{\xi+1} \leftarrow \cdots$$

is a tower of trivial fibrations indexed by some limit ordinal λ then the projection

$$\varprojlim_{\xi<\lambda} \mathbf{C}_\xi \to \mathbf{C}_0$$

is again a trivial fibration.

(b) There is a similar model structure on the category of small groupoids, denoted **Grpd**, with the three classes of maps defined exactly as for **Cat**.

(c) The category **Op** of operads carries a 'naive' model structure generalizing that of example (a) above. Indeed, say a morphism $\varphi\colon \mathbf{P} \to \mathbf{Q}$ is a *weak equivalence* if it is an equivalence of operads, meaning

$$\mathbf{P}(c_1, \ldots, c_n; d) \to \mathbf{Q}(\varphi(c_1), \ldots, \varphi(c_n); \varphi(d))$$

is a bijection for every tuple of colours $c_1, \ldots, c_n, d$ of $\mathbf{P}$ and φ is essentially surjective (meaning the functor $j^*\varphi$ of underlying categories is essentially surjective). Cofibrations are the morphisms which are injective on colours and fibrations are the maps which give an isofibration on the underlying categories. The axioms are easily verified. Let us illustrate this by checking one of the lifting axioms. Consider a diagram of operads

$$\begin{tikzcd} \mathbf{A} \arrow[r, "u"] \arrow[d, "f"'] & \mathbf{P} \arrow[d, "p"] \\ \mathbf{B} \arrow[r, "v"] & \mathbf{Q} \end{tikzcd}$$

in which f is injective on objects and p is a trivial fibration, i.e., an equivalence of operads which is moreover an isofibration. Equivalently, p is an equivalence of operads which is surjective on colours. To find a lift $g\colon \mathbf{B} \to \mathbf{P}$, first pick for every colour b of $\mathbf{B}$ a colour $g(b)$ of $\mathbf{P}$ satisfying $v(b) = p(g(b))$ and moreover $g(b) = u(a)$ whenever $b = f(a)$. There is now a unique way to define g on operations in a way that is compatible with v, since

$$\mathbf{P}(c_1, \ldots, c_n; d) \to \mathbf{Q}(p(c_1), \ldots, p(c_n); p(d))$$

is bijective for every tuple of colours of $\mathbf{P}$. The map g constructed in this way will satisfy $pg = v$ by construction. To check that $gf = u$, note that this is true on colours by construction. On operations, it follows from the equation $vf = pu$ and the fact that p acts bijectively on sets of operations, as above.

(d) Let k be a field and consider the category of non-negatively graded chain complexes $V = (V_*, d)$ of vector spaces over k,

$$V = (V_0 \xleftarrow{d} V_1 \xleftarrow{d} V_2 \xleftarrow{d} \cdots).$$

This category carries a model structure in which a map is a *weak equivalence* if it is a quasi-isomorphism, i.e., if it induces isomorphisms in homology. Furthermore, the cofibrations are the maps which are injective in each degree, and the fibrations are the maps which are surjective in each strictly positive degree. Again, it is not too hard to verify that the axioms hold. In the literature one finds many extensions and variations on this example, notably where the field k is replaced by a ring R (and vector spaces by modules, of course). In this case, a cofibration is a degreewise injective map which also has degreewise projective cokernel.

(e) Later in this book we will encounter many constructions of new model categories from given ones. For now, let us just observe the following cases:

(e.i) If $\mathcal{E}$ is a model category, then so is its opposite category $\mathcal{E}^{\mathrm{op}}$, with the same weak equivalences but cofibrations and fibrations reversed.

(e.ii) If $\mathcal{E}$ is a model category and X is an object of $\mathcal{E}$, then so are the slice categories $\mathcal{E}/X$ and $X/\mathcal{E}$. The forgetful functors $\mathcal{E}/X \to \mathcal{E}$ and $X/\mathcal{E} \to \mathcal{E}$ define the three classes of maps. For example, a map

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & X & \end{tikzcd}$$

is a weak equivalence in $\mathcal{E}/X$ precisely if $A \to B$ is a weak equivalence in $\mathcal{E}$. In particular, if 1 denotes the terminal object one finds that the category $1/\mathcal{E} = \mathcal{E}_*$ of *pointed objects* of $\mathcal{E}$ carries a natural model structure. (The same fact and terminology apply when 1 is replaced by the unit for some tensor product on $\mathcal{E}$.)

(e.iii) If $\{\mathcal{E}_i\}_{i\in I}$ is a family of model categories indexed by a set I, then the product category $\prod_i \mathcal{E}_i$ again carries a model structure in which the three relevant classes of morphisms are simply defined componentwise. More precisely, a map $(X_i)_{i\in I} \to (Y_i)_{i\in I}$ is a fibration precisely if each $X_i \to Y_i$ is, and similarly for the cofibrations and weak equivalences.

7.2 Some Background on Topological Spaces

In the next section we will construct a model structure on the category of topological spaces. The proof uses some elementary facts from algebraic topology, which can be found in any of the standard textbooks and which we briefly recall here.

We begin with a discussion of cellular spaces and maps, and of CW-complexes. We write

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

for the n-disk and $\partial D^n \subseteq D^n$ for its boundary. One can of course use any homeomorphic model for the inclusion, e.g., the geometric realization $|\partial\Delta[n]| \to |\Delta[n]|$ of the boundary inclusion of the n-simplex. We say that a space Y is obtained from another space X by *attaching a family of cells* $\{D^{n_i} \mid i \in I\}$ if it can be written as a pushout

$$\begin{tikzcd} \coprod_{i\in I} \partial D^{n_i} \arrow[r] \arrow[d, tail] & X \arrow[d, tail] \\ \coprod_{i\in I} D^{n_i} \arrow[r] & Y. \end{tikzcd}$$

A map $X \to Y$ is called a *cellular extension* if it can be factored as

$$\begin{tikzcd} X = X_0 \arrow[r] \arrow[drrr] & X_1 \arrow[r] \arrow[drr] & X_2 \arrow[r] \arrow[dr] & \cdots \\ & & & Y \end{tikzcd}$$

with $X_\infty = \varinjlim_k X_k \to Y$ an isomorphism and each X_k obtained from X_{k-1} by attaching a family of cells. More briefly, a cellular extension is a directed colimit of cell attachments. We also say Y is a cellular extension of X. By definition of the colimit, X_∞ has the weak topology with respect to the inclusions $X_k \to X_\infty$. A *relative CW-complex* is a cellular extension $X \to Y$ equipped with a factorization as above in which $X_{n-1} \to X_n$ is obtained by attaching cells of dimension n only. An important property of cellular extensions is that 'compact subsets are contained in finitely many cells'. We state this as follows:

Lemma 7.6 *Let A be a topological space and let $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ be a sequence of subspaces such that*

- *$A = \bigcup_n A_n$ and A has the weak topology with respect to the A_n,*
- *A_n is closed in A_{n+1},*
- *$A_n - A_{n-1}$ is a T_1-space.*

Then any compact subset $K \subseteq A$ is contained in some A_n.

Proof For a contradiction, suppose $K \subseteq A$ is compact and not contained in any A_n. Replacing $(A_n)_{n\geq 0}$ by a subsequence if necessary, we may suppose that there are points $x_n \in A_n - A_{n-1}$ which belong to K. Since K is compact, the collection $S = \{x_n\}_{n\geq 0}$ must have an accumulation point. On the other hand we claim that S is closed and discrete, which is a contradiction. Indeed, it suffices to prove that each $S \cap A_n$ is closed and discrete for every n. Clearly $S \cap A_0$ is. Now suppose the same is true of $S \cap A_{n-1}$. Then

$$S \cap A_n = (S \cap A_{n-1}) \cup (S \cap (A_n - A_{n-1}))$$

and $S \cap (A_n - A_{n-1})$ is open and closed in $S \cap A_n$ by the assumptions. This proves the lemma. $\square$

The next basic notion we need is that of a *Serre fibration*. Recall that a map $E \longrightarrow B$ is a Serre fibration if any commutative square if of the form

$$\begin{tikzcd} I^n \times \{0\} \arrow[r] \arrow[d] & E \arrow[d] \\ I^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

has a diagonal filling as indicated. Here $I = [0, 1] \subseteq \mathbb{R}$ is the unit interval. A diagonal in such a square is at the same time an *extension* of the map $I^n \times \{0\} \longrightarrow E$ and a *lift* of the homotopy $I^n \times I \longrightarrow B$, and one refers to this property of $E \longrightarrow B$ as the 'homotopy extension and lifting property' (HELP). Of course one can replace $I^n \times \{0\} \subseteq I^n \times I$ by any homeomorphic inclusion. Convenient and often used models are the inclusion

$$A^{n+1} = I^n \times \{0\} \cup \partial I^n \times I \longrightarrow I^{n+1}$$

(of a box without a lid into a solid cube) and the realizations of the simplicial horn inclusions

$$|\Lambda^k[n]| \longrightarrow |\Delta[n]| = \Delta^n.$$

In fact, since the class of maps with respect to which $E \longrightarrow B$ has the right lifting property is saturated, it follows that for any anodyne extension of simplicial sets $M \longrightarrow N$ (see Section 5.3), the Serre fibration $E \longrightarrow B$ has the right lifting property with respect to $|M| \longrightarrow |N|$. Notice in particular that this means that the singular complex functor Sing maps a Serre fibration to a Kan fibration.

A basic property of Serre fibrations is the *long exact sequence of homotopy groups*. Let $p : E \longrightarrow B$ be a Serre fibration, and let $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$ be a choice of basepoints. Write $F = p^{-1}(b_0)$ for the fibre and $i : F \longrightarrow E$ for the inclusion. There is a long exact sequence

$$\cdots \longrightarrow \pi_n F \xrightarrow{i_*} \pi_n E \xrightarrow{p_*} \pi_n B \xrightarrow{\delta} \pi_{n-1} F \longrightarrow \cdots$$

where we have omitted the basepoints from the notation. In low degrees this sequence passes from abelian groups to groups to pointed sets, but exactness continues to make sense. The readers unfamiliar with this sequence can certainly prove this for themselves, once provided with the definition of the map $\delta : \pi_n B \longrightarrow \pi_{n-1} F$. Given $\alpha \in \pi_n B$, represent α by a map $a : I^n \longrightarrow B$ sending the boundary to b_0, and lift in

$$\begin{tikzcd} A^n \arrow[r, "e_0"] \arrow[d] & E \arrow[d, "p"] \\ I^n \arrow[r, "a"] \arrow[ur, dashed, "b"] & B. \end{tikzcd}$$

Then $\delta\alpha$ is the element in $\pi_{n-1} F$ represented by the restriction of b to $I^{n-1} \times \{1\}$.

Lemma 7.7 *Let $p\colon E \to B$ be a map between topological spaces. Then the following properties are equivalent:*

(1) *The map p is a Serre fibration inducing isomorphisms $\pi_n E \to \pi_n B$ for each $n \geq 0$ and each (compatible) choice of basepoints.*
(2) *The map p has the right lifting property with respect to the boundary inclusion $\partial D^n \to D^n$ for each $n \geq 0$.*
(3) *The map p has the right lifting property with respect to all cellular maps.*

Proof The implication (2) ⇒ (3) is clear because the class of maps with respect to which p has the right lifting property is saturated. Conversely (2) is a special case of (3). We now prove (3) implies (1). Clearly p must be a Serre fibration. To prove that the maps $\pi_n E \to \pi_n B$ are isomorphisms, fix $n \geq 0$ and basepoints $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$. Let $a : D^n \to B$ represent an element $\alpha \in \pi_n B$ and lift in the diagram

$$\begin{tikzcd} \partial D^n \arrow[r, "e_0"] \arrow[d] & E \arrow[d] \\ D^n \arrow[r, "a"] \arrow[ur, dashed] & B \end{tikzcd}$$

to see that $\pi_n(E, e_0) \to \pi_n(B, b_0)$ is surjective. To see that it is injective, let $\alpha, \beta : D^n \rightrightarrows E$ be two maps sending ∂D^n to e_0, which become homotopic relative to the boundary once composed with b, say by a homotopy h. Then a lift in

$$\begin{tikzcd} D^n \times \{0,1\} \cup \partial D^n \times I \arrow[r, "\alpha\cup\beta\cup e_0"] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r, "b"] \arrow[ur, dashed] & B \end{tikzcd}$$

shows that α and β are also homotopic relative to the boundary. Finally, we prove (1) ⇒ (2). Let $p\colon E \to B$ be a Serre fibration inducing isomorphisms on all homotopy groups and consider a commutative square

$$\begin{tikzcd} \partial D^n \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ D^n \arrow[r, "g"] \arrow[ur, dashed, "\ell"] & B, \end{tikzcd}$$

in which we need to find a diagonal lift ℓ. Pick a basepoint $d_0 \in \partial D_n$ and write $e_0 = f(d_0)$ and $b_0 = p(e_0)$. Since D^n is contractible, there exists a homotopy

$$h\colon D^n \times I \to B$$

from $g = h_0$ to the constant map h_1 with value b_0. Since p is a Serre fibration, we can lift the restriction of this homotopy to the boundary ∂D^n to a homotopy $\overline{h}$ as in the following square:

$$
\begin{tikzcd}
\partial D^n \times \{0\} \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\
\partial D^n \times I \arrow[r, "h"'] \arrow[ur, dashed, "\overline{h}"] & B.
\end{tikzcd}
$$

Then the map $\overline{h}_1 \colon \partial D^n \to E$ has image contained in the fibre $p^{-1}(b_0)$. By the long exact sequence of the fibration p this fibre has vanishing homotopy groups, so that there must exist an extension of $\overline{h}_1$ to a map $k \colon D^n \to p^{-1}(b_0)$. Now choose a lift L in the following diagram:

$$
\begin{tikzcd}
\partial D^n \times I \cup D^n \times \{1\} \arrow[r, "\overline{h} \cup k"] \arrow[d] & E \arrow[d, "p"] \\
D^n \times I \arrow[r, "h"'] \arrow[ur, dashed, "L"] & B
\end{tikzcd}
$$

Then the restriction of L to $D^n \times \{0\}$ provides a solution ℓ to our original lifting problem. $\square$

Remark 7.8 We already observed that applying Sing to a Serre fibration gives a Kan fibration of simplicial sets. Lemma 7.7 also shows that Sing of a Serre fibration which is also a weak homotopy equivalence gives a trivial fibration of simplicial sets.

For later use, we record the fact that being a Serre fibration is a local property:

Proposition 7.9 *Let $p \colon E \to B$ be a map with the property that every $b \in B$ has a neighbourhood U such that the restriction*

$$p^{-1}U \xrightarrow{p} U$$

is a Serre fibration. Then p is itself a Serre fibration.

Proof Let p be as in the statement of the proposition and consider a lifting problem

$$
\begin{tikzcd}
I^n \times \{0\} \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\
I^{n+1} \arrow[r, "g"] \arrow[ur, dashed, "h"] & B.
\end{tikzcd}
$$

By compactness of I^{n+1} and the Lebesgue covering lemma, there exists a natural number N such that if we subdivide I^{n+1} into a grid of N^{n+1} little cubes with side lengths $1/N$, the image under p of each little cube is contained in an open set U over which p is a Serre fibration. Order these cubes lexicographically and label them $C_1, \ldots, C_{N^{n+1}}$. Write $I_r := I^n \times \{0\} \cup C_1 \cup \cdots \cup C_r$. We will now define lifts

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ I_r \arrow[r, "g"] \arrow[ur, dashed, "h_r"] & B. \end{tikzcd}$$

by induction on r. To extend a given lift h_r from I_r to I_{r+1}, we have to extend h_r over the cube C_r. Thus it suffices to solve the lifting problem

$$\begin{tikzcd} C_r \cap I_r \arrow[r, "h_r|_{C_r}"] \arrow[d] & E \arrow[d, "p"] \\ C_r \arrow[r, "g"] \arrow[ur, dashed] & B. \end{tikzcd}$$

This can be done because $C_r \cap I_r \to C_r$ is itself homeomorphic to the inclusion $I^n \times \{0\} \to I^{n+1}$. Indeed, the intersection $C_r \cap I_r$ is essentially a variation of the 'box without a lid'

$$I^n \times \{0\} \cup \partial I^n \times I,$$

but possibly with ∂I^n replaced by a smaller union of faces of I^n. □

Corollary 7.10 *A locally trivial fibre bundle $p\colon E \to B$ is a Serre fibration.*

Proof By definition every point in B has a neighbourhood on which p is homeomorphic to the projection of a product onto one of its factors, which is evidently a Serre fibration. □

7.3 A Model Structure for Topological Spaces

In this section we shall use the basic facts from the previous section to construct a model structure on the category **Top** of (compactly generated weak Hausdorff) topological spaces. The relevant classes of maps are defined as follows. A map $f : X \to Y$ is called a *weak equivalence* if it induces a bijection $\pi_0 X \to \pi_0 Y$ and an isomorphism $\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for any $n \geq 1$ and any choice of basepoint $x_0 \in X$. A map $f : X \to Y$ is called a *fibration* if it is a Serre fibration and a *cofibration* if it is a retract of a cellular extension.

Theorem 7.11 (Quillen) *These classes of maps constitute a model structure on the category* **Top**.

Proof The axioms (M1) (finite limits and colimits), (M2) (two-out-of-three for weak equivalences) and (M3) (retracts) are obviously satisfied. For (M5), observe that any map $f : X \to Y$ can be factored as a cellular map $i : X \to Z$ followed by a map $p : Z \to Y$ having the right lifting property with respect to all cellular maps. Indeed, one applies the small object argument (cf. Remark 3.38) to the set of boundary inclusions of cells $\partial D^n \to D^n$ ($n \geq 0$). In a bit more detail, one sets $Z_0 := X$ and for $k \geq 1$ one inductively defines factorizations

$$X \xrightarrow{i_k} Z_k \xrightarrow{p_k} Y$$

of f as follows. Suppose Z_k, i_k, and p_k have been defined. Write $\mathcal{S}_k$ for the set of all commutative squares of the form

$$\begin{tikzcd} \partial D^n \arrow[r] \arrow[d] & Z_k \arrow[d, "p_k"] \\ D^n \arrow[r] & Y. \end{tikzcd}$$

Define Z_{k+1} by a pushout square

$$\begin{tikzcd} \coprod_{S \in \mathcal{S}_k} \partial D^n \arrow[r] \arrow[d] & Z_k \arrow[d] \\ \coprod_{S \in \mathcal{S}_k} D^n \arrow[r] & Z_{k+1} \end{tikzcd}$$

and define i_{k+1} to be the composite of i_k followed by the right-hand vertical map in the square. The universal property of the pushout defines the map $p_{k+1} : Z_{k+1} \to Y$. Taking the colimit over k gives a factorization

$$X \xrightarrow{i_\infty} Z_\infty \xrightarrow{p_\infty} Y$$

of f in which i_∞ is a cellular extension. To see that p_∞ has the desired right lifting property with respect to cellular maps, we should solve lifting problems of the form

$$\begin{tikzcd} \partial D^n \arrow[r] \arrow[d] & Z_\infty \arrow[d, "p_\infty"] \\ D^n \arrow[r] \arrow[ur, dashed, "\ell"] & Y. \end{tikzcd}$$

Observe that the map $\partial D^n \to Z_\infty$ must factor through some Z_k by Lemma 7.6. The resulting square

$$\begin{tikzcd} \partial D^n \arrow[r] \arrow[d] & Z_k \arrow[d, "p_k"] \\ D^n \arrow[r] & Y. \end{tikzcd}$$

defines an element of $\mathcal{S}_k$ and a corresponding map $\ell_k : D^n \to Z_{k+1}$. The composite

$$D^n \xrightarrow{\ell_k} Z_{k+1} \to Z_\infty$$

gives the desired lift. To conclude that p_∞ is a trivial fibration, we use that a map having the right lifting property with respect to all cellular maps is a Serre fibration and a weak equivalence by Lemma 7.7.

The other factorization, into a trivial cofibration followed by a fibration, is proved in exactly the same way, now using the inclusions

$$I^n \times \{0\} \to I^n \times I \tag{7.1}$$

instead of the boundary inclusions $\partial D^n \to D^n$. These inclusions are all cellular and part of strong deformation retracts (hence in particular weak equivalences). Hence if $\{A_i \to B_i\}$ is a family of maps of the form (7.1), then their coproduct is again cellular and a strong deformation retract and so is any pushout of it. Finally, if

$$X_0 \to X_1 \to X_2 \to \cdots$$

is a sequence of cellular strong deformations retracts, then each $A_i \to \varinjlim_k A_k$ is as well, as the reader can easily verify. Hence applying the small object argument to factor a map $X \to Y$ into a (transfinite) composition of pushouts of maps of the form (7.1) (for varying $n \geq 0$) followed by a map having the right lifting property with respect to such finishes the argument.

It remains to prove the lifting axiom (M4). So consider a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i"] & X \arrow[d, two heads, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

where i is a cofibration and p is a fibration. If p is also a weak equivalence, then p has the right lifting property with respect to all cellular extensions by Lemma 7.7. But then it also has the right lifting property with respect to any retract of a cellular extension, i.e., with respect to any cofibration. If instead i is also a weak equivalence, then we can factor i as $v \circ j$ with $j : A \to C$ in the saturation of the class of maps of type (7.1) and v a Serre fibration, as in the second factorization for (M5) just discussed. Observe that j has the left lifting property with respect to any Serre fibration. Also, since i and j are weak equivalences, so is v by the two-out-of-three axiom. Thus by Lemma 7.7 again, v has the right lifting property with respect to all cellular maps. We can now find a diagonal filling in the square above by lifting in two steps:

$$\begin{tikzcd} A \arrow[r, tail, "j"] \arrow[d, tail, "i"] & C \arrow[d, two heads, "v"] \\ B \arrow[r, equal] \arrow[ur, dashed, "r"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "j"] & X \arrow[d, two heads, "p"] \\ C \arrow[r, "gv"'] \arrow[ur, dashed, "h"'] & Y. \end{tikzcd}$$

Then hr is the required lift and the proof is complete. □

Remark 7.12 One can reinterpret the last step in the proof above in the following way. Write $\mathcal{J}$ for the class of maps described in (7.1). Then the left-hand square above shows that any trivial cofibration i is a retract of a $\mathcal{J}$-cellular map (see Definition 5.18). Indeed, one can redraw that diagram as follows:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, tail, "i"] & A \arrow[r, equal] \arrow[d, tail, "j"] & A \arrow[d, tail, "i"] \\ B \arrow[r, "r"] & C \arrow[r, "v"] & B. \end{tikzcd}$$

Since Serre fibrations have the right lifting property with respect to $\mathcal{J}$ (by definition), they also have the right lifting property with respect to any retract of a $\mathcal{J}$-cellular map.

Generally, if $\mathcal{J}$ is a set of maps in a model category with the property that fibrations have the right lifting property with respect to $\mathcal{J}$, then the argument above shows that any trivial cofibration is a retract of a $\mathcal{J}$-cellular map. This is often called the 'retract argument'. It follows that the class of trivial cofibrations is the saturation of $\mathcal{J}$. Indeed, we just argued that any trivial cofibration is in this saturation. Conversely, any map in $\mathcal{J}$ is a trivial cofibration (it has the left lifting property with respect to fibrations), hence so is any map in the saturation of $\mathcal{J}$.

Remark 7.13 The proof of Theorem 7.11 just given uses the small object argument in two ways: to construct the cofibrations as the saturation of the *set* of boundary inclusions $\partial D^n \to D^n$, $n \geq 0$, and to construct the trivial cofibrations as the saturation of the *set* of inclusions $I^n \times \{0\} \to I^n \times I$, $n \geq 0$. The crucial properties that made our argument work were

(i) the inclusions $I^n \times \{0\} \to I^n \times I$ are themselves cellular extensions,
(ii) the saturation of these $I^n \times \{0\} \to I^n \times I$ is contained in the class of weak equivalences,
(iii) the domains ∂D^n and I^n are compact, so factor through a finite stage of a colimit of a sequence
$$X_0 \to X_1 \to X_2 \to \cdots$$
of cellular extensions.

More generally, a model category $\mathcal{E}$ is said to be *cofibrantly generated* if there are *sets* of maps $\mathcal{I}$ and $\mathcal{J}$ whose saturations are the classes of cofibrations and of trivial cofibrations, respectively, satisfying conditions (i')–(iii') listed below. These sets $\mathcal{I}$ and $\mathcal{J}$ are then called the sets of *generating cofibrations* and *generating trivial cofibrations*. In turn, in the process of trying to establish a model structure on a category $\mathcal{E}$ (which possesses all the necessary colimits), one can use two such sets $\mathcal{I}$ and $\mathcal{J}$ to construct factorizations as in (M5). The saturation of $\mathcal{I}$ will then be defined to be the class of cofibrations and one hopes to prove that the saturation of $\mathcal{J}$ acts as the class of trivial cofibrations. One can prove (M5) as we did if conditions analogous to (i)–(iii) above hold, namely

(i') $\mathcal{J}$ is contained in the saturation of $\mathcal{I}$,
(ii') the saturation of $\mathcal{J}$ is contained in the class of weak equivalences,
(iii') if

$$X_0 \to X_1 \to X_2 \to \cdots$$

is a sequence of $\mathcal{I}$-cellular maps and K is the domain of a map in $\mathcal{I}$ or $\mathcal{J}$, then any map

$$K \to \varinjlim_k X_k$$

factors through some X_i.

This argument also works if condition (iii') is satisfied for colimits of continuous sequences indexed over a fixed regular cardinal κ. However, for most of the model structures we construct in this book, either the countable case described above or the case where κ is the first uncountable cardinal suffices.

Remark 7.14 The model structure of Theorem 7.11 satisfies several additional useful properties. One of these is that in a pullback square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r, "v"] & Y \end{tikzcd}$$

in which g is a fibration and v is a weak equivalence, the map u is again a weak equivalence. If this additional property holds in a model structure, then one says it is *right proper*. We will come back to this property in a more general context in Section 7.6 below. For now, let us give a proof which is somewhat particular to the case of topological spaces. Write F and G for the respective fibres of f and g (for chosen basepoints $b \in B$ and $v(b) = y \in Y$) and $w : F \to G$ for the restriction of u. Then w is a homeomorphism, so in particular a weak equivalence. It now follows by the five lemma that u is a weak equivalence whenever v is, simply comparing the long exact sequences of homotopy groups induced by f and g.

$$\begin{tikzcd} \cdots \arrow[r] & \pi_n F \arrow[r] \arrow[d] & \pi_n A \arrow[r] \arrow[d] & \pi_n B \arrow[r] \arrow[d] & \pi_{n-1} F \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \pi_n G \arrow[r] & \pi_n X \arrow[r] & \pi_n Y \arrow[r] & \pi_{n-1} G \arrow[r] & \cdots \end{tikzcd}$$

Remark 7.15 Dually to the previous remark, the model structure of Theorem 7.11 is also *left proper*, meaning that the pushout of a weak equivalence along a cofibration is again a weak equivalence. To see this, recall that a cofibration is a retract of a cellular extension. Since weak equivalences of topological spaces are preserved under direct limits, it suffices to prove that for a weak equivalence $f : X \to Y$ and a cell attachment

$$\begin{tikzcd} S^n \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \\ D^{n+1} \arrow[r] & X \cup_\alpha D^{n+1}, \end{tikzcd}$$

the corresponding map $\widetilde{f}\colon X \cup_\alpha D^{n+1} \to Y \cup_{f\alpha} D^{n+1}$ is a weak equivalence. To see this, note that $\widetilde{f}$ still gives a bijection on path components and moreover an isomorphism on fundamental groups (at any basepoint) by van Kampen's theorem. A standard Mayer–Vietoris argument shows that $\widetilde{f}$ induces an isomorphism on homology (with any local coefficient system). These facts together imply that $\widetilde{f}$ is a weak equivalence.

7.4 Homotopies Between Morphisms in a Model Category

Recall that an object in a model category is called fibrant if it maps to the terminal object by a fibration and cofibrant if the map from the initial object into it is a cofibration. In this section we will describe a notion of homotopy for morphisms in a model category $\mathcal{E}$ which behaves well as long as the domain is cofibrant and the codomain is fibrant. This leads to a well-defined category $\mathrm{Ho}(\mathcal{E})$ of objects which are both fibrant and cofibrant, and homotopy classes of maps between them as morphisms. The main result of this section is Proposition 7.27, stating that a morphism between fibrant-cofibrant objects is a weak equivalence if and only if it is a homotopy equivalence. For the rest of this section we work with a fixed model category $\mathcal{E}$.

Definition 7.16 (a) Let A be an object of $\mathcal{E}$. A *cylinder on A* is a factorization of the fold map $\nabla : A \amalg A \to A$ into a cofibration followed by a trivial fibration:

$$\begin{tikzcd} A \amalg A \arrow[r, tail, "(i_0, i_1)"] \arrow[dr, "\nabla"'] & \mathrm{Cyl}(A) \arrow[d, two heads, "\varepsilon"', "\wr"] \\ & A. \end{tikzcd}$$

If ε is only a weak equivalence, we speak of a *weak cylinder*.

(b) Dually, for an object X, a *path object for X* if is a factorization of its diagonal into a trivial cofibration followed by a fibration:

$$\begin{tikzcd} X \arrow[r, tail, "\sim", "c"'] \arrow[dr, "\Delta"'] & PX \arrow[d, two heads, "(p_0, p_1)"] \\ & X \times X. \end{tikzcd}$$

We speak of a *weak path object* if c is only a weak equivalence.

Remark 7.17 (a) In a weak cylinder object, the maps i_0 and i_1 are trivial cofibrations if A is cofibrant. Indeed, each of the two coproduct inclusions $A \to A \amalg A$ is a cofibration because

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & A \amalg A \end{tikzcd}$$

is a pushout. Composition of (i_0, i_1) with each of these coproduct inclusions shows that i_0 and i_1 are cofibrations. By two-out-of-three they are weak equivalences, using that $\varepsilon i_0 = \mathrm{id}_A = \varepsilon i_1$ and ε is a weak equivalence by assumption.

(b) Dually, the maps p_0 and p_1 in a path object are both trivial fibrations, provided X is fibrant.

Example 7.18 For the category of topological spaces with the model structure of the previous section, the usual cylinder

$$X \amalg X \xrightarrow{(i_0, i_1)} X \times I \xrightarrow{\mathrm{pr}_1} X$$

of a space X provides a cylinder object. Dually, the path space

$$X \xrightarrow{\text{const}} X^I \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} X \times X$$

provides a weak path object. In the category of small categories equipped with the naive model structure, one can use the functors

$$\mathbf{C} \amalg \mathbf{C} \xrightarrow{(i_0, i_1)} \mathbf{C} \times \tau(J) \xrightarrow{\mathrm{pr}_1} \mathbf{C}$$

to provide a cylinder object for a category $\mathbf{C}$. Recall that $\tau(J)$ is the groupoid consisting of two objects and an isomorphism between them. Similarly to the case of spaces, the functor category $C^{\tau(J)}$ gives a path object for $\mathbf{C}$.

Definition 7.19 Let A and X be objects of $\mathcal{E}$.

(a) Two maps $f, g : A \to X$ are called *left homotopic* if there exists a weak cylinder $\mathrm{Cyl}(A)$ and a map

$$h : \mathrm{Cyl}(A) \to X$$

for which $hi_0 = f$ and $hi_1 = g$. We write $f \sim_L g$. We call such an h a *left homotopy* between f and g.

(b) Dually, f and g are called *right homotopic* if there is a weak path object PX and a *right homotopy*

$$k : A \to PX$$

for which $p_0 k = f$ and $p_1 k = g$. We write $f \sim_R g$.

Proposition 7.20 *Let A be a cofibrant object and X a fibrant object.*

(a) *Being left homotopic is independent of the choice of weak cylinder and defines an equivalence relation on the set of morphisms from A to X.*
(b) *The dual statement holds for right homotopic maps.*
(c) *The relations of being left and right homotopic coincide.*

Proof The proofs are elementary applications of the axioms. We present a few observations concerning (a) and (c) and leave further details to the reader.

First of all, notice that $f \sim_L g$ via a left homotopy h using a weak cylinder object $A \amalg A \to \mathrm{Cyl}(A) \to A$, then there is also a left homotopy from a cylinder. Indeed, we can factor $\varepsilon : \mathrm{Cyl}(A) \to A$ as a trivial cofibration followed by a trivial fibration

$$\begin{tikzcd} \mathrm{Cyl}(A) \arrow[r, tail, "\sim"] & \mathrm{Cyl}'(A) \arrow[r, two heads, "\sim", "\varepsilon'"'] & A \end{tikzcd}$$

and lift in

$$\begin{tikzcd} \mathrm{Cyl}(A) \arrow[r, "h"] \arrow[d, tail, "\wr"] & X \arrow[d] \\ \mathrm{Cyl}'(A) \arrow[r] \arrow[ur, dashed] & 1. \end{tikzcd}$$

The dashed map is now a left homotopy from the actual cylinder $\mathrm{Cyl}'(A)$.

Next, if $\mathrm{Cyl}_1(A)$ and $\mathrm{Cyl}_2(A)$ with their associated maps are two cylinders on A, then a lift in

$$\begin{tikzcd} A \amalg A \arrow[r] \arrow[d, tail] & \mathrm{Cyl}_1(A) \arrow[d, two heads, "\wr"] \\ \mathrm{Cyl}_2(A) \arrow[r] \arrow[ur, dashed] & A \end{tikzcd}$$

shows that if two maps f and g are left homotopic via $\mathrm{Cyl}_1(A)$, then they also are via $\mathrm{Cyl}_2(A)$. This shows that being left homotopic is independent of the cylinder.

If $A \amalg A \xrightarrow{(i_0,i_1)} \mathrm{Cyl}(A) \xrightarrow{\varepsilon} A$ is a cylinder, then so is $A \amalg A \xrightarrow{(i_1,i_0)} \mathrm{Cyl}(A) \xrightarrow{\varepsilon} A$, which shows the symmetry of the left homotopy relation. Transitivity is proved by gluing two copies of a cylinder, as in the pushout

$$\begin{tikzcd} A \arrow[r, tail, "i_0"] \arrow[d, tail, "i_1"] & \mathrm{Cyl}(A) \arrow[d, tail, "j_1"] \\ \mathrm{Cyl}(A) \arrow[r, tail, "j_0"] & \mathrm{Cyl}'(A). \end{tikzcd}$$

Then $A \amalg A \xrightarrow{(i_0',i_1')} \mathrm{Cyl}'(A) \xrightarrow{\varepsilon'} A$ is a weak cylinder, where ε' is the unique map with $\varepsilon' j_0 = \varepsilon = \varepsilon' j_1$, and $i_0' = j_0 i_0$ while $i_1' = j_1 i_1$.

For part (c), suppose $\mathrm{Cyl}(A)$ and PX are cylinder and path objects respectively, and suppose $h : \mathrm{Cyl}(A) \to X$ is a left homotopy from f to g as above. The map $f\varepsilon$ is a left homotopy from f to itself and we can lift in

$$\begin{tikzcd} A \arrow[r, "cf"] \arrow[d, tail, "i_0"', "\wr"] & PX \arrow[d, two heads] \\ \mathrm{Cyl}(A) \arrow[r, "{(h,f\varepsilon)}"'] \arrow[ur, dashed, "l"] & X \times X \end{tikzcd}$$

to obtain a map l for which li_1 is a right homotopy from g to f. The argument that right homotopic implies left homotopic is dual. □

Definition 7.21 For a cofibrant object A and a fibrant object X, we shall write $[A, X]$ for the set of equivalence classes of the equivalence relation of Proposition 7.20 and call its elements *homotopy classes* of maps from A to X.

The following basic 'homotopy lifting lemma' is very useful:

Lemma 7.22 *Let B be a cofibrant object and $p : X \to Y$ a fibration. If in a diagram*

$$\begin{tikzcd} & X \arrow[d, two heads, "p"] \\ B \arrow[r, "g"] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists a map $k : B \to X$ such that pk is left homotopic to g, then there also exists a map $l : B \to X$ with $pl = g$ and l left homotopic to k. In other words, a lift up to left homotopy is left homotopic to an actual lift. A dual statement applies to diagrams of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "j"'] & X \\ B \arrow[ur, dashed] & \end{tikzcd}$$

where j is a cofibration and X is fibrant.

Proof Suppose $h : \mathrm{Cyl}(B) \to Y$ is a homotopy from $hi_0 = pk$ to $hi_1 = g$. Then a lift H exists in the diagram

$$\begin{tikzcd} B \arrow[r, "k"] \arrow[d, tail, "i_0"', "\wr"] & X \arrow[d, two heads, "p"] \\ \mathrm{Cyl}(B) \arrow[r, "h"'] \arrow[ur, dashed, "H"'] & Y. \end{tikzcd}$$

Setting $l = Hi_1$ gives the desired lift $B \to X$. □

Remark 7.23 The lifting axiom (M4) merely requires the existence of a lift and says nothing about its possible uniqueness. However, these lifts are always unique up to fibrewise relative homotopy. More precisely, if $f : X \to Y$ is a fibration we can form a *fibrewise path object* by factoring $\Delta : X \to X \times_Y X$ as

$$\begin{tikzcd} X \arrow[r, tail, "\sim", "c"'] \arrow[dr, "\Delta"'] & P_Y X \arrow[d, two heads, "{(p_0,p_1)}"] \\ & X \times_Y X. \end{tikzcd}$$

Then if k and l are two lifts in a commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, tail, "i"', "\wr"] & X \arrow[d, two heads, "f"] \\ B \arrow[r, "v"] \arrow[ur, dashed] & Y \end{tikzcd}$$

where i is a trivial cofibration and f a fibration, we can lift in

$$\begin{tikzcd} A \arrow[r, "cu"] \arrow[d, tail, "i"', "\wr"] & P_Y X \arrow[d, two heads, "(p_0,p_1)"] \\ B \arrow[r, "(k,l)"'] \arrow[ur, dashed, "h"] & X \times_Y X \end{tikzcd}$$

to find a *fibrewise homotopy h relative to A*. Composition with a lift in

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail, "c"', "\wr"] & PX \arrow[d, two heads] \\ P_Y X \arrow[r] \arrow[ur, dashed] & X \times X \end{tikzcd}$$

shows that such fibrewise relative homotopies in particular yield ordinary right homotopies. There is of course a dual statement for squares involving a cofibration i and a trivial fibration f.

Remark 7.24 In fact the fibrewise path object $P_Y X$ described above is simply a path object in the sense of Definition 7.16 for the (fibrant) object $X \to Y$ of the slice category $\mathcal{E}/Y$. Hence Remark 7.17 applies and the projections $p_0, p_1 : P_Y X \to X$ are trivial fibrations. Dual remarks apply to *relative cylinder objects* $\mathrm{Cyl}_A(B)$ associated to a cofibration $A \to B$.

Proposition 7.25 *Let $A \to B$ be a trivial cofibration between cofibrant objects and let $X \to Y$ be a trivial fibration between fibrant objects. Then composition with these maps induces bijections*

$$\begin{tikzcd} {[B, X]} \arrow[r, "\cong"] \arrow[d, "\cong"] & {[B, Y]} \arrow[d, "\cong"] \\ {[A, X]} \arrow[r, "\cong"] & {[A, Y]}\,. \end{tikzcd}$$

Proof Note first that postcomposition with $X \to Y$ obviously respects the left homotopy relation, while precomposition with $A \to B$ preserves the right homotopy relation, so these maps are well-defined. Let us prove that the map $[B, X] \to [A, X]$ on the left of the diagram is a bijection. The other cases are identical or dual. First of all, lifting in

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "\wr"] & X \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, dashed] & 1 \end{tikzcd}$$

shows that $\mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$ is surjective, so the same is true of $[B, X] \to [A, X]$. For injectivity, choose a weak cylinder $A \amalg A \to \mathrm{Cyl}(A) \to A$ for A and define a weak cylinder for B as the pushout in the left square of the following diagram.

$$\begin{tikzcd}
A \amalg A \arrow[r, tail] \arrow[d, tail, "\wr"] & \mathrm{Cyl}(A) \arrow[r, "\sim"] \arrow[d, tail, "\wr"] & A \arrow[d, tail, "\wr"] \\
B \amalg B \arrow[r, tail] & \mathrm{Cyl}(B) \arrow[r] & B
\end{tikzcd}$$

The vertical map on the left is a trivial cofibration since $A \to B$ is, so that $\mathrm{Cyl}(A) \to \mathrm{Cyl}(B)$ is a trivial cofibration as well. Similarly, the map $B \amalg B \to \mathrm{Cyl}(B)$ is a cofibration since $A \amalg A \to \mathrm{Cyl}(B)$ is. The universal property of the pushout gives the map $\mathrm{Cyl}(B) \to B$ on the lower right, which is necessarily a weak equivalence by the two-out-of-three axiom. With this particular choice of cylinder, one easily shows that if two maps $B \to X$ become left homotopic via $\mathrm{Cyl}(A)$ after composing with $A \to B$, they are themselves already left homotopic via $\mathrm{Cyl}(B)$. So $[B, X] \to [A, X]$ is injective as well. □

Note that the proof of Proposition 7.25 above starts with the observation that composition of homotopy classes of maps is well-defined. Hence we can make the following definition:

Definition 7.26 The *homotopy category* $\mathrm{Ho}(\mathcal{E})$ is the category with as objects those objects in $\mathcal{E}$ which are both fibrant and cofibrant, and as morphisms the homotopy classes of maps.

Let us say that a map $f : X \to Y$ between objects which are both fibrant and cofibrant is a *homotopy equivalence* if its homotopy class is an isomorphism in $\mathrm{Ho}(\mathcal{E})$. Equivalently, f is a homotopy equivalence if there exists a map $g : Y \to X$ with fg homotopic to id_Y and gf homotopic to id_X. Clearly homotopy equivalences satisfy the two-out-of-three property. We end this section by collecting some basic facts about homotopy equivalences in a model category. The main result is the following:

Proposition 7.27 *Let $f : X \to Y$ be a map between objects which are both fibrant and cofibrant.*

(i) *If f is a weak equivalence, then it is a homotopy equivalence.*
(ii) *If f is a homotopy equivalence, then it is a weak equivalence.*

Proof (of Proposition 7.27) (i). Suppose f is a weak equivalence. Then we may factor it is a trivial cofibration $j : X \to Z$ followed by a trivial fibration $p : Z \to Y$. So it suffices to prove that each of these two is a homotopy equivalence. This is immediate from Proposition 7.25 and the Yoneda lemma. In detail, precomposition with j induces bijections $[Z, X] \to [X, X]$ and $[Z, Z] \to [X, Z]$. The first gives the existence of a map $r : Z \to X$ with $ri \sim \mathrm{id}_X$. To prove that $ir \sim \mathrm{id}_Z$, it suffices by the second bijection to show that $iri \sim i$, which follows from $ri \sim \mathrm{id}_X$. The argument for the trivial fibration p is similar. □

To prove Proposition 7.27(ii) we will need some preparation.

Lemma 7.28 *Let X and Y be both fibrant and cofibrant and suppose $q : X \to Y$ is a fibration which is also a homotopy equivalence. Then there is a section $s : Y \to X$ homotopy inverse to q, i.e., $qs = \mathrm{id}_Y$ and sq is homotopic to id_X. In fact, sq is even fibrewise homotopic over Y to id_X.*

Proof Write $r : Y \to X$ for a homotopy inverse to q. Consider the lifting problem

$$\begin{tikzcd} & X \arrow[d, two heads, "q"] \\ Y \arrow[ur, dashed] \arrow[r, equal] & Y. \end{tikzcd}$$

Then r is a solution up to homotopy, so that the homotopy lifting lemma 7.22 implies the existence of a map s homotopic to r with $qs = \mathrm{id}_Y$. Giving a fibrewise homotopy over Y between sq and id_X amounts to providing a lift in the diagram

$$\begin{tikzcd} & P_Y X \arrow[d, two heads, "(p_0,p_1)"] \\ X \arrow[ur, dashed] \arrow[r, "(sq,\mathrm{id}_X)"'] & X \times_Y X. \end{tikzcd}$$

Note that $X \times_Y X$ is fibrant by virtue of q being a fibration between fibrant objects. Again by the homotopy lifting lemma 7.22 it suffices to solve this lifting problem up to homotopy. Since $s : Y \to X$ is a homotopy equivalence, it suffices to precompose by this map and provide a lift in the resulting diagram

$$\begin{tikzcd} & P_Y X \arrow[d, two heads, "(p_0,p_1)"] \\ Y \arrow[ur, dashed] \arrow[r, "(s,s)"'] & X \times_Y X. \end{tikzcd}$$

The composition

$$Y \xrightarrow{s} X \xrightarrow{c} P_Y X$$

does the job. □

Proof (of Proposition 7.27) (ii). Suppose that $f : X \to Y$ is a homotopy equivalence between objects which are both fibrant and cofibrant. Factor f as a trivial cofibration $i : X \to Z$ followed by a fibration $q : Z \to Y$. It suffices to show that q is a weak equivalence. The homotopy class of i is an isomorphism in $\mathrm{Ho}(\mathcal{E})$ by Proposition 7.27(i). It follows that the homotopy class of q is an isomorphism as well. Thus Lemma 7.28 implies the existence of a section s of q with a fibrewise homotopy $h : Z \to P_Y Z$ from sq to id_Z. We will show that q is in fact a trivial fibration by showing it has the right lifting property with respect to cofibrations. So consider a lifting problem

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, tail, "i"'] & Z \arrow[d, two heads, "q"] \\
B \arrow[r, "v"] \arrow[ur, dashed] & Y
\end{tikzcd}
$$

in which i is a cofibration. A first attempt to find a lift is to consider the map $k = sv : B \to Z$. It satisfies $qk = v$, but the composite ki is only fibrewise homotopic (rather than equal) to u. To fix this, consider the diagram

$$
\begin{tikzcd}
A \arrow[r, "hu"] \arrow[d, tail, "i"'] & P_Y Z \arrow[d, "p_0"] \\
B \arrow[r, "k"'] \arrow[ur, dashed, "l"] & Z.
\end{tikzcd}
$$

The dashed lift l exists because the first coordinate projection p_0 is a trivial fibration (see Remark 7.24). Finally, taking $p_1 l : B \to Z$ solves our original lifting problem, since $p_1 li = p_1 hu = u$ and $qp_1 l = qp_0 l = qk = v$. This completes the proof. □

7.5 The Homotopy Category of a Model Category

We will apply the results of the previous section to see that the homotopy category $\mathrm{Ho}(\mathcal{E})$ is the universal solution to turning the weak equivalences of $\mathcal{E}$ into isomorphisms. We can define a functor

$$\eta\colon \mathcal{E} \to \mathrm{Ho}(\mathcal{E})$$

by choosing for each object a fibrant and cofibrant replacement. More precisely, for each $X \in \mathcal{E}$ pick a fibrant replacement

$$X \overset{\sim}{\underset{i_X}{\rightarrowtail}} X_f$$

and take it to be the identity if X is already fibrant. Similarly choose a cofibrant replacement

$$X_c \overset{\sim}{\underset{q_X}{\twoheadrightarrow}} X$$

being the identity if X happens to be cofibrant. Then on objects we define $\eta(X) = (X_f)_c$. On morphisms, we use the lifting axiom (M4) to extend a given $\alpha\colon X \to Y$ to a map $\alpha_f\colon X_f \to Y_f$ and then lift to $(\alpha_f)_c$, as in

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, tail, "\wr"] & Y \arrow[r, tail, "\sim"] & Y_f \arrow[d, two heads] \\
X_f \arrow[rr] \arrow[urr, dashed, "\alpha_f"] & & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[rr, tail] \arrow[d, tail] & & (Y_f)_c \arrow[d, two heads, "\wr"] \\
(X_f)_c \arrow[r, two heads, "\sim"] \arrow[urr, dashed, "(\alpha_f)_c"] & X_f \arrow[r, "\alpha_f"'] & Y_f.
\end{tikzcd}
$$

Since these lifts are unique up to homotopy (cf. Remark 7.23), the resulting homotopy class $\eta(\alpha) = [(\alpha_f)_c]$ is independent of choices, which also shows that η is in fact a functor.

Remark 7.29 Note that we could as well have interchanged the order of fibrant and cofibrant replacement and defined $\eta(X) = (X_c)_f$, and similarly for morphisms.

The universal property of $\mathrm{Ho}(\mathcal{E})$ we will phrase is really a universal property in the (large) 2-category of categories:

Definition 7.30 If $\mathbf{C}$ is a category and W a class of morphisms in $\mathbf{C}$, then we call a functor $F\colon \mathbf{C} \to \mathbf{D}$ a *(categorical) localization of* $\mathbf{C}$ *at* W if the following two properties hold:

(1) F sends every element of W to an isomorphism in $\mathbf{D}$.
(2) For any category $\mathbf{E}$, precomposition by F gives an equivalence of categories

$$F^* : \mathrm{Fun}(\mathbf{D}, \mathbf{E}) \to \mathrm{Fun}_W(\mathbf{C}, \mathbf{E}).$$

Here $\mathrm{Fun}(-,-)$ denotes the category of functors and natural transformations, whereas $\mathrm{Fun}_W(\mathbf{C}, \mathbf{E})$ denotes the full subcategory of $\mathrm{Fun}(\mathbf{C}, \mathbf{E})$ on functors sending elements of W to isomorphisms in $\mathbf{E}$.

Theorem 7.31 *The functor* $\eta\colon \mathcal{E} \to \mathrm{Ho}(\mathcal{E})$ *is a localization of* $\mathcal{E}$ *at the class of weak equivalences.*

Proof First observe that η indeed sends weak equivalences to isomorphisms, by Proposition 7.27(i) and the fact that $(\alpha_f)_c$ is a weak equivalence whenever α is. Suppose that $\varphi\colon \mathcal{E} \to \mathbf{C}$ is any functor mapping weak equivalences to isomorphisms. Define a functor $\psi\colon \mathrm{Ho}(\mathcal{E}) \to \mathbf{C}$ by $\psi(X) = \varphi(X)$ on objects and $\psi([\alpha]) = \varphi(\alpha)$ on morphisms. The latter is indeed well-defined on homotopy classes, for if

$$A \amalg A \xrightarrow{(i_0,i_1)} \mathrm{Cyl}(A) \xrightarrow{\varepsilon} A$$

is a cylinder, the functor φ maps ε to an isomorphism, so $\varphi(i_0) = \varphi(i_1)$ because $\varphi(\varepsilon i_0) = \mathrm{id}_{\varphi(A)} = \varphi(\varepsilon i_1)$. Furthermore,

$$\tau_X = (\varphi(X) \xrightarrow{\varphi(i_X)} \varphi(X_f) \xrightarrow{\varphi(q_{X_f})^{-1}} \varphi((X_f)_c))$$

defines a natural isomorphism $\varphi \to \psi\eta = \eta^*\psi$. Indeed, the map τ_X is natural in X since the maps i_X and q_{X_f} are unique up to homotopy (Remark 7.23 again). Also, note that the definitions of ψ and τ are functorial in φ. If $\theta\colon \mathrm{Ho}(\mathcal{E}) \to \mathbf{C}$ is any functor, then the functor ψ associated to $\theta\eta$ is simply θ again. We conclude that the assignment $\varphi \mapsto \psi$ is a pseudo-inverse to η^*. □

Remark 7.32 It is useful to observe that in fact a map $\alpha\colon X \to Y$ is a weak equivalence *if and only if* $\eta(\alpha)$ is an isomorphism in $\mathrm{Ho}(\mathcal{E})$. Indeed, $\eta(\alpha)$ is an isomorphism if and only if the right-most vertical map in the following diagram is a weak equivalence (by Proposition 7.27):

$$\begin{tikzcd}
X \arrow[r, tail, "\sim"] \arrow[d, "\alpha"] & X_f \arrow[d, "\alpha_f"] & (X_f)_c \arrow[l, two heads, "\sim"'] \arrow[d, "(\alpha_f)_c"] \\
Y \arrow[r, tail, "\sim"] & Y_f & (Y_f)_c . \arrow[l, two heads, "\sim"']
\end{tikzcd}$$

But by the two-out-of-three property this is the case if and only if α is a weak equivalence.

Remark 7.33 If A and X are arbitrary objects of $\mathcal{E}$, with arbitrary cofibrant and fibrant replacements $A_c \xrightarrow{\sim} A$ and $X \xrightarrow{\sim} X_f$ respectively, then one easily checks that

$$\mathrm{Ho}(\mathcal{E})(\eta A, \eta X) \cong [A_c, X_f].$$

Thus $\mathrm{Ho}(\mathcal{E})$ is equivalent to the category with the same objects as $\mathcal{E}$ and as morphisms from A to X the set $[A_c, X_f]$.

These remarks lead to the following useful observation:

Proposition 7.34 *In a model category $\mathcal{E}$, the three classes of morphisms are entirely determined by the cofibrations and the fibrant objects.*

Proof The fibrations are determined by the trivial cofibrations, so it suffices to check that the cofibrations and the fibrant objects determine the weak equivalences. Also, the cofibrations determine the trivial fibrations, so we are free to use those. To check whether a morphism $A \to B$ is a weak equivalence, factor $0 \to A$ as a cofibration followed by a trivial fibration:

$$0 \rightarrowtail A_c \overset{\sim}{\twoheadrightarrow} A.$$

Similarly factor the composite of the maps $A_c \to A \to B$ to obtain a square

$$\begin{tikzcd}
A_c \arrow[r] \arrow[d, two heads, "\wr"] & B_c \arrow[d, two heads, "\wr"] \\
A \arrow[r] & B
\end{tikzcd}$$

in which the vertical arrows are trivial fibrations and A_c and B_c are cofibrant. Observe that $A \to B$ is a weak equivalence if and only if $A_c \to B_c$ is. By the previous remark the latter is the case if and only if for each fibrant object X, the map

$$[B_c, X] \to [A_c, X]$$

is an isomorphism, because that would mean that $A_c \to B_c$ is an isomorphism in $\mathrm{Ho}(\mathcal{E})$. □

We finish this section by observing the ‘two-out-of-six property’ for weak equivalences, which will occasionally be useful:

Proposition 7.35 *Consider a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[d] & X \arrow[d] \\ B \arrow[r, "\sim"] \arrow[ur] & Y \end{tikzcd}$$

in a model category $\mathcal{E}$ such that the horizontal morphisms are weak equivalences as indicated. Then every map in the diagram is a weak equivalence.

Remark 7.36 The term 'two-out-of-six' refers to the fact that the diagram above really contains six morphisms: the sixth one is the composite arrow $A \to Y$. Also, observe that the diagram shape above is encoded by a functor $[3] \to \mathcal{E}$, recording the three composable morphisms $A \to B \to X \to Y$.

Proof By Remark 7.32 it suffices to check that the image of this diagram in the homotopy category $\mathrm{Ho}(\mathcal{E})$ consists entirely of isomorphisms. The upper triangle in the square shows that the image of the arrow $B \to X$ in $\mathrm{Ho}(\mathcal{E})$ admits a right inverse, whereas the lower triangle shows that it admits a left inverse. Therefore $B \to X$ gives an isomorphism in $\mathrm{Ho}(\mathcal{E})$. But then by two-out-of-three, the same is true for the vertical arrows. □

7.6 Brown's Lemma and Proper Model Categories

Recall that a model category is called *right proper* if its weak equivalences are stable under pullback along fibrations. Dually, a model category is *left proper* if its weak equivalences are stable under pushouts under cofibrations. In Section 7.3 we showed that the model category of topological spaces is left proper and also right proper, using the long exact sequence of a fibration. In this section we will make several useful observations about left or right proper model categories. In particular we prove *Brown's lemma*, which implies that any model category in which all objects are fibrant is right proper, but will also be useful in other contexts. All the general statements in this section for right proper model categories of course have a dual form for left proper model categories and vice versa.

Lemma 7.37 (Brown's lemma) *Let $f\colon X \to Y$ be a weak equivalence between fibrant objects in a model category. Then f factors as a trivial cofibration $i\colon X \to Z$ followed by a trivial fibration $p\colon Z \to Y$, where moreover there exists a trivial fibration $q\colon Z \to X$ with $qi = \mathrm{id}_X$.*

Proof Factor $(\mathrm{id}_X, f)\colon X \to X \times Y$ as a trivial cofibration followed by a fibration:

$$\begin{tikzcd} X \arrow[rr, "{(1,f)}"] \arrow[dr, tail, "\sim"'] & & X \times Y \\ & Z \arrow[ur, two heads, "{(q,p)}"'] & \end{tikzcd}$$

Since X and Y are fibrant the projections $X \times Y \to X$ and $X \times Y \to Y$ are fibrations, hence so are p and q. Since $qi = \mathrm{id}_X$ and $pi = f$, it follows by two-out-of-three that p and q are weak equivalences as well. □

The following is a typical application of Brown's lemma:

Proposition 7.38 *Any functor $R : \mathcal{F} \to \mathcal{E}$ between model categories which preserves trivial fibrations also preserves weak equivalences between fibrant objects.*

Proof Given any such weak equivalence $f : X \to Y$, consider the maps p, i, and q of Brown's lemma. Since R preserves trivial fibrations, the maps Rp and Rq are weak equivalences. But then by two-out-of-three, so is Ri. We conclude that $Rf = Rp \circ Ri$ is a weak equivalence as well. □

Corollary 7.39 *Let $f : X \to Y$ be a morphism in a model category $\mathcal{E}$. Then the pullback functor*

$$f^* : \mathcal{E}/Y \to \mathcal{E}/X$$

preserves weak equivalences between fibrations over Y.

Proof The fibrations over Y are the fibrant objects of $\mathcal{E}/Y$ and f^* is a functor which preserves trivial fibrations, simply because these are stable under pullback. □

Another application of Brown's lemma is the following:

Proposition 7.40 *In a pullback square*

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, two heads] & W \arrow[d, two heads, "p"] \\ X \arrow[r, "\sim", "f"'] & Y \end{tikzcd}$$

where p is a fibration and f is a weak equivalence between fibrant objects, the map g is also a weak equivalence.

Proof Factor f as in Brown's lemma. Since the pullback of a trivial fibration is again a trivial fibration, it suffices to consider the case where f is a section of a trivial fibration $q : Y \to X$. Consider the diagram of pullback squares

$$\begin{tikzcd} T \arrow[r, "k"'] \arrow[d, two heads] & U \arrow[r, two heads, "\sim", "h"'] \arrow[d, two heads, "u"] & W \arrow[d, two heads, "qp"] \\ X \arrow[r, "\sim", "f"'] & Y \arrow[r, two heads, "\sim", "q"'] & X. \end{tikzcd}$$

Then h is a pullback of q, hence a trivial fibration, and hk is an isomorphism because it is a pullback of $qf = \mathrm{id}_X$. Therefore all horizontal maps in the diagram are weak equivalences and all vertical maps are fibrations. From the commutative square

$$\begin{tikzcd} W \arrow[r, "\mathrm{id}_W"] \arrow[d, "p"] & W \arrow[d, "qp"] \\ Y \arrow[r, "q"] & X \end{tikzcd}$$

and the universal property of the pullback, there exists a $t : W \to U$ with $ht = \mathrm{id}_W$ and $ut = p$. The first equation gives that t is also a weak equivalence, since h is. In fact, it is a weak equivalence between fibrations over Y, namely p and u. By Corollary 7.39 the pullback of t along f is still a weak equivalence between fibrations over X:

$$\begin{tikzcd} X \times_Y W \arrow[rr, "\sim", "f^*t"'] \arrow[dr, two heads] & & X \times_Y U \arrow[dl, two heads] \\ & X. & \end{tikzcd}$$

The top horizontal map may be identified with the map $g : V \to W$, which completes the proof. □

Corollary 7.41 *A model category in which every object is fibrant is right proper.*

Remark 7.42 The dual forms of the preceding statements will often be used as well. To be explicit, any weak equivalence $X \to Y$ between cofibrant objects can be factored as a trivial cofibration $X \to Z$ followed by a trivial fibration $Z \to Y$ having a trivial cofibration as a section. Furthermore, any functor which preserves trivial cofibrations also preserves weak equivalences between cofibrant objects. Also, weak equivalences between cofibrant objects are stable under pushout along a cofibration. As a particular case, a model category in which every object is cofibrant is automatically left proper.

Another useful fact for proper model categories is the following lemma:

Lemma 7.43 *Let $\mathcal{E}$ be a left proper model category, $p\colon Y \to X$ a fibration and $i\colon A \to B$ a cofibration in $\mathcal{E}$. Consider a cofibrant replacement of i, given by a diagram*

$$\begin{tikzcd} A_c \arrow[r, "\sim"] \arrow[d, tail, "i_c"] & A \arrow[d, tail, "i"] \\ B_c \arrow[r, "\sim"] & B \end{tikzcd}$$

in which i_c is a cofibration between cofibrant objects. If p has the right lifting property with respect to i_c, then it also has the right lifting property with respect to i.

Proof Form the commutative diagram

$$\begin{tikzcd} A_c \arrow[r, "\sim"] \arrow[d, tail, "i_c"] & A \arrow[rr] \arrow[d, tail] & & Y \arrow[d, two heads] \\ B_c \arrow[r, "\sim"] & P \arrow[r] & B \arrow[r] & X, \end{tikzcd}$$

where the square on the left is a pushout. The map $B_c \to P$ is a weak equivalence because $\mathcal{E}$ is left proper. By assumption there exists a lift $B_c \to Y$ in the outer rectangle, hence also a lift $\delta\colon P \to Y$ in the smaller rectangle by the universal property of the pushout P. As $B_c \to B$ is a weak equivalence, so is $P \to B$ by two-out-of-three. Factor $P \to B$ as a trivial cofibration $P \to Q$ followed by a trivial fibration $Q \to B$. Now we can lift successively in the following two squares:

$$
\begin{tikzcd}
P \arrow[r, "\delta"] \arrow[d, tail, "\wr"] & Y \arrow[d, two heads] \\
Q \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, tail] & Q \arrow[d, two heads, "\wr"] \\
B \arrow[r, equal] \arrow[ur, dashed] & B.
\end{tikzcd}
$$

Composing the two lifts gives the required map $B \to Y$. □

7.7 Transfer of Model Structures

Let $\mathcal{E}$ be a cofibrantly generated model category (see Remark 7.13), so $\mathcal{E}$ has a set $\mathcal{I}$ of morphisms generating the cofibrations as a saturated class and similarly a generating set $\mathcal{J}$ for the trivial cofibrations. Let $\mathcal{A}$ be another category related to $\mathcal{E}$ by a pair of adjoint functors (left adjoint on top)

$$\mathcal{E} \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathcal{A}.$$

We will assume that $\mathcal{A}$ (like $\mathcal{E}$) has all small limits and colimits. For example, the reader might think of the objects of $\mathcal{A}$ as objects of $\mathcal{E}$ equipped with some algebraic structure and f^* as the forgetful functor. Its left adjoint then assigns to each object X of $\mathcal{E}$ the 'free algebraic structure generated by $\mathcal{E}$'. More specifically, $\mathcal{E}$ could be the category of topological spaces equipped with the model structure of Section 7.3 and $\mathcal{A}$ could be the category of topological monoids. We will consider more examples of this kind in Section 13.4.

Our aim in this section is to describe a method to transfer the model structure from $\mathcal{E}$ to $\mathcal{A}$. More precisely, we simply define a map $A \to B$ in $\mathcal{A}$ to be a fibration or a weak equivalence precisely if $f^*A \to f^*B$ is a fibration or a weak equivalence, respectively, in $\mathcal{E}$. We then ask ourselves when this gives a model structure again. The cofibrations will be the maps having the right lifting property with respect to the trivial fibrations. Note that $f^*A \to f^*B$ is a fibration in $\mathcal{E}$ if and only if it has the right lifting property with respect to every generating trivial cofibration $U \to V$, so that $A \to B$ is a fibration in $\mathcal{A}$ if and only if it has the right lifting property with respect to every corresponding $f_!U \to f_!V$. Thus, by the retract argument of Remark 7.12, the class of trivial cofibrations in $\mathcal{A}$ has to be the saturation of the set of generating trivial cofibrations of the form $f_!U \to f_!V$, with $U \to V$ ranging over the generating trivial cofibrations in $\mathcal{E}$. Similarly, the set of cofibrations in $\mathcal{A}$ must be the saturation of the set of maps $f_!U \to f_!V$ now with $U \to V$ ranging over the generating cofibrations in $\mathcal{E}$.

The main step in proving the existence of the desired model structure will be to find conditions that guarantee that in the case of generating trivial cofibrations, the saturation of this set is indeed contained in the class of weak equivalences in $\mathcal{A}$ we defined above. Any morphism $A \to B$ of $\mathcal{A}$ in this saturation can be obtained as a retract of a (transfinite) colimit of a sequence of morphisms which occur in a pushout of the form

$$\begin{tikzcd} f_!U \arrow[r] \arrow[d] & A \arrow[d] \\ f_!V \arrow[r] & B, \end{tikzcd}$$

where $U \to V$ is a generating trivial cofibration in $\mathcal{E}$. So if the functor f^* maps each such $A \to B$ to a weak equivalence, as well as any transfinite composition of such morphisms, then f^* will send any morphism in this saturation to a weak equivalence. This leads to the following formulation:

Theorem 7.44 *Let $\mathcal{E}$ be a cofibrantly generated model category and let $f_! : \mathcal{E} \rightleftarrows \mathcal{A} : f^*$ be an adjoint pair of functors, with f^* preserving filtered colimits. Assume $\mathcal{A}$ has finite limits and all small colimits. Call a morphism $A \to B$ a fibration (or a weak equivalence) if $f^*A \to f^*B$ is one in $\mathcal{E}$. Then these fibrations and weak equivalences form part of a cofibrantly generated model structure on $\mathcal{A}$ provided the following conditions are satisfied:*

(1) *For every generating trivial cofibration $i: U \to V$ in $\mathcal{E}$, any pushout of the morphism $f_!i$ in $\mathcal{A}$ is a weak equivalence.*
(2) *Any transfinite composition of such pushouts as in (1) is a weak equivalence.*

The model structure on $\mathcal{A}$ obtained in this way is referred to as the transferred model structure.

Remark 7.45 If p is a pushout of a morphism $f_!i$, for a generating trivial cofibration i, it will often happen that f^*p is a *trivial cofibration* in $\mathcal{E}$ (rather than just a weak equivalence). In this case, condition (2) of the theorem will be automatic. Indeed, any transfinite composition of such maps f^*p, being a transfinite composition of trivial cofibrations, will itself be a trivial cofibration and in particular a weak equivalence. Notice that in the situation above, if f^* preserves *all* colimits, then condition (1) is also automatically satisfied.

Proof (of Theorem 7.44) As explained before the statement of the theorem, we can take the generating (trivial) cofibrations in $\mathcal{A}$ to be the maps of the form $f_!U \to f_!V$ where $U \to V$ is a generating (trivial) cofibration in $\mathcal{E}$. Axioms (M1–3) are evidently satisfied. Moreover, the small object argument provides two factorizations of a map $A \to B$ in $\mathcal{A}$: one factorization into maps $i: A \to C$ and $p: C \to B$ in which p has the right lifting property with respect to all generating cofibrations and i lies in the saturation of the class of generating cofibrations; another into maps $j: A \to D$

and $q: D \to B$ where q has the right lifting property with respect to all generating trivial cofibrations and j lies in the saturation of these generating trivial cofibrations. (Here we have used that the domains $f_!U$ of generators are still compact, since f^* preserves filtered colimits.) Then f^*p has the right lifting property with respect to all the (generating) cofibrations in $\mathcal{E}$, so f^*p is a trivial fibration in $\mathcal{E}$. Hence p is a trivial fibration in $\mathcal{A}$ by definition. Similarly, f^*q is a fibration in $\mathcal{E}$, so q is a fibration in $\mathcal{A}$. Moreover, i and j are cofibrations in $\mathcal{A}$ by definition and j is also a weak equivalence precisely by conditions (1) and (2) of the theorem, as we explained above. It remains to verify the lifting axiom (M4). So consider a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{A}$, where p is a fibration and i is a cofibration. If p is also a weak equivalence then a lift exists simply by definition of the cofibrations. In the other case where i is a weak equivalence, we already discussed above that the retract argument (Remark 7.12) implies that it i lies in the saturation of the generating trivial cofibrations $f_!U \to f_!V$. Since p has the right lifting property with respect to these, it also has the right lifting property with respect to i. □

We conclude this section with some first examples of transferred model structures.

Example 7.46 Let $\mathcal{E}$ be a cofibrantly generated model category. Let $\mathbf{C}$ be a small category and consider the category $\mathcal{E}^{\mathbf{C}}$ of functors from $\mathbf{C}$ to $\mathcal{E}$, i.e., diagrams of 'shape' $\mathbf{C}$ in $\mathcal{E}$. Then $\mathcal{E}^{\mathbf{C}}$ carries a model structure in which a morphism $f : X \to Y$ (i.e., a natural transformation between functors) is a weak equivalence, resp. a fibration, if and only if $f(c)$ is a weak equivalence, resp. a fibration, for every object c of $\mathbf{C}$. This is usually called the *projective model structure*. As its description suggests, it is obtained by transfer from the model structure on the product

$$\mathcal{E}^{\mathbf{C}_0} = \prod_{c \in \mathbf{C}_0} \mathcal{E},$$

where $\mathbf{C}_0$ denotes the set of objects of $\mathbf{C}$ and the model structure on the product is the evident one described in Example (d)(iii) of Section 7.1. There is a pair of adjoint functors

$$u_! : \mathcal{E}^{\mathbf{C}_0} \rightleftarrows \mathcal{E}^{\mathbf{C}} : u^*$$

induced by the inclusion $u : \mathbf{C}_0 \to \mathbf{C}$, where $\mathbf{C}_0$ is viewed as a discrete category having identity arrows only. Thus u^* is simply the forgetful functor remembering only the values of a functor, while for an object X of $\mathcal{E}^{\mathbf{C}_0}$, its value under $u_!$ is described by

$$(u_!X)(c) = \coprod_{d \to c} X(d)$$

with the coproduct ranging over all arrows $d \to c$ in $\mathbf{C}$. The 'action' of $\mathbf{C}$ on $u_!X$ is by composition. The conditions for transfer given in Theorem 7.44 are easily seen to hold. Indeed, since u^* preserves colimits, it suffices to check that for a trivial cofibration $X \to Y$ in $\mathcal{E}^{\mathbf{C}_0}$, the map $u^*u_!X \to u^*u_!Y$ is again a trivial cofibration, which is obvious.

The construction of the projective model structure is functorial in $\mathbf{C}$, in the following sense. If $\varphi : \mathbf{C} \to \mathbf{D}$ is a functor between small categories, it induces an adjoint pair

$$\varphi_! : \mathcal{E}^{\mathbf{C}} \rightleftarrows \mathcal{E}^{\mathbf{D}} : \varphi^*.$$

It is clear that φ^* preserves fibrations and weak equivalences for the two projective model structures on $\mathcal{E}^{\mathbf{C}}$ and $\mathcal{E}^{\mathbf{D}}$. In the terminology of Section 8.3, the pair $(\varphi_!, \varphi^*)$ is a *Quillen adjunction*.

Example 7.47 A special case of the previous example is where $\mathbf{C}$ is the 'arrow', i.e., the category of shape

$$\bullet \to \bullet.$$

The functor category $\mathcal{E}^{\mathbf{C}}$ is then usually called the *arrow category* and denoted $\mathrm{Ar}(\mathcal{E})$. Its objects are morphisms $f : X \to Y$ in $\mathcal{E}$, its morphisms

$$(u, v) : (X \xrightarrow{f} Y) \to (X' \xrightarrow{g} Y')$$

are commutative squares

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & X' \arrow[d, "g"] \\ Y \arrow[r, "v"] & Y'. \end{tikzcd}$$

If we equip $\mathrm{Ar}(\mathcal{E})$ with the projective model structure, then a pair (u, v) is a weak equivalence (or a fibration) precisely if both u and v are in $\mathcal{E}$, and a cofibration if and only if $X \to X'$ and the map from the pushout $Y \cup_X X' \to Y'$ are both cofibrations. This last fact can be checked directly by contemplating what it means for a morphism in $\mathrm{Ar}(\mathcal{E})$ to have the left lifting property with respect to trivial fibrations. In particular, an object $X \to Y$ of $\mathrm{Ar}(\mathcal{E})$ is cofibrant if both X and Y are cofibrant and if $X \to Y$ is a cofibration in $\mathcal{E}$.

Example 7.48 In the next section we will consider the example where $\mathbf{C}$ is the 'span'

$$\bullet \leftarrow \bullet \to \bullet$$

for which we write S. So an object of $\mathcal{E}^S$ is a diagram $B \leftarrow A \to C$ in $\mathcal{E}$. We write $\mathrm{Span}(\mathcal{E})$ for this diagram category $\mathcal{E}^S$. Much like the previous example, it is simple to verify that a span is cofibrant in the projective model structure on $\mathrm{Span}(\mathcal{E})$ precisely if all three objects are cofibrant and both arrows are cofibrations in $\mathcal{E}$.

7.8 Homotopy Pushouts and the Cube Lemma

We conclude this chapter with a short section about the interaction between pushouts and weak equivalences. In particular we will prove the 'cube lemma' (cf. Lemma 7.51 and Corollary 7.50), which will be a useful tool for inductive arguments involving CW-structures, or skeletal filtrations of simplicial and dendroidal sets.

A basic problem is that weak equivalences in a model category $\mathcal{E}$ are generally not preserved under pushout. For example, consider the following two diagrams of topological spaces:

$$
\begin{tikzcd}
\partial D^n \arrow[r] \arrow[d] & * \\
* &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\partial D^n \arrow[r] \arrow[d, tail] & * \\
D^n &
\end{tikzcd}
$$

Clearly the two spans are weakly equivalent. However, the two pushouts are *not* weakly equivalent; on the left one gets the one-point space, but on the right the result is (homeomorphic to) the n-sphere S^n. It will be useful to have conditions on diagrams of the above shape which guarantee that the pushout *is* invariant up to weak equivalence.

A general way to approach the problem is via the concept of the *homotopy colimit*, which is in a precise sense the best approximation to the functor of taking colimits by a functor which preserves weak equivalences (a 'derived functor' in the language of Section 8.3). We will have more to say on this perspective in Section 10.5, but here we will give a concrete treatment in the specific case of pushouts.

We begin with the following preliminary version of the cube lemma. It states that the pushout of a span

$$B \leftarrow A \rightarrow C$$

is invariant up to weak equivalence if we assume that all three objects are cofibrant and both arrows are cofibrations; more briefly, if the span is cofibrant in the projective model structure of Example 7.48.

Lemma 7.49 *Consider a cubical diagram*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & C \arrow[dd] \arrow[dr] & \\
& B \arrow[rr, crossing over] & & D \arrow[dd] \\
A' \arrow[rr] \arrow[dr] & & C' \arrow[dr] & \\
& B' \arrow[rr] \arrow[from=uu, crossing over] & & D'
\end{tikzcd}$$

in a model category $\mathcal{E}$. Suppose that the top and bottom faces are pushout squares and that the vertical maps $A \to A'$, $B \to B'$, and $C \to C'$ are weak equivalences. Then the remaining vertical map $D \to D'$ is also a weak equivalence if both of the spans

$$B \leftarrow A \to C \quad \textit{and} \quad B' \leftarrow A' \to C'$$

are cofibrant in the projective model structure, i.e., consist of cofibrations between cofibrant objects.

Proof The cube of the lemma describes a weak equivalence between the two spans in the model category $\mathrm{Span}(\mathcal{E})$. The map $D \to D'$ is the image of this weak equivalence under the colimit functor

$$\mathrm{colim} : \mathrm{Span}(\mathcal{E}) \to \mathcal{E}.$$

Its right adjoint is the constant diagram functor, which obviously preserves fibrations. By adjunction, the functor colim preserves trivial cofibrations and hence weak equivalences between cofibrant objects, by (the dual of) Proposition 7.38. Hence the map $D \to D'$ is also a weak equivalence. □

In practice it will be very useful to have the conclusion of Lemma 7.49 above under weaker conditions. Let us introduce a bit of terminology. A pushout square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

in $\mathcal{E}$ is said to be a *homotopy pushout* if, after choosing a cofibrant replacement

$$\begin{tikzcd}
B_0 \arrow[d, "\wr"] & A_0 \arrow[l, tail] \arrow[r, tail] \arrow[d, "\wr"] & C_0 \arrow[d, "\wr"] \\
B & A \arrow[l] \arrow[r] & C
\end{tikzcd}$$

in the category of spans, the induced map of pushouts

$$B_0 \amalg_{A_0} C_0 \to B \amalg_A C$$

is a weak equivalence in $\mathcal{E}$. This definition is easily seen to be independent of the choice of cofibrant replacement. By the two-out-of-three property of weak equivalences and Lemma 7.49, we immediately conclude the following:

Corollary 7.50 *Consider a cubical diagram*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & C \arrow[dr] \arrow[dd] & \\
& B \arrow[rr] \arrow[dd] & & D \arrow[dd] \\
A' \arrow[rr] \arrow[dr] & & C' \arrow[dr] & \\
& B' \arrow[rr] & & D'
\end{tikzcd}$$

in a model category $\mathcal{E}$. Suppose that the top and bottom faces are homotopy pushout squares and that the vertical maps $A \to A'$, $B \to B'$, and $C \to C'$ are weak equivalences. Then the remaining vertical map $D \to D'$ is also a weak equivalence.

Of course any pushout square arising from a cofibrant span $B \leftarrow A \to C$ is a homotopy pushout square. However, the following suffices:

Lemma 7.51 *Consider a pushout square*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

in a model category $\mathcal{E}$. Then it is a homotopy pushout if at least one of the following two conditions is satisfied:

(1) *The model category $\mathcal{E}$ is left proper and the map $A \to C$ is a cofibration.*
(2) *The objects A, B, and C are cofibrant and the map $A \to C$ is a cofibration.*

Proof (of Lemma 7.51) Choose a cofibrant replacement of spans

$$\begin{tikzcd}
B_0 \arrow[d, "\wr"] & A_0 \arrow[l, tail] \arrow[r, tail] \arrow[d, "\wr"] & C_0 \arrow[d, "\wr"] \\
B & A \arrow[l] \arrow[r] & C.
\end{tikzcd}$$

Write D_0 for the pushout of the top row; our goal is to prove that the natural map $D_0 \to D$ is a weak equivalence. First assume that condition (1) is satisfied. Form the pushout square

$$\begin{tikzcd}
C_0 \arrow[r, tail] \arrow[d, "\wr"] & D_0 \arrow[d] \\
C \arrow[r, tail] & C \amalg_{C_0} D_0.
\end{tikzcd}$$

Since $\mathcal{E}$ is left proper, the map $D_0 \to C \amalg_{C_0} D_0$ is a weak equivalence, so by two-out-of-three it will suffice to show that the natural map

$$C \amalg_{C_0} D_0 \to D$$

is a weak equivalence. Since $C_0 \to D_0$ is the pushout of $A_0 \to B_0$, we have an isomorphism $C \amalg_{C_0} D_0 \cong C \amalg_{A_0} B_0$. Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & A \amalg_{A_0} B_0 \arrow[r] \arrow[d, tail] & B \arrow[d, tail] \\ C \arrow[r] & C \amalg_{A_0} B_0 \arrow[r] & D. \end{tikzcd}$$

Since the square on the left and the rectangle are pushouts, the square on the right is a pushout as well. We need to show that the lower right horizontal map is a weak equivalence. All the vertical maps are cofibrations, so by left properness it now suffices to prove that the map

$$A \amalg_{A_0} B_0 \to B$$

is a weak equivalence. For this, consider the pushout square

$$\begin{tikzcd} A_0 \arrow[r, tail] \arrow[d, "\wr"] & B_0 \arrow[d] \\ A \arrow[r, tail] & A \amalg_{A_0} B_0 \end{tikzcd}$$

and invoke left properness once again to conclude that the vertical map on the right is a weak equivalence. Applying two-out-of-three and the assumption that $B_0 \to B$ is a weak equivalence completes the proof in this case.

If one assumes condition (2) instead, the proof proceeds in the same way. One replaces the use of left properness everywhere by an application of the dual of Proposition 7.40, which works by the assumption that all objects under consideration are cofibrant. $\square$

Remark 7.52 Of course the statements of this section have evident duals, replacing pushouts by pullbacks, cofibrations by fibrations, and cofibrant objects by fibrant objects.

Historical Notes

As already indicated in the introduction to this chapter, the notion of model category arose in Quillen's work [123, 124] as a tool to formalize the idea that two homotopy theories can be equivalent. Brown's lemma was also formulated early on in the development of abstract homotopy theory, namely in [33]. However, the development of the theory in the shape we use it in later chapters of this book took some time, one reason being that it was difficult to show without further conditions that the axioms were preserved under certain important constructions such as forming the category of presheaves with values in a model category. The crucial step here is a systematic use of the small object argument (already employed by Quillen in [123]), for example as in Joyal's construction of a model structure for simplicial sheaves in his letter to Grothendieck [91]. The notion of a cofibrantly generated model category very clearly brings out the role of the small object argument in producing factorizations. This is evident in the transfer of model structures explained in Section 7.7, which in this form is due to Crans [46]. In the 1990s several accounts of cofibrantly generated model categories (and their implications for the theory of localization) appeared: the standard references were written by Hovey [88] and Hirschhorn [84]. Another very accessible introduction to model categories from this time is the paper of Dwyer–Spalinski [52]. Alternative references for the basic theory of model categories are [69, 82].

Chapter 8
Model Structures on the Category of Simplicial Sets

In this chapter we will apply the formalism of Quillen model categories discussed in the previous chapter to the category of simplicial sets. In his first exposition of the theory of model categories, Quillen already showed that the category of simplicial sets carries a model structure for which the fibrant objects are the Kan complexes and the fibrations are the Kan fibrations (see Chapter 5 for a discussion of these notions). We will refer to it as the *Kan–Quillen model structure*. Quillen defined the weak equivalences to be the maps between simplicial sets for which the geometric realization functor induces isomorphisms in homotopy groups. These are exactly the weak equivalences of the model structure on topological spaces discussed in Section 7.3, and the adjoint pair of functors given by the geometric realization and the singular complex in fact induces an equivalence between the associated homotopy categories. In this sense, simplicial sets and topological spaces are models for the same homotopy theory. This correspondence between simplicial sets and topological spaces on the one hand guides the development of the theory of simplicial sets by providing topological intuition, while on the other hand it shows that simplicial sets form a complete combinatorial framework for the study of the homotopy theory of spaces.

The category of simplicial sets carries another model structure, discovered much later, for which the fibrant objects are precisely the ∞-categories. This model structure, called the *Joyal model structure* or *categorical model structure*, has the same cofibrations as the older model structure mentioned above, but a smaller class of weak equivalences. We will choose an anachronistic approach and begin this chapter with a development of the categorical model structure, capturing the homotopy theory of ∞-categories. Then we will construct Quillen's model structure described above in the same fashion, highlighting the relation between the two model structures. Although non-standard, our approach to these model structures has the appealing feature that it is uniform and internal to the category of simplicial sets. In particular, the relation to topological spaces is not needed. Apart from building on the results of Chapter 5, our treatment is completely self-contained. Perhaps more importantly, the methods we present here can be adapted to the category of dendroidal sets, as we will see in the next chapter.

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The contents of this chapter, then, are as follows. In the first two sections we will construct these two model category structures for ∞-categories and Kan complexes, respectively. After a general interlude about a suitable notion of adjoint functors between model categories, we will establish the equivalence between the Kan–Quillen model structure on simplicial sets and the one on topological spaces introduced in the previous chapter. The proof of this equivalence is somewhat involved; it makes use of minimal fibrations and the related fact, also established by Quillen, that the geometric realization of a Kan fibration is a Serre fibration. We return to the categorical model structure in Section 8.7; we will use the results from Chapter 5 to give several equivalent characterizations of the maps between ∞-categories which are weak equivalences in the Joyal model structure. In particular, we will see that they can be described in a categorical way, as 'functors' which are essentially surjective and fully faithful in the appropriate sense. In the final section 8.8, we describe the *covariant model structure* on the slice category $\mathbf{sSets}/V$, for some fixed base simplicial set V. This structure serves as a model for the homotopy theory of V-indexed diagrams of simplicial sets, in a sense to be made precise in Section 14.8.

8.1 The Categorical Model Structure on Simplicial Sets

In this section we will prove that the category of simplicial sets carries a model structure for which the cofibrations are the monomorphisms and the fibrant objects are precisely the inner Kan complexes, i.e., the ∞-categories. (Recall from Proposition 7.34 that this determines the model structure uniquely.) This model structure is known as the *categorical model structure* or the *Joyal model structure*. The proof of the existence of this model structure relies on the properties of inner Kan complexes and inner anodyne maps already established in Chapter 5. In particular, we will again use the small object argument which gave a factorization of an arbitrary morphism into a monomorphism followed by a map having the right lifting property with respect to monomorphisms. Similarly, it will provide a factorization of a map into a trivial cofibration followed by a fibration in the sense of this model structure about to be defined. But in order to be able to use this argument, we need to find a *set* of generating trivial cofibrations of which the saturation is precisely the class of all trivial cofibrations. Given what we have done already in Chapter 5, this will turn out to be the main difficulty of the proof.

Let us first define the weak equivalences in the categorical model structure. We use the functor $\tau\colon \mathbf{sSets} \to \mathbf{Cat}$, left adjoint to the nerve, discussed in Section 2.4.

Definition 8.1 A map $A \to B$ between simplicial sets is a *categorical weak equivalence* if for any ∞-category X, the induced map

$$\tau(X^B) \to \tau(X^A)$$

is an equivalence of categories.

Recall from 5.38 that if X is an ∞-category, then so are X^B and X^A, so the description of τ applied to ∞-categories discussed in Lemma 5.6 applies here.

Theorem 8.2 *The category of simplicial sets carries a model structure in which the cofibrations are precisely the monomorphisms and the weak equivalences are the categorical weak equivalences. Its fibrant objects are precisely the ∞-categories.*

We will refer to the fibrations in this model structure (i.e., the maps having the right lifting property with respect to all trivial cofibrations) as *categorical fibrations*.

Remark 8.3 This model structure has the property that every object is cofibrant, so it is left proper (cf. Remark 7.42). As in any model structure, the fibrations are the maps having the right lifting property with respect to the trivial cofibrations. Notice that for a cofibration (i.e., monomorphism) $A \to B$, the functor $\tau X^B \to \tau X^A$ is an isofibration between categories (cf. Theorem 5.55 and Corollary 5.54), so that $A \to B$ is a trivial cofibration if and only if, for every ∞-category X, the map $\tau X^B \to \tau X^A$ is a trivial fibration between categories in the model structure of Example (a) at the end of Section 7.1. It does not seem possible to give a more explicit and workable description of the fibrations in the model category of the theorem, but for fibrations between fibrant objects more can be said. We will come back to this point in Proposition 8.16 below.

Let us now turn to the proof of Theorem 8.2. We begin with a few easy lemmas.

Lemma 8.4 (a) *If $X \to Y$ is a J-homotopy equivalence between ∞-categories, then $\tau X \to \tau Y$ is an equivalence of categories.*
(b) *In particular, if $X \to Y$ is a map between ∞-categories which has the right lifting property with respect to all monomorphisms, then $\tau X \to \tau Y$ is an equivalence of categories.*

Proof (a) This is clear from the fact that τJ is the groupoid with two objects and an isomorphism between them. Indeed, since τ preserves products, it follows that applying τ to J-homotopic maps between ∞-categories gives naturally isomorphic functors.

(b) If $f : X \to Y$ has the right lifting property with respect to all monomorphisms, then it is a J-homotopy equivalence, as one sees by lifting successively in the following two squares:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, tail] & X \arrow[d, "f"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "s"] & Y, \end{tikzcd} \qquad \begin{tikzcd} J \times Y \cup \partial J \times X \arrow[r, "s\pi_2 \cup (sf, \mathrm{id}_X)"] \arrow[d, tail] & X \arrow[d, "f"] \\ J \times X \arrow[r, "f\pi_2"] \arrow[ur, dashed] & Y. \end{tikzcd}$$

□

Lemma 8.5 *Any J-homotopy equivalence between simplicial sets is a categorical equivalence.*

Proof Suppose $f : A \leftrightarrows B : g$ are maps of simplicial sets, $h : J \times A \to A$ is a J-homotopy between gf and id_A, and $k : J \times B \to B$ is a J-homotopy between fg and id_B. Then for any other simplicial set X, the induced maps

$$g^* : X^A \leftrightarrows X^B : f^*$$

are also J-homotopic. Indeed, a homotopy $h^* : J \times X^A \to X^A$ between f^*g^* and the identity is constructed as the adjoint of the composition

$$J \times X^A \times A \cong X^A \times (J \times A) \xrightarrow{\mathrm{id}\times h} X^A \times A \xrightarrow{\mathrm{ev}} X.$$

Another homotopy k^* is constructed similarly. The result follows by Lemma 8.4(a). □

Corollary 8.6 *Any map between simplicial sets having the right lifting property with respect to monomorphisms is a categorical equivalence.*

Proof As in the proof of Lemma 8.4(b), any map with this lifting property is in particular a J-homotopy equivalence. □

Lemma 8.7 *Any inner anodyne map between simplicial sets is a trivial cofibration.*

Proof If $A \to B$ is inner anodyne and X is an ∞-category, then $X^B \to X^A$ has the right lifting property with respect to all monomorphisms by Corollary 5.39(ii), so the conclusion follows from Lemma 8.4. □

Corollary 8.8 *For X a simplicial set, the map $X \to \Delta[0]$ is a categorical fibration if and only if X is an ∞-category.*

Proof If the stated map has the right lifting property with respect to trivial cofibrations, then Lemma 8.7 implies that X is an ∞-category. Conversely, if X is an ∞-category and $A \to B$ a trivial cofibration, then

$$\tau X^B \to \tau X^A$$

is a trivial fibration of categories and in particular surjective on objects. Thus any map $A \to X$ admits an extension to B. □

Lemma 8.9 *A monomorphism $A \to B$ between ∞-categories is a trivial cofibration if and only if it is a strong J-deformation retract.*

Proof The fact that a strong J-deformation retract is a categorical equivalence is immediate from Lemma 8.5. Conversely, suppose $i : A \to B$ is a trivial cofibration between ∞-categories. Then

$$i_A^* = i^* : \tau A^B \to \tau A^A$$

is a trivial fibration between categories, as we noted in Remark 8.3, hence is surjective on objects. So we find a map $r\colon B \longrightarrow A$ with $ri = \mathrm{id}_A$. But

$$i_B^* = i^*\colon \tau B^B \longrightarrow \tau B^A$$

is also a trivial fibration between categories and $i_B^*(\mathrm{id}_B) = i_B^*(ir)$, so id_B and ir are isomorphic objects of $\tau(B^B)$ by an isomorphism lying in the fibre of i_B^* over i. This means that there is a map $h : J \longrightarrow B^B$ with $h_0 = \mathrm{id}_B$ and $h_1 = ir$, for which $i_B^* \circ h\colon J \longrightarrow B^A$ is J-homotopic relative to the endpoints of J to the constant map $J \longrightarrow B^A$ with value i (see Corollary 5.57). Write $H\colon J \times J \longrightarrow B^A$ for such a homotopy, so (not to confuse the two coordinates of $J \times J$)

$$H(0,-) = i_B^* \circ h, \quad H(1,-) = H(-,1) = H(-,0) = \mathrm{const}_i.$$

We can now modify h by picking a lift L in the diagram

$$\begin{tikzcd} V \arrow[r, "G"] \arrow[d] & k(B^B) \arrow[d, "i^*"] \\ J \times J \arrow[r, "H"] & k(B^A), \end{tikzcd}$$

where $V = (J \times \partial J) \cup (\{0\} \times J)$ and $G(0,-) = h$, $G(-,0) = \mathrm{id}_B$, and $G(-,1) = ir$. Such a lift indeed exists because $k(B^B) \longrightarrow k(B^A)$ is a Kan fibration by Theorem 5.55 and the morphism $V \longrightarrow J$ is anodyne. Then $l := L(1,-)$ is a homotopy from id_B to ir with $i_B^*(l) = H(1,-) = \mathrm{const}_i$, meaning l is a homotopy relative to A, as required. □

The small object argument allows us to functorially replace any simplicial set A by an ∞-category $\widehat{A}$. To be precise, define A' to be the pushout

$$\begin{tikzcd} \coprod \Lambda^k[n] \arrow[r] \arrow[d] & A \arrow[d, tail] \\ \coprod \Delta[n] \arrow[r] & A', \end{tikzcd}$$

where the coproduct ranges over all $0 < k < n$ and all maps $\Lambda^k[n] \longrightarrow A$. Then form a countable sequence

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

by $A_0 = A$ and $A_{n+1} = (A_n)'$ and let $\widehat{A} = \varinjlim A_n$ be its colimit. The following properties are clear from the construction:

Lemma 8.10 (a) *The map $A \longrightarrow \widehat{A}$ is inner anodyne and $\widehat{A}$ is an ∞-category. Moreover, the construction of $\widehat{A}$ is functorial and preserves monomorphisms.*
(b) *If A is countable, then so is $\widehat{A}$.*
(c) *If $B \subseteq \widehat{A}$ is countable, then there is a countable $U \subseteq A$ such that $B \subseteq \widehat{U} \subseteq \widehat{A}$.*

This leads to the following observation:

Corollary 8.11 *A monomorphism $i : A \to B$ is a categorical equivalence if and only if it fits into a diagram*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"] & A' \arrow[d, "j"] \\ B \arrow[r, "v"] & B' \end{tikzcd}$$

where u and v are inner anodyne maps into ∞-categories A' and B' respectively, and j is a strong J-deformation retract.

Proof Immediate from Lemmas 8.7, 8.9, and 8.10. □

Corollary 8.12 *The class of trivial cofibrations is saturated, i.e., it is closed under pushouts, retracts, and (possibly transfinite) composition.*

Proof If $A \to B$ is a retract of $C \to D$, then $\tau X^B \to \tau X^A$ is a retract of $\tau X^D \to X^C$, so the class is clearly closed under retracts. Similarly, a transfinite composition of trivial cofibrations

$$A_0 \to A_1 \to A_2 \to \cdots \to A_\xi \to \cdots$$

with colimit A gives, for any ∞-category X, a tower

$$\tau X^{A_0} \leftarrow \tau X^{A_1} \leftarrow \cdots \leftarrow \tau X^{A_\xi} \leftarrow \cdots$$

of trivial fibrations between categories with limit τX^A. Then each projection $\tau X^A \to \tau X^{A_\xi}$ is again a trivial fibration (as already observed in Example (a) at the end of Section 7.1). This proves that each $A_\xi \to A$ is a trivial cofibration. Finally, consider a pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

in which u is a trivial cofibration. By Corollary 8.11 we can complete this into a diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dd, "u"] \arrow[dr] & & C \arrow[dd] \arrow[dr] & \\ & \widehat{A} \arrow[rr] \arrow[dd] & & C' \arrow[dd] \\ B \arrow[rr] \arrow[dr] & & D \arrow[dr] & \\ & \widehat{B} \arrow[rr] & & D' \end{tikzcd}$$

where $A \to \widehat{A}$ and $B \to \widehat{B}$ are inner anodyne and $\widehat{u} : \widehat{A} \to \widehat{B}$ is a strong J-deformation retract, while C' and D' are constructed as pushouts of the top and bottom faces. Then the front face is also a pushout. The map $C' \to D'$ is a pushout of a strong deformation retract and hence itself a trivial cofibration. The maps $C \to C'$

and $D \to D'$ are pushouts of inner anodynes, therefore also trivial cofibrations. It follows (by two-out-of-three for the weak equivalences in **Cat**) that $C \to D$ is also a trivial cofibration. $\square$

Lemma 8.13 *Consider a strong deformation retract consisting of maps $u : A \to B$, $r : B \to A$ with $ru = \mathrm{id}_A$ and a homotopy $h : J \times B \to B$ from ur to id_B relative to A. Then for any countable $U \subseteq A$ and $V \subseteq B$, there are countable U' and V' with $U \subseteq U' \subseteq A$ and $V \subseteq V' \subseteq B$, such that $U' = u^{-1}(V')$, r maps V' into U', and h restricts to a map $J \times V' \to V'$. So in particular, U' is a strong J-deformation retract of V'.*

Proof We will construct a sequence of countable simplicial subsets

$$\begin{tikzcd} U = U_0 & \subseteq U_1 & \subseteq U_2 & \subseteq \cdots & \subseteq A \arrow[d, shift right, "u"'] \\ V = V_0 & \subseteq V_1 & \subseteq V_2 & \subseteq \cdots & \subseteq B \arrow[u, shift right, "r"'] \end{tikzcd}$$

such that

$$u(U_n) \subseteq V_{n+1},\ r(V_n) \subseteq U_{n+1},\ h(J \times V_n) \subseteq V_{n+1}.$$

Start with $U = U_0$ and $V = V_0$. Now pick $V_1 \subseteq B$ to be a countable simplicial subset containing $u(U_0)$ and $h(J \times V_0)$. Next, let $U_1 = r(V_1)$. Now repeat this with U_0 and V_0 replaced by U_n and V_n. Having done this for all $n \geq 0$, let $U' = \cup_n U_n$ and $V' = \cup_n V_n$. Then $u(U') \subseteq V'$ and $r(V') \subseteq U'$ (and hence $u^{-1}(V') = U'$) and h restricts to a map $J \times V' \to V'$. $\square$

Lemma 8.14 *Let $u\colon A \to B$ be a trivial cofibration and let $U \subseteq A$ and $V \subseteq B$ be countable simplicial subsets. Then there are countable U' and V' with $U \subseteq U' \subseteq A$ and $V \subseteq V' \subseteq B$ such that $U' = u^{-1}(V')$ and u restricts to a trivial cofibration $U' \to V'$.*

Proof The map $u\colon A \to B$ fits into a diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail, "u"] & \widehat{A} \arrow[d, shift right, "\widehat{u}"'] \\ B \arrow[r, tail, "j"] & \widehat{B} \arrow[u, shift right, "r"'] \end{tikzcd}$$

as in Corollary 8.11. By repeated application of Lemmas 8.10 and 8.13, we can now construct two sequences of maps between countable subobjects

$$\begin{tikzcd} U = U_0 & \subseteq U_1 \arrow[d, "u"'] & \subseteq U_2 \arrow[d, "u"'] & \subseteq \cdots & \subseteq A \arrow[d, "u"'] \\ V = V_0 & \subseteq V_1 & \subseteq V_2 & \subseteq \cdots & \subseteq B \end{tikzcd}$$

and

$$\begin{tikzcd}
\overline{U}_1 \arrow[d, shift right] & \subseteq \overline{U}_2 \arrow[d, shift right] & \subseteq \cdots & \subseteq \widehat{A} \arrow[d, shift right, "\widehat{u}"'] \\
\overline{V}_1 \arrow[u, shift right] & \subseteq \overline{V}_2 \arrow[u, shift right] & \subseteq \cdots & \subseteq \widehat{B} \arrow[u, shift right, "r"']
\end{tikzcd}$$

such that the strong deformation retraction between $\widehat{A}$ and $\widehat{B}$ restricts to strong deformation retracts between $\overline{U}_n$ and $\overline{V}_n$, and such that

$$\widehat{U}_n \subseteq \overline{U}_n \subseteq \widehat{U}_{n+1}, \quad \widehat{V}_n \subseteq \overline{V}_n \subseteq \widehat{V}_{n+1}.$$

Moreover, we can arrange that $\widehat{u}^{-1}(\overline{V}_n) = \overline{U}_n$ and $\widehat{u}^{-1}(V_n) = U_n$ as in Lemmas 8.10 and 8.13. Let $U' = \cup_n U_n$ and $V' = \cup V_n$. Then $\widehat{U}' = \cup_n \widehat{U}_n = \cup_n \overline{U}_n$ and $\widehat{V}' = \cup_n \widehat{V}_n = \cup_n \overline{V}_n$, so $\widehat{U}'$ is a strong deformation retract of $\widehat{V}'$. Thus, we have found a diagram of subsets of the first square of this proof as follows,

$$\begin{tikzcd}
U' \arrow[r] \arrow[d, "u"] & \widehat{U}' \arrow[d, shift right] \\
V' \arrow[r] & \widehat{V}' \arrow[u, shift right]
\end{tikzcd}$$

with horizontal inner anodyne maps and the map on the right part of a strong deformation retract. In particular $U' \to V'$ is a trivial cofibration, proving the lemma. □

Corollary 8.15 *Any trivial cofibration is the transfinite composition of pushouts of trivial cofibrations between countable simplicial sets.*

Proof Let $A \to B$ be a trivial cofibration. Identifying A with its image we may assume that $A \subseteq B$ and the map is the inclusion. Consider the set E of all non-degenerate simplices in B which do not belong to A, and fix a well-ordering on E. By induction, we will construct a sequence

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_\xi \subseteq A_{\xi+1} \subseteq \cdots$$

of subobjects of B, such that each inclusion $A_\xi \subseteq B$ is a trivial cofibration, such that each $A_\xi \to A_{\xi+1}$ is the pushout of a trivial cofibration between countable simplicial sets, and such that $\cup A_\xi = B$. Let $A_0 = A$. If A_ζ have been constructed for all $\zeta < \xi$ and ξ is a limit ordinal, we let $A_\xi = \cup_{\zeta<\xi} A_\zeta$. If $\xi = \zeta + 1$ and $A_\zeta = B$ then we are done. If not, let $b \in B$ be the first element in E which does not belong to A_ζ, let $V_b \subseteq B$ be the simplicial subset generated by b, and let $U_b = A_\zeta \cap V_b$. By Lemma 8.14 there are larger but still countable subsets $U_b' \subseteq A_\zeta$ and $V_b' \subseteq B$ such that $A_\zeta \to B$ restricts to a trivial cofibration $U_b' \to V_b'$, and moreover $V_b' \cap A_\zeta = U_b'$. Now define $A_{\zeta+1}$ as the pushout in the following square:

$$\begin{tikzcd} U'_b \arrow[r] \arrow[d, tail, "\wr"] & A_\zeta \arrow[d] \\ V'_b \arrow[r] & A_{\zeta+1}. \end{tikzcd}$$

Then the pushout map $A_\zeta \to A_{\zeta+1}$ is again a trivial cofibration. Moreover, $A_{\zeta+1}$ can be identified with the subset $A_\zeta \cup V'_b$ of B because $V'_b \cap A_\zeta = U'_b$. This completes the inductive step and the proof. □

We are now ready to prove the existence of the model structure of Theorem 8.2. The characterization of the fibrant objects will be given immediately after that.

Proof (of Theorem 8.2) Axiom (M1) is clear, as well as (M2) and (M3), which follow directly from the corresponding facts for categories. The small object argument provides two factorizations of each map $f: X \to Y$, once into a cofibration $u: X \to Z$ followed by a map $p: Z \to Y$ having the right lifting property with respect to all cofibrations, and once into a trivial cofibration $v: X \to W$ followed by a map $q: W \to Z$ having the right lifting property with respect to all trivial cofibrations (and thus, by definition, a fibration). Indeed, applying the small object argument here is possible because the cofibrations form the saturation of the *set* of boundary inclusions

$$\{\partial\Delta[n] \to \Delta[n] \,|\, n \geq 0\},$$

while the trivial cofibrations are generated by the *set* of those between countable simplicial sets (Corollary 8.15). Since the map $p: Z \to Y$ above is a weak equivalence by Corollary 8.6, this proves the factorization axiom (M5). It remains to check the lifting axiom (M4). So consider a square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ B \arrow[r] & Y, \end{tikzcd}$$

where i is a cofibration and f is a fibration. If i is also a weak equivalence then a lift $B \to X$ exists by definition of the fibrations. If f is a weak equivalence, the existence of a lifting is shown by the following standard retract argument: factor f as a cofibration $u : X \to Z$ followed by a map $p : Z \to Y$ having the right lifting property with respect to cofibrations. Then u is a weak equivalence because f and p are, and p has the right lifting property with respect to i. Then two liftings in the squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & Z \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, equal] \arrow[d, tail, "u"', "\wr"] & X \arrow[d, "f"] \\ Z \arrow[r, "p"] \arrow[ur, dashed] & Y \end{tikzcd}$$

compose to give the required lift $B \to X$. (Alternatively, notice that the square on the right exhibits f as a retract of p, hence the name of the argument.) This proves the theorem. □

We conclude this section by giving the promised characterizations of fibrant objects and of fibrations between them.

Proposition 8.16 *Let $f: X \to Y$ be an inner fibration between ∞-categories. Then the following properties are equivalent:*

(1) *The map f is a fibration in the Joyal model structure.*
(2) *The map f has J-path lifting, i.e., it has the right lifting property with respect to $\{0\} \to J$.*
(3) *The functor $\tau X \to \tau Y$ is an isofibration.*
(4) *For every monomorphism of simplicial sets $A \to B$, the map f has the right lifting property with respect to*

$$(J \times A) \cup (\{0\} \times B) \to J \times B.$$

(5) *The map f has the right lifting property with respect to any strong J-deformation retract.*

Before we prove Proposition 8.16, we observe the following.

Corollary 8.17 *The fibrant objects in the Joyal model structure are precisely the ∞-categories. A map $X \to Y$ between ∞-categories is a fibration if and only if it is an inner fibration with J-path lifting.*

Proof The first sentence is precisely Corollary 8.8. The second is immediate from Proposition 8.16. □

Proof (of Proposition 8.16) The equivalence between (2) and (3) is part of Corollary 5.54. We will show (2) ⇒ (4) ⇒ (5) ⇒ (1) ⇒ (2).

(2) ⇒ (4): The map f has the right lifting property with respect to $(J \times A) \cup (\{0\} \times B) \to J \times B$ if and only if

$$X^B \to Y^A \times_{X^A} X^B$$

has the right lifting property with respect to $\{0\} \to J$ and for this it suffices that the induced map on maximal Kan complexes

$$k(X^B) \to k(Y^A) \times_{k(X^A)} k(X^B)$$

has this lifting property. This is Theorem 5.56.

(4) ⇒ (5): Let $i : A \to B$ be part of a strong J-deformation retract, with retraction $r : B \to A$ and homotopy $h : J \times B \to B$ (rel A) between ir and id_B. Given a square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"] & X \arrow[d, "f"] \\ B \arrow[r, "\beta"] & Y, \end{tikzcd}$$

we can attempt to construct a lift by considering the map $\gamma = \alpha r$. This might not make the lower triangle involving β commute. As usual (by now), this can be fixed by lifting in the diagram

$$\begin{tikzcd}
(J \times A) \cup (\{0\} \times B) \arrow[r, "\alpha\pi_2 \cup \gamma"] \arrow[d] & X \arrow[d, "f"] \\
J \times B \arrow[r, "\beta h"'] \arrow[ur, dashed, "k"] & Y
\end{tikzcd}$$

and replacing γ by k_1.

(5) $\Rightarrow$ (1): We prove that $f : X \to Y$ has the right lifting property with respect to any trivial cofibration $u : A \to B$. Indeed, by Corollary 8.11 the map u fits into a square

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "u"] & \widehat{A} \arrow[d, "\widehat{u}"] \\
B \arrow[r, "j"] & \widehat{B}
\end{tikzcd}$$

with i and J inner anodyne and $\widehat{u}$ a strong deformation retract between ∞-categories. Now given any square as in the back of

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dd, "u"'] \arrow[dr] & & X \arrow[dd, "f"] \\
& \widehat{A} \arrow[ur, dashed] \arrow[dd] & \\
B \arrow[rr] \arrow[dr] & & Y, \\
& \widehat{B} \arrow[ur, dashed] &
\end{tikzcd}$$

we can complete this into a commutative diagram as indicated because X and Y are ∞-categories and f is an inner fibration. One then finds a diagonal filling in the right-hand face by assumption (5), which composes with $B \to \widehat{B}$ to give the required lift in the back face.

(1) $\Rightarrow$ (2): Clearly $\{0\} \to J$ is a trivial cofibration; it is a J-deformation retract. So $X \to Y$ has the right lifting property with respect to it. $\square$

8.2 The Kan–Quillen Model Structure on Simplicial Sets

In the previous section we established a model structure on the category of simplicial sets for which the cofibrations are the monomorphisms and the fibrant objects are exactly the ∞-categories. We defined a map $A \to B$ to be a categorical equivalence if for every ∞-category X, the map $\tau X^B \to \tau X^A$ is an equivalence of categories. The proof was based on some good properties of this functor $\tau : \mathbf{sSets} \to \mathbf{Cat}$, as well as the fact that every simplicial set admits an inner anodyne map into an ∞-category in a 'size-controlled' way.

We will not attempt to axiomatize the general pattern of this proof, but we do point out that exactly the same proof works in several other contexts. The first of these is very similar to the one in the previous section; we will now construct a model structure with (again) the monomorphisms as cofibrations, but now with Kan complexes as the fibrant objects. Thus, we adapt Definition 8.1 as follows.

Definition 8.18 A map $A \to B$ is a *weak homotopy equivalence* if for every Kan complex K, the induced map $\tau K^B \to \tau K^A$ is an equivalence of categories.

Notice that the simplicial sets K^B and K^A occurring in this definition are again Kan complexes (cf. Corollary 5.38), so that the categories τK^B and τK^A are in fact groupoids (cf. Lemma 5.48). If $A \to B$ is a monomorphism, then $K^B \to K^A$ is a Kan fibration and $\tau K^B \to \tau K^A$ is a fibration between groupoids (cf. Corollary 5.39 and Lemma 5.48 again). Thus, $A \to B$ is a trivial cofibration if and only if $\tau K^B \to \tau K^A$ is a trivial fibration between groupoids for every Kan complex K.

The analogue of Theorem 8.2 can now be stated as follows.

Theorem 8.19 *The category of simplicial sets admits a model structure in which the cofibrations are the monomorphisms and the weak equivalences are the weak homotopy equivalences just defined. Its fibrations are precisely the Kan fibrations; in particular, the fibrant objects are the Kan complexes.*

We will refer to the model structure of Theorem 8.19 as the *Kan–Quillen model structure*. Notice that since any Kan complex is in particular an ∞-category, every categorical equivalence is also a weak homotopy equivalence. Thus, when we compare the Joyal model structure to the Kan–Quillen model structure, we see that they have the same cofibrations (and hence the same trivial fibrations), but that the Kan–Quillen model structure has *more* weak equivalences and hence *fewer* fibrant objects. We will come back to this pattern at the end of Section 8.3. Note also that as in Remark 8.3, the model structure of the theorem is left proper because every object is cofibrant. We will see later that it is in fact also right proper, unlike the categorical model structure. Observe that the direct analogue of Theorem 8.2 would state that the fibrations between Kan complexes in the Kan–Quillen model structure are precisely the Kan fibrations. This is correct, but the statement of Theorem 8.19 is stronger: the fibrations between *arbitrary* objects are precisely the Kan fibrations.

Corollary 8.20 *A map of simplicial sets is a trivial cofibration in the Kan–Quillen model structure if and only if it is anodyne. In particular, the horn inclusions* $\Lambda^k[n] \to \Delta[n]$ *with* $0 \leq k \leq n$ *form a set of generating trivial cofibrations in this model structure.*

Proof A map is a trivial cofibration if and only if it has the left lifting property with respect to fibrations, giving the first sentence. The second sentence follows from Corollary 5.21. $\square$

Before we set to work and prove the theorem, let us point out again that in the context of Kan complexes and Kan fibrations, we might as well replace the 'interval' J by the much smaller representable simplicial set $\Delta[1]$. Indeed, for any simplicial set A, the maps $i_0, i_1 \colon A \to \Delta[1] \times A$ are anodyne (so in particular trivial cofibrations). Hence the projection $\Delta[1] \times A \to A$ is a weak equivalence by two-out-of-three. In particular, for any simplicial set A, the diagram

$$A \amalg A \to \Delta[1] \times A \to A$$

is a weak cylinder object in this model structure, so suffices to define the homotopy relation on morphisms. Therefore, in mimicking the strategy of the previous section, we will use $\Delta[1]$ rather than J.

We will now list the variations of the lemmas and propositions from the previous section, which can all be proved in exactly the same way and are sometimes consequences of the statements of the previous section.

Lemma 8.21 (a) *If $X \to Y$ is a $\Delta[1]$-homotopy equivalence between Kan complexes, then $\tau X \to \tau Y$ is an equivalence of groupoids.*
(b) *In particular, if $X \to Y$ is a map between Kan complexes having the right lifting property with respect to all monomorphisms, then $\tau X \to \tau Y$ is an equivalence of groupoids.*

The proof of this lemma is identical to that of Lemma 8.4. But in fact, since every $\Delta[1]$-homotopy equivalence between Kan complexes can be extended to a J-homotopy equivalence (as a consequence of Lemma 5.22), part (a) of this lemma is in fact a consequence of part (a) of Lemma 8.4. The same applies to (b).

Lemma 8.22 *Any $\Delta[1]$-homotopy equivalence is a weak homotopy equivalence.*

Lemma 8.23 *Any anodyne map is a weak homotopy equivalence.*

Lemma 8.24 *A monomorphism between Kan complexes is a trivial cofibration if and only if it is a strong $\Delta[1]$-deformation retract (and also if and only if it is a strong J-deformation retract).*

Using the small object argument as in the previous section, but now with respect to the set of all horn inclusions (rather than just inner horn inclusions), one finds an assignment $A \mapsto \widehat{A}$ satisfying the following:

Lemma 8.25 (a) *The map $A \to \widehat{A}$ is anodyne and $\widehat{A}$ is a Kan complex. Moreover, the construction of $\widehat{A}$ is functorial and preserves monomorphisms.*
(b) *If A is countable, then so is $\widehat{A}$.*
(c) *If $B \subseteq \widehat{A}$ is countable, then there is a countable $U \subseteq A$ such that $B \subseteq \widehat{U} \subseteq \widehat{A}$.*

Remark 8.26 In the current context there is an alternative to the abstract construction of $\widehat{A}$ used above. Kan's Ex^∞-functor gives an explicit method of replacing a simplicial set up to weak equivalence by a Kan complex. It is defined by iterating the functor Ex, which is right adjoint to barycentric subdivision. Since we will not need its construction, we omit further details.

Corollary 8.27 *A monomorphism $i\colon A \to B$ is a weak homotopy equivalence if and only if it fits into a diagram*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"] & X \arrow[d, "j"] \\ B \arrow[r, "v"] & Y \end{tikzcd}$$

where u and v are anodyne maps into Kan complexes X and Y respectively, and j is a strong $\Delta[1]$-deformation retract.

The statements of Corollary 8.12 and Lemmas 8.13 and 8.14 carry over verbatim. Again we obtain:

Corollary 8.28 *Any trivial cofibration (i.e., monomorphism which is also a weak homotopy equivalence) is the transfinite composition of pushouts of trivial cofibrations between countable simplicial sets.*

The proof of Theorem 8.19, except the last statement about fibrations, now proceeds exactly as the proof of Theorem 8.2 in Section 8.1.

We now turn to a discussion of the fibrations in the Kan–Quillen model structure. First, we state the analogue of Proposition 8.16, which can be proved in exactly the same way.

Proposition 8.29 *Let $f : X \to Y$ be an inner fibration between Kan complexes. Then the following properties are equivalent:*

(1) *The map f is a fibration in the Kan–Quillen model structure.*
(2) *The map f has path lifting, i.e., it has the right lifting property with respect to* $\{0\} \to \Delta[1]$.
(3) *The functor $\tau X \to \tau Y$ is an isofibration.*
(4) *For every monomorphism of simplicial sets $A \to B$, the map f has the right lifting property with respect to*

$$(\Delta[1] \times A) \cup (\{0\} \times B) \to \Delta[1] \times B.$$

(5) *The map f has the right lifting property with respect to any strong* $\Delta[1]$-*deformation retract.*

In the previous section we deduced from the analogue of this proposition a characterization of the categorical fibrations between ∞-categories. Here we establish the following, which completes the proof of Theorem 8.19:

Proposition 8.30 *The fibrations in the Kan–Quillen model structure are precisely the Kan fibrations.*

Proof Lemma 8.23 implies that every fibration is a Kan fibration. For the converse, recall from Section 5.8 that every Kan fibration can be factored as a trivial fibration followed by a minimal fibration. Trivial fibrations are of course, in particular, fibrations. Moreover, fibrations in any model structure are closed under pullback. Therefore, by appealing to Proposition 8.29, Lemma 8.31 below completes the proof. □

Lemma 8.31 *Any minimal fibration is the pullback of a minimal fibration between Kan complexes.*

Proof Let $M \to X$ be a minimal fibration. By Corollary 5.76, it is locally trivial. Moreover, working with one connected component of X at a time, we may assume that X is connected. So we can fix a minimal Kan complex F representing the fibre of $M \to X$.

Now consider for an arbitrary map $\alpha \colon \Lambda^k[n] \to X$ from a horn the pushout

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \\
\Delta[n] \arrow[r] & Y.
\end{tikzcd}$$

It suffices to extend $M \to X$ to a minimal Kan fibration $N \to Y$ which fits into a pullback

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & N \arrow[d] \\
X \arrow[r] & Y.
\end{tikzcd}$$

Indeed, we can the iterate this process (in the style of the small object argument) and obtain a minimal fibration fitting in a pullback diagram of the form above where $X \to Y$ is an anodyne map into a Kan complex Y. To find such an extension $N \to Y$, note that since $M \to X$ is locally trivial, the pullback along α is isomorphic to a product, as in

$$\begin{tikzcd}
\Lambda^k[n] \times F \arrow[r, "\cong"] \arrow[d] & \alpha^* M \arrow[r] \arrow[d] & M \arrow[d] \\
\Lambda^k[n] \arrow[r, equal] & \Lambda^k[n] \arrow[r, "\alpha"] & X.
\end{tikzcd}$$

Now form the pushout as on the top face of the cube

$$\begin{tikzcd}
\Lambda^k[n] \times F \arrow[rr] \arrow[dr] \arrow[dd] & & M \arrow[dr] \arrow[dd] & \\
& \Delta[n] \times F \arrow[rr, crossing over] & & N \arrow[dd, dashed] \\
\Lambda^k[n] \arrow[rr] \arrow[dr] & & X \arrow[dr] & \\
& \Delta[n] \arrow[rr] \arrow[from=uu, crossing over] & & Y.
\end{tikzcd}$$

Then there is a unique map $N \to Y$ as indicated which makes the cubical diagram commute. We claim that this map is again a minimal fibration. Indeed, for a diagram

$$\begin{tikzcd} \Lambda^i[m] \arrow[r, "\gamma"] \arrow[d] & N \arrow[d] \\ \Delta[m] \arrow[r, "\beta"] & Y, \end{tikzcd}$$

the map β either lands in $X \subseteq Y$ or it factors through $\Delta[n] \to Y$. But then the existence of a lift $\Delta[m] \to N$ (and its uniqueness properties as required for a minimal fibration) follow immediately from the fact that all three vertical maps other than $N \to Y$ are already known to be minimal fibrations. We leave it to the reader to check that the right-hand face of the cube is a pullback square. □

8.3 Quillen Adjunctions and Derived Functors

Now that we have presented a number of examples of model categories, it seems a good time for a brief intermezzo on 'morphisms' between model categories. The relevant notion is that of a Quillen pair of adjoint functors, briefly *Quillen pair* or *Quillen adjunction*, which derives its importance from the fact that it induces a pair of adjoint functors between the associated homotopy categories (Section 7.5). We will also discuss the notion of *Quillen equivalence*, the latter being an appropriate definition of 'equivalence between homotopy theories' which we promised in the introduction to this chapter.

Definition 8.32 Let $\mathcal{E}$ and $\mathcal{D}$ be model categories. A *Quillen adjunction* or *Quillen pair* between them is a pair of adjoint functors (left adjoint on the left)

$$f_! : \mathcal{E} \rightleftarrows \mathcal{D} : f^*$$

with the property that $f_!$ preserves cofibrations and f^* preserves fibrations. We will refer to $f_!$ as a *left Quillen functor* and f^* as a *right Quillen functor*.

Remark 8.33 (a) Because $f_!$ and f^* are adjoint, there are various equivalent ways to phrase this last condition. Indeed, f^* preserves fibrations if and only if $f_!$ preserves trivial cofibrations. Dually, $f_!$ preserves cofibrations if and only f^* preserves trivial fibrations.

(b) Since f^* preserves trivial fibrations, it follows from Brown's lemma that it preserves weak equivalences between fibrant objects (cf. Proposition 7.38). Dually, $f_!$ preserves weak equivalences between cofibrant objects.

(c) If $\mathcal{E}$ is a cofibrantly generated model category, the classes of cofibrations and trivial cofibrations are the saturations of sets of generating cofibrations and trivial cofibrations, respectively. Since any left adjoint preserves the constructions involved in the process of saturation (pushouts, colimits of sequences, retracts), it suffices to check that $f_!$ sends *generating* (trivial) cofibrations in $\mathcal{E}$ to (trivial) cofibrations in $\mathcal{D}$ in order for $f_!$ to be a left Quillen functor.

Before constructing the adjunction between homotopy categories induced by a Quillen pair, we will describe when a functor $G\colon \mathcal{E} \to \mathcal{D}$ between model categories induces a corresponding functor $\mathrm{Ho}(\mathcal{E}) \to \mathrm{Ho}(\mathcal{D})$ between homotopy categories, usually called a *derived functor* of G. Theorem 7.31 says that it is sufficient for G to preserve weak equivalences, but in cases of practical interest this is often not the case. We will identify convenient conditions which guarantee that a derived functor of G exists. The most common of these is that G is part of a Quillen pair, as defined above.

We will use the notation $\eta_{\mathcal{E}}\colon \mathcal{E} \to \mathrm{Ho}(\mathcal{E})$ for the functor introduced in Section 7.5.

Definition 8.34 Let $G\colon \mathcal{E} \to \mathcal{D}$ be a functor between model categories and write $\overline{G} = \eta_{\mathcal{D}} \circ G\colon \mathcal{E} \to \mathrm{Ho}(\mathcal{D})$.

(i) A *left derived functor* of G is a functor $\mathbf{L}G : \mathrm{Ho}(\mathcal{E}) \to \mathrm{Ho}(\mathcal{D})$ equipped with a natural transformation
$$\varepsilon\colon \mathbf{L}G \circ \eta_{\mathcal{E}} \Rightarrow \overline{G}$$
which is universal in the following sense: for any $H\colon \mathrm{Ho}(\mathcal{E}) \to \mathrm{Ho}(\mathcal{D})$ and any natural transformation $\nu\colon H \circ \eta_{\mathcal{E}} \Rightarrow \overline{G}$, there is a unique natural transformation $\mu\colon H \Rightarrow \mathbf{L}G$ with $\nu = \varepsilon(\mu \circ \eta_{\mathcal{E}})$.

(ii) Dually, a *right derived functor* of G is a functor $\mathbf{R}G : \mathrm{Ho}(\mathcal{E}) \to \mathrm{Ho}(\mathcal{D})$ equipped with a natural transformation
$$\varepsilon\colon \overline{G} \Rightarrow \mathbf{R}G \circ \eta_{\mathcal{E}}.$$
which is universal in the evident sense dual to (i).

Remark 8.35 The universal properties stated in the definition above are exactly those of Kan extensions. More precisely, a left derived functor $\mathbf{L}G$ is precisely a *right* Kan extension of $\overline{G}$ along $\eta_{\mathcal{E}}$, whereas a right derived functor is a *left* Kan extension of $\overline{G}$ along $\eta_{\mathcal{E}}$. In particular, left and right derived functors (if they exist) are unique up to natural isomorphism. We will therefore often speak of *the* left/right derived functor of a given G, when it exists.

Lemma 8.36 *If G preserves cofibrations and trivial cofibrations between cofibrant objects, then a left derived functor of G exists.*

Proof We begin by constructing a suitable functor $g\colon \mathcal{E} \to \mathrm{Ho}(\mathcal{D})$ that sends weak equivalences to isomorphisms. For every $X \in \mathcal{E}$, choose a cofibrant replacement
$$X_c \xrightarrow[q_X]{\sim} X$$
and define $g(X) = \overline{G}(X_c)$. For a morphism $\alpha\colon X \to Y$ one finds a corresponding $\alpha_c\colon X_c \to Y_c$ as before, simply by lifting $\alpha \circ q_X\colon X_c \to Y$ along the trivial fibration q_Y. The choice of α_c is unique up to left homotopy. Define $g(\alpha) = \overline{G}(\alpha_c)$. To see that g is a well-defined functor, it suffices to check that it sends two left homotopic maps with domain X_c to the same morphism in $\mathrm{Ho}(\mathcal{D})$. Since G preserves cofibrations and

weak equivalences between cofibrant objects (the latter by the dual of Proposition 7.38), it sends a weak cylinder object for X_c to a weak cylinder object for $G(X_c)$. In particular, it will send left homotopic maps out of X_c to left homotopic maps out of $G(X_c)$.

Observe that g sends weak equivalences to isomorphisms, again because G preserves weak equivalences between cofibrant objects. Theorem 7.31 implies that (up to natural isomorphism) g must then factor as a composition of functors

$$\mathcal{E} \xrightarrow{\eta_{\mathcal{E}}} \mathrm{Ho}(\mathcal{E}) \to \mathrm{Ho}(\mathcal{D}).$$

We will denote the second one by $\mathbf{L}G$.

To complete the proof, we should supply a universal natural transformation ε from $\mathbf{L}G \circ \eta_{\mathcal{E}}$ to $\overline{G}$, or equivalently a natural transformation $g \Rightarrow \overline{G}$. Recall that $\overline{G}(X) = (G(X)_c)_f$, for some choice of cofibrant and fibrant approximation in $\mathcal{D}$. Lift in the diagram

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d, tail] & G(X)_c \arrow[r, tail, "\sim"] \arrow[d, two heads, "\wr"] & (G(X)_c)_f \\
G(X_c) \arrow[r, "G(q_X)"'] \arrow[ur, dashed] & G(X) &
\end{tikzcd}$$

to find a map $\varepsilon_X : g(X) = G(X_c) \to (G(X)_c)_f$. This gives a well-defined natural transformation after composing with $\eta_{\mathcal{E}} : \mathcal{E} \to \mathrm{Ho}(\mathcal{E})$, since the choice of lift is unique up to homotopy. It is straightforward to verify that ε is universal, using that any composition of functors

$$\mathcal{E} \xrightarrow{\eta_{\mathcal{E}}} \mathrm{Ho}(\mathcal{E}) \to \mathrm{Ho}(\mathcal{D})$$

sends the map $q_X : X_c \to X$ to an isomorphism in $\mathrm{Ho}(\mathcal{D})$. □

Corollary 8.37 *If $f_! : \mathcal{E} \to \mathcal{D}$ is a left Quillen functor, then it admits a left derived functor $\mathbf{L}f_!$. Similarly, a right Quillen functor f^* admits a right derived functor $\mathbf{R}f^*$.*

Remark 8.38 By the proof of Lemma 8.36, the value of $\mathbf{L}f_! \circ \eta_{\mathcal{E}}$ on a morphism $\alpha : X \to Y$ is the image of $f_!(\alpha_c) : F_!(X_c) \to f_!(Y_c)$ in $\mathrm{Ho}(\mathcal{D})$, for any choice of cofibrant replacement α_c of α. In practice one often has a *functorial* cofibrant replacement $Q : \mathcal{E} \to \mathcal{E}$, equipped with a natural transformation $Q \to \mathrm{id}_{\mathcal{E}}$ which is a trivial fibration when evaluated at any object of $\mathcal{E}$. (Indeed, many authors even require the existence of functorial factorizations in the axioms for a model structure, although we, following Quillen, have not done this.) For example, factorizations constructed using the small object argument can be made functorial. One can then consider the functor

$$f_! \circ Q : \mathcal{E} \to \mathcal{D},$$

and the composite $\eta_{\mathcal{D}} \circ f_! \circ Q$ is then naturally isomorphic to $\mathbf{L}f_! \circ \eta_{\mathcal{E}}$. Incidentally, in the presence of functorial factorizations the statement of Lemma 8.36 only needs that $f_!$ sends trivial cofibrations between cofibrant objects to weak equivalences. Indeed, it will then preserve weak equivalences between cofibrant objects by Brown's lemma, so that $\eta_{\mathcal{D}} \circ f_! \circ Q$ is a functor sending weak equivalences to isomorphisms. Theorem 7.31 then guarantees that this composite factors through a functor defined on $\mathrm{Ho}(\mathcal{E})$, namely the left derived functor $\mathbf{L}f_!$.

Remark 8.39 It sometimes happens that all objects in $\mathcal{E}$ are cofibrant, for example in the Joyal and Kan–Quillen model structures on the category of simplicial sets. In that case $f_!$ preserves weak equivalences between arbitrary objects and one can simply take $\mathbf{L}f_!(X)$ to be the image in $\mathrm{Ho}(\mathcal{D})$ of $f_!(X)$, i.e., one does not have to 'derive' $f_!$ at all. The dual remark applies when all objects of $\mathcal{D}$ are fibrant. An example is the model structure on **Top** we discussed in the previous chapter.

Lemma 8.40 *If $(f_!, f^*)$ is a Quillen pair, then $(\mathbf{L}f_!, \mathbf{R}f^*)$ is an adjoint pair of functors between homotopy categories.*

Proof If $\eta\colon \mathrm{id}_{\mathcal{E}} \to f^* f_!$ is the unit of the adjoint pair $(f_!, f^*)$, then one constructs a unit for the pair $(\mathbf{L}f_!, \mathbf{R}f^*)$ as follows. For X an object of $\mathrm{Ho}(\mathcal{E})$ (i.e., a fibrant and cofibrant object of $\mathcal{E}$), one considers

$$X \xrightarrow{\eta_X} f^* f_!(X) \xrightarrow{f^* i_{f_!(X)}} f^*(f_!(X)_f),$$

where $i_{f_!(X)}$ is a fibrant replacement of $f_!(X)$. Now note that the right-hand side computes the value of $\mathbf{R}f^*$. Since the choice of $i_{f_!(X)}$ is unique up to homotopy, this gives a well-defined natural map in $\mathrm{Ho}(\mathcal{E})$. Similarly one constructs a counit and one easily verifies the triangle identities. $\square$

Remark 8.41 The unit and counit for the adjoint pair $(\mathbf{L}f_!, \mathbf{R}f^*)$ are often called the *derived unit* and *derived counit* respectively.

The following 'recognition lemma' for Quillen adjunctions is often useful:

Lemma 8.42 *Let $f_!\colon \mathcal{E} \to \mathcal{D}$ be a functor between model categories which admits a right adjoint f^*. If $f_!$ preserves cofibrations and f^* preserves fibrations between fibrant objects, then $(f_!, f^*)$ is a Quillen adjunction.*

This lemma follows immediately from the following statement, which we record for future reference:

Lemma 8.43 *A cofibration $i\colon A \to B$ in a model category $\mathcal{E}$ is a trivial cofibration if and only if it has the left lifting property with respect to fibrations between fibrant objects.*

Proof The only nontrivial direction of the statement is where we assume that i has the stated lifting property. Choose a fibrant replacement $B \to B_f$ and factor the composite $A \to B \to B_f$ as a trivial cofibration $A \to A_f$ followed by a fibration $A_f \to B_f$ to get a commutative square as follows:

$$\begin{tikzcd} A \arrow[r, tail, "\sim"] \arrow[d, tail, "i"'] & A_f \arrow[d, two heads] \\ B \arrow[r, tail, "\sim"] \arrow[ur, dashed] & B_f. \end{tikzcd}$$

A lift l exists by our assumption on i. Since the horizontal maps are weak equivalences, the diagram shows that the image of l in the homotopy category $\mathrm{Ho}(\mathcal{E})$ has both a right and a left inverse. But then it must be an isomorphism in $\mathrm{Ho}(\mathcal{E})$, so that l is a weak equivalence. By two-out-of-three it follows that every map in the square is a weak equivalence, including i. □

Definition 8.44 A Quillen adjunction $(f_!, f^*)$ between model categories $\mathcal{E}$ and $\mathcal{D}$ is a *Quillen equivalence* if the pair $(\mathbf{L}f_!, \mathbf{R}f^*)$ is an adjoint equivalence between the homotopy categories $\mathcal{E}$ and $\mathcal{D}$.

Remark 8.45 The notion of Quillen equivalence is not symmetric (since it depends on the direction of the left adjoint). Thus, we say that two model categories $\mathcal{E}$ and $\mathcal{D}$ are *Quillen equivalent* if there exists a zig-zag

$$\mathcal{E} \to \cdot \leftarrow \cdots \to \mathcal{D}$$

of left Quillen functors, each of which is part of a Quillen equivalence. More loosely, one might say that $\mathcal{E}$ and $\mathcal{D}$ 'model the same homotopy theory'.

We record several characterizations of Quillen equivalences:

Lemma 8.46 *For a left Quillen functor $f_! \colon \mathcal{E} \to \mathcal{D}$ with right adjoint f^*, the following are equivalent:*

(1) *The pair $(f_!, f^*)$ is a Quillen equivalence.*
(2) *For any cofibrant $X \in \mathcal{E}$ there exists a fibrant replacement $i_{f_!(X)} : f_!(X) \to f_!(X)_f$ such that the composite $X \to f^* f_!(X) \to f^*(f_!(X)_f)$ is a weak equivalence, and for any fibrant $Y \in \mathcal{D}$ there exists a cofibrant replacement $q_{f^*(Y)} : f^*(Y)_c \to f^*(Y)$ such that the composite $f_!(f^*(Y)_c) \to f_! f^*(Y) \to Y$ is a weak equivalence.*
(3) *For any cofibrant $X \in \mathcal{E}$ and fibrant $Y \in \mathcal{D}$, any map $\alpha \colon f_!(X) \to Y$ is a weak equivalence if and only if its adjoint $\widehat{\alpha} \colon X \to f^*(Y)$ is a weak equivalence.*

Proof Condition (2) implies that the derived unit and counit are equivalences and therefore implies (1). Conversely, (1) implies (2) when X (resp. Y) is additionally assumed to be fibrant (resp. cofibrant). But this is no loss of generality, since for general X one can always choose a fibrant replacement $X \to X_f$ and consider the following commuting square:

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail, "\wr"] & f^*(f_!(X)_f) \arrow[d] \\ X_f \arrow[r] & f^*(f_!(X_f)_f). \end{tikzcd}$$

The bottom horizontal map gives the derived unit at X_f and is thus a weak equivalence by assumption. The map $f_!(X) \to f_!(X_f)$ is a weak equivalence because $f_!$ preserves trivial cofibrations. Since f^* preserves weak equivalences between fibrants, the right vertical map in the square is a weak equivalence. Therefore all maps in the diagram are weak equivalences. The argument for Y is similar but dual.

Condition (3) implies that for objects $X \in \mathrm{Ho}(\mathcal{E})$ and $Y \in \mathrm{Ho}(\mathcal{D})$, any map $\alpha\colon \mathbf{L}f_!(X) \to Y$ is an isomorphism if and only if its adjoint $X \to \mathbf{R}f^*(Y)$ is an isomorphism. Unit and counit of the pair $(\mathbf{L}f_!, \mathbf{R}f^*)$, being adjoint to identity maps, are therefore isomorphisms and $(\mathbf{L}f_!, \mathbf{R}f^*)$ is an adjoint equivalence. Finally, we need to argue that (2) implies (3). So suppose X is cofibrant, Y is fibrant, and $\alpha\colon f_!(X) \to Y$ is an equivalence. Choose a fibrant replacement of $f_!(X)$ and pick a lift β in the following diagram:

$$\begin{tikzcd} f_!(X) \arrow[r, "\alpha"] \arrow[d, tail, "\wr"] & Y. \\ f_!(X)_f \arrow[ur, dashed, "\beta"'] & \end{tikzcd}$$

Then β is a weak equivalence of fibrant objects, so that $f^*(\beta)$ is a weak equivalence as well. The map $\widehat{\alpha}$ is the composition of the maps

$$X \to f^* f_!(X) \to f^*(f_!(X)_f) \xrightarrow{f^*(\beta)} f^*(Y).$$

The composition of the first two maps is a weak equivalence by assumption, so that the total composition $\widehat{\alpha}$ is a weak equivalence as well. The converse direction, from $\widehat{\alpha}$ to α, is proved analogously. □

Example 8.47 (a) Let $i^*\colon \mathbf{Gpd} \to \mathbf{Cat}$ denote the inclusion of the category of (small) groupoids into the category of (small) categories. It has both adjoints; the right adjoint i_* maps a small category $\mathbf{C}$ to its maximal subgroupoid $i_*\mathbf{C}$ of all isomorphisms in $\mathbf{C}$, whereas the left adjoint $i_!$ maps $\mathbf{C}$ to the category $i_!\mathbf{C} = \mathbf{C}[\mathbf{C}^{-1}]$ obtained by inverting all morphisms in $\mathbf{C}$. (Another way to describe this groupoid is as the fundamental groupoid $\pi_1(B\mathbf{C}, \mathrm{ob}(\mathbf{C}))$ of the classifying space of $\mathbf{C}$ with vertices the set $\mathrm{ob}(\mathbf{C})$ of objects of $\mathbf{C}$.) Clearly i^* preserves weak equivalences, fibrations, and cofibrations. So both $(i_!, i^*)$ and (i^*, i_*) are Quillen pairs. In both model categories, all objects are fibrant as well as cofibrant. Thus $\mathrm{Ho}(\mathbf{Gpd})$ has the same objects as $\mathbf{Gpd}$ and natural isomorphism classes of functors as morphisms. A similar description applies to $\mathrm{Ho}(\mathbf{Cat})$.

(b) Consider the adjoint pair of functors

$$|\cdot| : \mathbf{sSets} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$$

given by geometric realization and the singular complex. We already observed that the functor Sing maps Serre fibrations to Kan fibrations (e.g. in Example 5.15, as well as in Section 7.2). Also, the geometric realization of any boundary inclusion $\partial\Delta[n] \to \Delta[n]$ is a cofibration of topological spaces. Hence the functor $|\cdot|$ preserves cofibrations and the adjoint pair above is a Quillen adjunction when we equip **sSets** with the Kan–Quillen model structure. Every simplicial set is cofibrant and every topological space is fibrant, so the functors $|\cdot|$ and Sing are 'already derived'. In particular, they preserve weak equivalences between arbitrary objects. Moreover, the unit and counit maps

$$A \to \mathrm{Sing}|A| \quad \text{and} \quad |\mathrm{Sing}(X)| \to X$$

for a simplicial set A and topological space X also represent the derived unit and counit maps. We will prove later in this chapter that these two maps are weak equivalences. In other words, we will prove that this Quillen pair is a Quillen equivalence (see Theorem 8.65).

(c) Consider the adjoint pair

$$\tau : \mathbf{sSets} \rightleftarrows \mathbf{Cat} : N.$$

The nerve functor N sends fibrations of categories, i.e. isofibrations, to fibrations in the Joyal model structure. It also preserves trivial fibrations (and in fact all weak equivalences), so the pair above is a Quillen adjunction. All objects in **Cat** are fibrant and all objects in the Joyal model structure on **sSets** are cofibrant, so these two functors τ and N are already derived. Thus for a small category $\mathbf{C}$ and simplicial set A, the derived unit and counit are represented by the ordinary unit and counit maps

$$A \to N\tau(A) \quad \text{and} \quad \tau N\mathbf{C} \to \mathbf{C}.$$

The counit map is an isomorphism, so the full embedding $N : \mathbf{Cat} \to \mathbf{sSets}$ remains fully faithful when viewed as a functor $\mathrm{Ho}(\mathbf{Cat}) \to \mathrm{Ho}(\mathbf{sSets}_{\mathrm{cat}})$, where the subscript cat indicates the categorical (or Joyal) model structure. Also, the fact that τ is already derived implies that it sends categorical equivalences between arbitrary simplicial sets to equivalences of categories.

(d) Let us write the two model categories of simplicial sets we have constructed so far as $\mathbf{sSets}_{\mathrm{cat}}$ (as above) and $\mathbf{sSets}_{\mathrm{KQ}}$. These have the same cofibrations, while $\mathbf{sSets}_{\mathrm{KQ}}$ has more weak equivalences (and consequently fewer fibrations). The identity functor on **sSets** is of course adjoint to itself. In order not to confuse the left and right adjoint notationally, we write this as a pair

$$\mathrm{id}_! : \mathbf{sSets}_{\mathrm{cat}} \rightleftarrows \mathbf{sSets}_{\mathrm{KQ}} : \mathrm{id}^*.$$

It is clear that this is a Quillen pair. The functor $\mathrm{id}_!$ is already derived (because every object is cofibrant), but id^* is not. For a simplicial set A, the derived unit $A \to \mathbf{R}\mathrm{id}^*(\mathrm{id}_! A)$ is represented by an anodyne map $A \to \widehat{A}$ into a Kan complex $\widehat{A}$. For a fibrant object in $\mathbf{sSets}_{\mathrm{KQ}}$, i.e. for a Kan complex K, the derived counit map $\mathbf{L}\mathrm{id}_!\mathbf{R}\mathrm{id}^*(K) \to K$ is represented by the identity map. Thus, at the level of homotopy categories, $\mathrm{Ho}(\mathbf{sSets}_{\mathrm{KQ}})$ is a full subcategory of $\mathrm{Ho}(\mathbf{sSets}_{\mathrm{cat}})$ for which the inclusion has a left adjoint. After this example we will discuss this situation a bit more generally; it is an instance of *left Bousfield localization*, a concept which will feature often in this book.

(e) Left Quillen functors between model categories can be composed. Thus, from the previous examples one can construct the left Quillen functor

$$i_! \circ \tau : \mathbf{sSets}_{\mathrm{cat}} \to \mathbf{Gpd},$$

and several other such composites.

(f) The composed adjoint pair

$$\mathbf{sSets} \underset{N}{\overset{\tau}{\rightleftarrows}} \mathbf{Cat} \underset{i^*}{\overset{i_!}{\rightleftarrows}} \mathbf{Gpd}$$

is in fact also a Quillen pair for the Kan–Quillen model structure on **sSets** (although τ itself is *not* a left Quillen functor from $\mathbf{sSets}_{\mathrm{KQ}}$ to **Cat**). Indeed, one way to see this is simply to check that the nerve functor sends a fibration of groupoids to a Kan fibration. Another way to look at it is to observe that the composed left Quillen functor $\mathbf{sSets}_{\mathbf{cat}} \to \mathbf{Gpd}$ of example (e) above factors through $\mathbf{sSets}_{\mathrm{KQ}}$, as in the following diagram of left Quillen functors:

$$\begin{tikzcd}
\mathbf{sSets}_{\mathbf{cat}} \arrow[r, "\tau"] \arrow[d, "\mathrm{id}_!"] & \mathbf{Cat} \arrow[d, "i_!"] \\
\mathbf{sSets}_{\mathbf{KQ}} \arrow[r, dashed] & \mathbf{Gpd}.
\end{tikzcd}$$

This is the case simply because the nerve of a groupoid is a Kan complex (and we do not have to consider fibrations of groupoids). Indeed, to check that the factorization exists, it suffices to prove that $i_!\tau$ sends a weak homotopy equivalence $A \to B$ between simplicial sets to a weak equivalence of groupoids. For each Kan complex K, the induced map $[B, K] \to [A, K]$ is a bijection. In particular, this applies to $K = N\mathbf{G}$, for $\mathbf{G}$ any groupoid. By adjunction the map $[\tau B, \mathbf{G}] \to [\tau A, \mathbf{G}]$ is also an isomorphism, so that the Yoneda lemma applied to the category Ho(**Gpd**) implies that $\tau A \to \tau B$ is a weak equivalence of groupoids.

(g) More generally, the same argument shows that a Quillen pair

$$f_! : \mathbf{sSets}_{\mathbf{cat}} \rightleftarrows \mathcal{E} : f^*$$

into a model category $\mathcal{E}$ factors through $\mathrm{id}_! : \mathbf{sSets}_{\mathbf{cat}} \to \mathbf{sSets}_{\mathbf{KQ}}$ precisely when f^* sends fibrant objects of $\mathcal{E}$ to Kan complexes. This is an instance of the general theory of Bousfield localizations. A first discussion follows right after this example, a more thorough treatment is given in Chapter 11.

(h) The construction of model structures via transfer always gives Quillen pairs. More precisely, suppose $\mathcal{E}$ is a cofibrantly generated model category and

$$\mathcal{E} \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathcal{A}$$

is an adjunction which meets the conditions for transfer (see Theorem 7.44). If we equip $\mathcal{A}$ with the transferred model structure, then this adjoint pair becomes a Quillen pair, essentially by construction.

(i) Any morphism $\varphi\colon X \to Y$ in a model category $\mathcal{E}$ gives a Quillen adjunction

$$\varphi_! : \mathcal{E}/X \rightleftarrows \mathcal{E}/Y : \varphi^*$$

between the respective slice categories. Here $\varphi_!$ is the functor that composes a morphism $A \to X$ with φ and φ^* is the functor that takes the pullback along φ. It follows immediately from the definitions that $\varphi_!$ preserves cofibrations and trivial cofibrations, making it a left Quillen functor. If φ is a trivial fibration in $\mathcal{E}$, then the pair above is in fact a Quillen equivalence. To see this, note that for a cofibrant object $A \to X$, the derived unit may be described by the first arrow in the top row of the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \times_Y A \arrow[r, two heads, "\sim"] \arrow[d] & A \arrow[d] \\
X \arrow[r, equal] & X \arrow[r, two heads, "\sim", "\varphi"'] & Y.
\end{tikzcd}$$

By two-out-of-three, that arrow is a weak equivalence. For a fibrant object $B \to Y$, the derived counit may be computed as the top horizontal arrow in the following square, which is a trivial fibration:

$$\begin{tikzcd}
X \times_Y B \arrow[r, two heads, "\sim"] \arrow[d] & B \arrow[d] \\
X \arrow[r, two heads, "\sim", "\varphi"'] & Y.
\end{tikzcd}$$

Now Brown's Lemma 7.37 also implies that whenever φ is a weak equivalence between fibrant objects in $\mathcal{E}$, the associated Quillen adjunction between slice categories is a Quillen equivalence.

(j) The previous example can be sharpened a bit if $\mathcal{E}$ is a right proper model category. Indeed, if $\varphi\colon X \to Y$ is a weak equivalence between arbitrary objects, the adjunction

$$\varphi_! : \mathcal{E}/X \rightleftarrows \mathcal{E}/Y : \varphi^*$$

is a Quillen equivalence. To see this, pick a cofibrant object $A \to X$ of $\mathcal{E}/X$ and factor the composite map $A \to Y$ as a weak equivalence $A \to A'$ followed by a fibration $A' \to Y$. The derived unit may be described by the arrow $A \to X \times_Y A'$ in the following diagram:

$$\begin{tikzcd}
A \arrow[dr] \arrow[drr, bend left, "\sim"] \arrow[ddr, bend right] & & \\
& X \times_Y A' \arrow[r] \arrow[d] & A' \arrow[d, two heads] \\
& X \arrow[r, "\sim"] & Y.
\end{tikzcd}$$

The top horizontal arrow is a weak equivalence by right properness, hence also the arrow $A \to X \times_Y A'$ by two-out-of-three. For a fibrant object $B \to Y$ of $\mathcal{E}/Y$, the fact that the derived counit is a weak equivalence follows directly from the following pullback square:

$$\begin{tikzcd} X \times_Y B \arrow[r, two heads, "\sim"] \arrow[d] & B \arrow[d, two heads] \\ X \arrow[r, "\sim", "\varphi"'] & Y. \end{tikzcd}$$

(k) The following variation on (i) will also be useful. Suppose that

$$f_! : \mathcal{E} \rightleftarrows \mathcal{F} : f^*$$

is a Quillen pair between model categories $\mathcal{E}$ and $\mathcal{F}$. Let Y be an object of $\mathcal{F}$. Then the adjoint pair above induces another Quillen pair

$$\overline{f}_! : \mathcal{E}/f^*Y \rightleftarrows \mathcal{F}/Y. : \overline{f}^*$$

The functor $\overline{f}^*$ just applies f^* to a morphism, whereas $\overline{f}_!(X \to f^*Y)$ forms the composition

$$f_!X \to f_!f^*Y \xrightarrow{\varepsilon} Y.$$

If Y is fibrant and the original pair $(f_!, f^*)$ is a Quillen equivalence, then we claim that this new pair $(\overline{f}_!, \overline{f}^*)$ is also a Quillen equivalence. First of all, $f_!$ detects weak equivalences between cofibrant objects, so that the same is true of $\overline{f}_!$. It follows that the derived functor $\mathbf{L}\overline{f}_!$ detects isomorphisms. It then suffices to check that the derived counit

$$\mathbf{L}\overline{f}_!\mathbf{R}\overline{f}^* \xrightarrow{\varepsilon} \mathrm{id}_{\mathrm{Ho}(\mathcal{F})}$$

is an isomorphism. Indeed, the derived unit is then an isomorphism by virtue of the commutative triangle

$$\begin{tikzcd} & \mathbf{L}\overline{f}_!\mathbf{R}\overline{f}^*\mathbf{L}\overline{f}_! \arrow[dr, "\varepsilon\mathbf{L}\overline{f}_!", "\cong"'] & \\ \mathbf{L}\overline{f}_! \arrow[ur, "\mathbf{L}\overline{f}_!\eta"] \arrow[rr, equal] & & \mathbf{L}\overline{f}_!. \end{tikzcd}$$

To analyse the derived counit, consider a fibration $Z \to Y$ in $\mathcal{F}$ and a cofibrant replacement $W \xrightarrow{\sim} f^*Z$ in $\mathcal{E}$. Then the derived counit is described by the adjoint map $f_!W \to Z$ over Y. But at the same time this is just the derived counit of the original adjunction $(f_!, f^*)$ evaluated at the fibrant object Z, hence a weak equivalence by assumption.

Remark 8.48 As a sort of converse to part (h) of the previous, it is possible to recognize when a Quillen adjunction arises from transfer in the following manner. Suppose we start with a Quillen pair $f_! : \mathcal{E} \rightleftarrows \mathcal{A} : f^*$ between model categories. If, ignoring that $\mathcal{A}$ has a model structure already, the conditions for transfer are met, this defines a new model category $\mathcal{A}_{\mathrm{tr}}$ with the same underlying category $\mathcal{A}$, and a diagram of Quillen pairs (in which we only draw the left Quillen functors):

$$\begin{tikzcd} \mathcal{E} \arrow[r, "f_!"] \arrow[d, "g_!"'] & \mathcal{A} \\ \mathcal{A}_{\mathrm{tr}} \arrow[ur, "\mathrm{id}_A"'] & \end{tikzcd}$$

Here $\mathrm{id}_{\mathcal{A}}$ and its right adjoint are both the identity functor, which is left Quillen when viewed as a functor $\mathcal{A}_{\mathrm{tr}} \to \mathcal{A}$. The vertical functor is of course also $f_!$, but we have relabelled it to make the distinction in codomain. If f^* preserves and detects weak equivalences and trivial fibrations (as g^* does, by definition of transfer), then the weak equivalences and trivial fibrations in $\mathcal{A}$ and in $\mathcal{A}_{\mathrm{tr}}$ coincide and hence so do the cofibrations, being the maps having the left lifting property with respect to the trivial fibrations. Since any two of the three classes in a model structure determine the third, it follows that $\mathcal{A}_{\mathrm{tr}}$ is identical to $\mathcal{A}$. In particular, the (trivial) cofibrations in $\mathcal{A}$ are generated by the images under $f_!$ of (trivial) cofibrations in $\mathcal{E}$.

We end this section with a brief discussion of *left Bousfield localization*, to which we will return in more detail in Section 11.3. Let $\mathcal{E}$ be a model category. We will frequently encounter the situation, already illustrated by Example 8.47(d) above, where there is another model structure on $\mathcal{E}$ with the same cofibrations, but with a larger class of weak equivalences (and hence a smaller class of fibrations). Let us refer to this larger class of weak equivalences as the *local weak equivalences* and the corresponding class of fibrations as the *local fibrations*. So every weak equivalence is a local weak equivalence and every local fibration is a fibration. Write $\mathcal{E}_{\mathrm{loc}}$ for this second model structure on $\mathcal{E}$. Then the identity functor defines a Quillen pair

$$\mathrm{id}_! : \mathcal{E} \rightleftarrows \mathcal{E}_{\mathrm{loc}} : \mathrm{id}^*.$$

The induced adjoint pair

$$\mathbf{L}\mathrm{id}_! : \mathrm{Ho}(\mathcal{E}) \rightleftarrows \mathrm{Ho}(\mathcal{E}_{\mathrm{loc}}) : \mathbf{R}\mathrm{id}^*$$

has the property that the counit $\mathbf{L}\mathrm{id}_!\mathbf{R}\mathrm{id}^*(E) \to E$ is an isomorphism, for any $E \in \mathcal{E}$. Indeed, for a locally fibrant and cofibrant object E, we have $\mathbf{R}\mathrm{id}^*(E) = \mathrm{id}^*E = E$ and $\mathbf{L}\mathrm{id}_!(E) = E$ as well. For a fibrant and cofibrant object E in $\mathcal{E}$, the unit $E \to \mathbf{R}\mathrm{id}^*\mathbf{L}\mathrm{id}_!(E)$ is a fibrant replacement of E in $\mathcal{E}_{\mathrm{loc}}$, i.e., a local weak equivalence $E \to E_{\mathrm{loc}}$ into a locally fibrant object. In particular, the right adjoint functor $\mathbf{R}\mathrm{id}^*$ is fully faithful, so that we can regard $\mathrm{Ho}(\mathcal{E}_{\mathrm{loc}})$ as a full subcategory of $\mathrm{Ho}(\mathcal{E})$. The left adjoint $\mathbf{L}\mathrm{id}_!$ is a localization at the class of local weak equivalences in the sense of Definition 7.30. Indeed, it sends every local weak equivalence to an isomorphism in $\mathrm{Ho}(\mathcal{E}_{\mathrm{loc}})$. It is universal with this property as a consequence of the fact that the derived unit is a local weak equivalence, as just discussed. We will refer to $\mathcal{E}_{\mathrm{loc}}$ as a *left Bousfield localization* of $\mathcal{E}$. We will come back to this concept in more detail in Chapter 11; for now we record some first properties.

Although $\mathcal{E}_{\mathrm{loc}}$ has more weak equivalences than $\mathcal{E}$, it is important to realize that the weak equivalences between fibrant objects have not changed. For easy reference, we formulate this explicitly:

Lemma 8.49 *For a left Bousfield localization $\mathcal{E}_{\mathrm{loc}}$ of $\mathcal{E}$ as above, every local weak equivalence between locally fibrant objects is already a weak equivalence.*

Proof By Brown's lemma, in the form of Proposition 7.38, it suffices to show that every trivial fibration between fibrant objects in $\mathcal{E}_{\mathrm{loc}}$ is also a trivial fibration between fibrant objects in $\mathcal{E}$. But $\mathcal{E}$ and $\mathcal{E}_{\mathrm{loc}}$ have the same cofibrations, hence the same trivial fibrations. □

In fact, we can deduce that also the fibrations between fibrant objects do not change:

Lemma 8.50 *A map $f\colon X \to Y$ between locally fibrant objects is a fibration in $\mathcal{E}$ if and only if it is a local fibration, i.e., a fibration in $\mathcal{E}_{\mathrm{loc}}$.*

Proof Clearly every local fibration is in particular a fibration (since $\mathcal{E}_{\mathrm{loc}}$ has more trivial cofibrations than $\mathcal{E}$). Conversely, assume that f is a fibration between locally fibrant objects. Consider a lifting problem

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "v"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

in which i is a cofibration and a local weak equivalence. Factor u as a local trivial cofibration $j\colon A \to A'$ followed by a local fibration $p\colon A' \to X$. Then A' is a locally fibrant object. Push out j along i to obtain a larger diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & A' \arrow[r, "p"] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] & P \arrow[r] & Y. \end{tikzcd}$$

Now factor $P \to Y$ as a local trivial cofibration $k\colon P \to B'$ followed by a local fibration $q\colon B' \to Y$. Then B' is also locally fibrant and composing the maps $A' \to P \to B'$ gives a further diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & A' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] & B' \arrow[r, "q"'] & Y. \end{tikzcd}$$

In this diagram, the middle vertical map is a local trivial cofibration between locally fibrant objects. But then it is also a trivial cofibration in $\mathcal{E}$, by Lemma 8.49 above. Hence a lift in the square on the right exists. Precomposing this lift with the map $B \to B'$ gives a lift in the original square. □

Recall from Example 8.47(i) above that any morphism $f\colon X \to Y$ in a model category $\mathcal{E}$ induces a Quillen pair $f_! : \mathcal{E}/X \rightleftarrows \mathcal{E}/Y : f^*$ and that this Quillen pair is a Quillen equivalence whenever f is a weak equivalence between fibrant objects or a trivial fibration between arbitrary objects. Since not every fibrant object is locally fibrant, the following assertion is slightly stronger. It is again an easy application of Brown's lemma:

Proposition 8.51 *Let $\mathcal{E}_{\mathrm{loc}}$ be a left Bousfield localization of $\mathcal{E}$. Then any weak equivalence $f\colon X \to Y$ between fibrant objects in $\mathcal{E}$ induces a Quillen equivalence*

$$f_! : \mathcal{E}_{\mathrm{loc}}/X \rightleftarrows \mathcal{E}_{\mathrm{loc}}/Y : f^*.$$

Proof Exactly as in Example 8.47(i) above, we can use Brown's lemma to reduce this to the case where f is a trivial fibration in $\mathcal{E}$. But then f is also a trivial fibration in $\mathcal{E}_{\mathrm{loc}}$ and we are back in the situation of Example 8.47(i). □

8.4 Homotopy Groups of Simplicial Sets

Throughout this section we will work with the Kan–Quillen model structure on simplicial sets, so that 'weak equivalence' and 'fibration' will always mean weak homotopy equivalence and Kan fibration respectively. The aim of this section is to prove Theorem 8.58, which states that a map of simplicial sets is a weak homotopy equivalence if and only if it induces an isomorphism on homotopy groups (at all possible basepoints).

A *pointed* simplicial set is a pair (X, x_0) where X is a simplicial set and $x_0 \in X_0$ a vertex in X ('the basepoint'). A morphism between pointed simplicial sets f : $(X, x_0) \to (Y, y_0)$ is a map $f\colon X \to Y$ between simplicial sets with $f(x_0) = y_0$. We shall often leave the basepoint implicit and delete it from the notation when it is clear from the context. We sometimes refer to morphisms between pointed simplicial sets as *pointed maps*. The category of pointed simplicial sets will be denoted $\mathbf{sSets}_*$.

This category is a slice category, namely $\mathbf{sSets}_* = \Delta[0]/\mathbf{sSets}$, so it carries a model structure induced by the Kan–Quillen model structure on **sSets**, with the same weak equivalences, fibrations, and cofibrations, cf. Example (d)(ii) of Section 7.1.

In this section we will write

$$S^n = \Delta[n]/\partial\Delta[n]$$

for the *simplicial n-sphere*. Its geometric realization is homeomorphic to the usual topological n-sphere. This simplicial set S^n is naturally pointed, its basepoint being the image of the boundary $\partial\Delta[n]$ which has been collapsed to a single vertex. For any pointed simplicial set (X, x_0) we then define

$$\pi_n(X, x_0) = \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sSets}_*)}(S^n, (X, x_0))$$

to be the set of morphisms in the homotopy category of $\mathbf{sSets}_*$. Thus, this set does not change if we replace S^n and (X, x_0) by weakly equivalent objects. Moreover, this set is the set of homotopy classes in $\mathbf{sSets}_*$ (the *pointed* homotopy classes of maps) whenever X is a Kan complex. Explicitly, for a pointed Kan complex (X, x_0) one has

$$\begin{aligned} \pi_n(X, x_0) &= [S^n, (X, x_0)] \\ &= [(\Delta[n], \partial\Delta[n]), (X, x_0)] \end{aligned}$$

where the first set of brackets denotes homotopy classes of pointed maps and the second those of maps of pairs. A pair of simplicial sets (M, A) consists of a simplicial set M and a simplicial subset $A \subseteq M$. A map of pairs $f : (M, A) \to (N, B)$ is a morphism $f : M \to N$ for which $f(A) \subseteq B$. We call such a morphism a weak equivalence if both $f : M \to N$ and its restriction $f : A \to B$ are weak equivalence in the Kan–Quillen model structure. For greater flexibility, it is important to observe that in the description above, we can replace the pair $(\Delta[n], \partial\Delta[n])$ by any weakly equivalent pair, as expressed in the following lemma.

Lemma 8.52 *Let $f : (M, A) \to (N, B)$ be a weak equivalence of pairs. Then for any pointed Kan complex (X, x_0), the map*

$$f^* : [(N, B), (X, x_0)] \to [(M, A), (X, x_0)]$$

defined by precomposition with f is a bijection.

Proof The easiest way to understand this lemma is to interpret f as a weak equivalence between cofibrant objects in the projective model structure on the arrow category $\mathrm{Ar}(\mathbf{sSets})$, see Example 7.47 below. The homotopy classes of maps above are then simply sets of homomorphisms in the corresponding homotopy category. $\square$

As a consequence of the lemma, there is also a 'cubical' description of the set $\pi_n(X, x_0)$, which we state explicitly as follows.

Proposition 8.53 *For a pointed Kan complex (X, x_0), there is a natural bijection*

$$\pi_n(X, x_0) \cong [(\Delta[1]^n, \partial(\Delta[1]^n)), (X, x_0)],$$

where

$$\partial(\Delta[1]^n) = \bigcup_{i=1}^{n} \Delta[1]^{i-1} \times \partial\Delta[1] \times \Delta[1]^{n-i}$$

is the boundary of the cube.

For $n = 0, 1$, the two descriptions of $\pi_n(X, x_0)$ coincide, of course. For $n > 1$, the proposition is a consequence of Lemma 8.52 together with the following observation.

Lemma 8.54 *The pairs $(\Delta[n], \partial\Delta[n])$ and $(\Delta[1]^n, \partial(\Delta[1]^n))$ are weakly equivalent. More precisely, there exists a zig-zag of weak equivalences*

$$(\Delta[n], \partial\Delta[n]) \xrightarrow{\sim} \cdot \xleftarrow{\sim} (\Delta[1]^n, \partial(\Delta[1]^n)).$$

Remark 8.55 The geometric realizations of these two pairs are obviously homeomorphic, so the lemma would be evident if we knew that geometric realization detects weak equivalences. However, we are still on our way to establishing this relation between the homotopy theories of simplicial sets and topological spaces, so that we are forced to give the rather elaborate combinatorial proof below. The reader is encouraged to work out some low-dimensional cases for himself using some pictures.

Proof It will be convenient to use the notation

$$(M, A) \wedge (N, B) := (M \times N, M \times B \cup A \times N).$$

Then

$$\begin{aligned}(\Delta[1]^n, \partial(\Delta[1]^n)) &= (\Delta[1], \partial\Delta[1]) \wedge (\Delta[1]^{n-1}, \partial(\Delta[1]^{n-1})) \\ &= (\Delta[1], \partial\Delta[1]) \wedge \cdots \wedge (\Delta[1]^n, \partial(\Delta[1]^n)),\end{aligned}$$

with n factors occurring on the right. We now establish a zig-zag as in the statement of the lemma by induction on n. For $n = 0, 1$ there is nothing to prove. So let us prove for $n > 1$ that there is a zig-zag of weak equivalences

$$(\Delta[n], \partial\Delta[n]) \xrightarrow{\sim} \cdot \xleftarrow{\sim} (\Delta[1], \partial\Delta[1]) \wedge (\Delta[n-1], \partial\Delta[n-1]).$$

We can write $\Delta[1] \times \Delta[n-1] = \bigcup_{i=0}^{n-1} A_i$ where $A_i \subseteq \Delta[1] \times \Delta[n-1]$ is (the image of) the ith shuffle

$$\alpha_i : \Delta[n] \to \Delta[1] \times \Delta[n-1]$$

characterized by its effect on vertices by

$$\alpha_i(j) = \begin{cases} (0, j) & \text{for } j \leq i, \\ (1, j-1) & \text{for } j > i. \end{cases}$$

Also write

$$B := \partial(\Delta[1] \times \Delta[n-1]) = (\partial\Delta[1] \times \Delta[n-1]) \cup (\Delta[1] \times \partial\Delta[n-1]).$$

For the copy $A_i \subseteq \Delta[1] \times \Delta[n-1]$ of the n-simplex, we will use some notation as for the standard n-simplex and write

$$\partial_j A_i \quad j = 0, \ldots, n$$

for its jth face and similarly

$$\Lambda^k A_i = \bigcup_{j \neq k} \partial_j A_i \quad \text{and} \quad \partial A_i = \bigcup_j \partial_j A_i.$$

We will also have occasion to use the notation $\Lambda^E A_i$ for the union of all the faces $\partial_j A_i$ except those with $j \in E$. Notice that

$$\begin{aligned} B \cap A_0 &= \Lambda^1 A_0, \\ B \cap A_i &= \Lambda^{i,i+1} A_i, \quad 0 < i < n-1, \\ B \cap A_{n-1} &= \Lambda^{n-1} A_{n-1}, \end{aligned}$$

and

$$\partial_{i+1} A_i = \partial_{i+1} A_{i+1},$$

while for $i < j$,

$$\begin{aligned} A_i \cap A_j &= \partial_{i+1} A_i \cap \cdots \cap \partial_j A_i \\ &= \partial_{i+1} A_j \cap \cdots \cap \partial_j A_j. \end{aligned}$$

Now observe that $\partial A_{n-1} \subseteq B \cup A_{n-2}$, so there are maps

$$\begin{tikzcd} (\Delta[n], \partial\Delta[n]) \arrow[r, "\alpha_{n-1}"] & (\Delta[1] \times \Delta[n-1], B \cup A_{n-2} \cup A_{n-3} \cup \cdots A_0) \\ & (\Delta[1] \times \Delta[n-1], B). \arrow[u, "\mathrm{incl}"] \end{tikzcd}$$

The map $\alpha_{n-1} : \Delta[n] \to \Delta[1] \times \Delta[n-1]$ is obviously a weak homotopy equivalence (since both domain and codomain are $\Delta[1]$-homotopy equivalent to $\Delta[0]$), so it now suffices to prove that the inclusions

$$\partial A_{n-1} \xrightarrow{u} B \cup A_{n-2} \cup \cdots \cup A_0 \xleftarrow{v} B$$

are both weak homotopy equivalences.

The case of u. It follows from the identities above that

$$B \cup A_{n-2} \cup \cdots \cup A_0 = \partial A_{n-1} \cup A_{n-2} \cup \cdots \cup A_0.$$

We show by downward induction on i (starting at $i = n-2$) that each inclusion

$$\partial A_{n-1} \to \partial A_{n-1} \cup A_{n-2} \cup \cdots \cup A_i$$

is a trivial cofibration. The conclusion then follows by setting $i = 0$. For the first step $i = n-2$, consider the pushout

$$\begin{tikzcd}
\partial A_{n-1} \cap A_{n-2} \arrow[r] \arrow[d] & A_{n-2} \arrow[d] \\
\partial A_{n-1} \arrow[r] & \partial A_{n-1} \cup A_{n-2}
\end{tikzcd}$$

and observe that $\partial A_{n-1} \cap A_{n-2} = \partial_{n-1} A_{n-1} \cong \Delta[n-2]$ is contractible. Thus, the top horizontal map is a trivial cofibration (the codomain is contractible as well) and hence so is the bottom one. For the induction step, consider the pushout

$$\begin{tikzcd}
(\partial A_{n-1} \cup A_{n-2} \cup \cdots \cup A_{i+1}) \cap A_i \arrow[r] \arrow[d] & A_i \arrow[d] \\
\partial A_{n-1} \cup A_{n-2} \cup \cdots \cup A_{i+1} \arrow[r] & \partial A_{n-1} \cup A_{n-2} \cup \cdots \cup A_i
\end{tikzcd}$$

and observe that

$$(\partial A_{n-1} \cup A_{n-2} \cup \cdots \cup A_{i+1}) \cap A_i = \partial_{i+1} A_i = \partial_{i+1} A_{i+1}$$

is again contractible, showing that both horizontal maps in the pushout are trivial cofibrations, as before.

The case of v. We show that each of the maps

$$B \to B \cup A_0 \to B \cup A_0 \cup A_1 \to \cdots \to B \cup A_0 \cup \cdots \cup A_{n-2}$$

is a weak equivalence. The first is a pushout of $B \cap A_0 \to A_0$ and $B \cap A_0 = \Lambda^1 A_0$ so this is an anodyne map, hence a trivial cofibration. For $0 < i < n-1$, the map

$$B \cup A_0 \cup \cdots \cup A_{i-1} \to B \cup A_0 \cup \cdots \cup A_i$$

is a pushout of

$$(B \cup A_0 \cup \cdots \cup A_{i-1}) \cap A_i \to A_i.$$

The object on the left can be rewritten as $\Lambda^{i+1} A_i$, which is weakly contractible. So the latter map is anodyne again, and hence so is its pushout. This completes the proof. (Notice that this argument fails for $i = n-1$, as it should.) □

Let us return to the cubical description of the sets $\pi_n(X, x_0)$ for a pointed Kan complex (X, x_0) as in Proposition 8.53. For $n = 0$, the set $\pi_0(X, x_0)$ is the set of connected components of X. This is the quotient of the set X_0 of vertices of X by the equivalence relation $x \sim y$ if and only if there is a 1-simplex $\alpha \in X_1$ with $\alpha_0 = d_1\alpha = x$ and $\alpha_1 = d_0\alpha = y$. For $n = 1$, the set $\pi_1(X, x_0)$ is the set of morphisms from x_0 to itself in the category τX. This category is a groupoid if X is a Kan complex and hence $\pi_1(X, x_0)$ is a group. For $n > 1$, we can form the *loop space* $\Omega(X, x_0)$ as the pullback

$$\begin{tikzcd}
\Omega(X, x_0) \arrow[r] \arrow[d] & X^{\Delta[1]} \arrow[d] \\
\Delta[0] \arrow[r] & X \times X = X^{\partial\Delta[1]}.
\end{tikzcd}$$

Here the bottom map is given by the pair (x_0, x_0). The restriction map on the right is a Kan fibration by Corollary 5.39, so $\Omega(X, x_0)$ is a Kan complex. Now the isomorphism

$$\pi_n(X, x_0) = \pi_{n-1}\Omega(X, x_0)$$

is immediate from the cubical description of these homotopy groups. For $n > 1$, the group $\pi_n(X, x_0)$ is abelian, as one shows exactly as in topology. Moreover, with the cubical description of Proposition 8.53 one can mimic one proof of the long exact sequence of a Serre fibration between topological spaces and obtain the following.

Proposition 8.56 *Let $p : E \to B$ be a Kan fibration between pointed Kan complexes, with fibre F over the basepoint of B. Then there is an induced long exact sequence of homotopy groups*

$$\cdots \to \pi_n F \to \pi_n E \to \pi_n B \xrightarrow{\delta} \pi_{n-1} F \to \cdots .$$

Proof As in Section 7.2 we only describe the morphism δ and leave the remaining details to the reader. Let $[\alpha] \in \pi_n B$ be an element represented by $\alpha : \Delta[1]^n \to B$ mapping $\partial(\Delta[1]^n)$ to the basepoint b_0 of B. Let

$$U = \Delta[1] \times \partial(\Delta[1]^{n-1}) \cup (\{0\} \times \Delta[1]^{n-1}),$$

where of course $\{0\}$ denotes the image of $\partial_1 : \Delta[0] \to \Delta[1]$. By Corollary 5.28 the map $U \to \Delta[1]^n$ is left anodyne, so in particular a trivial cofibration. Hence we can find a lift in

$$\begin{tikzcd}
U \arrow[r, "e_0"] \arrow[d] & E \arrow[d] \\
\Delta[1]^n \arrow[r, "\alpha"'] \arrow[ur, dashed, "\beta"] & B
\end{tikzcd}$$

where e_0 denotes the constant map with value the basepoint e_0 in E. Then the restriction of β to $\{1\} \times \Delta[1]^{n-1}$ defines a map $\Delta[1]^{n-1} \to E$ sending the boundary to the basepoint, which represents $\delta[\alpha]$. One has to check that the map δ is well-defined, that it is a group homomorphism (for $n \geq 2$), and that it renders the sequence exact. All this is completely analogous to the topological case of a Serre fibration, but is a good exercise in lifting properties of Kan fibrations for those readers less familiar with these. $\square$

Lemma 8.57 *Let M be a minimal Kan complex and let $x_0 \in M$ be any basepoint. If $\pi_n M \cong \{0\}$ for every $n \geq 0$, then M is isomorphic to $\Delta[0]$.*

Proof Since $\pi_0 M = \{0\}$, the simplicial set M is connected. But a connected minimal Kan complex has only one vertex, i.e., $M_0 = \{x_0\}$. Suppose that for a given $n > 0$ we have proved that M_k consists of a single element for all $k < n$. The set M_n is non-empty because it contains a degenerate simplex coming from M_{n-1}. If $x, y \in M_n$ are two n-simplices, then they are constant (with the same value) on $\partial\Delta[n]$, so represent elements of $\pi_n(M, x_0)$. The assumption that this homotopy group is trivial implies that x and y are homotopic relative to the boundary $\partial\Delta[n]$. Because M is minimal, we must have $x = y$. The lemma now follows by induction on n. $\square$

We are now ready to state the main result of this section:

Theorem 8.58 *Let $f : X \to Y$ be a map between simplicial sets. Then f is a weak homotopy equivalence if and only if it induces an isomorphism $\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for every basepoint $x_0 \in X_0$ and every $n \geq 0$.*

Remark 8.59 It suffices to check the condition of the theorem for one basepoint x_0 in every path component of X_0. Indeed, if x_0 and x_1 are distinct vertices connected by a path $\alpha : \Delta[1] \to X$, then

$$(X, x_0) \to (X, \mathrm{im}(\alpha)) \leftarrow (X, x_1)$$

are weak equivalences in Ar(**sSets**) and one finds an isomorphism $\pi_n(X, x_0) \cong \pi_n(X, x_1)$.

Proof The 'only if' part of the theorem is clear from the definitions. For the converse, first notice that we may choose a trivial cofibration $Y \to Y'$ with Y' a Kan complex, and by factoring the resulting composite $X \to Y \to Y'$ as a trivial cofibration followed by a fibration we obtain a square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, tail, "\wr"] & Y \arrow[d, tail, "\wr"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

where the vertical maps are trivial cofibrations and the bottom horizontal map f' is a Kan fibration between Kan complexes. Moreover, f' will still satisfy the hypotheses of the theorem. Thus, it suffices to prove the theorem for f'. Factor it as a trivial fibration $p : X' \to Z$ followed by a minimal fibration $q : Z \to Y'$. Then p certainly induces isomorphisms in homotopy groups and hence so does q. This reduces the problem to proving the theorem for a minimal fibration between Kan complexes.

From the assumption that $q : \pi_0(Z) \cong \pi_0(Y')$, we find that $Z \to Y'$ must be surjective on vertices. Choose any basepoint $y_0 \in Y$ and another one $z_0 \in Z$ with $q(z_0) = y_0$. Let M be the fibre over y_0. Then from the long exact sequence associated to q we find that $\pi_n(M, z_0)$ is trivial for all $n \geq 0$. By Lemma 8.57, M is a point. Now local triviality of q implies that for any simplex $\alpha : \Delta[n] \to Y$ in the same path component as y_0, one has a pullback square

$$\begin{tikzcd} \Delta[n] \times M \arrow[r] \arrow[d] & Z \arrow[d, "q"] \\ \Delta[n] \arrow[r, "\alpha"] & Y. \end{tikzcd}$$

Since $M \cong \Delta[0]$, the vertical map on the left is an isomorphism. Since y_0 and α were arbitrary, the map q itself must be an isomorphism, so in particular a weak homotopy equivalence. □

8.5 Geometric Realizations and Fibrations

In this short section we will prove the following important fact, first observed by Quillen. It will be a crucial tool in the proof of the Quillen equivalence between the category of simplicial sets (with the Kan–Quillen model structure) and the category of topological spaces, which we will give in the next section.

Theorem 8.60 *The geometric realization of a Kan fibration is a Serre fibration.*

Every Kan fibration can be decomposed as a trivial fibration followed by a minimal fibration and minimal fibrations are locally trivial. Since locally trivial fibre bundles of topological spaces are Serre fibrations, the theorem follows from the following two statements.

Proposition 8.61 *The geometric realization of a trivial fibration between simplicial sets is a trivial Serre fibration between topological spaces.*

Proposition 8.62 *The geometric realization of a minimal fibration is a locally trivial fibre bundle.*

Let us once more make explicit the two meanings of the phrase ‘locally trivial’ used above. A map $f : X \to Y$ between simplicial sets is locally trivial if for every n-simplex $\alpha : \Delta[n] \to X$ the pullback of f along α is isomorphic to a projection, as in

$$\begin{tikzcd} \Delta[n] \times F \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r, "\alpha"] & Y. \end{tikzcd}$$

On the other hand, a map between topological spaces $p : A \to B$ is locally trivial if every point $a \in A$ admits a neighbourhood U for which the restriction of p to $p^{-1}U \to U$ is homeomorphic over U to a projection, as in the pullback

$$\begin{tikzcd} U \times p^{-1}\{a\} \arrow[r] \arrow[d] & A \arrow[d, "p"] \\ U \arrow[r, "\alpha"] & B. \end{tikzcd}$$

Proof (of Proposition 8.61) Let $f \colon X \to Y$ be a trivial fibration of simplicial sets. We claim that f is a retract of the projection $Y \times X \to Y$. Then $|f|$ is a retract of the projection $|Y| \times |X| \to |Y|$, because geometric realization preserves products. So in particular, $|f|$ will be a retract of a Serre fibration, hence itself a Serre fibration. To prove the claim, consider the square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "{(f,\mathrm{id})}"'] & X \arrow[d, "f"] \\ Y \times X \arrow[r, "\pi_1"'] \arrow[ur, dashed] & Y. \end{tikzcd}$$

The left-hand vertical map is a monomorphism (so a cofibration), hence a lift as indicated exists. This lift exhibits f as a retract of $\pi_1 \colon Y \times X \to Y$. $\square$

The proof of Proposition 8.62 is a bit delicate, since we have to carefully build up open neighbourhoods and trivializations over these. Let us introduce some terminology for a map $p : A \to B$ between topological spaces. A trivialization (with fibre F) over a subset $U \subseteq A$ is a homeomorphism over U of the form

$$\begin{tikzcd} U \times F \arrow[rr, "\alpha"] \arrow[dr, "\pi_1"'] & & p^{-1}U \arrow[dl] \\ & U. & \end{tikzcd}$$

Lemma 8.63 *Let $U \subseteq V \subseteq A$ be open subsets and let $\alpha : U \times F \to B$ and $\beta : V \times F \to B$ be two trivializations with fibre F. If the inclusion $U \to V$ admits a retraction, then α extends to a trivialization over V. In other words, β can be modified so as to agree with α on U.*

Proof Consider the diagram

$$\begin{tikzcd} U \times F \arrow[r, "\alpha"] \arrow[dr, "\pi_1"'] & A \arrow[d] & V \times F \arrow[l, "\beta"'] \arrow[dl, "\pi_1"] \\ & B & \end{tikzcd}$$

and write $\theta : U \to \mathrm{Aut}(F)$ for the map

$$\theta(a)(x) = \pi_2\beta^{-1}\alpha(a, x),$$

where $\mathrm{Aut}(F)$ is the set of homeomorphisms of F onto itself. Then θ extends to a map $\tau : V \longrightarrow \mathrm{Aut}(F)$ since $U \to V$ is assumed to be a retract. Now let $\gamma : V \times F \longrightarrow B$ be the map defined by $\gamma(a, x) = \beta(a, \tau(a)(x))$. Then γ is the required trivialization over V which extends α. □

For the next lemma, recall that the geometric realization $|X|$ of a simplicial set X is a CW-complex, whose CW-decomposition

$$|X|^{(0)} \subseteq |X|^{(1)} \subseteq |X|^{(2)} \subseteq \cdots$$

corresponds to the skeletal filtration

$$\mathrm{sk}_0 X \subseteq \mathrm{sk}_1 X \subseteq \mathrm{sk}_2 X \subseteq \cdots$$

of X, meaning

$$|\mathrm{sk}_n X| = |X|^{(n)}.$$

Each point $\xi \in |X|$ can be uniquely written as $\xi = x \otimes t$, where $x \in X_n$ is a non-degenerate n-simplex and $t \in \Delta^n$ is an interior point. Thus $\xi \in |X|^{(n)} - |X|^{(n-1)}$. For such a point ξ, the following lemma and its proof tell us how to build up specific neighbourhoods of ξ.

Lemma 8.64 *Let $\xi \in |X|$ be a point in the geometric realization of a simplicial set X and assume $\xi \in |X|^{(n)} - |X|^{(n-1)}$. Then ξ has arbitrarily small closed neighbourhoods $V \subseteq |X|$ with the following properties for $V_m := V \cap |X|^{(m)}$:*

- *$V_m = \varnothing$ for $m < n$,*
- *V_n is homeomorphic to an n-dimensional disk,*
- *V_m is a deformation retract of V_{m+1} for $m \geq n$; in particular, each V_m is contractible for $m \geq n$.*

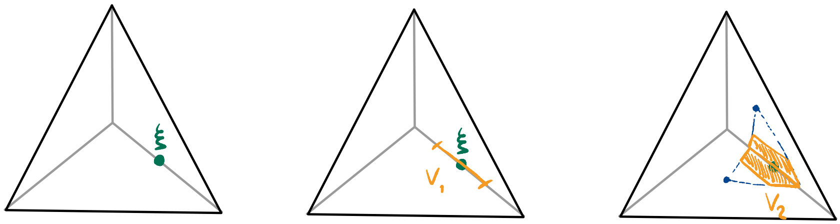
Proof To start, write $\xi = x \otimes t$ as above and let V_n be the image of a small closed ball around $t \in \mathrm{int}(\Delta^n)$ under the map $\hat{x} : \Delta^n \longrightarrow |X|^{(n)}$, which is an open embedding on the interior of Δ^n. We will extend this neighbourhood V_n to $V_{n+1} \subseteq |X|^{(n+1)}$ as follows. Consider a non-degenerate $n+1$-simplex $z \in X_{n+1}$ and the corresponding map $\hat{z} : \Delta^{n+1} \longrightarrow |X|^{(n+1)}$. Then

$$V_n(z) := \hat{z}^{-1}(V_n) \subseteq \partial\Delta^{n+1}$$

is a (possibly empty) disjoint union of copies of V_n, each lying on a face of Δ^{n+1}. Let b be the barycentre of Δ^{n+1} and write

$$V_{n+1}(z) = \{tb + (1-t)a \mid a \in V_n(z),\ 0 \leq t \leq \varepsilon\}$$

for some small $\varepsilon > 0$ (which we can keep fixed if we are not concerned with the size of V). Said more informally, $V_{n+1}(z)$ is a union of strips, one for each component of $V_n(z)$, lying in the triangle whose base is that component and whose apex is the barycentre b.



Now let V_{n+1} be the union of all the images of the $V_{n+1}(z)$ where z ranges over the non-degenerate simplices in X:

$$V_{n+1} = \bigcup_z \hat{z}(V_{n+1}(z)) \subseteq |X|^{(n+1)}.$$

Then $V_{n+1} \cap |X|^{(n)} = V_n$ and $V_n \subseteq V_{n+1}$ obviously has the properties stated in the lemma. We can now proceed in exactly the same way to construct the entire sequence $V_n \subseteq V_{n+1} \subseteq V_{n+2} \subseteq \cdots$ and let $V = \cup_n V_n$. □

Proof (of Proposition 8.62) Let $f: X \to Y$ be a locally trivial map of simplicial sets. By working on one connected component of Y (and hence of $|Y|$) at a time, we may assume that Y is connected and we can fix a single model F for the fibre. Then by the assumption of local triviality, for each non-degenerate m-simplex z in Y there is a pullback square of simplicial sets

$$\begin{tikzcd} \Delta[m] \times F \arrow[r, "\tau_z"] \arrow[d] & X \arrow[d, "f"] \\ \Delta[m] \arrow[r, "z"] & Y. \end{tikzcd}$$

We will slightly abuse notation and also write τ_z for the geometric realization of the trivialization in the diagram above:

$$\tau_z : \Delta^m \times |F| \to |X|.$$

Consider a point $\xi \in |Y|$, say $\xi = x \otimes t \in |Y|^{(n)} - |Y|^{(n-1)}$, and pick a neighbourhood V of it as in Lemma 8.64. We will construct a trivialization of $|X| \to |Y|$ over V. This V is the union of $V_n \subseteq V_{n+1} \subseteq \cdots$ and it suffices to construct a compatible sequence of trivializations

$$\alpha_m : V_m \times |F| \to |X|$$

of $|X| \to |Y|$ over each of the V_m, with $m \geq n$. First, V_n is the image of a small ball in the interior of Δ^n under $\hat{x}: \Delta^n \to |Y|^{(n)}$, which is a homeomorphism onto its image, and we can take α_n to be the restriction of τ_x to this small ball. Next, suppose that

α_m has been constructed, defining a homeomorphism

$$V_m \times |F| \to |f|^{-1}(V_m) \subseteq X$$

over V_m. Then for each non-degenerate $m+1$-simplex $z \in Y$, α_m restricts and pulls back to a trivialization over $V_m(z) \subseteq \partial\Delta^{m+1}$. We can extend this trivialization over $V_m(z)$ to one over $V_{m+1}(z)$ by applying Lemma 8.64 (for the trivialization over $V_m(z)$ induced by α_m and the one over $V_{m+1}(z)$ given by τ_z). Since the images of the $V_{m+1}(z)$ for varying z only intersect in V_m, these together give a well-defined trivialization over V_{m+1}. This completes the induction. □

8.6 The Equivalence Between Simplicial Sets and Topological Spaces

In modern homotopy theory, the word 'space' is often used as a synonym for 'simplicial set'. The reason for this lies in the following theorem, which this short section aims to prove.

Theorem 8.65 *The geometric realization and singular complex functors*

$$|\cdot| : \mathbf{sSets} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$$

form a Quillen equivalence between the category of simplicial sets with the Kan–Quillen model structure and the category of topological spaces with the model structure of Section 7.3.

Thus, the theorem asserts in particular that the stated Quillen adjunction induces an equivalence of homotopy categories

$$\mathrm{Ho}(\mathbf{sSets}_{\mathrm{KQ}}) \rightleftarrows \mathrm{Ho}(\mathbf{Top}).$$

Since every simplicial set is cofibrant and every topological space is fibrant, the functors involved are 'already derived', so another way to state this equivalence is to assert that

(1) for each Kan complex K, the unit $\eta : K \to \mathrm{Sing}(|K|)$ is weak homotopy equivalence of simplicial sets; and
(2) for each topological space X, the counit $|\mathrm{Sing}(X)| \to X$ is a weak homotopy equivalence of topological spaces.

Now clearly, for a topological space X and a basepoint $x_0 \in X$, the homotopy groups of X coincide with the simplicial homotopy groups of $\mathrm{Sing}(X)$,

$$\pi_n(X, x_0) \xrightarrow{\cong} \pi_n(\mathrm{Sing}(X), x_0).$$

Thus, it follows from Theorem 8.58 that a map $X \to Y$ of topological spaces is a weak homotopy equivalence if and only if $\mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ is one. One says that the functor Sing *detects* weak equivalences. As a consequence, the triangle identities

$$\begin{tikzcd} {|K|} \arrow[r, "|\eta_K|"] \arrow[dr, equal] & \mathrm{Sing}(|K|) \arrow[d, "\varepsilon_{|K|}"] \\ & {|K|} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sing}(X) \arrow[r, "\eta_{\mathrm{Sing}(X)}"] \arrow[dr, equal] & \mathrm{Sing}|\mathrm{Sing}(X)| \arrow[d, "\mathrm{Sing}(\varepsilon_X)"] \\ & \mathrm{Sing}(X) \end{tikzcd}$$

imply that it now suffices to prove that the unit $\eta : K \to \mathrm{Sing}(|K|)$ is a weak homotopy equivalence for each Kan complex K. Indeed, then the second triangle shows that for each topological space X the map $\mathrm{Sing}(\varepsilon_X)$ is a weak equivalence as well, so that the same is true of ε_X itself. The following lemma shows that η indeed induces isomorphisms in homotopy groups for any vertex x_0 of K:

$$\pi_n(K, x_0) \xrightarrow{\pi_n \eta} \pi_n(|K|, x_0) \cong \pi_n(\mathrm{Sing}(|K|), x_0).$$

Hence it completes the proof of Theorem 8.65.

Lemma 8.66 *For any Kan complex K, any vertex $x_0 \in K_0$, and any $n \geq 0$, the natural map $\pi_n(K, x_0) \to \pi_n(|K|, x_0)$ from simplicial to topological homotopy groups (or pointed sets if $n = 0$) is an isomorphism.*

Proof First note that $|K|$ is connected whenever K is and that $|\cdot|$ preserves coproducts, so the assertion is clear for $n = 0$. Proceeding by induction, suppose we have proved the lemma for $m < n$ and all K. Consider the Kan fibration

$$\mathrm{ev} = (\mathrm{ev}_0, \mathrm{ev}_1) : K^{\Delta[1]} \to K^{\partial\Delta[1]} = K \times K$$

and write $PK = P(K, x_0)$ for the pullback

$$\begin{tikzcd} PK \arrow[r] \arrow[d] & K^{\Delta[1]} \arrow[d, "\mathrm{ev}"] \\ K = \Delta[0] \times K \arrow[r, "x_0 \times \mathrm{id}"'] & K \times K. \end{tikzcd}$$

Then $PK \to K$ is a Kan fibration with fibre $\Omega(K, x_0)$. Moreover, PK is itself a contractible Kan complex because it is also the pullback of a trivial fibration

$$\begin{tikzcd} PK \arrow[r] \arrow[d] & K^{\Delta[1]} \arrow[d, "\mathrm{ev}_0"] \\ \Delta[0] \arrow[r] & K. \end{tikzcd}$$

By Theorem 8.60, $|PK| \to |K|$ is a Serre fibration with fibre $|\Omega(K, x_0)|$. Since PK is a contractible Kan complex, $|PK|$ is a contractible space, and the long exact sequences of these two fibrations give isomorphisms which we can compare by the natural map in the lemma, as in

$$\begin{tikzcd}
\pi_n(K, x_0) \arrow[r, "\cong"] \arrow[d] & \pi_{n-1}(\Omega(K, x_0)) \arrow[d] \\
\pi_n(|K|, x_0) \arrow[r, "\cong"] & \pi_{n-1}(|\Omega(K, x_0)|).
\end{tikzcd}$$

Here we used that the map of the lemma is also compatible with the boundary δ in the long exact sequence. Since the map on the right is an isomorphism by the induction hypothesis, so is the map on the left. □

We conclude this section with the following consequence:

Proposition 8.67 *The Kan–Quillen model structure on the category of simplicial sets is right proper, i.e., the pullback of any weak homotopy equivalence along a fibration is again a weak homotopy equivalence.*

Proof The left adjoint of a Quillen equivalence detects weak equivalences between cofibrant objects; since *every* object is cofibrant in the Kan–Quillen model structure, the geometric realization functor detects arbitrary weak equivalences. By Proposition 2.6 it preserves pullbacks and by Theorem 8.60 it sends Kan fibrations to Serre fibrations. The conclusion of the proposition now follows from the fact that the model category **Top** is right proper, cf. Remark 7.14 or Corollary 7.41. □

8.7 Categorical Weak Equivalences Between ∞-Categories

The Kan–Quillen model structure on the category of simplicial sets is in many ways easier to work with than the Joyal model structure. For example, one has an explicit description of the fibrations and consequently an explicit set of generating trivial cofibrations. Moreover, the Quillen equivalence to topological spaces helps to transfer results (and intuition!) from the homotopy theory of topological spaces to the Kan–Quillen model structure.

In this section we will return to the Joyal model structure and describe its weak equivalences between fibrant objects in various ways. These descriptions and their proofs will make use of Kan complexes and of our knowledge of weak homotopy equivalences. The main result will be the characterization (Theorem 8.74 below) of the categorical equivalences between ∞-categories as those maps which are essentially surjective and fully faithful, just like for ordinary categories. We begin with the following easy consequences of previous results. For a small category **C**, we write $\pi_0\mathbf{C}$ for the set of isomorphism classes of objects of **C**. Alternatively, it is the set of connected components of the maximal groupoid contained in **C**.

Lemma 8.68 *For a simplicial set A and an ∞-category X, there is a natural bijection*

$$[A, X] \cong \pi_0\tau(X^A).$$

Here the left-hand side denotes the set of morphisms $A \to X$ in the homotopy category $\mathrm{Ho}(\mathbf{sSets}_{\mathrm{cat}})$. *In different words, it is the set of homotopy classes of maps $A \to X$ with respect to the categorical model structure.*

Proof Observe that the maps $\partial J \times A \to J \times A \to A$ exhibit $J \times A$ as a cylinder object for A. Thus, two maps $f, g \colon A \to X$ are homotopic with respect to the categorical model structure if and only if they are J-homotopic. By Theorem 5.55, this is the case if and only if f and g are equivalent objects of the ∞-category X^A. By definition, this means that f and g are isomorphic objects of the category $\tau(X^A)$. □

Proposition 8.69 *For a map $X \to Y$ between ∞-categories, the following properties are equivalent:*

(1) *The map $X \to Y$ is a categorical equivalence.*
(2) *The map $X \to Y$ is a J-homotopy equivalence.*
(3) *For each simplicial set A, the map $X^A \to Y^A$ is a categorical equivalence.*
(4) *For each simplicial set A, the map $k(X^A) \to k(Y^A)$ is weak homotopy equivalence between Kan complexes.*
(5) *For each simplicial set A, the map $\tau(X^A) \to \tau(Y^A)$ is an equivalence of categories.*

Proof In any model structure, the weak equivalences between fibrant-cofibrant objects are the same as the homotopy equivalences, by Proposition 7.27. As we observed in the proof of Lemma 8.68 above, we may identify such homotopy equivalences with J-homotopy equivalences, showing that (1) and (2) are equivalent. Moreover, if $X \to Y$ is a J-homotopy equivalence, the same is true for the maps of (3) and (4). Indeed, for (4) one observes that for a homotopy inverse $Y^A \to X^A$ to the map of (3), the image of the Kan complex $k(Y^A)$ must be contained in the maximal Kan complex $k(X^A)$. That (3) implies (5) follows from the fact that τ sends categorical equivalences between simplicial sets to equivalences of categories, which we already observed in Example 8.47(c). Finally, if either (4) or (5) holds, then Lemma 8.68 and the Yoneda lemma imply that $X \to Y$ represents an isomorphism in the homotopy category $\mathrm{Ho}(\mathbf{sSets}_{\mathrm{cat}})$. Indeed, we may identify homotopy classes of maps $A \to X$ with isomorphism classes of objects in the category $\tau(X^A)$, or connected components of $k(X^A)$. But then $X \to Y$ is a categorical equivalence (cf. Remark 7.32). □

Lemma 8.70 *Suppose that for every $n \geq 0$, condition (4) of Proposition 8.69 holds if A is the simplex $\Delta[n]$. Then (4) holds for every simplicial set A.*

Proof We can factor a general $f \colon X \to Y$ between ∞-categories as a trivial cofibration followed by a categorical fibration. Property (4) holds for any trivial cofibration between ∞-categories, so that without loss of generality we may assume that f is a categorical fibration. Then $k(X) \to k(Y)$ is in fact a trivial fibration, by applying condition (4) with $A = \Delta[0]$. By a skeletal induction we will prove that (4) holds for any simplicial set A of dimension n (meaning $A = \mathrm{sk}_n A$), for any $n \geq 0$. If A is of dimension 0 (i.e. discrete), observe that $X^A \to Y^A$ is just a product of copies of f indexed by the vertices of A. In particular, it gives a trivial fibration $k(X^A) \to k(Y^A)$. For $n > 0$, we have a pushout

$$\begin{tikzcd}
\coprod \partial\Delta[n] \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}A \arrow[d] \\
\coprod \Delta[n] \arrow[r] & A
\end{tikzcd}$$

with the coproduct ranging over the non-degenerate n-simplices of A. Write $\partial N \to N$ for the vertical map on the left and consider the cube

$$\begin{tikzcd}
k(X^A) \arrow[rr] \arrow[dr] \arrow[dd] & & k(X^N) \arrow[dr] \arrow[dd] & \\
& k(Y^A) \arrow[rr] \arrow[dd] & & k(Y^N) \arrow[dd] \\
k(X^{\mathrm{sk}_{n-1}A}) \arrow[rr] \arrow[dr] & & k(X^{\partial N}) \arrow[dr] & \\
& k(Y^{\mathrm{sk}_{n-1}A}) \arrow[rr] & & k(Y^{\partial N})
\end{tikzcd}$$

in which the front and back faces are pullbacks. The three maps from back to front corresponding to $\mathrm{sk}_{n-1}A$, ∂N, and N are weak homotopy equivalences; the first two by the inductive hypothesis, the third one by assumption. Since the vertical maps are fibrations, it follows by (the duals of) Lemma 7.51 and Corollary 7.50 that the fourth map corresponding to A itself is also a weak homotopy equivalence, completing the induction.

For a general simplicial set A, writing $A = \cup_n \mathrm{sk}_n A$ shows that $k(X^A) \to k(Y^A)$ is the inverse limit of the trivial fibrations $k(X^{\mathrm{sk}_n A}) \to k(Y^{\mathrm{sk}_n A})$, hence itself a trivial fibration. □

Corollary 8.71 *Let $f: X \to Y$ be a map between ∞-categories. Then f is a categorical weak equivalence if and only if $k(X^{\Delta[1]}) \to k(Y^{\Delta[1]})$ and $k(X) \to k(Y)$ are both weak homotopy equivalences of Kan complexes.*

Proof Since the inclusion $\mathrm{Sp}[n] \to \Delta[n]$ of the spine of the n-simplex is inner anodyne, the map $X^{\Delta[n]} \to X^{\mathrm{Sp}[n]}$ is a trivial fibration, and similarly for Y. Now

$$k(X^{\mathrm{Sp}[n]}) = k(X^{\Delta[1]}) \times_{k(X)} k(X^{\Delta[1]}) \times_{k(X)} \cdots \times_{k(X)} k(X^{\Delta[1]})$$

and similarly for Y. Moreover, since the two evaluation maps $k(X^{\Delta[1]}) \to k(X)$ are Kan fibrations, these pullbacks are homotopy pullbacks (cf. Lemma 7.51). So $k(X^{\Delta[n]}) \to k(Y^{\Delta[n]})$ is a weak homotopy equivalence if and only if $k(X^{\mathrm{Sp}[n]}) \to$

$k(Y^{\mathrm{Sp}[n]})$ is a weak homotopy equivalence, which by the above description is the case if and only if $k(X^{\Delta[1]}) \to k(Y^{\Delta[1]})$ and $k(X) \to k(Y)$ are weak homotopy equivalences. □

Definition 8.72 (a) For an ∞-category X and two vertices x and y of X, the *space of morphisms* $X(x, y)$ from x to y is the pullback

$$\begin{tikzcd} X(x,y) \arrow[r] \arrow[d] & X^{\Delta[1]} \arrow[d] \\ \Delta[0] \arrow[r, "{(x,y)}"] & X^{\partial\Delta[1]} = X \times X. \end{tikzcd}$$

(b) A map $f\colon X \to Y$ between ∞-categories is *fully faithful* if for any two vertices x and y of X, the induced map $X(x, y) \to Y(f(x), f(y))$ is a weak homotopy equivalence.

(c) A map $f\colon X \to Y$ between ∞-categories is *essentially surjective* if $\tau f\colon \tau X \to \tau Y$ is an essentially surjective functor.

Remark 8.73 Notice that since $\partial\Delta[1] \to \Delta[1]$ is bijective on vertices, the right-hand square in

$$\begin{tikzcd} X(x,y) \arrow[r] \arrow[d] & k(X^{\Delta[1]}) \arrow[r] \arrow[d] & X^{\Delta[1]} \arrow[d] \\ \Delta[0] \arrow[r] & k(X^{\partial\Delta[1]}) \arrow[r] & X^{\partial\Delta[1]} \end{tikzcd}$$

is a pullback (since the equivalences in $X^{\Delta[1]}$ are precisely the pointwise equivalences, see Theorem 5.55). It follows that the left-hand square is a pullback as well. Since the middle vertical map is Kan fibration, each $X(x, y)$ is a Kan complex. Also notice that it easily follows from Corollary 5.57 that

$$(\tau X)(x, y) \cong \pi_0 X(x, y).$$

Theorem 8.74 *A map $f\colon X \to Y$ between ∞-categories is a categorical weak equivalence if and only if it is fully faithful and essentially surjective.*

Proof The implication from left to right is clear from Proposition 8.69. For the other implication, suppose f is essentially surjective and fully faithful. We will verify the conditions of Corollary 8.71. First we will prove that $k(X) \to k(Y)$ induces isomorphisms in homotopy groups $\pi_n k(X) \cong \pi_n k(Y)$ for all $n \geq 0$ and all choices of basepoint. Since $(\tau X)(x, y) = \pi_0 X(x, y)$, and similarly for Y, we find first of all that $\tau X \to \tau Y$ is an equivalence of categories. But the maximal subgroupoid of τX is precisely $\tau k(X)$, and similarly for Y, so that $\pi_0 k(X) \to \pi_0 k(Y)$ must be a bijection. Indeed, the set of components $\pi_0 k(X)$ is the same as the set of components of the groupoid $\tau k(X)$, and similarly for Y. To show that $\pi_n(k(X), x) \to \pi_n(k(Y), f(x))$ is also an isomorphism for all $n \geq 1$ and $x \in X_0$, observe that $\Omega(k(X), f(x)) \subseteq X^{\Delta[1]}$ consists precisely of the loops at x which are equivalences in X. In other words, the front and back faces of the cube

$$\begin{tikzcd}
\Omega(k(X),x) \arrow[rr, tail] \arrow[dr] \arrow[dd] & & \Omega(X,x) \arrow[dr] \arrow[dd] & \\
& \Omega(k(Y),f(x)) \arrow[rr, tail] \arrow[dd] & & \Omega(Y,f(x)) \arrow[dd] \\
(\tau k(X))(x,x) \arrow[rr, tail] \arrow[dr, "\cong"] & & (\tau X)(x,x) \arrow[dr, "\cong"] & \\
& (\tau k(Y))(f(x),f(x)) \arrow[rr, tail] & & (\tau Y)(f(x),f(x))
\end{tikzcd}$$

are pullbacks. Notice that the bottom face is a square of sets, so the horizontal inclusions to the right in the top face are inclusions of a set of connected components. If $X \to Y$ is fully faithful, then $\Omega(X, x) \to \Omega(Y, f(x))$ is a weak homotopy equivalence and hence so is $\Omega(k(X), x) \to \Omega(k(Y), f(x))$. This proves that $k(X) \to k(Y)$ is indeed a weak homotopy equivalence.

It remains to be proved that $k(X^{\Delta[1]}) \to k(Y^{\Delta[1]})$ is a weak homotopy equivalence as well. The vertical arrows in the square

$$\begin{tikzcd}
k(X^{\Delta[1]}) \arrow[r] \arrow[d] & k(Y^{\Delta[1]}) \arrow[d] \\
k(X) \times k(X) \arrow[r] & k(Y) \times k(Y)
\end{tikzcd}$$

are fibrations. The bottom horizontal map is a weak homotopy equivalence, as we just saw. For any point $(x, y) \in k(X) \times k(X)$, the corresponding map of fibres of the vertical maps (over (x, y) and $(f(x), f(y))$) is the map

$$X(x, y) \to Y(f(x), f(y))$$

which is a weak homotopy equivalence by assumption. It follows (by the long exact sequence of a fibration) that the top horizontal map in the square is a weak homotopy equivalence, as desired. □

8.8 The Covariant Model Structure

Let V be a simplicial set. Then the Joyal and Kan–Quillen model structures induce corresponding model structures on the slice category **sSets**$/V$, by defining a map

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\
& V &
\end{tikzcd}$$

of simplicial sets over V to be a fibration, cofibration, or weak equivalence precisely if $X \to Y$ itself is one of these in the corresponding model structure on the category **sSets**. With respect to the Joyal model structure, this leads to a model structure on $\mathbf{sSets}/V$ in which the fibrant objects are the categorical fibrations $X \to V$. Working with respect to the Kan–Quillen model structure gives a model structure with fibrant objects the Kan fibrations $X \to V$. The purpose of this section is to discuss a third model structure on $\mathbf{sSets}/V$ of which the fibrant objects are the left fibrations $X \to V$. This model structure generally does *not* arise from slicing a model structure on the category **sSets** over V. The relevance of the covariant model structure will become clear later, when we exhibit a Quillen equivalence to the homotopy theory of 'diagrams of spaces on V' (see Section 14.8).

To avoid too much duplication of arguments, we postpone most proofs until Section 9.5, where we treat the corresponding model structure for dendroidal sets. All results of that section imply their simplicial counterparts after observing that for a simplicial set V, the slice category $\mathbf{dSets}/i_!V$ can be identified with the category $\mathbf{sSets}/V$. For two objects $f: A \to V$ and $g: X \to V$ of the latter category, we define a mapping object

$$\mathbf{hom}_V(A, X)$$

as follows: it is the simplicial set whose n-simplices are maps $\Delta[n] \times A \to X$ for which the diagram

$$\begin{tikzcd} \Delta[n] \times A \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ A \arrow[r, "f"] & V \end{tikzcd}$$

commutes. In other words, $\mathbf{hom}_V(A, X)$ is the pullback

$$\begin{tikzcd} \mathbf{hom}_V(A, X) \arrow[r] \arrow[d] & X^A \arrow[d, "-\circ g"] \\ \Delta[0] \arrow[r, "f"] & V^A. \end{tikzcd}$$

If $X \to V$ is a left fibration, then the simplicial set $\mathbf{hom}_V(A, X)$ is in fact a Kan complex (cf. Remark 9.60). We will refer to the model structure of the following theorem as the *covariant model structure over* V. As explained above, it is a special case of the corresponding Theorem 9.59.

Theorem 8.75 *Let V be a simplicial set. The category* $\mathbf{sSets}/V$ *carries a left proper cofibrantly generated model structure with the following properties:*

(a) *The cofibrations are the monomorphisms over V.*
(b) *The fibrant objects are the left fibrations $X \to V$.*
(c) *The fibrations between fibrant objects are the left fibrations.*
(d) *A map $A \to B$ between simplicial sets over V is a weak equivalence if and only if for any left fibration $X \to V$, the map*

$$\mathbf{hom}_V(B, X) \to \mathbf{hom}_V(A, X)$$

is a weak homotopy equivalence of Kan complexes.

The weak equivalences between fibrant objects in the covariant model structure can be conveniently characterized as the fibrewise weak homotopy equivalences. Indeed, the following is the specialization of Theorem 9.63 to the context of simplicial sets:

Theorem 8.76 *Consider a map*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & V & \end{tikzcd}$$

between left fibrations $X \to V$ and $Y \to V$ of simplicial sets. Then the following are equivalent:

(1) *The map f is a weak equivalence in the covariant model structure over V.*
(2) *The map f is a fibrewise homotopy equivalence over V.*
(3) *For any map of simplicial sets $A \to V$, the map*

$$\mathbf{hom}_V(A, X) \to \mathbf{hom}_V(A, Y)$$

is a weak homotopy equivalence of Kan complexes.
(4) *For every vertex $v \in V_0$, the map $X_v \to Y_v$ between fibres over v is a weak homotopy equivalence of Kan complexes.*

Although the covariant model structure on $\mathbf{sSets}/V$ generally need not arise as a 'sliced' model structure from one on the category $\mathbf{sSets}$, we do note the following special case:

Proposition 8.77 *If V is a Kan complex, then the covariant model structure on* $\mathbf{sSets}/V$ *coincides with the Kan–Quillen model structure. In other words, the fibrant objects of the covariant model structure are the Kan fibrations $X \to V$.*

Proof Corollary 5.50 implies that a left fibration over a Kan complex $X \to V$ is in fact a Kan fibration. □

We conclude this section by proving several useful facts about the class of left anodyne morphisms, which play an important role in the covariant model structure. We begin by observing that they are in particular trivial cofibrations with respect to this model structure:

Lemma 8.78 *Consider maps of simplicial sets $X \xrightarrow{i} Y \to V$ and suppose that i is left anodyne. Then i is a trivial cofibration in the covariant model structure over V.*

Proof By Lemma 8.43 it suffices to check that i has the left lifting property with respect to fibrations between fibrant objects. The conclusion now follows from Theorem 8.75(c), stating that the fibrations between fibrant objects are precisely the left fibrations. □

Conversely, left anodynes can often be recognized by virtue of the following:

Lemma 8.79 *Consider maps of simplicial sets $X \xrightarrow{i} Y \xrightarrow{f} V$. Suppose that f is a left fibration and i is a trivial cofibration in the covariant model structure over V. Then i is left anodyne.*

Proof Factor i as a left anodyne $u: X \to Z$ followed by a left fibration $p: Z \to Y$. Then p is a fibration between fibrant objects in the covariant model structure over V, by items (b) and (c) of Theorem 8.75. Hence there exists a lift as indicated in the following diagram:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "i"'] & Z \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, dashed] & Y. \end{tikzcd}$$

This lift exhibits i as a retract of u, so that i is left anodyne. □

Corollary 8.80 *The class of left anodyne morphisms has the right cancellation property among monomorphisms: if $A \xrightarrow{i} B \xrightarrow{j} C$ are monomorphisms such that i and ji are left anodyne, then j is left anodyne as well.*

Proof Lemma 8.78 implies that i and ji are trivial cofibrations in the covariant model structure over C. By two-out-of-three for weak equivalences, j must then be a trivial cofibration as well. When thought of as a map in $\mathbf{sSets}/C$, its codomain is the identity map of C, which is in particular fibrant in the covariant model structure over C. Therefore Lemma 8.79 implies that j is left anodyne. □

Left anodyne morphisms often arise from 'left deformation retracts', as in the following:

Lemma 8.81 *Consider a monomorphism of simplicial sets $i: A \to B$ and a left deformation retract of it, i.e., a map $r: B \to A$ with $ri = \mathrm{id}_A$ and a homotopy $h: \Delta[1] \times B \to B$ relative to A from ir to id_B. In other words, h satisfies*

$$h|_{\{0\}\times B} = ir, \quad h|_{\{1\}\times B} = \mathrm{id}_A, \quad h|_{\Delta[1]\times A} = i \circ \pi_A,$$

with $\pi_A: \Delta[1] \times A \to A$ the projection. Then i is left anodyne.

Remark 8.82 As will be clear from the proof, the direction of the homotopy *from* the map ir and *to* the identity id_B is essential.

Proof The pushout-product of i with the inclusion $\{0\} \to \Delta[1]$ gives a map

$$\Delta[1] \times A \cup_{\{0\}\times A} \{0\} \times B \to \Delta[1] \times B$$

which is left anodyne by Lemma 6.29 (or rather just its simplicial version, which is much easier to prove). Now the diagram

$$\begin{tikzcd}
A \arrow[r, "\{1\}\times \mathrm{id}_A"] \arrow[d, "i"] & \Delta[1]\times A \cup_{\{0\}\times A} \{0\}\times B \arrow[r, "\pi_A \cup r"] \arrow[d] & A \arrow[d, "i"] \\
B \arrow[r, "\{1\}\times \mathrm{id}_B"] & \Delta[1]\times B \arrow[r, "h"] & B
\end{tikzcd}$$

exhibits i as a retract of the left anodyne map in the middle. □

A typical application of the previous lemma is the following. Recall that for a simplicial set X and vertex $x \in X_0$, we defined the slice $X_{x/}$ to be the simplicial set whose n-simplices are the $(n+1)$-simplices of X with initial vertex equal to x. This slice has a distinguished vertex $\mathrm{id}_x\colon \Delta[0] \to X_{x/}$ corresponding to the degenerate 1-simplex at x.

Lemma 8.83 *Let X be a simplicial set and $x \in X_0$ a vertex. Then the inclusion $\Delta[0] \xrightarrow{\mathrm{id}_x} X_{x/}$ is left anodyne.*

Proof There is a unique retraction $r\colon X_{x/} \to \Delta[0]$. We will now define a homotopy

$$h\colon \Delta[1]\times X_{x/} \to X_{x/}$$

making $\Delta[0]$ a left deformation retract of $X_{x/}$. An n-simplex α of $\Delta[1]\times X_{x/}$ is a pair (τ, f) consisting of a map $\tau\colon \Delta[n] \to \Delta[1]$ and a map $f\colon \Delta[0]\star\Delta[n] \to X$ mapping the first vertex $\Delta[0]$ to x. Label that first vertex -1 and label the vertices of $\Delta[n]$ by $0, \dots, n$ as usual. Then define a map of simplicial sets $\varphi_\tau\colon \Delta[0]\star\Delta[n] \to \Delta[0]\star\Delta[n]$ by letting its action on vertices be as follows: $\varphi_\tau(-1) = -1$ and for $i \geq 0$,

$$\varphi_\tau(i) = \begin{cases} -1 & \text{if } \tau(i) = 0 \\ i & \text{if } \tau(i) = 1 \end{cases}.$$

Then $h(\tau, f) = f \circ \varphi_\tau$ defines the desired homotopy. □

Historical Notes

The notions of Quillen adjunction and Quillen equivalence originate in [123], although of course with different terminology. In that book it was already proved that the categories of simplicial sets and of topological spaces both carry a model structure, and that the geometric realization and singular complex provide a Quillen equivalence between these. Our proof of this Quillen equivalence largely follows [61]. The fact that the geometric realization of a Kan fibration is a Serre fibration was proved by Quillen.

The categorical model structure on simplicial sets was established by Joyal. Although most of the technical results needed to establish this model structure already occur in [92], the first published account of many aspects of the Joyal model structure is Lurie's book [105]. The results on equivalences between ∞-categories of Section 8.7 and on the covariant model structure in Section 8.8 appear both in [105] and in [90].

The Kan–Quillen model structure on simplicial sets is one of the most central ones in the literature. Most proofs of its existence follow Quillen in using geometric realization. Our presentation is anachronistic in that it presents the categorical model structure first and deduces the Kan–Quillen model structure from it. Rather than relying on Smith's general theory of combinatorial model categories or Cisinski's theory of model structures on presheaf categories (as is done by Joyal and Lurie), we build the categorical model structure by hand in a rather direct fashion. The crucial step that replaces these general theories here is the 'ladder argument' appearing in Lemma 8.14. We have chosen this direct approach in order to be self-contained and because it applies in the context of dendroidal sets as well, as we will see in the next chapter. Similarly, we will use this argument in our treatment of Bousfield localization later on.

Chapter 9
Three Model Structures on the Category of Dendroidal Sets

In this chapter we will construct several model structures on the category of dendroidal sets. For the different model structures, the fibrant objects and fibrations will be the different types of dendroidal Kan complexes and dendroidal fibrations introduced in Chapter 6. For example, we will construct the *operadic model structure*, for which the fibrant objects will be the ∞-operads and the fibrations between fibrant objects will be the J-fibrations, so that this model structures describes the homotopy theory of ∞-operads. Later, we will prove that this model structure is Quillen equivalent to a model structure on simplicial or topological operads. It is in this precise sense that dendroidal sets form a combinatorial model for the theory of topological operads. The other model structures that we will construct, namely the covariant and Picard model structures, will be shown to model $\mathbf{E}_\infty$-spaces and infinite loop spaces, respectively.

The construction of these dendroidal model structures extends what we did in the previous chapter for simplicial sets in two ways. First of all, for the Joyal model structure we used a method of proof which we will adapt to apply to dendroidal sets as well. Secondly, under the equivalence of categories between **sSets** and the slice category **dSets**$/\eta$, one recovers the model structures on simplicial sets presented in the previous chapter. In particular, in this way the operadic model structure reduces to the Joyal model structure and the Picard model structure reduces to the Kan–Quillen one. Slicing the dendroidal covariant model structure over a simplicial set will provide proofs of the results we only outlined for the simplicial case in Section 8.8.

We will begin the chapter with the construction of a generic model structure on dendroidal sets defined in terms of a set $\mathcal{A}$ of cofibrations satisfying some conditions. The operadic and Picard model structures will then be constructed as special cases of this generic one in Sections 9.2 and 9.7, respectively. A full treatment of the covariant model structure in Section 9.5 requires a relative version of the generic construction, which we will explain in Section 9.4. Recall the functor $i_!$ embedding the category **sSets** into the category **dSets**. Throughout this chapter we will often omit it from our notation to avoid cluttering, simply regarding the category of simplicial sets as a subcategory of that of dendroidal sets.

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9.1 The $\mathcal{A}$-Model Structure for Dendroidal Sets

In this section we will present a general type of model structure on the category **dSets** of dendroidal sets, depending on a chosen set $\mathcal{A}$ of normal monomorphisms. We will then discuss particular choices for the set $\mathcal{A}$ in later sections. To carry out the necessary constructions, the set $\mathcal{A}$ should satisfy the following condition:

Definition 9.1 A set $\mathcal{A}$ of normal monomorphisms between normal dendroidal sets is called *admissible* if for any morphism $A \to B$ in $\mathcal{A}$ and any $n > 0$, the pushout-product map

$$\Delta[n] \otimes A \cup \partial\Delta[n] \otimes B \to \Delta[n] \otimes B$$

also belongs to $\mathcal{A}$.

Note that the pushout-product map of the definition is again a normal monomorphism, by Corollary 4.21. Recall that for two dendroidal sets A and X, we write $\mathbf{hom}(A, X)$ for the simplicial set defined by

$$\mathbf{hom}(A, X)_n = \mathbf{dSets}(\Delta[n] \otimes A, X).$$

Definition 9.2 A dendroidal set X is called $\mathcal{A}$-*local* if the following two conditions are satisfied:

(1) For any normal monomorphism $A \to B$ between dendroidal sets, the map

$$\mathbf{hom}(B, X) \to \mathbf{hom}(A, X)$$

is a categorical fibration of simplicial sets.

(2) Moreover, it is a trivial fibration whenever $A \to B$ belongs to $\mathcal{A}$.

Notice in particular that the first condition implies that $\mathbf{hom}(B, X)$ is an ∞-category for any $\mathcal{A}$-local object X and any normal dendroidal set B. One can of course express the property of being $\mathcal{A}$-local in terms of lifting properties. Using the characterization of categorical fibrations between ∞-categories given in Corollary 8.17, the definition of being $\mathcal{A}$-local translates into the following two lemmas, which do not require any further proof.

Lemma 9.3 *A dendroidal set X is $\mathcal{A}$-local if and only if it has the extension property with respect to the following three sets of maps:*

(i) *The maps*

$$\Lambda^k[n] \otimes \Omega[T] \cup \Delta[n] \otimes \partial\Omega[T] \to \Delta[n] \otimes \Omega[T],$$

for any tree T in $\mathbf{\Omega}$ and any $0 < k < n$.

(ii) *The maps*

$$\{0\} \otimes \Omega[T] \cup J \otimes \partial\Omega[T] \to J \otimes \Omega[T],$$

for any tree T in $\mathbf{\Omega}$ and $\{0\} \to J$ the inclusion of the first vertex.

(iii) *The maps*

$$\partial\Delta[n] \otimes B \cup \Delta[n] \otimes A \to \Delta[n] \otimes B,$$

for any $A \to B$ in $\mathcal{A}$ and $n \geq 0$.

Remark 9.4 In (iii) of the lemma, it would suffice to write only the case $n = 0$. Indeed, admissibility of $\mathcal{A}$ then guarantees that this includes all the maps described in (iii).

Writing $\overline{\mathcal{A}}$ for the saturation of $\mathcal{A}$ and using admissibility of $\mathcal{A}$, we can reformulate Lemma 9.3 as follows:

Lemma 9.5 *A dendroidal set X is $\mathcal{A}$-local if and only if it has the extension property with respect to the following three classes of maps:*

(i) *The maps*

$$M \otimes B \cup N \otimes A \to N \otimes B,$$

for any normal monomorphism $A \to B$ of dendroidal sets and any inner anodyne map $M \to N$ of simplicial sets.

(ii) *The maps*

$$\{0\} \otimes B \cup J \otimes A \to J \otimes B,$$

for any normal monomorphism $A \to B$ of dendroidal sets.

(iii) *The maps*

$$M \otimes B \cup N \otimes A \to N \otimes B,$$

for any monomorphism $M \to N$ of simplicial sets and any $A \to B$ in the saturation $\overline{\mathcal{A}}$ of $\mathcal{A}$.

Note that all of the maps occurring in (i)–(iii) of the lemma are normal monomorphisms by Corollary 4.21. Recall from Section 3.7 that a *normalization* of a dendroidal set A is a map $p\colon A' \to A$ with the property that A' is normal and p is a trivial fibration (i.e., has the right lifting property with respect to all normal monomorphisms). We extend this terminology to morphisms and refer to a commutative square

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d] & B' \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

as a *normalization* of f if each of the vertical maps is a normalization. Notice that by choosing a normalization $A' \to A$ of A first and then factoring the composition $A' \to A \to B$ as a normal monomorphism followed by a trivial fibration, we find that any map $f\colon A \to B$ admits a normalization, and even one in which $f'\colon A' \to B'$ is a normal monomorphism.

Lemma 9.6 *Normalizations are unique up to J-homotopy equivalence. More precisely,*

(a) *If $\varphi_1 : A_1 \longrightarrow A$ and $\varphi_2 : A_2 \longrightarrow A$ are normalizations of A, then there exists a J-homotopy equivalence $A_1 \longrightarrow A_2$ over A.*
(b) *If $f_1 : A_1 \longrightarrow B_1$ and $f_2 : A_2 \longrightarrow B_2$ are two normalizations of a morphism $f : A \longrightarrow B$, then there is a diagram*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ A_2 \arrow[r] & B_2 \end{tikzcd}$$

which commutes up to J-homotopy, in which the vertical maps are J-homotopy equivalences over A and over B, respectively.

Proof For (a), choose lifts k and l in the diagrams

$$\begin{tikzcd} & A_2 \arrow[d] \\ A_1 \arrow[r] \arrow[ur, dashed, "k"] & A \end{tikzcd} \qquad \begin{tikzcd} & A_1 \arrow[d] \\ A_2 \arrow[r] \arrow[ur, dashed, "l"] & A, \end{tikzcd}$$

which exist by virtue of the fact that A_1, A_2 are normal and the vertical maps are trivial fibrations. To see that lk is J-homotopic to the identity of A_1 (fibrewise over A), choose a lift in the diagram

$$\begin{tikzcd} \partial J \otimes A_1 \arrow[r, "{(lk, \mathrm{id})}"] \arrow[d] & A_1 \arrow[d, "\varphi_1"] \\ J \otimes A_1 \arrow[r, "\varphi_1 \circ \pi"'] \arrow[ur, dashed] & A. \end{tikzcd}$$

Such a lift exists because the left vertical map is a normal monomorphism and φ_1 is a trivial fibration. The same argument applies to the composition kl.

For (b) one constructs J-homotopy equivalences $k_A : A_1 \longrightarrow A_2$ over A and $k_B : B_1 \longrightarrow B_2$ over B by applying item (a). To see that the resulting square commutes up to J-homotopy, pick a lift in the square

$$\begin{tikzcd} \partial J \otimes A_1 \arrow[r, "{(f_2 k_A, k_B f_1)}"] \arrow[d] & B_2 \arrow[d] \\ J \otimes A_1 \arrow[r] \arrow[ur, dashed] & B. \end{tikzcd}$$

This completes the proof. □

The following definition is the crucial ingredient in defining the $\mathcal{A}$-model structure.

Definition 9.7 A morphism $A \to B$ is an $\mathcal{A}$-*weak equivalence* if it has a normalization $A' \to B'$ with the property that for any $\mathcal{A}$-local object X, the map

$$\mathbf{hom}(B', X) \to \mathbf{hom}(A', X)$$

is a categorical equivalence between ∞-categories.

Remark 9.8 (i) The first thing to observe is that Lemma 9.6 implies that this definition is independent of the chosen normalization, because any J-homotopy, say $h : J \otimes A \to B$, induces a J-homotopy

$$J \times \mathbf{hom}(B, X) \to \mathbf{hom}(A, X)$$

by transposing the map

$$h^* : \mathbf{hom}(B, X) \to \mathbf{hom}(J \otimes A, X) \cong \mathbf{hom}(A, X)^J$$

given by composition with h.

(ii) If the normalization $A' \to B'$ is chosen to be a normal monomorphism (which, as already remarked, is always possible), then $\mathbf{hom}(B', X) \to \mathbf{hom}(A', X)$ is a categorical fibration between ∞-categories. Hence it is in fact a trivial fibration whenever $A \to B$ is an $\mathcal{A}$-weak equivalence.

(iii) In Section 8.7 we presented various characterizations of the categorical equivalences between ∞-categories, which of course each lead to an alternative formulation of Definition 9.7 above. In particular, $A \to B$ is an $\mathcal{A}$-weak equivalence if and only if for each $\mathcal{A}$-local object X, the map

$$\tau\mathbf{hom}(B', X) \to \tau\mathbf{hom}(A', X)$$

is an equivalence of categories, a formulation which parallels Definition 8.1. Indeed, to check that

$$\mathbf{hom}(B', X) \to \mathbf{hom}(A', X)$$

is a categorical equivalence, Proposition 8.69 guarantees that it suffices to check that for any simplicial set S, the functor

$$\tau(\mathbf{hom}(B', X)^S) \to \tau(\mathbf{hom}(A', X)^S)$$

is an equivalence of categories. This map may be identified with the map

$$\tau(\mathbf{hom}(B', X^S)) \to \tau(\mathbf{hom}(A', X^S))$$

and X^S is another $\mathcal{A}$-local object whenever X is $\mathcal{A}$-local.

We are now ready to formulate a general existence theorem for model structures on dendroidal sets, as announced at the start of this section.

Theorem 9.9 *Let $\mathcal{A}$ be an admissible set of normal monomorphisms between dendroidal sets. There exists a model structure on the category* **dSets** *in which the cofibrations are the normal monomorphisms, the weak equivalences are the $\mathcal{A}$-weak equivalences, and the fibrant objects are precisely the $\mathcal{A}$-local objects. This model structure is left proper.*

Anticipating the proof of this theorem, we will refer to a map which is both a normal monomorphism and an $\mathcal{A}$-weak equivalence as an *$\mathcal{A}$-trivial cofibration*, and to a map having the right lifting property with respect to all these $\mathcal{A}$-trivial cofibrations as an *$\mathcal{A}$-fibration*.

The theorem asserts that each admissible set of monomorphisms between normal dendroidal sets gives a model structure. If $\mathcal{A} \subseteq \mathcal{B}$ are two such sets, then clearly every $\mathcal{B}$-local object is also $\mathcal{A}$-local, and hence any $\mathcal{A}$-weak equivalence is also a $\mathcal{B}$-weak equivalence. Thus, writing $\mathbf{dSets}_{\mathcal{A}}$ for the category of dendroidal sets equipped with the model structure given by $\mathcal{A}$, and similarly for $\mathcal{B}$, we obtain a Quillen pair

$$\mathrm{id}_! : \mathbf{dSets}_{\mathcal{A}} \rightleftarrows \mathbf{dSets}_{\mathcal{B}} : \mathrm{id}^*,$$

where $\mathrm{id}_!$ and id^* are both the identity functor on the underlying category. The cofibrations are the same in the two model categories. This is therefore an example of a left Bousfield localization, as discussed at the end of Section 8.3. The induced adjunction between homotopy categories

$$\mathbf{L}\mathrm{id}_! : \mathrm{Ho}(\mathbf{dSets}_{\mathcal{A}}) \rightleftarrows \mathrm{Ho}(\mathbf{dSets}_{\mathcal{B}}) : \mathbf{R}\mathrm{id}^*$$

has the property that the counit $\mathbf{L}\mathrm{id}_!\mathbf{R}\mathrm{id}^*(X) \to X$ is an isomorphism for each object X. This makes $\mathrm{Ho}(\mathbf{dSets}_{\mathcal{B}})$ into a reflective subcategory of $\mathrm{Ho}(\mathbf{dSets}_{\mathcal{A}})$.

The proof of Theorem 9.9 follows much the same pattern as that of Theorem 8.2, although some further technicalities arise because in the category of dendroidal sets (unlike for simplicial sets) we have to distinguish between monomorphisms and normal monomorphisms. To avoid unnecessary notational complications and to emphasize some analogies with the arguments of Chapter 8, we will assume that the domains and codomains of morphisms in the admissible set $\mathcal{A}$ are finite dendroidal sets (meaning they have finitely many non-degenerate dendrices). We refer to Remark 9.31 below for an explanation of how to avoid this assumption.

Lemma 9.10 *The class of $\mathcal{A}$-weak equivalences satisfies the two-out-of-three property for a composable pair of morphisms. Moreover, this class is closed under transfinite composition.*

Proof If $A \xrightarrow{f} B \xrightarrow{g} C$ is a composable pair of morphisms, one can choose normalizations fitting into a commutative diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C. \end{tikzcd}$$

It is then clear from the two-out-of-three property for weak equivalences between ∞-categories that if two out of f, g, and gf are $\mathcal{A}$-weak equivalences, then so is the third. For transfinite composition, let

$$A_1 \to A_2 \to A_3 \to \cdots$$

be a sequence of $\mathcal{A}$-weak equivalences and write A_∞ for the colimit. We claim that each $A_i \to A_\infty$ is again an $\mathcal{A}$-weak equivalence. To see this, choose successive normalizations as in

$$\begin{tikzcd}
A'_1 \arrow[r] \arrow[d] & A'_2 \arrow[r] \arrow[d] & A'_3 \arrow[r] \arrow[d] & \cdots \\
A_1 \arrow[r] & A_2 \arrow[r] & A_3 \arrow[r] & \cdots
\end{tikzcd}$$

for which each $A'_i \to A'_{i+1}$ is a normal monomorphism. Write A'_∞ for the colimit of the top row, which is a normalization of A_∞ by the fact that a filtered colimit of trivial fibrations is another trivial fibration. Then for any $\mathcal{A}$-local object X, one obtains a tower of trivial fibrations of ∞-categories

$$\mathbf{hom}(A'_1, X) \leftarrow \mathbf{hom}(A'_2, X) \leftarrow \cdots$$

with inverse limit $\mathbf{hom}(A'_\infty, X)$. Hence each projection $\mathbf{hom}(A'_\infty, X) \to \mathbf{hom}(A'_i, X)$ is again a trivial fibration. This shows that each $A_i \to A_\infty$ is an $\mathcal{A}$-weak equivalence. The same argument applies to a longer sequence indexed by an arbitrary ordinal. □

In Definition 3.34 we defined a trivial fibration of dendroidal sets to be a map which has the right lifting property with respect to normal monomorphisms. We warn the reader that at this point we have not yet proved that this is equivalent to being an $\mathcal{A}$-fibration and an $\mathcal{A}$-weak equivalence (although this will turn out to be the case). We will use this terminology in the following lemma:

Lemma 9.11 (i) *Any J-homotopy equivalence of dendroidal sets is an $\mathcal{A}$-weak equivalence.*

(ii) *Suppose $f: X \to Y$ is a trivial fibration between normal dendroidal sets. Then f is a J-homotopy equivalence.*

(iii) *Any trivial fibration between dendroidal sets is an $\mathcal{A}$-weak equivalence.*

Proof (i) Suppose $f: A \to B$ is a J-homotopy equivalence, given by a homotopy inverse $g: B \to A$ and homotopies $h: J \otimes A \to A$ and $k: J \otimes B \to B$ between the composites gf and fg and the two identities, respectively. Choose normalizations as in the commutative diagram

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "p"] & B' \arrow[r, "g'"] \arrow[d, "q"] & A' \arrow[d, "p"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & A.
\end{tikzcd}$$

Then a lift in

$$\begin{tikzcd}
\partial J \otimes B' \arrow[r, "{(f'g', \mathrm{id}_{B'})}"] \arrow[d, tail] & B' \arrow[d, "q"', "\wr"] \\
J \otimes B' \arrow[r, "k \circ (\mathrm{id} \otimes q)"'] & B
\end{tikzcd}$$

and a similar one for A' show that f' and g' are part of a J-homotopy equivalence. But then so are

$$\mathbf{hom}(B', X) \to \mathbf{hom}(A', X) \to \mathbf{hom}(B', X)$$

for any X, showing that f is a weak equivalence.

(ii) Suppose $f : X \to Y$ has the right lifting property with respect to normal monomorphisms. If Y is normal, then a lift in the square on the left shows that f has a section s:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d, tail] & X \arrow[d, "f"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\partial J \otimes X \cup J \otimes Y \arrow[r, "(sf, \mathrm{id}) \cup s\pi"] \arrow[d] & X \arrow[d, "f"] \\
J \otimes X \arrow[r, "\pi \circ (\mathrm{id} \otimes f)"] & Y.
\end{tikzcd}$$

If X is normal as well, then the monomorphism $s : Y \to X$ is necessarily normal. The left-hand vertical map of the square on the right is the pushout-product of $\partial J \to J$ with s, hence also a normal monomorphism. A lift in that square shows that sf is homotopic to the identity (fibrewise over Y). Here π denotes the projection $M \otimes Z \to Z$, which exists for any simplicial set M and any dendroidal set Z.

(iii) Suppose $f : X \to Y$ is a trivial fibration. Let $Y' \to Y$ be a normalization of Y and let $X' \to X \times_Y Y'$ be one of the pullback. Then all morphisms in the diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[dr] & X \times_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$

are trivial fibrations. In particular, this shows that $X \to Y$ has a normalization which is a trivial fibration. The statement now follows from (i) and (ii). □

Lemma 9.12 (i) *Any $\mathcal{A}$-trivial cofibration between normal $\mathcal{A}$-local objects is a strong J-deformation retract.*

(ii) *Any $\mathcal{A}$-weak equivalence between normal $\mathcal{A}$-local objects is a J-homotopy equivalence.*

Proof (i) Let $u : A \to B$ be an $\mathcal{A}$-trivial cofibration between normal $\mathcal{A}$-local objects. Then

$$\mathbf{hom}(B, A) \to \mathbf{hom}(A, A)$$

is a trivial fibration of simplicial sets, hence a surjection on vertices. So we find a retraction $r : B \to A$ with $ru = \mathrm{id}_A$. Next,

$$\mathbf{hom}(B, B) \to \mathbf{hom}(A, B)$$

is a trivial fibration as well. A lift in

$$\begin{tikzcd}
\partial J \arrow[r, "{(\mathrm{id}, ur)}"] \arrow[d] & \mathbf{hom}(B, B) \arrow[d] \\
J \arrow[r, "\mathrm{const}_u"] & \mathbf{hom}(A, B)
\end{tikzcd}$$

gives the necessary homotopy.

(ii) Any map $A \to C$ of dendroidal sets can be factored as a normal monomorphism followed by a trivial fibration, say as $A \xrightarrow{u} B \xrightarrow{p} C$. Since C is $\mathcal{A}$-local, so is B. Also, the fact that A is normal implies that B is normal. By Lemma 9.11, p is a J-homotopy equivalence and an $\mathcal{A}$-weak equivalence. Lemma 9.10 then implies that u is also an $\mathcal{A}$-weak equivalence. Then (i) implies that u is a J-homotopy equivalence. We conclude that the composite pu is a J-homotopy equivalence as well. □

Lemma 9.13 *The class of $\mathcal{A}$-trivial cofibrations is saturated, i.e., it is closed under pushouts, transfinite composition, and retracts.*

Proof As in the proof of Corollary 8.12, the cases of retracts and transfinite compositions are straightforward. We focus on pushouts. Consider a pushout square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "u"] & C \arrow[d, "v"] \\
B \arrow[r] & D
\end{tikzcd}$$

in which u is an $\mathcal{A}$-trivial cofibration. Then u is in particular a normal monomorphism, hence so is v. Let $D' \to D$ be a normalization of D. Pulling back this map to all the other objects in the square gives a cube in which the bottom face is a pushout and all vertical faces are pullbacks:

$$\begin{tikzcd}
A' \arrow[rr] \arrow[dr] \arrow[dd] & & C' \arrow[dr] \arrow[dd] & \\
& B' \arrow[rr] \arrow[dd] & & D' \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & C \arrow[dr] & \\
& B \arrow[rr] & & D.
\end{tikzcd}$$

It follows that the top face is again a pushout. Moreover, all of the vertical maps are trivial fibrations and the objects in the top face are normal (since they admit maps to D'). Hence every vertical map is a normalization. Upon mapping into a $\mathcal{A}$-local object X, the top face gives a pullback diagram

$$
\begin{tikzcd}
\mathbf{hom}(D', X) \arrow[r] \arrow[d] & \mathbf{hom}(B', X) \arrow[d] \\
\mathbf{hom}(C', X) \arrow[r] & \mathbf{hom}(A', X)
\end{tikzcd}
$$

in which the right-hand vertical map is a trivial fibration of simplicial sets by assumption. But then so is the left-hand vertical map, showing that v is an $\mathcal{A}$-weak equivalence. □

In exactly the same way, one proves the following lemma.

Lemma 9.14 *The pushout of an $\mathcal{A}$-weak equivalence along a normal monomorphism is again an $\mathcal{A}$-weak equivalence.*

Proof Consider a pushout square as in the previous proof, but now with a normal monomorphism $A \to B$ and an $\mathcal{A}$-weak equivalence $A \to C$. Construct normalizations in the same way, resulting in the final pullback square of that proof, now with the property that the map $\mathbf{hom}(C', X) \to \mathbf{hom}(A', X)$ is a weak equivalence between ∞-categories and $\mathbf{hom}(B', X) \to \mathbf{hom}(A', X)$ is a categorical fibration. It follows from Proposition 7.40 that $\mathbf{hom}(D', X) \to \mathbf{hom}(B', X)$ is again a weak equivalence between ∞-categories. We conclude that $B \to D$ is an $\mathcal{A}$-weak equivalence. □

Lemma 9.15 *The classes (i)–(iii) listed in Lemmas 9.3 and 9.5 consist of $\mathcal{A}$-trivial cofibrations.*

Proof We will prove that each of the maps in Lemma 9.5 is an $\mathcal{A}$-trivial cofibration under the assumption that all the objects involved are normal (so that there is no need for normalization). This will in particular include the maps of Lemma 9.3. But since the classes of maps occurring in Lemma 9.5 are in the saturations of the corresponding classes of Lemma 9.3, this will also imply the general case.

If $U \to V$ is one of the maps in Lemma 9.5, with U and V normal, we need to prove for any $\mathcal{A}$-local object X that $\mathbf{hom}(V, X) \to \mathbf{hom}(U, X)$ is a trivial fibration of simplicial sets. In other words, it suffices to prove that it has the right lifting property with respect to each monomorphism $K \to L$ of simplicial sets or, equivalently, that the pushout-product map

$$L \otimes U \cup K \otimes V \to L \otimes V$$

is again in one of the classes (i)–(iii) of Lemma 9.5. This follows from the symmetry and partial (!) associativity properties of the tensor product. Indeed, if L and M are simplicial sets and A is a dendroidal set, then

$$L \otimes (M \otimes A) \cong M \otimes (L \otimes A),$$

as discussed in Section 4.4. This implies that for $U \to V$ in each of the classes (i)–(iii), the pushout-product map above is in the same class. For example, if $U \to V$ is of the form $M \otimes B \cup N \otimes A \to N \otimes B$, then the relevant pushout-product map is

$$M \otimes (L \otimes B) \cup N \otimes (L \otimes A \cup K \otimes B) \to N \otimes (L \otimes B),$$

which is of the same form, but with B replaced by $L \otimes B$ and A replaced by $L \otimes A \cup K \otimes B$. The other two cases are similar. For (iii) one uses admissibility of $\mathcal{A}$. □

Definition 9.16 A map of dendroidal sets is $\mathcal{A}$-*anodyne* if it lies in the saturation of the classes (i)–(iii) of Lemma 9.5 (or equivalently of Lemma 9.3).

With this definition we can rephrase (the proof of) Lemma 9.15 by saying that if $U \to V$ is $\mathcal{A}$-anodyne, then so is the pushout-product

$$L \otimes U \cup K \otimes V \to L \otimes V$$

for any monomorphism $K \to L$ of simplicial sets. Also, Lemmas 9.13 and 9.15 together show that any $\mathcal{A}$-anodyne map is an $\mathcal{A}$-weak equivalence.

We now wish to prove that any $\mathcal{A}$-trivial cofibration lies in the saturation of the *set* of $\mathcal{A}$-trivial cofibrations between *countable* dendroidal sets, in the case where $\mathcal{A}$ consists of morphisms between finite dendroidal sets, as we will assume from now on. (See Remark 9.31 for the general case.) The proof is mostly analogous to the one for simplicial sets in Chapter 8, leading via Lemmas 8.10, 8.13, and 8.14 to Corollary 8.15 there. However, the need for normalizations makes the argument here a little more involved.

We use the small object argument with respect to the maps of Lemma 9.3 to produce, for any dendroidal set A, a map

$$A \to \widehat{A}$$

which is $\mathcal{A}$-anodyne and such that $\widehat{A}$ is $\mathcal{A}$-local. This assignment is functorial and has the following properties, which are clear from the construction:

Lemma 9.17 (a) *The functor $A \mapsto \widehat{A}$ preserves normal monomorphisms.*
(b) *If A is countable, then so is $\widehat{A}$. If $B \subseteq \widehat{A}$ is countable, then there exists a countable $U \subseteq A$ such that $B \subseteq \widehat{U} \subseteq \widehat{A}$ and $\widehat{U} \cap A = U$.*

Corollary 9.18 *A normal monomorphism $i : A \to B$ between normal dendroidal sets is an $\mathcal{A}$-weak equivalence if and only if it fits into a diagram*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"] & A' \arrow[d, "j"] \\ B \arrow[r, "v"] & B' \end{tikzcd}$$

where u and v are $\mathcal{A}$-anodyne maps into $\mathcal{A}$-local objects A' and B' respectively, and j is a strong J-deformation retract.

Proof Since $\mathcal{A}$-anodyne maps and J-homotopy equivalences are $\mathcal{A}$-weak equivalences, a diagram as in the corollary will show that i is an $\mathcal{A}$-weak equivalence. For the converse, suppose i is an $\mathcal{A}$-weak equivalence. Consider the following diagram, in which the square is a pushout:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & \widehat{A} \arrow[d] \\
B \arrow[r] \arrow[dr] & P \arrow[d] \\
& \widehat{P}.
\end{tikzcd}$$

The map $\widehat{A} \to P$ is an $\mathcal{A}$-trivial cofibration by Lemma 9.13. Since $P \to \widehat{P}$ is $\mathcal{A}$-anodyne, the composite $\widehat{A} \to \widehat{P}$ is an $\mathcal{A}$-trivial cofibration between normal $\mathcal{A}$-local objects. Lemma 9.12 then guarantees that it is a strong J-deformation retract. □

The statement of the following lemma and its proof are the same as for simplicial sets, cf. Lemma 8.13.

Lemma 9.19 *Consider a strong deformation retract of dendroidal sets consisting of maps $u : A \to B$, $r : B \to A$ with $ru = \mathrm{id}_A$ and a homotopy $h : J \times B \to B$ from ur to id_B relative to A. Then for any countable $U \subseteq A$ and $V \subseteq B$, there are countable U' and V' with $U \subseteq U' \subseteq A$ and $V \subseteq V' \subseteq B$, such that $U' = u^{-1}(V')$, r maps V' into U', and h restricts to a map $J \times V' \to V'$. So in particular, U' is a strong J-deformation retract of V'.*

Lemma 9.20 *Each dendroidal set X admits a normalization $X' \to X$ with countable fibres. In particular, if X is countable then we may take X' countable as well.*

Proof Upon inspection of the small object argument, one sees that it suffices to prove that if $p \colon X \to Y$ is a map with countable fibres, then so is the map $p' \colon X' \to Y$ obtained as the pushout

$$\begin{tikzcd}
\coprod_i \partial\Omega[T] \arrow[r, "b"] \arrow[d] & X \arrow[d] \arrow[ddr, bend left, "p"] & \\
\coprod_i \Omega[T] \arrow[r, "a'"] \arrow[drr, bend right, "a"'] & X' \arrow[dr, dashed, "p'"] & \\
& & Y,
\end{tikzcd}$$

where i ranges over all commutative squares

$$\begin{tikzcd}
\partial\Omega[T] \arrow[r, "b_i"] \arrow[d] & X \arrow[d, "p"] \\
\Omega[T] \arrow[r, "a_i"] & Y,
\end{tikzcd}$$

while $a = \{a_i\}$ and $b = \{b_i\}$. Let $y : \Omega[S] \to Y$ represent a non-degenerate element of $Y(S)$. It suffices to show that there are only countably many 'new' non-degenerate elements $x \in X'(S)$ with $p'(x) = y$, new in the sense of not belonging to $X(S) \subseteq X'(S)$. Any such x must arise as a composition

$$\Omega[S] \xrightarrow{\varphi} \Omega[T] \xrightarrow{a'_i} X'$$

for some i. Since x is assumed non-degenerate as well as new, the morphism φ must be injective as well as surjective (respectively), hence an isomorphism. Since $\mathrm{Aut}(T)$ is finite, it thus suffices to count the x's for which φ is the identity. But then $a_i = y$ and for a given y there are only countable many b_i which fit into a commutative square

$$\begin{tikzcd} \partial\Omega[T] \arrow[r, "b_i"] \arrow[d] & X \arrow[d] \\ \Omega[T] \arrow[r, "y"] & Y, \end{tikzcd}$$

showing there are only countable many of the x's we were trying to count. □

Remark 9.21 There is a more appealing construction of a normalization with countable fibres, namely the projection $X \times w^*\mathbf{P} \to X$, where $\mathbf{P}$ is the simplicial Barratt–Eccles operad with $\mathbf{P}(n) = E\Sigma_n$ (cf. Section 2.7.6). Indeed, the map $w^*\mathbf{P} \to 1$ is a trivial fibration of dendroidal sets. Although easily shown by hand, this will also become clear when we discuss the fact that w^* is a right Quillen functor with respect to a suitable model structure on the category of simplicial operads (cf. Section 14.6).

Lemma 9.22 *Let $u\colon A \to B$ be an $\mathcal{A}$-trivial cofibration between dendroidal sets and let $U \subseteq A$ and $V \subseteq B$ be countable dendroidal subsets. Then there are countable U' and V' with $U \subseteq U' \subseteq A$ and $V \subseteq V' \subseteq B$ such that $U' = u^{-1}(V')$ and u restricts to an $\mathcal{A}$-trivial cofibration $U' \to V'$.*

Proof First of all, if $u\colon A \to B$ is a map between *normal* dendroidal sets, the proof is exactly the same as that of Lemma 8.14, now using Corollary 9.18, Lemma 9.17, and Lemma 9.19. Let us explain how to reduce the general case to the case where A and B are normal. Consider a map $u\colon A \to B$ as in the statement of the lemma. Let $q\colon B' \to B$ be a normalization with countable fibres (as in Lemma 9.20) and construct the pullback

$$\begin{tikzcd} A' \arrow[r, "u'"] \arrow[d, "p"] & B' \arrow[d, "q"] \\ A \arrow[r, "u"] & B. \end{tikzcd}$$

Then p is a normalization of A with countable fibres. Now take $U \subseteq A$ and $V \subseteq B$ as in the lemma. Set $U' = p^{-1}U$ and $V' = q^{-1}V$. Then U' and V' are again countable. By the case of A and B normal, there are countable $U' \subseteq U'_1$ and $V' \subseteq V'_1$ such that $u'\colon A' \to B'$ restricts to an $\mathcal{A}$-weak equivalence $U'_1 \to V'_1$ and $(u')^{-1}V'_1 = U'_1$. Let $U_1 = p(U'_1)$ and $V_1 = q(V'_1)$. Then $u^{-1}V_1 = U_1$. Now repeat the argument with U and

V replaced by U_1 and V_1. Continuing in this way, we build a ladder of dendroidal sets

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U_1 \arrow[r] \arrow[d] & U_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & A \arrow[d] \\ V \arrow[r] & V_1 \arrow[r] & V_2 \arrow[r] & \cdots \arrow[r] & B \end{tikzcd}$$

for which the normalizations, obtained by restricting $q\colon B' \to B$, interpolate $\mathcal{A}$-weak equivalences $U_i' \to V_i'$ as in

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U_1' \arrow[r] \arrow[d] & p^{-1}U_1 \arrow[r] \arrow[d] & U_2' \arrow[r] \arrow[d] & p^{-1}U_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & A' \arrow[d] \\ V' \arrow[r] & V_1' \arrow[r] & q^{-1}V_1 \arrow[r] & V_2' \arrow[r] & q^{-1}V_2 \arrow[r] & \cdots \arrow[r] & B' \end{tikzcd}$$

Then $\varinjlim_i p^{-1}U_i \to \varinjlim_i q^{-1}V_i$ is a normalization of $\varinjlim_i U_i \to \varinjlim_i V_i$. It is an $\mathcal{A}$-weak equivalence because it coincides with $\varinjlim_i U_i' \to \varinjlim_i V_i'$ and each $U_i' \to V_i'$ is an $\mathcal{A}$-trivial cofibration. This proves the lemma. □

Corollary 9.23 *Any $\mathcal{A}$-trivial cofibration is a transfinite composition of pushouts of $\mathcal{A}$-trivial cofibrations between countable dendroidal sets.*

Proof The proof is the same as that of Corollary 8.15, with an appeal to Lemma 8.14 replaced by one to Lemma 9.22. □

With all these preparations out of the way, we are now ready to prove Theorem 9.9 stated in the beginning of this section. As mentioned earlier, we restrict our attention to the case where the domains of the morphisms in $\mathcal{A}$ are finite dendroidal sets. Although the general case can be proved in exactly the same way, all our examples satisfy this assumption. See also Remark 9.31 below.

Proof (of Theorem 9.9) The proof follows the same pattern as that of Theorem 8.2. Axiom (M1) (existence of limits and colimits) is clear and axiom (M2) (two-out-of-three) was verified in Lemma 9.10. Axiom (M3) (retracts) is clear from the fact that if A is a retract of B then it has a normalization $A' \to A$ which is a retract of a normalization of B', as in the diagram

$$\begin{tikzcd} A' \arrow[r, dashed] \arrow[d] & B' \arrow[r] \arrow[d] & A' \arrow[d] \\ A \arrow[r] & B \arrow[r] & A, \end{tikzcd}$$

in which the square on the right is a pullback. For the factorization axiom (M5), note that the small object argument provides the following two factorizations of a morphism $f\colon X \to Y$. The first one as $X \xrightarrow{i} Z \xrightarrow{p} Y$ where i is a normal monomorphism while p has the right lifting property with respect to all normal monomorphisms, the second as $X \xrightarrow{j} W \xrightarrow{q} Y$ where j is an $\mathcal{A}$-trivial cofibration and q has the right lifting property with respect to all $\mathcal{A}$-trivial cofibrations between countable objects, and hence by Corollary 9.23 with respect to all $\mathcal{A}$-trivial cofibrations. In these factorizations, q is an $\mathcal{A}$-fibration by definition and p is a fibration as well as a weak equivalence by Lemma 9.11. This proves that (M5) holds. Finally, for the lifting axiom (M4), consider a square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

in which j is a cofibration (i.e., a normal monomorphism) and f is a fibration. If j is also an $\mathcal{A}$-weak equivalence, then a lift exists by definition of the $\mathcal{A}$-fibrations. If f is also an $\mathcal{A}$-weak equivalence, one factors f as $f = pi$ with p a trivial fibration and i an $\mathcal{A}$-trivial cofibration and uses the same retract argument we have already applied several times, for example at the end of the proof of Theorem 8.2, to see that f is a retract of p. Hence f has the right lifting property with respect to normal monomorphisms. The fact that the $\mathcal{A}$-model structure is left proper is Lemma 9.14. We will characterize the $\mathcal{A}$-fibrant objects (and $\mathcal{A}$-fibrations between them) in Proposition 9.25 below. □

To characterize the $\mathcal{A}$-fibrant objects, we will need the following preliminary observation.

Lemma 9.24 *A map $X \to Y$ of dendroidal sets is a fibration in the $\mathcal{A}$-model structure if and only if it has the right lifting property with respect to all trivial cofibrations between normal objects.*

Proof Let $e\colon E \to 1$ be a normalization of the terminal object in dendroidal sets. Then for any X, the map $E \times X \to X$ is a trivial fibration. So a map $X \to Y$ is an $\mathcal{A}$-weak equivalence if and only if $E \times X \to E \times Y$ is. Moreover, since any representable object $\Omega[T]$ is normal, it admits a map to E. So $X \to Y$ has the right lifting property with respect to a boundary inclusion $\partial T \to T$ if and only if $E \times X \to E \times Y$ has. Thus in the Quillen pair

$$e_! : \mathbf{dSets}/E \rightleftarrows \mathbf{dSets} : e^*,$$

where e^* is the product with E, the functor e^* preserves and detects weak equivalences and trivial fibrations. Remark 8.48 then implies that the model structure on **dSets** agrees with the model structure transferred along this adjunction from the slice category $\mathbf{dSets}/E$. In particular, a map f of dendroidal sets is a fibration if and only if e^*f is a fibration. The lemma follows, because the normal objects are precisely the objects admitting a map to E. □

Proposition 9.25 *Let $f\colon X \to Y$ be a map between $\mathcal{A}$-local objects. Then f is an $\mathcal{A}$-fibration if and only if it has the right lifting property with respect to all $\mathcal{A}$-anodyne maps.*

Before we prove the proposition, we observe the following.

Corollary 9.26 *The fibrant objects in the $\mathcal{A}$-model structure are precisely the $\mathcal{A}$-local objects.*

Proof Any fibrant object is $\mathcal{A}$-local by Lemma 9.15. Conversely, if X is $\mathcal{A}$-local, Proposition 9.25 guarantees that the map $X \to 1$ to the terminal object is an $\mathcal{A}$-fibration. □

Proof (of Proposition 9.25) Any fibration has the right lifting property with respect to $\mathcal{A}$-anodynes by Lemma 9.15. For the converse, let $p\colon X \to Y$ be a map between $\mathcal{A}$-local objects having the right lifting property with respect to $\mathcal{A}$-anodyne maps. Consider a lifting problem of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] \arrow[ur, dashed] & Y, \end{tikzcd}$$

where i is an $\mathcal{A}$-trivial cofibration. By Lemma 9.24 above it suffices to consider the case where A and B are normal objects. Then as in Corollary 9.18 we may form a square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & A' \arrow[d, "j"] \\ B \arrow[r, "v"] & B', \end{tikzcd}$$

where u and v are $\mathcal{A}$-anodyne and j is part of a strong deformation retract between normal $\mathcal{A}$-local objects, with retraction $r\colon B' \to A'$ and homotopy $h : J \otimes B' \to B'$ between jr and $\mathrm{id}_{B'}$. Now first extend g to a map $g' : B' \to Y$, which is possible since v is $\mathcal{A}$-anodyne, and next lift f to f' as in

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "u"'] & & X \arrow[d, "p"] \\ A' \arrow[r, "j"'] \arrow[urr, dashed, "f'"] & B' \arrow[r, "g'"'] & Y \end{tikzcd}$$

using the assumption on p. Then $k = f'r\colon B' \to X$ is a map with $kj = f'rj = f'$, but it is not quite a lift in

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ B' \arrow[r, "g'"] \arrow[ur, dashed] & Y \end{tikzcd}$$

because $pk = pf'r = g'jr$ is only homotopic to g' (relative to A'). To fix this, choose a lift in

$$\begin{tikzcd}
\{0\} \otimes B' \cup J \otimes A' \arrow[rrr, "k \cup f'\pi_2"] \arrow[d] & & & X \arrow[d, "p"] \\
J \otimes B' \arrow[rr, "h"] \arrow[urrr, dashed, "l"] & & B' \arrow[r, "g'"] & Y
\end{tikzcd}$$

using that the left-hand vertical map is $\mathcal{A}$-anodyne. Then $l_1 : B' \to Y$ has $l_1 j = f'$ and $pl_1 = g'h_1 = g'$, so $l_1 v : B \to X$ is the required lift in the original lifting problem. Indeed, $l_1 vi = l_1 ju = f'u = f$ and $pl_1 v = g'v = g$. This proves the proposition. □

Proposition 9.25 also gives the following useful criterion for recognizing left Quillen functors out of the $\mathcal{A}$-model structure:

Lemma 9.27 *Suppose $\mathcal{E}$ is a model category and $f_! : \mathbf{dSets} \to \mathcal{E}$ is a left adjoint functor. If $f_!$ preserves cofibrations and sends every $\mathcal{A}$-anodyne map to a trivial cofibration in $\mathcal{E}$, then $f_!$ is left Quillen.*

Proof By Lemma 8.42 it suffices to check that the right adjoint f^* preserves fibrations between fibrant objects. The assumption of the lemma guarantees that f^* sends every fibration to a map having the right lifting property with respect to $\mathcal{A}$-anodynes; the desired conclusion then follows from Proposition 9.25. □

Another useful consequence of Proposition 9.25 is that the $\mathcal{A}$-model structure on the category of dendroidal sets is compatible with the Joyal model structure on the category of simplicial sets, in the sense of Proposition 9.28 below. Roughly speaking, it says that the $\mathcal{A}$-model is 'enriched' over the Joyal model structure; this statement is imprecise though, because the simplicial mapping objects $\mathbf{hom}(-,-)$ do not provide an actual enrichment of **dSets** over **sSets** because of the subtle associativity properties of the tensor product of dendroidal sets.

Proposition 9.28 *For a monomorphism $i : M \to N$ of simplicial sets and a normal monomorphism of dendroidal sets $j : A \to B$, the pushout-product*

$$N \otimes A \cup M \otimes B \to N \otimes B$$

is a normal monomorphism, which is moreover an $\mathcal{A}$-weak equivalence whenever i is a categorical equivalence or j is an $\mathcal{A}$-weak equivalence.

Dually, if $p : X \to Y$ is an $\mathcal{A}$-fibration, then the induced map

$$\mathbf{hom}(B, X) \to \mathbf{hom}(B, Y) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(A, X)$$

is a categorical fibration of simplicial sets, which is a trivial fibration whenever j or p is an $\mathcal{A}$-weak equivalence.

The proof of the proposition depends on the following, which is an easy consequence of the definition of $\mathcal{A}$-anodynes:

Lemma 9.29 *For i and j (normal) monomorphisms as in the statement of Proposition 9.28, the pushout-product*

$$N \otimes A \cup M \otimes B \to N \otimes B$$

is $\mathcal{A}$-anodyne whenever one of the following three conditions is satisfied:

(1) *The map j is $\mathcal{A}$-anodyne.*
(2) *The map i is inner anodyne.*
(3) *The map i is the inclusion $\{0\} \to J$.*

Proof We covered case (1) in the proof of Lemma 9.15 (cf. the remarks immediately after Definition 9.16). The argument for the other two cases is entirely analogous, simply rewriting the relevant pushout-products in one of the forms (i)–(iii) of Lemma 9.5. □

Corollary 9.30 *If $p\colon X \to Y$ is a map of dendroidal sets which has the right lifting property with respect to $\mathcal{A}$-anodynes and $j\colon A \to B$ is a normal monomorphism between dendroidal sets, then the map*

$$\mathbf{hom}(B, X) \to \mathbf{hom}(B, Y) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(A, X)$$

is an inner fibration with J-path lifting. If X and Y are $\mathcal{A}$-local and j is an $\mathcal{A}$-weak equivalence, then this map is even a trivial fibration.

Proof Everything but the last sentence follows by adjunction from cases (2) and (3) of Lemma 9.29. For the last part, consider the diagram

$$\begin{tikzcd}
\mathbf{hom}(B, X) \arrow[r] \arrow[dr] & \mathbf{hom}(B, Y) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(A, X) \arrow[d] \\
& \mathbf{hom}(A, X).
\end{tikzcd}$$

The slanted map is a categorical equivalence by definition of the $\mathcal{A}$-weak equivalences. The vertical map is a pullback of the trivial fibration $\mathbf{hom}(B, Y) \to \mathbf{hom}(A, Y)$ and hence itself a trivial fibration. By two-out-of-three it follows that the horizontal map is a categorical equivalence between ∞-categories. Since we already know that it is a categorical fibration, it must therefore be a trivial fibration. □

Proof (of Proposition 9.28) The second half follows from the first by the adjunction between $- \otimes A$ and $\mathbf{hom}(A, -)$. For the pushout-product, we have already proved (and used many times) that the map is a normal monomorphism (cf. Proposition 4.21). Now suppose that i is also a categorical equivalence or j is also an $\mathcal{A}$-weak equivalence. To show that the pushout-product is an $\mathcal{A}$-trivial cofibration, it suffices to check that it has the left lifting property with respect to $\mathcal{A}$-fibrations between $\mathcal{A}$-fibrant objects by virtue of Lemma 8.43. By Proposition 9.25, such fibrations are precisely the maps between $\mathcal{A}$-local objects with the right lifting property against $\mathcal{A}$-anodyne maps. Let $p\colon X \to Y$ be such a map and consider the lifting problem

$$\begin{tikzcd} N \otimes A \cup M \otimes B \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ N \otimes B \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

By adjunction, it is equivalent to a lifting problem

$$\begin{tikzcd} M \arrow[rr] \arrow[d, "i"] & & \mathbf{hom}(B, X) \arrow[d] \\ N \arrow[r] \arrow[urr, dashed] & \mathbf{hom}(B, Y) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(A, X). & \end{tikzcd}$$

The right-hand map is a categorical fibration between ∞-categories by Corollary 9.30. Hence a lift exists whenever i is a categorical trivial cofibration. On the other hand, if $j\colon A \to B$ is an $\mathcal{A}$-weak equivalence, then the right-hand map is a trivial fibration (again by Corollary 9.30) and therefore a lift exists in that case as well. $\square$

Remark 9.31 We have constructed a model structure associated with an admissible set $\mathcal{A}$ of normal monomorphisms between finite dendroidal sets, by concluding that the $\mathcal{A}$-trivial cofibrations are generated by the set of $\mathcal{A}$-trivial cofibrations between countable objects. A similar argument works without the finiteness assumptions on the morphisms in $\mathcal{A}$, but one should replace countability by the use of an inaccessible cardinal λ exceeding the size of all the domains and codomains of the morphisms in $\mathcal{A}$. In this book, this more general version will not play a role.

9.2 The Operadic Model Structure

In this section we introduce the model structure on the category of dendroidal sets that has as its fibrant objects the ∞-operads, called the *operadic model structure*. This is the analogue of the Joyal (or categorical) model structure from the previous chapter. We will see later that the resulting model category is Quillen equivalent to that of simplicial or topological operads.

Recall from Chapter 6 the inner horn inclusions of trees, i.e., the inclusions of the form

$$\Lambda^e[T] \to \Omega[T]$$

where T is a tree and e an inner edge of T. We will write $\mathcal{I}$ for the set of inner horn inclusions and $\overline{\mathcal{I}}$ for its saturation, the class of inner anodyne maps. Recall that if $M \to N$ is a monomorphism between simplicial sets and $A \to B$ a normal monomorphism between dendroidal sets, then the pushout-product map

$$M \otimes B \cup_{M \otimes A} N \otimes A \to N \otimes B$$

is again a normal monomorphism, which belongs to $\overline{\mathcal{I}}$ if either $A \to B$ or $M \to N$ is inner anodyne (cf. Corollary 6.26).

We also recall that the maps between dendroidal sets having the right lifting property with respect to $\mathcal{J}$ are called *inner fibrations* and that an ∞-operad is a dendroidal set for which the map $X \to 1$ is an inner fibration. Then dual to our previous statement about the pushout-product map, if $A \to B$ is a normal monomorphism and $X \to Y$ is an inner fibration, then the map

$$\mathbf{hom}(B, X) \to \mathbf{hom}(A, X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B, Y)$$

is an inner fibration between ∞-categories. It is a trivial fibration if $A \to B$ is inner anodyne. Finally, we recall that if in addition $X \to Y$ has J-path lifting then so does the map above (cf. Theorem 6.52), while the map $X \to 1$ has J-path lifting for any ∞-operad X. All these properties come together in the following statement:

Theorem 9.32 *There exists a model structure on the category* **dSets** *of dendroidal sets with the following properties:*

(a) *The cofibrations are the normal monomorphisms.*
(b) *The fibrant objects are the ∞-operads.*
(c) *The fibrations between fibrant objects are the inner fibrations having J-path lifting.*
(d) *A map $A \to B$ between normal dendroidal sets is a weak equivalence if and only if for every ∞-operad X, the map*

$$\mathbf{hom}(B, X) \to \mathbf{hom}(A, X)$$

is a categorical equivalence between ∞-categories.

Moreover, this model structure is left proper and cofibrantly generated.

Remark 9.33 As stated before, we will refer to this model structure as the *operadic model structure*. Similarly, we will refer to its weak equivalences and fibrations as *operadic weak equivalences* and *operadic fibrations*, respectively.

Remark 9.34 As a special case of Proposition 9.28, the simplicial 'tensoring' of the category **dSets** makes the operadic model structure on **dSets** compatible with the categorical model structure on **sSets**.

Proof We wish to apply Theorem 9.9 about the existence of $\mathcal{A}$-model structures. For this, take for $\mathcal{A}$ any set of morphisms with $\mathcal{J} \subseteq \mathcal{A} \subseteq \overline{\mathcal{J}}$ which is admissible. For example, we can take all maps $A \to B$ in $\overline{\mathcal{J}}$ for which A and B are finite (i.e., have finitely many non-degenerate dendrices). The $\mathcal{A}$-anodynes may now be described in the following way. Corollary 6.26 implies that the maps of type (i) and (iii) in Lemma 9.3 are inner anodyne. Thus, the class of $\mathcal{A}$-anodyne morphisms is the smallest saturated class containing the inner anodyne maps as well as the maps

$$\{0\} \otimes \Omega[T] \cup J \otimes \partial\Omega[T] \to J \otimes \Omega[T]$$

for each tree T. By the results from Chapter 6 we recalled above, together with Proposition 9.25, the $\mathcal{A}$-local objects are precisely the ∞-operads and the fibrations between $\mathcal{A}$-local objects are the inner fibrations having J-path lifting. $\square$

We introduce the following terminology for the $\mathcal{A}$-anodyne morphisms for the specific choice of $\mathcal{A}$ we introduced above:

Definition 9.35 The class of *J-anodyne morphisms* of dendroidal sets is the smallest saturated class containing the inner horn inclusions of trees as well as the inclusion $\{0\} \to J$.

Remark 9.36 This definition looks like it describes a slightly smaller class of maps than the $\mathcal{A}$-anodynes described in the proof of Theorem 9.32. However, the map

$$j\colon \{0\} \otimes \Omega[T] \cup J \otimes \partial\Omega[T] \to J \otimes \Omega[T]$$

is also J-anodyne. To see this, construct a square

$$\begin{tikzcd}
\{0\} \otimes \Omega[T] \cup J \otimes \partial\Omega[T] \arrow[r, "u"] \arrow[d, "j"] & X \arrow[d, "p"] \\
J \otimes \Omega[T] \arrow[r, equal] & J \otimes \Omega[T],
\end{tikzcd}$$

by factoring j into a J-anodyne map u followed by a map p having the right lifting property with respect to J-anodynes, using the small object argument. Since $J \otimes \Omega[T]$ is an ∞-operad (it is the dendroidal nerve of an operad) and p is an inner fibration with J-path lifting, the dendroidal set X is also an ∞-operad. Then by Theorem 9.32(c), p is a fibration between fibrant objects. Since j is a trivial cofibration, there exists a lift in the square. Such a lift exhibits j as a retract of u, so that j is also J-anodyne.

The operadic model structure can be related to several others already discussed:

Proposition 9.37 *The adjoint pair τ : **dSets** $\rightleftarrows$ **Op** : N is a Quillen pair, where **Op** is equipped with the naive model structure (cf. the end of Section 7.1). Moreover, the functor τ preserves weak equivalences between arbitrary objects.*

Proof It is clear from the definitions that N preserves fibrations while τ preserves cofibrations, proving the first sentence of the proposition. Since τ is a left Quillen functor, it preserves weak equivalences between cofibrant objects. For an arbitrary operadic equivalence $f\colon X \to Y$ between dendroidal sets we argue as follows. Take a normalization of the terminal object $E \to 1$. Then the map of operads $\tau E \to \tau(1) = \mathbf{Com}$ is an equivalence, as can be seen directly from the description of the homotopy operad τE of Lemma 6.7. Consider the square

$$\begin{tikzcd}
E \times X \arrow[r, "E\times f"] \arrow[d, two heads, "\wr"] & E \times Y \arrow[d, two heads, "\wr"] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

By our previous observation and the fact that τ preserves products, τ will send the vertical maps to equivalences. The top horizontal map is a weak equivalence between cofibrant objects and therefore also preserved by τ. It follow by two-out-of-three that $\tau(f)$ is an equivalence of operads. □

Proposition 9.38 *The adjoint pair $i_! : \mathbf{sSets} \rightleftarrows \mathbf{dSets} : i^*$ is a Quillen pair between the Joyal model structure and the operadic one. Moreover, $i_!$ detects weak equivalences between arbitrary objects.*

Proof All of these claims follow easily from the identification $\mathbf{sSets} \simeq \mathbf{dSets}/\eta$, combined with the observation that the operadic model structure on the slice category $\mathbf{dSets}/\eta$ agrees with the Joyal model structure on the category of simplicial sets. In these terms, the adjunction of the proposition can be thought of as the usual adjoint pair for a slice category:

$$\mathbf{dSets}/\eta \underset{\eta^*}{\overset{\eta_!}{\rightleftarrows}} \mathbf{dSets}.$$

□

We will spend the rest of this section studying the operadic equivalences between ∞-operads, with the aim of characterizing them as the fully faithful and essentially surjective maps. This discussion parallels the one for simplicial sets in Section 8.7. We begin with the analogue of Proposition 8.69:

Proposition 9.39 *For a map $f\colon X \to Y$ between ∞-operads, the following properties are equivalent:*

(1) *The map f is an operadic equivalence.*
(2) *Any normalization f' of f is a J-homotopy equivalence.*
(3) *For any normal dendroidal set A, the map $\mathbf{hom}(A, X) \to \mathbf{hom}(A, Y)$ is a categorical equivalence between ∞-categories.*
(4) *For any normal dendroidal set A, the map $k\mathbf{hom}(A, X) \to k\mathbf{hom}(A, Y)$ is a homotopy equivalence between Kan complexes.*
(5) *For any normal dendroidal set A, the map $\tau\mathbf{hom}(A, X) \to \tau\mathbf{hom}(A, Y)$ is an equivalence of categories.*

Proof For a normal dendroidal set A, observe that

$$A \amalg A \cong A \otimes \partial J \to A \otimes J \to A$$

gives a cylinder object for A. Indeed, the first map is a normal monomorphism by Proposition 4.21 and the second map is an operadic equivalence, because it admits a section $A \cong A \otimes \{0\} \to A \otimes J$ which is a trivial cofibration by Corollary 6.30 and the fact that $\{0\} \to J$ is left anodyne (see Proposition 5.22). As a consequence, we may identify left homotopies of maps out of A with J-homotopies.

By definition, (1) is equivalent to the statement that any normalization $f'\colon X' \to Y'$ of f is an operadic equivalence between normal ∞-operads. Since weak equivalences between fibrant-cofibrant objects coincide with homotopy equivalences, the remarks above imply that f' is an operadic equivalence if and only if it is a J-homotopy equivalence, showing the equivalence between (1) and (2). To see that (2) implies (3), consider the square

$$\begin{tikzcd}
\mathbf{hom}(A, X') \arrow[r] \arrow[d] & \mathbf{hom}(A, Y') \arrow[d] \\
\mathbf{hom}(A, X) \arrow[r] & \mathbf{hom}(A, Y).
\end{tikzcd}$$

The vertical maps are trivial fibrations between ∞-categories by Theorem 6.33, hence J-homotopy equivalences. Assuming (2), the top horizontal map is a J-homotopy equivalence. It follows that the same is true for the bottom horizontal map. A J-homotopy equivalence of ∞-categories also gives a homotopy equivalent of their respective maximal Kan complexes, showing that (3) implies (4). Also, (3) implies (5) since τ sends categorical equivalences between simplicial sets to equivalences of categories. Finally, both (4) and (5) imply that f induces a natural isomorphism between the functors represented by X and Y on the homotopy category $\mathbf{Ho}(\mathbf{dSets})$ (taken with respect to the operadic model structure). Indeed, as in Lemma 8.68 we can identify the homotopy classes of maps $A \to X$ with the isomorphism classes of objects in the category $\tau\mathbf{hom}(A, X)$, or with the connected components of $k\mathbf{hom}(A, X)$. We conclude that f is an operadic equivalence. □

Lemma 9.40 *Consider a map $f : X \to Y$ between ∞-operads. If condition (4) of Proposition 9.39 holds for all representable dendroidal sets $\Omega[T]$, then it holds for all normal dendroidal sets A.*

Proof Without loss of generality we may assume f is an operadic fibration. Indeed, a general f may be factored as a trivial cofibration i followed by an operadic fibration p. Then i admits a normalization which is a trivial cofibration between normal ∞-operads, hence a J-homotopy equivalence. But then $\mathbf{hom}(A, i)$ is a J-homotopy equivalence between simplicial sets by the same argument as in the proof of Proposition 9.39 above. Hence it suffices to prove the proposition for the operadic fibration p.

The proof proceeds by skeletal induction on the normal dendroidal set A. The base of the induction is the 0-skeleton $\mathrm{sk}_0 A$, which can be written as a coproduct $\coprod \eta$ indexed by the elements of A_η. The map of (4) is then the corresponding product of maps of the form $k\mathbf{hom}(\eta, X) \to k\mathbf{hom}(\eta, Y)$, which are homotopy equivalences by assumption. For the induction step, consider a pushout square of the form

$$\begin{tikzcd}
\coprod_S \partial\Omega[T] \arrow[r] \arrow[d] & A \arrow[d] \\
\coprod_S \Omega[T] \arrow[r] & B
\end{tikzcd}$$

and the associated cube of Kan complexes

$$\begin{tikzcd}
k\mathbf{hom}(B,X) \arrow[rr] \arrow[dr] \arrow[dd] & & \prod_S k\mathbf{hom}(T,X) \arrow[dr] \arrow[dd] & \\
& k\mathbf{hom}(B,Y) \arrow[rr] \arrow[dd] & & \prod_S k\mathbf{hom}(T,Y) \arrow[dd] \\
k\mathbf{hom}(A,X) \arrow[rr] \arrow[dr] & & \prod_S k\mathbf{hom}(\partial T,X) \arrow[dr] & \\
& k\mathbf{hom}(A,Y) \arrow[rr] & & \prod_S k\mathbf{hom}(\partial T,Y).
\end{tikzcd}$$

In this cube, the front and back faces are pullbacks and the vertical maps are Kan fibrations. Then those faces are homotopy pullbacks in the Kan–Quillen model structure by the dual of Lemma 7.51. The three maps from back to front other than $k\mathbf{hom}(B,X) \to k\mathbf{hom}(B,Y)$ are homotopy equivalences by the inductive hypothesis and the assumption of the lemma. Corollary 7.50 guarantees that the remaining map is also a homotopy equivalence. For a general normal dendroidal set A, writing $A = \cup_n \mathrm{sk}_n A$ shows that $k\mathbf{hom}(A,X) \to k\mathbf{hom}(A,Y)$ is the inverse limit of the trivial fibrations $k\mathbf{hom}(\mathrm{sk}_n A, X) \to k\mathbf{hom}(\mathrm{sk}_n A, Y)$, hence itself a trivial fibration. □

Corollary 9.41 *Let $f: X \to Y$ be a map between ∞-operads. Then f is an operadic equivalence if and only if $k(i^*X) \to k(i^*Y)$ and $k\mathbf{hom}(C_n, X) \to k\mathbf{hom}(C_n, Y)$, for each $n \geq 0$, are homotopy equivalences between Kan complexes.*

Proof The implication from left to right follows from Proposition 9.39, where we have identified $\mathbf{hom}(\eta, X)$ with i^*X. For the converse, the preceding lemma shows that it suffices to prove that $k\mathbf{hom}(T,X) \to k\mathbf{hom}(T,Y)$ is a homotopy equivalence for each tree T. Consider the spine inclusion $\mathrm{Sp}[T] \to T$. This map is inner anodyne by Lemma 6.37, so the vertical maps in the square

$$\begin{tikzcd}
k\mathbf{hom}(T,X) \arrow[r] \arrow[d] & k\mathbf{hom}(T,Y) \arrow[d] \\
k\mathbf{hom}(\mathrm{Sp}[T],X) \arrow[r, "g"] & k\mathbf{hom}(\mathrm{Sp}[T],Y)
\end{tikzcd}$$

are trivial fibrations. It thus suffices to show that the lower map g is a homotopy equivalence. Observe that $k\mathbf{hom}(\mathrm{Sp}[T], X)$ can be written as a pullback of Kan complexes of the form $k\mathbf{hom}(C_n, X)$ for all the vertices of T and $k\mathbf{hom}(\eta, X)$ for all the inner edges of T. Moreover, since the evaluation maps $k\mathbf{hom}(C_n, X) \to k\mathbf{hom}(\eta, X)$ at the various edges of C_n are Kan fibrations (and similarly for Y), these pullbacks are homotopy pullbacks (cf. Lemma 7.51). Using the assumption of the corollary we conclude that g is a homotopy equivalence. □

For an ∞-operad X, we sometimes refer to the elements of X_η as the *colours* (or the *objects*) of X. Note that these coincide with the objects of the ∞-category i^*X, i.e., the elements of the set $(i^*X)_0$.

Definition 9.42 (a) For an ∞-operad X and colours $x_1, \ldots x_n, y$ of X, the *space of operations* $X(x_1, \ldots, x_n; y)$ from $x_1, \ldots, x_n$ to y is the pullback

$$\begin{tikzcd}
X(x_1, \ldots, x_n; y) \arrow[r] \arrow[d] & \mathbf{hom}(C_n, X) \arrow[d] \\
\Delta[0] \arrow[r, "(x_1, \ldots, x_n, y)"] & \mathbf{hom}(\partial C_n, X) = (i^*X)^{n+1},
\end{tikzcd}$$

where the map on the bottom is given by $x_1, \ldots, x_n$ for the leaves of C_n and y for the root edge of C_n.

(b) A map $f\colon X \to Y$ between ∞-operads is *fully faithful* if for any tuple of colours $x_1, \ldots, x_n, y$ of X, the induced map

$$X(x_1, \ldots, x_n; y) \to Y(f(x_1), \ldots f(x_n); f(y))$$

is a weak homotopy equivalence.

(c) A map $f\colon X \to Y$ between ∞-operads is *essentially surjective* if the map of underlying ∞-categories $i^*X \to i^*Y$ is essentially surjective, or in other words if $\tau(i^*f)\colon \tau(i^*X) \to \tau(i^*Y)$ is an essentially surjective functor between ordinary categories.

Remark 9.43 As was the case with mapping spaces in ∞-categories, we observe that the simplicial sets $X(x_1, \ldots, x_n; y)$ are in fact Kan complexes. Indeed, they fit into pullback squares

$$\begin{tikzcd}
X(x_1, \ldots, x_n; y) \arrow[r] \arrow[d] & k\mathbf{hom}(C_n, X) \arrow[d] \\
\Delta[0] \arrow[r, "(x_1, \ldots, x_n, y)"] & k\mathbf{hom}(\partial C_n, X),
\end{tikzcd}$$

where the right-hand vertical map is a Kan fibration. Also, these mapping spaces are related to the sets of operations in the homotopy operad τX by natural isomorphisms

$$(\tau X)(x_1, \ldots, x_n; y) \cong \pi_0 X(x_1, \ldots, x_n; y).$$

Remark 9.44 Let $y_1, \ldots, y_m, z$ and $x_1, \ldots, x_n, y_i$, with $1 \le i \le m$, be two sequences of colours in an ∞-operad X with common element y_i. Write $C_m \circ_i C_n$ for the tree obtained by grafting the n-corolla C_n onto the ith leaf of C_n. Then the maps

$$\Omega[C_{n+m-1}] \xrightarrow{\partial_i} \Omega[C_m \circ_i C_n] \xleftarrow{j} \Omega[C_m] \cup_i \Omega[C_n]$$

induce maps

$$\mathbf{hom}(C_{n+m-1}, X) \leftarrow \mathbf{hom}(C_m \circ_i C_n) \xrightarrow{j^*} \mathbf{hom}(C_m, X) \times_{i^*X} \mathbf{hom}(C_n, X),$$

and j^* is a trivial fibration, because j is inner anodyne. A section of this map then yields a map

$$\begin{array}{c} X(y_1, \ldots, y_m; z) \times X(x_1, \ldots, x_n; y_i) \\ \downarrow \\ X(x_1, \ldots, x_{i-1}, y_1, \ldots, y_m, x_{i+1}, \ldots, x_n; z), \end{array}$$

unique up to homotopy. This can be thought of as a composition operation (up to homotopy) for the ∞-operad X. In particular, upon taking π_0 of the Kan complexes involved, we retrieve the composition of the homotopy operad τX.

The proof of the following is very similar to that of Theorem 8.74.

Theorem 9.45 *A map $f: X \to Y$ between ∞-operads is an operadic equivalence if and only if it is fully faithful and essentially surjective.*

Proof If f is an operadic equivalence, then it follows easily from Proposition 9.39 that f is indeed fully faithful and essentially surjective. Conversely, suppose f is fully faithful and essentially surjective. Then in particular $i^* f$ is an equivalence of ∞-categories by Theorem 8.74, so that $k(i^* f): k(i^* X) \to k(i^* Y)$ is a weak homotopy equivalence of Kan complexes. By Corollary 9.41, it now suffices to show that for each $n \geq 0$ the maps $k\mathbf{hom}(C_n, X) \to k\mathbf{hom}(C_n, Y)$ induced by f are weak homotopy equivalences. These maps fit into a square

$$\begin{tikzcd} k\mathbf{hom}(C_n, X) \arrow[r] \arrow[d] & k\mathbf{hom}(C_n, Y) \arrow[d] \\ k\mathbf{hom}(\partial C_n, X) \arrow[r] & k\mathbf{hom}(\partial C_n, Y) \end{tikzcd}$$

in which the vertical arrows are Kan fibrations. The bottom map can be identified with the product $k(i^* X)^{n+1} \to k(i^* Y)^{n+1}$ and is therefore a weak homotopy equivalence. For any point $(x_1, \ldots x_n, y) \in k(i^* X)^{n+1}$, the corresponding map of fibres of the vertical maps over $(x_1, \ldots, x_n, y)$ and $(f(x_1), \ldots, f(x_n), f(y))$ is the map

$$X(x_1, \ldots, x_n; y) \to Y(f(x_1), \ldots, f(x_n); f(y)),$$

which is a weak homotopy equivalence by assumption. From the long exact sequence of homotopy groups of a Kan fibration we now conclude that the top horizontal arrow in the square must be a weak equivalence as well, completing the proof. □

9.3 Open and Uncoloured Dendroidal Sets

The goal of this section is to show how the operadic model structure induces similar model structures on the categories of open and of uncoloured dendroidal sets. The first of these two cases is very simple. Recall from Section 3.5.4 that the category

odSets of open dendroidal sets is the category of presheaves on the full subcategory $\mathbf{\Omega}^{\circ} \subseteq \mathbf{\Omega}$ consisting of open trees. This category can be identified with a slice category of the category of dendroidal sets itself,

$$\mathbf{odSets} \simeq \mathbf{dSets}/O,$$

where O is the dendroidal set defined by

$$O_T = \begin{cases} * & \text{if } T \text{ is open,} \\ \varnothing & \text{otherwise.} \end{cases} \tag{9.1}$$

Recall also the associated pair of adjoint functors

$$o_! : \mathbf{odSets} \rightleftarrows \mathbf{dSets} : o^*$$

with $o_!$ fully faithful, and the fact that under the equivalence 9.1 above the functor $o_!$ can be identified with the forgetful functor. Using the induced model structure on slice categories (cf. Example (d)(ii) at the end of Section 7.1), we immediately obtain the following consequence of Theorem 9.32:

Corollary 9.46 *The category* **odSets** *of open dendroidal sets carries a left proper cofibrantly generated model structure, in which the cofibrations are the normal monomorphisms and in which the weak equivalences are the operadic equivalences. The fibrant objects are the open ∞-operads and the fibrations between fibrant objects are the inner fibrations with J-path lifting.*

Proof All the statements of the corollary are general facts about model structures on slice categories, except the one about fibrant objects. Indeed, by definition a fibrant object of $\mathbf{dSets}/O$ is an operadic fibration $p\colon X \longrightarrow O$. Note that for an open dendroidal set X the map $p\colon X \longrightarrow O$ is unique, and that O itself is an ∞-operad. Hence p is an inner fibration if and only if X is an ∞-operad. The fact that p has J-path lifting is trivial, because $i^*O \cong \Delta[0]$. □

We have seen in Section 6.3 that the tensor product of open dendroidal sets behaves well with respect to inner anodyne morphisms. In fact, we have the following:

Proposition 9.47 *For normal monomorphisms $u\colon A \longrightarrow B$ and $v\colon C \longrightarrow D$ between open dendroidal sets, the pushout-product*

$$B \otimes C \cup A \otimes D \to B \otimes D$$

is a normal monomorphism, which is moreover a trivial operadic cofibration whenever u or v is.

Proof The pushout-product is a normal monomorphism by Proposition 4.26. We claim that the pushout-product is J-anodyne whenever one of the two maps (say u) is J-anodyne (see Definition 9.35). By the usual arguments involving saturated classes it suffices to check this when u is inner anodyne or u is the inclusion $\{0\} \to J$. The case of inner anodynes is covered by Corollary 6.26. The case $\{0\} \to J$ is covered by Remark 9.36.

To get the case of a general trivial operadic cofibration u from the case of J-anodynes treated above, we reason as in the proof of Proposition 9.28. Instead of the simplicial mapping objects $\mathbf{hom}(-,-)$ we now use the 'inner hom' $\mathbf{Hom}(-,-)$ of dendroidal sets, characterized by the property that $\mathbf{Hom}(A,-)$ is right adjoint to $A\otimes -$ as a functor from the category **dSets** to itself. To show that the pushout-product of u and v is a trivial cofibration, it suffices to show that any lifting problem

$$\begin{tikzcd} B \otimes C \cup A \otimes D \to B \otimes D \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B \otimes D \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

admits a solution, where p is an inner fibration with J-path lifting between ∞-operads. The right-hand vertical map in the adjoint lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathbf{Hom}(D,X) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & \mathbf{Hom}(D,Y) \times_{\mathbf{Hom}(C,Y)} \mathbf{Hom}(C,X) \end{tikzcd}$$

is still an inner fibration with J-path lifting between ∞-operads, as a consequence of the first part of this proof. Hence it is a categorical fibration and a lift exists by the assumption that u is a trivial cofibration. $\square$

As a consequence of Proposition 9.47, the tensor product functor

$$A \otimes - \colon \mathbf{odSets} \to \mathbf{odSets}$$

preserves cofibrations and trivial cofibrations for the operadic model structure, whenever A is a cofibrant (i.e., normal) dendroidal set. By Brown's lemma, the functor $A \otimes -$ therefore preserves weak equivalences between cofibrant objects. Consequently, the restriction of the tensor product functor

$$- \otimes - \colon \mathbf{odSets} \times \mathbf{odSets} \to \mathbf{odSets}$$

to cofibrant objects respects weak equivalences. Therefore it descends to give a well-defined tensor product on the homotopy category:

$$- \otimes - \colon \mathrm{Ho}(\mathbf{odSets}) \times \mathrm{Ho}(\mathbf{odSets}) \to \mathrm{Ho}(\mathbf{odSets}).$$

Remark 9.48 Although the tensor product of dendroidal sets itself is not quite associative, this tensor product on the homotopy category *does* give rise to a symmetric monoidal structure. This follows from Proposition 6.32, which states that the relevant 'associator maps' for the tensor product, when evaluated on corollas, are inner anodyne. The case of general normal dendroidal sets follows by first reducing to representables, using skeletal induction and the cube lemma, and then reducing from general trees to corollas using the spine.

We now turn to the case of uncoloured dendroidal sets. Recall from Section 3.5.6 that a dendroidal set X is *uncoloured* if X_η is a single point. The uncoloured dendroidal sets form a full subcategory **udSets** of the category **dSets**, related by adjoint functors to the category $\mathbf{dSets}_* = \eta/\mathbf{dSets}$ of pointed dendroidal sets, of which the objects are pairs (X, x_0) with $x_0 \in X_\eta$ a chosen basepoint. These adjoint functors are denoted

$$\begin{tikzcd} \mathbf{dSets}_* \arrow[r, bend left, "r_!"] \arrow[r, bend right, "r_*"'] & \mathbf{udSets}. \arrow[l, "r^*"'] \end{tikzcd}$$

The functor r^* is simply the forgetful functor. Its left and right adjoints $r_!$ and r_* can be described explicitly as in Section 3.5.6; $r_!$ collapses all of X_η to a single point, whereas r_* restricts to the basepoint, meaning it retains only those dendrices in X_T for which each edge has colour $x_0 \in X_\eta$.

Uncoloured dendroidal sets relate to uncoloured operads just like dendroidal sets relate to operads. In particular, there is an adjoint pair of functors

$$\mathbf{udSets} \underset{N}{\overset{\tau}{\rightleftarrows}} \mathbf{uOp},$$

and a homotopy-coherent version for uncoloured simplicial operads

$$\mathbf{udSets} \underset{w^*}{\overset{w_!}{\rightleftarrows}} \mathbf{usOp},$$

which we will later show to be a Quillen equivalence. For now, we will show that the operadic model structure on **dSets** restricts to a model structure on **udSets**. Recall that the category $\mathbf{dSets}_* = \eta/\mathbf{dSets}$ of pointed dendroidal sets inherits a model structure from **dSets**, for which the forgetful functor $\mathbf{dSets}_* \to \mathbf{dSets}$ detects cofibrations, fibrations, and weak equivalences. Note also that an object (X, x_0) is cofibrant if $\eta \to X$ is a normal monomorphism, which is the case precisely if X itself is a normal dendroidal set.

Theorem 9.49 *The category* **udSets** *of uncoloured dendroidal sets carries a model structure in which a map $X \to Y$ is a cofibration (resp. a weak equivalence) if and only if its image under r^* is a cofibration (resp. a weak equivalence) in* $\mathbf{dSets}_*$*, or equivalently in* **dSets**. *This model structure is cofibrantly generated and left proper. Moreover, the fibrant objects are precisely the uncoloured ∞-operads. In other words, r^* preserves and detects fibrant objects.*

Remark 9.50 The reader should be warned that r^* does *not* preserve fibrations in general. Equivalently, $r_!$ does not preserve arbitrary trivial cofibrations. Indeed, the image of the trivial cofibration $\{0\} \to J$ under $r_!$ is the map of simplicial sets $\{0\} \to J/\partial J$. The simplicial set $J/\partial J$ is not weakly contractible (it has the weak homotopy type of the circle), so this map cannot be an operadic equivalence.

Proof (of Theorem 9.49) We take the cofibrations and weak equivalences on **udSets** as in the statement of the theorem and define the fibrations to be the maps having the right lifting property with respect to trivial cofibrations. Let us verify the axioms for a model structure. Axioms (M1–3) are evident, as is one half of the lifting axiom (M4): for a square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

in which i is a trivial cofibration and p is a fibration, a lift exists by definition of the fibrations. The other half, meaning the case where i is a cofibration and p is a fibration and a weak equivalence, follows by the retract argument. Let us briefly summarize it again. Factor p as $q \circ u$ where u is a normal monomorphism and q has the right lifting property with respect to normal monomorphisms. (We will establish the existence of this factorization in the category of uncoloured dendroidal sets below.) Then q is a weak equivalence, so u is a trivial cofibration by two-out-of-three. The fact that p has the right lifting property with respect to u now implies that p is a retract of q, and therefore also has the right lifting property with respect to normal monomorphisms.

It remains to verify the factorization axiom (M5). The factorization of an arbitrary morphism into a normal monomorphism followed by a trivial fibration was already used above; to construct it for a general morphism $f : X \to Y$ of uncoloured dendroidal sets, first factor it as a normal monomorphism i followed by a trivial fibration p in the category $\mathbf{dSets}_*$ of pointed dendroidal sets, and subsequently apply the functor r_*. Clearly $r_*X = X$ and $r_*Y = Y$. Furthermore, r_* preserves normal monomorphisms, so r_*i is a normal monomorphism. To see that r_*p is a trivial fibration, note that for any normal monomorphism u between uncoloured dendroidal sets, p has the right lifting property with respect to the normal monomorphism r^*u. Consequently, r_*p has the right lifting property with respect to u.

To obtain the other factorization, into a trivial cofibration followed by a fibration, we use the small object argument again. To do this we need to know that the trivial cofibrations are generated, as a saturated class, by a *set*. As in Section 9.1 one may take the trivial cofibrations between countable uncoloured dendroidal sets as such a generating set.

Finally, we will identify the fibrant objects as the uncoloured ∞-operads. If X is an uncoloured ∞-operad, then it is fibrant as a dendroidal set and thus has the right lifting property with respect to all trivial cofibrations; in particular, it has the right lifting property with respect to trivial cofibrations between uncoloured dendroidal sets and is therefore a fibrant object of **udSets**. Conversely, assume that $X \in$ **udSets** is fibrant. Observe that any inner horn inclusion $\Lambda^e[T] \to T$ is bijective on edges, from which it easily follows that the square

$$\begin{tikzcd}
\Lambda^e[T] \arrow[r] \arrow[d] & r_!\Lambda^e[T] \arrow[d] \\
T \arrow[r] & r_!T
\end{tikzcd}$$

is a pushout and the vertical map on the right is a trivial cofibration of uncoloured dendroidal sets. Thus X has the right lifting property with respect to such a map. Consequently, r^*X has the right lifting property with respect to the inner horn inclusion on the left and r^*X is indeed an uncoloured ∞-operad. □

9.4 The Relative $\mathcal{A}$-Model Structure

In this section we will discuss a variation on the general method from Section 9.1 in order to construct relative versions of $\mathcal{A}$-model structures on the slice category $\mathbf{dSets}/V$ of dendroidal sets over a fixed 'base' V. For two objects $f\colon A \to V$ and $g\colon X \to V$ of this category, the simplicial set

$$\mathbf{hom}_V(A, X)$$

is defined as follows: its n-simplices are maps $\Delta[n] \otimes A \to X$ for which the diagram

$$\begin{tikzcd}
\Delta[n] \otimes A \arrow[r] \arrow[d] & X \arrow[d, "g"] \\
A \arrow[r, "f"] & V
\end{tikzcd}$$

commutes. In other words, $\mathbf{hom}_V(A, X)$ is the pullback

$$\begin{tikzcd}
\mathbf{hom}_V(A, X) \arrow[r] \arrow[d] & \mathbf{hom}(A, X) \arrow[d, "-\circ g"] \\
\Delta[0] \arrow[r, "f"] & \mathbf{hom}(A, V).
\end{tikzcd}$$

Then any map $A \to B$ of dendroidal sets over V induces a map of simplicial sets

$$\mathbf{hom}_V(B, X) \to \mathbf{hom}_V(A, X).$$

Now let $\mathcal{A}$ be a set of normal monomorphisms between normal objects of the category $\mathbf{dSets}/V$. As in Section 9.1, we call $\mathcal{A}$ *admissible* if for any $A \to B$ in $\mathcal{A}$ and any $n \geq 0$, the map

$$\Delta[n] \otimes A \cup \partial\Delta[n] \otimes B \to \Delta[n] \otimes B$$

(with its evident map to V) again belongs to $\mathcal{A}$. We define an object $X \to V$ over V to be $\mathcal{A}$-*local* if it has the following two properties:

(1) For any normal monomorphism $A \to B$ over V, the map

$$\mathbf{hom}_V(B, X) \to \mathbf{hom}_V(A, X).$$

is a categorical fibration of simplicial sets.

(2) Moreover, it is a trivial fibration whenever $A \to B$ belongs to $\mathcal{A}$.

Thus in particular, if A is a normal dendroidal set over V and $X \to V$ is $\mathcal{A}$-local, then $\mathbf{hom}_V(A, X)$ is an ∞-category. As in Section 9.1, we can reformulate the property of being $\mathcal{A}$-local over V in terms of lifting properties:

Lemma 9.51 *An object $X \to V$ is $\mathcal{A}$-local if its has the right lifting property with respect to the following maps over V:*

(i) *The maps*

$$\Lambda^k[n] \otimes B \cup \Delta[n] \otimes A \to \Delta[n] \otimes B,$$

for any normal monomorphism $A \to B$ over V and any $0 < k < n$.

(ii) *The maps*

$$\{0\} \otimes B \cup J \otimes A \to J \otimes B,$$

for any normal monomorphism $A \to B$ over V.

(iii) *The maps*

$$\partial\Delta[n] \otimes B \cup \Delta[n] \otimes A \to \Delta[n] \otimes B,$$

for any $A \to B$ over V contained in the set $\mathcal{A}$ and any $n \geq 0$.

The following terminology extends that of Section 9.1 to the relative case:

Definition 9.52 A normal monomorphism $A \to B$ over V is $\mathcal{A}$-*anodyne* (over V) if it lies in the saturation of the classes of maps (i)–(iii) above.

So an object $X \to V$ is $\mathcal{A}$-local if and only if it has the right lifting property with respect to all $\mathcal{A}$-anodyne maps over V.

Definition 9.53 A morphism $A \to B$ over V is an $\mathcal{A}$-*weak equivalence* if it has a normalization $A' \to B'$ with the property that for any $\mathcal{A}$-local object $X \to V$, the map

$$\mathbf{hom}_V(B', X) \to \mathbf{hom}_V(A', X)$$

is a categorical equivalence between ∞-categories.

Remark 9.54 The uniqueness of normalizations up to J-homotopy of Lemma 9.6 implies that the choice of normalization in the above definition is irrelevant.

Theorem 9.55 *Let $\mathcal{A}$ be an admissible set of normal monomorphisms over V. Then there exists a model structure on* $\mathbf{dSets}/V$ *with the following properties:*

(a) *The cofibrations are the normal monomorphisms over V.*
(b) *The weak equivalences are the $\mathcal{A}$-weak equivalences just defined.*
(c) *The fibrant objects are the $\mathcal{A}$-local objects over V.*
(d) *A map between $\mathcal{A}$-local objects over V is a fibration if and only if it has the right lifting property with respect to $\mathcal{A}$-anodyne maps over V.*

Moreover, this model structure is left proper and cofibrantly generated.

We will refer to the fibrations in the model structure of the theorem as $\mathcal{A}$*-fibrations* (over V, if necessary).

Proof The proof proceeds along the same lines as that of Theorem 9.9. As we did then, we define a map in $\mathbf{dSets}/V$ to be a fibration precisely if it has the right lifting property with respect to all normal monomorphisms over V which are also $\mathcal{A}$-weak equivalences. With these definitions in place, axioms (M1–3) are clearly satisfied. (For the two-out-of-three axiom (M2) one uses the evident version of Lemma 9.10 for maps over V.) The factorization axiom (M5) is proved by the small object argument. The factorization into a normal monomorphism followed by a map having the right lifting property with respect to normal monomorphisms proceeds as in Section 9.1, using the fact that the normal monomorphisms are generated as a saturated class by boundary inclusions of trees. Moreover, a map over V having the right lifting property with respect to all monomorphisms will have a normalization which is a J-homotopy equivalence over V, hence a weak equivalence, as in Lemma 9.11. For the other factorization into a trivial cofibration followed by a fibration, we assume again that for all maps $A \to B$ in $\mathcal{A}$ the dendroidal sets A and B are finite. (This assumption is unnecessary, but the general case requires a bit more set theory and induction over larger ordinals, cf. Remark 9.31). We can then use the small object argument again, taking as a set of generating trivial cofibrations the $\mathcal{A}$-trivial cofibrations between countable dendroidal sets over V. The justification of this is identical to that in Section 9.1.

Finally we should prove the lifting axiom (M4). Half of it is automatic from the definition of the fibrations, the other half is proved by the retract argument, of which we have seen several examples by now (cf. the proof of Theorem 9.49 in the previous section). The characterization of the fibrant objects and fibrations between them proceeds exactly as in Section 9.1. □

We will see several examples of $\mathcal{A}$-model structures relative to a base V in the remainder of this chapter. For now, we will limit ourselves to some remarks about various functors relating to change of base.

Suppose $\mathcal{A}$ is an admissible set of normal monomorphisms over V. Then for any map $f: V \to W$ between dendroidal sets, we obtain a set $f_!\mathcal{A}$ of maps over W, simply by regarding each map $A \to B$ over V belonging to $\mathcal{A}$ as a map over W by composing with f. Then clearly $f_!\mathcal{A}$ is again admissible. Let us denote the relative $\mathcal{A}$-model structure on $\mathbf{dSets}/V$ by $(\mathbf{dSets}/V)_{\mathcal{A}}$ and the relative $f_!\mathcal{A}$-model structure on $\mathbf{dSets}/W$ by $(\mathbf{dSets}/W)_{f_!\mathcal{A}}$. There is an adjoint pair of functors

$$\mathbf{dSets}/V \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathbf{dSets}/W$$

given by composition with f and pullback along f. The functor $f_!$ is simplicial, in the sense that for any map of dendroidal sets $A \to V$ and any simplicial set M, the canonical map

$$f_!(M \otimes A) \to M \otimes f_!(A)$$

is an isomorphism over W (indeed, it is essentially the identity map). It follows by adjunction that for every $X \to W$ the canonical map

$$\mathbf{hom}_W(f_!A, X) \to \mathbf{hom}_W(A, f^*X)$$

is an isomorphism as well. In particular, $X \to W$ is $f_!\mathcal{A}$-local over W if and only if $f^*X \to V$ is $\mathcal{A}$-local over V. Using this observation, we easily conclude the following:

Proposition 9.56 *The adjoint pair*

$$(\mathbf{dSets}/V)_{\mathcal{A}} \underset{f^*}{\overset{f_!}{\rightleftarrows}} (\mathbf{dSets}/W)_{f_!\mathcal{A}}$$

is a Quillen pair. Moreover, f^* detects fibrant objects and fibrations between them.

Proof Clearly $f_!$ preserves normal monomorphisms, hence cofibrations. Also, we just observed that an object $X \to W$ is $f_!\mathcal{A}$-local if and only if $f^*X \to V$ is $\mathcal{A}$-local. Thus, f^* preserves and detects fibrant objects and for the same reason fibrations between fibrant objects. The fact that $(f_!, f^*)$ is a Quillen pair now follows from Lemma 8.42. □

This concludes our brief discussion of the pushforward of an admissible set of morphisms over V along a map $f: V \to W$. Now let us consider the case where we pull back $\mathcal{A}$ along a map $f: W \to V$, giving a set of maps $f^*\mathcal{A}$ over W consisting of all pullbacks

$$\begin{tikzcd} A \times_V W \arrow[rr] \arrow[dr] & & B \times_V W \arrow[dl] \\ & W & \end{tikzcd}$$

of maps $A \to B$ in $\mathcal{A}$. The $f^*\mathcal{A}$-local objects are defined in terms of the simplicial sets $\mathbf{hom}_W(f^*A, X)$. Now note that since $f_!$ is simplicial, there is a natural map

$$f_!(\Delta[n] \otimes f^*A) \cong \Delta[n] \otimes f_!f^*A \to \Delta[n] \otimes A$$

for each $n \geq 0$ and hence by adjunction a map

$$\Delta[n] \otimes f^*A \to f^*(\Delta[n] \otimes A)$$

natural in n and in A. These maps together induce a natural map

$$\mathbf{hom}_V(A, f_*X) \to \mathbf{hom}_W(f^*A, X).$$

Proposition 9.57 *For any map of dendroidal sets $f\colon W \to V$ and maps $A \to V$, $X \to W$, the natural map*

$$\mathbf{hom}_V(A, f_*X) \to \mathbf{hom}_W(f^*A, X)$$

is an isomorphism.

Proof An n-simplex of $\mathbf{hom}_W(f^*A, X)$ is a map

$$\Delta[n] \otimes (W \times_V A) \to X$$

over W, while an n-simplex of $\mathbf{hom}_V(A, f_*X)$ is a map $\Delta[n] \otimes A \to f_*X$ over V or, equivalently, a map $f^*(\Delta[n] \otimes A) \to X$ over W. Consider the diagram

$$\begin{tikzcd}
\Delta[n] \otimes (W \times_V A) \arrow[r] \arrow[d] & \Delta[n] \otimes A \arrow[d] \\
W \times_V A \arrow[r] \arrow[d] & A \arrow[d] \\
W \arrow[r] & V.
\end{tikzcd}$$

By Corollary 4.17, the top square is a pullback and hence so is the rectangle. Thus

$$\Delta[n] \otimes (W \times_V A) \cong f^*(\Delta[n] \otimes A),$$

from which the proposition follows. □

Having established the previous proposition, we can now proceed as for Proposition 9.56 above and conclude the following by exactly the same argument:

Proposition 9.58 *Let $\mathcal{A}$ be an admissible set of morphisms over V and let $f\colon W \to V$ be any map. Then $f^*\mathcal{A}$ is again admissible and the adjoint pair*

$$(\mathbf{dSets}/V)_{f^*\mathcal{A}} \underset{f_*}{\overset{f^*}{\rightleftarrows}} (\mathbf{dSets}/W)_{\mathcal{A}}$$

is a Quillen pair. Moreover, f_* detects fibrant objects and fibrations between fibrant objects.

9.5 The Covariant Model Structure on Dendroidal Sets

In this section we will discuss a special case of the relative $\mathcal{A}$-model structure of the previous section, namely that of the *covariant model structure* on $\mathbf{dSets}/V$ for some fixed dendroidal set V. For the set $\mathcal{I}/V$ of inner horn inclusions over V there is a relative operadic model structure on $\mathbf{dSets}/V$. The covariant model structure over V is a localization of the relative operadic one, where besides the inner horn inclusions one also takes the inclusions of leaf horns into account. The use of this model structure will be that if V is an ∞-operad, the covariant model structure on $\mathbf{dSets}/V$ describes the homotopy theory of V*-algebras*. We will come to this relation in Section 14.8.

Recall that $\Lambda^x[T] \to \Omega[T]$ denotes the dendroidal subset of the representable dendroidal set $\Omega[T]$, given by the union of all the faces of T except $\partial_x T$. In Section 9.2 we focused on the case where x is an inner edge in T, so that $\Lambda^x[T] \to \Omega[T]$ is an inner horn inclusion. Here we will include the case where x is a leaf vertex of T, so that $\partial_x T$ is the face of T obtained by chopping off the vertex x and the leaves immediately above it (if any). If T is a corolla with a unique vertex v, we interpret $\Lambda^x[T]$ as the disjoint union of the leaves of T or, more precisely, the disjoint union of copies of η indexed by the leaves of T. It will be convenient to use a concise notation for this, which we introduced before; we will write $\lambda(T)$ for the set of leaves of T and $\ell[T] \subset \Omega[T]$ for the corresponding subobject of the dendroidal set represented by T. With this notation, the leaf horn of the n-corolla C_n is the inclusion

$$\ell[C_n] \to \Omega[C_n].$$

Let us denote the set of all horn inclusions $\Lambda^x[T] \to \Omega[T]$ where x is either an inner edge or a leaf vertex by $\mathcal{L}$. As in Definition 6.14, we call a normal monomorphism *leaf anodyne* if it belongs to the saturation $\overline{\mathcal{L}}$ of $\mathcal{L}$.

We recall from Corollary 6.30 that if $M \to N$ is a monomorphism between simplicial sets and $A \to B$ is a normal monomorphism between dendroidal sets, then the pushout-product map

$$M \otimes B \cup_{M \otimes A} N \otimes A \to N \otimes B$$

is a normal monomorphism, which is leaf anodyne whenever $A \to B$ or $M \to N$ is. (Recall that for simplicial sets, the notion of leaf anodyne map reduces to that of a left anodyne map.) We also recall that the maps between dendroidal sets having the right lifting property with respect to $\mathcal{L}$ are called left fibrations. For a normal monomorphism $A \to B$ and an inner fibration $X \to Y$, the map

$$\mathbf{hom}(B, X) \to \mathbf{hom}(A, X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B, Y)$$

is an inner fibration of simplicial sets, which is a left fibration if $X \to Y$ is, and a trivial fibration if in addition the map $A \to B$ is leaf anodyne. This fact is dual to our previous statement about pushout-products.

Theorem 9.55 now specializes to the following result:

Theorem 9.59 *Let V be a dendroidal set. The category* $\mathbf{dSets}/V$ *carries a left proper cofibrantly generated model structure with the following properties:*

(a) *The cofibrations are the normal monomorphisms over V.*
(b) *The fibrant objects are the left fibrations $X \to V$.*

(c) *The fibrations between fibrant objects are the left fibrations.*
(d) *A map $A \to B$ between normal objects over V is a weak equivalence if for any left fibration $X \to V$, the map*

$$\mathbf{hom}_V(B, X) \to \mathbf{hom}_V(A, X)$$

is a categorical equivalence of ∞-categories.

We will refer to the model structure of the theorem as the *covariant model structure over V* and denote the corresponding model category by $(\mathbf{dSets}/V)_{\mathrm{cov}}$.

Remark 9.60 The ∞-categories $\mathbf{hom}_V(B, X)$ and $\mathbf{hom}_V(A, X)$ featuring in item (d) of the theorem are in fact Kan complexes. Indeed, take $0 \leq k < n$ and consider a lifting problem as on the left, which is equivalent to the one depicted on the right:

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & \mathbf{hom}_V(A, X) \arrow[d] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & \Delta[0]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\Lambda^k[n] \otimes A \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta[n] \otimes A \arrow[r] \arrow[ur, dashed] & V
\end{tikzcd}$$

The left vertical arrow in the square on the right is leaf anodyne by Corollary 6.30, so that a lift exists by the assumption that $X \to V$ is a left fibration. We conclude that $\mathbf{hom}_V(A, X) \to \Delta[0]$ is a left fibration. But then it is also a Kan fibration (cf. Corollary 5.51).

Proof (of Theorem 9.59) Consider the set $\mathcal{L}/V$ of leaf horn inclusions over V, i.e. the commutative diagrams of the form

$$\begin{tikzcd}
\Lambda^x[T] \arrow[rr] \arrow[dr] & & \Omega[T] \arrow[dl] \\
& V, &
\end{tikzcd}$$

where x is an inner edge or a leaf vertex of the tree T. Let $\overline{\mathcal{L}}/V$ be the saturation in the category $\mathbf{dSets}/V$ and let $\mathcal{A}$ be an admissible set with $\mathcal{L}/V \subseteq \mathcal{A} \subseteq \overline{\mathcal{L}}/V$. For example, $\mathcal{A}$ could consist of all the map $A \to B$ in $\overline{\mathcal{L}}/V$ between finite dendroidal sets. Then by Theorem 9.55 there is a left proper cofibrantly generated model structure on $\mathbf{dSets}/V$ in which the cofibrations are the normal monomorphisms over V. The $\mathcal{A}$-local objects are those $X \to V$ which have the right lifting property with respect to the following three classes of maps over V:

(i) The maps

$$N \otimes A \cup M \otimes B \to N \otimes B,$$

for any normal monomorphism $A \to B$ over V and any inner anodyne map $M \to N$ between simplicial sets.

(ii) The maps

$$J \otimes A \cup \{0\} \otimes B \to J \otimes B,$$

for any normal monomorphism $A \to B$ over V.

(iii) The maps

$$N \otimes A \cup M \otimes B \to N \otimes B$$

for any map $A \to B$ in $\overline{\mathcal{L}}/V$ and any monomorphism $M \to N$ between simplicial sets.

But the classes (i) and (ii) are contained in (iii). For (ii), this follows from the fact that $\{0\} \to J$ is left anodyne (cf. Proposition 5.22). Also, (iii) is contained in $\overline{\mathcal{L}}/V$. This shows that the local objects are precisely the left fibrations. The rest of the theorem is now clear from Theorem 9.55. □

Corollary 9.61 *The covariant model structure on* $\mathbf{dSets}/V$ *is a left Bousfield localization of the relative operadic model structure on* $\mathbf{dSets}/V$.

Proof The two model structures have the same cofibrations and the class of J-anodyne morphisms over V is (by definition) contained in the class of leaf anodyne morphisms over V. □

Next, we observe the following statements concerning change of base:

Proposition 9.62 (a) *Let* $f: V \to W$ *be a map of dendroidal sets. Then* f *induces a Quillen pair*

$$f_! : (\mathbf{dSets}/V)_{\mathrm{cov}} \longrightarrow (\mathbf{dSets}/W)_{\mathrm{cov}} : f^*.$$

(b) *If* $f: V \to W$ *is a left fibration, then the covariant model structure* $(\mathbf{dSets}/V)_{\mathrm{cov}}$ *agrees with the slice model structure* $(\mathbf{dSets}/W)_{\mathrm{cov}}/f$.

(c) *If* $f: V \to W$ *is an operadic equivalence between* ∞*-operads then the pair in (a) is a Quillen equivalence.*

Proof (a) This is a special case of Proposition 9.56.

(b) The fibrant objects in the slice model structure on $(\mathbf{dSets}/W)_{\mathrm{cov}}/f$ are the commutative diagrams

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr] & & V \arrow[dl] \\ & W & \end{tikzcd}$$

in which $g: X \to V$ is a fibration in the covariant model structure on $\mathbf{dSets}/W$. Since f is a left fibration, $V \to W$ is a fibrant object of $(\mathbf{dSets}/W)_{\mathrm{cov}}$. Hence g is a fibration in the covariant model structure over W precisely if it is a left fibration, by Theorem 9.59(c). It follows that $(\mathbf{dSets}/W)_{\mathrm{cov}}/f$ and $(\mathbf{dSets}/V)_{\mathrm{cov}}$ have the same cofibrations and the same fibrant objects, so that the two model structures must coincide.

(c) By Brown's lemma it suffices to prove this if $f: V \to W$ is a trivial fibration in the operadic model structure. But then f is a trivial fibration in $(\mathbf{dSets}/W)_{\mathrm{cov}}$ as well, so that

$$(\mathbf{dSets}/W)_{\mathrm{cov}}/f \underset{f^*}{\overset{f_!}{\rightleftarrows}} (\mathbf{dSets}/W)_{\mathrm{cov}}$$

is a Quillen equivalence (cf. Example 8.47(i)). The result follows by part (b). □

The weak equivalences between fibrant objects in the covariant model structure can be characterized very efficiently as those maps which induce fibrewise weak homotopy equivalences, as in part (5) of the following:

Theorem 9.63 *Consider a map*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & V & \end{tikzcd}$$

between left fibrations $X \to V$ and $Y \to V$ of dendroidal sets. Then the following are equivalent:

(1) *The map f is a weak equivalence in the covariant model structure over V.*
(2) *Any normalization f' of f is a fibrewise J-homotopy equivalence over V.*
(3) *For any normal dendroidal set A over V, the map*

$$\mathbf{hom}_V(A, X) \to \mathbf{hom}_V(A, Y)$$

is a weak homotopy equivalence of Kan complexes.
(4) *For every colour $v \in V_\eta$, the map $X_v \to Y_v$ between fibres over v is a weak homotopy equivalence of Kan complexes.*

In part (4), X_v denotes the pullback

$$\begin{tikzcd} X_v \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[0] \arrow[r, "v"] & V. \end{tikzcd}$$

Note that X_v is indeed a Kan complex, because a left fibration over $\eta = \Delta[0]$ is automatically a Kan fibration (recall that we are suppressing the inclusion $i_!$ of simplicial sets into dendroidal sets from the notation). To prove the theorem it will be convenient to have the following characterization of left fibrations:

Proposition 9.64 *An inner fibration $f : X \to Y$ of dendroidal sets is a left fibration if and only if for any $n \geq 0$, the map*

$$\mathbf{hom}(C_n, X) \to \mathbf{hom}(\ell[C_n], X) \times_{\mathbf{hom}(\ell[C_n], Y)} \mathbf{hom}(C_n, Y)$$

is a trivial fibration of simplicial sets.

Proof If f is a left fibration, then the map of the lemma is a trivial fibration by Theorem 6.33. Conversely, suppose f is an inner fibration for which the map of the lemma is a trivial fibration for any $n \geq 0$. For a normal monomorphism $A \longrightarrow B$ of dendroidal sets, consider the map

$$\mathbf{hom}(B, X) \to \mathbf{hom}(A, X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B, Y).$$

It is an inner fibration between simplicial sets by Theorem 6.33. Consider the class $\mathcal{C}$ of normal monomorphisms $A \longrightarrow B$ for which this map is a trivial fibration. Then $\mathcal{C}$ is saturated and closed under-two-out-three among normal monomorphisms. It contains the inner anodynes, again by Theorem 6.33, and the maps $\ell[C_n] \longrightarrow \Omega[C_n]$ by assumption. Then Proposition 6.41 guarantees that $\mathcal{C}$ contains all leaf anodyne maps. Since any trivial fibration is in particular surjective on vertices, it follows that $X \longrightarrow Y$ has the right lifting property with respect to leaf anodyne maps, so that it is a left fibration. □

Proof (of Theorem 9.63) The equivalence between statements (1)-(3) is proved exactly as for Proposition 9.39. Moreover, as in Corollary 9.41, f is a weak equivalence between left fibrations if and only if the following two statements are true:

(3a) For any map $\eta \longrightarrow X$, the map

$$\mathbf{hom}_V(\eta, X) \to \mathbf{hom}_V(\eta, Y)$$

is a weak homotopy equivalence between Kan complexes.

(3b) For any $n \geq 0$ and any map $C_n \longrightarrow X$, the map

$$\mathbf{hom}_V(C_n, X) \to \mathbf{hom}_V(C_n, Y)$$

is a weak homotopy equivalence between Kan complexes.

But by Proposition 9.64, the vertical maps in the square

$$\begin{tikzcd}
\mathbf{hom}_V(C_n, X) \arrow[r] \arrow[d] & \mathbf{hom}_V(C_n, Y) \arrow[d] \\
\mathbf{hom}_V(\ell[C_n], X) \arrow[r] & \mathbf{hom}_V(\ell[C_n], Y)
\end{tikzcd}$$

are trivial fibrations. Since $\mathbf{hom}_V(\ell[C_n], X) = \mathbf{hom}_V(\eta, X)^n$ and similarly for Y, (3b) follows from (3a). Furthermore (3a) is the same assertion as (4), because for a given map $v \colon \eta \longrightarrow X$, the simplicial set $\mathbf{hom}_V(\eta, X)$ is precisely the Kan complex X_v (and similarly for Y). □

Corollary 9.65 *Consider a diagram*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\
& V &
\end{tikzcd}$$

of dendroidal sets. If $X \to V$ and $Y \to V$ are left fibrations and the map of fibres $X_v \to Y_v$ is a weak homotopy equivalence of Kan complexes for each $v \in V_\eta$, then f is an operadic equivalence.

Proof The map f is an equivalence in the covariant model structure over V by Theorem 9.63. It is then also a weak equivalence in the relative operadic model structure over V by Corollary 9.61 (the covariant model structure is a localization of the operadic one) and Lemma 8.49 (the local weak equivalences between fibrants are the usual weak equivalences). The forgetful functor $(\mathbf{dSets}/V)_{\mathrm{cov}} \to (\mathbf{dSets})_{\mathrm{cov}}$ is left Quillen and therefore preserves weak equivalences between cofibrant objects; since it also preserves normalizations, it preserves weak equivalences between arbitrary objects. In particular, f is an operadic equivalence. □

We have seen in Proposition 9.28 that the $\mathcal{A}$-model structure is 'enriched' over the Joyal model structure in an appropriate sense. The same is true of the covariant model structure, but in fact we have the following stronger result in this particular case, replacing the Joyal model structure with the Kan–Quillen model structure. It can be seen as a strengthening of Remark 9.60, which states that $\mathbf{hom}_V(A, X)$ is a Kan complex whenever A is a normal dendroidal set over V and $X \to V$ is a left fibration.

Proposition 9.66 *For a monomorphism $i\colon M \to N$ of simplicial sets and a normal monomorphism of dendroidal sets $j\colon A \to B$ over V, the pushout-product*

$$N \otimes A \cup M \otimes B \to N \otimes B$$

is a normal monomorphism over V, which is moreover a covariant weak equivalence over V whenever i is a weak homotopy equivalence or j is a covariant weak equivalence over V.

Dually, if $p\colon X \to Y$ is a fibration over V in the covariant model structure, then the induced map

$$\mathbf{hom}_V(B, X) \to \mathbf{hom}_V(B, Y) \times_{\mathbf{hom}_V(A,Y)} \mathbf{hom}_V(A, X)$$

is a Kan fibration of simplicial sets, which is a trivial fibration whenever j or p is a covariant weak equivalence over V.

Proof For the first part, fix $j\colon A \to B$ and consider the class $\mathcal{C}$ of all monomorphisms $i\colon M \to N$ for which the pushout-product of i with j is a covariant trivial cofibration. This class is saturated, closed under two-out-of-three among monomorphisms, and contains the left anodyne maps of simplicial sets by Corollary 6.30. But then $\mathcal{C}$ contains all trivial cofibrations of simplicial sets in the Kan–Quillen model structure. Indeed, if $u\colon A \to B$ is such a trivial cofibration, factor the unique map $B \to \Delta[0]$ as a left anodyne $B \to C$ followed by a left fibration $C \to \Delta[0]$. Then C is a Kan complex (see Corollary 5.51). By two-out-of-three it suffices to show that the composite trivial cofibration $A \to C$ is in $\mathcal{C}$. Factor it again as a left anodyne $v\colon A \to D$ followed by a left fibration $q\colon D \to C$. Then q is in fact a Kan fibration by Corollary 5.50. Thus a lift in the square

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, tail, "\wr"'] & D \arrow[d, two heads, "q"] \\ C \arrow[ur, dashed] \arrow[r, equal] & C \end{tikzcd}$$

exists and exhibits the map $A \to C$ as a retract of v, completing the argument.

For the other case, fix a monomorphism i of simplicial sets and consider the class $\mathcal{C}$ of normal monomorphisms $j\colon A \to B$ over V for which the pushout-product with i is a covariant trivial cofibration. It contains the leaf anodyne morphisms, again by Corollary 6.30, and satisfies the same closure properties as above. A completely analogous factorization and lifting argument shows that $\mathcal{C}$ must contain all the covariant trivial cofibrations over V.

The second half of the proposition follows from the first by adjunction. □

We conclude this section with an example. It concerns operads and their algebras in **Sets**, but the observation to be made (Proposition 9.67 below) lies at the basis of an equivalence of homotopy theories between $(\mathbf{dSets}/w^*\mathbf{P})_{\mathrm{cov}}$ and a model category structure on simplicial $\mathbf{P}$-algebras for a simplicial operad $\mathbf{P}$. For a precise statement and proof we refer the reader to Section 14.8.

Let $\mathbf{P}$ be an operad in **Sets** and write C for its set of colours. The nerve construction defines a functor on the category of $\mathbf{P}$-algebras (in **Sets**)

$$N(\mathbf{P}, -)\colon \mathrm{Alg}_{\mathbf{P}} \to \mathbf{dSets}/N\mathbf{P}$$

sending a $\mathbf{P}$-algebra A to the left fibration $N(\mathbf{P}, A) \to N\mathbf{P}$ (cf. Example 6.10(d)). Recall that for a dendrex $\Omega[T] \to N\mathbf{P}$, a lift

$$\begin{tikzcd} & N(\mathbf{P}, A) \arrow[d] \\ \Omega[T] \arrow[ur] \arrow[r] & N\mathbf{P} \end{tikzcd}$$

consists of a labelling of the edges of T by elements of A, compatible with the colours and operations that the given map $\Omega[T] \to N\mathbf{P}$ assigns to the edges and vertices of T. Since such a labelling is completely determined by its values on the leaves of T, we find that the functor $N(\mathbf{P}, -)$ has a left adjoint

$$F_{\mathbf{P}}\colon \mathbf{dSets}/N\mathbf{P} \to \mathrm{Alg}_{\mathbf{P}}$$

determined on representables over $N\mathbf{P}$ by the formula

$$F_{\mathbf{P}}(\Omega[T] \xrightarrow{\xi} N\mathbf{P}) = \mathrm{Free}_{\mathbf{P}}(\lambda(T) \xrightarrow{\lambda(\xi)} C).$$

Here the right-hand side is the free $\mathbf{P}$-algebra generated by the leaves of T, where a leaf l is considered as a generator of colour $\xi(l)$. We will often simply write $F_{\mathbf{P}}(T)$ or $F_{\mathbf{P}}(\xi)$ for this free algebra in what follows, if ξ or T is clear from the context.

We will prove the following proposition for a family of colours of **P** indexed by a set U. Such a family is a map $\varphi\colon U \to C$ between sets or, equivalently, an object

$$U \cdot \eta = \coprod_U \eta \to N\mathbf{P}$$

in the category $\mathbf{dSets}/N\mathbf{P}$.

Proposition 9.67 *Suppose* **P** *is a* Σ*-free operad in* **Sets***, so that* $N\mathbf{P}$ *is a normal dendroidal set. Let* φ *be a family of colours of* **P** *as above. Then the unit map*

$$U \cdot \eta \to N(\mathbf{P}, F_{\mathbf{P}}(\varphi))$$

is a leaf anodyne map, so in particular a trivial cofibration in the covariant model structure over $N\mathbf{P}$.

Before embarking on the proof of the proposition, we observe that it easily implies the following more general statement:

Corollary 9.68 *Let* **P** *be a* Σ*-free operad in* **Sets** *and let* $\{\xi_u\colon \Omega[T_u] \to N\mathbf{P}\}_{u\in U}$ *be a family of dendrices of* $N\mathbf{P}$ *indexed by a set* U *and write* $\xi\colon \coprod_{u\in U}\Omega[T_u] \to N\mathbf{P}$ *for the induced map from the coproduct. Then the unit map*

$$\xi \to N(\mathbf{P}, F_{\mathbf{P}}(\xi))$$

is a trivial cofibration in the covariant model structure over $N\mathbf{P}$.

Proof Consider the diagram

$$\begin{tikzcd}
\coprod \ell[T_u] \arrow[r] \arrow[d] & N(\mathbf{P}, F_{\mathbf{P}}(\coprod \ell[T_u] \to N\mathbf{P})) \arrow[d] \\
\coprod \Omega[T_u] \arrow[r] & N(\mathbf{P}, F_{\mathbf{P}}(\coprod \Omega[T_u] \to N\mathbf{P}))
\end{tikzcd}$$

in which all coproducts are over U. Since $F_{\mathbf{P}}$ commutes with coproducts and

$$F_{\mathbf{P}}(\Omega[T_u] \to N\mathbf{P}) = F_{\mathbf{P}}(\ell[T_u] \to N\mathbf{P})$$

by definition, the right-hand map in the diagram is an isomorphism. The left-hand map is leaf anodyne and hence a trivial cofibration in the covariant model structure over $N\mathbf{P}$. So the top map is a covariant weak equivalence over $N\mathbf{P}$ if and only if the bottom map is. In particular, the corollary follows from the proposition. □

Proof (of Proposition 9.67) We begin by considering the dendroidal set $N(\mathbf{P}, F_{\mathbf{P}}(U \cdot \eta \to N\mathbf{P}))$ more closely. A dendrex (ξ, a) of shape T is an element $\xi \in N\mathbf{P}_T$ together with a labelling of each leaf l of T by an element a_l of the free algebra $F_{\mathbf{P}}(U \cdot \eta)$ of colour l. Such an element a_l is given by applying an operation

$$p_l \in \mathbf{P}(\varphi u_1^l, \ldots, \varphi u_{n_l}^l; \xi l)$$

to generators $u_1^l, \ldots, u_{n_l}^l \in U$. So, we can enlarge T by grafting an n_l-corolla on top of the leaf l, labelling its inputs by these generators $u_1^l, \ldots, u_{n_l}^l \in F_{\mathbf{P}}(U \cdot \eta)$ and the vertex of the corolla by p_l. (This extension is not unique, but Σ_n acts freely on the set of extensions of this type because $\mathbf{P}$ is assumed Σ-free.) Doing this for each leaf, we see that T is a face of another dendrex

$$(\widetilde{x}, \widetilde{a}) \colon \widetilde{T} \to N(\mathbf{P}, F_{\mathbf{P}}(U \cdot \eta)),$$

where $\widetilde{T}$ is obtained from T by grafting corollas onto its leaves and the labelling $\widetilde{a}$ of the leaves of $\widetilde{T}$ is by generators of $F_{\mathbf{P}}(U \cdot \eta)$, i.e., by elements of U. Let us call a dendrex of this kind *special*.

Since $\mathbf{P}$ is assumed to be Σ-free, the dendroidal sets $N\mathbf{P}$ and $N(\mathbf{P}, F_{\mathbf{P}}(U \cdot \eta))$ are normal and we can build up $N(\mathbf{P}, F_{\mathbf{P}}(U \cdot \eta))$ from $U \cdot \eta$ by successively attaching non-degenerate special dendrices, by induction on the size of the dendrex. More precisely, consider the filtration

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq A_{n+1} \subseteq \cdots \subseteq N(\mathbf{P}, F_{\mathbf{P}}(U \cdot \eta)),$$

where A_n is generated by all the non-degenerate special dendrices indexed by trees with at most n vertices. Then $A_0 = U \cdot \eta$. Moreover, since $N\mathbf{P}$ is normal, $A_{n-1} \subseteq A_n$ fits into a pushout square

$$\begin{tikzcd}
\coprod_{(T,\xi,a)} \Omega[T] \times_{A_n} A_{n-1} \arrow[r] \arrow[d, tail] & A_{n-1} \arrow[d, tail] \\
\coprod_{(T,\xi,a)} \Omega[T] \arrow[r] & A_n,
\end{tikzcd}$$

where the coproduct is over isomorphism classes of trees with exactly n vertices and non-degenerate special dendrices (ξ, a) of $N(\mathbf{P}, F_{\mathbf{P}}(U \cdot \eta))_T$. For such a (ξ, a), no leaf face can be special, while each inner face or potential root face is still special. So the intersection $\Omega[T] \times_{A_n} A_{n-1}$ is of the form $\Lambda^V[T]$ where V is the set of all leaf vertices of T. By Lemma 6.21, the map $\Lambda^V[T] \to \Omega[T]$ is leaf anodyne. Therefore $A_{n-1} \to A_n$, as well as $A_0 \to \varinjlim A_n$, are leaf anodyne maps. This completes the proof. □

9.6 The Absolute Covariant Model Structure

In this section we will discuss some aspects of the 'absolute' covariant model structure on the category **dSets** itself, i.e., on **dSets**$/V$ where $V = 1$ is the terminal object. This serves as preparation and motivation for the next section, in which we will consider a further localization of the absolute covariant model structure, called the Picard model structure. The relevance of these two model structures will become apparent later, when we prove that the associated homotopy categories are equivalent to those of $\mathbf{E}_\infty$-spaces and of infinite loop spaces, respectively.

As a special case of Theorem 9.59 (or, in fact, already of Theorem 9.9), we see that the absolute covariant model structure on **dSets** has the following characteristics:

(1) The cofibrations are the normal monomorphisms.
(2) The fibrant objects are the dendroidal left Kan complexes, i.e., the dendroidal sets having the right lifting property with respect to all inclusions $\Lambda^x[T] \to \Omega[T]$ where x is an inner edge or a leaf vertex in the tree T.

Furthermore, if X is such a fibrant object, then the simplicial set i^*X is a Kan complex by Corollary 5.51. By Theorem 9.63, a map $X \to Y$ between fibrant objects is a covariant weak equivalence if and only if $i^*X \to i^*Y$ is a weak homotopy equivalence between Kan complexes.

Example 9.69 Let $\mathbf{M}$ be a symmetric monoidal category. Recall that $\mathbf{M}$ can naturally be viewed as an operad $\mathbf{M}^\otimes$: its colours are the objects of $\mathbf{M}$ and its operations $(c_1, \ldots, c_n) \to c$ are the morphisms $c_1 \otimes \cdots \otimes c_n \to c$. Taking the nerve, we obtain a dendroidal set $N\mathbf{M}^\otimes$. This dendroidal is an ∞-operad; it is even a strict inner Kan complex. Moreover, $N\mathbf{M}^\otimes$ clearly has the right lifting property with respect to $\ell[C_n] \to \Omega[C_n]$, as for any objects $c_1, \ldots, c_n$ the identity is an operation $(c_1, \ldots, c_n) \to c_1 \otimes \cdots \otimes c_n$ in $\mathbf{M}^\otimes$. (For $n = 0$, the object $c_1 \otimes \cdots \otimes c_n$ is the monoidal unit of $\mathbf{M}$.) For $N\mathbf{M}^\otimes$ to be covariantly fibrant, the nerve $N\mathbf{M} = i^*N\mathbf{M}^\otimes$ of the underlying category must in particular be a Kan complex, so $\mathbf{M}$ has to be a groupoid. We claim that if this is the case, then $N\mathbf{M}^\otimes$ is indeed covariantly fibrant. To check this, consider an extension problem of the kind

$$\begin{tikzcd} \Lambda^x[T] \arrow[r, "f"] \arrow[d] & N\mathbf{M}^\otimes, \\ \Omega[T] \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

where x is a leaf vertex (we already know the extension exists for any inner edge x). We have just seen that an extension exists if T is a corolla (so that $\Lambda^x[T] \to \Omega[T]$ is the inclusion of its leaves). We claim that for a larger tree T, there exists a *unique* extension. If T has two vertices, then it consists of the corolla with vertex x grafted onto a leaf, say e, of another corolla with vertex y. Write $c_1, \ldots, c_n$ for the inputs of x and $d_1, \ldots, d_m$ for the inputs of y, where necessarily $d_i = e$ for some i. Then $f\colon \Lambda^x[T] \to N\mathbf{M}^\otimes$ sends the vertex x to a map $\alpha\colon f(c_1) \otimes \cdots \otimes f(c_n) \to f(e)$ and the inner face $\partial_e T$ to a map

$$\beta\colon f(d_1) \otimes \cdots \otimes f(d_{i-1}) \otimes f(c_1) \otimes \cdots \otimes f(c_n) \otimes f(d_{i+1}) \otimes \cdots \otimes f(d_m) \to f(r),$$

with r the root edge of T. Since $\mathbf{M}$ is a groupoid, both of these maps are isomorphisms and there exists a unique isomorphism $\gamma\colon f(d_1) \otimes \cdots \otimes f(d_m) \to f(r)$ making the following diagram commute:

$$\begin{tikzcd}
f(d_1)\otimes\cdots\otimes f(d_{i-1})\otimes f(c_1)\otimes\cdots\otimes f(c_n)\otimes f(d_{i+1})\otimes\cdots\otimes f(d_m) \arrow[r, "\beta"] \arrow[d, "\mathrm{id}\otimes\alpha\otimes\mathrm{id}"'] & f(r). \\
f(d_1)\otimes\cdots\otimes f(d_m) \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

This defines the desired extension of f to all of T. For a larger tree T with more than two vertices, its spine $\mathrm{Sp}[T]$ is already contained in $\Lambda^x[T]$. Since $N\mathbf{M}^{\otimes}$ is strict inner Kan, there is a unique extension of $f|_{\mathrm{Sp}[T]}$ to a map $g\colon \Omega[T] \to N\mathbf{M}^{\otimes}$. It remains to check that g agrees with f on $\Lambda^x[T]$. Write e for the outgoing edge of x again. For any face $\partial_y T$ other than the inner face $\partial_e T$, the spine of $\partial_y T$ is already contained in $\Lambda^x[T]$, so that f and g must agree on $\partial_y T$. To argue that f and g agree on the spine of $\partial_e T$, it remains to check that they agree on the leaf vertex arising as the composition of x and the vertex below it, to which it is attached via e. This follows exactly as in the case of a tree with two vertices, which we treated above.

The previous example shows that the dendroidal nerve of a symmetric monoidal groupoid is a fibrant object in the covariant model structure on **dSets**. In fact the converse also holds, as we will now show.

Proposition 9.70 *Let X be a covariantly fibrant dendroidal set. Then τX is isomorphic to an operad of the form $\mathbf{M}^{\otimes}$ associated to a symmetric monoidal groupoid* **M**.

Applying the proposition to a dendroidal set of the form $N\mathbf{P}$, for an operad **P**, gives the following:

Corollary 9.71 *Let* **P** *be an operad in* **Sets**. *If $N\mathbf{P}$ is covariantly fibrant, then* **P** *is isomorphic to an operad of the form $\mathbf{M}^{\otimes}$ associated to a symmetric monoidal groupoid* **M**.

Remark 9.72 The unary operations of **P** constitute a category $j^*\mathbf{P}$. The proposition asserts that this category carries a symmetric monoidal structure for which the morphisms $c_1 \otimes \cdots \otimes c_n \to d$ are in natural bijective correspondence with the operations in $\mathbf{P}(c_1, \ldots, c_n; d)$. It follows from the Yoneda lemma that this tensor product is unique up to unique isomorphism.

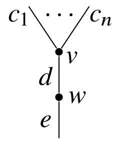
Proof (of Proposition 9.70) Suppose X is covariantly fibrant. Then in particular i^*X is a Kan complex, so $\tau(i^*X)$ is a groupoid. For any sequence of colours $c_1, \ldots, c_n$, choose a lift in

$$\begin{tikzcd}
\ell[C_n] \arrow[r] \arrow[d] & X. \\
\Omega[C_n] \arrow[ur, dashed] &
\end{tikzcd}$$

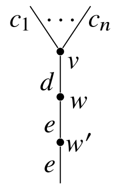
This gives another colour d corresponding to the root and an operation $\delta = \delta_{c_1,\ldots,c_n} \in (\tau X)(c_1, \ldots, c_n; d)$. Note that any two choices yield isomorphic results under $c_1, \ldots, c_n$. More precisely, if $\varepsilon \in (\tau X)(c_1, \ldots, c_n; e)$ is induced by another choice of lift, then we can find a lift in

$$\begin{tikzcd} \Lambda^v[T] \arrow[r, "\delta\cup\varepsilon"] \arrow[d] & X, \\ \Omega[T] \arrow[ur, dashed] & \end{tikzcd}$$

where T is the tree



and $\delta \cup \varepsilon$ is δ on $\partial_w T$ and ε on $\partial_d T$. Then an extension to T gives a map $\varphi\colon d \to e$ in the groupoid $\tau(i^* X)$ with $\varphi \circ \delta = \varepsilon$. Such a φ is unique, because if ψ is another map arising in this way, then φ and ψ together define a map on the leaf horn of the tree



which corresponds to φ on the root face and to ψ on the lowest inner face, and the degenerate dendrex given by ε and the identity on e on the top inner face. An extension in particular shows that the diagram

$$\begin{tikzcd} d \arrow[r, "\varphi"] \arrow[d, "\psi"'] & e \arrow[dl, equal] \\ e & \end{tikzcd}$$

commutes in $\tau(i^* X)$, so $\varphi = \psi$.

So, let us fix choices for δ and d and denote these by $c_1 \otimes \cdots \otimes c_n := d$ and

$$\delta = \delta_{c_1,\ldots,c_n} \in (\tau X)(c_1, \ldots, c_n; c_1 \otimes \cdots \otimes c_n).$$

Then composition with δ defines for each colour a of $\tau(X)$ a map

$$\delta^*\colon (\tau X)(c_1 \otimes \cdots \otimes c_n; a) \to (\tau X)(c_1, \ldots, c_n; a),$$

natural in a. This map δ^* is a bijection, because we have just seen that any two operations $\delta \in (\tau X)(c_1, \ldots, c_n; d)$ and $\varepsilon \in (\tau X)(c_1, \ldots, c_n; e)$ differ by a unique isomorphism $\varphi\colon d \to e$. In particular, any $\alpha \in (\tau X)(c_1, \ldots, c_n; a)$ is uniquely of the form $\delta^* \varphi$ for such a map $\varphi\colon c_1 \otimes \cdots \otimes c_n \to a$. Having established that δ^* is a natural isomorphism, the functoriality and associativity properties of the chosen tensor product follow from those of τX and the Yoneda lemma. $\square$

Remark 9.73 The example and proposition above provide an illustration of the characterization of the covariant weak equivalences between fibrant objects. Indeed, for a map $\mathbf{P} \to \mathbf{Q}$ of operads, it is in general not enough to be an equivalence of underlying categories in order for $N\mathbf{P} \to N\mathbf{Q}$ to be an operadic equivalence. But it is enough if $\mathbf{P}$ and $\mathbf{Q}$ arise from symmetric monoidal categories.

Later, we will refine the relation between fibrant objects in the absolute covariant model structure and symmetric monoidal groupoids by showing that in fact any such fibrant object gives rise to an $\mathbf{E}_\infty$-space. For now we make a more basic observation, namely that for a covariantly fibrant object X the tensor product operation defined in the proof of Proposition 9.70 can be upgraded to a map of Kan complexes

$$- \otimes - \colon i^*X \times i^*X \to i^*X.$$

To do this, first consider the restriction map

$$\mathbf{hom}(\Omega[C_2], X) \to \mathbf{hom}(\ell[C_2], X) \cong i^*X \times i^*X.$$

Since $\ell[C_2] \to C_2$ is leaf anodyne and X is covariantly fibrant, this map is a trivial fibration between Kan complexes. Hence we may choose a section $s\colon i^*X \times i^*X \to \mathbf{hom}(\Omega[C_2], X)$. Now precomposing with the root inclusion $\eta \to C_2$ in the first variable gives a further map

$$\mathbf{hom}(\Omega[C_2], X) \to \mathbf{hom}(\eta, X) \cong i^*X.$$

Composing these maps

$$i^*X \times i^*X \to \mathbf{hom}(\Omega[C_2], X) \to i^*X$$

now gives the desired 'tensor product'. Of course this construction depends on a choice, namely that of the section s. However, since any two such sections are fibrewise homotopic, the resulting maps will also be homotopic. Also note that replacing 2 by n gives a similar construction of an n-fold tensor product

$$\otimes_n \colon (i^*X)^n \to i^*X.$$

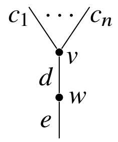
Using these choices, we can relate the spaces of operations $X(x_1, \ldots, x_n; y)$ in the ∞-operad X to mapping spaces in the underlying ∞-category i^*X as follows:

Lemma 9.74 *Let X be a dendroidal left Kan complex and let $x_1, \ldots, x_n, y$ be a sequence of colours of X. Then the choice of n-fold tensor product above defines a diagram of Kan complexes*

$$X(x_1, \ldots, x_n; y) \leftarrow M \to i^*X(x_1 \otimes \cdots \otimes x_n; y)$$

where both arrows are trivial Kan fibrations. In particular, the Kan complexes $X(x_1, \ldots, x_n; y)$ and $i^*X(x_1 \otimes \cdots \otimes x_n; y)$ are homotopy equivalent.

Proof Write $\xi\colon \Omega[C_n] \to X$ for the n-corolla of X defined by $s(x_1, \dots, x_n)$, with s as above. Thus ξ is a corolla with leaves $x_1, \dots, x_n$ and root $x_1 \otimes \cdots \otimes x_n$. Consider the tree $T = C_1 \circ C_n$, depicted as follows:



Define the simplicial set M by the following pullback square:

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & \mathbf{hom}(\Omega[T], X) \arrow[d] \\
\{\xi\} \times \{y\} \arrow[r] & \mathbf{hom}(\Omega[C_n], X) \times \mathbf{hom}(\eta, X).
\end{tikzcd}$$

Here the right-hand vertical map is the product of the restriction along the inclusion of the leaf corolla $v\colon C_n \to T$ and the restriction along the root inclusion $\{e\} \to T$. We can expand this diagram as follows, where both squares are pullbacks:

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & \mathbf{hom}(\Omega[T], X) \arrow[d, two heads, "\wr"] \\
K \arrow[r] \arrow[d] & \mathbf{hom}(\Omega[C_n], X) \times_{i^*X} \mathbf{hom}(\Omega[C_1], X) \arrow[d] \\
\{\xi\} \times \{y\} \arrow[r] & \mathbf{hom}(\Omega[C_n], X) \times \mathbf{hom}(\eta, X).
\end{tikzcd}$$

Here the upper vertical map on the right is the restriction along the spine inclusion $\Omega[C_n] \cup_{\{d\}} \Omega[C_1] \to \Omega[T]$ (and hence a trivial fibration, since X is an ∞-operad) and the lower vertical map on the right restricts further along the root inclusion $\{e\} \to C_1$. It follows that $M \to K$ is also a trivial fibration. Moreover, by restricting to the root d of C_n, the simplicial set K is clearly isomorphic to the pullback of the span

$$\begin{tikzcd}
 & \mathbf{hom}(\Omega[C_1], X) \arrow[d] \\
\{x_1 \otimes \cdots \otimes x_n\} \times \{y\} \arrow[r] & \mathbf{hom}(\eta, X) \times \mathbf{hom}(\eta, X).
\end{tikzcd}$$

By definition this pullback is the simplicial set $i^*X(x_1 \otimes \cdots \otimes x_n; y)$, establishing one of the trivial fibrations claimed in the lemma. For the other map, consider the leaf horn

$$\Lambda^v[T] \cong \Omega[C_n] \cup_{\ell[C_n]} \Omega[C_n].$$

It is the union of the n-corolla with vertex v and the n-corolla arising as the inner face $\partial_d T$. Construct a similar diagram of pullbacks

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & \mathbf{hom}(\Omega[T], X) \arrow[d, two heads, "\wr"] \\
L \arrow[r] \arrow[d] & \mathbf{hom}(\Omega[C_n], X) \times_{(i^*X)^{\times n}} \mathbf{hom}(\Omega[C_n], X) \arrow[d] \\
\{\xi\} \times \{y\} \arrow[r] & \mathbf{hom}(\Omega[C_n], X) \times \mathbf{hom}(\eta, X),
\end{tikzcd}$$

where now the upper vertical map on the right restricts along the horn inclusion $\Lambda^v[T] \to \Omega[T]$. Since that inclusion is a leaf anodyne, the restriction (and hence also its pullback $M \to L$) is a trivial fibration. By an argument similar to the one above, we identify the pullback L as $X(x_1, \ldots, x_n; y)$. □

For future use we also observe the following 'global' version of Lemma 9.74. Indeed, the following implies the conclusion of the previous lemma by taking the fibre over a vertex $(x_1, \ldots, x_n, y) \in (i^*X)^{n+1}$.

Lemma 9.75 *Let X be a dendroidal left Kan complex. The Kan fibration*

$$\mathbf{hom}(\Omega[C_n], X) \to \mathbf{hom}(\partial\Omega[C_n], X) \cong (i^*X)^{n+1}$$

is homotopy equivalent (via a zig-zag of trivial fibrations over $(i^*X)^{n+1}$, as before) to the fibration φ_X defined by the following pullback square:

$$\begin{tikzcd}
T_n X \arrow[r] \arrow[d, "\varphi_X"] & \mathbf{hom}(\Delta[1], i^*X) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
(i^*X)^n \times i^*X \arrow[r, "(\otimes_n, \mathrm{id})"] & i^*X \times i^*X.
\end{tikzcd}$$

Proof The strategy of proof is very similar to that of Lemma 9.74. Consider the same tree $T = C_1 \circ C_n$ as in that proof. The inner face inclusion $C_n \cong \partial_d T \to T$ and spine inclusion $\mathrm{Sp}[T] \to T$, which are both covariant trivial cofibrations, induce trivial fibrations as in the following diagram:

$$\begin{tikzcd}
\mathbf{hom}(\Omega[C_n], X) \arrow[dr] & \mathbf{hom}(\Omega[T], X) \arrow[l, two heads, "\sim"'] \arrow[r, two heads, "\sim"] \arrow[d] & \mathbf{hom}(\mathrm{Sp}[T], X) \arrow[dl] \\
& (i^*X)^n \times i^*X. &
\end{tikzcd}$$

Here the vertical map is induced by restriction along the inclusion $\coprod_{n+1} \eta \to \Omega[T]$ of all the external edges of T. Thus it suffices to consider the slanted map on the right. This map fits into a further commutative diagram

$$\begin{tikzcd}
& \mathbf{hom}(\mathrm{Sp}[T], X) \arrow[r] \arrow[dl] \arrow[d] & \mathbf{hom}(\Omega[C_1], X) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
(i^*X)^n \times i^*X & \mathbf{hom}(\Omega[C_v] \cup \{e\}, X) \arrow[l, two heads, "\sim"'] \arrow[r] & i^*X \times i^*X,
\end{tikzcd}$$

where the square is a pullback. Here C_v denotes the n-corolla with vertex v in T and e is the root edge of T. The lower left horizontal map is a trivial fibration between Kan complexes because $\ell[C_v] \to \Omega[C_v]$ is a covariant trivial cofibration. Hence it admits a section s (which is a homotopy equivalence) and moreover the composition of such an s with the lower right horizontal map precisely defines the map $(\otimes_n, \mathrm{id})$. This completes the proof. □

The tensor products on i^*X constructed above are unital and associative up to homotopy. To be precise, applying our construction of the n-fold tensor product in the particular case $n = 0$ defines a colour $\mathbf{1}_X$ of X which serves as a unit. Indeed:

Lemma 9.76 *Let X be a dendroidal left Kan complex, with tensor products on i^*X defined as above. The map*

$$\mathbf{1}_X \otimes -\colon i^*X \to i^*X$$

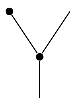
is homotopic to the identity of i^*X. Moreover, the tensor product is associative up to homotopy, in the sense that the two assignments

$$(x, y, z) \mapsto (x \otimes y) \otimes z \quad \textit{and} \quad (x, y, z) \mapsto x \otimes (y \otimes z)$$

are homotopic maps from $(i^*X)^3$ to i^*X.

Remark 9.77 Of course the statement above concerning three-fold tensor products admits a generalization to n-fold tensor products and associator maps as in our discussion of unbiased tensor products in Section 4.4. The reader is invited to check that a version of the argument below provides this generalization.

Proof Let T be the tree



obtained by grafting a 0-corolla onto one of the leaves of C_2. Write l for the unique leaf of T and r for its root. By construction, the map $\mathbf{1}_X \otimes -$ is defined by picking a section of the trivial fibration

$$\mathbf{hom}(\Omega[T], X) \to \mathbf{hom}(\{l\}, X) \cong i^*X$$

and composing with the map

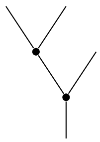
$$\mathbf{hom}(\Omega[T], X) \to \mathbf{hom}(\{r\}, X) \cong i^*X.$$

Consider the unique inner face map $C_1 \to T$ and the resulting commutative diagram

$$\begin{tikzcd}
\mathbf{hom}(\{l\}, X) \arrow[d, equal] & \mathbf{hom}(\Omega[T], X) \arrow[l] \arrow[r] \arrow[d] & \mathbf{hom}(\{r\}, X) \arrow[d, equal] \\
\mathbf{hom}(\{l\}, X) & \mathbf{hom}(\Omega[C_1], X) \arrow[l] \arrow[r] & \mathbf{hom}(\{r\}, X).
\end{tikzcd}$$

The leftward map on the bottom row admits a section, defined by the degeneracy map $C_1 \to \eta$, so that the resulting composite $\mathbf{hom}(\{l\}, X) \to \mathbf{hom}(\{r\}, X)$ is the identity, after identifying domain and codomain with i^*X. It follows that $\mathbf{1}_X \otimes -$ is indeed homotopic to the identity.

The argument for associativity is entirely analogous; one compares the tree



and its inner face, which is a 3-corolla, to show that the maps corresponding to $(x \otimes y) \otimes z$ and the 3-fold tensor product $x \otimes y \otimes z$ are homotopic. The conclusion then follows by symmetry. □

9.7 The Picard Model Structure

In Section 9.6 we considered the absolute covariant model structure on the category of dendroidal sets, which is a left Bousfield localization of the operadic model structure. Eventually we will show that the corresponding homotopy category is equivalent to that of $\mathbf{E}_\infty$-spaces. Anticipating this result, we will study a further left Bousfield localization of the covariant model structure in this section, called the *Picard model structure*. We will see in Section 13.5 that the homotopy category of the latter is equivalent to that of *grouplike* $\mathbf{E}_\infty$-spaces, which in turn is equivalent to that of infinite loop spaces or connective spectra. In this section we will restrict ourselves to proving some basic properties of the Picard model structure and giving some examples of fibrant objects in it, in analogy with the examples of the previous section.

A *Picard groupoid* is a symmetric monoidal groupoid $\mathbf{C}$ in which for every object c of $\mathbf{C}$, the functor

$$c \otimes -\colon \mathbf{C} \to \mathbf{C}$$

is an equivalence of categories. In other words, every object should be *invertible* with respect to the tensor product. We will often write $c^\vee$ for an inverse to c, i.e., an object such that there exists an isomorphism between $c \otimes c^\vee$ and the monoidal unit $\mathbf{1}$. A typical example (and the reason for the name) of a Picard groupoid is the groupoid of line bundles on an algebraic variety or a manifold.

The relation between Picard groupoids and dendroidal sets lies in the following observation, building on Corollary 9.71:

Proposition 9.78 *Let* $\mathbf{P}$ *be a symmetric monoidal category. Then* $\mathbf{P}$ *is a Picard groupoid if and only if the nerve* $N\mathbf{P}^\otimes$ *of the operad associated to* $\mathbf{P}$ *has the right lifting property with respect to any horn inclusion* $\Lambda^x[T] \to \Omega[T]$*; or, in other words, if the map* $N\mathbf{P}^\otimes \to 1$ *to the terminal dendroidal set is a dendroidal Kan fibration.*

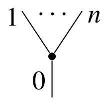
Remark 9.79 Recall that in the special case where T is a corolla C_n, a horn inclusion $\Lambda^x[C_n] \to \Omega[C_n]$ is of the form

$$\coprod_{e \in E(C_n)-\{x\}} \eta \to \Omega[C_n],$$

where the disjoint union is over all but one of the edges of C_n. In the previous section we only considered the 'leaf horn' of the corolla, which is the inclusion of all leaves of C_n. This is of course the horn where one omits the root edge.

Proof Corollary 9.71 states that $\mathbf{P}$ is a symmetric monoidal groupoid if and only if $N\mathbf{P}^{\otimes}$ is covariantly fibrant. So it suffices to prove that every object of $\mathbf{P}$ is invertible with respect to the tensor product if and only if $N\mathbf{P}^{\otimes}$ additionally has the right lifting property with respect to the following two types of inclusions:

(i) The inclusion
$$\coprod_{i \neq n} \eta \to \Omega[C_n]$$
of all edges in a corolla C_n except for one of its leaves, which we may as well assume to be the one labelled n:



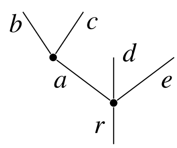
(ii) The inclusion $\Lambda^r[T] \to \Omega[T]$ of the root horn into a tree T for which the root face exists (i.e., there is exactly one inner edge attached to its root vertex).

If $\mathbf{P}$ is a Picard groupoid, then the right lifting property with respect to (i) is satisfied, because given objects $c_0, \ldots, c_{n-1}$ in $\mathbf{P}$, an extension to $\Omega[C_n] \to N\mathbf{P}^{\otimes}$ of the required kind consists of an object c_n and a map $c_1 \otimes \cdots \otimes c_n \to c_0$. Taking

$$c_n = c_{n-1}^{\vee} \otimes \cdots \otimes c_1^{\vee} \otimes c_0$$

will clearly make this possible.

For the right lifting property with respect to (ii), first consider the case that T has only one inner edge; for example, T is



Then a map $\Lambda^r[T] \to N\mathbf{P}^{\otimes}$ is given by maps $f\colon a \otimes d \otimes e \to r$ and $g\colon b \otimes c \otimes d \otimes e \to r$ (where we have already labelled edges of T according to their images in $\mathbf{P}$), and we need to find a map $h\colon b \otimes c \to a$ such that

$$f \circ (h \otimes d \otimes e) = g.$$

Such a map indeed exists, because $f^{-1}g : b \otimes c \otimes d \otimes e \to a \otimes d \otimes e$ is of the form $h \otimes d \otimes e$ for a unique h, using that all the objects involved are invertible with respect to the tensor product. If T is a larger tree, then $\Lambda^r[T]$ contains the spine of T and one easily finds an extension to T in the same way as in Example 9.69.

Conversely, suppose $\mathbf{P}$ is a symmetric monoidal groupoid having the lifting properties corresponding to (i) and (ii). Write $\Lambda^2[C_2]$ for the horn of C_2 consisting of the root edge and the leaf edge 1. Define a map $f : \Lambda^2[C_2] \to N\mathbf{P}^{\otimes}$ by sending the root to the unit $\mathbf{1}$ and the one leaf edge to c. Then an extension of f to all of C_2 will send the other leaf edge to an object $c^{\vee}$ with the property that there is an isomorphism $c \otimes c^{\vee} \cong \mathbf{1}$, showing that $\mathbf{P}$ is Picard. □

Remark 9.80 The reader will observe that in the last part of the proof, we have only used a fraction of the conditions corresponding to the inclusions of types (i) and (ii). This is indicative of a more general fact, which we will express in Proposition 9.84 below.

Our purpose in this section is to describe the left Bousfield localization of the (absolute) covariant model structure of the previous section for which the fibrant objects are precisely the dendroidal Kan complexes, i.e., the dendroidal sets having the right lifting property with respect to all horn inclusions $\Lambda^x[T] \to \Omega[T]$. Thus, by Proposition 9.78, the nerves of Picard groupoids will be particular examples of fibrant objects of this model structure.

We will establish the existence of this model structure in the same way as for the covariant model structure, but using a larger class of anodynes. Write $\mathcal{P}_0$ for the class of anodyne maps of dendroidal sets, i.e., the saturation of the set of *all* horn inclusions of trees. We write $\mathcal{P}$ for the smallest class containing $\mathcal{P}_0$, closed under pushouts and composition, and satisfying the right cancellation property among normal monomorphisms: if $u : A \to B$, $v : B \to C$ are normal monomorphisms such that u and vu are in $\mathcal{P}$, then v is in $\mathcal{P}$. We call the elements of $\mathcal{P}$ *Picard anodyne maps*. The following lemma expresses that $\mathcal{P}$ is an admissible class:

Lemma 9.81 *For any $n \geq 0$ and any morphism $A \to B$ in $\mathcal{P}$, the pushout-product map*

$$\partial\Delta[n] \otimes B \cup \Delta[n] \otimes A \to \Delta[n] \otimes B$$

is Picard anodyne.

Remark 9.82 An explicit construction of the class $\mathcal{P}$ from $\mathcal{P}_0$ is as follows. Define a class $\mathcal{P}_1'$ containing $\mathcal{P}_0$ by declaring a normal monomorphism v to be in $\mathcal{P}_1'$ if $v \in \mathcal{P}_0$ or if there exists a $u \in \mathcal{P}_0$ such that vu exists and is in $\mathcal{P}_0$. Then define $\mathcal{P}_1$ to be the closure of $\mathcal{P}_1'$ under composition and pushouts. In the same way, construct a class $\mathcal{P}_2$ from $\mathcal{P}_1$, etc. Then

$$\bigcup_{n \geq 0} \mathcal{P}_n$$

is closed under composition and pushouts and satisfies the right cancellation property. Moreover, it is the smallest such class containing $\mathcal{P}_0$.

Lemma 9.81 is very similar to our results about pushout-products of various kinds of anodynes in Section 6.3. Indeed, the cases where $A \to B$ is an inner horn, leaf horn, or root horn corresponding to a *unary* root vertex are covered by Lemma 6.24 and Corollary 6.30. What remains is to analyze the case of root horns of trees whose root vertex is not unary, as well as the behaviour of the right cancellation property under pushout-products. We postpone the somewhat laborious proof of Lemma 9.81 to the end of this section and first consider its consequences.

Inspired by Proposition 9.78 we will also refer to dendroidal Kan complexes as *Picard ∞-groupoids*. Any Picard ∞-category X is in particular an ∞-operad, of course. Also, its underlying ∞-category i^*X is a Kan complex, justifying the use of the term Picard ∞-groupoid. The methods of Section 9.1, in particular Theorem 9.9, now give most of the following:

Theorem 9.83 *There exists a left proper, cofibrantly generated model structure on the category* **dSets** *with the following properties:*

(a) *The cofibrations are the normal monomorphisms.*
(b) *The fibrant objects are the Picard ∞-groupoids.*
(c) *The fibrations between Picard ∞-groupoids are the dendroidal Kan fibrations.*
(d) *A map $X \to Y$ between Picard ∞-groupoids is a weak equivalence if and only if the map of Kan complexes $i^*X \to i^*Y$ is a weak homotopy equivalence.*

We will refer to the model structure of the theorem as the *Picard model structure* and to its weak equivalences as the *Picard equivalences*. Given Lemma 9.81 above, the model structure of the theorem is constructed as for Theorem 9.59, now using for $\mathcal{A}$ an admissible set of Picard anodyne maps containing the horn inclusions of trees; for example, the set of Picard anodyne maps between finite dendroidal sets. Part (a) of the theorem is then satisfied. It is clear that the Picard model structure is a left Bousfield localization of the absolute covariant model structure on dendroidal sets. Therefore item (d) of Theorem 9.83 follows from Theorem 9.63 (characterizing covariant weak equivalences between fibrant objects) and Lemma 8.49 (weak equivalences between local objects do not change in a localization). We will prove part (b) as Corollary 9.87 below and (c) in Lemma 9.88.

First, let us observe the following analogue of Proposition 6.41, giving several equivalent characterizations of the class of Picard anodyne morphisms:

Proposition 9.84 *Let $\mathcal{C}$ be a saturated class of normal monomorphisms between dendroidal sets which is closed under two-out-of-three amongst normal monomorphisms. The following three classes of maps are Picard anodyne. If $\mathcal{C}$ contains any one of these three, then $\mathcal{C}$ contains all Picard anodyne morphisms:*

(1) *For each tree T with at least one vertex, the inclusion*

$$\coprod \eta \to \Omega[T],$$

where the coproduct is over all but one of the outer edges (leaves and root) of T. (This includes the morphism $\varnothing \to \Omega[C_0]$.)

(2) *The spine inclusion* $\mathrm{Sp}[T] \to \Omega[T]$ *of any tree, as well as the inclusions of type (1) in the case where* T *is a corolla.*
(3) *The spine inclusion* $\mathrm{Sp}[T] \to \Omega[T]$ *of any tree, the leaf inclusions* $\ell[C_n] \to \Omega[C_n]$ *of any corolla, as well as the inclusion*

$$\eta \amalg \eta \to \Omega[C_2]$$

of one of the leaves and the root of the 2-corolla.

As part of the proof, let us first show the following:

Lemma 9.85 *The morphisms listed in (1)–(3) of Proposition 9.84 are all Picard anodyne.*

Proof For (2) and (3) this is clear from the definitions and the fact that spine inclusions are inner anodyne (Lemma 6.37). The morphisms of (1) are leaf anodyne in case the missing edge is the root of T, by Proposition 6.41. If the missing outer edge is a leaf l, we reason by induction on the size of T. If T is a corolla, we are back in case (2). For a larger tree, write T as a grafting $T' \circ_e C_v$, where v is a leaf vertex of T with outgoing edge e, and C_v denotes the corolla with vertex v. Then

$$\Omega[T'] \cup_e \Omega[C_v] \to \Omega[T]$$

is inner anodyne by Lemma 6.37. Write $\mathrm{Out}(T)$ for the disjoint union of external edges of T (though of as a subobject of $\Omega[T]$) and similarly for T'. We consider two cases:

(a) The missing leaf l is a leaf of C_v.
(b) The missing leaf l is not a leaf of C_v.

In case (a), consider the maps

$$\mathrm{Out}(T) - \{l\} \xrightarrow{f} \Omega[T'] \cup (\mathrm{Out}(T) - \{l\}) \xrightarrow{g} \Omega[T'] \cup \Omega[C_v] \to \Omega[T].$$

The last one is inner anodyne, so it suffices to show that f and g are Picard anodyne. This follows from the pushout squares

$$\begin{tikzcd}
\mathrm{Out}(T') - \{e\} \arrow[r] \arrow[d] & \mathrm{Out}(T) - \{l\} \arrow[d, "f"] \\
\Omega[T'] \arrow[r] & \Omega[T'] \cup (\mathrm{Out}(T) - \{l\}),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Out}(C_v) - \{l\} \arrow[r] \arrow[d] & \Omega[T'] \cup (\mathrm{Out}(T) - \{l\}) \arrow[d, "g"] \\
\Omega[C_v] \arrow[r] & \Omega[T'] \cup \Omega[C_v]
\end{tikzcd}$$

and the inductive hypothesis on T'. In case (b), we consider the maps

$$\mathrm{Out}(T) - \{l\} \xrightarrow{f} (\mathrm{Out}(T) - \{l\}) \cup \Omega[C_v] \xrightarrow{g} \Omega[T'] \cup \Omega[C_v] \to \Omega[T].$$

Again it suffices to check that f and g are Picard anodyne. This can be seen from the observation that

$$(\mathrm{Out}(T) - \{l\}) \cup \Omega[C_v] = (\mathrm{Out}(T') - \{l\}) \cup \Omega[C_v]$$

and the pushout squares

$$\begin{tikzcd}
\mathrm{Out}(C_v) - \{e\} \arrow[r] \arrow[d] & \mathrm{Out}(T) - \{l\} \arrow[d, "f"] \\
\Omega[C_v] \arrow[r] & (\mathrm{Out}(T) - \{l\}) \cup \Omega[C_v],
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Out}(T') - \{l\} \arrow[r] \arrow[d] & (\mathrm{Out}(T') - \{l\}) \cup \Omega[C_v] \arrow[d, "g"] \\
\Omega[T'] \arrow[r] & \Omega[T'] \cup \Omega[C_v].
\end{tikzcd}$$

This completes the proof. □

Proof (of Proposition 9.84) It remains to prove that any class $\mathcal{C}$ as in the statement of the proposition contains all Picard anodyne maps. Proposition 6.41 implies that $\mathcal{C}$ contains all leaf anodynes. So it suffices to prove the following statements, for $\mathcal{C}$ a saturated class of normal monomorphisms between dendroidal sets which is closed under two-out-of-three amongst normal monomorphisms and contains the leaf anodynes:

(a) If $\mathcal{C}$ contains the inclusions of type (1) for the 2-corolla, then $\mathcal{C}$ contains those inclusions for any corolla C_n.
(b) If $\mathcal{C}$ contains those inclusions for all corollas C_n, then it contains the inclusions $\amalg\eta \to \Omega[T]$ as in (1).
(c) If $\mathcal{C}$ contains all the inclusions of (1), then it contains the root horn $\Lambda^r[T] \to \Omega[T]$ for each tree T which admits a root face.

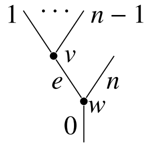
For (a), denote the leaf edges of the n-corolla by $1, \ldots, n$ and its root by 0. It suffices to consider the inclusions

$$\coprod_{0 \le i < n} \eta \to \Omega[C_n]$$

of all the edges except the nth leaf. (The case where the root edge is missing is of course leaf anodyne.) For $n = 1$ this is the root inclusion of the 1-corolla. To see that it is contained in $\mathcal{C}$, consider the maps

$$\eta_0 \to \Omega[C_1] \to J.$$

The composition, as well as the second map, are leaf anodyne as a consequence of Lemma 5.22. By the two-out-of-three property, the first map is also in $\mathcal{C}$. We now consider the case $n \geq 2$, working by induction on n. The case $n = 2$ is contained in $\mathcal{C}$ by assumption. Suppose we have already dealt with the corollas C_k for $2 \leq k < n$. Consider the inclusion $\mathrm{Out}(C_n) - \{n\} \to \Omega[C_n]$. Let T be the tree obtained by grafting an $(n-1)$-corolla C_v on top of a 2-corolla C_w, with edges labelled as in the following picture:



Then $\Omega[C_n] = \Omega[\partial_e T] \to \Omega[T]$ belongs to $\mathcal{C}$, because both horizontal inclusions in

$$\begin{tikzcd} \ell[C_n] \arrow[r] \arrow[d, equal] & \Omega[\partial_e T] \arrow[d] \\ \ell[T] \arrow[r] & \Omega[T] \end{tikzcd}$$

are leaf anodyne. Consider the maps

$$\mathrm{Out}(T) - \{n\} \to (\mathrm{Out}(T) - \{n\}) \cup \Omega[C_v] \to \mathrm{Sp}[T] \to \Omega[T].$$

The first one is a pushout of the leaf anodyne $\ell[C_v] \to \Omega[C_v]$ and therefore in $\mathcal{C}$. The second is a pushout of $\eta_0 \amalg \eta_e \to \Omega[C_w]$ and hence in $\mathcal{C}$ by assumption. The last map is inner anodyne, so we conclude that the composition of all three maps is in $\mathcal{C}$. But that map is also the composition of the maps

$$\coprod_{0 \leq i < n} \eta \to \Omega[C_n] \to \Omega[T]$$

and we already observed that the second map is in $\mathcal{C}$. Hence the first is in $\mathcal{C}$ as well, establishing (a).

For (b), consider a general tree T and an inclusion $\amalg \eta \to \Omega[T]$ of all but one of its outer edges. If the missing edge is the root, then the inclusion is leaf anodyne and thus contained in $\mathcal{C}$. If it is a leaf of T, number the leaves of T by $1, \ldots, n$ such that n is the missing leaf and consider the inclusions

$$\mathrm{Out}(T) - \{n\} \to \Omega[C_n] \to \Omega[T],$$

where the second map is the unique map preserving the root and sending leaves of C_n to leaves of T; it is the 'maximal inner face' of T obtained by contracting all inner edges. The first map is in $\mathcal{C}$ by (a), so by the two-out-of-three property it suffices to show that the second map is in $\mathcal{C}$ as well. This follows from the diagram

$$\begin{tikzcd}
\ell[C_n] \arrow[r] \arrow[d, equal] & \Omega[C_n] \arrow[d] \\
\ell[T] \arrow[r] & \Omega[T]
\end{tikzcd}$$

and the two-out-of-three property, noting that the horizontal maps are leaf anodyne and therefore in $\mathcal{C}$.

Finally we prove (c). Note that the final part of the argument for (b) actually proves the following, to be used below: if $f\colon S \to T$ is any map of trees which gives a bijection between $\mathrm{Out}(S)$ and $\mathrm{Out}(T)$, then f is in $\mathcal{C}$. This follows from the diagram

$$\begin{tikzcd}
\ell[S] \arrow[r] \arrow[d, equal] & \Omega[S] \arrow[d] \\
\ell[T] \arrow[r] & \Omega[T]
\end{tikzcd}$$

in which the horizontal arrows are leaf anodyne. Now consider a general tree T, larger than a corolla, for which the root face exists. Let V be a collection of external vertices containing the root vertex. We will show by downward induction on V that $\Lambda^V[T] \to \Omega[T]$ belongs to $\mathcal{C}$. In the minimal case where V consists of only the root vertex, this yields the desired result. Observe that if V is the set of *all* external vertices, then $\Lambda^V[T]$ is the union of all inner faces of T. Let $e_1, \ldots, e_n$ be all the inner edges of T. The face map $\partial_{e_1}T \to \Omega[T]$ is a bijection on external edges and therefore in $\mathcal{C}$. Note that there are pushout squares

$$\begin{tikzcd}
\bigcup_{i=1}^{k-1} \Omega[\partial_{e_i}\partial_{e_k}T] \arrow[r] \arrow[d] & \Omega[\partial_{e_k}T] \arrow[d] \\
\bigcup_{i=1}^{k-1} \Omega[\partial_{e_i}T] \arrow[r] & \bigcup_{i=1}^{k} \Omega[\partial_{e_i}T].
\end{tikzcd}$$

Working by induction on the number of inner faces and the size of T, we may assume that the top horizontal map is in $\mathcal{C}$. Hence the bottom map is in $\mathcal{C}$ as well. Composing these maps for $k = 2, \ldots, n$ it follows that the first of the maps

$$\Omega[\partial_{e_1}T] \to \Lambda^V[T] \to \Omega[T]$$

is in $\mathcal{C}$, and we already noted that the composition is in $\mathcal{C}$. Hence the remaining map $\Lambda^V[T] \to \Omega[T]$ is in $\mathcal{C}$ as well. This completes the base case where V consists of all external vertices. For the inductive step on V, assume that $V = W \cup \{v\}$ for some external vertex v. Consider the pushout square

$$\begin{tikzcd}
\Lambda^W[\partial_v T] \arrow[r] \arrow[d] & \Omega[\partial_v T] \arrow[d] \\
\Lambda^V[T] \arrow[r] & \Lambda^W[T].
\end{tikzcd}$$

If T is so small that $\partial_v T$ is a corolla, then T has only two vertices and is obtained by grafting the corolla C_v onto the root corolla C_r along an inner edge e. Then $\partial_v T = C_r$ and the square above should be read as

$$\begin{array}{ccc} \mathrm{Out}(C_r) - \{e\} & \longrightarrow & \Omega[C_r] \\ \downarrow & & \downarrow \\ \Lambda^V[T] & \longrightarrow & \Lambda^r[T]. \end{array}$$

The top arrow in this second square is in $\mathcal{C}$ by assumption and working by induction on the size of T, we may assume that the top arrow in the first squares is in $\mathcal{C}$ as well. Hence the bottom arrows in both squares are as well, completing the proof. □

We are now ready to establish the promised characterization of the fibrant objects of the Picard model structure. Note that a dendroidal set X which is fibrant is at least a dendroidal Kan complex, since horn inclusions of trees are Picard anodyne (and hence Picard trivial cofibrations). Proposition 9.70 implies that $\tau(i^*X)$ is canonically a symmetric monoidal groupoid. The main step towards this characterization is the following property of this symmetric monoidal groupoid:

Proposition 9.86 *A dendroidal set X is fibrant for the Picard model structure if and only if it is a dendroidal left Kan complex for which the symmetric monoidal groupoid $\tau(i^*X)$ is a Picard groupoid, i.e., if every object is invertible with respect to the tensor product.*

Proof Recall that the fibrant objects of the $\mathcal{A}$-model structure are precisely the $\mathcal{A}$-local objects. In the case at hand, this means that a dendroidal set X is fibrant if and only if for every Picard anodyne morphism $f: A \to B$, the map

$$\mathbf{hom}(B, X) \to \mathbf{hom}(A, X)$$

is a trivial fibration of simplicial sets.

First assume that X is fibrant in the Picard model structure. Then X is in particular fibrant in the absolute covariant model structure and therefore a dendroidal left Kan complex. Using that X has the extension property with respect to the inclusion $\eta \amalg \eta \to \Omega[C_2]$ of one leaf and the root of C_2, it is straightforward to see that any object x of the symmetric monoidal groupoid $\tau(i^*X)$ has an inverse with respect to the tensor product. Indeed, one picks a lift in the diagram

$$\begin{tikzcd} \eta \amalg \eta \arrow[r, "{(x,\mathbf{1}_X)}"] \arrow[d] & X, \\ \Omega[C_2] \arrow[ur, dashed] & \end{tikzcd}$$

where the top horizontal map sends one leaf of C_2 to x and the root to $\mathbf{1}_X$. Evaluating the lift on the other leaf of C_2 gives the desired inverse $x^\vee$.

For the converse, assume X is a dendroidal left Kan complex for which $\tau(i^*X)$ is a Picard groupoid. Write $\mathcal{C}$ for the collection of normal monomorphisms $f\colon A \to B$ for which the map above is a trivial fibration. It is saturated and closed under two-out-of-three. Moreover, it contains all leaf anodyne morphisms by the assumption that X is a dendroidal left Kan complex and hence fibrant in the absolute covariant model structure. By Proposition 9.84 it suffices to show that $\mathcal{C}$ contains the inclusion $\eta \amalg \eta \to \Omega[C_2]$ of one of the two leaves and the root of the 2-corolla. Thus, we should check that the Kan fibration

$$\mathbf{hom}(\Omega[C_2], X) \to \mathbf{hom}(\eta \amalg \eta, X) \cong i^*X \times i^*X$$

is in fact a trivial fibration. We will do this by checking that the fibre of this map over any vertex (x_1, y) of $i^*X \times i^*X$ is contractible. By Lemma 9.75, this fibre is homotopy equivalent to the simplicial set Y defined by the following pullback square:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & P_y i^*X \arrow[d, "\mathrm{ev}_0"] \\ \{x_1\} \times i^*X \arrow[r, "\otimes"] & i^*X \end{tikzcd}$$

Here $P_y i^*X$ denotes the fibre of $\mathrm{ev}_1 : \mathbf{hom}(\Delta[1], i^*X) \to i^*X$ over the vertex y. We claim that the bottom horizontal map is a homotopy equivalence. It follows that the top map is a homotopy equivalence as well; since $P_y i^*X$ is contractible, this proves the lemma.

To establish our claim, recall that $\tau(i^*X)$ is a Picard groupoid, so there exists a vertex $x_1^\vee$ such that $x_1 \otimes x_1^\vee$ and $x_1^\vee \otimes x_1$ are equivalent to the unit $\mathbf{1}_X$. But since the tensor product on i^*X is unital and associative up to homotopy (Lemma 9.76), it follows that $x_1 \otimes -$ has a left and right homotopy inverse, hence is a homotopy equivalence. □

Corollary 9.87 *A dendroidal set X is fibrant for the Picard model structure if and only if it is a Picard ∞-groupoid, i.e., has the extension property with respect to all horn inclusions of trees.*

Proof If X is fibrant then it has the extension property with respect to all horn inclusions, because those inclusions are Picard anodyne. Conversely, in the proof of Proposition 9.86 we already saw that for any dendroidal left Kan complex X which has the extension property with respect to the inclusion $\eta \amalg \eta \to C_2$ of a leaf and the root, the groupoid $\tau(i^*X)$ is Picard. In particular this holds true if X is a dendroidal Kan complex. But then Proposition 9.86 implies that X is fibrant. □

We can now also characterize the fibrations between fibrant objects:

Lemma 9.88 *A map $f\colon X \to Y$ between Picard ∞-groupoids is a fibration in the Picard model structure if and only if it is a dendroidal Kan fibration, i.e., has the right lifting property with respect to all horn inclusions.*

Proof We will use the fact that the Picard model structure is a left Bousfield localization of the absolute covariant model structure. By Lemma 8.50 (fibrations between fibrants do not change under localization), the map f is a Picard fibration if and only if it is a fibration in the covariant model structure. By Theorem 9.59(c), we see that this is the case if and only if f is a left fibration. Thus in particular, every dendroidal Kan fibration between Picard ∞-groupoids is a Picard fibration. For the converse, suppose that f is a left fibration between Picard ∞-groupoids. We will argue that f also has the right lifting property with respect to any root horn inclusion $\Lambda^r[T] \to \Omega[T]$. In other words, we will show that the map of simplicial sets

$$\mathbf{hom}(\Omega[T], X) \to \mathbf{hom}(\Lambda^r[T], X) \times_{\mathbf{hom}(\Lambda^r[T], Y)} \mathbf{hom}(\Omega[T], Y)$$

is surjective on vertices. In fact, we can even argue that this map is a trivial fibration. Indeed, it is a Kan fibration by Proposition 9.66. To see that it is a weak homotopy equivalence, consider the diagram

$$\begin{tikzcd}
\mathbf{hom}(\Omega[T], X) \arrow[dr] \arrow[drr, bend left] \arrow[ddr, two heads, bend right, "\sim"'] & & \\
& \mathbf{hom}(\Lambda^r[T], X) \times_{\mathbf{hom}(\Lambda^r[T], Y)} \mathbf{hom}(\Omega[T], Y) \arrow[r] \arrow[d, two heads, "\wr"] & \mathbf{hom}(\Omega[T], Y) \arrow[d, two heads, "\wr"] \\
& \mathbf{hom}(\Lambda^r[T], X) \arrow[r] & \mathbf{hom}(\Lambda^r[T], Y).
\end{tikzcd}$$

The left slanted and right vertical maps are trivial fibrations because X and Y are assumed to be fibrant objects. The left vertical map, being a pullback of a trivial fibration, is then also a trivial fibration. The map into the upper left corner of the square is then a weak equivalence by two-out-of-three. □

To conclude this section we prove Lemma 9.81. We first observe the following general fact about pushout-products:

Lemma 9.89 *Let $\mathcal{C}$ be a class of maps which is closed under pushout and composition and satisfies the right cancellation property. For a fixed map $i: A \to B$, consider the class $\mathcal{B}$ of maps $j: C \to D$ such that the pushout-product*

$$B \otimes C \cup A \otimes D \to B \otimes D$$

is contained in $\mathcal{C}$. Then $\mathcal{B}$ is also closed under pushout and composition and satisfies the right cancellation property.

Proof The case of pushouts and composition was already covered in Lemma 4.23. For the right cancellation property, consider $j: C \to D$ and $k: D \to E$ such that j and kj are in $\mathcal{B}$. Consider the diagram

$$\begin{tikzcd}
A \otimes D \cup_{A \otimes C} B \otimes C \arrow[rr] \arrow[d] & & B \otimes D \arrow[d] & \\
A \otimes E \cup_{A \otimes C} B \otimes C \arrow[r] & A \otimes E \cup_{A \otimes D} B \otimes D \arrow[r] & B \otimes E. &
\end{tikzcd}$$

The square is a pushout. The top horizontal map is in $\mathcal{C}$ by assumption, so that the bottom horizontal map in the square is in $\mathcal{C}$ as well. The composite of the lower two horizontal maps is in $\mathcal{C}$ by assumption. Since $\mathcal{C}$ has the right cancellation property, the remaining horizontal map is in $\mathcal{C}$. Hence k is in $\mathcal{B}$. $\square$

Recall that $\mathcal{P}_0$ denotes the class of anodyne maps of dendroidal sets and $\mathcal{P}$ denotes the smallest class containing $\mathcal{P}_0$ that is moreover closed under pushouts and composition and satisfies the right cancellation property. We explained in Remark 9.82 how to construct $\mathcal{P}$ from $\mathcal{P}_0$ explicitly by defining intermediate classes $\mathcal{P}_n$ for $n \geq 0$. Now suppose that the pushout-product of any element of $\mathcal{P}_0$ with a normal monomorphism is contained in $\mathcal{P}$. Then Lemma 9.89 implies that the same is true for elements of $\mathcal{P}_1$ and, by induction, for elements of any $\mathcal{P}_n$. Hence, to prove Lemma 9.81, it suffices to treat the generators of $\mathcal{P}_0$, i.e., the case where $A \longrightarrow B$ is a horn inclusion of a tree. As already remarked immediately after the statement of the lemma, the only case we still have to cover is that of a root horn of a tree T whose root vertex is not unary. We split the statement into two parts:

Lemma 9.90 *Let T be a non-unary corolla and let l be a leaf of T. Then for any $n \geq 0$, the map*

$$\partial\Delta[n] \otimes \Omega[T] \cup \Delta[n] \otimes (\partial T - \{l\}) \longrightarrow \Delta[n] \otimes \Omega[T]$$

is Picard anodyne.

Lemma 9.91 *Let T be a tree with at least two vertices, such that T admits a non-unary root face. Then for any $n \geq 0$, the map*

$$\partial\Delta[n] \otimes \Omega[T] \cup \Delta[n] \otimes \Lambda^r[T] \longrightarrow \Delta[n] \otimes \Omega[T]$$

is Picard anodyne.

Remark 9.92 Note that the lemmas above do *not* state that these pushout-product maps are root anodyne. In that sense, these lemmas are qualitatively different from our previous results (namely Lemma 6.24 and Corollary 6.30) about pushout-products of other inner, leaf, and unary root anodynes.

In the proofs of these lemmas it will be convenient to use the following:

Lemma 9.93 *Let T be a tree and let $V \subseteq \Omega[T]$ be a union of subtrees S of T such that for each subtree S occurring in the union, $\ell[S] \subseteq \ell[T]$. In words, each leaf of S should also be a leaf of T. Then the inclusion $V \cup \ell[T] \subseteq \Omega[T]$ is in $\mathcal{P}$.*

Proof Since $\mathcal{P}$ satisfies the right cancellation property and $\ell[T] \longrightarrow \Omega[T]$ is in $\mathcal{P}$, it suffices to show that $\ell[T] \longrightarrow V \cup \ell[T]$ is in $\mathcal{P}$. We work by induction on the number of subtrees S constituting V. If there is only one, then $\ell[T] \longrightarrow \Omega[S] \cup \ell[T]$ is a pushout of $\ell[S] \longrightarrow \Omega[S]$ and hence leaf anodyne. For a larger number of trees $S_1, \ldots, S_n$, consider the pushout square

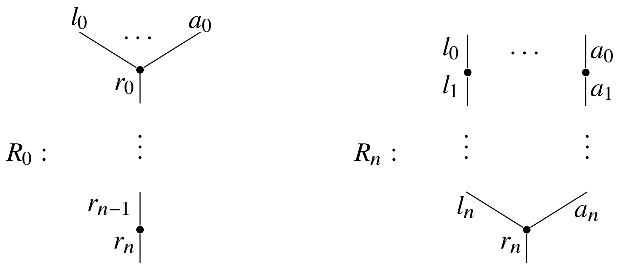
$$\begin{tikzcd}
\big(S_n \cap (S_1 \cup \cdots \cup S_{n-1})\big) \cup \ell[S_n] \arrow[r] \arrow[d] & S_1 \cup \cdots \cup S_{n-1} \cup \ell[T] \arrow[d] \\
S_n \arrow[r] & T.
\end{tikzcd}$$

(As usual we have abbreviated $\Omega[T]$ by T and similarly for the S_i.) Then it suffices to show that the left vertical map is in $\mathcal{P}$. That map is of the same form but for fewer subtrees, with S_n playing the role of T. This establishes the inductive step. □

Proof (of Lemma 9.90) For $n = 0$ there is nothing to prove, so assume $n > 0$. The map of the lemma is a normal monomorphism by Proposition 4.21 and we will accordingly regard

$$A_0 := \partial\Delta[n] \otimes T \cup \Delta[n] \otimes (\partial T - \{l\})$$

as a subobject of $\Delta[n] \otimes T$. Throughout this proof we write v for the unique vertex of T. The shuffles of the tensor product $\Delta[n] \otimes T$ are linearly ordered, where the initial shuffle R_0 is obtained by grafting T on top of the linear tree $[n]$ and the final shuffle R_n has copies of $[n]$ grafted onto the leaves of T:



We have only explicitly drawn one leaf of T other than l and called it a, but of course there could be many. The ith shuffle R_i is characterized by the fact that i is the smallest number for which the edge $(i, r) = r_i$ occurs in R_i. Alternatively, it is the unique shuffle containing the vertex $(i, v) = v_i$. Setting

$$A_i := A_0 \cup R_0 \cup \cdots \cup R_{i-1}$$

gives a filtration

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{n+1} = \Delta[n] \otimes T$$

and it will suffice to show that each inclusion $A_i \subseteq A_{i+1}$, adjoining a shuffle R_i, is Picard anodyne. We treat the cases $i = 0$, $0 < i < n$, and $i = n$ separately.

First, for $i = 0$, all faces of R_0 are contained in A_0 with the exception of the inner face contracting r_0. Thus $A_0 \to A_1$ is a pushout of the inner horn inclusion $\Lambda^{r_0}[R_0] \to R_0$ and therefore inner (in particular Picard) anodyne.

For $0 < i < n$, we define a further filtration

$$A_i =: A_i^0 \subseteq A_i^1 \subseteq A_i^2 \subseteq \cdots, \quad \bigcup_j A_i^j = A_{i+1},$$

by adjoining the prunings P of R_i one by one, in an order that extends the partial ordering of inclusion of prunings. Consider a step $A_i^j \subseteq A_i^{j+1}$ adjoining some pruning P. If P does not contain the vertex v_i, then it is only some linear part of R_i containing the root r_n and is contained in A_0, leaving nothing to prove. If P does contain v_i, write $J(P)$ for the set of all inner edges of P *except* for the outgoing edge r_i of the vertex v_i. (Compared to our earlier proofs of this kind, the edge r_i will play the role of the 'special edge'.) For a subset $H \subseteq J(P)$ we write P^H for the tree obtained from P by contracting all edges in $J(P) - H$. Pick a linear order on the collection of subsets of $J(P)$ extending the partial order of inclusion and adjoin the trees P^H to A_i^j in this order to obtain a further filtration

$$A_i^j =: A_i^{j,0} \subseteq A_i^{j,1} \subseteq A_i^{j,2} \subseteq \cdots, \quad \bigcup_k A_i^{j,k} = A_i^j \cup P = A_i^{j+1}.$$

We claim that each inclusion $A_i^{j,k} \subseteq A_i^{j,k+1}$ is inner anodyne. Indeed, say this inclusion is given by adjoining some tree P^H. If P^H is already contained in $A_i^{j,k}$ then there is nothing to prove. If it does not, we observe the following:

(1) Any leaf face of P^H is contained in a smaller pruning and therefore in A_i^j.
(2) The root face of P^H is contained in $\partial_n \Delta[n] \otimes T$, so in particular in A_0.
(3) Any face of P^H contracting an inner edge e other than r_i is contained in $P^{H'}$ with $H' = H - \{e\}$ and therefore in $A_i^{j,k}$ by our induction on the size of H.
(4) The inner face of P^H contracting the inner edge r_i cannot be contained in $A_i^{j,k}$, cf. (a)-(c) below.

It follows that $A_i^{j,k} \subseteq A_i^{j,k+1}$ is a pushout of the inner horn inclusion $\Lambda^{r_i}[P^H] \to P^H$ and hence inner anodyne as desired. It remains to verify the following:

(a) The inner face $\partial_{r_i} P^H$ still contains a non-unary vertex and is therefore clearly not contained in $\Delta[n] \otimes (\partial T - \{l\})$.
(b) The face $\partial_{r_i} P^H$ is not contained in $\partial\Delta[n] \otimes T$. Indeed if it were, then it would be contained in $\partial_i \Delta[n] \otimes T$, meaning that r_i is the *only* occurrence of i in P^H. In particular, all edges of the form l_i and a_i above v_i in P must have been inner and must already have been contracted to form P^H. But then P^H factors through the earlier shuffle R_{i-1}, contradicting our assumption that P^H is not contained in $A_i^{j,k}$.
(c) It follows from (a) and (b) that $\partial_{r_i} P^H$ is not contained in A_0. Also, it cannot be contained in an earlier shuffle (only in later ones, where v is shuffled down further), not in an earlier pruning, nor in some previous P^H. We conclude that indeed $\partial_{r_i} P^H$ cannot be contained in $A_i^{j,k}$.

We now proceed to the remaining case $i = n$, adjoining the shuffle R_n. It is here that the right cancellation property of $\mathcal{P}$ will play an important role. The map $A_n \subseteq A_n \cup R_n = A_{n+1}$ is a pushout of the inclusion $A_n \cap R_n \to R_n$ and hence it suffices to show that the latter map is Picard anodyne. First observe that the intersection $A_n \cap R_n$ is a union $E \cup L$ of the following two subobjects:

(1) L is the union of the leaf branches $\Delta[n] \otimes \{a\}$, for a ranging over the leaves of T other than l. Of course L is already contained in A_0.
(2) E is a union $E = \bigcup_{0 \leq i \leq n} E_i$, where E_i is the inner face of R_n obtained by contracting all edges of the form l_i and a_i (if $i > 0$, so that these are inner) or the outer face chopping off the leaves of the form l_0 and a_0 (for $i = 0$). Note that $E_i \subseteq A_0$ for $i < n$, whereas the inner face E_n is contained in the previous shuffle R_{n-1}. □

We will work by induction on the dimension n of the simplex $\Delta[n]$. For $n = 0$ the map under consideration is

$$E_0 \cup L \to R_0.$$

This is just a horn inclusion of the corolla associated with the leaf l and hence Picard anodyne by definition. Indeed, E_0 is the root edge of the corolla T, whereas L is the union of all leaf edges other than l.

For $n > 0$ we consider the diagram

$$\begin{tikzcd}
E_0 \cap (E_1 \cup \cdots \cup E_n \cup L) \arrow[r] \arrow[d] & E_1 \cup \cdots \cup E_n \cup L \arrow[d] \arrow[dr] & \\
E_0 \arrow[r] & E_0 \cup \cdots \cup E_n \cup L \arrow[r] & R_n
\end{tikzcd}$$

The slanted map is Picard anodyne by Lemma 9.93. Applying the right cancellation property to the triangle in the diagram, we see that it suffices to show that the right-hand vertical map is Picard anodyne. In turn, this map is a pushout of the left vertical map. Thus it suffices to show that the latter is Picard anodyne. But this map is exactly of the general form $E \cup L \to R$ again, but now for smaller n; indeed, the outer face E_0 is the last shuffle of the tensor product $\partial_0\Delta[n] \otimes T$. This establishes the inductive step and completes the proof.

Proof (of Lemma 9.91) The strategy is very similar to the proof of Lemma 9.90 above. Write e for the unique inner edge attached to the root vertex v of T and list the other incoming edges of v (which are necessarily leaf edges) as $a_1, \ldots, a_k$. Again we regard

$$A_0 := \partial\Delta[n] \otimes T \cup \Delta[n] \otimes \Lambda^r[T]$$

as a subobject of $\Delta[n] \otimes T$, and consider a linear order on the shuffles of the tensor product $\Delta[n] \otimes T$ compatible with the partial order starting with the shuffle R_0 obtained by grafting T on top of the linear tree $[n]$ and ending with the shuffle R_N having copies of $[n]$ grafted onto the leaves of T. Adjoining the R_i one by one gives a filtration

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_N = \Delta[n] \otimes T$$

with

$$A_i := A_0 \cup R_0 \cup \cdots \cup R_{i-1}.$$

We will show that each inclusion $A_i \subseteq A_{i+1}$ is Picard anodyne.

We will distinguish two cases: one where the shuffle R_i being adjoined does *not* have the root vertex v of T occurring at the bottom and one where it does. In the first case, the inclusion $A_i \subseteq A_{i+1}$ adjoining R_i is inner anodyne. The argument is essentially the same as before and we summarize it briefly. One refines the inclusion $A_i \subseteq A_{i+1}$ by adjoining prunings P of R_i one by one and then filtering further by adjoining inner faces P^H of such prunings, contracting inner edges of P other than the 'special edge' r_i. Then each step of the filtration is either an identity or a pushout of the inner horn inclusion

$$\Lambda^{r_i}[P^H] \to P^H$$

by exactly the same argument as in our previous proof.

For the case where the root vertex v of T occurs at the root of the shuffle R_i, the argument will also be similar to that of our previous proof. As before it suffices to show that the inclusion $A_i \cap R_i \to R_i$ is Picard anodyne. The intersection $A_i \cap R_i$ may now be described as a union $E \cup L$ of the following subobjects:

(1) L is the union of the leaf branches $\Delta[n] \otimes \{a_j\}$, for $1 \leq j \leq k$. All of these are already contained in A_0.
(2) E is a union $E_0 \cup E'$, where E' is the union of inner faces obtained as the intersection of R_i with all the previous shuffles already adjoined. The face E_0 is the pruning of R_i obtained by removing all edges of which the first coordinate is 0. In particular, it is a shuffle of $\partial_0 \Delta[n] \otimes T$.

Again, we work by induction on the dimension n of the simplex $\Delta[n]$. For $n = 0$ the map under consideration is isomorphic to $\Lambda^r[T] \to T$ and hence Picard anodyne by definition. For $n > 0$ we consider the diagram

$$\begin{tikzcd}
E_0 \cap (E' \cup L) \arrow[r] \arrow[d] & E' \cup L \arrow[d] \arrow[dr] & \\
E_0 \arrow[r] & E_0 \cup E' \cup L \arrow[r] & R_n
\end{tikzcd}$$

The slanted map is Picard anodyne by Lemma 9.93. As before it suffices to show that the left-hand vertical map is in $\mathcal{P}$. Again it is of the same general form as the map $A_i \cap R_i \to R_i$, but now for the smaller tensor product $\partial_0 \Delta[n] \otimes T$. This establishes the induction. $\square$

Historical Notes

What we call the operadic model structure on dendroidal sets originates in [40], as does the its version for uncoloured dendroidal sets. The covariant model structure for dendroidal sets was first studied in [77]. The Picard model structure first appears in work of Bašić–Nikolaus [13], who called it the 'stable model structure' because of its relation to infinite loop spaces. (We have chosen a different term, not only because of the relation to Picard categories, but also to avoid possible confusion with the notion of a stable model category.)

Our presentation here is different from all of these sources. We have tried to give uniform proofs of the existence of these various model structures by introducing the $\mathcal{A}$-model structure. As the reader will have noticed, the existence of 'countable approximations' of a trivial cofibration (Lemma 9.22) proved by means of the ladder argument plays a central role in establishing the model structures.

Part III
The Homotopy Theory of Simplicial and Dendroidal Spaces

Chapter 10
Reedy Categories and Diagrams of Spaces

In later chapters we will develop the homotopy theory of simplicial and dendroidal spaces, i.e., diagrams of simplicial sets indexed on the categories $\mathbf{\Delta}^{\mathrm{op}}$ and $\mathbf{\Omega}^{\mathrm{op}}$. There is a standard way of equipping such diagram categories with a model structure other than the projective one, but with the same weak equivalences, called the *Reedy model structure*. This structure has the advantage that both the fibrations and cofibrations are easy to control. In this short chapter we will introduce the theory of Reedy categories and the Reedy model structure, as well as its basic features. Moreover, this model structure will be a very useful tool for studying (co)simplicial objects in general model categories. It provides a flexible framework for various notions of 'geometric realization' for such objects, as well as a general theory of 'resolutions' of objects of an arbitrary model category, which will be important in the next chapter.

We illustrate the usefulness of cosimplicial objects in model categories by giving a general construction (due to Bousfield–Kan) of *homotopy colimits* in Section 10.5. The concluding Section 10.6 is a discussion of Quillen's Theorems A and B; these are fundamental tools in the study of categories from the point of view of simplicial homotopy theory and their proofs serve as a demonstration of the use of bisimplicial sets and homotopy colimits.

10.1 Reedy Categories

Recall from Example 7.46 that if $\mathcal{E}$ is a cofibrantly generated model category and $\mathbf{C}$ is a small category, then the category $\mathcal{E}^{\mathbf{C}}$ of $\mathbf{C}$-indexed diagrams in $\mathcal{E}$ carries a projective model structure. The fibrations are easily described as those natural transformations of diagrams which give a fibration in $\mathcal{E}$ after evaluation at any object in $\mathbf{C}$ (and the weak equivalences are described similarly). The cofibrations, by contrast, are less explicit: they are defined simply by having the left lifting property with respect to trivial fibrations. As a consequence, the projective model structure has relatively few cofibrations, which can be inconvenient. In this section we will introduce a class of indexing categories, the so-called *Reedy categories*, for which model structures on

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the functor category $\mathcal{E}^{\mathbf{C}}$ exist in which the fibrations and cofibrations play a more symmetric role. In particular, these model structures will have more cofibrations than the projective model structure. The weak equivalences will remain the same.

Definition 10.1 A *Reedy category* is a category $\mathbf{R}$ equipped with the following structure:

- a function $d\colon \mathrm{ob}(\mathbf{R}) \to \mathbb{N}$, called the *degree function*,
- two subcategories $\mathbf{R}^+$ and $\mathbf{R}^-$, called the subcategories of *positive* and of *negative* morphisms, respectively.

These data should satisfy the following axioms:

(a) Every morphism $f\colon r \to s$ can be factored as $f = p \circ n$, where p is positive and n is negative, and this factorization is unique up to isomorphism.
(b) If $f\colon r \to s$ belongs to $\mathbf{R}^+$ then $d(r) \leq d(s)$. This inequality is strict unless f is an isomorphism. Similarly, if $f\colon r \to s$ belongs to $\mathbf{R}^-$ then $d(r) \geq d(s)$ and moreover $d(r) > d(s)$ unless f is an isomorphism.
(c) Conversely, any isomorphism is both positive and negative.
(d) If $f\colon r \to s$ belongs to $\mathbf{R}^+$ and θ is an automorphism of r, then $f\theta = f$ implies that $\theta = \mathrm{id}$. Dually, if $f\colon r \to s$ belongs to $\mathbf{R}^-$ and $\theta f = f$ for some automorphism θ of s, then $\theta = \mathrm{id}$. (Informally, 'isomorphisms regard positive morphisms as monos and negative morphisms as epis'.)

Remark 10.2 (1) We will often denote the degree function as

$$d(r) = |r|$$

and write $r \xrightarrow{-} s$ or $r \xrightarrow{+} s$ to denote a morphism that is negative or positive, respectively.

(2) The uniqueness condition (a) means that in a solid commutative diagram

$$\begin{tikzcd} q \arrow[r, "-"] \arrow[d, "-"'] & r \arrow[d, "+"] \\ s \arrow[r, "+"] \arrow[ur, dashed, "\simeq"] & t \end{tikzcd}$$

there exists an isomorphism as indicated by the dashed arrow, making both triangles commute. This isomorphism is unique by condition (d).

(3) If $\mathbf{R}$ is a Reedy category, then the opposite category $\mathbf{R}^{\mathrm{op}}$ also acquires the structure of a Reedy category by interchanging the roles of the positive and negative morphisms. (However, we shall try to avoid using both structures, as it is easier to keep a single notion of positive and negative morphisms in mind for the examples below.)

Example 10.3 (1) The original example motivating the definition is the simplex category $\mathbf{\Delta}$, where $\mathbf{\Delta}^+$ consists of injections and $\mathbf{\Delta}^-$ of surjections, while the degree function d is defined by $d([n]) = n$. This category has no nontrivial isomorphisms, of course.

(2) Every group or groupoid is a Reedy category, where every morphism is both positive and negative.

(3) The category $\mathbf{\Omega}$ is a Reedy category as follows. For a tree T, we define its degree $d(T)$ to be the number of vertices in T. The subcategory $\mathbf{\Omega}^+$ consists of compositions of faces and isomorphisms, while $\mathbf{\Omega}^-$ consists of compositions of degeneracies and isomorphisms. The factorization axiom (a) holds by Proposition 3.9.

(4) The category $\mathbf{F}$ of finite sets is a Reedy category, where monomorphisms are positive and surjections are negative, while the degree function d counts the number of elements of a set.

(5) The category $\mathbf{F}_{\text{part}}$ of finite sets and partial maps is a Reedy category. Recall that a partial map $f\colon A \to B$ is a pair $(A' \subseteq A, f\colon A' \to B)$, with φ a map of finite sets. We refer to the subset A' as the *domain of definition* of f and denote it $\text{dom}(f)$. Every such partial map $f\colon A \to B$ can be factored as a composition of partial maps

$$A \longrightarrow \text{dom}(f) \longrightarrow\!\!\!\!\rightarrow \text{im}(f) \rightarrowtail B,$$

where the first map $A \to \text{dom}(f)$ is the 'inert' map which is the identity on $\text{dom}(f) \subseteq A$ and undefined outside $\text{dom}(f)$. As before we define the degree function d as the cardinality, partially defined surjections as negative maps and totally defined injections as positive morphisms. This category $\mathbf{F}_{\text{part}}$ is isomorphic to the category $\mathbf{F}_*$ of finite *pointed* sets via the functor adding to each set A a disjoint basepoint and extending each partial map $f\colon A \to B$ to an actual map

$$f_+\colon A \amalg \{*\} \to B \amalg \{*\}$$

sending every element outside $\text{dom}(f)$ to the basepoint.

(6) The category $\mathbf{\Gamma}$ introduced in Section 3.5.7 is the opposite of the category $\mathbf{F}_{\text{part}}$ (or $\mathbf{F}_*$). Therefore it becomes a Reedy category by defining a morphism in $\mathbf{\Gamma}$ to be positive (resp. negative) if its opposite is negative (resp. positive) in $\mathbf{F}_{\text{part}}$. Recall that taking the set of vertices of a tree defines a functor

$$V\colon \mathbf{\Omega} \to \mathbf{\Gamma}.$$

This functor preserves the Reedy structure. For example, if $S \to T$ is a face map in $\mathbf{\Omega}$, every vertex v in S is mapped to a subtree T_v of T and this defines a partial surjection $V(T) \to V(S)$, sending the vertices in T_v back to v.

Example 10.4 There is a collection of examples of a slightly different nature, which will be useful in deriving some general properties of homotopy limits and colimits, cf. Section 10.5 below. Among them are the following.

(1) The 'span category'

$$\bullet \longleftarrow \bullet \longrightarrow \bullet$$

can be made into a Reedy category in three distinct ways, depicted as

$$\bullet \xleftarrow{+} \bullet \xrightarrow{+} \bullet, \qquad \bullet \xleftarrow{+} \bullet \xrightarrow{-} \bullet, \qquad \bullet \xleftarrow{-} \bullet \xrightarrow{-} \bullet.$$

The same applies to its opposite.

(2) The category $\mathbb{N}$ corresponding to the poset of natural numbers, as well as its opposite, can be made into Reedy categories in a unique way so as to make the degree function the identity:

$$0 \xrightarrow{+} 1 \xrightarrow{+} 2 \xrightarrow{+} \cdots$$

and

$$0 \xleftarrow{-} 1 \xleftarrow{-} 2 \xleftarrow{-} \cdots.$$

Proposition 10.5 *If $\varphi\colon \mathbf{R} \to \mathbf{S}$ is an equivalence of categories and $\mathbf{S}$ has the structure of a Reedy category, then $\mathbf{R}$ can also be equipped with the structure of a Reedy category in a unique way for which φ preserves the Reedy structure.*

The proof of the proposition is obvious and implies the following useful fact.

Corollary 10.6 *Every Reedy category is equivalent to a skeletal Reedy category.*

10.2 Reedy Fibrations

In this section $\mathcal{E}$ is a fixed model category. For a Reedy category $\mathbf{R}$, we consider the category $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$ of '$\mathcal{E}$-valued presheaves on $\mathbf{R}$'. For example, if $\mathcal{E}$ is the category of simplicial sets with the Kan–Quillen model structure, then the categories of bisimplicial sets

$$\mathbf{sSets}^{\mathbf{\Delta}^{\mathrm{op}}}$$

and of dendroidal simplicial sets

$$\mathbf{sSets}^{\mathbf{\Omega}^{\mathrm{op}}}$$

are of this form. Since we think of simplicial sets with the Kan–Quillen model structure as a substitute for the homotopy theory of topological spaces, we will often refer to these as the categories of *simplicial spaces* and of *dendroidal spaces*, respectively.

For a simplicial set X, the face maps on X_n define a map

$$X_n \to X(\partial\Delta[n]).$$

Here $X(\partial\Delta[n])$ is the set of maps from $\partial\Delta[n] \to X$, which can also be written as the limit

$$\varprojlim_{\alpha\colon [m]\to[n]} X_m,$$

where the limit ranges over all proper injections $\alpha\colon [m] \to [n]$ in $\mathbf{\Delta}$. The same definition makes sense for arbitrary Reedy categories $\mathbf{R}$. Call a morphism $r \to s$ in $\mathbf{R}$ *strictly positive* if it is positive and *not* an isomorphism. If X is an $\mathcal{E}$-valued presheaf on $\mathbf{R}$, we will write

$$X(\partial r) = \varprojlim_{\alpha\colon s \xrightarrow{+} r} X_s$$

where the limit ranges over all strictly positive morphisms into r. More precisely, this is the limit over the full subcategory of $\mathbf{R}/r$ consisting of strictly positive morphisms into r. Equivalently (by an evident cofinality argument), one can take the limit over the full subcategory of $\mathbf{R}/r$ consisting of those $t \to r$ which factor through some strictly positive $s \to r$. From the latter description it is clear that $X(\partial r)$ is functorial in r and in particular carries a (right) action by the group $\mathrm{Aut}(r)$. Moreover, there is an $\mathrm{Aut}(r)$-equivariant map $X(r) \to X(\partial r)$. More generally, if $Y \to X$ is a morphism in $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$, we obtain for each object r in $\mathbf{R}$ an $\mathrm{Aut}(r)$-equivariant map in $\mathcal{E}$,

$$Y(r) \to Y(\partial r) \times_{X(\partial r)} X(r),$$

natural in r. These are usually called *matching maps* in the literature.

Definition 10.7 A map $Y \to X$ in $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$ is called a *Reedy (trivial) fibration* if for each object r in $\mathbf{R}$, the matching map above is a (trivial) fibration in $\mathcal{E}$.

We will see later that these are indeed the (trivial) fibrations for a model structure on $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$. The results of this section are preparing for that.

In order to characterize these fibrations by a lifting property we introduce some notation. If P is a presheaf of sets on $\mathbf{R}$ and A is an object of $\mathcal{E}$, we can define an object $A \boxtimes P$ of $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$ by

$$(A \boxtimes P)(r) := A \times P(r) = \coprod_{x \in P(r)} A.$$

In particular, we will use this notation for P the representable presheaf $\mathbf{R}(-, r)$ and for the subpresheaf $\mathbf{R}^+(-, r)$ of morphisms $t \to r$ which factor through a strictly positive morphism into r. We will usually abbreviate the presheaf $\mathbf{R}(-, r)$ by just r and the subpresheaf $\mathbf{R}^+(-, r)$ by ∂r. Thus, there are natural bijective correspondences between sets of morphisms

$$\mathrm{Hom}_{\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}}(A \boxtimes r, X) = \mathrm{Hom}_{\mathcal{E}}(A, X(r))$$

and

$$\mathrm{Hom}_{\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}}(A \boxtimes \partial r, X) = \mathrm{Hom}_{\mathcal{E}}(A, X(\partial r)),$$

for any A in $\mathcal{E}$ and X in $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$. From these correspondences we immediately obtain the following lemma.

Lemma 10.8 *A map $Y \to X$ in $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$ is a Reedy fibration (or a Reedy trivial fibration) if and only if for every object r in $\mathbf{R}$ and every trivial cofibration $A \to B$ in $\mathcal{E}$ (or every cofibration, respectively) the map $Y \to X$ has the right lifting property with respect to*

$$A \boxtimes r \cup_{A \boxtimes \partial r} B \boxtimes \partial r \to B \boxtimes r.$$

Remark 10.9 If $\mathcal{E}$ is cofibrantly generated, the same is of course true when one only considers *generating* (trivial) cofibrations $A \to B$.

We shall need to consider natural maps of the form

$$Y(r) \to Y(V) \times_{X(V)} X(r)$$

for 'subobjects of the boundary' $V \subseteq \partial r$. More precisely, let us call a subpresheaf $V \subseteq \mathbf{R}(-, r)$ *positive* if each $s \to r$ in V factors through some strictly positive map $t \to r$ belonging to V. Thus, ∂r is the maximal such positive subpresheaf of $\mathbf{R}(-, r)$. We will need the following simple lemma for such V.

Lemma 10.10 *Let $V \subseteq \mathbf{R}(-, r)$ be a positive subpresheaf. Then for any strictly positive $f: s \to r$ not belonging to V, the pullback $f^{-1}(V)$ is a positive subpresheaf of $\mathbf{R}(-, s)$.*

Proof This subpresheaf $f^{-1}(V)$ consists of those $t \to s$ for which the composition $t \to s \to r$ with f lies in V, so in particular factors through some strictly positive $m: x \to r$ in V. We can take for m the positive part in the negative-positive factorization of $t \to r$, as indicated in the outer square of the following diagram.

$$\begin{tikzcd}
s \arrow[rr, "+", "f"'] & & r \\
& y \arrow[ul, dashed, "+"'] & \\
t \arrow[uu] \arrow[ur, dashed, "-"] \arrow[rr, "-"] & & x \arrow[uu, "m", "+"'] \arrow[ul, "\simeq"]
\end{tikzcd}$$

We claim that $t \to s$ must factor through a strictly positive map in $f^{-1}(V)$ as well. Indeed, factor $t \to s$ as

$$t \xrightarrow{-} y \xrightarrow{+} s$$

as in the diagram. Then by uniqueness of factorizations, there exists an isomorphism $x \xrightarrow{\simeq} y$ making the diagram commute. If $y \to s$ would *not* be strictly positive, then it would be an isomorphism. But then f would be isomorphic to m and hence belong to V. This contradicts the assumption. □

For V as in the lemma, we will write

$$X(V) := \varprojlim_{(s \to r) \in V} X(s),$$

extending the notation $X(\partial r)$. Thus $X(V)$ is characterized by the bijective correspondence

$$\mathrm{Hom}_{\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}}(A \boxtimes V, X) = \mathrm{Hom}_{\mathcal{E}}(A, X(V)),$$

for every A in $\mathcal{E}$.

Proposition 10.11 (i) *If $Y \to X$ is a Reedy (trivial) fibration, then for each r in $\mathbf{R}$ and each positive subpresheaf $V \subseteq \partial r$, the generalized matching map*

$$Y(r) \to Y(V) \times_{X(V)} X(r)$$

is a (trivial) fibration in $\mathcal{E}$. In particular, so is $Y(r) \to X(r)$.

(ii) *If $Y \to X$ is a Reedy fibration, then it is a Reedy trivial fibration if and only if $Y(r) \to X(r)$ is a weak equivalence in $\mathcal{E}$ for every r in $\mathbf{R}$.*

Proof (i) First note that the 'in particular' part is simply the special case $V = \varnothing$. The proof of the other statement proceeds by induction on the degree of r and the size of V. For $|r| = 0$ there is nothing to prove, as $\partial r = \varnothing$.

For a given r, suppose the statement is true for all s of smaller degree and all $V \subseteq \partial s$. If $V \subset W \subseteq \partial r$ and W is the presheaf generated by adjoining a single strictly positive morphism $f : s \to r$ to V, we can consider the diagram

$$\begin{tikzcd}
Y(W) \times_{X(W)} X(r) \arrow[r] \arrow[d] & Y(V) \times_{X(V)} X(r) \arrow[d] \\
Y(W) \arrow[r] \arrow[d, equal] & Y(V) \times_{X(V)} X(W) \arrow[d, equal] \\
Y(V) \times_{Y(f^{-1}V)} Y(s) \arrow[r] \arrow[d] & Y(V) \times_{X(f^{-1}V)} X(s) \arrow[d] \\
Y(s) \arrow[r] & Y(f^{-1}V) \times_{X(f^{-1}V)} X(s)
\end{tikzcd}$$

in which each square is a pullback. The bottom horizontal map is a (trivial) fibration by the inductive hypothesis and Lemma 10.10, hence so is the top one.

Now let us suppose that $\mathbf{R}$ is countable for the moment. Then the maximal positive subpresheaf $\partial r \subseteq r$ can be written as a union

$$V = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \subseteq \cdots, \qquad \bigcup_n V_n = \partial r,$$

where V_{n+1} is obtained by adjoining a single strictly positive arrow $s \to r$ to V_n. Thus $Y(\partial r) \times_{X(\partial r)} X(r)$ is the limit of a tower

$$\begin{tikzcd} Y(r) \arrow[r] & Y(\partial r) \times_{X(\partial r)} X(r) \arrow[d] \\ & \vdots \arrow[d] \\ & Y(V_n) \times_{X(V_n)} X(r) \arrow[d] \\ & Y(V_{n-1}) \times_{X(V_{n-1})} X(r) \arrow[d] \\ & \vdots \arrow[d] \\ & Y(V_0) \times_{X(V_0)} X(r). \end{tikzcd}$$

Since all the maps in the tower have just been shown to be (trivial) fibrations, so is the projection from the limit,

$$Y(\partial r) \times_{X(\partial r)} X(r) \to Y(V) \times_{X(V)} X(r).$$

But then so is the composition

$$Y(r) \to Y(V) \times_{X(V)} X(r),$$

proving the statement in the case where **R** is countable.

All examples in this book concern countable Reedy categories, so this countability assumption is no restriction. Nonetheless, the same argument works for any Reedy category, except that one has to replace the tower of countable height by a taller one indexed by a suitable ordinal (not exceeding the cardinality of **R**).

(ii) The direction $\Rightarrow$ follows from part (i). For the reverse implication, assume $Y \to X$ is a Reedy fibration for which each $Y(r) \to X(r)$ is a weak equivalence. Then reading the same induction used to prove part (i) backwards, we can show by induction on the degree of r and the size of V that the maps in the tower are in fact trivial fibrations. But then so is the projection from the limit. Thus, taking $V_0 = \varnothing$ and applying two-out-of-three to the diagram

$$\begin{tikzcd} Y(r) \arrow[r] \arrow[d, "\wr"] & Y(\partial r) \times_{X(\partial r)} X(r) \arrow[d, two heads, "\wr"] \\ X(r) \arrow[r, equal] & Y(\varnothing) \times_{X(\varnothing)} X(r), \end{tikzcd}$$

we find that $Y(r) \to Y(\partial r) \times_{X(\partial r)} X(r)$ is a weak equivalence as well. □

Corollary 10.12 *Every Reedy (trivial) fibration in $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$ is a (trivial) fibration in the projective model structure.*

10.3 The Reedy Model Structure

As before, let $\mathcal{E}$ be a model category. If $\mathcal{E}$ is cofibrantly generated, then the category $\mathcal{E}^G$ of objects in $\mathcal{E}$ with a right G-action carries a projective model structure. A map in this model structure is a fibration or a weak equivalence if and only if it is one in $\mathcal{E}$. The cofibrations generally have a further freeness property.

Example 10.13 For the case $\mathcal{E} = \mathbf{sSets}$, equipped with the Kan–Quillen model structure, a map $X \to Y$ in $\mathbf{sSets}^G$ is a projective cofibration if and only if it is a monomorphism with the property that G acts freely on the complement of the image. Indeed, such a monomorphism can be obtained as the transfinite composition of pushouts of the maps

$$\partial\Delta[n] \times G \to \Delta[n] \times G,$$

which are precisely the generating cofibrations.

Theorem 10.14 *Let $\mathcal{E}$ be a cofibrantly generated model category and let $\mathbf{R}$ be a Reedy category. Then the category $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$ of $\mathcal{E}$-valued presheaves on $\mathbf{R}$ carries a cofibrantly generated model structure in which the fibrations are the Reedy fibrations and the weak equivalences are those maps $X \to Y$ such that $X(r) \to Y(r)$ is a weak equivalence for every r in $\mathbf{R}$. In other words, they coincide with the projective weak equivalences.*

We will explicitly describe the cofibrations in this model structure as well (cf. Proposition 10.15), but let us first prove the theorem.

Proof Notice first that Proposition 10.11(ii) asserts that a map is a Reedy fibration and a weak equivalence if and only if it is a Reedy trivial fibration as defined in Definition 10.7. For the moment, let us define a map to be a cofibration if it has the left lifting property with respect to the Reedy trivial fibrations. Then the axioms (M1–3) for a model structure are evidently satisfied. Furthermore, the factorization axiom follows from Lemma 10.8 and the small object argument once we observe that the saturation of the collection of maps of the form

$$A \boxtimes r \cup_{A \boxtimes \partial r} B \boxtimes \partial r \to B \boxtimes r,$$

for $A \to B$ a trivial cofibration in $\mathcal{E}$, is contained in the class of weak equivalences. This is indeed the case, because at a fixed object s in $\mathbf{R}$, this is the map

$$\Big(\coprod_{s \to r} A\Big) \cup \Big(\coprod_{s \xrightarrow{+} r} B\Big) \to \coprod_{s \to r} B,$$

where the three coproducts are over all the maps $s \to r$, the maps $s \to r$ factoring through a strictly positive map into r, and all the maps $s \to r$, respectively. As usual, the small object argument shows something stronger, namely that any map factors as a (trivial) fibration preceded by a map having the left lifting property with respect to all (trivial) fibrations.

Finally, one half of the lifting axiom (M4) holds by definition, the other half follows from the standard retract argument. This proves the theorem. $\square$

A feature of the Reedy model structure that we have already alluded to is that the cofibrations admit an explicit description dual to that of the Reedy fibrations. In order to state it, we define for an $\mathcal{E}$-valued presheaf U on $\mathbf{R}$ and an object $r \in \mathbf{R}$ an 'object of degenerate elements'

$$\deg(U)(r) = \varinjlim_{r \xrightarrow{-} s} U(s)$$

where the colimit ranges over all strictly negative morphisms out of r. For the case where $R = \mathbf{\Delta}$, the object $\deg(U)(n)$ consists precisely of the degenerate simplices in $U(n)$, which motivates the notation. There is an evident map

$$\deg(U)(r) \to U(r)$$

which is natural in r and in particular $\mathrm{Aut}(r)$-equivariant. More generally, if $U \to V$ is a morphism in $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$, we obtain for each object r in $\mathbf{R}$ a map

$$U(r) \cup_{\deg(U)(r)} \deg(V)(r) \to V(r),$$

again natural in r. In the literature these maps are called *latching maps*, dual to the matching maps we described before.

Proposition 10.15 *A morphism $U \to V$ in $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$ is a (trivial) cofibration in the Reedy model structure of Theorem 10.14 if and only if for each object r in $\mathbf{R}$, the latching map is a (trivial) cofibration in the projective model structure on $\mathcal{E}^{\mathrm{Aut}(r)}$. In particular, if r has no nontrivial automorphisms, the latching map should simply be a (trivial) cofibration in $\mathcal{E}$.*

Proof We prove the proposition for the cofibrations. The case of trivial cofibrations proceeds in exactly the same way. Let $\mathcal{C}$ be the class of morphisms $U \to V$ in $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$ for which each latching map is a cofibration in $\mathcal{E}^{\mathrm{Aut}(r)}$. Using the fact that colimits in $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$ are computed 'pointwise', for each r separately, one readily checks that this class is saturated. Moreover, each of the generating cofibrations

$$A \boxtimes r \cup B \boxtimes \partial r \to B \boxtimes r,$$

for a cofibration $A \to B$ in $\mathcal{E}$ (cf. Lemma 10.8), belongs to $\mathcal{C}$. Indeed, at a fixed object s in $\mathbf{R}$, the union

$$(\partial r \cup \deg(r))(s)$$

is precisely the set of maps $s \to r$ that are *not* isomorphisms. Therefore the latching map

$$(A \boxtimes r \cup B \boxtimes \partial r \cup B \boxtimes \deg(r))(s) \to (B \boxtimes r)(s)$$

is a pushout of the map

$$A \times \mathrm{Iso}(s, r) \to B \times \mathrm{Iso}(s, r),$$

which is a cofibration in $\mathcal{E}^{\mathrm{Aut}(r)}$. This proves that the Reedy cofibrations belong the class $\mathcal{C}$.

For the converse, we may as well assume that $\mathbf{R}$ is skeletal (cf. Corollary 10.6). Consider a morphism $i\colon U \to V$ in $\mathcal{C}$. To show that it is a Reedy cofibration, take any commutative square

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "i"] & Y \arrow[d, "p"] \\ V \arrow[r, "\beta"] & X \end{tikzcd}$$

where p is a Reedy trivial fibration. Let $\mathrm{sk}_n(X)$ be the restriction of X to the full subcategory $\mathbf{R}_{\leq n}$ of $\mathbf{R}$ of degree $\leq n$ and similarly for the other objects involved. Then it suffices to define compatible liftings

$$\begin{tikzcd} \mathrm{sk}_n(U) \arrow[r] \arrow[d] & \mathrm{sk}_n(Y) \arrow[d] \\ \mathrm{sk}_n(V) \arrow[r] \arrow[ur, dashed, "\varphi_n"] & \mathrm{sk}_n(X) \end{tikzcd}$$

by induction on n. For $n = 0$, the category $\mathbf{R}_{\leq 0}$ consists of isomorphisms only. Since $\mathbf{R}$ is assumed to be skeletal, it falls apart into a disjoint union of the groups $\mathrm{Aut}(r)$, for r ranging over the objects of degree 0. By the assumption on i, the map $\mathrm{sk}_0(i)\colon \mathrm{sk}_0(U) \to \mathrm{sk}_0(V)$ corresponds to a collection of projective cofibrations in the categories $\mathcal{E}^{\mathrm{Aut}(r)}$ and similarly $\mathrm{sk}_0(p)$ corresponds to a collection of trivial fibrations. Therefore a lift φ_0 exists.

For the inductive step, suppose that the lift φ_{n-1} has been defined and consider an object r in $\mathbf{R}$ of degree n. Working in the model category $\mathcal{E}^{\mathrm{Aut}(r)}$, we can find a lift φ_n in

$$\begin{tikzcd} U(r) \cup_{\deg(U)(r)} \deg(V)(r) \arrow[r, "(\alpha,\varphi)"] \arrow[d, tail] & Y(r) \arrow[d, two heads, "\wr"] \\ V(r) \arrow[r, "(\varphi,\beta)"'] \arrow[ur, dashed, "\varphi_n"] & Y(\partial r) \times_{X(\partial r)} X(r). \end{tikzcd}$$

Here the map $\varphi\colon V(r) \to Y(\partial r)$ on the bottom is the composition

$$V(r) \to V(\partial r) \xrightarrow{\varphi_{n-1}} Y(\partial r)$$

and similarly for $\varphi\colon \deg(V)(r) \longrightarrow Y(r)$ on top. By construction, this φ_n is natural for all automorphisms of r, all strictly negative $r \xrightarrow{-} s$ and all strictly positive $t \xrightarrow{+} r$. If we do this for all objects r of degree n, we obtain a map $\varphi_n\colon \mathrm{sk}_n(V) \longrightarrow \mathrm{sk}_n(Y)$ which is natural on all of $\mathbf{R}_{\leq n}$. Indeed, any map $r \longrightarrow s$ in $\mathbf{R}_{\leq n}$ belongs either to $\mathbf{R}_{<n}$, or is an automorphism, or factors as $r \xrightarrow{-} t \xrightarrow{+} s$, where the first is strictly negative or the second is strictly positive. This completes the induction and the proof of the proposition. □

Corollary 10.16 *Let $U \longrightarrow V$ be a Reedy cofibration in $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$. Then $U \longrightarrow V$ is a weak equivalence in the Reedy model structure if and only if each of the latching maps is a trivial cofibration.*

Indeed, by Proposition 10.15 these are just two ways of describing the trivial cofibrations in the Reedy model structure.

Remark 10.17 We have observed in Section 10.1 that for every Reedy category $\mathbf{R}$, its opposite is a Reedy category as well. So for any cofibrantly generated model category $\mathcal{E}$, the category $\mathcal{E}^{\mathbf{R}}$ of covariant functors also carries a Reedy model structure. To emphasize this distinction, we will sometimes refer to the two model structures appearing in this way as the *contravariant Reedy model structure* (on $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$) and the *covariant Reedy model structure* (on $\mathcal{E}^{\mathbf{R}}$). Rather than switching the meaning of ‘positive’ and ‘negative’ in $\mathbf{R}$, it will be easier to keep only on interpretation of these adjectives in mind for a given $\mathbf{R}$. The covariant Reedy fibrations and cofibrations can be described explicitly as follows:

- A map $Y \longrightarrow X$ in $\mathcal{E}^{\mathbf{R}}$ is a (covariant) Reedy fibration if and only if for each r in $\mathbf{R}$, the map
$$Y(r) \longrightarrow \varprojlim Y(s) \times_{\varprojlim X(s)} X(r)$$
is a fibration in $\mathcal{E}$ (and similarly for trivial fibrations), where the limits are taken over the strictly negative maps $r \xrightarrow{-} s$ out of r.
- A map $U \longrightarrow V$ in $\mathcal{E}^{\mathbf{R}}$ is a (covariant) Reedy cofibration if and only if for each r in $\mathbf{R}$, the map
$$U(r) \cup_{\varinjlim U(s)} \varinjlim V(s) \longrightarrow V(r)$$
is a cofibration in the projective model structure on $\mathcal{E}^{\mathrm{Aut}(r)}$ (and similarly for the trivial cofibrations), where the colimits are taken over the strictly positive maps $s \xrightarrow{+} r$.

Remark 10.18 We have assumed throughout that $\mathcal{E}$ is a *cofibrantly generated* model category, partly because this applies to all of our examples and partly to make sure that the projective model structure on each $\mathcal{E}^{\mathrm{Aut}(r)}$ exists. However, assuming the latter to exist, it is also possible to define not only the Reedy fibrations as in Section 10.2 but also the Reedy cofibrations as in Proposition 10.15. Then one uses the inductive argument in the proof of that proposition to show that these definitions yield a model structure. This applies in particular to Reedy categories in which the automorphism group of each object is trivial, in which case the assumption that $\mathcal{E}$ is cofibrantly generated becomes redundant. The Reedy model structure will then exist generally; of course, it might not be cofibrantly generated if $\mathcal{E}$ is not.

10.4 Simplicial Objects and Geometric Realization

Let $\mathcal{E}$ be a model category. Recall that we write

$$s\mathcal{E} = \mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$$

for the category of simplicial objects in $\mathcal{E}$. Since $\mathbf{\Delta}$ is a Reedy category, $s\mathcal{E}$ carries a Reedy model structure (in addition to the projective one, if it exists). The goal of this section is to present some basic properties of this particular Reedy model structure.

To begin with, let us observe that in many examples the Reedy cofibrant objects and the Reedy cofibrations are easy to recognize. We say that colimits in $\mathcal{E}$ are *universal* if for any morphism $A \to B$ in $\mathcal{E}$ the pullback functors $\mathcal{E}/B \to \mathcal{E}/A$ preserves colimits. This property holds in the category of sets and hence in all categories where colimits and pullbacks are computed 'as in **Sets**', in particular in categories of presheaves of sets or of simplicial sets.

Lemma 10.19 *Suppose that colimits in $\mathcal{E}$ are universal. Then for any object X in $s\mathcal{E}$, the map* $\deg(X)_n \to X_n$ *is a monomorphism. More generally, if $X \to Y$ is a monomorphism in $s\mathcal{E}$, then so is the latching map*

$$\deg(Y)_n \cup_{\deg(X)_n} X_n \to Y_n.$$

Proof We prove the first assertion; the proof of the second works the same way, but requires a bit more notation. Consider the pullback square

$$\begin{tikzcd} P \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & \deg(X)_n \arrow[d] \\ \deg(X)_n \arrow[r] & X_n \end{tikzcd}$$

in $\mathcal{E}$. We will prove that the evident map $p\colon \deg(X)_n \to P$ with $\pi_1 \circ p = \pi_2 \circ p = \mathrm{id}$ is an isomorphism. This will give the desired conclusion; indeed, if $f, g\colon V \to \deg(X)_n$ are two maps in $\mathcal{E}$ which become equal after composition with $\deg(X)_n \to X_n$, then together they define a map $v\colon V \to P$. Since p is an isomorphism, f and g must have been equal to begin with.

Universality of colimits implies that

$$\deg(X)_n \times_{X_n} \deg(X)_n = \varinjlim_{\sigma_1,\sigma_2} X_{m_1} \times_{X_n} X_{m_2},$$

where the colimit ranges over proper surjections $\sigma_1 : [n] \to [m_1]$ and $\sigma_2 : [n] \to [m_2]$ in $\mathbf{\Delta}$. In Proposition 2.1 we observed that for any two such σ_1 and σ_2, their pushout

$$\begin{tikzcd} {[n]} \arrow[r, "\sigma_1"] \arrow[d, "\sigma_2"'] & {[m_1]} \arrow[d] \\ {[m_2]} \arrow[r] & {[k]} \end{tikzcd}$$

exists in $\mathbf{\Delta}$ and is absolute. In particular, the pushout induces a pullback

$$\begin{tikzcd} X_n & X_{m_1} \arrow[l, "\sigma_1^*"'] \\ X_{m_2} \arrow[u, "\sigma_2^*"] & X_k . \arrow[l] \arrow[u] \end{tikzcd}$$

It follows that the map

$$\varinjlim_{\tau} X_k \to \varinjlim_{\sigma_1,\sigma_2} X_{m_1} \times_{X_n} X_{m_2}$$

is an isomorphism, where the colimit on the left ranges over proper surjections $\tau : [n] \to [k]$. But this is precisely the map

$$p : \deg(X)_n \to \deg(X)_n \times_{X_n} \deg(X)_n$$

described in the first part of the proof. □

Corollary 10.20 *Suppose that colimits in $\mathcal{E}$ are universal and that the cofibrations of the model structure on $\mathcal{E}$ are precisely the monomorphisms. Then the same is true for the Reedy model structure on $s\mathcal{E}$.*

Proof With the stated hypotheses, Lemma 10.19 implies that every monomorphism in $s\mathcal{E}$ is a Reedy cofibration. Conversely, for any Reedy cofibration $f : X \to Y$ in $s\mathcal{E}$, the maps $f_n : X_n \to Y_n$ are cofibrations (hence monomorphisms) in $\mathcal{E}$ for every n. But then f itself is a monomorphism in $s\mathcal{E}$. □

Example 10.21 Let $\mathcal{E}$ be the category **sSets**. Then $s\mathcal{E}$ is the category of bisimplicial sets, or simplicial spaces since we suggestively refer to the objects of $\mathcal{E}$ as 'spaces'. We will also denote this category **bisSets**. If one equips $\mathcal{E}$ with the Kan–Quillen or categorical model structure, then the Reedy cofibrations $s\mathcal{E}$ are precisely the monomorphisms.

Remark 10.22 Recall that the projective model structure on $s\mathcal{E}$, when it exists, has as its fibrations (resp. weak equivalences) the maps of simplicial objects $X_\bullet \to Y_\bullet$ such that for each $n \geq 0$, the morphism $X_n \to Y_n$ is a fibration (resp. weak equivalence). In other words, the fibrations and weak equivalences are detected 'pointwise'. By contrast, Corollary 10.20 describes a case in which the cofibrations and weak equivalences are detected pointwise. Such a model structure, when it exists, is called the *injective* model structure. Thus Corollary 10.20 can be interpreted as saying that if colimits in $\mathcal{E}$ are universal and the cofibrations of the model structure on $\mathcal{E}$ are precisely the monomorphisms, then the Reedy and injective model structures coincide on $s\mathcal{E}$.

The same principle applies when the cofibrations are the monomorphisms with some additional freeness conditions. Rather than formalize a general statement, we mention two relevant examples:

Example 10.23 (1) Let G be a group and let $\mathcal{E} = \mathbf{sSets}^G$ be the category of simplicial sets equipped with a right G-action, equipped with the projective model structure. Then the Reedy cofibrations in $s\mathcal{E}$ are the monomorphisms $f\colon X \to Y$ for which G acts freely on the complement of the image.

(2) Let $\mathcal{E}$ be the category **dSets** of dendroidal sets, with one of the model structures discussed in Chapter 9. A map $X \to Y$ is a Reedy cofibration with respect to any of these model structures if and only if for each tree T in $\mathbf{\Omega}$ and each $n \geq 0$, the map $X(T)_n \to Y(T)_n$ of sets is a monomorphism with a free action of $\mathrm{Aut}(T)$ on the complement of the image. In other words, each $X_n \to Y_n$ is a normal monomorphism of dendroidal sets.

The next topic to discuss is a general form of 'geometric realization'. Recall that for a simplicial set or simplicial space X, its geometric realization $|X|$ is a colimit of copies of $X_n \times \Delta^n$, where Δ^n is the standard topological n-simplex. These n-simplices form a *cosimplicial object*, i.e., a functor

$$\Delta^\bullet\colon \mathbf{\Delta} \to \mathbf{Top}\colon [n] \mapsto \Delta^n.$$

For a simplicial space $X \in \mathbf{sTop} = \mathbf{Top}^{\mathbf{\Delta}^{\mathrm{op}}}$, one can view $|X|$ as a 'tensor product' of the right $\mathbf{\Delta}$-module X and the left $\mathbf{\Delta}$-module $\Delta^\bullet$. This formulation applies more generally. Indeed, if $\mathcal{E}$ is a category with a suitable tensor product, one can form for each simplicial object X in $s\mathcal{E}$ and each cosimplicial object $C \in \mathcal{E}^{\mathbf{\Delta}}$ such a colimit in $\mathcal{E}$ (assuming colimits in $\mathcal{E}$ exist), which we will denote by

$$|X|_C$$

and refer to as the *realization of X with respect to $C^\bullet$*. To be precise, $|X|_C$ can be constructed as the coequalizer

$$\coprod_{[n]\xrightarrow{\alpha}[m]} X_m \otimes C^n \overset{\alpha^*\otimes \mathrm{id}}{\underset{\mathrm{id}\otimes\alpha_*}{\rightrightarrows}} \coprod_{n\geq 0} X_n \otimes C^n \longrightarrow |X|_C.$$

Let us assume that the tensor product in $\mathcal{E}$ preserves colimits in each variable separately. Just like for the geometric realization of simplicial sets, the good behaviour of the tensor product with respect to colimits implies that this general realization $|X|_C$ carries a skeletal filtration. In other words, the coequalizer above can be constructed in stages, as

$$|X|_C = \varinjlim_n |X|_C^{(n)}$$

where $|X|_C^{(0)} = X_0 \otimes C^0$ and $|X|_C^{(n)}$ is constructed from $|X|_C^{(n-1)}$ as the pushout

$$\begin{tikzcd} \deg(X)_n \otimes C^n \cup X_n \otimes \partial C^n \arrow[r] \arrow[d] & {|X|_C^{(n-1)}} \arrow[d] \\ X_n \otimes C^n \arrow[r] & {|X|_C^{(n)}.} \end{tikzcd}$$

Here $\partial C^n = \varinjlim_{[m]\to[n]} C^m$, where the colimit ranges over strict monomorphisms $[m] \to [n]$ in $\mathbf{\Delta}$. (In other words, $\partial C^n \to C^n$ is the latching object for the covariant Reedy model structure on $\mathcal{E}^{\mathbf{\Delta}}$, if $\mathcal{E}$ is a model category.) The union symbol in the top left corner of the square denotes the pushout over $\deg(X)_n \otimes \partial C^n$. The vertical map on the left comes from the maps $X_m \otimes C^n \to X_n \otimes C^n$, for proper surjections $[n] \to [m]$ involved in the definition of $\deg(X)_n$, and the maps $X_n \otimes C^m \to X_n \otimes C^n$ for proper injections $[m] \to [n]$ involved in ∂C^n. (Thus, the construction of the left vertical map already uses the assumption that the tensor product preserves colimits in each variable separately.)

Let us moreover assume that there exists an 'internal hom' with respect to the tensor product of $\mathcal{E}$, characterized by the usual adjunction property

$$\mathcal{E}(A \otimes B, C) \cong \mathcal{E}(A, \mathbf{Hom}(B, C)),$$

for objects A, B, and C in $\mathcal{E}$. Then just like for topological spaces or simplicial sets, the 'geometric realization functor'

$$|-|_C \colon s\mathcal{E} \to \mathcal{E}$$

has a right adjoint

$$\mathrm{Sing}_C \colon \mathcal{E} \to s\mathcal{E},$$

the 'singular complex functor' defined by

$$\mathrm{Sing}_C(E)_n = \mathbf{Hom}(C^n, E).$$

Example 10.24 If E is an object of $\mathcal{E}$ and M is a simplicial set, one can form an object $E \boxtimes M$ in $s\mathcal{E}$ (as in previous sections for $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$) and hence a realization $|E \boxtimes M|_C$. Now the terms C^n of the cosimplicial object C reoccur as realizations of standard simplices, as

$$|E \boxtimes \Delta[n]|_C = E \otimes C^n.$$

One can identify several other such realizations, for example

$$|E \boxtimes \partial\Delta[n]|_C = E \otimes \partial C^n.$$

Let us interpret this general geometric realization in the context of a model category $\mathcal{E}$. A cosimplicial object C is cofibrant for the covariant Reedy model structure on $\mathcal{E}^{\mathbf{\Delta}}$ if each of the boundary inclusions $\partial C^n \to C^n$ considered above is a cofibration. This implies that each object C^n is cofibrant as well. We say that $C^\bullet$ *has the pushout-product property* if for any cofibration $U \to V$ in $\mathcal{E}$, the map

$$U \otimes C^n \cup_{U \otimes \partial C^n} V \otimes \partial C^n \to V \otimes C^n$$

is another cofibration in $\mathcal{E}$, which is moreover a trivial cofibration whenever $U \to V$ is. Notice that we have come across conditions of this type in Chapters 8 and 9. In particular, if the model structure on $\mathcal{E}$ itself has the property that the pushout-product of two cofibrations is another cofibration, which is trivial if one of the factors is, then the assumption that $C^\bullet$ is Reedy cofibrant automatically implies that it also has the pushout-product property.

Proposition 10.25 *Let $\mathcal{E}$ be a model category equipped with a tensor product and internal hom as above. If $C^\bullet$ is a Reedy cofibrant object in $\mathcal{E}^{\mathbf{\Delta}}$ satisfying the pushout-product condition, then the associated adjoint pair*

$$|-|_C : s\mathcal{E} \rightleftarrows \mathcal{E} : \mathrm{Sing}_C$$

is a Quillen pair.

Proof We will check that the right adjoint is a right Quillen functor. To this end, consider a map $f: E \to D$ in $\mathcal{E}$. Then $\mathrm{Sing}_C(f)$ is a Reedy fibration (resp. a Reedy trivial fibration) if and only if for each trivial cofibration (resp. each cofibration) $i: U \to V$ in $\mathcal{E}$, the map $\mathrm{Sing}_C(f)$ has the right lifting property with respect to the maps

$$U \boxtimes \Delta[n] \cup V \boxtimes \partial\Delta[n] \to V \boxtimes \Delta[n], \quad n \geq 0,$$

cf. Lemma 10.8. By adjunction, Example 10.24 shows that this is equivalent to $E \to D$ having the right lifting property with respect to the maps

$$U \otimes C^n \cup V \otimes \partial C^n \to V \otimes C^n.$$

and each of these maps is a (trivial) cofibration whenever $U \to V$ is, by the pushout-product assumption on $C^\bullet$. This shows that $\mathrm{Sing}_C(-)$ preserves fibrations and trivial fibrations, so proves the proposition. □

Example 10.26 Consider for each $n \geq 0$ the representable simplicial set $\Delta[n]$. These form a cosimplicial object $\Delta[\bullet]$ in **sSets**, which is easily seen to be Reedy cofibrant in $\mathbf{sSets}^{\mathbf{\Delta}}$. (Indeed, this just means that $\partial\Delta[n] \to \Delta[n]$ is a monomorphism of simplicial sets for $n \geq 0$.) Moreover, $\Delta[\bullet]$ satisfies the pushout-product property for the cartesian product, both for the Kan–Quillen and the categorical model structures on the category of simplicial sets. Thus we obtain a geometric realization functor with respect to the cartesian product and this cosimplicial object,

$$|-|_\Delta : \mathbf{bisSets} \to \mathbf{sSets},$$

which is left Quillen for both of these model structures. In this special case, this realisation functor takes a particularly simple form, because it can be identified with the diagonal. In more detail, consider the diagonal functor

$$\delta\colon \mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}.$$

It induces adjoint functors (cf. Section 2.4)

$$\begin{tikzcd} \mathbf{sSets} \arrow[r, bend left, "\delta_!"] \arrow[r, bend right, "\delta_*"'] & \mathbf{bisSets} \arrow[l, "\delta^*"'] \end{tikzcd}$$

and we claim that there is a natural isomorphism

$$|X|_\Delta \cong \delta^* X$$

for each bisimplicial set X. Indeed, since both functors preserve colimits, it suffices to check this for representable bisimplicial sets X. If X is represented by $([p], [q]) \in \mathbf{\Delta} \times \mathbf{\Delta}$, one usually writes $X = \Delta[p, q]$. Then $\delta^*\Delta[p, q] = \Delta[p] \times \Delta[q]$. On the other hand, X is also of the form $E \boxtimes M$ considered in Example 10.24 above, namely $X = \Delta[p] \boxtimes \Delta[q]$. So

$$|X|_\Delta = \Delta[p] \times \Delta[q]$$

as well. This identification is evidently natural in p and q.

As a consequence, we obtain the following fundamental property of bisimplicial sets, which we record explicitly:

Corollary 10.27 *Let $X \to Y$ be a map of bisimplicial sets. If each $X_n \to Y_n$ is a (classical or categorical) weak equivalence of simplicial sets, then so is the diagonal $\delta^*X \to \delta^*Y$.*

Proof Consider the Reedy model structure on **bisSets**. Then $X \to Y$ is a Reedy weak equivalence between Reedy cofibrant objects (cf. Example 10.21) and these are preserved by any left Quillen functor. The result then follows from Proposition 10.25 and the identification of $|-|_\Delta$ with the diagonal. □

Example 10.28 Let $J[n]$ be the nerve of the groupoid on the objects $\{0, \ldots, n\}$ with exactly one isomorphism between any two objects. Then there is a map of simplicial sets $\Delta[n] \to J[n]$ which is the identity on objects, and which is natural in $[n]$. In other words, it is a map of cosimplicial objects $\Delta[\bullet] \to J[\bullet]$. The maps $\Delta[n] \to J[n]$ are weak equivalences of simplicial sets for the Kan–Quillen model structure (both are weakly contractible), but not for the categorical one. Thus, we obtain an essentially different realization

$$|-|_J : \mathbf{sSets}^{\Delta^{\mathrm{op}}} \to \mathbf{sSets}$$

for the categorical model structure and the associated Reedy model structure.

Example 10.29 Each cosimplicial object $C^\bullet$ in simplicial sets gives a cosimplicial object $i_!C^\bullet$ in dendroidal sets via the embedding $i_!\colon \mathbf{sSets} \to \mathbf{dSets}$. In particular, we obtain such objects $i_!\Delta[\bullet]$ and $i_!J[\bullet]$ in $\mathbf{dSets}^{\boldsymbol{\Delta}}$, which are again Reedy cofibrant and have the pushout-product property. Thus, we obtain two geometric realizations for any dendroidal space X, related by a map

$$|X|_\Delta \to |X|_J,$$

and both define left Quillen functors $\mathbf{sdSets} \to \mathbf{dSets}$. In particular, both preserve weak equivalences between cofibrant (i.e., normal) objects.

We conclude this section with the following useful application of the fundamental property of bisimplicial sets, cf. Corollary 10.27 above. For a category $\mathbf{C}$, consider the category

$$\mathrm{sPSh}(\mathbf{C}) = \mathbf{sSets}^{\mathbf{C}^{\mathrm{op}}}$$

of simplicial presheaves on $\mathbf{C}$. This category can be equipped with the projective model structure (with respect to the Kan–Quillen model structure on $\mathbf{sSets}$).

Proposition 10.30 *For each simplicial presheaf X on $\mathbf{C}$ there exists a weak equivalence $P \to X$ such that for every $n \geq 0$, the presheaf P_n is a coproduct of representable presheaves.*

Remark 10.31 The construction in the proof below in fact supplies an explicit cofibrant replacement of X in the projective model structure on the category $\mathrm{sPSh}(\mathbf{C})$ of simplicial presheaves, cf. Example 13.36(a).

Proof Write $\mathcal{E} = \mathbf{sSets}^{\mathbf{C}^{\mathrm{op}}}$, considered as a model category with the projective model structure. Let

$$P_n = \coprod_{c_0 \to \cdots \to c_n} \mathbf{C}(-, c_0) \times X(c_n).$$

Then $P_\bullet$ is an object of $s\mathcal{E}$, the category of bisimplicial presheaves on $\mathbf{C}$. (The simplicial structure of $P_\bullet$ comes from that of the nerve of $\mathbf{C}$ and the fact that $\mathbf{C}(-, c_0)$ is covariant in c_0, while $X(c_n)$ is contravariant in c_n.) As a bisimplicial presheaf, we can write more explicitly

$$P_{n,m}(d) = \coprod_{c_0 \to \cdots \to c_n} \mathbf{C}(d, c_0) \times X(c_n)_m,$$

for every object d in $\mathbf{C}$. There is a canonical map of bisimplicial presheaves

$$\varphi\colon P \to \mathrm{con}(X),$$

where $\mathrm{con}(X)$ as a constant simplicial object in simplicial presheaves. In bidegree (n, m) this map is defined as

$$\varphi(d \to c_0 \to \cdots \to c_n, x \in X(c_n)_m) = x|_d \in X(d)_m,$$

where $x|_d$ denotes the restriction of x along the composition $d \to c_n$ determined by the presheaf structure of X. For fixed m and d, this map can be viewed as

$$\varphi\colon N(d/X_m) \to X(d)_m,$$

where d/X_m is the category whose objects are pairs $(d \xrightarrow{\alpha} c, x \in X(c)_m)$ and whose morphisms $(\alpha, x) \to (\alpha', x')$ are $\beta\colon c \to c'$ in $\mathbf{C}$ with $\beta^* x' = x$. This category has a 'discrete' subcategory given by objects (α, x) where α is the identity on d, which can clearly be identified with just the set $X(d)_m$. Moreover, the corresponding inclusion $X(d)_m \to d/X_m$ has a right adjoint, mapping $(d \xrightarrow{\alpha} c, x)$ to $\alpha^* x$. This adjunction gives a weak equivalence of nerves, showing that φ is a weak equivalence for fixed m and for each d. By Corollary 10.27 it follows that $\delta^* P(d) \to X(d)$ is a weak equivalence for each d, i.e., $\delta^* P \to X$ is a projective weak equivalence. Now observe that for each n, the presheaf $(\delta^* P)_n$ of sets is a sum of representables,

$$(\delta^* P)_n = \coprod \mathbf{C}(-, c_0),$$

where the sum ranges over all $c_0 \to \cdots \to c_n$ in $\mathbf{C}$ and all $x \in X(c_n)_n$. This proves the proposition. $\square$

10.5 Homotopy Colimits

In this short section we make some remarks on homotopy colimits, which will be of use later on. We will leave it to the reader to explicitly formulate the dual remarks for homotopy limits.

Let $\mathcal{E}$ be a cofibrantly generated model category. Then for any small category $\mathbf{C}$, there are adjoint functors

$$\varinjlim : \mathcal{E}^{\mathbf{C}} \rightleftarrows \mathcal{E} : \mathrm{con}.$$

The projective model structure on $\mathcal{E}^{\mathbf{C}}$ makes this into a Quillen pair. One generally writes hocolim $:= \mathbf{L}\varinjlim$ for the left derived functor of the colimit functor and refers to it as the *homotopy colimit functor*. Thus, one way to compute $\mathrm{hocolim}_{\mathbf{C}} X$ for a diagram X in $\mathcal{E}^{\mathbf{C}}$ is as $\varinjlim_{\mathbf{C}} X'$, where $X' \to X$ is a cofibrant replacement of X in the projective model structure.

Example 10.32 Consider the poset of natural numbers $\mathbb{N}$, regarded as a category. Thus, there is a unique morphism $i \to j$ whenever $i \leq j$. A diagram

$$X = (X_0 \to X_1 \to X_2 \to \cdots)$$

is projectively cofibrant if each X_i is cofibrant and $X_i \to X_{i+1}$ is a cofibration for every $i \geq 0$. Thus, for a general diagram

$$Y_0 \to Y_1 \to Y_2 \to \cdots$$

one can compute its homotopy colimit as follows. First construct a diagram

$$\begin{tikzcd} Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \\ Y_0^c \arrow[u, "\wr"] \arrow[r] & Y_1^c \arrow[u, "\wr"] \arrow[r] & Y_2^c \arrow[u, "\wr"] \arrow[r] & \cdots \end{tikzcd}$$

in which the vertical maps are weak equivalences and the bottom row consists of cofibrations between cofibrant objects. (One can inductively define Y_{n+1}^c by factoring the composite $Y_n^c \to Y_n \to Y_{n+1}$ as a cofibration followed by a trivial fibration.) Then

$$\mathrm{hocolim}_{\mathbb{N}} Y_\bullet \simeq \varinjlim_{\mathbb{N}} Y_\bullet^c.$$

In the example above, cofibrant objects in the projective model structure are easy to understand. In other cases it can be convenient to be able to switch to an equivalent model structure with more cofibrations, so that it is possible to find another (usually smaller) cofibrant resolution. Indeed, suppose we have a model structure (with the same weak equivalences) on $\mathcal{E}^{\mathbf{C}}$ for which the adjoint pair $(\varinjlim, \mathrm{con})$ still forms a Quillen pair. Then taking the left derived functor of $\varinjlim$ results in an equivalent homotopy colimit functor (or, more precisely, a naturally isomorphic functor $\mathrm{Ho}(\mathcal{E}^{\mathbf{C}}) \to \mathrm{Ho}(\mathcal{E})$). Let us illustrate the idea by another example.

Example 10.33 Consider the span category S,

$$1 \xleftarrow{-} 0 \xrightarrow{+} 2,$$

interpreted as a Reedy category as indicated by the labels of the arrows. Equip the category $\mathcal{E}^S$ with the covariant Reedy model structure. Then a map of spans

$$\alpha\colon (X_1 \leftarrow X_0 \to X_2) \to (Y_1 \leftarrow Y_0 \to Y_2)$$

is a Reedy cofibration if the maps $X_0 \to Y_0$, $X_1 \to Y_1$, and

$$X_2 \cup_{X_0} Y_0 \to Y_2$$

are cofibrations in $\mathcal{E}$. Dually, it is a Reedy fibration if $X_1 \to Y_1$, $X_2 \to Y_2$, and

$$X_0 \to Y_0 \times_{Y_1} X_1$$

are fibrations. From this description it is easily checked that the constant diagram functor

$$\mathrm{con}\colon \mathcal{E} \to \mathcal{E}^S$$

preserves Reedy fibrations. Since it also preserves weak equivalences, it is right Quillen. Hence, the left derived functor

$$\mathbf{L}\varinjlim\colon \mathrm{Ho}(\mathcal{E}^S) \to \mathrm{Ho}(\mathcal{E})$$

with respect to the Reedy model structure is a model for the homotopy colimit. Thus, to compute the homotopy pushout of a span

$$X_1 \leftarrow X_0 \rightarrow X_2,$$

it suffices to replace it by a Reedy cofibrant weakly equivalent span. This is one in which all three objects are cofibrant and the rightward arrow is a cofibration. Observe that this recovers Lemma 7.51(2) from our earlier discussion of homotopy pushouts.

Finally, we will use Reedy model structures to establish a general formula for homotopy colimits, which is essentially the classical formula of Bousfield–Kan.

Example 10.34 Let $\mathbf{D}$ be a small category and consider the adjoint pair

$$\varinjlim : \mathcal{E}^{\mathbf{D}} \rightleftarrows \mathcal{E} : \mathrm{con}.$$

for the projective model structure on $\mathcal{E}^{\mathbf{D}}$, as in the beginning of this section. There is a standard technique for computing $\mathrm{hocolim}_{\mathbf{D}} X$ for a diagram $X \in \mathcal{E}^{\mathbf{D}}$ with the property that each $X(d)$ is a cofibrant object of $\mathcal{E}$, as we will now explain.

For an object d of $\mathbf{D}$, write

$$Y_n(d) = \coprod_{c_0 \to \cdots \to c_n \to d} X(c_0).$$

Then $Y_\bullet(d)$ is an object of $\mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$, with simplicial structure coming from that of the nerve of $\mathbf{D}/d$ and the action by $\mathbf{D}$ on X (involved in the face $d_0 \colon Y_n(d) \to Y_{n-1}(d)$.) This simplicial object $Y_\bullet(d)$ is clearly covariantly functorial in d; i.e., we have constructed a functor

$$Y_\bullet \colon \mathbf{D} \to \mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

Moreover, for the constant simplicial object $\mathrm{con}X(d)$, there are simplicial maps

$$Y_\bullet(d) \underset{\nu}{\overset{\pi}{\rightleftarrows}} \mathrm{con}X(d).$$

For a string of morphisms $c_0 \to \cdots \to c_n \to d$, the morphism π maps $X(c_0)$ to $X(d)$ by acting via the composed morphism $c_0 \to d$. The map ν sends $X(d)$ to the summand where all of the arrows $c_0 \to \cdots \to c_n \to d$ are the identity map of d. The map π is natural in d, but ν evidently is not.

Now assume that $\mathcal{E}$ has a tensor product that preserves colimits in each variable separately, satisfies the pushout-product property, and has cofibrant tensor unit $\mathbf{1}$. Recall that to any cosimplicial object $C^\bullet$ of $\mathcal{E}$, we can associate a 'geometric realization functor'

$$|\cdot|_C \colon \mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{E}.$$

We claim that under some natural conditions on the cosimplicial object $C^\bullet$ to be specified below, the homotopy colimit of X can be computed as

$$\mathrm{hocolim}_{\mathbf{D}} X \cong \varinjlim |Y_\bullet|_C.$$

The conditions we assume are as follows (cf. Section 10.4):

(1) The cosimplicial object $C^\bullet \in \mathcal{E}^{\mathbf{\Delta}}$ is Reedy cofibrant.
(2) The object $C^\bullet$ is a homotopically constant resolution of the tensor unit $\mathbf{1}$, in the sense that there is a weak equivalence

$$C^0 \to \mathbf{1}$$

and every map in the cosimplicial object

$$C^0 \rightleftarrows C^1 \mathrel{\substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow}} C^2 \cdots$$

is a weak equivalence.

The basic examples to keep in mind are $\mathcal{E} = \mathbf{sSets}$ (say with the Kan–Quillen model structure) or $\mathcal{E} = \mathbf{Top}$. In this case condition (2) can be phrased as saying that each C^n is weakly contractible, so that the standard simplices form a suitable choice. To establish our claim that $\varinjlim |Y_\bullet|_C$ is a model for the homotopy colimit, it suffices to verify the following two properties:

(a) The functor

$$d \mapsto |Y_\bullet(d)|_C$$

is a projectively cofibrant object of $\mathcal{E}^{\mathbf{D}}$.
(b) There is a weak equivalence

$$|Y_\bullet|_C \to X$$

in $\mathcal{E}^{\mathbf{D}}$.

Indeed, (a) and (b) combined give that $|Y_\bullet|_C$ is a cofibrant replacement for the diagram X. Property (a) readily follows from the definition of $|Y(d)|_C$ by considering its skeletal filtration. Indeed, one starts by observing that

$$|Y_\bullet|_C^{(0)} = \coprod_{c_0} \mathbf{D}(c_0, -) \boxtimes (X_{c_0} \otimes C^0).$$

(For a set S and object E of $\mathcal{E}$, we use the notation $S \boxtimes E$ to denote the S-fold coproduct of copies of E.) The expression above is a projectively cofibrant functor, using that $X_{c_0} \otimes C^0$ is a cofibrant object of $\mathcal{E}$. Proceeding by induction, we have that $|Y_\bullet|_C^{(n)}$ is obtained from $|Y_\bullet|_C^{(n-1)}$ by a pushout along

$$\coprod_{c_0 \to \cdots \to c_n} \mathbf{D}(c_n, -) \boxtimes (X_{c_0} \otimes \partial C^n) \to \coprod_{c_0 \to \cdots \to c_n} \mathbf{D}(c_n, -) \boxtimes (X_{c_0} \otimes C^n),$$

where the coproduct ranges over all nondegenerate simplices $c_0 \to \cdots \to c_n$. This is again a projective cofibration by our assumptions on X and $C^\bullet$, completing the proof of (a).

To establish (b), let us examine the two maps

$$Y_\bullet(d) \underset{\nu}{\overset{\pi}{\rightleftarrows}} \mathrm{con}X(d)$$

a little more closely. First observe that $\pi\nu$ is simply the identity of $\mathrm{con}X(d)$. In the other direction, the fact that the category $\mathbf{D}/d$ has a terminal object easily leads to the construction of a simplicial homotopy between $\nu\pi$ and the identity on $Y_\bullet(d)$. In other words, the simplicial set $\mathrm{con}X(d)$ is a simplicial deformation retract of $Y_\bullet(d)$ (although the retraction is not natural in d). We will demonstrate that simplicial homotopies yield left homotopies in $\mathcal{E}$ upon applying the realization $|\cdot|_C$. Property (b) then follows immediately from this.

For a simplicial set K and a simplicial object $Z_\bullet$ of $\mathcal{E}$, we write $K \boxtimes Z_\bullet$ (generalizing our previous notation) for the object of $\mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$ having $(K \boxtimes Z_\bullet)_n = K_n \boxtimes Z_n$. A simplicial homotopy between $\nu\pi$ and the identity on $Y_\bullet(d)$ can be thought of as a map of simplicial objects in $\mathcal{E}$

$$\Delta[1] \boxtimes Y_\bullet(d) \to Y_\bullet(d).$$

We should show that its realization is a weak equivalence in $\mathcal{E}$. More generally, we will demonstrate that for a general Reedy cofibrant simplicial object $Z_\bullet$ of $\mathcal{E}$, the projection map

$$\Delta[1] \boxtimes Z_\bullet \to \Delta[0] \boxtimes Z_\bullet \cong Z_\bullet$$

becomes a weak equivalence in $\mathcal{E}$ upon applying $|\cdot|_C$. (Said differently, this will prove that $|\Delta[1] \boxtimes Z_\bullet|_C$ serves as a cylinder object for $|Z_\bullet|_C$.) First consider the special case where $Z_\bullet$ is of the form $\Delta[n] \boxtimes E$ for some $n \geq 0$ and some cofibrant object E in $\mathcal{E}$. Observe that

$$|\Delta[n] \boxtimes E|_C \cong C^n \otimes E \xrightarrow{\simeq} C^0 \otimes E \xrightarrow{\simeq} E,$$

using Remark 10.24 for the isomorphism and assumption (2) for the weak equivalences. Now

$$|\Delta[1] \boxtimes (\Delta[n] \boxtimes E)|_C \cong |(\Delta[1] \times \Delta[n]) \boxtimes E|_C \cong \varinjlim_{\sigma} C^{n+1} \otimes E,$$

where σ ranges over all the shuffles $[n+1] \to [1] \times [n]$. Assumption (2) on our cosimplicial object $C^\bullet$ and an easy induction on the shuffles now implies that the maps

$$\varinjlim_{\sigma} C^{n+1} \otimes E \to C^0 \otimes E \to E$$

are weak equivalences. This completes the proof in the special case $Z_\bullet = \Delta[n] \boxtimes E$. For general Reedy cofibrant $Z_\bullet$ we argue by induction on its skeleta $\mathrm{sk}_n Z$. We can write $\mathrm{sk}_0 Z = \Delta[0] \boxtimes Z_0$, while $\mathrm{sk}_n Z$ is obtained from $\mathrm{sk}_{n-1} Z$ as the pushout

$$\begin{tikzcd}
\partial\Delta[n] \boxtimes Z_n \cup \Delta[n] \boxtimes \deg(Z)_n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1} Z \arrow[d] \\
\Delta[n] \boxtimes Z_n \arrow[r] & \mathrm{sk}_n Z.
\end{tikzcd}$$

The union in the top left denotes the pushout over $\partial\Delta[n] \boxtimes \deg(Z)_n$. With this notation, we have $Z \cong \varinjlim_n \mathrm{sk}_n Z$. The vertical map on the left is a Reedy cofibration in $\mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$. Now Corollary 7.50 and the cube lemma 7.51 give the inductive step from $\mathrm{sk}_{n-1} Z$ to $\mathrm{sk}_n Z$, using our previous argument for the object $\Delta[n] \boxtimes Z_n$ in the lower left corner.

10.6 A Version of Quillen's Theorem B

In the preceding sections we have seen some first applications of bisimplicial sets or, more generally, of simplicial objects in a model category $\mathcal{E}$, for example to the construction of homotopy colimits. To further illustrate the usefulness of these notions we devote this section to proving some fundamental facts about homotopy colimits of functors from a (simplicial) category $\mathbf{C}$ to the category $\mathbf{sSets}$ of simplicial sets. In particular we prove Quillen's Theorems A and B; these theorems are central to many applications of the theory of simplicial sets and belong in any homotopy theorist's toolkit. Also, the version of Quillen's Theorem B we treat here will be useful to us in Section 14.1.

Let $\mathbf{C}$ be a category and $f: \mathbf{C} \to \mathbf{sSets}$ a simplicial diagram on it. According to the results of the previous section, the homotopy colimit of f may be described as the diagonal of the bisimplicial set $h(f)$ with (p, q)-simplices

$$h(f)_{p,q} = \coprod_{c_0 \to \cdots \to c_p} f(c_0)_q.$$

Here the coproduct is indexed over the p-simplices of the nerve $N\mathbf{C}$. In particular, there is an evident projection map

$$\mathrm{hocolim}_{\mathbf{C}} f \xrightarrow{\pi} N\mathbf{C}.$$

Note that the fibre of π over a vertex $c \in C$ is precisely the simplicial set $f(c)$. Thus it is tempting to think of π as a kind of fibration that encodes the family of simplicial sets $f(c)$ indexed over the points of $\mathbf{C}$. The following makes this idea more precise:

Proposition 10.35 *Suppose that for every morphism $\alpha: c \to d$ of $\mathbf{C}$, the map $f(\alpha)$ is a weak homotopy equivalence of simplicial sets. Then there is a natural weak homotopy equivalence from $f(c)$ to the homotopy fibre of π at c.*

Proof Factor the map $c: \Delta[0] \to N\mathbf{C}$ as an anodyne map $i: \Delta[0] \to U$ followed by a Kan fibration $p: U \to N\mathbf{C}$. Then consider the following diagram of pullback squares:

$$\begin{tikzcd}
f(c) \arrow[r] \arrow[d] & \mathrm{hofib}_c(\pi) \arrow[r] \arrow[d] & \mathrm{hocolim}_{\mathbf{C}} f \arrow[d, "\pi"] \\
\Delta[0] \arrow[r, "i"] & U \arrow[r] & N\mathbf{C}.
\end{tikzcd}$$

The square on the right is a homotopy pullback square by the dual of Lemma 7.51 and therefore its top left corner is indeed a model for the homotopy fibre, as indicated by the notation. The map i is a (transfinite) composition of pushouts of horn inclusions $\Lambda^k[n] \to \Delta[n]$ over $N\mathbf{C}$ and therefore $f(c) \to \mathrm{hofib}_c(\pi)$ is a composition of pushouts of maps of the form

$$\Lambda^k[n] \times_{N\mathbf{C}} \mathrm{hocolim}_{\mathbf{C}} f \to \Delta[n] \times_{N\mathbf{C}} \mathrm{hocolim}_{\mathbf{C}} f.$$

Hence it suffices to show that each such map is a trivial cofibration of simplicial sets (with respect to the Kan–Quillen model structure). Clearly it is a monomorphism, so we will check that it is a weak homotopy equivalence. Note that the codomain is the diagonal of the bisimplicial set with (p, q)-simplices

$$\coprod_{k_0 \to \cdots \to k_p} f(\sigma(k_0))_q,$$

where the disjoint union ranges over p-simplices of $\Delta[n]$ and $\sigma\colon \Delta[n] \to N\mathbf{C}$ is the n-simplex we are working over. The domain can be described similarly, but now only taking p-simplices of the horn $\Lambda^k[n]$ as an indexing set. The inequality $0 \leq k_0$ in $[n]$ induces a map $f(\sigma(0)) \to f(\sigma(k_0))$ of simplicial sets that is a weak homotopy equivalence by assumption. Upon taking diagonals and applying the fundamental property of bisimplicial sets (Corollary 10.27), we conclude that the map above is weakly equivalent to the map

$$\Lambda^k[n] \times f(\sigma(0)) \to \Delta[n] \times f(\sigma(0)),$$

which is anodyne. This completes the proof. □

Proposition 10.35 can be generalized to a statement about a *simplicial* category $\mathbf{C}$ and a simplicial functor $f\colon \mathbf{C} \to \mathbf{sSets}$. Such a functor can be described explicitly by a collection of simplicial sets $\{f(c)\}_{c \in \mathbf{C}}$ together with maps

$$f_{c,d}\colon \mathbf{C}(c, d) \times f(c) \to f(d).$$

Alternatively, it is a system of (ordinary) functors $f_p\colon \mathbf{C}_p \to \mathbf{Sets}$, for $p \geq 0$, appropriately compatible with the simplicial structure maps. For every $\alpha\colon \Delta[p] \to \mathbf{C}(c, d)$ we get a corresponding map

$$\alpha_*\colon \Delta[p] \times f(c) \to f(d).$$

We say $\mathbf{C}$ *acts by weak equivalences* (via f) if each of these maps is a weak homotopy equivalence. Note that it suffices to check this for $p = 0$, since any map $\Delta[0] \to \Delta[p]$ is a weak homotopy equivalence.

As before we can define a bisimplicial set $h(f)$ with (p, q)-simplices the pairs

$$\big((c_0 \to \cdots \to c_p) \in N(\mathbf{C}_q)_p, x \in f(c_0)_q\big).$$

Let us define $\mathrm{hocolim}_{\mathbf{C}} f$ to be the diagonal of this $h(f)$. Equivalently, $\mathrm{hocolim}_{\mathbf{C}} f$ is the diagonal of the bisimplicial set $(\mathrm{hocolim}_{\mathbf{C}}(f_p))_q$. Clearly there is a natural projection map

$$\pi\colon \mathrm{hocolim}_{\mathbf{C}} f \to \delta^* N\mathbf{C},$$

with $N\mathbf{C}$ the bisimplicial set given by $N\mathbf{C}_{p,q} = N(\mathbf{C}_p)_q$.

Proposition 10.36 *Let $f\colon \mathbf{C} \to$ **sSets** be a simplicial functor and suppose $\mathbf{C}$ acts by weak equivalences. Then there is a natural weak homotopy equivalence from $f(c)$ to the homotopy fibre of π at c.*

Proof Reasoning as in the proof of Proposition 10.35, it suffices to show that for every map $\sigma\colon \Delta[n] \to \delta^* N\mathbf{C}$ and every horn $\Lambda^k[n]$ of the n-simplex, the pullback

$$\Lambda^k[n] \times_{\delta^* N\mathbf{C}} \mathrm{hocolim}_{\mathbf{C}} f \to \Delta[n] \times_{\delta^* N\mathbf{C}} \mathrm{hocolim}_{\mathbf{C}} f$$

is a weak homotopy equivalence. Now observe that the unit of the adjoint pair $(\delta_!, \delta^*)$ evaluated on a simplex $\Delta[n]$ is precisely the diagonal map

$$\Delta[n] \to \delta^* \delta_! \Delta[n] = \Delta[n] \times \Delta[n].$$

This map admits a retraction induced by the map of posets $[n] \times [n] \to [n]\colon (a, b) \mapsto \max(a, b)$. Furthermore, this retraction exhibits the horn inclusion $\Lambda^k[n] \to \Delta[n]$ as a retract of the map $\delta^* \delta_! \Lambda^k[n] \to \delta^* \delta_! \Delta[n]$. Hence, to prove that the map in the first display is a weak homotopy equivalence, it suffices to prove that the corresponding map of bisimplicial sets

$$\delta_! \Lambda^k[n] \times_{N\mathbf{C}} h(f) \to \delta_! \Delta[n] \times_{N\mathbf{C}} h(f)$$

becomes a weak homotopy equivalence upon applying δ^*. For a given map $\sigma\colon \Delta[n] \to \delta^* N\mathbf{C}$, corresponding to a string of morphisms $\sigma(0) \to \cdots \to \sigma(n)$ in the category $\mathbf{C}_n$, the codomain of the map above can be described explicitly as follows: its (p, q)-simplices are triples

$$([p] \xrightarrow{\alpha} [n], [q] \xrightarrow{\beta} [n], x \in f(\sigma(\alpha(0)))_q).$$

The domain $\delta_! \Lambda^k[n] \times_{N\mathbf{C}} h(f)$ can be described similarly, with the additional requirement that α and β factor through the horn $\Lambda^k[n]$. The assumption that f acts by weak equivalences now guarantees that the 'action map'

$$f_*\colon \delta_! \Delta[n] \times f(\sigma(0)) \to \delta_! \Delta[n] \times_{N\mathbf{C}} h(f)$$

is a weak homotopy equivalence for fixed p, hence a weak homotopy equivalence upon applying δ^*. The same holds after replacing $\Delta[n]$ by $\Lambda^k[n]$, so that finally it suffices to prove that

$$\delta_! \Lambda^k[n] \times f(\sigma(0)) \to \delta_! \Delta[n] \times f(\sigma(0))$$

becomes a weak homotopy equivalence upon taking diagonals. But this is clear, since both $\delta^* \delta_! \Lambda^k[n]$ and $\delta^* \delta_! \Delta[n]$ are weakly contractible simplicial sets. □

With the previous two propositions in hand we can now prove a version of Quillen's Theorems A and B. These theorems allow one to analyze the homotopy fibre of a map $N\mathbf{C} \to N\mathbf{D}$ induced by a functor $f : \mathbf{C} \to \mathbf{D}$ under certain conditions. Usually they are stated for ordinary categories $\mathbf{C}$ and $\mathbf{D}$, but Proposition 10.36 will give us the added generality of simplicial categories.

To state these results we introduce some notation. Suppose $f : \mathbf{C} \to \mathbf{D}$ is a (simplicial) functor between simplicial categories. Then for any object d of $\mathbf{D}$ we can consider the slice category f/d, which is the simplicial category with p-simplices given by the slice category $\mathbf{C}_p \times_{\mathbf{D}_p} \mathbf{D}_p/d$ in simplicial degree p. Taking the nerves of these we obtain a simplicial diagram

$$\mathbf{D} \to \mathbf{sSets} : d \mapsto \delta^* N(f/d).$$

Theorem 10.37 (Quillen's Theorem B) *Suppose $f : \mathbf{C} \to \mathbf{D}$ is a functor between simplicial categories. If $\mathbf{D}$ acts by weak equivalences on the diagram $\delta^* N(f/-)$, then there is a natural weak homotopy equivalence between $\delta^* N(f/d)$ and the homotopy fibre of the map $\delta^* Nf : \delta^* N\mathbf{C} \to \delta^* N\mathbf{D}$ at d.*

Proof We define a trisimplicial set $\mathbf{C}//\mathbf{D}$ with (p, q, r)-simplices the pairs of simplices

$$(c_0 \to \cdots \to c_p) \in N(\mathbf{C}_r)_p, \quad (f(c_p) \to d_0 \to \cdots \to d_q) \in N(f(c_p)/\mathbf{D}_r)_q.$$

There are natural maps

$$N\mathbf{C}_{p,r} \xleftarrow{\pi_1} (\mathbf{C}//\mathbf{D})_{p,q,r} \xrightarrow{\pi_2} N\mathbf{D}_{q,r}$$

projecting onto the two factors. Note that π_1 may be written as

$$\coprod_{c_0 \to \cdots \to c_p} N(f(c_p)/\mathbf{D}_r)_q \to \coprod_{c_0 \to \cdots \to c_p} \Delta[0],$$

where the coproducts range over p-simplices $c_0 \to \cdots \to c_p$ of the nerve of $\mathbf{C}_r$. For fixed c_p and r, the map $N(f(c_p)/\mathbf{D}_r) \to \Delta[0]$ is a weak homotopy equivalence, because the category $f(c_p)/\mathbf{D}_r$ has an initial object. Hence the diagonal of π_1 is a weak homotopy equivalence of simplicial sets as well. The map π_2 may be written

$$\coprod_{d_0 \to \cdots \to d_q} N((f/d_0)_r)_p \to \coprod_{d_0 \to \cdots \to d_q} \Delta[0],$$

where this time the coproducts range over q-simplices of the nerve of $\mathbf{D}_r$. Viewed in this way, we may reinterpret its diagonal as the projection

$$\pi\colon \operatorname{hocolim}_{\mathbf{D}}\delta^*N(f/-) \to \delta^*N\mathbf{D}.$$

Now consider the diagram

$$\begin{tikzcd}
\delta^*N\mathbf{C} \arrow[d, "\delta^*Nf"] & \delta^*(\mathbf{C}//\mathbf{D}) \arrow[l, "\pi_1"', "\sim"] \arrow[r, equal] \arrow[d] & \operatorname{hocolim}_{\mathbf{D}}\delta^*N(f/-) \arrow[d, "\pi"] \\
\delta^*N\mathbf{D} & \delta^*(\mathbf{D}//\mathbf{D}) \arrow[l, "\pi_1"', "\sim"] \arrow[r, "\pi_2", "\sim"'] & \delta^*N\mathbf{D}.
\end{tikzcd}$$

Here $\mathbf{D}//\mathbf{D}$ is formed with respect to the identity functor; moreover, the map $\pi_2\colon \delta^*(\mathbf{D}//\mathbf{D}) \to \delta^*N\mathbf{D}$ is a weak homotopy equivalence by an evident variation of the argument we used for π_1 above. According to the diagram above, determining the homotopy fibre of the left vertical map is equivalent to determining the homotopy fibre of π. But Proposition 10.36 identifies this homotopy fibre as $\delta^*N(f/d)$. □

The following special case has its own name and is very useful in practice:

Corollary 10.38 (Quillen's Theorem A) *Suppose $f\colon \mathbf{C} \to \mathbf{D}$ is a functor between simplicial categories. If $\delta^*N(f/d)$ is weakly contractible for every object d of $\mathbf{D}$, then f induces a weak homotopy equivalence of nerves $\delta^*N\mathbf{C} \to \delta^*N\mathbf{D}$.*

Historical Notes

Generalizing work of Bousfield and Kan for simplicial spaces, Reedy showed that simplicial objects in a model category carry what is now called a Reedy model structure; the properties of the category $\mathbf{\Delta}$ he used in his proof gave rise to the notion of a Reedy category. Reedy's work [127] remains unpublished; accounts of it appear in the books of Hovey [88] and Hirschhorn [84]. In order to include categories in which objects have nontrivial automorphisms, like $\mathbf{\Omega}$ and Segal's category $\mathbf{\Gamma}$, the paper [18] introduced the notion of a 'generalized Reedy category' and proved the existence of a model structure in this context. For $\mathbf{\Gamma}$-spaces, this model structure already occurs in the work of Bousfield–Friedlander [31]. In this book we have simply used the term 'Reedy category' for this more general notion. The model we have given for homotopy colimits is the one described by Bousfield–Kan [32]. A first version of Quillen's Theorem B occurs in his work on algebraic K-theory [126]. Several generalizations subsequently appeared in the literature, notably in the context of the group completion theorem of McDuff–Segal [113].

Chapter 11
Mapping Spaces and Bousfield Localizations

Recall from Section 8.3 that a left Bousfield localization of a model category $\mathcal{E}$ is a different model structure on the same category with more weak equivalences, but the same cofibrations. We have seen several examples of these already, such as the Kan–Quillen model structure as a localization of the categorical model structure on simplicial sets, as well as the various model structures on the category of dendroidal sets. In the next chapter, it will be necessary to have a general method of constructing such localizations, starting only from a 'basic' model structure and a set of morphisms which one would like to make weak equivalences. We will establish the technique to do so in this chapter. It requires a general notion of 'mapping space' in an arbitrary model category, which we will discuss first.

11.1 Mapping Spaces

Let $\mathcal{E}$ be a model category. For objects X and Y in $\mathcal{E}$ we will define a *mapping space*

$$\mathrm{Map}_{\mathcal{E}}(X, Y),$$

or just $\mathrm{Map}(X, Y)$ if $\mathcal{E}$ is clear from context. This mapping space will be a simplicial set, functorial up to homotopy in X and Y, respecting weak equivalences, and with the property that there is a natural isomorphism (as functors on $\mathrm{Ho}(\mathcal{E})$) as follows:

$$\pi_0 \mathrm{Map}_{\mathcal{E}}(X, Y) \cong \mathrm{Ho}(\mathcal{E})(X, Y).$$

In fact, we will provide several different but weakly equivalent ways of constructing such a mapping space and, strictly speaking, the use of the phrase '*the* mapping space' $\mathrm{Map}_{\mathcal{E}}(X, Y)$ is somewhat misleading. It would be better to speak of 'models for the mapping space'. We will come back to this point in Remark 11.8 below.

Before turning to the definitions, recall that the constant functors

$$\mathrm{con}\colon \mathcal{E} \to \mathcal{E}^{\mathbf{\Delta}} \quad \text{and} \quad \mathrm{con}\colon \mathcal{E} \to \mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$$

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behave in opposite ways with respect to the Reedy model structures: the constant functor $\mathcal{E} \to \mathcal{E}^{\mathbf{\Delta}}$ is right Quillen, whereas the constant functor $\mathcal{E} \to \mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$ is left Quillen. On the other hand, the functor $\mathrm{con}\colon \mathcal{E} \to \mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$ is rarely right Quillen. For this to be the case, the diagonal $X \to X \times X$ of any fibrant object in $\mathcal{E}$ would have to be a fibration, for example.

Definition 11.1 A *simplicial resolution* of an object X in $\mathcal{E}$ is a fibrant replacement

$$\mathrm{con}(X) \to \widehat{X}_\bullet$$

in the (contravariant) Reedy model structure on $\mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$. Dually, a *cosimplicial resolution* is a cofibrant replacement

$$\check{X}^\bullet \to \mathrm{con}(X)$$

in the (covariant) Reedy model structure on $\mathcal{E}^{\mathbf{\Delta}}$.

Remark 11.2 (a) By the axioms for a model structure, these resolutions always exist. In fact, one can always arrange $\mathrm{con}(X) \to \widehat{X}_\bullet$ to be a trivial cofibration in $\mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$ and $\check{X}^\bullet \to \mathrm{con}(X)$ to be a trivial fibration in $\mathcal{E}^{\mathbf{\Delta}}$, although the definition only requires these maps to be weak equivalences.

(b) A cosimplicial resolution $\check{X}^\bullet \to \mathrm{con}(X)$ in particular contains a cofibrant replacement $\check{X}^0 \xrightarrow{\sim} X$ of X in $\mathcal{E}$. Conversely, if $X_c \to X$ is a cofibrant replacement of X in $\mathcal{E}$, one can always construct $\check{X}^\bullet$ in such a way as to have $\check{X}^0 = X_c$. In particular, if X is already cofibrant, we can find a cosimplicial resolution $\check{X}^\bullet \to \mathrm{con}(X)$ which starts with $\check{X}^0 = X$. Dual remarks apply to simplicial resolutions $\mathrm{con}(X) \to \widehat{X}_\bullet$.

(c) If $X \to Y$ is a weak equivalence, any cosimplicial resolution of X is also one of Y. Conversely, a simple application of the factorization axiom for the model category $\mathcal{E}^{\mathbf{\Delta}}$ shows that for any map $X \to Y$ in $\mathcal{E}$ and any cosimplicial resolution $\check{X}^\bullet \to \mathrm{con}(X)$ there is a cosimplicial resolution of Y fitting into a commutative diagram in $\mathcal{E}^{\mathbf{\Delta}}$ as follows:

$$\begin{tikzcd} \check{X}^\bullet \arrow[r] \arrow[d] & \mathrm{con}(X) \arrow[d] \\ \check{Y}^\bullet \arrow[r] & \mathrm{con}(Y). \end{tikzcd}$$

If $X \to Y$ is a weak equivalence, then of course so is $\check{X}^\bullet \to \check{Y}^\bullet$. Again, dual remarks apply to simplicial resolutions in $\mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$.

Let $C^\bullet$ be a cosimplicial object in $\mathcal{E}$. Recall that, just like for standard topological simplices, we write ∂C^n for the colimit of the C^k for all monomorphisms $[k] \to [n]$ in $\mathbf{\Delta}$ except the identity. Similarly, we write

$$\Lambda^i C^n \to C^n$$

for the colimit of all the C^k for monomorphisms $[k] \to [n]$ which miss a $j \neq i$ from their image, exactly as for $\Lambda^i[n] \subseteq \Delta[n]$ in simplicial sets. If $C^\bullet$ is Reedy cofibrant, then $\Lambda^i C^n \to C^n$ is a cofibration by the dual of Proposition 10.11(i).

Lemma 11.3 (a) *Let $C^\bullet$ be a Reedy cofibrant cosimplicial object in $\mathcal{E}$. If all the cosimplicial structure maps $\alpha_* \colon C^m \to C^n$ are weak equivalences, then $\Lambda^i C^n \to C^n$ is a trivial cofibration for all $n > 0$ and $0 \leq i \leq n$.*
(b) *Dually, for a Reedy fibrant simplicial object $Y_\bullet$ in $\mathcal{E}$ in which all structure maps are weak equivalences, the maps*

$$Y_n \to Y(\Lambda^i[n])$$

are trivial fibrations for all $0 \leq i \leq n$.

Proof We prove the statements for the cosimplicial object $C^\bullet$; the second half of the lemma is dual. By induction we will argue that for $n > 0$ and $k \leq n$, the union of the first $k + 1$ faces $\partial_0, \cdots, \partial_k$ with ∂_i omitted gives a trivial cofibration

$$\bigcup_{i \neq j \leq k} C^{n-1} \xrightarrow{\cup \partial_j} C^n.$$

For $n = 1$ this is one of the two maps $C^0 \to C^1$ and hence a trivial cofibration by assumption. If the assertion has been proved for all smaller n, then we proceed for fixed n by induction on k, using that the square in the diagram

$$\begin{tikzcd}
\bigcup_{i\neq j\leq k-1} C^{n-2} \arrow[r] \arrow[d] & C^{n-1} \arrow[d] \arrow[ddr, bend left, "\partial_k"] & \\
\bigcup_{i\neq j\leq k-1} C^{n-1} \arrow[r] \arrow[drr, bend right, "\cup\partial_j"'] & \bigcup_{i\neq j\leq k} C^{n-1} \arrow[dr, "\cup\partial_j"] & \\
& & C^n
\end{tikzcd}$$

is a pushout. The top horizontal map is a trivial cofibration by induction, hence so is the lower horizontal map. Now the two-out-of-three property of weak equivalences and the inductive hypothesis on k imply that the slanted map is a weak equivalence as well. □

Proposition 11.4 (a) *Let $\check{A}^\bullet$ be a cosimplicial resolution of an object A in $\mathcal{E}$. If $X \to Y$ is a (trivial) fibration in $\mathcal{E}$, then*

$$\mathcal{E}(\check{A}^\bullet, X) \to \mathcal{E}(\check{A}^\bullet, Y)$$

is a (trivial) Kan fibration between simplicial sets.
(b) *Dually, let $\widehat{X}_\bullet$ be a simplicial resolution of an object X in $\mathcal{E}$. If $A \to B$ is a (trivial) cofibration in $\mathcal{E}$, then*

$$\mathcal{E}(B, \widehat{X}_\bullet) \to \mathcal{E}(A, \widehat{X}_\bullet)$$

is a (trivial) Kan fibration of simplicial sets.

Proof Again we only prove (a), the proof of (b) being dual. A lifting problem in **sSets** as in the square on the left is equivalent to one in $\mathcal{E}$ as on the right:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & \mathcal{E}(\check{A}^\bullet, X) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathcal{E}(\check{A}^\bullet, Y) \end{tikzcd} \qquad \begin{tikzcd} \partial\check{A}^n \arrow[r] \arrow[d] & X \arrow[d] \\ \check{A}^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

From this, the case of a trivial fibration $X \to Y$ is clear. The proof for a fibration is similar, replacing $\partial\Delta[n]$ by $\Lambda^i[n]$ and $\partial\check{A}^n$ by $\Lambda^i\check{A}^n$ and using that $\Lambda^i\check{A}^n \to \check{A}^n$ is a trivial cofibration in $\mathcal{E}$, as in the preceding lemma. □

Corollary 11.5 (a) *Let $\check{A}^\bullet$ be a cosimplicial resolution of an object A in $\mathcal{E}$. Then a weak equivalence $X \to Y$ between fibrant objects in $\mathcal{E}$ induces a weak equivalence between Kan complexes*

$$\mathcal{E}(\check{A}^\bullet, X) \to \mathcal{E}(\check{A}^\bullet, Y).$$

(b) *Dually, let $\widehat{X}_\bullet$ be a simplicial resolution of an object X in $\mathcal{E}$. Then a weak equivalence $A \to B$ between cofibrant objects in $\mathcal{E}$ induces a weak equivalence between Kan complexes*

$$\mathcal{E}(B, \widehat{X}_\bullet) \to \mathcal{E}(A, \widehat{X}_\bullet).$$

Proof This is clear from Proposition 11.4 and Brown's lemma (cf. Proposition 7.38). □

Corollary 11.6 *Let $\check{A}^\bullet$ be a cosimplicial resolution of an object A in $\mathcal{E}$ and $\widehat{X}_\bullet$ a simplicial resolution of an object X. Then there are weak equivalences of simplicial sets*

$$\mathcal{E}(\check{A}^0, \widehat{X}_\bullet) \xrightarrow{\sim} \delta^*\mathcal{E}(\check{A}^\bullet, \widehat{X}_\bullet) \xleftarrow{\sim} \mathcal{E}(\check{A}^\bullet, \widehat{X}_0),$$

where δ^* is the diagonal of a bisimplicial set (as in Example 10.26).

Proof View $\mathcal{E}(\check{A}^\bullet, \widehat{X}_0)$ as a bisimplicial set which is constant in the second coordinate, i.e., as a bisimplicial set with (p,q)-simplices $\mathcal{E}(\check{A}^p, \widehat{X}_0)$. Then the map on the right in the statement of the lemma arises from a map of bisimplicial sets

$$\mathcal{E}(\check{A}^p, \widehat{X}_0) \to \mathcal{E}(\check{A}^p, \widehat{X}_q).$$

All of the degeneracy maps $X_0 \to X_q$ are weak equivalences by the definition of a simplicial resolution. Thus for fixed q, Proposition 11.4 implies that the map

$$\mathcal{E}(\check{A}^\bullet, \widehat{X}_0) \to \mathcal{E}(\check{A}^\bullet, \widehat{X}_q)$$

is a weak equivalence of simplicial sets. By Corollary 10.27 we obtain a weak equivalence on the diagonals of these bisimplicial sets, which concludes the proof for the right-hand map of the lemma. The argument for the map on the left is analogous. □

Definition 11.7 For two objects A and X in a model category $\mathcal{E}$, a *mapping space* $\mathrm{Map}_{\mathcal{E}}(A, X)$ is a simplicial set weakly equivalent to each of the three simplicial sets in the previous corollary.

A specific choice of such a simplicial set will sometimes be referred to as 'a model for the mapping space'. As stated before, if $\mathcal{E}$ is clear from context we just write $\mathrm{Map}(A, X)$.

Remark 11.8 (a) This definition might strike the reader as somewhat odd in that the notation $\mathrm{Map}_{\mathcal{E}}(A, X)$ does not necessarily refer to any object in particular, but it captures the common uses in the literature. In practice, however, one generally works with a specific model of $\mathrm{Map}_{\mathcal{E}}(A, X)$. For example, one takes a fibrant replacement $X \to X_f$ of X in $\mathcal{E}$ and a convenient cosimplicial resolution $\check{A}^\bullet \to A$, and interprets $\mathrm{Map}_{\mathcal{E}}(A, X)$ as the Kan complex $\mathcal{E}(\check{A}^\bullet, X_f)$. On the other hand, it will sometimes be efficient to make 'model-independent' statements about mapping spaces, a first example being Proposition 11.9 below. Our use of the expression $\mathrm{Map}_{\mathcal{E}}(A, X)$ will hopefully always be clear from context.

(b) If A is cofibrant and X is fibrant, one can always choose resolutions $\check{A}^\bullet$ and $\widehat{X}_\bullet$ with $\check{A}^0 = A$ and $\widehat{X}_0 = X$. Doing so, the models of $\mathrm{Map}_{\mathcal{E}}(A, X)$ just described will have as their set of vertices precisely the set $\mathcal{E}(A, X)$ of morphisms from A to X.

(c) It is always possible to choose models of mapping spaces in such a way that for $A \to B$ and $Y \to X$ in $\mathcal{E}$, 'the' morphism

$$\mathrm{Map}(B, Y) \to \mathrm{Map}(A, Y) \times_{\mathrm{Map}(A,X)} \mathrm{Map}(B, X)$$

is a Kan fibration between Kan complexes, which is moreover a trivial fibration if $A \to B$ or $Y \to X$ is a weak equivalence. Indeed, replacing $A \to B$ and $Y \to X$ by weakly equivalent maps if necessary, we may assume that $A \to B$ is a cofibration between cofibrant objects and $Y \to X$ is a fibration between fibrant ones. Now choose simplicial resolutions which fit into a square

$$\begin{tikzcd}
\mathrm{con}(Y) \arrow[r, "\sim"] \arrow[d] & \widehat{Y}_\bullet \arrow[d] \\
\mathrm{con}(X) \arrow[r, "\sim"] & \widehat{X}_\bullet
\end{tikzcd}$$

in such a way that $\widehat{Y}_\bullet \to \widehat{X}_\bullet$ is a Reedy fibration between Reedy fibrant objects in $\mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$. Then a lifting problem as on the left translates into one on the right in the following diagrams:

$$\begin{tikzcd}
\partial\Delta[n] \arrow[rr] \arrow[d] & & \mathcal{E}(B, \widehat{Y}_\bullet) \arrow[d] \\
\Delta[n] \arrow[r] \arrow[urr, dashed] & \mathcal{E}(A, \widehat{Y}_\bullet) \times_{\mathcal{E}(A, \widehat{X}_\bullet)} \mathcal{E}(B, \widehat{X}_\bullet), &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr] \arrow[d, tail] & & \widehat{Y}_n \arrow[d] \\
B \arrow[r] \arrow[urr, dashed] & \widehat{Y}(\partial\Delta[n]) \times_{\widehat{X}(\partial\Delta[n])} \widehat{X}_n. &
\end{tikzcd}$$

The map on the very right is a fibration in $\mathcal{E}$ because $\widehat{Y}_\bullet \to \widehat{X}_\bullet$ is a Reedy fibration, so a lift exists if $A \to B$ is a trivial cofibration. The other cases ($Y \to X$ being a trivial fibration, or $\partial\Delta[n] \to \Delta[n]$ replaced by $\Lambda^i[n] \to \Delta[n]$) are similar.

The relation between the mapping space as defined above and homotopy classes of maps in the sense of model categories is as expected, by virtue of the following:

Proposition 11.9 *For objects A and X in a model category $\mathcal{E}$, there is an isomorphism*

$$\pi_0\mathrm{Map}_{\mathcal{E}}(A, X) \cong \mathrm{Ho}(\mathcal{E})(A, X).$$

Proof Note first that the assertion is independent of the choice of model for $\mathrm{Map}_{\mathcal{E}}(A, X)$. We may assume that A is cofibrant and X is fibrant. Then if $\widehat{X}_\bullet$ is a simplicial resolution of X, the maps

$$\widehat{X}_0 \xrightarrow{s_0} \widehat{X}_1 \xrightarrow{(d_0,d_1)} \widehat{X}_0 \times \widehat{X}_0$$

describe a path object for $\widehat{X}_0$. It follows that $\pi_0\mathcal{E}(A, \widehat{X}_\bullet)$ is exactly the set of (right) homotopy classes of maps from A to $\widehat{X}_0$. From this the proposition is clear. □

Corollary 11.10 *A map $A \to B$ in a model category $\mathcal{E}$ is a weak equivalence if and only if for every object X in $\mathcal{E}$, the map*

$$\mathrm{Map}_{\mathcal{E}}(B, X) \to \mathrm{Map}_{\mathcal{E}}(A, X)$$

is a weak equivalence of simplicial sets.

Proof If $A \to B$ is a weak equivalence, then the map of the corollary is a weak equivalence by Corollary 11.5. Conversely, if the map of the corollary is an equivalence for every X, then it follows from the previous proposition that the image of $A \to B$ in the homotopy category $\mathrm{Ho}(\mathcal{E})$ is an isomorphism. But then the map itself is a weak equivalence by Remark 7.32. □

Recall that a Quillen adjunction $f_! : \mathcal{E} \rightleftarrows \mathcal{F} : f^*$ induces an adjunction on the level of homotopy categories, by taking derived functors:

$$\mathrm{Ho}(\mathcal{E}) \underset{\mathbf{R}f^*}{\overset{\mathbf{L}f_!}{\rightleftarrows}} \mathrm{Ho}(\mathcal{F}).$$

In particular, for objects E of $\mathcal{E}$ and F of $\mathcal{F}$, the identification of Proposition 11.9 gives a bijection

$$\pi_0\mathrm{Map}_{\mathcal{F}}(\mathbf{L}f_!E, F) \cong \pi_0\mathrm{Map}_{\mathcal{E}}(E, \mathbf{R}f^*F).$$

The following proposition shows that this bijection can be lifted to a weak equivalence of mapping spaces.

Proposition 11.11 *Let $f_! : \mathcal{E} \rightleftarrows \mathcal{F} : f^*$ be a Quillen pair. Then for objects E and F as above there is a weak equivalence of simplicial sets*

$$\mathrm{Map}_{\mathcal{F}}(\mathbf{L}f_!E, F) \simeq \mathrm{Map}_{\mathcal{E}}(E, \mathbf{R}f^*F).$$

Proof We may assume E is cofibrant and F fibrant. If $\check{E}^{\bullet}$ is a cosimplicial resolution of E in $\mathcal{E}^{\mathbf{\Delta}}$, then $f_!\check{E}^{\bullet}$ is one of $f_!E$ in $\mathcal{F}^{\mathbf{\Delta}}$. The proposition is then immediate from the bijection

$$\mathcal{F}(f_!\check{E}^{\bullet}, F) \cong \mathcal{E}(\check{E}^{\bullet}, f^*F). \qquad \square$$

Corollary 11.12 (1) *If $f_! : \mathcal{E} \rightleftarrows \mathcal{F} : f^*$ is a Quillen equivalence, then for any two objects A and X in $\mathcal{E}$, the morphism*

$$\mathrm{Map}_{\mathcal{E}}(A, X) \to \mathrm{Map}_{\mathcal{F}}(\mathbf{L}f_!A, \mathbf{L}f_!X)$$

is a weak equivalence of simplicial sets (and similarly for $\mathbf{R}f^*$ applied to objects in $\mathcal{F}$).

(2) *If $f_! : \mathcal{E} \rightleftarrows \mathcal{F} : f^*$ is a left Bousfield localization (cf. the discussion at the end of Section 8.3), then for any objects B and Y in $\mathcal{F}$, the map*

$$\mathrm{Map}_{\mathcal{F}}(B, Y) \to \mathrm{Map}_{\mathcal{E}}(\mathbf{R}f^*B, \mathbf{R}f^*Y)$$

is a weak equivalence of simplicial sets. Thus, one might say that $\mathbf{R}f^*$ is homotopically fully faithful.

We conclude this section with a simple observation about mapping spaces in slice categories. Let $\mathcal{E}$ be a model category and A an object of $\mathcal{E}$. Then the slice category $A/\mathcal{E}$ is again a model category, as we have seen before. For objects $f\colon A \to X$ and $g\colon A \to Y$ of this category, let us write $\mathrm{Map}_A(X, Y)$ for the mapping space between them, defined with respect to the model category $A/\mathcal{E}$. We write $\mathrm{Map}(X, Y)$ for the mapping space between X and Y with respect to the model category $\mathcal{E}$.

Proposition 11.13 *Suppose A is a cofibrant object of $\mathcal{E}$ and f and g are as above. Then there is a homotopy pullback square of simplicial sets*

$$\begin{tikzcd}
\mathrm{Map}_A(X, Y) \arrow[r] \arrow[d] & \mathrm{Map}(X, Y) \arrow[d] \\
\Delta[0] \arrow[r, "g"] & \mathrm{Map}(A, Y).
\end{tikzcd}$$

Proof Without loss of generality we may assume that $f\colon A \to X$ is a cofibration (so that in particular X is cofibrant). Fix an arbitrary simplicial resolution $Y \to \widehat{Y}_{\bullet}$ of Y; note that this also provides a resolution in $A/\mathcal{E}$ by precomposing with the fixed map $g\colon A \to Y$. By construction, there is a pullback square of simplicial sets as follows:

$$\begin{tikzcd}
(A/\mathcal{E})(X, \widehat{Y}_\bullet) \arrow[r] \arrow[d] & \mathcal{E}(X, \widehat{Y}_\bullet) \arrow[d] \\
\Delta[0] \arrow[r, "g"] & \mathcal{E}(A, \widehat{Y}_\bullet).
\end{tikzcd}$$

These simplicial sets are (models for) the ones appearing the statement of the proposition; moreover, all of them are Kan complexes and the map on the right is a Kan fibration by Proposition 11.4. The square is therefore also a homotopy pullback by the dual of Lemma 7.51. □

11.2 Common Models for Mapping Spaces

The purpose of this section is to discuss some specific models for mapping spaces for the various model structures on the categories of simplicial and dendroidal sets we have defined. We begin with a general construction.

Suppose $\mathcal{E}$ comes equipped with a notion of tensor product, given by a functor

$$- \otimes - : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$$

which preserves colimits in each variable separately and admits a unit $\mathbf{1}$. We assume that this tensor product satisfies the pushout-product property and that $\mathbf{1}$ is cofibrant. Let $C^\bullet$ be a cosimplicial resolution of $\mathbf{1}$. The assumption that $\mathbf{1}$ is cofibrant implies that the weak equivalence $C^\bullet \xrightarrow{\sim} \mathbf{1}$ is preserved by tensoring with an arbitrary cofibrant object. For any cofibrant object A of $\mathcal{E}$, the map

$$C^\bullet \otimes A \to \mathrm{con}(\mathbf{1} \otimes A) \cong \mathrm{con}(A)$$

is a cosimplicial resolution of A. If tensoring with C^n has a right adjoint $\mathbf{Hom}(C^n, -)$ for each $n \geq 0$, then dually $\mathbf{Hom}(C^\bullet, X)$ is a simplicial resolution of any fibrant object X.

Example 11.14 (Simplicial sets) (a) Consider the category **sSets** of simplicial sets equipped with the Kan–Quillen model structure. Then the representable objects $\Delta[n]$ together form a cosimplicial resolution $\Delta[\bullet]$ of the terminal object $\Delta[0]$, because all the $\Delta[n]$ are weakly contractible (i.e., weakly equivalent to $\Delta[0]$) in this model structure. Every simplicial set is cofibrant, so that

$$\check{A}^n := \Delta[n] \times A$$

defines a cosimplicial resolution of any simplicial set A. Consequently, for a fibrant object (i.e., a Kan complex) X, we have a model for the mapping space given by

$$\mathrm{Map}(A, X)_n = \mathbf{sSets}(\Delta[n] \times A, X) = (X^A)_n.$$

In other words, the usual exponential X^A is a model for the mapping space.

(b) Now consider the categorical model structure for simplicial sets. Let $J[n]$ be the nerve of the groupoid with $\{0, \ldots, n\}$ as objects and exactly one isomorphism between any two objects. (In particular, $J[1]$ is the usual J we used many times in earlier chapters.) Then $J[\bullet]$ is a cosimplicial resolution of the terminal object $\Delta[0] = J[0]$ and we find the formula

$$\mathrm{Map}(A, X)_n = \mathbf{sSets}(J[n] \times A, X)$$

as a model for the mapping space, valid for any simplicial set A and any fibrant object (i.e., ∞-category) X. This simplicial set $\mathrm{Map}(A, X)$ is a Kan complex, whereas the exponential X^A used in (a) is only an ∞-category in general. In fact, the evident maps $\Delta[n] \to J[n]$ yield a map of simplicial sets

$$\mathrm{Map}(A, X) \to X^A.$$

Since the left-hand side is a Kan complex, this map must factor through the maximal Kan complex $k(X^A)$ in X^A. In fact the resulting map $\mathrm{Map}(A, X) \to k(X^A)$ is a trivial fibration. Indeed, writing Z for the exponential X^A, it suffices to show that for any ∞-category Z the map

$$\mathbf{sSets}(J[\bullet], Z) \to \mathbf{sSets}(\Delta[\bullet], kZ) = kZ$$

is a trivial fibration of simplicial sets. We may identify the left-hand side with $\mathbf{sSets}(J[\bullet], kZ)$, so in fact it suffices to show that

$$\mathbf{sSets}(J[\bullet], K) \to \mathbf{sSets}(\Delta[\bullet], K)$$

is a trivial fibration for any Kan complex K. But this follows easily from the fact that $\Delta[n] \to J[n]$ is a trivial cofibration in the Kan–Quillen model structure. We conclude that $k(X^A)$ is another model for the mapping space $\mathrm{Map}(A, X)$ with respect to the Joyal model structure.

(c) For any $n > 0$, the simplicial set $J[n]$ has infinitely many nondegenerate simplices. There exists a smaller cosimplicial resolution $Q^\bullet$ of $\Delta[0]$ in the categorical model structure, for which each Q^n is a finite simplicial set. As before, we take $Q^0 = \Delta[0]$. The simplicial set Q^1 is the quotient of $\Delta[3]$ obtained by forcing the two edges (02) and (13) to be degenerate:

$$\begin{tikzcd} & 0 \arrow[d] \arrow[ddl, Rightarrow] \arrow[ddr] & \\ & 1 \arrow[dl] \arrow[dr, Rightarrow] & \\ 2 \arrow[rr] & & 3. \end{tikzcd}$$

Formally, it is defined as the pushout

$$\begin{tikzcd}
\Delta[1] \amalg \Delta[1] \arrow[r, "(02)\amalg(13)"] \arrow[d] & \Delta[3] \arrow[d] \\
\Delta[0] \amalg \Delta[0] \arrow[r] & Q^1.
\end{tikzcd}$$

(Intuitively, Q^1 is the quotient of the nerve of [3] where we force the arrow $1 \to 2$ to have both a left and a right inverse.) Notice that Q^1 has a natural map to $J[1] = J$ with image the 3-simplex

$$0 \to 1 \to 0 \to 1.$$

The map $Q^1 \to J$ is a weak equivalence in the categorical model structure. Indeed, to prove this it suffices to prove that the inclusion of a vertex $\Delta[0] \to Q^1$ has the left lifting property with respect to any categorical fibration $Y \to X$ between ∞-categories. But clearly the image of any 1-simplex of Q^1 in an ∞-category X is an equivalence. Thus by Corollaries 5.53, 5.54, and 8.17, a commutative diagram

$$\begin{tikzcd}
\Delta[0] \arrow[r] \arrow[d] & Y \arrow[d] \\
Q^1 \arrow[r] & X
\end{tikzcd}$$

factors through the Kan fibration $kY \to kX$, so in fact we need only show that $\Delta[0] \to Q^1$ is a trivial cofibration in the Kan–Quillen model structure. This is clear because Q^1 is weakly contractible.

Generally, we take Q^n to be the simplicial set obtained from the nerve of the category

$$0 \to 1 \to \cdots \to n$$

by forcing each arrow $i \to i + 1$ to have a left and a right inverse. For example, for $n = 3$ it can be pictured as

$$\begin{tikzcd}
a_3 \arrow[d, equal] & a_2 \arrow[l] \arrow[d, "*"] \arrow[dl] & a_1 \arrow[l] \arrow[d, "*"] \arrow[dl] & a_0 \arrow[l] \arrow[d, "*"] \arrow[dl] \\
0 \arrow[r] \arrow[d, "*"] & 1 \arrow[r] \arrow[d, "*"] \arrow[dl] & 2 \arrow[r] \arrow[d, "*"] \arrow[dl] & 3 \arrow[d, equal] \arrow[dl] \\
b_3 & b_2 \arrow[l] & b_1 \arrow[l] & b_0, \arrow[l]
\end{tikzcd}$$

where each arrow marked $*$ has been made degenerate. Formally, it is defined as a quotient of Δ^{3n} formed by collapsing each of the 1-simplices as in the picture above to a copy of $\Delta[0]$. To be precise, thinking of Δ^{3n} as the nerve of the directed category

$$a_0 \to \cdots \to a_n = 0 \to \cdots \to n = b_0 \to \cdots \to b_n,$$

the 1-simplices to be collapsed correspond to the morphisms

$$a_{n-i} \to i \quad \text{and} \quad i \to b_{n-i}.$$

Then just as for $n = 1$, there are maps

$$\Delta^{3n} \to Q^n \to J[n],$$

where $Q^n \to J[n]$ is obtained by observing that the map $\Delta^{3n} \to J[n]$ given by the nondegenerate $3n$-simplex

$$0 \to 1 \to 0 \to \cdots \to 1 \to 0$$

for n even or

$$0 \to 1 \to 0 \to \cdots \to 0 \to 1$$

for n odd factors through the quotient $\Delta^{3n} \to Q^n$. The map $Q^n \to J[n]$ is again a categorical equivalence, by the same argument as for $n = 1$. Thus, for any ∞-category X and any simplicial set A,

$$\mathrm{Map}(A, X)_n := \mathbf{sSets}(Q^n \times X, Y)$$

is another model for the mapping space in the categorical model structure.

Example 11.15 (Simplicial presheaves) For a small category $\mathbf{C}$, consider the category $\mathrm{sPSh}(\mathbf{C}) = \mathbf{sSets}^{\mathbf{C}^{\mathrm{op}}}$ of simplicial presheaves on $\mathbf{C}$, equipped with the projective model structure. There is a functor

$$\mathrm{con}\colon \mathbf{sSets} \to \mathrm{sPSh}(\mathbf{C})$$

assigning to each simplicial set X the constant simplicial presheaf with value X. If $\mathbf{C}$ has a terminal object, then $\mathrm{con}(X)$ is the product of the presheaf represented by that terminal object with X, hence $\mathrm{con}(X)$ is projectively cofibrant. It follows that in this case $C^n := \mathrm{con}(\Delta[n])$ gives a cosimplicial resolution of the terminal object in $\mathrm{sPSh}(\mathbf{C})$. For general $\mathbf{C}$ this is not necessarily true. Nonetheless, if X is a cofibrant simplicial presheaf, then

$$\check{X}^n = \mathrm{con}(\Delta[n]) \times X$$

defines a cosimplicial resolution of X, as we will demonstrate below. Then a model for the mapping space between a cofibrant object X and a fibrant object Y is given by

$$\mathrm{Map}(X, Y)_n := \mathrm{sPSh}(\mathbf{C})(\mathrm{con}(\Delta[n]) \times X, Y).$$

To show that $\check{X}^n$ is indeed a cosimplicial resolution, for each cofibrant object X, we need to prove that

$$\mathrm{con}(\partial\Delta[n]) \times X \to \mathrm{con}(\Delta[n]) \times X)$$

is a projective cofibration. For this, consider the class of $\mathcal{A}$ of morphisms $A \to B$ in the category sPSh($\mathbf{C}$) for which

$$\mathrm{con}(U) \times B \cup \mathrm{con}(V) \times A \to \mathrm{con}(V) \times B$$

is a projective cofibration for any monomorphism $U \to V$ of simplicial sets. It then suffices to show that this class is saturated (which follows by the same standard arguments for pushout-products as before) and contains the generating cofibrations. If $A \to B$ is a generating projective cofibration of the form

$$\mathrm{con}(\partial\Delta[n]) \times \mathbf{C}(-, c) \to \mathrm{con}(\Delta[n]) \times \mathbf{C}(-, c),$$

where c is an object of $\mathbf{C}$, then the map under consideration is

$$\mathrm{con}(U \times \Delta[n] \cup V \times \partial\Delta[n]) \times \mathbf{C}(-, c) \to \mathrm{con}(V \times \Delta[n]) \times \mathbf{C}(-, c).$$

Clearly this is again a cofibration.

Example 11.16 (Dendroidal sets) The inclusion $i_! : \mathbf{sSets} \to \mathbf{dSets}$ is a left Quillen functor for the categorical model structure on **sSets** and the operadic model structure on **dSets**, as well as for the Kan–Quillen model structure on **sSets** and the covariant one on **dSets**. Thus, we can 'transfer' the cosimplicial resolutions of Example 11.14 via the functor $i_!$ to obtain cosimplicial resolutions of dendroidal sets. More specifically, if A is a normal dendroidal set and X is an ∞-operad, then the following are models for $\mathrm{Map}(A, X)$ for the operadic model structure:

$$\begin{aligned} \mathrm{Map}_J(A, X)_n &:= \mathbf{dSets}(i_! J[n] \otimes A, X) \\ \mathrm{Map}_Q(A, X)_n &:= \mathbf{dSets}(i_! Q^n \otimes A, X). \end{aligned}$$

As in the case of simplicial sets, the mapping space $\mathrm{Map}_J(A, X)$ is naturally equivalent to the maximal Kan complex in the ∞-category $i^*\mathbf{Hom}(A, X)$, with $\mathbf{Hom}(A, X)$ denoting the 'internal hom' of dendroidal sets adjoint to the tensor product. For the covariant model structure on **dSets** we can simply use the standard simplices and obtain another model

$$\mathrm{Map}_\Delta(A, X)_n := \mathbf{dSets}(i_! \Delta[n] \otimes A, X)$$

for the mapping space, this time for a normal dendroidal set A and a covariantly fibrant one X (i.e., a dendroidal left Kan complex).

Example 11.17 (Kites) If T is a tree, viewed as a representable dendroidal set, there is a very small and useful cofibrant resolution of T with respect to the operadic model structure, which we write as

$$\begin{tikzcd} \mathrm{kite}^0(T) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{kite}^1(T) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots . \end{tikzcd}$$

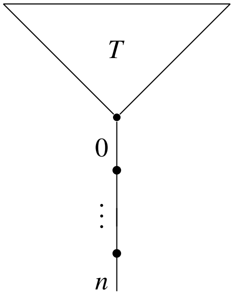
The object $\mathrm{kite}^n(T)$ is the tree T with a 'degenerate tail' of length n adjoined to it, defined as the pushout

$$\begin{tikzcd}
\Delta[n] \arrow[r] \arrow[d] & \Delta[0] \arrow[d] \\
{[n] \circ T} \arrow[r] & \mathrm{kite}^n(T).
\end{tikzcd}$$

Here $[n] \circ T$ is the grafting of T to the top of the linear tree $[n]$, which may be regarded as the pushout of the span

$$T \xleftarrow{\mathrm{root}} \eta \xrightarrow{0} [n]$$

in $\mathbf{\Omega}$. The object $\mathrm{kite}^n(T)$ derives its name from the following picture:



The cosimplicial structure of $\mathrm{kite}^\bullet(T)$ is given by interpreting it as a quotient of the grafting $[n] \circ T$. If T has leaves labelled $1, \ldots, k$, it is easy to see that $\mathrm{kite}^\bullet(T)$ is a Reedy cofibrant object of $\partial C_k / \mathbf{dSets}^{\mathbf{\Delta}}$, where the map $\partial C_k \to \mathrm{kite}^\bullet(T)$ picks out the leaves and root of T. To see that $\mathrm{kite}^\bullet(T)$ is indeed a cosimplicial resolution of T in this slice category, we need to check that each projection $\mathrm{kite}^n(T) \to T$, collapsing the tail to a single edge, is an operadic equivalence. To this end, consider the cube of dendroidal sets

$$\begin{tikzcd}
{[n]} \arrow[rr] \arrow[dr, equal] \arrow[dd] & & {[0]} \arrow[dr, equal] \arrow[dd] & \\
& {[n]} \arrow[rr] \arrow[dd] & & {[0]} \arrow[dd] \\
\Omega[T] \cup_\eta \Delta[n] \arrow[rr] \arrow[dr] & & \Omega[T] \cup_\eta \Delta[0] \arrow[dr] & \\
& \Omega[[n] \circ T] \arrow[rr] & & \mathrm{kite}^n(T).
\end{tikzcd}$$

in which the front and back faces are pushouts. The map $\Omega[T] \cup_\eta [n] \to \Omega[[n] \circ T]$, which grafts $[n]$ onto the root of T, is inner anodyne. Hence the cube lemma (cf. Corollary 7.50 and Lemma 7.51) guarantees that the map

$$\Omega[T] = \Omega[T] \cup_\eta \Delta[0] \to \mathrm{kite}^n(T)$$

is a weak equivalence. This map is a section of the projection $\mathrm{kite}^n(T) \to T$, which must then be a weak equivalence as well.

Example 11.18 (Spaces of operations) There is a close relation between the mapping spaces under consideration here and the 'space of operations' $X(x_1, \ldots, x_k; y)$ in an ∞-operad X as introduced in Definition 9.42. This simplicial set was defined by a pullback square

$$\begin{tikzcd} X(x_1, \ldots, x_k; y) \arrow[r] \arrow[d] & \mathbf{hom}(C_k, X) \arrow[d] \\ \Delta[0] \arrow[r, "(x_1, \ldots, x_k, y)"] & \mathbf{hom}(\partial C_k, X) = (i^* X)^{k+1}. \end{tikzcd}$$

In other words, the n-simplices of $X(x_1, \ldots, x_k; y)$ are the maps of dendroidal sets

$$C_k \otimes \Delta[n] \cup_{\partial C_k \otimes \Delta[n]} \partial C_k \to X$$

in the slice category $\partial C_k / \mathbf{dSets}$, where X is regarded as an object under ∂C_k via the map that sends the leaves of C_k to the $x_1, \ldots, x_k$ and its root to y. We claim that $X(x_1, \ldots, x_k; y)$ is (a model for) the mapping space $\mathrm{Map}(C_k, X)$ in the model category $\partial C_k / \mathbf{dSets}$. Let us write $\mathrm{Map}_{\partial C_k}(C_k, X)$ for this space to emphasize the fact that it is the mapping space for the slice category.

To verify our claim, recall from Proposition 11.13 that $\mathrm{Map}_{\partial C_k}(C_k, X)$ is the homotopy fibre of

$$\mathrm{Map}(C_k, X) \to \mathrm{Map}(\partial C_k, X),$$

where both mapping spaces refer to those in the model category $\mathbf{dSets}$, over the vertex of $\mathrm{Map}(\partial C_k, X)$ determined by the sequence $x_1, \ldots, x_k, y$. According to Example 11.16 we may identify $\mathrm{Map}(C_k, X)$ with the maximal Kan complex inside the ∞-category $\mathbf{hom}(C_k, X)$, and similarly for $\mathrm{Map}(\partial C_k, X)$. Thus, the space $\mathrm{Map}_{\partial C_k}(C_k, X)$ is the homotopy fibre of the left vertical map in the following square:

$$\begin{tikzcd} k\big(\mathbf{hom}(C_k, X)\big) \arrow[r] \arrow[d] & \mathbf{hom}(C_k, X) \arrow[d] \\ k\big(\mathbf{hom}(\partial C_k, X)\big) \arrow[r] & \mathbf{hom}(\partial C_k, X). \end{tikzcd}$$

The left vertical map is a Kan fibration according to Theorem 6.51(a), so that its actual fibre is a model for the homotopy fibre. Moreover, the square is a pullback: indeed, this follows from item (b) of the same theorem, stating that an edge of $\mathbf{hom}(C_k, X)$ is an equivalence if and only if it sends each colour of C_k to an equivalence in X. Hence the fibre of the left vertical map agrees with that of the right vertical map. The latter is precisely the space $X(x_1, \ldots, x_k; y)$, completing our argument.

The previous example describes a cosimplicial resolution of C_k in terms of kites; it follows that the mapping space $\mathrm{Map}_{\partial C_k}(C_k, X)$ may also be computed in terms of maps $\mathrm{kite}^\bullet(C_k) \to X$. The resulting space will then be weakly equivalent to the space $X(x_1, \ldots, x_k; y)$ discussed above.

11.3 Left Bousfield Localizations

In earlier chapters we have seen various examples of a *left Bousfield localization* of a model category $\mathcal{E}$: a new model structure $\mathcal{E}_{\mathrm{loc}}$ on the same category $\mathcal{E}$, with the same cofibrations but with a larger class of weak equivalences. (A first discussion was given at the end of Section 8.3.) In such a situation, the identity functor defines a Quillen pair

$$\mathcal{E} \underset{\mathrm{id}^*}{\overset{\mathrm{id}_!}{\rightleftarrows}} \mathcal{E}_{\mathrm{loc}}$$

that induces an adjoint pair at the level of homotopy categories

$$\mathrm{Ho}(\mathcal{E}) \underset{\mathbf{R}\mathrm{id}^*}{\overset{\mathbf{L}\mathrm{id}_!}{\rightleftarrows}} \mathrm{Ho}(\mathcal{E}_{\mathrm{loc}})$$

with the property that the counit is an isomorphism from $\mathbf{L}\mathrm{id}_! \circ \mathbf{R}\mathrm{id}^*$ to the identity of $\mathrm{Ho}(\mathcal{E}_{\mathrm{loc}})$. In other words, the right adjoint $\mathbf{R}i^*$ embeds $\mathrm{Ho}(\mathcal{E}_{\mathrm{loc}})$ as a full reflective subcategory of $\mathrm{Ho}(\mathcal{E})$.

Generally, given a model category $\mathcal{E}$ and a class of morphisms λ in $\mathcal{E}$, one can ask whether it is possible to construct a left Bousfield localization of $\mathcal{E}$ by forcing the elements of λ to be weak equivalences (and which is in an appropriate sense universal with this property). In the next section we will give a useful set of conditions making this possible, which we will apply many times in the remainder of this book. In this section we introduce some terminology and collect a few general observations.

To begin, consider a cofibrantly generated model category $\mathcal{E}$ and set of morphisms λ in $\mathcal{E}$. Notice that if $A \to B$ is a morphism in λ, we can use the factorization axiom to find a square

$$\begin{tikzcd} A' \arrow[r, two heads, "\sim"] \arrow[d, tail] & A \arrow[d] \\ B' \arrow[r, two heads, "\sim"] & B \end{tikzcd}$$

in which the horizontal maps are trivial fibrations, while $A' \to B'$ is a cofibration between cofibrant objects. By two-out-of-three, demanding that $A \to B$ be a weak equivalence is the same as demanding that $A' \to B'$ is a weak equivalence. Therefore, we may without loss of generality assume that λ consists of cofibrations between cofibrant objects. For such a set λ, we introduce the following terminology.

Definition 11.19 An object E in $\mathcal{E}$ is *λ-local* (or just *local* if λ is clear from context) if for any morphism $A \to B$ in λ, the morphism

$$\mathrm{Map}_{\mathcal{E}}(B, E) \to \mathrm{Map}_{\mathcal{E}}(A, E).$$

is a weak equivalence of simplicial sets. If E is in addition fibrant in $\mathcal{E}$, we say that E is *λ-fibrant*.

Remark 11.20 Many authors demand that E be fibrant in the above definition; however, for us it will occasionally be convenient to consider local objects which are not necessarily fibrant. Note that for us, the property of being λ-local is thus invariant under weak equivalence.

Recall that for a fibrant object E in $\mathcal{E}$, one can always arrange the map $\mathrm{Map}(B, E) \to \mathrm{Map}(A, E)$ to be a Kan fibration between Kan complexes, of which the restriction to vertices is the map of sets $\mathcal{E}(B, E) \to \mathcal{E}(A, E)$. Then E is λ-fibrant if and only if this map is a trivial fibration between these Kan complexes for every $A \to B$ in λ.

Definition 11.21 Let λ be a set of cofibrations between cofibrant objects in a cofibrantly generated model category $\mathcal{E}$. The *left Bousfield localization with respect to λ*, if it exists, is the model structure $\mathcal{E}_\lambda$ on the same underlying category $\mathcal{E}$, which is characterized by the fact that it has the same cofibrations as $\mathcal{E}$ and in which the fibrant objects are exactly the λ-fibrant objects.

We have seen various examples of left Bousfield localizations already. Indeed, the characterization of the fibrant objects shows that the Kan–Quillen model structure on the category of simplicial sets is the left Bousfield localization of the categorical model structure with respect to the horn inclusion $\Lambda^k[n] \to \Delta[n]$ for $k = 0, n$. For the category of dendroidal sets, the covariant (resp. Picard) model structure is the left Bousfield localization of the operadic one with respect to the leaf horn inclusions (resp. all horn inclusions). Before moving on to general existence results for Bousfield localizations, we observe some basic properties.

For emphasis we repeat the following facts, which were already proved as Lemmas 8.49 and 8.50.

Proposition 11.22 *A map between λ-fibrant objects is a weak equivalence (resp. fibration) in $\mathcal{E}_\lambda$ if and only if it is a weak equivalence (resp. fibration) in $\mathcal{E}$.*

Next, we record the following easy observation:

Proposition 11.23 *If $\mathcal{E}_\lambda$ is a left Bousfield localization of a left proper model category $\mathcal{E}$, then $\mathcal{E}_\lambda$ is again left proper.*

Proof Consider a pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & C \arrow[d, tail] \\ B \arrow[r] & D \end{tikzcd}$$

in which $A \to B$ is a cofibration and $A \to C$ is a weak equivalence in $\mathcal{E}_\lambda$. Factor the latter as $A \to X \to C$ where $A \to X$ is a trivial cofibration in $\mathcal{E}_\lambda$ and $X \to C$ is a fibration in $\mathcal{E}_\lambda$. Then $X \to C$ is in fact a trivial fibration in $\mathcal{E}_\lambda$, hence also a trivial fibration in $\mathcal{E}$. Thus $B \to D$ can be factored as the pushout of the trivial cofibration $A \to X$ and a pushout (along a cofibration) of the weak equivalence $X \to C$ in $\mathcal{E}$. Since $\mathcal{E}$ is assumed left proper, the latter pushout is also a weak equivalence (both in $\mathcal{E}$ and $\mathcal{E}_\lambda$). It follows that $B \to D$ is a weak equivalence in $\mathcal{E}_\lambda$. □

We now also state the universal property of left Bousfield localization alluded to at the beginning of this section.

Proposition 11.24 *Let $\mathcal{E}_\lambda$ be the left Bousfield localization of model category $\mathcal{E}$ with respect to a set λ of cofibrations between cofibrant objects, and let $f_! : \mathcal{E} \rightleftarrows \mathcal{F} : f^*$ be a Quillen pair. Then the following are equivalent:*

(i) *The pair above factors through $\mathcal{E}_\lambda$ by a Quillen pair $\mathcal{E}_\lambda \rightleftarrows \mathcal{F}$, as in*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\mathrm{id}_!"] \arrow[d, "f_!"'] & \mathcal{E}_\lambda \arrow[dl, dashed, "\overline{f}_!"] \\ \mathcal{F}. & \end{tikzcd}$$

(ii) *The functor f^* sends fibrant objects in $\mathcal{F}$ to λ-fibrant objects in $\mathcal{E}$.*
(iii) *The functor $f_!$ sends the morphisms in λ to weak equivalences in $\mathcal{F}$.*

Proof The implication (i) ⇒ (ii) is clear, as (ii) just says that f^*, as a functor $\mathcal{F} \to \mathcal{E}_\lambda$, preserves fibrant objects. For the equivalence (ii) ⇔ (iii), note that assumption (ii) says that

$$\mathrm{Map}_{\mathcal{E}}(B, f^*F) \to \mathrm{Map}_{\mathcal{E}}(A, f^*F)$$

is a weak equivalence for any $A \to B$ and any fibrant object F in $\mathcal{F}$. By adjunction, this is equivalent to

$$\mathrm{Map}_{\mathcal{F}}(f_!B, F) \to \mathrm{Map}_{\mathcal{F}}(f_!A, F)$$

being a weak equivalence. This means that $f_!$ maps $A \to B$ to a weak equivalence in $\mathcal{F}$. Finally, for the implication (ii) ⇒ (i), notice that when we view $f_!$ as a functor $\mathcal{E}_\lambda \to \mathcal{F}$, it still preserves cofibrations as these are the same in $\mathcal{E}$ as in $\mathcal{E}_\lambda$. To show that $f_!$ and f^* still define a Quillen pair, it thus suffices to prove that f^* preserves fibrations between fibrant objects (cf. Lemma 8.42). But this is clear from assumption (ii) and Lemma 8.50. □

11.4 Existence of Left Bousfield Localizations

Throughout this section $\mathcal{E}$ denotes a cofibrantly generated model category, which we will most of the time assume to be left proper. In the previous section we defined the notion of left Bousfield localization of $\mathcal{E}$ with respect to a set of morphisms λ, but left the question of whether such a localization exists open. There are several quite general criteria to be found in the literature; see the notes at the end of this chapter for references. These are all of a rather set-theoretic nature. Rather than proving one of these, we will begin by stating some general properties and then prove a more restrictive result which is sufficiently general to cover the examples in this book.

Recall that if $\mathbf{R}$ is a Reedy category, the projective and Reedy model structures are two Quillen equivalent model structures on the diagram category $\mathcal{E}^{\mathbf{R}^{\mathrm{op}}}$ with the same weak equivalences, while one (the projective model structure) has fewer cofibrations than the other (the Reedy model structure). In situations like this, the following observation is useful.

Proposition 11.25 *Let $\mathcal{E}$ be a model category and let λ be a set of morphisms in $\mathcal{E}$ for which the left Bousfield localization $\mathcal{E}_\lambda$ exists. Suppose $\mathcal{E}'$ is another model structure on the same category $\mathcal{E}$ with fewer cofibrations but the same weak equivalences, so that the identity functor $\mathcal{E}' \to \mathcal{E}$ is a left Quillen equivalence. Then the localization $\mathcal{E}'_\lambda$ also exists and $\mathcal{E}'_\lambda \to \mathcal{E}_\lambda$ is again a left Quillen equivalence.*

Proof We may assume that λ consists of cofibrations between cofibrant objects in $\mathcal{E}'$. Let us write $j_! : \mathcal{E}' \rightleftarrows \mathcal{E} : j^*$ for the Quillen equivalence given by the identity functors. By hypothesis, $j_!$ and j^* preserve and detect weak equivalences between arbitrary objects. Define a new model structure $\mathcal{E}'_\lambda$ by taking the weak equivalences to be those of $\mathcal{E}_\lambda$ and the cofibrations to be those of $\mathcal{E}'$. The fibrations in $\mathcal{E}'_\lambda$ are then defined as the morphisms having the right lifting property with respect to the trivial cofibrations in $\mathcal{E}'_\lambda$. Our task is to show that these classes of morphisms satisfy the axioms for a model structure. Assuming this for the moment, the claim that $\mathcal{E}'_\lambda \to \mathcal{E}_\lambda$ is a Quillen equivalence is clear from the fact that the weak equivalences of these model structures coincide.

So let us check the axioms (M1–5) for $\mathcal{E}'_\lambda$. As usual, the first three axioms are obvious. Moreover, the factorization of a morphism into a cofibration followed by a map having the right lifting property with respect to all the cofibrations can be performed in $\mathcal{E}'_\lambda$ as it is in $\mathcal{E}'$ (these two having the same cofibrations), and a map having this right lifting property is a weak equivalence in $\mathcal{E}'$, hence also in $\mathcal{E}'_\lambda$. For the other factorization, consider a map $X \to Y$. Factor it first as a trivial cofibration followed by a fibration in $\mathcal{E}_\lambda$, say $X \to V \to Y$. Next, factor $X \to V$ as a cofibration followed by a trivial fibration in $\mathcal{E}'_\lambda$, say as $X \to W \to V$. Then $X \to W$ is a weak equivalence in $\mathcal{E}'_\lambda$ because $X \to V$ and $W \to V$ are. Moreover, $W \to Y$ has the right lifting property with respect to all the trivial cofibrations in $\mathcal{E}'_\lambda$, because the same is true for both $W \to V$ and $V \to Y$. Thus $X \to W \to Y$ is the desired factorization into a trivial cofibration followed by a fibration in $\mathcal{E}'_\lambda$. Finally, one half of the lifting axiom (M5) holds by definition of the fibrations and the other half follows because the trivial fibrations in $\mathcal{E}'_\lambda$ are the same as those of $\mathcal{E}'$. Indeed, if $Y \to X$ is a fibration in $\mathcal{E}'_\lambda$ and a weak equivalence, we can factor it as $Y \to V \to X$ where $Y \to V$ is a cofibration and $V \to X$ has the right lifting property with respect to all the cofibrations. Then $Y \to V$ is also a weak equivalence, so $Y \to X$ has the right lifting property with respect to $Y \to V$. This makes $Y \to X$ a retract of $V \to X$, as in

$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d] & Y \arrow[d] \\ V \arrow[r] \arrow[ur, dashed] & X. \end{tikzcd}$$

Hence also $Y \to X$ has the right lifting property with respect to all cofibrations, as desired.

Finally, we need to prove that an object is fibrant in $\mathcal{E}'_\lambda$ if and only if it is fibrant in $\mathcal{E}'$ and λ-local. Take such a fibrant and λ-local object X in $\mathcal{E}'$ and factor $X \to 1$ into a trivial cofibration $f\colon X \to Y$ followed by a fibration $Y \to 1$ in $\mathcal{E}$. (In other words, Y is a fibrant replacement of X in the model structure $\mathcal{E}$.) Then $X \to Y$ is a weak equivalence between fibrant objects in $\mathcal{E}'$, so Y is also λ-local and hence fibrant in $\mathcal{E}_\lambda$. A fortiori, Y is fibrant in $\mathcal{E}'_\lambda$. Now, as in Brown's lemma, factor $(1, f)\colon X \to X \times Y$ as

$$\begin{tikzcd} X \arrow[rr, "{(1,f)}"] \arrow[dr, "u"'] & & X \times Y, \\ & Z \arrow[ur, "{(p,q)}"'] & \end{tikzcd}$$

where $X \xrightarrow{u} Z$ is a trivial cofibration in $\mathcal{E}'$ and $Z \to X \times Y$ is a fibration in $\mathcal{E}'$. Then all of u, p, and q are weak equivalences in $\mathcal{E}'$. Since X is fibrant in $\mathcal{E}'$, the map $Z \to Y$ is a fibration in $\mathcal{E}'$. Hence it is a trivial fibration in $\mathcal{E}'$ and also in $\mathcal{E}'_\lambda$. Since Y is fibrant in $\mathcal{E}'_\lambda$, so is Z. But X is a retract of Z, so X is fibrant in $\mathcal{E}'_\lambda$ as well. □

For a cofibrantly generated model category $\mathcal{E}$, again consider a set λ of cofibrations between cofibrant objects. If the localization $\mathcal{E}_\lambda$ were to exist, then its weak equivalences are determined by Corollary 11.10, i.e., a map $A \to B$ is a weak equivalence in $\mathcal{E}_\lambda$ if and only if

$$\mathrm{Map}_{\mathcal{E}_\lambda}(B, X) \to \mathrm{Map}_{\mathcal{E}_\lambda}(A, X)$$

is a weak equivalence of simplicial sets for every fibrant object X in $\mathcal{E}_\lambda$. Applying Corollary 11.12 to the Quillen pair $\mathcal{E} \rightleftarrows \mathcal{E}_\lambda$, this is the same as saying that

$$\mathrm{Map}_{\mathcal{E}}(B, X) \to \mathrm{Map}_{\mathcal{E}}(A, X)$$

is a weak equivalence for every λ-fibrant object X in $\mathcal{E}$. Thus, the following proposition shows that at least one criterion for the existence of $\mathcal{E}_\lambda$ is satisfied, namely that the (supposed) trivial cofibrations form a saturated class.

Proposition 11.26 *Let λ be a set of cofibrations between cofibrant objects in a left proper model category $\mathcal{E}$. Write $\widehat{\lambda}$ for the class of cofibrations $A \to B$ having the property that*

$$\mathrm{Map}_{\mathcal{E}}(B, X) \to \mathrm{Map}_{\mathcal{E}}(A, X)$$

is a weak equivalence for any λ-fibrant object X. Then $\widehat{\lambda}$ is a saturated class.

Proof It is clear that $\widehat{\lambda}$ is closed under retracts. For a given λ-fibrant object X, let us use the model for the mapping space $\mathrm{Map}_{\mathcal{E}}(A, X)$ given by

$$\mathcal{E}(A_c, \widehat{X}_\bullet),$$

where A_c is a cofibrant replacement of A and $\widehat{X}_\bullet$ is a simplicial resolution of X (and similarly for B in place of A). Then $A \to B$ belongs to $\widehat{\lambda}$ if and only if

$$\mathrm{Map}_{\mathcal{E}}(B, X) \to \mathrm{Map}_{\mathcal{E}}(A, X)$$

is a trivial fibration of simplicial sets (cf. Proposition 11.4), provided we arrange the map $A_c \to B_c$ of cofibrant replacements to be a cofibration (which we can always do). It then easily follows that $\widehat{\lambda}$ is closed under transfinite composition. For pushouts, observe that if a square as below on the left is a pushout in $\mathcal{E}$, then the one on the right is a pullback of simplicial sets:

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & C \arrow[d, tail] \\ B \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}(D, \widehat{X}_\bullet) \arrow[r] \arrow[d, two heads] & \mathcal{E}(B, \widehat{X}_\bullet) \arrow[d, two heads] \\ \mathcal{E}(C, \widehat{X}_\bullet) \arrow[r] & \mathcal{E}(A, \widehat{X}_\bullet). \end{tikzcd}$$

So if the right-hand map in the pullback is a trivial fibration then so is the one on the left. Hence it suffices to show that we can 'lift' such a pushout square to a pushout square of cofibrant replacements. To this end, first use the factorization axiom to lift $C \leftarrow A \to B$ to cofibrations between cofibrant objects as in

$$\begin{tikzcd} C' \arrow[d, two heads, "\wr"] & A' \arrow[l, tail] \arrow[r, tail] \arrow[d, two heads, "\wr"] & B' \arrow[d, two heads, "\wr"] \\ C & A \arrow[l] \arrow[r, tail] & B. \end{tikzcd}$$

Then take the pushout of the top row to get a cube

$$\begin{tikzcd}[row sep=small, column sep=small] A' \arrow[rr] \arrow[dd] \arrow[dr] & & B' \arrow[dd] \arrow[dr] & \\ & C' \arrow[rr, crossing over] & & D' \arrow[dd] \\ A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\ & C \arrow[rr] \arrow[from=uu, crossing over] & & D. \end{tikzcd}$$

in which bottom and top are pushouts. Then by Lemma 7.51 and the assumption that $\mathcal{E}$ is left proper, $D' \to D$ is again a weak equivalence. This concludes the proof that $\widehat{\lambda}$ is closed under pushouts. $\square$

Definition 11.27 Let λ be a set of morphisms as before and define the class $\widehat{\lambda}$ as in Proposition 11.26. Then λ is *localizable* if the class $\widehat{\lambda}$ admits the small object argument.

Recall that this means that $\widehat{\lambda}$ is the saturation of a *set* of morphisms, each having a domain which is 'small' in an appropriate sense. In all of the examples of this book 'small' can be taken to mean finite or countable (e.g. for simplicial or dendroidal sets). Generally, one fixes a regular cardinal κ and asks that the domains D are κ-compact, meaning that $\mathcal{E}(D, -)$ preserves κ-filtered colimits. Under these conditions, the small object argument provides a factorization of any map in $\mathcal{E}$ as a map in $\widehat{\lambda}$ followed by a map having the right lifting property with respect to all maps in $\widehat{\lambda}$. In the next section we provide a general way to prove that certain classes are localizable.

The localizability of λ allows us to use standard arguments to derive the following existence result.

Proposition 11.28 *Let $\mathcal{E}$ be a left proper cofibrantly generated model category. If λ is a localizable set of morphisms, then the left Bousfield localization $\mathcal{E}_\lambda$ exists.*

Proof As already described before, the definitions of the relevant classes of morphisms are as follows: the cofibrations are the same as those of $\mathcal{E}$ and a map $A \to B$ is a weak equivalence precisely if $\mathrm{Map}_{\mathcal{E}}(B, X) \to \mathrm{Map}_{\mathcal{E}}(A, X)$ is a weak equivalence of simplicial sets for every λ-fibrant object X. In particular, every weak equivalence in $\mathcal{E}$ is also one in $\mathcal{E}_\lambda$. The fibrations are defined as the maps having the right lifting property with respect to the trivial cofibrations, i.e., with respect to the maps in $\widehat{\lambda}$. With these definitions, axioms (M1–3) are clearly satisfied. Moreover, for the factorization axiom (M5), a map factors as a cofibration followed by a trivial fibration in $\mathcal{E}$ and this also gives the required factorization in $\mathcal{E}_\lambda$. As mentioned already, the assumption on $\widehat{\lambda}$ implies that any map in $\mathcal{E}$ factors as a map in $\widehat{\lambda}$ (i.e., a trivial cofibration) followed by one having the right lifting property with respect to all the maps in $\widehat{\lambda}$ (which is a fibration in $\mathcal{E}_\lambda$ by definition). Finally, one half of the lifting axiom (M4) holds by definition, while the other half follows by the usual retract argument. This proves that the classes of cofibrations, fibrations, and weak equivalences as just defined for $\mathcal{E}_\lambda$ indeed define a model structure.

It remains to be shown that the fibrant objects in $\mathcal{E}_\lambda$ are exactly the λ-fibrant ones, i.e., the fibrant objects X in $\mathcal{E}$ for which $\mathrm{Map}_{\mathcal{E}}(B, X) \to \mathrm{Map}_{\mathcal{E}}(A, X)$ is a weak equivalence for each $A \to B$ in λ. First suppose X is fibrant in $\mathcal{E}_\lambda$. Since the maps $A \to B$ in λ are obviously trivial cofibrations in $\mathcal{E}_\lambda$, it follows immediately that

$$\mathrm{Map}_{\mathcal{E}_\lambda}(B, X) \to \mathrm{Map}_{\mathcal{E}_\lambda}(A, X)$$

is a weak equivalence of simplicial sets. But $\mathcal{E}_\lambda$ can be replaced by $\mathcal{E}$ in this expression, as we already observed in Corollary 11.12, so X is λ-fibrant. For the converse, suppose X is λ-fibrant. We should show that $X \to 1$ has the right lifting property with respect to all cofibrations which are weak equivalences in $\mathcal{E}_\lambda$. Since X is fibrant in $\mathcal{E}$, Lemma 7.43 shows that it suffices to prove this for cofibrations between cofibrant objects. Let $A \to B$ be such a trivial cofibration between cofibrant objects in $\mathcal{E}_\lambda$. Then we can choose a model for the mapping spaces for which

$$\mathrm{Map}_{\mathcal{E}}(B, X) \to \mathrm{Map}_{\mathcal{E}}(A, X)$$

is a trivial fibration. Moreover, we may arrange $\mathrm{Map}(B, X)_0 = \mathcal{E}(B, X)$ and similarly for A. Any trivial fibration is surjective on vertices, so picking a preimage of the vertex corresponding to a given map $A \to X$ gives a lift $B \to X$. This completes the proof. □

11.5 Localizable Sets of Morphisms

Let λ be a set of cofibrations between cofibrant objects in a left proper model category $\mathcal{E}$. In order to apply Proposition 11.28 in practice to establish the existence of left Bousfield localizations, it is necessary to have a useful way of checking that λ is localizable in the sense of Definition 11.27. In this section we will give some criteria to do this, which are sufficiently general to cover the instances of left Bousfield localization used in this book (and many more). The arguments are modelled on those in Section 8.1.

We shall work with model structures on a category $\mathcal{E}$ of set-valued presheaves, $\mathcal{E} = \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, where $\mathbf{C}$ is a *countable* category. The categories of simplicial and dendroidal sets are examples of such $\mathcal{E}$. Moreover, if $\mathcal{E}$ is of this form then so is the category $s\mathcal{E} = \mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}$ of simplicial objects in $\mathcal{E}$. An object in $\mathcal{E}$ is called *finitely presented* (or briefly, *finite*) if it can be written as a finite colimit of representables. Similarly one defines *countable* objects. If K is finite and if

$$A_0 \to A_1 \to \cdots$$

is a countable sequence of monomorphisms, then any map $K \to \varinjlim_n A_n$ factors through A_n for some n.

Remark 11.29 Everything we do in this section works (with straightforward modification) with 'finite' replaced by 'κ-small' and considering κ-filtered colimits instead of just countable ones. Also, the hypothesis that $\mathcal{E}$ is a category of presheaves can be relaxed to only assume that it is 'presentable' (meaning a certain kind of localization of a presheaf category) at the cost of phrasing our arguments in a more abstract way. However, this added generality will not play a role for us and the essence of the arguments remains unchanged.

We will assume that $\mathcal{E}$ carries a model structure which is left proper and has the following four properties:

(1) Each cofibration is a monomorphism. Moreover, if $X \subseteq A \subseteq Y$ and $X \to Y$ is a cofibration, then so is $A \to Y$.
(2) The cofibrations are generated by a set of cofibrations between finite objects.
(3) An object X is fibrant if and only if $X \to 1$ has the right lifting property with respect to the trivial cofibrations between finite objects in $\mathcal{E}$.

(4) If A is a finite cofibrant object, then it admits a cosimplicial resolution $\check{A}^\bullet$ with each $\check{A}^n$ a finite object. Moreover, if $A \to B$ is a cofibration between finite cofibrant objects, one may arrange a compatible map of such finite cosimplicial resolutions $\check{A}^\bullet \to \check{B}^\bullet$ which is a Reedy cofibration.

It is useful to note that the set of all trivial cofibrations between finite objects is countable. All the model structures described so far in this book have the properties listed above. In fact it is possible to get rid of assumption (4) at the cost of working with a suitably large cardinal κ as in Remark 11.29. However, (4) is easily seen to hold in all of our examples, so we choose to include it.

Definition 11.30 A morphism $X \to Y$ is called *finite* (resp. *countable*) if for every object c in $\mathbf{C}$ and every morphism $\mathbf{C}(-, c) \to X$, the pullback $Y \times_X \mathbf{C}(-, c)$ is a finite (resp. countable) object.

The countability of $\mathbf{C}$ and property (2) ensure the following.

Lemma 11.31 *Suppose $X \to Y$ is a countable morphism. Then there exists a factorization $X \to Z \to Y$ into a cofibration followed by a trivial fibration, where $Z \to Y$ is again countable.*

Proof The factorization is constructed in the standard way from the small object argument, as we will explain. We may write Y as a filtered colimit of finite objects Y_i and define $X_i = Y_i \times_Y X$. Then X_i is countable and it suffices to construct the factorization for each $X_i \to Y_i$ and take the filtered colimit afterwards. In other words, we may assume Y is finite and X is countable. Then there are only countably many commutative squares

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

where $A \to B$ is a generating cofibration between finite objects. If we take the pushout along the coproduct of all these cofibrations as in the small object argument

$$\begin{tikzcd} \coprod A \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod B \arrow[r] & X', \end{tikzcd}$$

then X' is still countable. Repeating this construction countably many times and taking the colimit yields the required factorization. □

We will use the lemma in the following special case:

Example 11.32 If $X \to Y$ is a cofibration, then there exists a factorization of the fold map $Y \amalg_X Y \to Y$ as a cofibration followed by countable trivial fibration, giving a relative cylinder for Y with respect to X:

$$Y \amalg_X Y \rightarrowtail C_X(Y) \xrightarrow{\sim} \!\!\!\!\rightarrow Y.$$

The following definition reflects a property that we proved earlier for the categories of simplicial and dendroidal sets, cf. Sections 8.1 and 9.1, specifically Lemma 9.22.

Definition 11.33 Let λ be a set of cofibrations between cofibrant objects in a model category $\mathcal{E}$. We say that λ has the *countable approximation property* if for any cofibration $X \to Y$ in $\widehat{\lambda}$ and any commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

with $A \to B$ a monomorphism between countable objects, there exists an extension to a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & B' \arrow[r] & Y \end{tikzcd}$$

in which A' and B' are again countable, the square on the right is a pullback, and $A' \to B'$ also belongs to $\widehat{\lambda}$.

Proposition 11.34 *If λ has the countable approximation property, then $\widehat{\lambda}$ is generated by morphisms in $\widehat{\lambda}$ between countable objects. In particular, since there is only a set of such morphisms (up to isomorphism), the set λ is localizable.*

Proof The proof is the same as that of Corollary 8.15. More explicitly, let $f : X \to Y$ be a morphism in $\widehat{\lambda}$. We may assume that X is a subpresheaf of Y and that f is simply the inclusion. Enumerate the elements of Y which do not belong to X as $\{y_\xi \mid \xi < \alpha\}$ for some ordinal α. We will construct a sequence of subpresheaves

$$X = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\xi \subseteq M_{\xi+1} \subseteq \cdots \subseteq Y,$$

where each $M_\xi \to M_{\xi+1}$ belongs to $\widehat{\lambda}$ and where $y_\zeta \in M_\xi$ whenever $\zeta < \xi$. We proceed by induction. At limit ordinals ξ we define $M_\xi = \cup_{\zeta<\xi} M_\zeta$. If M_ξ has been defined then we define $M_{\xi+1}$ as follows. Let $B \subseteq Y$ be a countable subpresheaf with $y_\xi \in B$ (which exists because $\mathbf{C}$ is countable) and let $A = B \cap M_\xi$. The countable approximation property provides a diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & A' \arrow[r, tail] \arrow[d] & M_\xi \arrow[d] \\ B \arrow[r, tail] & B' \arrow[r, tail] & Y, \end{tikzcd}$$

where $A' \to B'$ belongs to $\widehat{\lambda}$. Let $M_{\xi+1} = B' \cup_{A'} M_\xi$ be the pushout. Then $M_\xi \to M_{\xi+1}$ belongs to $\widehat{\lambda}$ since $A' \to B'$ does, and $M_{\xi+1} \to Y$ is a monomorphism because the square is a pullback. So $M_{\xi+1} \to Y$ is a cofibration (using property (1) of $\mathcal{E}$), which must belong to $\widehat{\lambda}$ since $M_\xi \to Y$ and $M_\xi \to M_{\xi+1}$ do. This completes the induction. □

Proposition 11.35 *Let $\mathcal{E}$ be a model category satisfying properties (1)–(4). Let λ be a set of cofibrations between finite cofibrant objects. Then λ has the countable approximation property.*

If X is an object of $\mathcal{E}$, a λ-fibrant replacement of X is a cofibration $X \to X_\lambda$ which belongs to $\widehat{\lambda}$ and for which X_λ is a λ-fibrant object. Such a replacement can be constructed explicitly using the small object argument, using it to force $X_\lambda \to 1$ to have the right lifting property with respect to the set of trivial cofibrations between finite objects (using property (3) of $\mathcal{E}$) as well as with respect to the cofibrations

$$\breve{A}^n \cup_{\partial \breve{A}^n} \partial \breve{B}^n \to \breve{B}^n$$

for $A \to B$ in λ (using property (4) in order to assume that these are finite). This explicit construction will have the following obvious properties.

Lemma 11.36 (a) *If X is countable, then so is its λ-fibrant replacement X_λ.*
(b) *If X is arbitrary and $B \subseteq X_\lambda$ is countable, then there exists a countable $A \subseteq X$ for which $B \subseteq A_\lambda \subseteq X_\lambda$.*

Proof (of Proposition 11.35) Let $i\colon X \to Y$ be a cofibration in $\widehat{\lambda}$ and consider a diagram

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & X \arrow[d] \\ B_0 \arrow[r] & Y \end{tikzcd}$$

where A_0 and B_0 are countable. Suppose for the moment that X and Y are cofibrant. Let X_λ and Y_λ be λ-fibrant replacements, fitting into a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail, "i"] & X_\lambda \arrow[d, tail, "i_\lambda"] \\ Y \arrow[r] & Y_\lambda. \end{tikzcd}$$

Then i_λ is a morphism in $\widehat{\lambda}$ between cofibrant and λ-fibrant objects, which implies that it is a deformation retract. In other words, there exists a retraction $r_\lambda\colon Y_\lambda \to X_\lambda$ and a homotopy between $i_\lambda r_\lambda$ and the identity on Y_λ, which can be taken to be parametrized by a cylinder

$$Y_\lambda \cup_{X_\lambda} Y_\lambda \to C_{X_\lambda}(Y_\lambda) \xrightarrow{\pi} Y_\lambda$$

for which the projection is a countable morphism. Let us write $h\colon C_{X_\lambda}(Y_\lambda) \to Y_\lambda$ for this homotopy. Then there are countable $A' \subseteq X_\lambda$ and $B' \subseteq Y_\lambda$ with $A_0 \subseteq A'$ and $B_0 \subseteq B'$ for which r and h restrict to maps $r\colon B' \to A'$ and $h\colon C_{B'} \to B'$, where $C_{B'} = \pi^{-1}B'$ is the restriction of the cylinder to B'. Next, by the lemma, there are countable $A'' \subseteq X$ and $B'' \subseteq Y$ with $A' \subseteq (A_\lambda)''$, $B' \subseteq (B_\lambda)''$ and $A_0 \subseteq A''$, $B_0 \subseteq B''$. This gives a diagram

$$\begin{tikzcd}
A'_0 \arrow[rr] \arrow[dd] \arrow[dr] & & A'' \arrow[rr] \arrow[dd] \arrow[dr] & & X \arrow[dd] \arrow[dr] & \\
& A' \arrow[rr] \arrow[dd] & & A''_\lambda \arrow[rr] \arrow[dd] & & X_\lambda \arrow[dd] \\
B_0 \arrow[rr] \arrow[dr] & & B'' \arrow[rr] \arrow[dr] & & Y \arrow[dr] & \\
& B' \arrow[rr] & & B''_\lambda \arrow[rr] & & Y_\lambda
\end{tikzcd}$$

in which the two rightmost vertical maps in the front fact are part of deformation retracts, hence belong to $\widehat{\lambda}$. Now let $B_1 = B''$ and $A_1 = X \cap B_1$ (which contains A'') and repeat the construction with A_0 and B_0 replaced by A_1 and B_1. If we iterate this countably many times, we arrive at a sequence

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X \arrow[d] \\
B_0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & \cdots \arrow[r] & Y
\end{tikzcd}$$

for which the λ-fibrant replacements are interpolated by deformation retracts

$$\begin{tikzcd}
(A_1)_\lambda \arrow[r] \arrow[d] & A'_1 \arrow[r] \arrow[d, "\wr"] & (A_2)_\lambda \arrow[r] \arrow[d] & \cdots \\
(B_1)_\lambda \arrow[r] & B'_1 \arrow[r] & (B_2)_\lambda \arrow[r] & \cdots .
\end{tikzcd}$$

This shows that $\varinjlim (A_n)_\lambda \to \varinjlim (B_n)_\lambda$ belongs to $\widehat{\lambda}$ and hence so does $\varinjlim A_n \to \varinjlim B_n$. The diagram

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d] & \varinjlim A_n \arrow[r] \arrow[d] & X \arrow[d] \\
B_0 \arrow[r] & \varinjlim B_n \arrow[r] & Y
\end{tikzcd}$$

then shows that the countable approximation property holds.

In the preceding argument, we have assumed that X and Y are cofibrant. In the general case, we can use Lemma 11.31 to find a diagram

$$\begin{tikzcd} \widehat{X} \arrow[r, two heads, "p"] \arrow[d] & X \arrow[d] \\ \widehat{Y} \arrow[r, two heads, "q"] & Y \end{tikzcd}$$

where $\widehat{X}$ and $\widehat{Y}$ are cofibrant and the horizontal maps are trivial fibrations with countable fibres. Then $\widehat{X} \to \widehat{Y}$ also belongs to $\widehat{\lambda}$. Let $\widehat{A}_0 = p^{-1}(A_0)$ and $\widehat{B}_0 = q^{-1}(B_0)$. Then the proof above gives a diagram

$$\begin{tikzcd} \widehat{A}_0 \arrow[r] \arrow[d] & \widehat{A}_1 \arrow[r] \arrow[d] & \widehat{X} \arrow[d] \\ \widehat{B}_0 \arrow[r] & \widehat{B}_1 \arrow[r] & \widehat{Y}, \end{tikzcd}$$

where $\widehat{A}_1 \to \widehat{B}_1$ is a map between countable objects which belongs to $\widehat{\lambda}$. Let $A_1 \subseteq X$ and $B_1 \subseteq Y$ be countable subpresheaves with $\widehat{A}_1 \subseteq p^{-1}A_1$ and $\widehat{B}_1 \subseteq q^{-1}B_1$. Now iterate this countably many times to get a diagram

$$\begin{tikzcd} \widehat{A}_1 \arrow[r] \arrow[d] & p^{-1}(A_1) \arrow[r] \arrow[d] & \widehat{A}_2 \arrow[r] \arrow[d] & p^{-1}(A_2) \arrow[r] \arrow[d] & \cdots \arrow[r] & \widehat{X} \arrow[d] \\ \widehat{B}_1 \arrow[r] & q^{-1}(B_1) \arrow[r] & \widehat{B}_2 \arrow[r] & p^{-1}(B_2) \arrow[r] & \cdots \arrow[r] & \widehat{Y} \end{tikzcd}$$

in which each $\widehat{A}_n \to \widehat{B}_n$ belongs to $\widehat{\lambda}$. Then $p^{-1}(\cup A_n) \to q^{-1}(\cup B_n)$ does too, and hence so does $\cup A_n \to \cup B_n$. In this way, we deduce the case where X and Y are arbitrary from the one where they are both cofibrant, and the proof of the proposition is complete. $\square$

Let us summarize the conclusions:

Theorem 11.37 *Suppose $\mathcal{E} = \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is a category of presheaves on a countable category* $\mathbf{C}$. *Suppose that $\mathcal{E}$ carries a left proper model structure satisfying properties (1)–(4) listed at the start of this section. Then any set λ of cofibrations between finite cofibrant objects is localizable. Hence the left Bousfield localization $\mathcal{E}_\lambda$ of $\mathcal{E}$ with respect to λ exists.*

Historical Notes

The concept of localization (and the closely related notion of completion) was first introduced into homotopy theory to make sense of rationalization and p-completion of spaces; some of the earliest references are Quillen's paper [124], Sullivan's notes [136], and the book of Bousfield–Kan [32]. Localization with respect to general homology theories was described by Adams in [1]; the existence of such localizations (both in stable and unstable homotopy) was proved by Bousfield in [29, 28]; these two papers were crucial to the further development of the notion. The theory was generalized both by Bousfield [30] and by Dror-Farjoun [56] to localizations with respect to arbitrary maps between spaces. The first systematic and comprehensive exposition of localization in the context of model categories is the book of Hirschhorn [84].

In this chapter we have given a self-contained treatment of left Bousfield localization, trying to distinguish the general theory (see Section 11.3) and the problem of existence. We have proved existence only in the special case of a model structure on a category of simplicial presheaves whose cofibrations are monomorphisms. This suffices for our purposes in this book; moreover, many naturally occurring model categories are (closely related to one) of this form, as the work of Dugger [49] shows.

Chapter 12
Dendroidal Spaces and ∞-Operads

Recall that a dendroidal set X is an ∞-operad if it satisfies the inner Kan condition; i.e., it has the extension property with respect to all inner horn inclusions of trees. In particular, this condition guarantees that if we interpret the sets $X(T)$ as 'sets of operations' parametrized by the tree T, then there is a notion of composition of such operations (well-defined up to homotopy) when grafting trees T and T'. In Section 12.1 we introduce an analogous condition for dendroidal *spaces*, namely the *Segal condition*. A dendroidal space X satisfying this condition is called a dendroidal Segal space and again there exists a notion of 'composition of operations', well-defined up to homotopy. In Section 12.2 we introduce the notion of *completeness* of dendroidal Segal spaces. Essentially, this amounts to a localization of the homotopy theory of dendroidal Segal spaces in which the groupoid interval J is forced to become weakly contractible. With this localization in place, we will show that the homotopy theory of complete dendroidal Segal spaces is equivalent to that of ∞-operads, by exhibiting a Quillen equivalence between a certain model structure on the category of dendroidal spaces and the operadic model structure on the category of dendroidal sets. In Section 12.3 we further characterize completion as the localization of the homotopy theory of dendroidal Segal spaces at the fully faithful and essentially surjective maps. Section 12.4 aims to show that the Boardman–Vogt tensor product of dendroidal spaces (rather than of dendroidal sets) behaves well with respect to the homotopy theory we have introduced; it can be used to equip the homotopy category of ∞-operads with a symmetric monoidal structure. In particular, we will see that the subtle behaviour of the tensor product of dendroidal sets with respect to cofibrations does not pose such a problem in the context of dendroidal spaces. In the next two sections of this chapter we introduce closed and reduced dendroidal spaces, respectively. These two variants are designed to capture the somewhat simpler homotopy theory of ∞-operads in which the spaces of nullary operations are contractible and all unary operations are invertible, which is the case in many examples of interest. In the final Section 12.7, we discuss how the theory of dendroidal spaces specializes to that of simplicial spaces, in particular recovering Rezk's notion of complete Segal spaces.

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12.1 Dendroidal Segal Spaces

We begin this section by reviewing some notational conventions. A dendroidal space is, by definition, a contravariant functor from the category $\mathbf{\Omega}$ of trees to the category of simplicial sets,

$$X\colon \mathbf{\Omega}^{\mathrm{op}} \to \mathbf{sSets}.$$

Here and below, we use the term 'space' for a simplicial set, as suggested by the Quillen equivalence between the Kan–Quillen model structure and the classical model structure on the category of topological spaces. With natural transformations as morphisms, this defines a category

$$\mathbf{dSpaces} := \mathbf{sSets}^{\mathbf{\Omega}^{\mathrm{op}}}.$$

It is a category of simplicial presheaves of the type discussed several times already in this book. Before we go into specific aspects of dendroidal spaces, let us establish some notation. First of all, the category of dendroidal spaces is naturally tensored over that of simplicial sets: if X is a dendroidal space and M is a simplicial set, we denote by $X \boxtimes M$ the dendroidal space defined by

$$(X \boxtimes M)(T) = X(T) \times M,$$

where T ranges over $\mathbf{\Omega}$. For such X and M, we denote by X^M the dendroidal space defined by

$$X^M(T)_n = \mathbf{sSets}(M \times \Delta[n], X(T)) = \mathbf{dSpaces}(T \boxtimes (M \times \Delta[n]), X).$$

Here we identify T with the representable dendroidal set or the corresponding discrete dendroidal space, so $T \boxtimes M$ is the dendroidal space with

$$(T \boxtimes M)(S)_n = \mathbf{\Omega}(S, T) \times M_n.$$

We will continue to use this abbreviated notation for representables. As a further piece of notation, each dendroidal space X defines a functor (denoted by the same symbol)

$$X\colon \mathbf{dSets}^{\mathrm{op}} \to \mathbf{sSets}$$

by Kan extension: it is the unique functor (up to isomorphism) which agrees with the given $X\colon \mathbf{\Omega}^{\mathrm{op}} \to \mathbf{sSets}$ on representables and preserves all small limits. Thus, for a dendroidal set A, the simplicial set $X(A)$ is characterized by the identifications

$$\mathbf{sSets}(M, X(A)) \cong \mathbf{dSpaces}(A \boxtimes M, X) \cong \mathbf{dSpaces}(A, X^M).$$

Alternatively, writing A as a colimit of representables, we find the formula

$$X(A) = \varprojlim_{T \to A} X(T),$$

where the limit ranges over all the maps from representable dendroidal sets T to A. Finally, we observe that the inclusion $i\colon \mathbf{\Delta} \to \mathbf{\Omega}$ induces adjoint functors

$$\mathbf{sSpaces} \underset{i^*}{\overset{i_!}{\rightleftarrows}} \mathbf{dSpaces}$$

between the category of dendroidal spaces and that of simplicial spaces (alias bisimplicial sets), analogous to the usual adjunction between dendroidal and simplicial sets. As in this case (cf. Section 3.5.2), the category on the left can be identified with the slice category $\mathbf{dSpaces}/\eta$ and the adjunction then takes the form

$$\mathbf{dSpaces}/\eta \rightleftarrows \mathbf{dSpaces},$$

with left adjoint the forgetful functor.

With these notational conventions in place, we can now begin our discussion of the homotopy theory of dendroidal spaces. As a category of simplicial presheaves, **dSpaces** carries the projective model structure (with respect to the Kan–Quillen model structure on the category of simplicial sets). We denote this model category by

$$\mathbf{dSpaces}_P.$$

Recall that this projective model structure is defined by declaring a map $X \longrightarrow Y$ of dendroidal spaces to be a fibration (or weak equivalence) if for each tree T, the map $X(T) \longrightarrow Y(T)$ of simplicial sets is a Kan fibration (resp. a weak homotopy equivalence). The generating (trivial) cofibrations for the projective model structure are of the form

$$T \boxtimes M \longrightarrow T \boxtimes N,$$

where $M \longrightarrow N$ is a generating (trivial) cofibration in **sSets** and T is any object of $\mathbf{\Omega}$.

Since $\mathbf{\Omega}$ is a Reedy category, the category **dSpaces** also carries a Reedy model structure. We denote the corresponding model category by

$$\mathbf{dSpaces}_R.$$

The Reedy model structure has the same weak equivalences as the projective one, but fewer fibrations and hence more cofibrations. Explicitly, the Reedy cofibrations are generated by the maps of the form

$$T \boxtimes \partial\Delta[n] \cup \partial T \boxtimes \Delta[n] \longrightarrow T \boxtimes \Delta[n],$$

for all T in $\mathbf{\Omega}$ and all $n \geq 0$. A map $X \longrightarrow Y$ is a Reedy fibration if and only if for each tree T the map of simplicial sets

$$X(T) \longrightarrow Y(T) \times_{Y(\partial T)} X(\partial T)$$

is a Kan fibration. Here, according to the notational conventions explained above,

$$X(\partial T) = \varprojlim_{S \to T} X(S),$$

where the limit ranging over proper monomorphisms $S \longrightarrow T$. Recall from Proposition 10.11 that this implies that for any dendroidal subset $U \subseteq \partial T$, the map

$$X(T) \to Y(T) \times_{Y(U)} X(U)$$

is also a Kan fibration. In particular, taking $U = \varnothing$ we find that any Reedy fibration is a projective fibration, as already asserted above. All in all, the identity functor defines a Quillen equivalence

$$\mathbf{dSpaces}_P \underset{\mathrm{id}^*}{\overset{\mathrm{id}_!}{\rightleftarrows}} \mathbf{dSpaces}_R .$$

It will be convenient to have the following description of the Reedy cofibrations:

Lemma 12.1 *A map $X \longrightarrow Y$ of dendroidal spaces is a Reedy cofibration if and only if for each simplicial degree k, the map $X_k \longrightarrow Y_k$ of dendroidal sets is a normal monomorphism.*

We call a map with the properties of the lemma a *normal monomorphism* of dendroidal spaces.

Proof The class of degreewise normal monomorphisms is a saturated class that evidently contains the generating Reedy cofibrations listed above, so every Reedy cofibration is indeed a normal monomorphism of dendroidal spaces. For the converse, suppose $X \longrightarrow Y$ has the property that each $X_k \longrightarrow Y_k$ is a normal monomorphism of dendroidal sets. By definition of the Reedy cofibrations, we need to show that the map

$$\mathrm{deg}Y(T) \cup_{\mathrm{deg}X(T)} X(T) \to Y(T)$$

is a cofibration in $\mathbf{sSets}^{\mathrm{Aut}(T)^{\mathrm{op}}}$. Since $\mathrm{Aut}(T)$ acts freely on $Y(T)_k - X(T)_k$ for each k by assumption, it suffices to show that the map above is a monomorphism. This will follow if we demonstrate that for any element $x \in X(T)_k$ of which the image in $Y(T)_k$ is degenerate, x itself was already degenerate. To see this, consider a square of the form

$$\begin{tikzcd} T \boxtimes \Delta[k] \arrow[r, "x"] \arrow[d] & X \arrow[d] \\ S \boxtimes \Delta[k] \arrow[r] & Y, \end{tikzcd}$$

where the vertical map on the left arises from a degeneracy map $T \longrightarrow S$. There is such a square whenever x is degenerate as an element of $Y(T)_k$. Forming the pushout P in the square gives a factorization

$$X \to P \to Y,$$

where the first map is a surjection. Since the composition of the two maps is injective, the first map is in fact an isomorphism. It follows that x factors through $S \boxtimes \Delta[k]$, hence is already degenerate in X. □

Corollary 12.2 *The functor* **dSets** $\to$ **dSets**$_R$, *sending each dendroidal set to the corresponding discrete dendroidal space, preserves cofibrations.*

However, we note that this functor (even though it has a right adjoint) is *not* a left Quillen functor. Indeed, one readily verifies that it does not map an inner horn inclusion $\Lambda^e[T] \to T$ to a Reedy weak equivalence, for example. This situation will be 'corrected' by appropriately localizing the Reedy model structure, cf. Theorem 12.22 below.

Definition 12.3 A dendroidal space X is said to *satisfy the Segal condition* if for any tree T, the morphism

$$\mathrm{Map}(T, X) \to \mathrm{Map}(\mathrm{Sp}[T], X)$$

is a weak homotopy equivalence of simplicial sets. (Here, as before, we identify the tree T with the corresponding discrete dendroidal space.)

Remark 12.4 The expression 'Map' is defined in terms of (co)simplicial resolutions relative to a model structure on **dSpaces**. In our case, this is either the projective or the Reedy model structure. Since the two are related by a Quillen equivalence, the Segal condition is independent of which of the two model structures one takes.

Again identifying dendroidal sets with the corresponding discrete dendroidal spaces, the definition above has several equivalent formulations:

Lemma 12.5 *For a dendroidal space X, the following are equivalent:*

(1) *X satisfies the Segal condition.*
(2) *For any tree T and any inner edge e of T, the map*

$$\mathrm{Map}(T, X) \to \mathrm{Map}(\Lambda^e[T], X)$$

is a weak homotopy equivalence of simplicial sets.
(3) *For any inner anodyne morphism $A \to B$ of dendroidal sets, the map*

$$\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$$

is a weak homotopy equivalence of simplicial sets.

Proof The class of normal monomorphisms $A \to B$ of dendroidal sets having the property that

$$\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$$

is a weak homotopy equivalence is saturated and as the two-out-of-three property. Hence the lemma is clear from Proposition 6.39. □

The general theory of Bousfield localizations explained in Chapter 11 (and specifically Theorem 11.37) shows that there exists a left Bousfield localization of the Reedy model structure on dendroidal spaces for which the local objects are exactly the dendroidal spaces satisfying the Segal condition. Furthermore, Proposition 11.25 shows that a similar localization exists for the projective model structure on **dSpaces** and that it is Quillen equivalent to the previous localization of the Reedy model structure. We summarize these model structures in the following diagram, where the further subscript S denotes the localization with respect to the Segal condition and all the arrows are the identity, interpreted as a left Quillen functor:

$$\begin{tikzcd} \mathbf{dSpaces}_P \arrow[r, "\sim"] \arrow[d] & \mathbf{dSpaces}_R \arrow[d] \\ \mathbf{dSpaces}_{PS} \arrow[r, "\sim"] & \mathbf{dSpaces}_{RS}. \end{tikzcd}$$

Although the localized model categories $\mathbf{dSpaces}_{PS}$ and $\mathbf{dSpaces}_{RS}$ are Quillen equivalent, in a given context one may be easier to work with than the other. For example, consider the formulation of the Segal condition in terms of 'Map'. In the model category $\mathbf{dSpaces}_R$, the inner horn inclusions $\Lambda^e[T] \to T$ and spine inclusions $\mathrm{Sp}[T] \to T$ are cofibrations between cofibrant objects, by Lemma 12.1. (This is *not* the case in the projective model structure; the objects $\Lambda^e[T]$ and $\mathrm{Sp}[T]$ are usually not cofibrant.) Now for an arbitrary fibrant object X and a cofibrant one A, the simplicial set $\mathrm{Map}(A, X)$ can be calculated from the cosimplicial resolution $A \boxtimes \Delta[n]$, for $n \geq 0$ (cf. Example 11.15), as

$$\mathrm{Map}(A, X)_n = \mathrm{Hom}(A \boxtimes \Delta[n], X).$$

In particular, if A is a normal dendroidal *set*, A can be viewed as a Reedy cofibrant discrete dendroidal space, and we find for a Reedy fibrant dendroidal space X that as a model for Map we can take

$$\mathrm{Map}(A, X) = X(A)$$

as defined at the start of this section. Moreover, for such an X and a normal monomorphism $A \to B$ of dendroidal sets, the restriction map $X(B) \to X(A)$ is a Kan fibration. This applies in particular to the inclusions featuring in the previous lemma, which can therefore be reformulated as follows:

Lemma 12.6 *For a Reedy fibrant dendroidal space X, the following statements are equivalent:*

(1) *X satisfies the Segal condition.*
(2) *For each tree T, the map $X(T) \to X(\mathrm{Sp}[T])$ is a trivial fibration between Kan complexes.*
(3) *For each tree T and inner edge e of T, the map $X(T) \to X(\Lambda^e[T])$ is a trivial fibration between Kan complexes.*

(4) *For each inner anodyne map $A \to B$ between normal dendroidal sets, the map $X(B) \to X(A)$ is a trivial fibration between Kan complexes.*
(5) *X is a fibrant object in* $\mathbf{dSpaces}_{RS}$.

This lemma also leads to the following reformulation of the Segal condition, which is sometimes useful:

Lemma 12.7 *Suppose $T = T_1 \circ_e T_2$ is a tree arising from grafting a tree T_2 onto a leaf edge e of another tree T_1. Then a dendroidal space X satisfies the Segal condition if the square*

$$\begin{tikzcd} X(T) \arrow[r] \arrow[d] & X(T_1) \arrow[d, "e^*"] \\ X(T_2) \arrow[r, "e^*"'] & X(\eta) \end{tikzcd}$$

is a homotopy pullback, for any such T and a decomposition as a grafting of subtrees as above.

Remark 12.8 By induction on the size of T one also concludes that X satisfies the Segal condition if $X(T)$ is equivalent to the iterated homotopy pullback of the values of X on the corollas and edges making up the spine $\mathrm{Sp}[T]$.

Proof Assume that X satisfies the property described in the lemma. Since the property of being a homotopy pullback square is invariant under weak equivalence, we may take a Reedy fibrant replacement Y of X and prove that Y satisfies the Segal condition. Since Y is Reedy fibrant, the maps $Y(T_i) \to Y(\eta)$ are Kan fibrations. It follows from the dual of Lemma 7.51 that the relevant square for Y is a homotopy pullback if and only if the map

$$Y(T) \to Y(T_1) \times_{Y(\eta)} Y(T_2) \cong Y(T_1 \cup_e T_2)$$

to the actual pullback is a weak homotopy equivalence. Next observe that

$$\mathrm{Sp}[T] = \mathrm{Sp}[T_1] \cup_e \mathrm{Sp}[T_2].$$

Inductively assuming that $Y(T_i) \to Y(\mathrm{Sp}[T_i])$ is a trivial fibration for $i = 1, 2$ now shows that the map

$$Y(T) \to Y(\mathrm{Sp}[T])$$

is a also a trivial fibration. The previous lemma then implies that Y satisfies the Segal condition. □

These Reedy fibrant objects satisfying the Segal property are quite convenient to work with and it is helpful to single them out:

Definition 12.9 A *dendroidal Segal space* is a Reedy fibrant dendroidal space satisfying the equivalent conditions of the preceding lemma.

So, by definition, an arbitrary dendroidal space X has the Segal property if and only if it has a Reedy fibrant replacement which is a dendroidal Segal space.

Let us observe the following easy consequence of the definition.

Lemma 12.10 *Let X be a Reedy fibrant dendroidal space. Then X is a dendroidal Segal space if and only if for each monomorphism of simplicial sets $M \to N$, the map $(X^N)_0 \to (X^M)_0$ is an inner fibration between dendroidal sets. In particular, if X is a dendroidal Segal space then for each $k \geq 0$ the dendroidal set X_k is an ∞-operad.*

Proof For a map $A \to B$ between dendroidal sets and a map $M \to N$ between simplicial sets, $(X^N)_0 \to (X^M)_0$ has the right lifting property with respect to $A \to B$ in the category **dSets** if and only if $X(B) \to X(A)$ has the right lifting property with respect to $M \to N$ in the category **sSets**, as follows immediately from the definitions. Applying this to inner anodynes $A \to B$, the first statement is clear. The second statement follows by taking $M \to N$ to be the inclusion $\varnothing \to \Delta[k]$. □

We now list several fundamental examples of dendroidal Segal spaces.

Example 12.11 (i) Let $\mathbf{P}$ be a simplicial operad with set of colours C. If for each sequence $c_1, \ldots, c_n, c$ of colours the space $\mathbf{P}(c_1, \ldots, c_n; c)$ of operations is a Kan complex, then $N\mathbf{P}$ is a projectively fibrant dendroidal space. Indeed, for a tree T, fixing a planar structure on T gives an identification

$$N\mathbf{P}(T) = \coprod_{\alpha} \prod_{v} \mathbf{P}(\alpha(v)),$$

where the coproduct ranges over all functions $\alpha\colon E(T) \to C$ assigning colours to the edges of T and the product ranges over the vertices of v of T. Moreover, for a vertex v with input edges $e_1, \ldots, e_n$ and output edge e, we have written $\alpha(v)$ for the sequence $\alpha(e_1), \ldots, \alpha(e_n), \alpha(e)$. The right-hand side is clearly a Kan complex.

From this description of $N\mathbf{P}(T)$, we also see that

$$N\mathbf{P}(T) \to N\mathbf{P}(\mathrm{Sp}[T])$$

is in fact an isomorphism. Since $N\mathbf{P}$ is not necessarily Reedy fibrant, we cannot immediately conclude that $N\mathbf{P}$ satisfies the Segal condition. However, we may still apply the criterion of Lemma 12.7. So write $T = T_1 \circ_e T_2$ and consider the square

$$\begin{tikzcd} N\mathbf{P}(T) \arrow[r] \arrow[d] & N\mathbf{P}(T_1) \arrow[d] \\ N\mathbf{P}(T_2) \arrow[r] & N\mathbf{P}(\eta), \end{tikzcd}$$

which is a pullback by our discussion above. The fact that $N\mathbf{P}(\eta)$ is discrete implies that the bottom and right maps are Kan fibrations. But then Lemma 7.51 guarantees that the square is also a homotopy pullback.

(ii) Let X be an ∞-operad. Then we can define a dendroidal space $\widehat{X}$ by

$$\widehat{X}(T)_n = \mathbf{dSets}(J[n] \otimes T, X).$$

By Example 11.16, $J[-] \otimes T$ provides a cosimplicial resolution of a tree T (in the operadic model structure, say), and we can somewhat informally write

$$\widehat{X}(T) = \mathrm{Map}(T, X).$$

By the homotopy invariance properties of Map, it is then immediate that $\widehat{X}$ is a dendroidal Segal space, using that X itself is an ∞-operad.

(iii) Recall that for a simplicial operad $\mathbf{P}$ whose spaces of operations are Kan complexes, as in (i) above, we have defined an ∞-operad $w^*\mathbf{P}$ with

$$w^*\mathbf{P}(T) = \mathbf{sOp}(W(T), \mathbf{P}).$$

Here $W(T)$ is the Boardman–Vogt resolution of the free operad $\Omega(T)$ generated by the tree T. Thus, continuing example (ii), we can define for each simplicial operad a dendroidal Segal space

$$\widehat{N}\mathbf{P} := \widehat{w^*\mathbf{P}}.$$

The relation between this dendroidal space and the ordinary nerve $N\mathbf{P}$ is discussed in Section 14.6.

We conclude this section with a brief discussion of the weak equivalences (in the model category $\mathbf{dSpaces}_R$) between dendroidal Segal spaces and the notion of *homotopy operad* of a dendroidal Segal space. To this end, we first introduce some notation. Let X be a Reedy fibrant dendroidal space. An *object* of X is by definition a vertex of the simplicial set $X(\eta)$. For objects $x_1, \ldots, x_n, x$ of X, we write $X(x_1, \ldots, x_n; x)$ for the pullback

$$\begin{tikzcd} X(x_1, \ldots, x_n; x) \arrow[r] \arrow[d] & X(C_n) \arrow[d] \\ \Delta[0] \arrow[r, "(x_1, \ldots, x_n, x)"] & X(\partial C_n). \end{tikzcd}$$

Recall that ∂C_n is the coproduct of copies of η indexed by the edges of the n-corolla, so $X(\partial C_n) = \prod_{i=0}^n X(\eta)$. The lower map picks out the vertices $x_1, \ldots, x_n$ for the input edges of C_n and x for the output. Notice that the map $X(C_n) \to X(\partial C_n)$ is a Kan fibration between Kan complexes by the assumption that X is Reedy fibrant. Hence the fibre $X(x_1, \ldots, x_n; x)$ is also a Kan complex and the square is in fact a homotopy pullback. We will think of the simplicial set $X(x_1, \ldots, x_n; x)$ as the ‘space of operations’ in X from $x_1, \ldots, x_n$ to x.

Definition 12.12 A map $f\colon X \to Y$ between dendroidal Segal spaces is *fully faithful* if for each sequence $x_1, \ldots, x_n, x$ of objects in X, the induced map

$$X(x_1, \ldots, x_n; x) \to Y(f(x_1), \ldots, f(x_n); f(x))$$

is a weak homotopy equivalence of Kan complexes.

Proposition 12.13 *Let $f\colon X \to Y$ be a map between dendroidal Segal spaces. Then the following are equivalent:*

(1) *The map f is a weak equivalence in* **dSpaces**$_R$*, i.e., $X(T) \to Y(T)$ is a weak homotopy equivalence for each tree T.*
(2) *The map $X(T) \to Y(T)$ is a weak homotopy equivalence for each tree T with at most one vertex (i.e., T is η or a corolla).*
(3) *The map $f\colon X \to Y$ is fully faithful and $X(\eta) \to Y(\eta)$ is a weak homotopy equivalence.*

Proof It is clear that (1) implies (2). The equivalence between (2) and (3) can be seen as follows. Consider the square

$$\begin{tikzcd}
X(C_n) \arrow[r] \arrow[d] & Y(C_n) \arrow[d] \\
X(\partial C_n) \arrow[r] & Y(\partial C_n),
\end{tikzcd}$$

in which the bottom map is a weak homotopy equivalence by the assumption on $X(\eta) \to Y(\eta)$. Then the top map is an equivalence if and only if it induces an equivalence on fibres over any $(x_1, \ldots, x_n, x) \in X(\partial C_n)$ and $(f(x_1), \ldots, f(x_n), f(x)) \in Y(\partial C_n)$. Indeed, this is clear from the fact that the vertical maps are Kan fibrations and their resulting long exact sequences on homotopy groups. Finally, to see that (2) implies (1), observe that in the square

$$\begin{tikzcd}
X(T) \arrow[r] \arrow[d, two heads, "\wr"] & Y(T) \arrow[d, two heads, "\wr"] \\
X(\mathrm{Sp}[T]) \arrow[r] & Y(\mathrm{Sp}[T])
\end{tikzcd}$$

the vertical maps are trivial fibrations, as indicated. So the upper map is a weak equivalence if and only if the lower one is. But $X(\mathrm{Sp}[T])$ is an iterated pullback of the form

$$X(\mathrm{Sp}[T]) = X(C_{v_1}) \times_{X(\eta)} X(C_{v_2}) \times_{X(\eta)} \times \cdots \times_{X(\eta)} X(C_{v_n})$$

where $v_1, \ldots, v_n$ are the vertices of T and $C_{v_1}, \ldots, C_{v_n}$ denote the corresponding corollas. Each map $X(C_{v_i}) \to X(\eta)$ involved in this pullback is a Kan fibration by the Reedy condition, so the (iterated) pullback is also a homotopy pullback. The same applies to Y. So assumption (3) implies that $X(\mathrm{Sp}[T]) \to Y(\mathrm{Sp}[T])$ is a weak homotopy equivalence and hence so is $X(T) \to Y(T)$. □

For later use we now introduce the *homotopy operad* $\mathrm{ho}(X)$ of a dendroidal Segal space X. This $\mathrm{ho}(X)$ is an operad in the category of sets with colours given by the set $X(\eta)_0$ of objects of X. Given a tuple of colours $x_1, \ldots, x_n, x$, the corresponding set of operations in $\mathrm{ho}(X)$ is defined by

$$\mathrm{ho}(X)(x_1, \ldots, x_n; x) := \pi_0\big(X(x_1, \ldots, x_n; x)\big).$$

To make $\mathrm{ho}(X)$ into an operad we should describe the composition of operations, which is defined using the Segal property of X. To be precise, suppose $\xi_1 \in X(C_m)$ and $\xi_2 \in X(C_n)$ represent operations of $\mathrm{ho}(X)$, such that the root of the corolla ξ_1 corresponds to one of the leaves e of the corolla ξ_2. Then the Segal property of X guarantees the existence of an element $\zeta \in X(C_n \circ_e C_m)$ that restricts to ξ_1 and ξ_2 on the leaf and root corolla, respectively. The composite operation $\xi_2 \circ_e \xi_1$ is represented by the inner face $\partial_e \zeta$. The verification that $\mathrm{ho}(X)$ is a well-defined operad is straightforward (and similar to our earlier analysis of the operad τY associated to an ∞-operad Y). For example, associativity of the composition is guaranteed by the fact that $X(T) \to X(\mathrm{Sp}[T])$ is a trivial fibration for trees T built from three corollas. We leave the details to the reader.

12.2 Complete Dendroidal Segal Spaces

In the previous section we introduced dendroidal Segal spaces, which are the analogues in the category of dendroidal spaces of the ∞-operads in the category of dendroidal sets. However, it is not quite true that the model category **dSets** (with the operadic model structure) and the model category $\mathbf{dSpaces}_{RS}$ for dendroidal Segal spaces are Quillen equivalent. For this to work, one needs to localize $\mathbf{dSpaces}_{RS}$ further with respect to *completeness*. We will explain this concept in this section and prove the Quillen equivalence just alluded to in Theorem 12.22. Also, we will give various interpretations of the notion of completeness that will be technically convenient in the sequel.

Definition 12.14 A dendroidal space X is *complete* if it is local with respect to either of the two endpoint inclusions $\eta \to J$, interpreted as a morphism of discrete dendroidal spaces. In other words, X is complete if

$$\mathrm{Map}(J, X) \to \mathrm{Map}(\eta, X)$$

is a weak homotopy equivalence.

Remark 12.15 Notice that for a dendroidal space X, completeness depends only on the underlying *simplicial* space i^*X, since $\eta \to J$ lies in the image of the embedding $i_! \colon \mathbf{sSets} \to \mathbf{dSets}$. Notice also that if X is Reedy fibrant, we may identify $\mathrm{Map}(A, X)$ with $X(A)$, as discussed in the previous section. This applies in particular to $A = J$ and $A = \eta$.

According to Theorem 11.37, the model category $\mathbf{dSpaces}_{RS}$ admits a left Bousfield localization $\mathbf{dSpaces}_{RSC}$ with respect to either of the two morphisms $\eta \to J$, characterized by the property that its fibrant objects are the complete dendroidal Segal spaces. Recall that in the category of dendroidal sets, the morphisms $\eta \to J$ are operadic equivalences. Hence, in order to 'force' $\mathbf{dSpaces}_{RS}$ to become Quillen equivalent to the category **dSets** equipped with the operadic model structure, it is only natural to consider the localization $\mathbf{dSpaces}_{RSC}$ with respect to the completion just defined.

We will now characterize the complete dendroidal Segal spaces (i.e., the fibrant objects of $\mathbf{dSpaces}_{RSC}$) in terms of lifting properties.

Proposition 12.16 *A dendroidal space X is a complete dendroidal Segal space if and only if it has the extension property with respect to the following three classes of maps:*

(a) *The maps*
$$T \boxtimes \Lambda^k[n] \cup \partial T \boxtimes \Delta[n] \to T \boxtimes \Delta[n]$$
for each tree T and each $n \geq 1$, $0 \leq k \leq n$.

(b) *The maps*
$$T \boxtimes \partial\Delta[n] \cup \Lambda^e[T] \boxtimes \Delta[n] \to T \boxtimes \Delta[n]$$
for each tree T, each inner edge e of T, and each $n \geq 0$.

(c) *The maps*
$$J \boxtimes \partial\Delta[n] \cup \eta \boxtimes \Delta[n] \to J \boxtimes \Delta[n]$$
for each $n \geq 0$.

Proof A dendroidal space X is Reedy fibrant if and only if the maps $X(T) \to X(\partial T)$ are Kan fibrations for every tree T. This is equivalent to X having the extension property with respect to the maps (a). Such a Reedy fibrant X satisfies the Segal condition if and only if the fibration $X(T) \to X(\Lambda^e[T])$ is a weak homotopy equivalence, for each tree T and inner edge e of T. In other words, a Reedy fibrant X is a dendroidal Segal space if and only if it has the extension property with respect to the maps of (b). Finally, X is then complete if and only if the Kan fibration $X(J) \to X(\eta)$ is a weak equivalence, which is the same as having the extension property with respect to the maps of (c). □

Another way to reformulate these lifting conditions is as follows.

Lemma 12.17 *Let X be a dendroidal space. Then the following are equivalent:*

(1) *X is a complete dendroidal Segal space.*
(2) *For any monomorphism $M \to N$ of simplicial sets, $(X^N)_0 \to (X^M)_0$ is an operadic fibration of dendroidal sets. Moreover, this fibration is trivial if $M \to N$ is anodyne.*
(3) *For any $n \geq 0$, the map $X^{\Delta[n]} \to X^{\partial\Delta[n]}$ is an operadic fibration of dendroidal sets. Moreover, for any $n > 0$ and $0 \leq i \leq n$ the face map*

$$d_i \colon X_n \to X_{n-1}$$

is a trivial fibration.

Proof We first show the equivalence between (1) and (2). By definition, X is Reedy fibrant if $X(T) \to X(\partial T)$ is a Kan fibration for each T. In other words, for any anodyne map $M \to N$ there exists a lift in any square of the form

$$\begin{tikzcd} M \arrow[r] \arrow[d] & X(T) \arrow[d] \\ N \arrow[r] & X(\partial T). \end{tikzcd}$$

But this lifting problem is equivalent to

$$\begin{tikzcd} \partial T \arrow[r] \arrow[d] & (X^N)_0 \arrow[d] \\ T \arrow[r] & (X^M)_0, \end{tikzcd}$$

showing that $(X^N)_0 \to (X^M)_0$ is a trivial fibration. If X is Reedy fibrant, then the same argument shows that $(X^N)_0 \to (X^M)_0$ is an inner fibration between ∞-operads, for any monomorphism $M \to N$, if and only if X has the Segal property (as already observed in Lemma 12.10). Finally, it also shows that X is then complete if and only if $(X^N)_0 \to (X^M)_0$ has J-path lifting, for any monomorphism $M \to N$. By Theorem 9.32(c), the operadic fibrations between ∞-operads are precisely the inner fibrations with J-path lifting.

Clearly (2) implies (3). For the converse, observe that the class of monomorphisms of simplicial sets $M \to N$ such that $(X^N)_0 \to (X^M)_0$ is a (trivial) fibration is saturated and closed under two-out-of-three (among monomorphisms). The conclusion then follows from the fact that the boundary inclusions $\partial\Delta[n] \to \Delta[n]$ generate all monomorphisms as a saturated class, and that the face maps $\partial_i \colon \Delta[n-1] \to \Delta[n]$ generate all anodyne maps as a saturated class closed under two-out-of-three among monomorphisms. □

Up to now, we have been considering the category **dSpaces** of dendroidal spaces as the category $\mathbf{sSets}^{\mathbf{\Omega}^{\mathrm{op}}}$ of simplicial presheaves on the Reedy category $\mathbf{\Omega}$. As has already become clear above, it is sometimes useful to view this category as that of simplicial objects in **dSets**:

$$\mathbf{sSets}^{\mathbf{\Omega}^{\mathrm{op}}} = \mathbf{dSpaces} = \mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

In order to not confuse the two points of view, we will always write a dendroidal space X as a functor $\mathbf{\Omega}^{\mathrm{op}} \to \mathbf{sSets}$ as in the previous section and denote the corresponding simplicial object $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{dSets}$ by $\mathrm{tw}(X)$, the *twist* of X. So we have an equality of sets

$$\mathrm{tw}(X)_n(T) = X(T)_n.$$

We can regard $\mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}$ as a model category giving it the Reedy model structure with respect to the Reedy category $\mathbf{\Delta}$ and the operadic model structure on **dSets**. This model structure is different from the Reedy model structure $\mathbf{dSpaces}_R$ we have been considering. However, the descriptions of complete dendroidal Segal spaces above admit the following reformulation in these terms:

Corollary 12.18 *A dendroidal space X is a complete dendroidal Segal space if and only if both X and* $\mathrm{tw}(X)$ *are Reedy fibrant, in the categories* $\mathbf{sSets}^{\mathbf{\Omega}^{\mathrm{op}}}$ *and* $\mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}$, *respectively.*

Proof Being Reedy fibrant in $\mathbf{sSets}^{\mathbf{\Omega}^{\mathrm{op}}}$ means having the extension property with respect to the maps (a) of Proposition 12.16, whereas being Reedy fibrant in $\mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}$ means having the extension property with respect to the maps (b) and (c) of the same proposition. □

This gives yet another interpretation of complete dendroidal Segal spaces. Indeed, consider a dendroidal space X such that $\mathrm{tw}(X) \in \mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}$ is Reedy fibrant with respect to the Reedy structure of $\mathbf{\Delta}$. According to Lemma 12.17(3), X is then a complete dendroidal Segal space if and only if all face maps $d_i : X_n \to X_{n-1}$, which are already fibrations by the latter Reedy condition, are also weak equivalences. We record this observation for convenience:

Corollary 12.19 *A dendroidal space X is a complete dendroidal Segal space if and only if* $\mathrm{tw}(X)$ *is Reedy fibrant, interpreted as an object of* $\mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}$, *and* homotopically constant, *in the sense that for any $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$ the map*

$$\alpha^* : X_n \to X_m$$

is an operadic equivalence of dendroidal sets.

For our next few results it will be convenient to observe the following:

Lemma 12.20 *The two Reedy model structures on* **dSpaces**, *arising from its identification with* $\mathbf{sSets}^{\mathbf{\Omega}^{\mathrm{op}}}$ *and* $\mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}$ *respectively, have the same cofibrations.*

Proof For each of these two, the cofibrations are generated by the maps of the form

$$T \boxtimes \partial\Delta[n] \cup \partial T \boxtimes \Delta[n] \to T \boxtimes \Delta[n],$$

for trees T and $n \geq 0$. □

Since any model structure is uniquely characterized by its cofibrations and its fibrant objects, it follows from the lemma and Corollary 12.19 that the model category $\mathbf{dSpaces}_{RSC}$ for complete dendroidal Segal spaces may alternatively be characterized as the left Bousfield localization of the Reedy model structure on $\mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}$ for which the local objects are the homotopically constant ones. We record the following consequence for future reference:

Corollary 12.21 *Let X be a Reedy fibrant dendroidal space (with respect to the Reedy structure of $\mathbf{\Omega}$). Then in the model category $\mathbf{dSpaces}_{RSC}$, the object X admits a fibrant replacement by a complete dendroidal Segal space $\widehat{X}$ so that for each $n \geq 0$, the map $X_n \longrightarrow \widehat{X}_n$ is an operadic equivalence of dendroidal sets.*

Proof As already observed in the proof of Lemma 12.17, the fact that X is Reedy fibrant implies that the face maps

$$d_i \colon X_n \longrightarrow X_{n-1}$$

are trivial fibrations of dendroidal sets. In particular, the simplicial object $\mathrm{tw}(X) \in \mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}$ is homotopically constant. Any Reedy fibrant replacement $\mathrm{tw}(X) \longrightarrow Y$ in particular induces operadic equivalences $\mathrm{tw}(X)_n \longrightarrow Y_n$ for all $n \geq 0$, so that Y is also homotopically constant. But then Corollary 12.19 implies that Y corresponds to a complete dendroidal Segal space in $\mathbf{sSets}^{\mathbf{\Omega}^{\mathrm{op}}}$. □

Now consider the adjoint pair

$$\mathbf{dSets} \underset{\mathrm{dis}^*}{\overset{\mathrm{dis}_!}{\rightleftarrows}} \mathbf{dSpaces}.$$

For a dendroidal space Y, the right adjoint takes the degree zero part $\mathrm{dis}^* Y = Y_0$. The functor $\mathrm{dis}_!$ assigns to every dendroidal set X the corresponding discrete dendroidal space with $(\mathrm{dis}_! X)(T)_n = X(T)$. As a simplicial object

$$\mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{dSets} \colon [n] \mapsto (\mathrm{dis}_! X)_n = X,$$

it is constant in the variable n. By the discussion above, the fibrant objects of $\mathbf{dSpaces}_{RSC}$ are the *homotopically* constant dendroidal Segal spaces, which should make the following plausible:

Theorem 12.22 *The adjoint pair* $(\mathrm{dis}_!, \mathrm{dis}^*)$ *is a Quillen equivalence between the operadic model structure on* $\mathbf{dSets}$ *and the model category* $\mathbf{dSpaces}_{RSC}$ *for complete dendroidal Segal spaces.*

By the discussion preceding the theorem, it is really a special case of the following easy observation:

Lemma 12.23 *Let $\mathcal{E}$ be a model category and let $s\mathcal{E}$ be the category of simplicial objects in $\mathcal{E}$ equipped with the Reedy model structure. Then the constant simplicial object functor* $\mathrm{con} \colon \mathcal{E} \longrightarrow s\mathcal{E}$ *is a left Quillen functor, which becomes a Quillen equivalence for the left Bousfield localization of $s\mathcal{E}$ whose fibrant objects X are the homotopically constants ones, whenever that localization exists.*

Proof It is clear that for the Reedy model structure on $s\mathcal{E}$, the evaluation at 0 functor $\mathrm{ev}_0 \colon s\mathcal{E} \longrightarrow \mathcal{E}$ preserves fibrations and trivial fibrations, so con and ev_0 form a Quillen pair. For an object E in $\mathcal{E}$, write $\mathrm{con}(E) \longrightarrow \widehat{E}$ for a Reedy fibrant replacement in $s\mathcal{E}$. Then $E \longrightarrow \widehat{E}_n$ is a weak equivalence for each $n \geq 0$, so by two-out-of-three each face map $\widehat{E}_n \to \widehat{E}_{n-1}$ is a weak equivalence and $\widehat{E}$ is homotopically constant, hence local. The fact that $E \to \widehat{E}_0$ is a weak equivalence then shows that the derived unit

$$E \to \mathbf{R}\mathrm{ev}_0\mathbf{L}\mathrm{con}(E)$$

(which may be identified with $E \to \widehat{E}_0$ if E is cofibrant) is a weak equivalence. As to the derived counit, for a fibrant object X in $s\mathcal{E}$ this is the map $\mathrm{con}(X_0) \to X$ which is always a weak equivalence in degree 0, and hence a weak equivalences in all degrees precisely by the assumption that X is homotopically constant. $\square$

Example 12.24 Let X be an ∞-operad. In Example 12.11(ii) we constructed a corresponding dendroidal space $\widehat{X}$, for which $\widehat{X}(T)$ is a model of the mapping space $\mathrm{Map}(T, X)$. Since $\eta \to J$ is an operadic equivalence of dendroidal sets, $\mathrm{Map}(J, X) \to \mathrm{Map}(\eta, X)$ is a weak equivalence of simplicial sets. Hence $\widehat{X}$ is complete.

In a bit more detail, recall that we explicitly defined

$$\widehat{X}(T)_n = \mathbf{dSets}(J[n] \otimes T, X).$$

One can think of this construction as the 'geometric realization - singular complex' adjunction with respect to the cosimplicial object $J[\bullet]$.

$$\mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}} \underset{\mathrm{Sing}_J}{\overset{|\cdot|_J}{\rightleftarrows}} \mathbf{dSets}.$$

This is a Quillen adjunction by Proposition 10.25 and becomes a Quillen equivalence for the localization of $\mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}$ as in the lemma; indeed, the composition of $\mathrm{dis}_!$ followed by $|\cdot|_J$ is the identity. Since $\mathbf{L}\mathrm{dis}_!$ is an equivalence, the same is true for $\mathbf{L}|\cdot|_J$. When we identify the localization of the lemma with $\mathbf{dSpaces}_{RSC}$ as above, we can write this Quillen pair as

$$\mathbf{dSpaces}_{RSC} \underset{\mathrm{Sing}_J}{\overset{|\cdot|_J}{\rightleftarrows}} \mathbf{dSets}.$$

Example 12.25 Let X be a dendroidal Segal space. If $X(C_1) \to X(\eta)$ is a weak equivalence, then X is complete. Indeed, in order to show that $X(J) \to X(\eta)$ is also a weak equivalence, consider the collection of monomorphisms $M \to N$ of simplicial sets for which $X(i_!N) \to X(i_!M)$ is a weak equivalence. This collection is saturated and closed under two-out-of-three (among monomorphisms). Moreover, it contains all inner horn inclusions because X has the Segal property, as well as the two inclusions $\Delta[0] \to \Delta[1]$ by hypothesis. But then it must contain all simplicial anodyne maps, i.e., all trivial cofibrations in the Kan–Quillen model structures. In particular, it contains $\Delta[0] \to J$.

Remark 12.26 In some cases it is more convenient to work with the projective model structure $\mathbf{dSpaces}_P$, with the corresponding localizations $\mathbf{dSpaces}_{PS}$ by the Segal condition and $\mathbf{dSpaces}_{PSC}$ also with respect to $\eta \to J$. These localizations exist by

Proposition 11.25 and we get the following diagram of left Quillen functors and left Quillen equivalences (with all functors simply the identity, of course):

$$\begin{tikzcd} \mathbf{dSpaces}_P \arrow[r] \arrow[d, "\wr"] & \mathbf{dSpaces}_{PS} \arrow[r] \arrow[d, "\wr"] & \mathbf{dSpaces}_{PSC} \arrow[d, "\wr"] \\ \mathbf{dSpaces}_R \arrow[r] & \mathbf{dSpaces}_{RS} \arrow[r] & \mathbf{dSpaces}_{RSC}. \end{tikzcd}$$

Then a projectively fibrant object X is local for the PSC-localization if and only if a Reedy fibrant replacement of X is a dendroidal complete Segal space. The following example is an illustration of this situation.

Example 12.27 Let $\mathbf{P}$ be a simplicial operad and assume all its spaces of operations are Kan complexes. We noticed in Example 12.11 that $N\mathbf{P}$ is a projectively fibrant dendroidal space satisfying the Segal condition. In many examples, $\mathbf{P}$ has just one colour and the space $\mathbf{P}(1)$ of unary operations is contractible. This is the case for (simplicial) versions of the little n-cubes operad $\mathbf{E}_n$, for instance, and for the Barratt–Eccles operad. For such an operad $\mathbf{P}$, the map $N\mathbf{P}(C_1) \to N\mathbf{P}(\eta)$ is the map $\mathbf{P}(1) \to \Delta[0]$, which is a weak equivalence by hypothesis. The same is then true for a Reedy fibrant replacement of $N\mathbf{P}$, making it a dendroidal complete Segal space. Hence $N\mathbf{P}$ itself is a fibrant object in $\mathbf{dSpaces}_{PSC}$, but not necessarily in $\mathbf{dSpaces}_{RSC}$.

Example 12.28 Let $\mathbf{P}$ be a discrete operad, i.e., an operad in the category of sets, and write C for the set of colours of $\mathbf{P}$. One can construct a simplicial object $\widetilde{\mathbf{P}}$ in the category of discrete operads as follows. In simplicial degree n, the set of colours of $\widetilde{\mathbf{P}}_n$ is the set of strings of isomorphisms in (the underlying category of) $\mathbf{P}$ of the form

$$\overline{c}\colon c_0 \xrightarrow{\cong} c_1 \xrightarrow{\cong} \cdots \xrightarrow{\cong} c_n.$$

For such strings $\overline{c}^1, \ldots, \overline{c}^k$ and $\overline{c}$, an operation α in $\widetilde{\mathbf{P}}_n(\overline{c}^1, \ldots, \overline{c}^k; \overline{c})$ is a sequence of operations $\alpha_i \in \mathbf{P}(c_i^1, \ldots, c_i^k; c_i)$, for $i = 0, \ldots, n$, which are compatible with the specified isomorphisms. The simplicial structure maps relating these $\widetilde{\mathbf{P}}_n$ are defined exactly as for the nerves of categories. Thus, $d_i\colon \widetilde{\mathbf{P}}_n \to \widetilde{\mathbf{P}}_{n-1}$ deletes the ith component. Clearly, each such face map is surjective on colours and fully faithful on operations. For a fixed n, the dendroidal set $N(\widetilde{\mathbf{P}}_n)$ is the nerve of a discrete operad, hence a dendroidal strict inner Kan complex. The same is true for $N\widetilde{\mathbf{P}}(\partial\Delta[n])$ and one easily checks that

$$N\widetilde{\mathbf{P}}_n = N\widetilde{\mathbf{P}}(\Delta[n]) \to N\widetilde{\mathbf{P}}(\partial\Delta[n])$$

is a fibration between ∞-operads. The maps $N\widetilde{\mathbf{P}}_n \to N\widetilde{\mathbf{P}}_{n-1}$ are weak equivalences of ∞-operads because $\widetilde{\mathbf{P}}_n \to \widetilde{\mathbf{P}}_{n-1}$ is essentially surjective and fully faithful, so $N\widetilde{\mathbf{P}}$ is a fibrant object in $\mathbf{dSpaces}_{RSC}$. In other words, $N\widetilde{\mathbf{P}}$ is a complete dendroidal Segal space. As $\mathrm{con}(N\mathbf{P}) \to N\widetilde{\mathbf{P}}$ gives a weak equivalence of dendroidal sets in each simplicial degree, it is a weak equivalence in $\mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}$ and hence in $\mathbf{dSpaces}_{RSC}$.

Therefore our construction of $N\widetilde{\mathbf{P}}$ yields an explicit fibrant replacement (or 'completion') of $N\mathbf{P}$ in $\mathbf{dSpaces}_{RSC}$. In fact, the reader can unravel the definitions to find that

$$N\widetilde{\mathbf{P}}_n(T) = \mathbf{dSets}(J[n] \otimes T, N\mathbf{P}),$$

so this is just a special instance of Example 12.24.

12.3 Complete Weak Equivalences

In this section we discuss the weak equivalences in the model category $\mathbf{dSpaces}_{RSC}$ for complete dendroidal Segal spaces in more detail. We shall call a map of dendroidal spaces $f: X \longrightarrow Y$ a *complete weak equivalence* if it is a weak equivalence in $\mathbf{dSpaces}_{RSC}$. This model category is defined as a left Bousfield localization of the category $\mathbf{dSpaces}_R$ of dendroidal spaces with the Reedy model structure and we have seen in the previous section that it can also be viewed as a left Bousfield localization of the category $\mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}$ of simplicial objects in dendroidal sets, equipped with the Reedy model structure coming from $\mathbf{\Delta}$. In particular, the latter perspective immediately gives the following property of the complete weak equivalences:

Proposition 12.29 *Let $f: X \longrightarrow Y$ be a map of dendroidal spaces. If for each simplicial degree $n \geq 0$ the map $f_n: X_n \longrightarrow Y_n$ is an operadic weak equivalence between dendroidal sets, then $X \longrightarrow Y$ is a complete weak equivalence.*

Lemma 8.49 states that in a left Bousfield localization, the local weak equivalences between local objects are also weak equivalences in the original model category. Hence we can also immediately record the following.

Proposition 12.30 *Let $f: X \longrightarrow Y$ be a map between complete dendroidal Segal spaces. Then the following statements are equivalent:*

(1) *$f: X \longrightarrow Y$ is a complete weak equivalence.*
(2) *f induces a weak homotopy equivalence $X(T) \longrightarrow Y(T)$ between Kan complexes for each tree T in $\mathbf{\Omega}$.*
(3) *f induces an operadic equivalence $X_n \longrightarrow Y_n$ between ∞-operads for each $n \geq 0$.*
(4) *f induces an operadic equivalence $X_0 \longrightarrow Y_0$.*

Proof The equivalence between the first three statements follows by the comment preceding the proposition, interpreting $\mathbf{dSpaces}_{RSC}$ as a left Bousfield localization of the Reedy model structures on $\mathbf{sSets}^{\mathbf{\Omega}^{\mathrm{op}}}$ for statement (2) and on $\mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}$ for statement (3). Statement (4) is equivalent to (3), because each face map $d_i: X_n \longrightarrow X_{n-1}$ is a trivial fibration for a complete dendroidal Segal space X. □

In Theorem 9.45 we characterized the operadic equivalences between ∞-operads as those maps that are fully faithful and essentially surjective. We now aim for a similar description of weak equivalences between (complete) dendroidal Segal spaces. Recall that to define fully faithful maps between ∞-operads, we used the 'space of operations' in an ∞-operad X (which is a dendroidal *set*) defined by a pullback

$$\begin{tikzcd} X(x_1, \ldots, x_n; y) \arrow[r] \arrow[d] & \mathrm{Map}(C_n, X) \arrow[d] \\ \Delta[0] \arrow[r, "{(x_1, \ldots, x_n, y)}"] & \mathrm{Map}(\partial C_n, X). \end{tikzcd}$$

The functor sending a dendroidal space Y to its underlying dendroidal set Y_0 is right Quillen with respect to the model structure $\mathbf{dSpaces}_{RSC}$ (cf. Theorem 12.22), which makes it straightforward to compare this 'space of operations' to the one for dendroidal Segal spaces given in Section 12.1:

Lemma 12.31 (i) *Let Y be a complete dendroidal Segal space. Then for any sequence of objects $y_1, \ldots, y_n, y \in Y(\eta)_0$ there is a natural weak homotopy equivalence*

$$Y_0(y_1, \ldots, y_n; y) \simeq Y(y_1, \ldots, y_n; y).$$

(ii) *Consequently, a map $f\colon X \to Y$ between complete dendroidal Segal spaces is fully faithful if and only if $f_0\colon X_0 \to Y_0$ is a fully faithful map of ∞-operads.*

Proof This follows from Proposition 11.11 applied to the adjoint pair $(\mathrm{dis}_!, \mathrm{dis}^*)$. □

Remark 12.32 Note that item (i) in particular shows that for a complete dendroidal Segal space Y, the homotopy operad $\mathrm{ho}(Y)$ is naturally isomorphic to the homotopy operad $\tau(Y_0)$ associated to the dendroidal set Y_0.

Next, let us take a closer look at 'essential surjectivity'. Consider a map $f\colon X \to Y$ between complete dendroidal Segal spaces. Recall that the corresponding map $f_0\colon X_0 \to Y_0$ between ∞-operads is said to be essentially surjective if for any $y \in Y_0(\eta)$ there exists an $x \in X_0(\eta)$ and a 'path' $J \to Y_0$ connecting $f(x)$ to y. Another way to express this is by saying that there exists a lift as follows:

$$\begin{tikzcd} & Y(J) \arrow[d] \\ \Delta[0] \arrow[r, "{(f(x), y)}"'] \arrow[ur, dashed] & Y(\eta) \times Y(\eta). \end{tikzcd}$$

The completeness of Y implies that $Y(J)$ is a path object for the Kan complex $Y(\eta)$. More precisely, in the diagram

$$\begin{tikzcd} Y(\eta) \arrow[r] \arrow[dr, "\mathrm{diag}"'] & Y(J) \arrow[d] \\ & Y(\eta) \times Y(\eta), \end{tikzcd}$$

the horizontal map is a weak equivalence by completeness, whereas the vertical map is a fibration by Reedy fibrancy of Y. Another path object for $Y(\eta)$ would of course be just $Y(\eta)^{\Delta[1]}$. This shows the consistency of the following definition.

Definition 12.33 A map $f: X \to Y$ between complete dendroidal Segal spaces is *essentially surjective* if any of the following equivalent conditions holds:

(1) For any object $y \in Y(\eta)_0$ there exists an object $x \in X(\eta)_0$ and a path $\Delta[1] \to Y(\eta)$ connecting $f(x)$ to y.
(2) For any object $y \in Y(\eta)_0$ there exists an object $x \in X(\eta)_0$ and a path $J \to Y_0$ connecting $f(x)$ to y.
(3) The induced morphism of homotopy operads $\mathrm{ho}(X) \to \mathrm{ho}(Y)$ is essentially surjective.

Now that we have discussed the meanings of 'fully faithful' and 'essentially surjective' in the context of both ∞-operads and complete dendroidal Segal spaces, we can state the following variant of the previous proposition.

Corollary 12.34 *Let $f: X \to Y$ be a map between complete dendroidal Segal spaces. Then the following statements are equivalent:*

(1) *$f: X \to Y$ is a complete weak equivalence.*
(2) *$f: X \to Y$ is a fully faithful and essentially surjective map of complete dendroidal Segal spaces.*
(3) *$f_0: X_0 \to Y_0$ is a fully faithful and essentially surjective map of ∞-operads.*

Proof The fact that (1) is equivalent to (3) is clear from the previous proposition and the characterization of operadic equivalences between ∞-operads as the fully faithful and essentially surjective maps. By the discussion above, (2) is also equivalent to (3). Indeed, $f: X \to Y$ is fully faithful if and only if $f_0: X_0 \to Y_0$ is fully faithful by Lemma 12.31, while essential surjectivity of f is equivalent to that of f_0 by formulations (1) and (2) of the definition above. □

It is harder to show (and perhaps a bit surprising) that the equivalence between (1) and (2) of the corollary also holds if X and Y are dendroidal Segal spaces that are not necessarily complete. Without completeness, we cannot identify the space $X(x_1, \dots, x_n; x)$ with the space $X_0(x_1, \dots, x_n; x)$ associated to the ∞-operad X_0 and the different versions of Definition 12.33 are no longer equivalent. Thus, in addition to the notion of fully faithful map between Segal spaces already introduced in Section 12.1, we use version (3) of Definition 12.33 for essential surjectivity. Explicitly:

Definition 12.35 A map of dendroidal Segal spaces $f: X \to Y$ is *essentially surjective* if the induced morphism of homotopy operads $\mathrm{ho}(X) \to \mathrm{ho}(Y)$ is essentially surjective.

The main result of this section is as follows:

Theorem 12.36 *Let $f: X \to Y$ be a map of dendroidal Segal spaces. Then f is a complete weak equivalence if and only if f is fully faithful and essentially surjective.*

Remark 12.37 The theorem can be interpreted as saying that the model category $\mathbf{dSpaces}_{RSC}$ for complete dendroidal Segal spaces is the localization of the model category $\mathbf{dSpaces}_{RS}$ for dendroidal Segal spaces with respect to the fully faithful and essentially surjective maps.

A map f of dendroidal Segal spaces is fully faithful and essentially surjective if and only if f is fully faithful and induces an equivalence of homotopy operads; in particular, the class of such maps is clearly closed under two-out-of-three. We have already seen that the statement of the theorem is true for *complete* dendroidal Segal spaces, so in fact it suffices to prove that any dendroidal Segal space X has a completion (i.e., a fibrant replacement in $\mathbf{dSpaces}_{RSC}$) via a fully faithful and essentially surjective map. Thus, the theorem is a consequence of the following result.

Proposition 12.38 *Let X be a dendroidal Segal space. Then there exists a completion $u\colon X \to \widehat{X}_f$ by a map which is fully faithful and a bijection on objects:*

$$X(\eta)_0 \xrightarrow{\cong} \widehat{X}_f(\eta)_0.$$

Proof As in Example 12.24, we may associate to each of the ∞-operads X_n a complete dendroidal Segal space $\mathrm{Sing}_J(X_n)$, defined by

$$\mathrm{Sing}_J(X_n)(T)_m = \mathbf{dSets}(J[m] \otimes T, X_n).$$

Thus $\mathrm{Sing}_J(X_\bullet)$ is now a simplicial object in dendroidal spaces, i.e., a bisimplicial dendroidal set, and we define $\widehat{X}$ to be its diagonal:

$$\widehat{X}_n(T) := \mathrm{Sing}_J(X_n)(T)_n.$$

The map $J[n] \to J[0] = \eta$, which is an operadic equivalence, defines a map of dendroidal spaces $X \to \widehat{X}$, which in simplicial degree n is the operadic equivalence of dendroidal sets

$$X_n(-) \to \mathbf{dSets}(J[n] \otimes -, X_n).$$

By Proposition 12.29 it is then a complete weak equivalence. Let $\widehat{X} \to \widehat{X}_f$ be a Reedy fibrant replacement of dendroidal spaces and let $u\colon X \to \widehat{X}_f$ be the composition. Then u is also a complete weak equivalence. The construction of this Reedy fibrant replacement by the small object argument involves pushouts along maps of the form (a) in Proposition 12.16, which are isomorphisms in 'bidegrees' $(\eta, 0)$, i.e., do not add new objects. Since $X(\eta)_0 \to \widehat{X}(\eta)_0$ is an isomorphism, so is $X(\eta) \to \widehat{X}_f(\eta)_0$. It thus suffices to prove Lemmas 12.39 and 12.40 below. □

Lemma 12.39 *The dendroidal Segal space $\widehat{X}_f$ is complete.*

Lemma 12.40 *The map $u\colon X \to \widehat{X}_f$ is fully faithful.*

Proof (of Lemma 12.39) Observe that for fixed n and T, the face maps

$$\mathrm{Sing}_J(X_n)(T) \to \mathrm{Sing}_J(X_{n-1})(T)$$

are trivial fibrations of simplicial sets. Indeed, let us write $\partial J[m]$ for the union of the subobjects $\partial_i\colon J[m-1] \to J[m]$ (as we have previously been doing for cosimplicial objects). Then a lifting problem as in the square on the left is equivalent to a lifting problem as in the square on the right:

$$\begin{tikzcd} \partial\Delta[m] \arrow[r] \arrow[d] & \mathrm{Sing}_J(X_n)(T) \arrow[d] \\ \Delta[m] \arrow[r] \arrow[ur, dashed] & \mathrm{Sing}_J(X_{n-1})(T) \end{tikzcd} \qquad \begin{tikzcd} \Delta[n-1] \arrow[r] \arrow[d] & X(J[m]\otimes T) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & X(\partial J[m]\otimes T). \end{tikzcd}$$

The map on the very right is a Kan fibration because X is Reedy fibrant and $\partial J[m] \otimes T \to J[m] \otimes T$ is a normal monomorphism of dendroidal sets, so these lifting problems indeed have solutions. Then the fundamental property of bisimplicial sets (Corollary 10.27) guarantees that the simplicial set $\widehat{X}(T)$ is weakly equivalent to $\widehat{X}_0(T)$, for any tree T. The latter is local in $\mathbf{dSpaces}_{RSC}$ by Example 12.24. Hence any Reedy fibrant replacement of $\widehat{X}$ is a complete dendroidal Segal space. □

Proof (of Lemma 12.40) Recall the notation $\mathbf{Hom}(A, Y)$ for the 'internal hom' between dendroidal sets A and Y, defined by

$$\mathbf{Hom}(A, Y)(T) = \mathbf{dSets}(A \otimes T, Y).$$

The proof consists of three steps. First of all, we replace the dendroidal space $\widehat{X}$ by an equivalent object. Recall that $\mathrm{Sing}_J(X_n)(T)$ is a model for the mapping space $\mathrm{Map}(T, X_n)$. By Example 11.16 and 11.14(b) it is naturally equivalent (via a trivial fibration) to the mapping object $ki^*\mathbf{Hom}(T, X_n)$. Hence Corollary 10.27 (the fundamental property of bisimplicial sets) implies that the map

$$\widehat{X}(T) \to \mathrm{diag}(ki^*\mathbf{Hom}(T, X_\bullet))$$

is a weak equivalence of simplicial sets, for any T. We define $\overline{X}$ to be a Reedy fibrant replacement of the dendroidal space

$$T \mapsto \mathrm{diag}(ki^*\mathbf{Hom}(T, X_\bullet)).$$

Hence it suffices to prove that $X \to \overline{X}$ is fully faithful. Observe that the dendroidal space above is also naturally a subobject of the diagonal of the bisimplicial dendroidal set

$$\mathbf{Hom}(\Delta[\bullet], X_\bullet).$$

To be more precise, write $X^{E[m]}$ for the dendroidal space of which the n-simplices are described by

$$\big(X^{E[m]}\big)(T)_n = \big(ki^*\mathbf{Hom}(T, X_n)\big)_m.$$

Then $\overline{X}$ is, by construction, a Reedy fibrant replacement of the diagonal of $X_\bullet^{E[\bullet]}$. Since Theorem 6.51 states that 'equivalences are determined pointwise', one may also regard the elements of $X^{E[m]}(T)_n$ as the maps

$$\xi\colon \Delta[m] \otimes T \to X_n$$

such that for every edge e of T, the restriction $\xi_e\colon \Delta[m] \to X_n$ sends every edge of $\Delta[m]$ to an equivalence in X_n.

The second step is to reduce the problem to showing that the 'constant path map' $X \to X^{E[1]}$ is fully faithful. Indeed, if this is the case, then so is the iterated map

$$X \to X^{E[1]} \times_X \cdots \times_X X^{E[1]}$$

into an m-fold pullback. But the map

$$X^{E[m]} \to X^{E[1]} \times_X \cdots \times_X X^{E[1]},$$

arising from the inclusion of the 'intervals' with endpoints $(i, i+1)$ into $\Delta[m]$, is a trivial fibration by the Segal condition. So then $X \to X^{E[m]}$ is fully faithful as well. If this holds for every m, then $X \to \mathrm{diag}(X_\bullet^{E[\bullet]}) \xrightarrow{\sim} \overline{X}$ is also fully faithful, as follows easily from Corollary 10.27. This proves that it is indeed enough to show that $X \to X^{E[1]}$ is fully faithful.

For the third and final step, fix a sequence of objects $x_1, \ldots, x_k, x$ of X. Then $X(x_1, \ldots, x_k; x)$ is the pullback

$$\begin{tikzcd}
X(x_1, \ldots, x_k; x) \arrow[r] \arrow[d] & X(C_k) \arrow[d] \\
\Delta[0] \arrow[r] & X(\partial C_k).
\end{tikzcd}$$

Let us consider a similar pullback for $X^{E[1]}$, as in the left square of

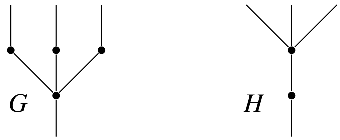
$$\begin{tikzcd}
X^{E[1]}(x_1, \ldots, x_k; x) \arrow[r] \arrow[d] & X^{E[1]}(C_k) \arrow[r] \arrow[d] & X(C_k \otimes C_1) \arrow[d] \\
\Delta[0] \arrow[r] & X^{E[1]}(\partial C_k) \arrow[r] & X(\partial C_k \otimes C_1),
\end{tikzcd}$$

where we have identified $x_1, \ldots, x_k, x$ with their images under $X \to X^{E[1]}$. The square on the right is a pullback as well, precisely because equivalences are determined pointwise as already noted above. Thus, if we show that the outer rectangle in the diagram

$$\begin{tikzcd}
X(x_1, \ldots, x_k; x) \arrow[r] \arrow[d] & X^{E[1]}(x_1, \ldots, x_k; x) \arrow[r] \arrow[d] & X(C_k \otimes C_1) \arrow[d] \\
\Delta[0] \arrow[r, equal] & \Delta[0] \arrow[r] & X(\partial C_k \otimes C_1),
\end{tikzcd}$$

is a homotopy pullback, it follows that the square on the left is a homotopy pullback and thus $X(x_1, \ldots, x_k; x) \to X^{E[1]}(x_1, \ldots, x_k; x)$ is a weak homotopy equivalence, as desired.

Now $C_k \otimes C_1$ is the union of two shuffles G and H,



whose intersection (which is just a k-corolla C_k) we denote by F. Thus

$$X(C_k \otimes C_1) = X(G) \times_{X(F)} X(H).$$

The Segal condition for X now gives homotopy equivalences

$$X(H) \simeq X(C_1) \times_{X(\eta)} X(C_k)$$

and

$$X(G) \simeq X(C_k) \times_{X(\eta)^k} X(C_1)^k.$$

Pulling back along the map from $\Delta[0]$ picking out $x_1, \ldots, x_k, x$ therefore gives homotopy pullback squares

$$\begin{tikzcd}
X(x_1, \ldots, x_k; x) \arrow[r] \arrow[d] & X(H) \arrow[d] \\
\Delta[0] \arrow[r] & X(C_1) \times_{X(\eta)} X(\partial C_k),
\end{tikzcd}$$

$$\begin{tikzcd}
X(x_1, \ldots, x_k; x) \arrow[r] \arrow[d] & X(G) \arrow[d] \\
\Delta[0] \arrow[r] & X(\partial C_k) \times_{X(\eta)^k} X(C_1)^k.
\end{tikzcd}$$

Combining these squares with the observation

$$\big(X(C_1) \times_{X(\eta)} X(\partial C_k)\big) \times_{X(\partial C_k)} \big(X(\partial C_k) \times_{X(\eta)^k} X(C_1)^k\big) = X(\partial C_k \otimes C_1)$$

then shows that

$$\begin{tikzcd}
X(x_1, \ldots, x_k; x) \arrow[r] \arrow[d] & X(G) \times_{X(F)} X(H) \arrow[d] \\
\Delta[0] \arrow[r] & X(\partial C_k \otimes C_1)
\end{tikzcd}$$

is a homotopy pullback square. But this is precisely the outer rectangle from our previous diagram. We conclude that $X \to X^{E[1]}$ is fully faithful and the proof of the lemma is complete. □

12.4 The Tensor Product of Dendroidal Spaces

The aim of this section is to show that the tensor product of dendroidal spaces behaves well with respect to the model structures we have introduced. In particular, we will see that it admits a derived functor and defines a symmetric monoidal structure on the homotopy category Ho(**dSpaces**$_{RSC}$). By Theorem 12.22 this category is equivalent to the homotopy category Ho(**dSets**) associated with the operadic model structure, which therefore inherits a symmetric monoidal structure as well. In particular, this result improves on our earlier statements that the tensor product gives a well-defined symmetric monoidal structure on the homotopy category of *open* dendroidal sets.

The tensor product of dendroidal spaces can be characterized as follows. For representable dendroidal spaces $S \boxtimes \Delta[m]$ and $T \boxtimes \Delta[n]$, with trees S and T, their tensor product is given by

$$(S \boxtimes \Delta[m]) \otimes (T \boxtimes \Delta[n]) = (S \otimes T) \boxtimes (\Delta[m] \times \Delta[n]).$$

Moreover, the tensor product preserves colimits in each variable separately.

For model categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, let us say that a functor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a *left Quillen bifunctor* if F preserves colimits in each variable separately and for cofibrations $i\colon K \to L$ in $\mathcal{C}$ and $j\colon M \to N$ in $\mathcal{D}$, the map

$$F(K, N) \amalg_{F(K,M)} F(L, M) \to F(L, N)$$

is a cofibration in $\mathcal{E}$ that is moreover trivial if i or j is trivial. With this terminology we can formulate the main technical result of this section as follows:

Theorem 12.41 *The tensor product of dendroidal spaces defines a Quillen bifunctor*

$$\mathbf{dSpaces}_{PSC} \times \mathbf{dSpaces}_{PSC} \to \mathbf{dSpaces}_{RSC}.$$

Observe that in the domain we are using the projective model structure for complete dendroidal Segal spaces, whereas the target has the Reedy version. At the level of homotopy categories this difference is irrelevant; in particular, we will deduce the following consequence of the theorem at the end of this section.

Corollary 12.42 *The tensor product of dendroidal spaces defines a symmetric monoidal structure on the homotopy category* Ho(**dSpaces**$_{RSC}$).

We start the proof of the theorem with a simple observation:

Lemma 12.43 *The tensor product of dendroidal spaces defines a left Quillen bifunctor*

$$\mathbf{dSpaces}_{P} \times \mathbf{dSpaces}_{P} \to \mathbf{dSpaces}_{R}.$$

Proof Generating (trivial) cofibrations for the projective model structure on dendroidal spaces are maps of the form $T \boxtimes M \to T \boxtimes N$ with T a tree and $M \to N$ a generating (trivial) cofibration for the Kan–Quillen model structure for simplicial sets. For another such map $S \boxtimes K \to S \boxtimes L$, their pushout-product is the map

$$(S \otimes T) \boxtimes (K \times N \cup_{K \times M} L \times M) \to (S \otimes T) \boxtimes (L \times N).$$

This is a Reedy cofibration (since $S \otimes T$ is normal and hence Reedy cofibrant) that is moreover a weak equivalence whenever $K \to L$ or $M \to N$ is. □

Remark 12.44 The preceding proof demonstrates why the Reedy model structure has to be used in the target; generally the dendroidal set $S \otimes T$ is not representable and there is no reason for it to be a projectively cofibrant dendroidal space. Still, the use of the projective model structure in the domain is necessary, because it makes the family of generating cofibrations small enough for the result to hold.

To see how the tensor product interacts with the localizations for the Segal and completeness conditions, we will again use the combinatorics of shuffles as in Section 6.3. Recall that for trees S and T, their set of shuffles is partially ordered with minimal element the shuffle obtained by grafting copies of S on top of T and maximal element the one where copies of T are grafted on top of S. The partial ordering is defined by 'shuffling down' the vertices of S through those of T.

The following is the crucial input to proving Theorem 12.41. Unlike the results of Section 6.3 it does not require any assumptions (such as openness) on our dendroidal spaces. Those assumptions were necessary in the case of dendroidal sets to guarantee that the relevant pushout-products were normal monomorphisms; we will see that the map featuring in the proof below is of a simpler form and can be checked to be a normal monomorphism without further hypotheses.

Lemma 12.45 *The tensor product of dendroidal spaces is compatible with the Segal condition in the sense that it defines a left Quillen bifunctor*

$$\mathbf{dSpaces}_{PS} \times \mathbf{dSpaces}_{PS} \to \mathbf{dSpaces}_{RS}.$$

Proof By Lemma 12.7, we can characterize the model category $\mathbf{dSpaces}_{PS}$ as obtained from $\mathbf{dSpaces}_{P}$ by localizing with respect to the 'grafting maps'

$$(S \cup_e R) \boxtimes \Delta[n] \cup (S \circ_e R) \boxtimes \partial\Delta[n] \to (S \circ_e R) \boxtimes \Delta[n]$$

where S and R are trees, e is a leaf of S and simultaneously the root of R, and $n \geq 0$. Given Lemma 12.43 above, it therefore suffices to prove that the pushout-product of a map as above with a generating cofibration $T \boxtimes M \to T \boxtimes N$ is a weak equivalence in the model category $\mathbf{dSpaces}_{RS}$. (Here $M \to N$ is a monomorphism of simplicial sets.) Such a pushout-product is of the form

$$X \boxtimes V \cup_{X \boxtimes U} Y \boxtimes U \to Y \boxtimes V$$

with $X = (S \cup_e R) \otimes T$, $Y = (S \circ_e R) \otimes T$, and $U \to V$ the monomorphism of simplicial sets

$$\partial\Delta[n] \times N \cup_{\partial\Delta[n] \times M} \Delta[n] \times M \to \Delta[n] \times N.$$

To show that this is a trivial cofibration in $\mathbf{dSpaces}_{RS}$, it suffices to show that the map

$$(S \cup_e R) \otimes T \to (S \circ_e R) \otimes T$$

is an inner anodyne map of dendroidal sets.

We begin by observing that this map is a normal monomorphism; indeed, the codomain is normal and the square

$$\begin{tikzcd}
\{e\} \otimes T \arrow[r] \arrow[d] & R \otimes T \arrow[d] \\
S \otimes T \arrow[r] & (S \circ_e R) \otimes T
\end{tikzcd}$$

is easily seen to be a pullback. We may therefore view $(S \cup_e R) \otimes T$ as a subobject of $(S \circ_e R) \otimes T$. (This is where the current proof is simpler, and applies more generally, than that of Lemma 6.24.)

Write v_e for the root vertex of R, which is also the top vertex of the edge e in the tree $S \circ_e R$. Any shuffle Q of $(S \circ_e R) \otimes T$ will have one or several vertices of the form $v_e \otimes t$, for t an edge of T, and we will call the outgoing edges $e \otimes t$ of such vertices *special edges*. Now consider a filtration

$$(S \cup_e R) \otimes T =: A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots, \qquad \bigcup_i A_i = (S \circ_e R) \otimes T$$

by adjoining the shuffles of $S \circ_e R$ and T one by one, in some order that is compatible with the partial ordering on shuffles described above the lemma. If A_{i+1} is obtained from A_i by adjoining some shuffle Q, we define a further filtration

$$A_i =: A_i^0 \subseteq A_i^1 \subseteq A_i^2 \subseteq \cdots,$$

by adjoining all of the outer faces P (possibly of high codimension) of Q one by one, in some order that extends the partial ordering of inclusion. Consider a step $A_i^j \subseteq A_i^{j+1}$ in this filtration, adjoining some outer face P of Q. As usual we write $I(P)$ for the set of inner edges of P. Furthermore, write $\Sigma_P \subseteq I(P)$ for the subset consisting of special edges $e \otimes t$ as defined above. Without loss of generality we may assume that Σ_P is nonempty; indeed, if it is not, then P is entirely contained in either $S \otimes T$ or $R \otimes T$ and the inclusion $A_i^j \subseteq A_i^{j+1}$ is the identity.

Now write $J(P) = I(P) - \Sigma_P$ and for every $H \subseteq J(P)$, write P^H for the tree obtained from P by contracting all edges in the complement $J(P) - H$. Adjoining the P^H to A_i^j in some order that extends the partial ordering of inclusion of subsets of $J(P)$, we obtain a further filtration

$$A_i^j =: A_i^{j,0} \subseteq A_i^{j,1} \subseteq A_i^{j,2} \subseteq \cdots, \qquad \bigcup_k A_i^{j,k} = A_i^j \cup P = A_i^{j+1}.$$

We will now argue that each inclusion $A_i^{j,k} \subseteq A_i^{j,k+1}$, adjoining some tree P^H, is inner anodyne. Suppose that P^H is not yet contained in $A_i^{j,k}$. Then we can argue the following:

(1) Any outer face of P^H is contained in A_i^j by our induction on the size of P.
(2) Any inner face of P^H contracting an edge of $J(P)$ is in $A_i^{j,k}$ by our induction on the size of H.
(3) Any inner face F contracting a special edge (or multiple such edges) in Σ_P cannot be contained in $A_i^{j,k}$. Indeed, it is clear that if such an F would be in A_0, then P^H itself would already have been contained in A_0. Also, F cannot factor through any earlier shuffle P'. Indeed, contracting special edges can only create overlap with other shuffles in which the vertex v_e has been shuffled further down. □

Now Lemma 6.22 implies that $A_i^{j,k} \subseteq A_i^{j,k+1}$ is a pushout of the inner anodyne map

$$\Lambda^{\Sigma_P}[P^H] \to P^H. \qquad \square$$

We are now almost done proving the main results of this section:

Proof (of Theorem 12.41) After Lemma 12.45 it only remains to show that the tensor product of dendroidal spaces also respects the localization with respect to completion. To be precise, it will suffice to show that the pushout-product of a map

$$J \boxtimes \partial\Delta[n] \cup \{0\} \boxtimes \Delta[n] \to J \boxtimes \Delta[n]$$

with a generating projective cofibration of the form $T \boxtimes M \to T \boxtimes N$ is a weak equivalence in the model category $\mathbf{dSpaces}_{RSC}$. As in the proof of the previous lemma, that pushout-product can be rewritten in the form

$$X \boxtimes V \cup_{X \boxtimes U} Y \boxtimes U \to Y \boxtimes V$$

with $X = \{0\} \otimes T$, $Y = J \otimes T$, and $U \to V$ the monomorphism of simplicial sets

$$\partial\Delta[n] \times N \cup_{\partial\Delta[n]\times M} \Delta[n] \times M \to \Delta[n] \times N.$$

Hence it suffices to show that $\{0\} \times T \to J \otimes T$ is trivial cofibration in the operadic model structure on dendroidal sets. This follows from Proposition 9.28. □

Proof (of Corollary 12.42) It follows from Theorem 12.41 that the tensor product indeed gives a well-defined functor on homotopy categories. To verify that this is a symmetric monoidal structure, it remains to check associativity. To do this, we will argue that the associator maps of the kind

$$(X \otimes Y) \otimes Z \to X \otimes Y \otimes Z$$

are weak equivalences whenever X, Y, and Z are cofibrant objects of $\mathbf{dSpaces}_{PSC}$ (cf. Remark 9.48). By the usual skeletal induction it suffices to do this for dendroidal sets represented by trees T_1, T_2, and T_3. Since $\mathrm{Sp}[T] \to T$ is a trivial cofibration, we can reduce further to the case of corollas. In that case we explicitly verified that these associators are inner anodyne in Proposition 6.32. □

12.5 Closed Dendroidal Spaces

For many (simplicial or topological) operads $\mathbf{P}$ of interest, the space $\mathbf{P}(0)$ of nullary operations is contractible, or even equal to a point. This is the case for the operads $\mathbf{E}_n$ of little n-cubes, for example. In this section we study variants of the theory of dendroidal spaces which take these conditions into account. Amongst other things, we will show that a dendroidal Segal space X for which all spaces of nullary operations are contractible may be replaced, up to weak equivalence, by one for which these spaces equal a point. In fact, we will prove that the homotopy theory of ∞-operads X with such contractible spaces of nullary operations may be presented in terms of presheaves on the smaller category $\overline{\mathbf{\Omega}}$ of closed trees.

We will see below that dendroidal Segal spaces X with $X(-;x)$ equal to a point, for each object x, can naturally be studied in terms of the following category:

Definition 12.46 The category of *closed dendroidal spaces* is the category of simplicial presheaves on the category $\overline{\mathbf{\Omega}}$ of closed trees:

$$\mathbf{cdSpaces} := \mathbf{sSets}^{\overline{\mathbf{\Omega}}^{\mathrm{op}}}.$$

To discuss dendroidal spaces Segal spaces X satisfying the weaker condition that $X(-;x)$ is contractible, we introduce the following terminology.

Definition 12.47 A dendroidal space X is *weakly closed* if for each tree T in $\mathbf{\Omega}$ with closure $T \subseteq \overline{T}$, the restriction map $X(\overline{T}) \to X(T)$ is a weak equivalence.

Remark 12.48 The property of being weakly closed is evidently invariant under weak equivalence in the projective or Reedy model structure on **dSpaces**. In particular, any weakly closed dendroidal space X has a Reedy fibrant replacement Y which is still weakly closed. For such a Y, the restriction maps $Y(\overline{T}) \to Y(T)$ are trivial fibrations.

We will mostly be interested in the property of being weakly closed for (complete) dendroidal Segal spaces. For these, we observe the following:

Proposition 12.49 *Let X be a dendroidal Segal space. Then the following properties are equivalent:*

(1) *X is weakly closed.*
(2) *For each nullary vertex v in a tree T, the map $X(T) \to X(\partial_v T)$ is a trivial fibration.*
(3) *The map $X(\overline{\eta}) \to X(\eta)$ is a trivial fibration.*

Proof Of course (3) is the special case of (2) where $T = \eta$. Also (1) follows from (2) by repeatedly applying the latter condition to the nullary vertices of $\overline{T} - T$ and (2) follows from (3) by noticing that

$$X(T) \to X(\partial_v T) \times_{X(\eta)} X(\overline{\eta})$$

is a weak equivalence by the Segal condition. □

By the above proposition, the weakly closed dendroidal Segal spaces are the fibrant objects in the left Bousfield localization of the model category $\mathbf{dSpaces}_{RS}$ with respect to the map $\eta \to \overline{\eta}$. We will denote this localization by

$$\mathbf{dSpaces}_{RS,\mathrm{wcl}}$$

and we use similar notations for the analogous localizations by $\eta \to \overline{\eta}$ of the model category $\mathbf{dSpaces}_{RSC}$ and the projective variants $\mathbf{dSpaces}_{PS}$ and $\mathbf{dSpaces}_{PSC}$. Notice in this context that $\eta \to \overline{\eta}$ is a Reedy cofibration, but not a projective one. We will now describe these localizations in a different, simpler way.

Recall the adjoint pair

$$\mathbf{\Omega} \underset{\mathrm{incl}}{\overset{\mathrm{cl}}{\rightleftarrows}} \overline{\mathbf{\Omega}}$$

between the categories of closed trees and all trees. As above, we often use the abbreviated notation $\overline{T}$ for the closure $\mathrm{cl}(T)$. The inclusion and closure functors induce adjoint pairs

$$\mathbf{cdSpaces} \underset{\mathrm{incl}^*}{\overset{\mathrm{incl}_!}{\rightleftarrows}} \mathbf{dSpaces} \underset{\mathrm{cl}^*}{\overset{\mathrm{cl}_!}{\rightleftarrows}} \mathbf{cdSpaces}$$

satisfying

$$\mathrm{cl}_!\mathrm{incl}_! = \mathrm{id}, \quad \mathrm{cl}^* = \mathrm{incl}_!.$$

The functor $\mathrm{incl}_!$ is fully faithful, so it embeds **cdSpaces** as a full subcategory of all dendroidal spaces. Its essential image consists precisely of those dendroidal spaces X for which the restriction $X(\overline{T}) \to X(T)$ is an isomorphism for each tree T.

These adjoint pairs are Quillen pairs for the projective model structure. The category **cdSpaces** also carries a Reedy model structure, where generating cofibrations are the maps of the form

$$\partial_{\mathrm{cl}} T \boxtimes \Delta[n] \cup T \boxtimes \partial\Delta[n] \to T \boxtimes \Delta[n].$$

Here $n \geq 0$, T is any closed tree, and $\partial_{\mathrm{cl}} T$ is the union of all its *closed* faces (which are the inner faces, together with the root face if it exists). These maps are also Reedy cofibrations in **dSpaces** and it is readily verified that $\mathrm{incl}_!$ is also a left Quillen functor with respect to the Reedy model structures. (The same is not true, however, for the functor $\mathrm{cl}_!$. Indeed, $\mathrm{cl}_!$ does not preserve normal monomorphisms, as we have seen before.)

For a closed tree T, the union of all the closed corollas indexed by the internal vertices of T forms the *closed spine* of T, denoted

$$\mathrm{Sp}_{\mathrm{cl}}[T] = \bigcup_v \overline{C_v} \to T.$$

The projective and Reedy model categories $\mathbf{cdSpaces}_P$ and $\mathbf{cdSpaces}_R$ can be localized with respect to these closed spine inclusions to produce model categories $\mathbf{cdSpaces}_{PS}$ and $\mathbf{cdSpaces}_{RS}$ respectively. The fibrant objects in $\mathbf{cdSpaces}_{RS}$ will be referred to as *closed dendroidal Segal spaces*.

Proposition 12.50 *The functors* $\mathrm{incl}_!$ *and* $\mathrm{cl}_!$ *define left Quillen functors*

$$\begin{aligned} \mathrm{incl}_! &: \mathbf{cdSpaces}_{PS} \to \mathbf{dSpaces}_{PS}, \\ \mathrm{incl}_! &: \mathbf{cdSpaces}_{RS} \to \mathbf{dSpaces}_{RS}, \\ \mathrm{cl}_! &: \mathbf{dSpaces}_{PS} \to \mathbf{cdSpaces}_{PS}. \end{aligned}$$

Proof We already observed that these are left Quillen functors for the projective and Reedy model structures, before localizing with respect to (closed) spine inclusions. So it suffices to prove that $\mathrm{incl}_!$ and $\mathrm{cl}_!$ send these localizing maps to weak equivalences. For $\mathrm{incl}_!$, we have to check that for a closed tree T, the inclusion $\mathrm{Sp}_{\mathrm{cl}}[T] \to T$ is a trivial cofibration in $\mathbf{dSpaces}_{RS}$. Using induction on the size of T, this follows from the following grafting property of a closed corolla onto a closed tree T. Consider a nullary vertex v of T attached to an edge e and a closed corolla $\overline{C_k}$. We can graft $\overline{C_k}$ onto $\partial_v T$ by identifying the root edge of $\overline{C_k}$ with the edge e to obtain a new tree T'. Then we may factor the closed spine inclusion of T' as

$$\mathrm{Sp}_{\mathrm{cl}}[T'] \to T \cup_{\overline{\eta}} \overline{C_k} \to T'.$$

The first map is a pushout of the closed spine inclusion $\mathrm{Sp}_{\mathrm{cl}}[T] \to T$ and hence a trivial cofibration by the inductive hypothesis. The second map fits into a square

$$\begin{tikzcd} (\partial_v T \cup_\eta \overline{\eta}) \cup_{\overline{\eta}} \overline{C_k} \arrow[r, equal] \arrow[d] & \partial_v T \cup_\eta \overline{C_k} \arrow[d] \\ T \cup_{\overline{\eta}} \overline{C_k} \arrow[r] & T'. \end{tikzcd}$$

The vertical maps are inner anodyne, so that the bottom horizontal map is a trivial cofibration.

Checking that $\mathrm{cl}_!$ is left Quillen is only slightly more involved since we have to work with the projective model structure. Again using an induction on the size of trees, we now consider a tree S with a leaf e and a corolla C_k to be grafted onto S to obtain a new tree S'. We have to prove that $\mathbf{L}\mathrm{cl}_!$ sends $S \cup_e C_k \to S$ to a weak equivalence in $\mathbf{cdSpaces}_{PS}$ (or equivalently in $\mathbf{cdSpaces}_{RS}$). The pushout A as in

$$\begin{tikzcd} \eta \boxtimes \partial\Delta[1] \arrow[r] \arrow[d] & S \amalg C_k \arrow[d] \\ \eta \boxtimes \Delta[1] \arrow[r] & A \end{tikzcd}$$

is a projectively cofibrant replacement of $S \cup_e C_k$, where the top map picks out the edge e in S and the root of C_k. Then $\mathrm{cl}_!$ sends A to a similar pushout $\overline{A}$ in $\mathbf{cdSpaces}$,

$$\begin{tikzcd} \overline{\eta} \boxtimes \partial\Delta[1] \arrow[r] \arrow[d] & \overline{S} \amalg \overline{C_k} \arrow[d] \\ \overline{\eta} \boxtimes \Delta[1] \arrow[r] & \overline{A}. \end{tikzcd}$$

This object $\overline{A}$ is a projectively cofibrant replacement of $\overline{S} \cup_{\overline{\eta}} \overline{C_k}$ and $\overline{S} \cup_{\overline{\eta}} \overline{C_k} \to \overline{S'}$ is a weak equivalence in $\mathbf{dSpaces}_{RS}$ by the closed analogue of Lemma 12.7. This completes the proof of the proposition. □

Theorem 12.51 (a) *The composition of the left Quillen functor* $\mathrm{incl}_! \colon \mathbf{cdSpaces}_{RS} \to \mathbf{dSpaces}_{RS}$ *with the localization* $\mathbf{dSpaces}_{RS} \to \mathbf{dSpaces}_{RS,\mathrm{wcl}}$ *is a Quillen equivalence.*

(b) *The left Quillen functor* $\mathrm{cl}_! \colon \mathbf{dSpaces}_{PS} \to \mathbf{cdSpaces}_{PS}$ *induces a Quillen equivalence* $\mathbf{dSpaces}_{PS,\mathrm{wcl}} \to \mathbf{cdSpaces}_{PS}$.

Proof Since the Reedy model structures in (a) are Quillen equivalent to the projective ones, both statements can be proved simultaneously by considering the diagram

$$\begin{tikzcd} \mathbf{cdSpaces}_{PS} \arrow[r, "\mathrm{incl}_!"] \arrow[dr, "f_!"'] & \mathbf{dSpaces}_{PS} \arrow[r, "\mathrm{cl}_!"] \arrow[d, "\mathrm{id}_!"] & \mathbf{cdSpaces}_{PS}. \\ & \mathbf{dSpaces}_{PS,\mathrm{wcl}} \arrow[ur, dashed, "g_!"'] & \end{tikzcd}$$

Here $f_!$ is the composition $\mathrm{id}_!\mathrm{incl}_!$. Since $\mathrm{cl}_!$ sends the localizing map $\eta \to \overline{\eta}$ to an isomorphism, it factors through a left Quillen functor $g_!$ as in the diagram. Now observe that $g_! f_!$ is the identity functor. Moreover, for a dendroidal space X and a tree T,

$$g^* f^* X(T) = X(\overline{T})$$

and the map $X(\overline{T}) \to X(T)$ (which is the counit of the adjoint pair $(\mathrm{incl}_!, \mathrm{incl}^*)$ in disguise) provides a natural map $g^* f^*(X) \to X$ which is a weak equivalence if X is a local object in $\mathbf{dSpaces}_{PS,\mathrm{wcl}}$. This proves the theorem. □

Corollary 12.52 *Every weakly closed dendroidal Segal space is weakly equivalent to a dendroidal Segal space X for which $X(\overline{T}) \to X(T)$ is an isomorphism for every tree T.*

Proof This follows from the theorem together with the fact that the X as in the corollary are exactly the objects in the essential image of the functor $\mathrm{incl}_! = \mathrm{cl}^*$. □

As observed in Remark 4.14, the tensor product of dendroidal sets restricts to one on the category of closed dendroidal sets and the functor $\mathrm{cl}_!$ respects the tensor product. By defining the tensor product of simplicial presheaves on $\overline{\mathbf{\Omega}}$ degreewise, we obtain a tensor product on **cdSpaces** for which $\mathrm{incl}_!$ and $\mathrm{cl}_!$ are again monoidal, in the sense that for dendroidal spaces (or closed dendroidal spaces, respectively) X and Y, there are natural isomorphisms

$$\mathrm{cl}_!(X \otimes Y) \cong \mathrm{cl}_!(X) \otimes \mathrm{cl}_!(Y)$$
$$\mathrm{incl}_!(X \otimes Y) \cong \mathrm{incl}_!(X) \otimes \mathrm{incl}_!(Y).$$

However, observe that $\mathrm{incl}_!$ does *not* preserve the unit, which is $\overline{\eta}$ in **cdSpaces**, but η in **dSpaces**. In Proposition 4.26 we observed that the tensor product behaves better on closed dendroidal sets than on general ones, because for normal monomorphisms $A \to B$ and $C \to D$ of closed dendroidal sets, the pushout-product map

$$A \otimes D \cup B \otimes C \to B \otimes D$$

is again a normal monomorphism. It follows that $\mathbf{cdSpaces}_{RS}$ behaves very much like a symmetric monoidal model category, the only failure being that the associativity maps for the tensor product are only trivial cofibrations (for cofibrant objects) rather than isomorphisms. Let us record this explicitly as follows.

Theorem 12.53 *The tensor product of closed dendroidal spaces has the following properties:*

(a) *If $A \to B$ and $C \to D$ are normal monomorphisms, then so is*

$$A \otimes D \cup B \otimes C \to B \otimes D.$$

If one of the two is in addition a weak equivalence, then so is the pushout-product map.

(b) *For normal closed dendroidal spaces $A_1, \ldots, A_n$, the associator maps (cf. Section 4.4)*

$$A_1 \otimes \cdots \otimes (A_i \otimes \cdots \otimes A_j) \otimes \cdots \otimes A_n \to A_1 \otimes \cdots \otimes A_n$$

are trivial cofibrations in $\mathbf{cdSpaces}_{RS}$.

12.6 Reduced Dendroidal Spaces

In the previous section we considered the homotopy theory of ∞-operads for which the spaces of nullary operations $X(-;x)$ are contractible, for each colour x of X, and showed that it can be presented in terms of simplicial presheaves on the category $\overline{\mathbf{\Omega}}$ of closed trees. In this section we will specialize further, considering operads for which in addition the unary operations are invertible. We will relate these to simplicial presheaves on the full subcategory $\overline{\mathbf{\Omega}}_{\mathrm{red}}$ of $\overline{\mathbf{\Omega}}$ on the closed trees without unary vertices. We will refer to such trees as *reduced closed trees*.

Definition 12.54 The category of *reduced dendroidal spaces* is the category of simplicial presheaves on the category of reduced closed trees:

$$\mathbf{rcdSpaces} := \mathbf{sSets}^{(\overline{\mathbf{\Omega}}_{\mathrm{red}})^{\mathrm{op}}}.$$

We will use such reduced dendroidal spaces as a model for the following kinds of dendroidal spaces:

Definition 12.55 A dendroidal space X is *weakly reduced* if it is weakly closed and for each degeneracy $\sigma\colon T \to S$ in $\mathbf{\Omega}$, the map $\sigma^*\colon X(S) \to X(T)$ is a weak equivalence.

Remark 12.56 The same terminology is used in a more restrictive sense for (uncoloured) operads. A simplicial or topological operad $\mathbf{P}$ is *reduced* (resp. *weakly reduced*) if $\mathbf{P}(0)$ and $\mathbf{P}(1)$ equal a point (resp. are weakly contractible). For example, the operads $\mathbf{E}_n$ are weakly reduced.

Remark 12.57 Observe that the property of being weakly reduced is invariant under weak equivalence.

The category **rcdSpaces** is simpler than that of all dendroidal spaces in several respects. First of all, the combinatorics of the category $\overline{\mathbf{\Omega}}_{\mathrm{red}}$ are simpler than those of $\mathbf{\Omega}$. Secondly, we will prove in this section that for weakly reduced dendroidal spaces, the notion of completeness is redundant and therefore plays no role in the theory.

For dendroidal Segal spaces, the property of being weakly reduced can be formulated as follows.

Proposition 12.58 *Let X be a weakly closed dendroidal Segal space. Then the following properties are equivalent:*

(1) *X is weakly reduced.*
(2) *The map $X(\eta) \to X(C_1)$ associated to the degeneracy $C_1 \to \eta$ is a weak equivalence.*
(3) *For any surjection $T \to S$ in $\mathbf{\Omega}$ from a tree T to a tree S without unary vertices, the map $X(S) \to X(T)$ is a weak equivalence.*

Proof A map $T \to S$ as in (3) is a composition of degeneracies, so (3) follows from (1). Furthermore, (2) is a special case of (3) and of (1), so it remains to prove that (2) implies (1). Let $\sigma_v : T \to S$ be degeneracy, deleting a unary vertex v of T. Then we can write T as a grafting $T_1 \circ C_v \circ T_2$, where C_v is the unary corolla with vertex v. By the Segal condition, the horizontal morphisms in the diagram

$$\begin{tikzcd} X(S) \arrow[r, "\sim"] \arrow[d] & X(T_1) \times_{X(\eta)} X(T_2) \arrow[d] \\ X(T) \arrow[r, "\sim"] & X(T_1) \times_{X(\eta)} X(C_1) \times_{X(\eta)} X(T_2) \end{tikzcd}$$

are weak equivalences. The map on the right is the pullback of the weak equivalence $X(\eta) \to X(C_1)$ along the projection

$$X(T_1) \times_{X(\eta)} X(C_1) \times_{X(\eta)} \times X(T_2) \to X(C_1).$$

Since this map is a fibration by the Reedy fibrancy of X, this pullback is also a weak equivalence. □

It follows from this proposition that the weakly reduced dendroidal Segal spaces are the fibrant objects in the left Bousfield localization of $\mathbf{dSpaces}_{RS,\mathrm{wcl}}$ by the map $C_1 \to \eta$. We will denote this localized model category by

$$\mathbf{dSpaces}_{RS,\mathrm{wred}}.$$

Let us observe that this model category is in fact a localization of the model category $\mathbf{dSpaces}_{RSC}$ for *complete* dendroidal Segal spaces:

Proposition 12.59 *Any weakly reduced dendroidal Segal space is complete.*

Proof Let X be a weakly reduced dendroidal Segal space. Consider the class of all monomorphisms $M \to N$ of simplicial sets for which the corresponding map $X(i_!N) \to X(i_!M)$ is a trivial fibration. It contains the inner anodynes, since X is Segal, as well as the two inclusions $\Delta[0] \to \Delta[1]$ by the proposition above. Since the class under consideration is saturated, it must then contain all anodyne maps (i.e., all trivial cofibrations in the Kan–Quillen model structure). In particular, it contains $\Delta[0] \to J$. Hence $X(J) \to X(\eta)$ is a trivial fibration and X is complete. □

We will now follow the same strategy as in the previous section and prove a Quillen equivalence between the model category $\mathbf{dSpaces}_{RS,\mathrm{wred}}$ and a smaller model category of reduced dendroidal spaces. The category $\overline{\mathbf{\Omega}}_{\mathrm{red}}$ inherits a Reedy structure from $\mathbf{\Omega}$, so that we may consider the projective and Reedy model structures on the category **rcdSpaces**. These can be localized for the closed spine inclusions of trees as in the previous section, so that we obtain a square of model categories and left Quillen functors (each of which is the identity) as follows:

$$\begin{array}{ccc} \mathbf{rcdSpaces}_P & \xrightarrow{\sim} & \mathbf{rcdSpaces}_R \\ \downarrow & & \downarrow \\ \mathbf{rcdSpaces}_{PS} & \xrightarrow{\sim} & \mathbf{rcdSpaces}_{RS}. \end{array}$$

Consider the inclusion functor

$$\mathrm{incl}\colon \overline{\mathbf{\Omega}}_{\mathrm{red}} \to \overline{\mathbf{\Omega}}$$

of reduced trees into closed trees. This functor has a left adjoint

$$\mathrm{red}\colon \overline{\mathbf{\Omega}} \to \overline{\mathbf{\Omega}}_{\mathrm{red}}$$

which sends a closed tree T to its 'reduction' $\mathrm{red}(T)$; the unit of this adjunction $T \to \mathrm{incl}(\mathrm{red}(T))$ is the maximal degeneracy which collapses all the unary vertices in T. Exactly as in the previous section, these functors induce Quillen pairs for the projective model structures on simplicial presheaves,

$$\mathbf{rcdSpaces}_P \underset{\mathrm{incl}^*}{\overset{\mathrm{incl}_!}{\rightleftarrows}} \mathbf{cdSpaces}_P,$$

$$\mathbf{cdSpaces}_P \underset{\mathrm{red}^*}{\overset{\mathrm{red}_!}{\rightleftarrows}} \mathbf{rcdSpaces}_P,$$

and $\mathrm{red}_! \circ \mathrm{incl}_!$ is the identity since $\mathrm{red} \circ \mathrm{incl}$ is. Moreover, $\mathrm{incl}_!$ is easily seen to preserve Reedy cofibrations (in fact, it preserves closed boundaries of closed trees), hence also induces a Quillen pair

$$\mathbf{rcdSpaces}_R \underset{\mathrm{incl}^*}{\overset{\mathrm{incl}_!}{\rightleftarrows}} \mathbf{cdSpaces}_R.$$

Theorem 12.60 *These Quillen pairs factor through the localizations with respect to the Segal condition and induce Quillen equivalences*

$$\mathbf{rcdSpaces}_{RS} \underset{\mathrm{incl}^*}{\overset{\mathrm{incl}_!}{\rightleftarrows}} \mathbf{cdSpaces}_{RS,\mathrm{wred}}$$

and

$$\mathbf{cdSpaces}_{PS,\mathrm{wred}} \underset{\mathrm{red}^*}{\overset{\mathrm{red}_!}{\rightleftarrows}} \mathbf{rcdSpaces}_{PS}.$$

Proof We begin by verifying that these functors are indeed compatible with localization with respect to the Segal condition. Clearly $\mathrm{incl}_!$ preserves closed spines and therefore gives a left Quillen functor

$$\mathrm{incl}_!\colon \mathbf{rcdSpaces}_{RS} \to \mathbf{cdSpaces}_{RS}.$$

To see that $\mathrm{red}_!$ similarly gives a left Quillen functor from $\mathbf{cdSpaces}_{PS}$ to $\mathbf{rcdSpaces}_{PS}$, we have to consider projectively cofibrant replacements as in the proof of Proposition 12.50. Using induction, it suffices to show that for a closed tree T and a grafting $T' = T \circ_{\overline{\eta}} \overline{C_n}$ obtained by replacing a nullary vertex v of T by a closed corolla $\overline{C_n}$, the functor $\mathbf{L}\mathrm{red}_!$ sends $T \cup_{\overline{\eta}} \overline{C_n} \to T'$ to a weak equivalence in $\mathbf{rcdSpaces}_{PS}$. To this end, consider the projectively cofibrant replacement of $A \to T \cup_{\overline{\eta}} \overline{C_n}$ defined as the pushout

$$\begin{tikzcd} \overline{\eta} \boxtimes \partial\Delta[1] \arrow[r] \arrow[d] & T \amalg \overline{C_n} \arrow[d] \\ \overline{\eta} \boxtimes \Delta[1] \arrow[r] & B. \end{tikzcd}$$

If $n = 1$ then $\mathrm{red}_!$ maps this to the pushout

$$\begin{tikzcd} \overline{\eta} \amalg \overline{\eta} \arrow[r] \arrow[d] & \mathrm{red}_! T \amalg \overline{\eta} \arrow[d] \\ \overline{\eta} \boxtimes \Delta[1] \arrow[r] & B'. \end{tikzcd}$$

Note that B' then also fits in a pushout square

$$\begin{tikzcd} \overline{\eta} \arrow[r] \arrow[d, "\wr"] & \mathrm{red}_! T \arrow[d, "\wr"] \\ \overline{\eta} \boxtimes \Delta[1] \arrow[r] & B', \end{tikzcd}$$

while $\mathrm{red}_!(T') = \mathrm{red}_!(T)$, so that the resulting map $B' \to \mathrm{red}_!(T')$ is indeed a weak equivalence.

For $n > 1$, the reduced dendroidal space $\mathrm{red}_! B$ is given by the pushout

$$\begin{tikzcd} \overline{\eta} \boxtimes \partial\Delta[1] \arrow[r] \arrow[d] & \mathrm{red}_! T \amalg \overline{C_n} \arrow[d] \\ \overline{\eta} \boxtimes \Delta[1] \arrow[r] & B''. \end{tikzcd}$$

Now observe that B'' exactly defines a projectively cofibrant replacement of $\mathrm{red}_!(T')$ in the same style; in particular, the map $B'' \to \mathrm{red}_!(T')$ defines a weak equivalence.

Now consider the diagram

$$\begin{tikzcd} \mathbf{rcdSpaces}_{PS} \arrow[r, "\mathrm{incl}_!"] \arrow[dr, "f_!"'] & \mathbf{cdSpaces}_{PS} \arrow[r, "\mathrm{red}_!"] \arrow[d, "\mathrm{id}_!"] & \mathbf{rcdSpaces}_{PS}. \\ & \mathbf{cdSpaces}_{PS,\mathrm{wred}} \arrow[ur, dashed, "g_!"'] & \end{tikzcd}$$

Then $\mathrm{red}_!$ factors through a left Quillen functor $g_!$ as indicated, because $\mathrm{red}_!$ sends the localizing map $\overline{C_1} \to \overline{\eta}$ to an isomorphism. Moreover, $\mathrm{red}_!\mathrm{incl}_!$ is the identity, while for X in **cdSpaces** and T a closed tree,

$$\mathrm{incl}_!\mathrm{incl}^*(X)(T) = \mathrm{red}^*\mathrm{incl}^*(X)(T) = X(\mathrm{red}(T)).$$

For local X, the counit $\mathrm{incl}_!\mathrm{incl}^*(X) \to X$ is a weak equivalence by definition. Hence $\mathbf{L}f_!$ and $\mathbf{L}g_!$ are mutually quasi-inverse functors, which completes the proof. □

Finally, we may combine Theorems 12.60 and 12.51 to obtain the following.

Corollary 12.61 *The pairs*

$$\mathbf{rcdSpaces}_{RS} \underset{\mathrm{incl}^*}{\overset{\mathrm{incl}_!}{\rightleftarrows}} \mathbf{dSpaces}_{RS,\mathrm{wred}}$$

and

$$\mathbf{dSpaces}_{PS,\mathrm{wred}} \underset{\mathrm{cl}^*\mathrm{red}^*}{\overset{\mathrm{red}_!\mathrm{cl}_!}{\rightleftarrows}} \mathbf{rcdSpaces}_{PS}.$$

are Quillen equivalences.

To conclude this section, we observe that all of the statements above have evident variants for open dendroidal spaces. Write $\mathbf{\Omega}^\circ_{\mathrm{red}}$ for the category of *open reduced trees*, defined as the full subcategory of the category $\mathbf{\Omega}^\circ$ of open trees on the trees without unary vertices. Then define the category of *reduced open dendroidal spaces* as the corresponding category of simplicial presheaves:

$$\mathbf{rodSpaces} := \mathbf{sSets}^{(\mathbf{\Omega}^\circ_{\mathrm{red}})^{\mathrm{op}}}.$$

We say an open dendroidal space $X \in \mathbf{odSpaces}$ is weakly reduced if for every degeneracy $\sigma\colon T \to S$ in $\mathbf{\Omega}$, the map $\sigma^*\colon X(S) \to X(T)$ is weak equivalence. Then we have the following analogue of Corollary 12.61, proved in precisely the same way:

Theorem 12.62 *The pairs*

$$\mathbf{rodSpaces}_{RS} \underset{\mathrm{incl}^*}{\overset{\mathrm{incl}_!}{\rightleftarrows}} \mathbf{odSpaces}_{RS,\mathrm{wred}}$$

and

$$\mathbf{odSpaces}_{PS,\mathrm{wred}} \underset{\mathrm{red}^*}{\overset{\mathrm{red}_!}{\rightleftarrows}} \mathbf{rodSpaces}_{PS}.$$

are Quillen equivalences.

12.7 Simplicial Spaces

In this section we summarize how the results of this chapter specialize once we restrict to simplicial (rather than dendroidal) spaces. Essentially we recover Rezk's theory of complete Segal spaces, which he proposed as a model for higher category theory. Theorem 12.66 below states that the category of simplicial spaces, equipped with a model structure in which the fibrant objects are the complete Segal spaces, is Quillen equivalent to the category of simplicial sets equipped with the Joyal model structure. In other words, the homotopy theory of complete Segal spaces is equivalent to that of ∞-categories.

Recall that the slice category $\mathbf{dSets}/\eta$ can be identified with the category $\mathbf{sSets}$ of simplicial sets. Similarly, regarding η as a discrete dendroidal space, we find an equivalence of categories

$$\mathbf{dSpaces}/\eta \simeq \mathbf{sSpaces},$$

where the right-hand side is of course just alternative notation for the category of bisimplicial sets. However, it is convenient (and important) to keep track of the two different simplicial directions; one of them is the 'categorical' direction, the other the 'space' direction. In particular, the model structures we consider in this section do not treat the two simplicial coordinates symmetrically. Although the results of this section follow trivially from the more general statements for dendroidal spaces proved before, we do collect them here for the convenience of the reader.

We briefly recall the relevant definitions. Throughout this section, we think of objects of the category $\mathbf{sSpaces}$ as functors

$$X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{sSets}$$

and use notation accordingly.

Definition 12.63 A simplicial space X *satisfies the Segal condition* if for every $n \geq 2$, the map

$$X(\Delta[n]) \longrightarrow \mathrm{Map}(\mathrm{Sp}[n], X)$$

is a weak homotopy equivalence of simplicial sets. A *Segal space* is a Reedy fibrant simplicial space X satisfying the Segal condition.

Alternatively, the definition can also be expressed as saying that $X(\Delta[n])$ is equivalent to the iterated homotopy pullback of the terms $X(\Delta[0,1]), \ldots, X(\Delta[n-1,n])$, and it is this form of the Segal condition one often finds in the literature. According to Lemma 12.7, the Segal condition admits the following reformulation. For $n \geq 2$ and $0 < k < n$, write $\Delta[0,k]$ and $\Delta[k,n]$ for the subsimplices of $\Delta[n]$ spanned by the vertices $0, \ldots, k$ and $k, \ldots, n$ respectively.

Lemma 12.64 *A simplicial space X satisfies the Segal condition if for any $n \geq 2$ and $0 < k < n$, the square*

$$\begin{tikzcd} X(\Delta[n]) \arrow[r] \arrow[d] & X(\Delta[0,k]) \arrow[d, "k^*"] \\ X(\Delta[k,n]) \arrow[r, "k^*"'] & X(\Delta[0]) \end{tikzcd}$$

is a homotopy pullback.

Definition 12.65 A simplicial space X is *complete* if the map of simplicial sets $J \to \Delta[0]$ induces a weak homotopy equivalence of simplicial sets

$$\mathrm{Map}(\Delta[0], X) \to \mathrm{Map}(J, X).$$

We write $\mathbf{sSpaces}_{RSC}$ for the category of simplicial spaces equipped with the left Bousfield localization of the Reedy model structure for which the fibrant objects are the complete Segal spaces. Below we record the simplicial versions of the two main theorems of this chapter. Of course most statements from the preceding sections have such a specialization; we leave it to the reader to formulate these explicitly.

Consider the adjoint pair

$$\mathbf{sSets} \underset{\mathrm{dis}^*}{\overset{\mathrm{dis}_!}{\rightleftarrows}} \mathbf{sSpaces}.$$

for which dis^* sends a simplicial space X to its degree zero part X_0, whereas $\mathrm{dis}_!$ assigns to a simplicial set the corresponding discrete simplicial space. Then Theorem 12.22 yields the following statement:

Theorem 12.66 *The adjoint pair* $(\mathrm{dis}_!, \mathrm{dis}^*)$ *is a Quillen equivalence between the Joyal model structure on* **sSets** *and the model category* $\mathbf{sSpaces}_{RSC}$ *for complete Segal spaces.*

Recall that a *complete weak equivalence* is a map $X \to Y$ of simplicial spaces which is a weak equivalence in the model category $\mathbf{sSpaces}_{RSC}$. We record the following special case of Theorem 12.36:

Theorem 12.67 *Let $f: X \to Y$ be a map of simplicial Segal spaces. Then f is a complete weak equivalence if and only if f is fully faithful and essentially surjective.*

To conclude this section we describe some important examples, emphasizing the fact that a complete Segal space should be thought of as a model for a homotopy theory.

Example 12.68 If $\mathbf{C}$ is a (small) category, its nerve $N\mathbf{C}$ is an ∞-category. As in Example 12.24, there is an associated simplicial space $\widehat{N\mathbf{C}}$ which can be explicitly described as follows:

$$\widehat{N\mathbf{C}}([n])_\bullet = \mathbf{sSets}(J[\bullet] \times \Delta[n], N\mathbf{C}).$$

The right-hand side can alternatively be written as $\mathbf{sSets}(J[\bullet], N(\mathbf{C}^{[n]}))$. For a category $\mathbf{D}$ let us write $\mathrm{iso}(\mathbf{D})$ for the maximal groupoid contained in $\mathbf{D}$. In other words, $\mathrm{iso}(\mathbf{D})$ is obtained from $\mathbf{D}$ by discarding all non-invertible morphisms. With this notation we may write

$$\widehat{N\mathbf{C}}([n]) = N\mathrm{iso}(\mathbf{C}^{[n]}).$$

The right-hand side is the nerve of a groupoid and hence a Kan complex. Of course much more is true: Example 12.24 guarantees that $\widehat{N\mathbf{C}}([n])$ is in fact a complete Segal space. This space is often called the *classifying diagram* of $\mathbf{C}$.

Example 12.69 The previous example may be generalized in the following way. Consider a small category $\mathbf{C}$ and a subcategory W, which will play the role of the 'weak equivalences'. Call a natural transformation α between functors $F, G\colon [n] \to \mathbf{C}$ a weak equivalence if each of the components $\alpha_c : F(c) \to G(c)$ of α is contained in W. Write $W(\mathbf{C}^{[n]})$ for the subcategory of $\mathbf{C}^{[n]}$ consisting of those morphisms (i.e., natural transformations) that are weak equivalences. Then we may form a simplicial space $N(\mathbf{C}, W)$ by setting

$$N(\mathbf{C}, W)([n]) = NW(\mathbf{C}^{[n]}).$$

In particular, if we take $W = \mathrm{iso}(\mathbf{C})$, then we retrieve the previous example:

$$N(\mathbf{C}, \mathrm{iso}(\mathbf{C})) = \widehat{N\mathbf{C}}.$$

In general, there is no reason to expect the simplicial space $N(\mathbf{C}, W)$ to be a complete Segal space. However, if $\mathcal{E}$ is a model category and W is its class of weak equivalences, then any Reedy fibrant replacement of $N(\mathcal{E}, W)$ is a complete Segal space. Moreover, for objects x and y of $\mathcal{E}$, the mapping spaces $\mathrm{Map}_{\mathcal{E}}(x, y)$ and $N(\mathcal{E}, W)(x, y)$ (as defined above Definition 12.12) are weakly equivalent, see [129].

Historical Notes

Most of the material in this chapter is a reworking of the papers [40, 41]. As explained, the results all specialize to analogous results for simplicial spaces proved earlier. In particular, the theory of complete Segal simplicial spaces was introduced by Rezk [129], who also proved the simplicial version of the characterization of complete weak equivalences between Segal spaces (Theorem 12.36). The equivalence between complete Segal spaces and the Joyal model structure on simplicial sets was demonstrated by Joyal–Tierney [94].

Chapter 13
Left Fibrations and the Covariant Model Structure

In Section 9.5 we introduced the covariant model structure on a slice category $\mathbf{dSets}/B$, for a fixed dendroidal set B. This model category represents the homotopy theory of 'B-algebras', as we will see in Section 13.5. The first aim of this chapter is to establish the analogous structure for the category of dendroidal *spaces* over a fixed dendroidal space B. We will do this in Section 13.1 and demonstrate that the covariant model structure on $\mathbf{dSpaces}/B$ is Quillen equivalent to the covariant model structure on $\mathbf{dSets}/B_0$ in the case when B is a complete dendroidal Segal space (cf. Proposition 13.7). In Section 13.3 we establish a form of 'homotopy invariance' for the covariant model structure. To be precise, we will show that a complete weak equivalence between dendroidal Segal spaces A and B induces a Quillen equivalence between the covariant model structures on $\mathbf{dSpaces}/A$ and $\mathbf{dSpaces}/B$. In order to prove this result in the stated generality we will need a digression on simplicial systems of model categories (Section 13.2), which helps to relate the covariant model structure on $\mathbf{dSpaces}/B$ to that on the categories $\mathbf{dSets}/B_n$, for $n \geq 0$.

The main aim of this chapter is to relate left fibrations to algebras for operads. If $\mathbf{P}$ is a simplicial operad, then the homotopy theory of $\mathbf{P}$-algebras can be described through a 'projective model structure', which we introduce in Section 13.4. Then in Section 13.5 we establish a Quillen equivalence between this model structure on the category of $\mathbf{P}$-algebras and the covariant model structure on the slice category $\mathbf{dSpaces}/N\mathbf{P}$. This result has a number of very useful consequences, including the 'straightening-unstraightening equivalence' for left fibrations over ∞-categories. We discuss these corollaries in Section 14.8 at the end of the next chapter, after we have established an equivalence of homotopy theories between ∞-operads and simplicial operads.

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13.1 The Covariant Model Structure on Dendroidal Spaces

In this section we will introduce the analogue of the covariant model structure on dendroidal sets (cf. Section 9.5) in the context of dendroidal spaces. Consider a fixed dendroidal space B. The slice category $\mathbf{dSpaces}/B$ inherits model structures from the Reedy model structure and the projective model structure on **dSpaces**, denoted $\mathbf{dSpaces}_R/B$ and $\mathbf{dSpaces}_P/B$, respectively. The covariant model structures will be left Bousfield localizations of these; to be precise, there will be one for the Reedy model structure and a Quillen equivalent one for the projective model structure.

Recall that a left fibration of dendroidal sets is a map that has the right lifting property with respect to inner horns and leaf horns. Alternatively, we characterized left fibrations over a dendroidal set B as fibrant objects in a localization of the operadic model structure on the slice category $\mathbf{dSets}/B$ with respect to the leaf inclusions of trees. Our first aim is to introduce the analogous notion for dendroidal spaces. For given objects $p\colon X \to B$ and $q\colon Y \to B$ of $\mathbf{dSpaces}/B$, let us write

$$\mathrm{Map}_B(X, Y)$$

for the mapping space between them, leaving the maps p and q implicitly understood. Recall that for a tree T, we write $\ell[T]$ for the coproduct of copies of η indexed by the leaves of T.

Definition 13.1 A map $p\colon X \to B$ of dendroidal spaces is a *left quasifibration* if for any tree T and any map $T \to B$, the map

$$\mathrm{Map}_B(T, X) \to \mathrm{Map}_B(\ell[T], X)$$

is a weak homotopy equivalence. In other words, p is a left quasifibration if it is local with respect to the leaf inclusions $\ell[T] \to T$ over B. We say $X \to B$ is a *Reedy left fibration* (resp. a *projective left fibration*) if it is a left quasifibration that is moreover a Reedy fibration (resp. projective fibration) of dendroidal spaces.

Remark 13.2 Since the Reedy and projective model structures on $\mathbf{dSpaces}/B$ are Quillen equivalent, the homotopy type of the mapping space $\mathrm{Map}_B(X, Y)$ with respect to either of these is the same. Hence the definition of left quasifibration is independent of which of the these two model structures on $\mathbf{dSpaces}/B$ is used.

The mapping space $\mathrm{Map}_B(X, Y)$, with respect to either the Reedy or the projective model structure, is straightforward to describe. If $Y \to B$ is a Reedy (resp. projective) fibration and X is Reedy (resp. projectively) cofibrant, we can use the cosimplicial resolution $X \boxtimes \Delta[\bullet]$ of X to compute (a model of) this mapping space as

$$\mathrm{Map}_B(X, Y)_n = (\mathbf{dSpaces}/B)(X \boxtimes \Delta[n], Y).$$

In other words, $\mathrm{Map}_B(X, Y)_n$ is the pullback

$$\begin{tikzcd}
\mathrm{Map}_B(X,Y)_n \arrow[r] \arrow[d] & \mathbf{dSpaces}(X \boxtimes \Delta[n], Y) \arrow[d] \\
\Delta[0] \arrow[r] & \mathbf{dSpaces}(X \boxtimes \Delta[n], B),
\end{tikzcd}$$

where the vertex defining the lower arrow is the composite of the projection morphism $X \boxtimes \Delta[n] \to X$ with the given map $X \to B$. Thus, if B itself is Reedy (resp. projectively) fibrant, we can rewrite this pullback as

$$\begin{tikzcd}
\mathrm{Map}_B(X,Y) \arrow[r] \arrow[d] & \mathrm{Map}(X,Y) \arrow[d] \\
\Delta[0] \arrow[r] & \mathrm{Map}(X,B),
\end{tikzcd}$$

where the two mapping spaces on the right are taken in the model category $\mathbf{dSpaces}_R$ (resp. $\mathbf{dSpaces}_P$). Notice that for this model of the mapping space, the morphism on the right is a Kan fibration, since $Y \to B$ is assumed to be a fibration and X is assumed cofibrant. Note that the objects T and $\ell[T]$ featuring in Definition 13.1 are both Reedy and projectively cofibrant, so that the preceding discussion applies to describe the mapping spaces featuring in that definition.

The following characterization of left quasifibrations is analogous to that of dendroidal Segal spaces in Lemma 12.7. It is a particularly convenient criterion, since it does not make reference to any particular model structure.

Lemma 13.3 *A map $X \to B$ of dendroidal spaces is a left quasifibration if and only if for every tree T, the square*

$$\begin{tikzcd}
X(T) \arrow[r] \arrow[d] & B(T) \arrow[d] \\
X(\ell[T]) \arrow[r] & B(\ell[T])
\end{tikzcd}$$

is a homotopy pullback.

Proof Without loss of generality we may assume that $X \to B$ is a Reedy fibration, since it can always be replaced by such without changing the homotopy type of $X(T)$. It follows from the discussion preceding the lemma that the map $\mathrm{Map}_B(T, X) \to \mathrm{Map}_B(\ell[T], X)$ may be identified with the map between the fibres of the horizontal maps in the square of the lemma over a specified vertex of $B(T)$, i.e., a fixed map $T \to B$. Since the horizontal maps are Kan fibrations under our assumptions, those fibres are also homotopy fibres. The lemma now follows from the observation that a square as above is a homotopy pullback if and only if the induced map of homotopy fibres (over any vertex of $B(T)$) of the horizontal maps is a weak homotopy equivalence. □

Example 13.4 Let $\mathbf{P}$ be a simplicial operad, with associated dendroidal space $N\mathbf{P}$. The following example will play a key role in our comparison between left fibrations over $N\mathbf{P}$ and $\mathbf{P}$-algebras in Section 13.5. Let A be a simplicial $\mathbf{P}$-algebra. Then we can define a dendroidal space $N(\mathbf{P}, A)$ over $N\mathbf{P}$ by applying the construction of Example 3.20(h) in each simplicial degree. More explicitly, if T is a tree, then an element of $N\mathbf{P}(T)_n$ consists of labellings of the edges of T by colours of $\mathbf{P}$ and the vertices of T by operations of $\mathbf{P}$ in simplicial degree n, compatible with the labelling of the edges by colours. An n-simplex of $N(\mathbf{P}, A)(T)$ consists of the same data, together with the assignment of an n-simplex of A to each leaf of T; more precisely, if a leaf l is labelled by a colour c of $\mathbf{P}$, then it should be assigned an n-simplex in $A(c)$. The map $N(\mathbf{P}, A) \to N\mathbf{P}$ is simply the projection forgetting this final assignment. It is clear from this description that

$$N(\mathbf{P}, A)(T) \to N(\mathbf{P}, A)(\ell[T]) \times_{N\mathbf{P}(\ell[T])} N\mathbf{P}(T)$$

is an isomorphism. In other words, $N(\mathbf{P}, A) \to N\mathbf{P}$ is a 'strict' left fibration. If the simplicial sets $\mathbf{P}(c_1, \ldots, c_n; c)$ of operations in $\mathbf{P}$ are Kan complexes, then the map $N\mathbf{P}(T) \to N\mathbf{P}(\ell[T])$ is a Kan fibration; indeed, the simplicial set $N\mathbf{P}(\ell[T])$ is discrete. It then follows from Lemma 7.51(1) and Proposition 8.67 that the pullback square

$$\begin{tikzcd}
N(\mathbf{P}, A)(T) \arrow[r] \arrow[d] & N\mathbf{P}(T) \arrow[d] \\
N(\mathbf{P}, A)(\ell[T]) \arrow[r] & N\mathbf{P}(\ell[T])
\end{tikzcd}$$

is also a homotopy pullback, so that Lemma 13.3 guarantees that $N(\mathbf{P}, A) \to N\mathbf{P}$ is a left quasifibration. If one assumes that all the simplicial sets A_c are Kan complexes, it is even a projective left fibration. We will see later that general (projective or Reedy) left fibrations $X \to N\mathbf{P}$ can be interpreted as $\mathbf{P}$-algebras up to coherent homotopy.

For later reference, we now review several equivalent formulations of the condition of being a Reedy left fibration.

Lemma 13.5 *For a Reedy fibration $X \to B$ between dendroidal spaces, the following are equivalent:*

(1) *$X \to B$ is a Reedy left fibration.*
(2) *For any tree T, the map*

$$X(T) \to X(\ell[T]) \times_{B(\ell[T])} B(T)$$

is a trivial fibration.
(3) *For any tree T and any inner or leaf horn $\Lambda^x[T] \to T$, the map*

$$X(T) \to X(\Lambda^x[T]) \times_{B(\Lambda^x[T])} B(T)$$

is a trivial fibration of simplicial sets.

Proof Since $X \to B$ is a Reedy fibration, the maps of items (2) and (3) are always Kan fibrations. The equivalence between (1) and (2) follows from Lemma 13.3 and the condition of Lemma 7.51 for homotopy pullback squares. The equivalence between (2) and (3) follows from Proposition 6.41, which states that the inclusions $\ell[T] \to T$ 'generate' the class of leaf anodynes in the appropriate sense. □

According to Theorem 11.37, the Bousfield localization of the Reedy model structure on $\mathbf{dSpaces}/B$ with respect to the leaf inclusions $\ell[T] \to T$ exists, with T ranging over all trees in $\mathbf{\Omega}$. We will call this the *Reedy covariant model structure* and denote the resulting model category by $(\mathbf{dSpaces}_R/B)_{\mathrm{cov}}$. Similarly, Proposition 11.25 guarantees the existence of the corresponding localization of the projective model structure, which we denote by $(\mathbf{dSpaces}_P/B)_{\mathrm{cov}}$ and refer to as the *projective covariant model structure*.

Theorem 13.6 *The fibrant objects of the Reedy covariant model structure on* $\mathbf{dSpaces}/B$ *are the Reedy left fibrations* $X \to B$. *If* B *is a dendroidal Segal space, then the covariant model structure is a left Bousfield localization of the model category* $\mathbf{dSpaces}_{RS}/B$. *If* B *is moreover complete, then it is a left Bousfield localization of* $\mathbf{dSpaces}_{RSC}/B$.

Proof The fibrant objects of $(\mathbf{dSpaces}_R/B)_{\mathrm{cov}}$ are the maps $X \to B$ which are fibrant as objects of $\mathbf{dSpaces}_R/B$ and local with respect to the maps $\ell[T] \to T$. By definition, these are the Reedy left fibrations. To argue that $(\mathbf{dSpaces}_R/B)_{\mathrm{cov}}$ is also a left Bousfield localization of $\mathbf{dSpaces}_{RS}/B$ in the case where B is a dendroidal Segal space, we should check that every Reedy left fibration $X \to B$ is fibrant as an object of $\mathbf{dSpaces}_{RS}/B$. The fibrant objects of the latter model category are the fibrations $X \to B$ in the model category $\mathbf{dSpaces}_{RS}$, which by Lemma 8.50 are exactly the Reedy fibrations $X \to B$ with X a local object, i.e., a dendroidal Segal space. But the fact that X is a dendroidal Segal space follows immediately from the fact that both $X \to B$ and $B \to *$ are inner fibrations, i.e., have the right lifting property with respect to the maps

$$T \boxtimes \partial\Delta[n] \cup \Lambda^e[T] \boxtimes \Delta[n] \to T \boxtimes \Delta[n]$$

for every tree T and inner edge e of T (cf. Lemma 13.5(3)). Similarly, for the final statement of the theorem it suffices to show that for a Reedy left fibration $X \to B$ with B a complete dendroidal Segal space, X is also complete. But $X \to B$ has the right lifting property with respect to all maps of the form

$$i_!N \boxtimes \partial\Delta[n] \cup i_!M \boxtimes \Delta[n] \to i_!N \boxtimes \Delta[n],$$

where $M \to N$ is a left anodyne map of simplicial sets. In particular, we may consider the map $\{0\} \to J$ and conclude that if B is complete, then so is X. □

The comparison with the covariant model structure on dendroidal sets (as opposed to spaces) is now quite straightforward:

Proposition 13.7 *Let B be a dendroidal complete Segal space, with associated ∞-operad B_0. Then there is a Quillen equivalence*

$$(\mathbf{dSets}/B_0)_{\mathrm{cov}} \underset{\mathrm{dis}^*}{\overset{\mathrm{dis}_!}{\rightleftarrows}} (\mathbf{dSpaces}_R/B)_{\mathrm{cov}}.$$

Proof For a dendroidal complete Segal space B, the previous theorem tells us that we can interpret $(\mathbf{dSpaces}_R/B)_{\mathrm{cov}}$ as a left Bousfield localization of $\mathbf{dSpaces}_{RSC}/B$. Now the Quillen equivalence

$$\mathbf{dSets} \underset{\mathrm{dis}^*}{\overset{\mathrm{dis}_!}{\rightleftarrows}} \mathbf{dSpaces}_{RSC}$$

of Theorem 12.22 gives another Quillen equivalence by slicing over B (cf. Example 8.47(j)), which we denote by

$$\mathbf{dSets}/B_0 \underset{\mathrm{dis}^*}{\overset{\mathrm{dis}_!}{\rightleftarrows}} \mathbf{dSpaces}_{RSC}/B.$$

Clearly, $\mathrm{dis}_!$ sends the localizing $\ell[T] \to T$ over B_0 to localizing maps over B, so it induces a further Quillen pair

$$(\mathbf{dSets}/B_0)_{\mathrm{cov}} \underset{\mathrm{dis}^*}{\overset{\mathrm{dis}_!}{\rightleftarrows}} (\mathbf{dSpaces}_{RSC}/B)_{\mathrm{cov}}.$$

In other words, the right Quillen functor dis^* preserves local objects. But for a complete Segal space X, the first Quillen equivalence above yields that

$$\mathrm{Map}_{\mathbf{dSpaces}_{RSC}}(T, X) \simeq \mathrm{Map}_{\mathbf{dSets}}(T, \mathrm{dis}^* X)$$

and similarly with B for X and/or $\ell[T]$ for T. Hence $X \to B$ is local with respect to the maps $\ell[T] \to T$ if and only if $X_0 \to B_0$ is; in other words, dis^* also *detects* local objects. This implies that the last Quillen pair above is in fact a Quillen equivalence. □

We include the following easy characterization of the weak equivalences between fibrant objects in the Reedy covariant model structure on $\mathbf{dSpaces}/B$, similar to the one in Theorem 9.63 for dendroidal sets. A similar characterization holds for the projective model structure and a projectively fibrant B, as will be discussed shortly. We will refer to weak equivalences in the Reedy covariant model structure as *covariant equivalences over B*.

Proposition 13.8 *For a map*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & B & \end{tikzcd}$$

between Reedy left fibrations in **dSpaces**$/B$*, the following statements are equivalent:*

(1) *The map is a covariant equivalence over* B.
(2) *For each vertex* $b \in B(\eta)_0$*, the map* $X_b \to Y_b$ *between the fibres is a weak homotopy equivalence of Kan complexes.*
(3) *The map* $X(\eta) \to Y(\eta)$ *is a weak homotopy equivalence of Kan complexes.*

Proof By Lemma 8.49, the weak equivalences between local objects in a Bousfield localization coincide with the weak equivalences in the original model category. In this case, this means that the covariant equivalences between Reedy left fibrations coincide with the usual equivalences in the Reedy model structure, which are the maps such that $X(T) \to Y(T)$ is a weak homotopy equivalence for each tree T. It follows that (1) implies (3). The maps in (2) arise from considering the fibres of the Kan fibrations $X(\eta) \to B(\eta)$ and $Y(\eta) \to B(\eta)$, so clearly also (2) and (3) are equivalent. To see that (3) implies (1), observe that since $X \to B$ and $Y \to B$ are Reedy left fibrations, the map $X(T) \to Y(T)$ is a weak homotopy equivalence if and only if

$$X(\ell[T]) \times_{B(\ell[T])} B(T) \to Y(\ell[T]) \times_{B(\ell[T])} B(T)$$

is a weak homotopy equivalence. This map is the pullback along $B(T) \to B(\ell[T])$ of the map

$$\begin{tikzcd} X(\ell[T]) \arrow[rr] \arrow[dr] & & Y(\ell[T]) \arrow[dl] \\ & B(\ell[T]) & \end{tikzcd}$$

between Kan fibrations over B. Since $B(\ell[T])$ is a product (indexed over the leaves of T) of copies of $B(\eta)$, and similarly for X and Y, it suffices to show that

$$\begin{tikzcd} X(\eta) \arrow[rr] \arrow[dr] & & Y(\eta) \arrow[dl] \\ & B(\eta) & \end{tikzcd}$$

is a weak equivalence between Kan fibrations over $B(\eta)$. But that is precisely the content of (3). □

We conclude this section with some remarks on the projective covariant model structure, which will sometimes be convenient to use. For a fixed projectively fibrant dendroidal space B and a Reedy fibrant replacement $f\colon B \to B'$ of it, consider the composition of left Quillen functors

$$\mathbf{dSpaces}_P/B \xrightarrow{f_!} \mathbf{dSpaces}_P/B' \xrightarrow{\mathrm{id}_!} \mathbf{dSpaces}_R/B'.$$

The first is composition with f whereas the second is just the identity, interpreted as a left Quillen functor from the projective to the Reedy model structure. Since f is a weak equivalence between projectively fibrant objects, the functor $f_!$ is part of a

Quillen equivalence by Example 8.47(i). Also, $\mathrm{id}_!$ is evidently a Quillen equivalence, since the weak equivalences in the projective and Reedy model structure coincide. Now localize the model category $\mathbf{dSpaces}_P/B$ to the projective covariant model structure $(\mathbf{dSpaces}_P/B)_{\mathrm{cov}}$ and observe that pushing forward this localization to $\mathbf{dSpaces}_R/B'$ gives the Reedy covariant model structure $(\mathbf{dSpaces}_R/B)_{\mathrm{cov}}$. Hence we find:

Proposition 13.9 *The adjoint pair*

$$(\mathbf{dSpaces}_P/B)_{\mathrm{cov}} \underset{f^*}{\overset{f_!}{\rightleftarrows}} (\mathbf{dSpaces}_R/B')_{\mathrm{cov}}$$

is a Quillen equivalence.

One can now use this Quillen equivalence to transfer results about the Reedy covariant model structure to the projective model structure, as we will for example see in Remark 13.27.

13.2 Simplicial Systems of Model Categories

In Chapter 12 we made good use of the fact that the category **dSpaces** of dendroidal spaces can be identified with the category $\mathbf{dSets}^{\mathbf{\Delta}^{\mathrm{op}}}$ of simplicial objects in **dSets** and therefore carries two different Reedy model structures: one with respect to the Reedy structure of $\mathbf{\Omega}$, the other one with respect to that of $\mathbf{\Delta}$. We wish to apply this point of view to the covariant model structure on $\mathbf{dSpaces}/B$ established in the previous section. However, for general B the category $\mathbf{dSpaces}/B$ is not directly identified with a category of simplicial objects. Indeed, for a constant dendroidal space $B = \mathrm{con}(B_0)$ one would have

$$\mathbf{dSpaces}/B = (\mathbf{dSets}/B_0)^{\mathbf{\Delta}^{\mathrm{op}}},$$

but if the assignment

$$\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{dSets}\colon [n] \to B_n$$

is not constant then there is no such formula. Thus, the way in which the covariant model structure on $\mathbf{dSpaces}/B$ is obtained from some kind of Reedy model structure with respect to $\mathbf{\Delta}$ is more subtle. The next section aims to explain how this can be understood, using a more flexible viewpoint on Reedy model structures developed in the current section.

We begin with the following general definition. Throughout this section, we will assume our model categories to be cofibrantly generated.

Definition 13.10 A *simplicial system of model categories* is a $\mathbf{\Delta}^{\mathrm{op}}$-shaped diagram

$$\mathcal{E}_0 \;\substack{\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow}\; \mathcal{E}_1 \;\substack{\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow}\; \mathcal{E}_2 \;\substack{\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow}\; \cdots$$

of model categories and left Quillen functors, for which the simplicial identities hold up to coherent natural isomorphism.

Remark 13.11 (1) For our purposes, the crucial example to have in mind is the following. If $X_\bullet$ is a simplicial object of a model category $\mathcal{E}$, then we obtain a simplicial system of model categories

$$\mathcal{E}/X_0 \leftrightarrows \mathcal{E}/X_1 \leftrightarrows \mathcal{E}/X_2 \leftrightarrows \cdots$$

for the slice model structures on the categories $\mathcal{E}/X_i$ or any modification of these model structures for which the adjoints pairs $\mathcal{E}/X_n \rightleftarrows \mathcal{E}/X_m$ induced by the simplicial structure maps $X_n \longrightarrow X_m$ remain Quillen pairs.

(2) One precise way of expressing the definition above is in terms of pseudofunctors from $\mathbf{\Delta}^{\mathrm{op}}$ into the 2-category of model categories, left Quillen functors, and natural isomorphisms between these. We will not spell this out in detail, as the only examples we will have to deal with are of the type above.

(3) If $\mathcal{E}_\bullet$ is a simplicial system of model categories, we shall denote the Quillen pair induced by a morphism $\alpha\colon [m] \longrightarrow [n]$ in $\mathbf{\Delta}$ by

$$\mathcal{E}_n \underset{\mathcal{E}(\alpha)^*}{\overset{\mathcal{E}(\alpha)_!}{\rightleftarrows}} \mathcal{E}_m.$$

The reader is advised to pay attention to the direction of the functors here.

For a simplicial system of model categories $\mathcal{E}_\bullet$, we can construct a new category $\Gamma(\mathcal{E}_\bullet)$, which we will refer to as the *totalization* of $\mathcal{E}_\bullet$, as follows. Its objects are pairs (X, θ) where $X = \{X_n\}_{n\geq 0}$ is a sequence of objects X_n in $\mathcal{E}_n$ and θ assigns to each $\alpha\colon [m] \longrightarrow [n]$ in $\mathbf{\Delta}$ a map

$$\theta_\alpha\colon X_n \longrightarrow \mathcal{E}(\alpha)^*(X_m).$$

These θ_α are required to be functorial in α and compatible with the coherence isomorphisms of $\mathcal{E}_\bullet$, in the sense that for $\alpha\colon [m] \longrightarrow [n]$ and $\beta\colon [n] \longrightarrow [l]$, the diagram

$$\begin{tikzcd}
X_l \arrow[rr, "\theta_\beta"] \arrow[d, "\theta_{\beta\alpha}"'] & & \mathcal{E}(\beta)^*(X_n) \arrow[d, "\mathcal{E}(\beta)^*(\theta_\alpha)"] \\
\mathcal{E}(\beta\alpha)^*(X_m) \arrow[rr, "\cong"] & & \mathcal{E}(\beta)^*\mathcal{E}(\alpha)^*(X_m)
\end{tikzcd}$$

commutes. By adjunction, the maps θ_α correspond to maps

$$\widehat{\theta}_\alpha\colon \mathcal{E}(\alpha)_!(X_n) \longrightarrow X_m$$

and the objects of $\Gamma(\mathcal{E}_\bullet)$ could of course also have been expressed in this form. For two objects (X, θ) and (Y, τ) in $\Gamma(\mathcal{E}_\bullet)$, a morphism $f\colon (X, \theta) \longrightarrow (Y, \tau)$ is a sequence of morphisms

$$f_n \colon X_n \longrightarrow Y_n,$$

compatible with θ and τ in the sense that each square of the form

$$\begin{tikzcd}
X_n \arrow[r, "f_n"] \arrow[d, "\theta_\alpha"] & Y_n \arrow[d, "\tau_\alpha"] \\
\mathcal{E}(\alpha)^*(X_m) \arrow[r, "\mathcal{E}(\alpha)^* f_m"] & \mathcal{E}(\alpha)^*(Y_m)
\end{tikzcd}$$

commutes.

Example 13.12 For a simplicial object $X_\bullet$ in a model category $\mathcal{E}$, the category $\Gamma(\mathcal{E}/X_\bullet)$ can be identified with $\mathcal{E}^{\mathbf{\Delta}^{\mathrm{op}}}/X$.

There is an evident forgetful functor

$$\varphi^* \colon \Gamma(\mathcal{E}_\bullet) \longrightarrow \prod_{n \geq 0} \mathcal{E}_n$$

sending an object (X, θ) to the sequence of objects $\{X_n\}_{n \geq 0}$. This functor has a left adjoint

$$\varphi_! \colon \prod_{n \geq 0} \mathcal{E}_n \longrightarrow \Gamma(\mathcal{E}_\bullet).$$

It can be described explicitly as follows: for an object $X = \{X_n\}_{n \geq 0}$, we have

$$\varphi_!(X)_n = \coprod_{\beta \colon [n] \to [k]} \mathcal{E}(\beta)_!(X_k),$$

together with (adjoint) structure maps

$$\widehat{\theta}_\alpha \colon \mathcal{E}(\alpha)_!(\varphi_!(X)_n) \longrightarrow \varphi_!(X)_m$$

for each $\alpha \colon [m] \to [n]$ being defined by

$$\begin{tikzcd}
\mathcal{E}(\alpha)_!(\varphi_!(X)_n) \arrow[rr, dashed, "\widehat{\theta}_\alpha"] \arrow[d, "\cong"] & & \varphi_!(X)_m \arrow[d, equal] \\
\coprod_{\beta \colon [n] \to [k]} \mathcal{E}(\alpha)_! \mathcal{E}(\beta)_!(X_k) \arrow[dr, "\cong"] & & \coprod_{\gamma \colon [m] \to [l]} \mathcal{E}(\gamma)_!(X_l) \\
& \coprod_{\beta \colon [n] \to [k]} \mathcal{E}(\beta\alpha)_!(X_k) \arrow[ur] &
\end{tikzcd}$$

where the morphism on the lower right sends the summand for β to the one for $\gamma = \beta\alpha$ via the identity map.

We will refer to the model structure of the following proposition as the *projective model structure* on $\Gamma(\mathcal{E}_\bullet)$.

Proposition 13.13 *The category $\Gamma(\mathcal{E}_\bullet)$ carries a model structure for which*

$$\prod_{n\geq 0} \mathcal{E}_n \underset{\varphi^*}{\overset{\varphi_!}{\rightleftarrows}} \Gamma(\mathcal{E}_\bullet)$$

is a Quillen adjunction with the property that φ^* detects fibrations and weak equivalences. This model structure is left proper whenever each $\mathcal{E}_n$ is.

Proof The product $\prod_{n\geq 0} \mathcal{E}$ carries an evident model structure for which the classes of fibrations, cofibrations, and weak equivalences are defined coordinatewise. The projective model structure on $\Gamma(\mathcal{E}_\bullet)$ is now obtained by transfer along the adjoint pair $\varphi_!$ and φ^*. The conditions for transfer are easily verified, using that φ^* preserves colimits. □

We now wish to refine this projective model structure on $\Gamma(\mathcal{E}_\bullet)$ and describe a Quillen equivalent 'Reedy model structure'. To this end, we introduce notation analogous to the one in Section 10.1 and define for an object (X, θ) in $\Gamma(\mathcal{E}_\bullet)$ and $n \geq 0$ the objects $X(\partial\Delta[n])$ and $\deg(X_n)$ in $\mathcal{E}_n$ by

$$X(\partial\Delta[n]) = \varprojlim_{\beta\colon [k]\to[n]} \mathcal{E}(\beta)^*(X_k),$$
$$\deg(X)_n = \varinjlim_{\sigma\colon [n]\to[k]} \mathcal{E}(\sigma)_!(X_k),$$

where β and σ range over proper injections and surjections in $\boldsymbol{\Delta}$, respectively. Note that the structure maps θ_β and θ_σ of (X, θ) together induce maps

$$\deg(X)_n \longrightarrow X_n \longrightarrow X(\partial\Delta[n])$$

in $\mathcal{E}_n$. We then define a morphism $(X, \theta) \longrightarrow (Y, \tau)$ in $\Gamma(\mathcal{E}_\bullet)$ to be a *Reedy fibration* if for each $n \geq 0$ the map

$$X_n \longrightarrow X(\partial\Delta[n]) \times_{Y(\partial\Delta[n])} Y_n$$

is a fibration in $\mathcal{E}_n$. Similarly, it is a *Reedy cofibration* if for each $n \geq 0$ the map

$$\deg(Y)_n \cup_{\deg(X)_n} X_n \longrightarrow Y_n$$

is a cofibration in $\mathcal{E}_n$.

Theorem 13.14 *The Reedy fibrations and cofibrations defined above are part of a model structure on the category $\Gamma(\mathcal{E}_\bullet)$ in which the weak equivalences are the same as for the projective model structure. The identity functor is a left Quillen equivalence from the projective model structure of Proposition 13.13 to this Reedy model structure. Finally, the Reedy model structure is left proper whenever each $\mathcal{E}_n$ is.*

A first step towards the proof of the theorem is the following fact, which is analogous to Proposition 10.11.

Lemma 13.15 *A map $X \to Y$ in $\Gamma(\mathcal{E}_\bullet)$ is both a Reedy fibration and a weak equivalence if and only if for each $n \geq 0$ the map*

$$X_n \to X(\partial\Delta[n]) \times_{Y(\partial\Delta[n])} Y_n$$

is a trivial fibration in $\mathcal{E}_n$.

Proof For $n \geq 0$ let us write $\mathbf{M}_n$ for the full subcategory of $\mathbf{\Delta}/[n]$ whose objects are the monomorphisms $[k] \to [n]$ in $\mathbf{\Delta}$. Then $\mathbf{M}_n$ is a Reedy category in which every morphism is positive. An object (X, θ) in $\Gamma(\mathcal{E}_\bullet)$ yields a functor

$$X^{(n)} \colon \mathbf{M}_n^{\mathrm{op}} \to \mathcal{E}_n$$

for each n, defined on objects $\alpha \colon [k] \to [n]$ by

$$X^{(n)}(\alpha) = \mathcal{E}(\alpha)^*(X_k)$$

and on morphisms γ as in

$$\begin{tikzcd} {[k]} \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] & & {[l]} \arrow[dl, "\beta"] \\ & {[n]} & \end{tikzcd}$$

by the map $X^{(n)}(\beta) \to X^{(n)}(\alpha)$ given by the commutative diagram

$$\begin{tikzcd} X^{(n)}(\beta) \arrow[r, equal] \arrow[dd, dashed] & \mathcal{E}(\beta)^*(X_l) \arrow[dr, "\mathcal{E}(\beta)^*(\theta_\gamma)"] & \\ & & \mathcal{E}(\beta)^*\mathcal{E}(\gamma)^*(X_k) \arrow[dl, "\cong"] \\ X^{(n)}(\alpha) \arrow[r, equal] & \mathcal{E}(\alpha)^*(X_k). & \end{tikzcd}$$

If $X \to Y$ is a Reedy fibration in $\Gamma(\mathcal{E}_\bullet)$ then for each $n \geq 0$ the map $X^{(n)} \to Y^{(n)}$ is a Reedy fibration in $\mathcal{E}_n^{\mathbf{M}_n^{\mathrm{op}}}$. This puts us in a position to use the analogue of the lemma for ordinary Reedy model structures, namely Proposition 10.11. First of all, if

$$X_n \to X(\partial\Delta[n]) \times_{Y(\partial\Delta[n])} Y_n$$

is a trivial fibration for each $n \geq 0$, then $X^{(n)} \to Y^{(n)}$ is a trivial Reedy fibration in the Reedy model structure with respect to $\mathbf{M}_n$, hence a weak equivalence there. So $X^{(n)}(\alpha) \to Y^{(n)}(\alpha)$ is a weak equivalence for each n and each $\alpha \colon [k] \to [n]$ and in particular so is $X_n \to Y_n$ itself. This proves one direction of the lemma.

For the converse, suppose $X_n \longrightarrow Y_n$ is a weak equivalence for each $n \geq 0$. Since $X^{(n)} \longrightarrow Y^{(n)}$ is a Reedy fibration in $\mathcal{E}_n^{\mathbf{M}_n^{\mathrm{op}}}$, it is also a projective fibration. In other words, $X^{(n)}(\alpha) \to Y^{(n)}(\alpha)$ is a fibration for each α, or $\mathcal{E}(\alpha)^*(X_k) \longrightarrow \mathcal{E}(\alpha)^*(Y_k)$ is a fibration for each $n \geq 0$ and each $\alpha\colon [k] \longrightarrow [n]$. In particular so is $X_n \longrightarrow Y_n$, so that it is a trivial fibration for each $n \geq 0$. But trivial fibrations are preserved by right Quillen functors, so $\mathcal{E}(\alpha)^*(X_k) \longrightarrow \mathcal{E}(\alpha)^*(Y_k)$ is a trivial fibration for each α as above. So $X^{(n)} \longrightarrow Y^{(n)}$ is a weak equivalence in the Reedy model structure with respect to $\mathbf{M}_n$, so that

$$X^{(n)}(\alpha) \to X^{(n)}(\partial\alpha) \times_{Y^{(n)}(\partial\alpha)} Y^{(n)}(\alpha)$$

is a trivial fibration for each α. For $\alpha = \mathrm{id}$ this precisely means that

$$X_n \longrightarrow X(\partial\Delta[n]) \times_{Y(\partial\Delta[n])} Y_n$$

is a trivial fibration. □

As a next preparation for the proof of the theorem, we consider two classes of morphisms in $\Gamma(\mathcal{E}_\bullet)$, which are intended to be the generating cofibrations and trivial cofibrations. For an object A in $\mathcal{E}_n$, define two objects in $\Gamma(\mathcal{E}_\bullet)$, heuristically denoted $A \boxtimes \Delta[n]$ and $A \boxtimes \partial\Delta[n]$, by setting

$$\begin{aligned}(A \boxtimes \Delta[n])_k &= \coprod_{\beta \in \Delta[n]_k} \mathcal{E}(\beta)_!(A),\\ (A \boxtimes \partial\Delta[n])_k &= \coprod_{\beta \in \partial\Delta[n]_k} \mathcal{E}(\beta)_!(A).\end{aligned}$$

In other words, in the first sum β ranges over all maps $[k] \longrightarrow [n]$, and in the second sum only over the nonsurjective ones. For $\alpha\colon [k] \longrightarrow [l]$, the adjoint structure map

$$\widehat{\theta}_\alpha\colon \mathcal{E}(\alpha)_!((A \boxtimes \Delta[n])_l) \to (A \boxtimes \Delta[n])_k$$

sends the summand $\mathcal{E}(\alpha)_!\mathcal{E}(\beta)_!(A)$ of $\mathcal{E}(\alpha)_!(A \boxtimes \Delta[n])_l$ for $\beta\colon [l] \longrightarrow [n]$ to the summand of $(A \boxtimes \Delta[n])_k$ for the composition $\beta\alpha\colon [l] \longrightarrow [n]$ via the isomorphism $\mathcal{E}(\alpha)_!\mathcal{E}(\beta)_! \cong \mathcal{E}(\beta\alpha)_!$. These structure maps evidently restrict to structure maps

$$\widehat{\theta}_\alpha\colon \mathcal{E}(\alpha)_!((A \boxtimes \partial\Delta[n])_l) \to (A \boxtimes \partial\Delta[n])_k$$

and we obtain a morphism

$$A \boxtimes \partial\Delta[n] \to A \boxtimes \Delta[n]$$

in $\Gamma(\mathcal{E}_\bullet)$.

Next, let $\mathcal{J}$ be the saturation in $\Gamma(\mathcal{E}_\bullet)$ of the set of morphisms of the form

$$A \boxtimes \Delta[n] \cup_{A \boxtimes \partial\Delta[n]} B \boxtimes \partial\Delta[n] \longrightarrow B \boxtimes \Delta[n],$$

where $A \to B$ ranges over generating cofibrations in $\mathcal{E}_n$ for all $n \geq 0$, and let $\mathcal{J}$ be defined similarly but with $A \to B$ ranging over generating trivial cofibrations.

Lemma 13.16 *For a (trivial) cofibration $A \to B$ in $\mathcal{E}_n$ the map defined above is a (trivial) Reedy cofibration in $\Gamma(\mathcal{E}_\bullet)$.*

Proof In simplicial degree k, we can write $\deg(B \boxtimes \Delta[n])_k$ as

$$\deg(B \boxtimes \Delta[n])_k = \coprod_{\sigma} \coprod_{\beta} \mathcal{E}(\sigma)_! \mathcal{E}(\beta)_!(B),$$

where σ ranges over all proper surjections $\sigma \colon [k] \to [l]$ and β over all maps $[l] \to [n]$ in $\mathbf{\Delta}$. Since any $\alpha \colon [k] \to [n]$ factors uniquely as

$$[k] \xrightarrow{\sigma} [l] \xrightarrow{\beta} [n]$$

where σ is a proper surjection if and only if α is not a monomorphism, we find that

$$\deg(B \boxtimes \Delta[n])_k = \coprod_{\alpha \colon [k] \to [n]} \mathcal{E}(\alpha)_!(B),$$

with α ranging over non-injective maps. So for $A \to B$ in $\mathcal{E}_n$, the degree k part of

$$(A \boxtimes \Delta[n] \cup_{A \boxtimes \partial\Delta[n]} B \boxtimes \partial\Delta[n]) \cup \deg(B \boxtimes \Delta[n]) \to B \boxtimes \Delta[n]$$

is the coproduct indexed by maps $\alpha \colon [k] \to [n]$ of maps $\mathcal{E}(\alpha)_! B \xrightarrow{\sim} \mathcal{E}(\alpha)_! B$ if α is not the identity, and $A \to B$ if α is the identity on $[n]$. This shows that

$$A \boxtimes \Delta[n] \cup_{A \boxtimes \partial\Delta[n]} B \boxtimes \partial\Delta[n] \to B \boxtimes \Delta[n]$$

is a Reedy cofibration if $A \to B$ is a cofibration, and clearly a weak equivalence for each k if $A \to B$ is moreover a weak equivalence. $\square$

Lemma 13.17 *A map $X \to Y$ in $\Gamma(\mathcal{E}_\bullet)$ is a Reedy fibration (respectively a trivial Reedy fibration) if and only if it has the right lifting property with respect to $\mathcal{I}$ (respectively $\mathcal{J}$).*

Proof For an object X in $\Gamma(\mathcal{E}_\bullet)$, there are natural bijective correspondences between maps $A \boxtimes \Delta[n] \to X$ in $\Gamma(\mathcal{E}_\bullet)$ and $A \to X_n$ in $\mathcal{E}_n$, and similarly for $A \boxtimes \partial\Delta[n] \to X$ and $A \to X(\partial\Delta[n])$. From this it is clear that $X \to Y$ is a Reedy fibration in $\Gamma(\mathcal{E}_\bullet)$ if and only if it has the right lifting property with respect to all the generating trivial cofibrations of the form

$$A \boxtimes \Delta[n] \cup_{A \boxtimes \partial\Delta[n]} B \boxtimes \partial\Delta[n] \to B \boxtimes \Delta[n]$$

where $A \to B$ is a generating trivial cofibration in $\mathcal{E}_n$. Using Lemma 13.15, a similar statement holds for a trivial Reedy fibration. This proves the lemma. $\square$

Lemma 13.18 *The classes of Reedy cofibrations and trivial Reedy cofibrations in* $\Gamma(\mathcal{E}_\bullet)$ *are saturated. In particular, they contain* $\mathcal{I}$ *and* $\mathcal{J}$ *respectively.*

Proof Dual to the objects $A \boxtimes \Delta[n]$ and $A \boxtimes \partial\Delta[n]$ defined above, we can define, for each object A in $\mathcal{E}_n$, two objects $A^{[n]}$ and $A^{\deg[n]}$ in $\Gamma(\mathcal{E}_\bullet)$ by

$$
\begin{aligned}
(A^{[n]})_k &= \prod_{\beta\colon [n]\to[k]} \mathcal{E}(\beta)^*(A),\\
(A^{\deg[n]})_k &= \prod_{\beta\colon [n]\xrightarrow{+}[k]} \mathcal{E}(\beta)^*(A),
\end{aligned}
$$

with the obvious structure maps θ. Here the first product is over all maps $\beta\colon [n] \to [k]$, the second product only over the non-identity maps β. Then for an object X in $\Gamma(\mathcal{E}_\bullet)$ there are bijective correspondences between maps $X \to A^{[n]}$ in $\Gamma(\mathcal{E}_\bullet)$ and $X_n \to A$ in $\mathcal{E}_n$. Similarly, maps $X \to A^{\deg[n]}$ correspond to maps $\deg(X)_n \to A$. So $X \to Y$ is a Reedy cofibration if and only if for any trivial fibration $B \to A$ in $\mathcal{E}_n$, the map $X \to Y$ has the left lifting property with respect to

$$B^{[n]} \to A^{[n]} \times_{B^{\deg[n]}} B^{\deg[n]}.$$

This shows that the class of Reedy cofibrations is saturated. In particular, by Lemma 13.16, it contains the class $\mathcal{I}$. Since the saturation of the maps

$$A \boxtimes \Delta[n] \cup_{A\boxtimes\partial\Delta[n]} B \boxtimes \partial\Delta[n] \to B \boxtimes \Delta[n]$$

where $A \to B$ is a trivial cofibration is obviously contained in the weak equivalences, the class $\mathcal{J}$ is contained in the trivial Reedy cofibrations. $\square$

Based on the lemmas above, the proof of Theorem 13.14 is now straightforward:

Proof (of Theorem 13.14) We prove the existence of the model structure; the verification of the further claims in the theorem is completely straightforward and left to the reader. As usual, axioms (M1–3) obviously hold. For the factorization axioms, the small object argument gives for each morphism $X \to Y$ in $\Gamma(\mathcal{E}_\bullet)$ a factorization into a morphism in $\mathcal{I}$ followed by a morphism having the right lifting property with respect to $\mathcal{I}$ and a similar factorization for $\mathcal{J}$. By Lemmas 13.17 and 13.18, these are in particular a factorization into a Reedy cofibration followed by a trivial Reedy fibration and by a trivial Reedy cofibration followed by a Reedy fibration. So it remains to prove the lifting axiom. To this end, consider a commutative square

$$
\begin{tikzcd}
U \arrow[r] \arrow[d] & X \arrow[d] \\
V \arrow[r] & Y
\end{tikzcd}
$$

where $U \to V$ is a Reedy cofibration and $X \to Y$ is a Reedy fibration. If $X \to Y$ is also a weak equivalence, we can use Lemma 13.15 to find lifts $V_n \to X_n$ compatible with the structure maps by induction on n. Indeed, having found such compatible lifts for all $k < n$ yields a commutative square

$$\begin{tikzcd}
U_n \cup_{\deg(U)_n} \deg(V)_n \arrow[r] \arrow[d] & X_n \arrow[d] \\
V_n \arrow[r] & X(\partial\Delta[n]) \times_{Y(\partial\Delta[n])} Y_n,
\end{tikzcd}$$

in which a diagonal gives the required lift for n. This provides the inductive step.

If on the other hand $U \to V$ is a weak equivalence, we factor $U \to V$ as $U \to W \to V$, where $U \to W$ is in $\mathcal{J}$ and $W \to V$ has the right lifting property with respect to all maps in $\mathcal{J}$. Then by two-out-of-three, the map $W \to V$ is also a weak equivalence, hence a trivial fibration. Then by the first part of the lifting axiom just proved, $U \to V$ is a retract of $U \to W$. Moreover $X \to Y$ has the right lifting property with respect to $U \to W$ by Lemma 13.17. Composing these two lifts (as in the retract argument) provides the required lift $V \to X$:

$$\begin{tikzcd}
U \arrow[rr] \arrow[dd] \arrow[dr] & & X \arrow[dd] \\
& W \arrow[ur, dashed] \arrow[dl, shift left] & \\
V \arrow[rr] \arrow[ur, dashed, shift left] & & Y.
\end{tikzcd}$$

This completes the proof of the theorem. □

Remark 13.19 The argument above indeed shows that $\mathcal{I}$ and $\mathcal{J}$ are classes of generating cofibrations and trivial cofibrations, respectively.

We wish to emphasize a particular left Bousfield localization of $\Gamma(\mathcal{E}_\bullet)$. Let us assume that each $\mathcal{E}_n$ is a left proper cofibrantly generated model category whose cofibrations are monomorphisms, satisfying the conditions for the existence of localizations. Then the same is true for the Reedy model structure on $\Gamma(\mathcal{E}_\bullet)$ and in this case we observe the following.

Proposition 13.20 *Under the above assumptions, the category $\Gamma(\mathcal{E}_\bullet)$ carries a model structure in which the cofibrations are the Reedy cofibrations and the fibrant objects are the Reedy fibrant objects (X, θ) with the additional property that for each $\alpha\colon [n] \to [m]$, the map $\theta_\alpha\colon X_m \to \mathcal{E}(\alpha)^*(X_n)$ is a weak equivalence.*

We denote the model category of the proposition by $\Gamma_w(\mathcal{E}_\bullet)$.

Proof By two-out-of-three for the weak equivalences, the condition on the Reedy fibrant objects (X, θ) is equivalent to the condition that for each face map $\partial_i\colon [n-1] \to [n]$, the corresponding face map

$$X_n \to \mathcal{E}(\partial_i)^*(X_{n-1})$$

is a weak equivalence. This map is a fibration if X is Reedy fibrant and a trivial fibration if and only if X has the right lifting property with respect to the maps of the form

$$A \boxtimes \Delta[n] \cup_{A \boxtimes \partial_i \Delta[n]} B \boxtimes \partial_i \Delta[n] \longrightarrow B \boxtimes \Delta[n]$$

for each generating cofibration $A \longrightarrow B$ in $\mathcal{E}_n$. So taking the left Bousfield localization with respect to these maps gives the desired result. □

The construction of the Reedy model structure is functorial in the simplicial system of model categories $\mathcal{E}_\bullet$, in the following sense. Define a Quillen pair

$$\mathcal{E}_\bullet \underset{\varphi^*}{\overset{\varphi_!}{\rightleftarrows}} \mathcal{F}_\bullet$$

between simplicial systems to be a sequence of Quillen pairs

$$\mathcal{E}_n \underset{\varphi_n^*}{\overset{(\varphi_n)_!}{\rightleftarrows}} \mathcal{F}_n$$

which are compatible with the simplicial structure maps up to coherent natural isomorphism. More precisely, as part of the structure of the Quillen pair $(\varphi_!, \varphi^*)$, we require for each $\alpha\colon [n] \longrightarrow [m]$ in $\mathbf{\Delta}$ a natural isomorphism between the two compositions in the square

$$\begin{tikzcd} \mathcal{E}_m \arrow[r, "(\varphi_m)_!"] \arrow[d, "\mathcal{E}(\alpha)_!"] & \mathcal{F}_m \arrow[d, "\mathcal{F}(\alpha)_!"] \\ \mathcal{E}_n \arrow[r, "(\varphi_n)_!"] & \mathcal{F}_n \end{tikzcd}$$

and these are required to be compatible with the isomorphisms for $\mathcal{E}$ and $\mathcal{F}$ associated to each composition $[n] \xrightarrow{\alpha} [m] \xrightarrow{\beta} [l]$.

Proposition 13.21 *A Quillen pair $\varphi_! : \mathcal{E}_\bullet \rightleftarrows \mathcal{F}_\bullet : \varphi^*$ induces a Quillen pair between the Reedy model categories*

$$\Gamma(\varphi)_! : \Gamma(\mathcal{E}_\bullet) \rightleftarrows \Gamma(\mathcal{F}_\bullet) : \Gamma(\varphi)^*.$$

The Quillen pair is compatible with the localizations introduced above and induces a similar pair $\Gamma_w(\mathcal{E}_\bullet) \rightleftarrows \Gamma_w(\mathcal{F}_\bullet)$. Moreover, if each pair $((\varphi_n)_!, \varphi_n^*)$ is a Quillen equivalence, then so is $(\Gamma(\varphi)_!, \Gamma(\varphi)^*)$.

Proof The functor $\Gamma(\varphi)_!$ is defined on an object of (X, θ) of $\Gamma(\mathcal{E}_\bullet)$ by

$$(\Gamma(\varphi)_!(X, \theta))_n = (\varphi_n)_!(X_n),$$

with structure maps defined by

$$\mathcal{F}(\alpha)_!((\varphi_m)_!(X_m)) \cong (\varphi_n)_! \mathcal{E}(\alpha)_!(X_m) \xrightarrow{(\varphi_n)_! \widehat{\theta}_\alpha} (\varphi_n)_!(X_n),$$

and similarly for $\Gamma(\varphi)^*$. It is readily verified that this is indeed a Quillen pair for the Reedy model structures. Moreover, $\Gamma(\varphi)^*$ clearly maps the fibrant objects of the localization $\Gamma_w(\mathcal{F}_\bullet)$ to those of $\Gamma_w(\mathcal{E}_\bullet)$, hence restricts to the claimed Quillen pair between localizations. $\square$

13.3 Homotopy Invariance of the Covariant Model Structure

In this section we will apply the construction of the total model category $\Gamma(\mathcal{E}_\bullet)$ of a simplicial system of model categories, as in the previous section, to the problem of homotopy invariance of the covariant model structure. To be precise, we aim to show that a complete weak equivalence of dendroidal Segal spaces $A \to B$ induces a Quillen equivalence between the covariant model structures on the slice categories $\mathbf{dSpaces}/A$ and $\mathbf{dSpaces}/B$ (cf. Corollary 13.26 below).

Recall from Proposition 9.62 that if $A \to B$ is an operadic equivalence of ∞-operads, then the induced Quillen pair

$$(\mathbf{dSets}/A)_{\mathrm{cov}} \rightleftarrows (\mathbf{dSets}/B)_{\mathrm{cov}}$$

is a Quillen equivalence. Exactly the same argument applies to dendroidal spaces and yields the following proposition. We remark right away that we will prove a stronger version of this result in Corollary 13.26 below.

Proposition 13.22 *Let $A \to B$ be a map between dendroidal Segal spaces. If $A \to B$ is a weak equivalence in the model category $\mathbf{dSpaces}_{RS}$ (i.e., if $A(T) \to B(T)$ is a weak homotopy equivalence for each tree T), then the induced Quillen pair*

$$(\mathbf{dSpaces}_R/A)_{\mathrm{cov}} \rightleftarrows (\mathbf{dSpaces}_R/B)_{\mathrm{cov}}$$

is a Quillen equivalence.

Proof By Brown's lemma it suffices to show this for a trivial fibration $f\colon A \to B$ between dendroidal Segal spaces. Such an f induces a Quillen equivalence

$$\mathbf{dSpaces}_{RS}/A \rightleftarrows \mathbf{dSpaces}_{RS}/B$$

by Example 8.47(i). To see that this pair also induces a Quillen equivalence between the covariant localizations, note that the left adjoint sends localizing morphisms $\ell[T] \to T$ over A to localizing maps over B. Conversely, any localizing morphism

$$\begin{tikzcd} \ell[T] \arrow[rr] \arrow[dr] & & T \arrow[dl] \\ & B & \end{tikzcd}$$

over B is in the image of the functor $\mathbf{dSpaces}_{RS}/A \to \mathbf{dSpaces}_{RS}/B$ because any map $T \to B$ may be lifted along the trivial fibration $A \to B$. □

To prove a sharper result, we will now identify the model category $(\mathbf{dSpaces}_R/B)_{\mathrm{cov}}$ with a model category of the form $\Gamma_w(\mathcal{E}_\bullet)$ discussed in the previous section. This identification will explain how the covariant model structure on $\mathbf{dSpaces}/B$ is related to the different covariant model structures on the categories $\mathbf{dSets}/B_n$ for $n \geq 0$.

Theorem 13.23 *Under the identification of the category* $\mathbf{dSpaces}/B$ *with* $\Gamma(\mathbf{dSets}/B_\bullet)$ *(cf. Example 13.12), the model structures* $\Gamma_w((\mathbf{dSets}/B_\bullet)_{\mathrm{cov}})$ *and* $(\mathbf{dSpaces}_R/B)_{\mathrm{cov}}$ *coincide.*

Proof Let us denote the adjoint functors between the categories involved by

$$\Gamma(\mathbf{dSets}/B_\bullet) \underset{\psi^*}{\overset{\psi_!}{\rightleftarrows}} \mathbf{dSpaces}/B.$$

This adjoint pair is an equivalence of categories. The functor ψ^* sends an object $X \to B$ to the pair $(X_\bullet, \theta)$ of objects $X_n \to B_n$ in $\mathbf{dSets}/B_n$ for $n \geq 0$ and structure maps

$$\theta_\alpha \colon X_n \to X_m \times_{B_m} B_n$$

for $\alpha \colon [m] \to [n]$ given by the simplicial structure of X and of B.

First of all, it is straightforward to verify that under the equivalence $\psi_!$ the cofibrations in the Reedy model structure on $\Gamma(\mathbf{dSets}/B_\bullet)$ associated to the operadic or covariant model structures on the categories $\mathbf{dSets}/B_n$ correspond to the Reedy cofibrations on $\mathbf{dSpaces}/B$. It thus suffices to prove that this equivalence identifies the fibrant objects of one model category with those of the other. The following arguments are very similar to those proving Lemma 12.17 and the reader might like to compare them.

The fibrant objects in $(\mathbf{dSpaces}_R/B)_{\mathrm{cov}}$ are the Reedy left fibrations $X \to B$. By Lemma 13.5, these are the maps of dendroidal spaces $p \colon X \to B$ having the right lifting property with respect to the following two kinds of morphisms:

(a) The maps
$$T \boxtimes \Lambda^k[n] \cup \partial T \boxtimes \Delta[n] \to T \boxtimes \Delta[n]$$
for any tree T and $0 \leq k \leq n$.

(b) The maps
$$T \boxtimes \partial\Delta[n] \cup \Lambda^x[T] \boxtimes \Delta[n] \to T \boxtimes \Delta[n]$$
for any tree T and any inner or leaf horn $\Lambda^x[T]$ of T.

Indeed, (a) encodes the Reedy condition, whereas the right lifting property with respect to the maps of (b) then makes p a left fibration. Condition (b) can equivalently be stated as saying that for any monomorphism $M \to N$ of simplicial sets, the resulting map

$$X(N) \to X(M) \times_{B(M)} B(N)$$

is a left fibration of dendroidal sets. In particular, if (b) is satisfied, then the maps

$$X_n \longrightarrow B_n \quad \text{and} \quad X(\partial\Delta[n]) \times_{B(\partial\Delta[n])} B_n \longrightarrow B_n$$

are left fibrations, the second one being a pullback of the left fibration $X(\partial\Delta[n]) \to B(\partial\Delta[n])$. Moreover, $X_n \to X(\partial\Delta[n]) \times_{B(\partial\Delta[n])} B_n$ is also a left fibration and hence a fibration in the covariant model structure on the category $\mathbf{dSets}/B_n$ by Theorem 9.59. Equivalently, $\psi^* p$ is a Reedy fibrant object of $\Gamma((\mathbf{dSets}/B_\bullet)_{\mathrm{cov}})$ in the sense of Theorem 13.14.

Conversely, if $\psi^* p$ is Reedy fibrant, then the maps

$$X_n \longrightarrow X(\partial\Delta[n]) \times_{B(\partial\Delta[n])} B_n$$

are covariant fibrations over B_n (in particular, left fibrations) and therefore p has the right lifting property with respect to the maps (b). For the remainder of this proof, assume that p has this lifting property.

It remains to show that p additionally has the right lifting property with respect to the maps of (a) if and only if $\psi^* p$ is a local object of $\Gamma_w((\mathbf{dSets}/B_\bullet)_{\mathrm{cov}})$, i.e., if and only if for each $\alpha\colon [m] \to [n]$, the associated map

$$\theta_\alpha\colon X_n \longrightarrow X_m \times_{B_m} B_n$$

is a weak equivalence. By two-out-of-three, it suffices to consider monomorphisms α, in which case the map above is a left fibration by the Reedy condition. Now consider the class $\mathcal{A}$ of monomorphisms $M \to N$ of simplicial sets for which the map

$$X(N) \longrightarrow X(M) \times_{B(M)} B(N)$$

is a trivial fibration. Clearly, $\mathcal{A}$ is saturated and closed under two-out-of-three among monomorphisms. If p has the right lifting property with respect to the maps of (a), then $\mathcal{A}$ contains all anodyne maps between simplicial sets, so in particular each θ_α is a weak equivalence. Conversely, if the maps θ_α are weak equivalences for the face inclusion $\alpha = \delta_i\colon [n-1] \to [n]$, then $\mathcal{A}$ must contain all anodyne maps. In particular, p will then have the right lifting property with respect to the maps of (a). This completes the proof. □

Corollary 13.24 *Let*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & B & \end{tikzcd}$$

be a map of dendroidal spaces over B. If $X_n \to Y_n$ is a covariant equivalence over B_n for every $n \geq 0$, then $X \to Y$ is a covariant equivalence over B.

Proof The assumption clearly implies that $\psi^*(X \to Y)$ is a weak equivalence in the Reedy model structure on $\Gamma((\mathbf{dSets}/B_\bullet)_{\mathrm{cov}})$. Hence it is also a weak equivalence in the localization $\Gamma_w((\mathbf{dSets}/B_\bullet)_{\mathrm{cov}})$ and thus a covariant equivalence over B by Theorem 13.23. □

Let us return to the invariance problem. In Proposition 13.22 our hypothesis was that $A \to B$ is a map of dendroidal Segal spaces for which $A(T) \to B(T)$ is a weak homotopy equivalence of simplicial sets for every tree T. The identification of model categories of Theorem 13.23 now yields a similar result if $A \to B$ has the property that $A_n \to B_n$ is an operadic equivalence of dendroidal sets for each $n \geq 0$:

Proposition 13.25 *Let $A \to B$ be a map between dendroidal Segal spaces. If $A_n \to B_n$ is an operadic equivalence of dendroidal sets for each $n \geq 0$, then the induced Quillen pair*

$$(\mathbf{dSpaces}_R/A)_{\mathrm{cov}} \rightleftarrows (\mathbf{dSpaces}_R/B)_{\mathrm{cov}}$$

is a Quillen equivalence.

Proof The dendroidal sets A_n and B_n are ∞-operads, so as already remarked at the beginning of this section, Proposition 9.62 shows that the Quillen pairs $\mathbf{dSets}/A_n \rightleftarrows \mathbf{dSets}/B_n$ are Quillen equivalences for the covariant model structures. It follows from Proposition 13.21 that the Quillen pair

$$\Gamma((\mathbf{dSets}/A_\bullet)_{\mathrm{cov}}) \rightleftarrows \Gamma((\mathbf{dSets}/B_\bullet)_{\mathrm{cov}})$$

is a Quillen equivalence as well. Moreover, this Quillen equivalence restricts to one between the localizations $\Gamma_w((\mathbf{dSets}/A_\bullet)_{\mathrm{cov}})$ and $\Gamma_w((\mathbf{dSets}/B_\bullet)_{\mathrm{cov}})$, so that the identification of Theorem 13.23 yields the result. □

As a consequence, we obtain the following sharpening of Proposition 13.22.

Corollary 13.26 *Let $A \to B$ be a complete weak equivalence between dendroidal Segal spaces. Then the induced Quillen pair*

$$(\mathbf{dSpaces}_R/A)_{\mathrm{cov}} \rightleftarrows (\mathbf{dSpaces}_R/B)_{\mathrm{cov}}$$

is a Quillen equivalence.

Proof If A and B are themselves complete dendroidal Segal spaces, then the corollary is a special case of Proposition 13.22 (and also of Proposition 13.25, for that matter). Thus it suffices to prove that each Reedy fibrant dendroidal space A admits a completion $\widehat{A}$ so that $A_n \to \widehat{A}_n$ is an operadic equivalence for every $n \geq 0$. This follows from Corollary 12.21. □

Remark 13.27 Using Proposition 13.9, we can also deduce a projective version of the previous corollary. To be precise, if $A \to B$ is a morphism of fibrant objects in $\mathbf{dSpaces}_{PS}$ which is a complete weak equivalence (i.e., is a weak equivalence in the localization $\mathbf{dSpaces}_{PSC}$), then the Quillen pair

$$(\mathbf{dSpaces}_P/A)_{\mathrm{cov}} \rightleftarrows (\mathbf{dSpaces}_P/B)_{\mathrm{cov}}$$

is a Quillen equivalence.

To conclude this section we observe a slight further sharpening of the previous remark. Indeed, it is not necessary to require A and B to be projectively fibrant:

Corollary 13.28 *Suppose $A \to B$ is a morphism of dendroidal spaces, both of which satisfy the Segal property. If this morphism is a complete weak equivalence, then the Quillen pair*

$$(\mathbf{dSpaces}_P/A)_{\mathrm{cov}} \rightleftarrows (\mathbf{dSpaces}_P/B)_{\mathrm{cov}}$$

is a Quillen equivalence.

Proof This corollary follows from Remark 13.27 if we can show that for any projective weak equivalence of dendroidal spaces $f\colon A \to A'$ the adjunction

$$(\mathbf{dSpaces}_P/A)_{\mathrm{cov}} \underset{f^*}{\overset{f_!}{\rightleftarrows}} (\mathbf{dSpaces}_P/A')_{\mathrm{cov}}$$

is a Quillen equivalence. For the projective model structures $\mathbf{dSpaces}_P/A$ and $\mathbf{dSpaces}_P/A'$ this follows from Example 8.47(j) and the fact that $\mathbf{dSpaces}_P$ is a right proper model category. To see that this equivalence respects the covariant localization, it suffices to show that the derived functors $\mathbf{R}f^*$ and $\mathbf{L}f_!$ preserve local objects. This is evident for f^*, since its left adjoint $f_!$ sends localizing morphisms $\ell[T] \to T$ to localizing morphisms. To see that $f_!$ preserves local objects we use the criterion of Lemma 13.3; indeed, that lemma makes it clear that a map $X \to A$ is a left quasifibration if and only if the same is true for the composite $X \to A'$. $\square$

13.4 The Homotopy Theory of Algebras

Let $\mathbf{P}$ be a simplicial operad. The main result of this chapter will come in the next section, when we establish a Quillen equivalence between the category of $\mathbf{P}$-algebras and the category $\mathbf{dSpaces}/N\mathbf{P}$, equipped with the covariant model structure. To prepare for this result we need a suitable model structure on the category of $\mathbf{P}$-algebras, which is what we establish in the present section. We also include several important examples, such as the category of simplicial categories (or operads) with fixed set of objects (or colours), which will be useful later.

Write C for the set of colours of $\mathbf{P}$. Recall that a $\mathbf{P}$-algebra is a collection of simplicial sets $\{A_c\}_{c\in C}$ equipped with maps of the form

$$\mathbf{P}(c_1,\dots,c_n;c) \times A_{c_1} \times \cdots \times A_{c_n} \to A_c$$

satisfying the usual axioms for associativity, symmetry, and unitality.

Theorem 13.29 *The category* $\mathrm{Alg}_{\mathbf{P}}$ *of* $\mathbf{P}$*-algebras carries a cofibrantly generated model structure in which a morphism $f\colon A \to B$ is a weak equivalence (resp. a fibration) if the morphism $f_c\colon A_c \to B_c$ is a weak homotopy equivalence (resp. a Kan fibration) of simplicial sets for every $c \in C$. This model structure is right proper.*

Remark 13.30 We will refer to the model structure of the theorem as the *projective model structure*. In the special case where $\mathbf{P}$ has only unary operations (so can be thought of as a simplicial category), the category of $\mathbf{P}$-algebras is precisely the category of simplicial functors $\mathbf{P} \to \mathbf{sSets}$. The model structure of the theorem is then the projective model structure already discussed in the case of ordinary categories in Example 7.46.

Note that once the model structure of Theorem 13.29 is established, the fact that it is right proper follows immediately from the corresponding fact for the Kan–Quillen model structure on the category of simplicial sets. Let us therefore focus on proving the existence of the projective model structure. As before, write U for the forgetful functor

$$\mathrm{Alg}_{\mathbf{P}} \to \prod_{c \in C} \mathbf{sSets} \colon A \mapsto \{A_c\}_{c \in C}$$

and $\mathrm{Free}_{\mathbf{P}}$ for its left adjoint, forming the free $\mathbf{P}$-algebra on a C-indexed collection of simplicial sets. Then the model structure of Theorem 13.29 is defined in such a way that f is a weak equivalence or fibration if and only if $U(f)$ is such. In the terminology of Section 7.7, the projective model structure on $\mathrm{Alg}_{\mathbf{P}}$ is transferred along U. We will call a morphism f of $\mathbf{P}$-algebras a *projective trivial cofibration* if it lies in the saturation of the class of morphisms of the form $\mathrm{Free}_{\mathbf{P}}(i)$, where i ranges over the trivial cofibrations of the category $\prod_{c \in C} \mathbf{sSets}$. To prove Theorem 13.29 it suffices to check that the conditions of Theorem 7.44 are satisfied; in other words, it will suffice to verify that every projective trivial cofibration is in particular a weak equivalence in $\mathrm{Alg}_{\mathbf{P}}$. Observe that (by construction) every projective trivial cofibration has the left lifting property with respect to fibrations in $\mathrm{Alg}_{\mathbf{P}}$, as defined in Theorem 13.29. The proof of that theorem is complete once we have settled the following:

Lemma 13.31 *A projective trivial cofibration $i \colon A \to B$ of $\mathbf{P}$-algebras is a weak equivalence.*

Proof The proof we present here is often referred to as 'Quillen's path object argument'. All it requires is the existence of a product-preserving fibrant replacement of simplicial sets, meaning a functor

$$R \colon \mathbf{sSets} \to \mathbf{sSets}$$

and a natural map $\alpha_X \colon X \to R(X)$ satisfying the following:

(1) For any X the simplicial set $R(X)$ is a Kan complex.
(2) For any X the map α_X is a weak homotopy equivalence.
(3) The functor E preserves finite products.

One example of such an R is the composite $\mathrm{Sing} \circ |-|$, relying on the fact that geometric realization preserves products, and α the unit map. (Readers familiar with Kan's Ex^∞-functor will realize that this is another such functor.)

The fact that R preserves products implies that for any $\mathbf{P}$-algebra A the collection $R(A)$ is naturally a $\mathbf{P}$-algebra again, using the structure maps

$$\begin{aligned}
&\mathbf{P}(c_1, \dots, c_n; c) \times R(A_{c_1}) \times \cdots \times R(A_{c_n}) \\
&\qquad \xrightarrow{\alpha} R(\mathbf{P}(c_1, \dots, c_n; c)) \times R(A_{c_1}) \times \cdots \times R(A_{c_n}) \\
&\qquad \cong R(\mathbf{P}(c_1, \dots, c_n; c) \times A_{c_1} \times \cdots \times A_{c_n}) \\
&\qquad \to R(A_c).
\end{aligned}$$

Write $R(B)^{\Delta[1]}$ for the 'path space' of the algebra $R(B)$: it is defined by

$$(R(B)^{\Delta[1]})_c := (R(B)_c)^{\Delta[1]}$$

with evident structure maps inherited from $R(B)$. Observe that the simplicial sets $(R(B)^{\Delta[1]})_c$ are still Kan complexes. Evaluation at 0 defines a trivial fibration

$$\mathrm{ev}_0 \colon R(B)^{\Delta[1]} \to R(B)$$

and, pulling back along the map $R(A) \xrightarrow{R(i)} R(B)$, a trivial fibration

$$R(A) \times_{R(B)} R(B)^{\Delta[1]} \to R(A).$$

This fibration admits a section using the 'constant path' map $R(B) \to R(B)^{\Delta[1]}$, formally defined by restriction along the map $\Delta[1] \to \Delta[0]$. This section is then a weak equivalence and defines the upper horizontal map in the following commutative square:

$$\begin{tikzcd}
A \arrow[r, "\sim"] \arrow[d, "i"'] & R(A) \times_{R(B)} R(B)^{\Delta[1]} \arrow[d, "\mathrm{ev}_1"] \\
B \arrow[r, "\alpha"] & R(B).
\end{tikzcd}$$

The vertical map on the right is given by evaluation at 1: in detail, it is the composite

$$R(A) \times_{R(B)} R(B)^{\Delta[1]} \to R(B)^{\Delta[1]} \xrightarrow{\mathrm{ev}_1} R(B).$$

Alternatively, this map can be factored as

$$R(A) \times_{R(B)} R(B)^{\Delta[1]} \to R(A) \times_{R(B)} R(B)^{\partial\Delta[1]} \cong R(A) \times R(B) \to R(B).$$

The first morphism is the pullback of the fibration $R(B)^{\Delta[1]} \to R(B)^{\partial\Delta[1]}$ and hence itself a fibration; the last morphism is a fibration because $R(A)_c$ is a Kan complex for any $c \in C$. It follows that the right vertical morphism in the square above is a fibration. Projective trivial cofibrations have the left lifting property with respect to fibrations, so a lift in the square exists. Since the horizontal maps in the square are weak equivalences, the two-out-of-six-property of weak homotopy equivalences (cf. Proposition 7.35) implies that all maps in the square are weak equivalences. In particular this proves that i is a weak equivalence, as desired. □

Example 13.32 (a) Let S be a set and consider the category $\mathbf{sOp}_S$ of S-coloured simplicial operads and morphisms between them that are the identity on colours. This category $\mathbf{sOp}_S$ is itself the category of simplicial algebras for a coloured operad $\mathbf{O}_S$ (cf. Example 1.22) and thus admits a projective model structure as in Theorem 13.29. Let us briefly recall this operad $\mathbf{O}_S$ and describe the weak equivalences and fibrations of $\mathbf{sOp}_S$ explicitly. The set of colours of $\mathbf{O}_S$ can be taken to be the set of S-coloured corollas (cf. Section 1.4), i.e., the set of corollas C equipped with a labelling $E(C) \to S$ of their edges by elements of S. An operation from a collection $C_1, \ldots, C_k$ of S-coloured corollas to a further S-coloured corolla C then consists of a tree T with k vertices $v_1, \ldots, v_k$, equipped with a labelling of the edges of T by elements of S, with the following extra data:

- An isomorphism of S-coloured corollas $C_{v_i} \cong C_i$ for each $1 \le i \le k$.
- An isomorphism of the S-coloured corolla C with the S-coloured corolla obtained from T by contracting all of its inner edges.

The tree T should be thought of as representing a way of composing k operations (represented by the coloured corollas C_i) into a single operation (represented by the coloured corolla C). Composition of operations in $\mathbf{O}_S$ is defined by grafting such S-coloured trees. One verifies that these data indeed define a coloured operad $\mathbf{O}_S$ and that its algebras are precisely (coordinate-free) S-coloured operads as defined in Section 1.4.

Now considering the projective model structure on $\mathbf{sOp}_S \cong \mathrm{Alg}_{\mathbf{O}_S}$, we see that a morphism $\varphi\colon \mathbf{P} \to \mathbf{Q}$ of S-coloured simplicial operads is a weak equivalence (resp. a fibration) precisely if the corresponding morphism of underlying S-coloured collections is a weak equivalence (resp. a fibration). In other words, φ is a weak equivalence (resp. a fibration) if and only if for every sequence of colours $c_1, \ldots, c_n, c$, the map

$$\varphi_{c_1,\ldots,c_n,c}\colon \mathbf{P}(c_1, \ldots, c_n; c) \to \mathbf{Q}(c_1, \ldots, c_n; c)$$

is a weak homotopy equivalence (resp. a Kan fibration) of simplicial sets.

(b) The previous example can be specialized to the category $\mathbf{sCat}_S$ of simplicial categories with set of objects S and functors between them that are the identity on objects. To be precise, there is an operad $\mathbf{P}_S$ with set of colours $S \times S$, and a unique operation

$$\big((c_1, d_1), \ldots, (c_k, d_k)\big) \to (c_1, d_k)$$

whenever $d_i = c_{i+1}$ for every $1 \le i \le k-1$. The category of simplicial $\mathbf{P}_S$-algebras is easily identified with $\mathbf{sCat}_S$, so that the latter admits a projective model structure by Theorem 13.29. A functor $\varphi\colon \mathbf{C} \to \mathbf{D}$ is a weak equivalence (resp. a fibration) in this model structure if and only if for every pair of objects (c, d), the induced map

$$\varphi_{c,d}\colon \mathbf{C}(c, d) \to \mathbf{D}(c, d)$$

is a weak homotopy equivalence (resp. a Kan fibration) of simplicial sets.

We conclude this section with a discussion of cofibrant objects in the category of **P**-algebras; a convenient characterization of such will be useful later. For simplicity (and because it covers all the examples we will need), take **P** to be an operad in **Sets**. By the construction of the projective model structure via transfer, a family of generating (trivial) cofibrations in $\mathrm{Alg}_{\mathbf{P}}$ can be obtained by applying the left adjoint functor $\mathrm{Free}_{\mathbf{P}}$ to the generating (trivial) cofibrations in the category $\prod_S \mathbf{sSets}$. Let us make this explicit. For $c \in S$ and a simplicial set A, write $c \otimes A$ for the S-indexed collection that takes the value A at c and $\varnothing$ elsewhere. Then the morphisms

$$\mathrm{Free}_{\mathbf{P}}(c \otimes (\partial\Delta[n] \to \Delta[n])) \qquad \text{and} \qquad \mathrm{Free}_{\mathbf{P}}(c \otimes (\Lambda^k[n] \to \Delta[n]))$$

for $c \in C$ and $0 \leq k \leq n$ form a set of generating cofibrations (resp. trivial cofibrations) for the projective model structure on $\mathrm{Alg}_{\mathbf{P}}$.

Definition 13.33 A simplicial **P**-algebra A is *s-free* if for each n there exists a collection of elements G_n in the algebra A_n such that the following hold:

(1) The **P**-algebra A_n is free on G_n, i.e., the canonical map $\mathrm{Free}_{\mathbf{P}}(G_n) \to A_n$ is an isomorphism.
(2) For any surjection $\alpha\colon [n] \to [m]$ in $\mathbf{\Delta}$, we have $\alpha^*(G_m) \subseteq G_n$. In words, a degeneracy of a generator is another generator.

Clearly any s-free algebra A can be built from the initial **P**-algebra by a composition of pushouts of morphisms of the kind $\mathrm{Free}_{\mathbf{P}}(c \otimes (\partial\Delta[n] \to \Delta[n]))$, one for every element of G_n that is 'non-degenerate', meaning not in the image of some $\alpha^*\colon G_m \to G_n$ with $m < n$. It follows that s-free simplicial **P**-algebras are cofibrant. Conversely, observe that anything built from the initial **P**-algebra by pushouts along $\mathrm{Free}_{\mathbf{P}}(c \otimes (\partial\Delta[n] \to \Delta[n]))$ is in particular s-free.

Lemma 13.34 *A simplicial* **P**-*algebra A is cofibrant if and only if it is a retract of an s-free algebra.*

Remark 13.35 If A is s-free, then in particular it is 'degreewise free', in the sense that A_n is a free **P**-algebra. Thus, any cofibrant **P**-algebra is a retract of a degreewise free algebra.

Proof The 'if' direction is clear from the discussion above the lemma. For the converse, write 0 for the initial **P**-algebra (which is just the collection of nullary operations $\{\mathbf{P}(-;c)\}_{c\in S}$) and suppose A is cofibrant. We may factor the morphism $i\colon 0 \to A$ as a composition of $j\colon 0 \to B$ and $p\colon B \to A$, with j a composition of pushouts of morphisms of generating cofibrations $\mathrm{Free}_{\mathbf{P}}(c \otimes (\partial\Delta[n] \to \Delta[n]))$ and p a morphism with the right lifting property with respect to those. Then B is s-free and p is a trivial fibration. Lifting in the square

$$\begin{tikzcd} 0 \arrow[r, "j"] \arrow[d, tail, "i"] & B \arrow[d, two heads, "p"', "\wr"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

shows that A is a retract of B. □

Example 13.36 (a) Consider a category $\mathbf{C}$ and the associated category $\mathbf{sSets}^{\mathbf{C}^{\mathrm{op}}}$ of simplicial presheaves. Interpreting $\mathbf{C}^{\mathrm{op}}$ as an operad with only unary operations, the free algebra on an object c is precisely the representable functor $\mathbf{C}(-, c)$. Thus, Lemma 13.34 in particular implies that any projectively cofibrant simplicial presheaf on $\mathbf{C}$ is a retract of a presheaf that is degreewise a coproduct of representables. (Observe that the resolution in Proposition 10.30 thus provides an explicit cofibrant replacement of simplicial presheaves.)

(b) Consider the operad $\mathbf{P}_S$ of Example 13.32(b) for which the algebras are precisely categories with set of objects S. The set of colours of $\mathbf{P}_S$ is $S \times S$. A collection of sets $\{A_{c,d}\}$ indexed by $S \times S$ is precisely a directed graph with set of vertices S, with the elements of $A_{c,d}$ representing edges from c to d. The free $\mathbf{P}_S$-algebra on A is then the free category on that directed graph. Lemma 13.34 characterizes the cofibrant simplicial categories; in particular, it implies that any cofibrant simplicial category is a retract of a simplicial category $\mathbf{C}$ with the property that for every n the category $\mathbf{C}_n$ is free on a directed graph.

13.5 Algebras and Left Fibrations

The goal of this section is to establish the promised Quillen equivalence between the covariant model structure on the category $\mathbf{dSpaces}/N\mathbf{P}$ and the projective model structure on the category of simplicial $\mathbf{P}$-algebras, for $\mathbf{P}$ a simplicial operad. We will do this in Theorem 13.37. At the end of this section we include some first consequences, namely the facts that the absolute covariant model structure on $\mathbf{dSets}$ is a 'model' for the homotopy theory of $\mathbf{E}_\infty$-spaces and the Picard model structure 'models' the homotopy theory of infinite loop spaces. Many more results, such as the 'straightening-unstraightening equivalence' for left fibrations over ∞-categories, can be deduced from Theorem 13.37. However, we postpone a discussion of these to Section 14.8, when we have a better grasp of the adjoint pair $(w_!, w^*)$ relating dendroidal sets and simplicial operads.

Let $\mathbf{P}$ be an operad in $\mathbf{Sets}$. Recall from Section 9.5 that the nerve construction for algebras defines an adjoint pair

$$\mathbf{dSets}/N\mathbf{P} \underset{N(\mathbf{P},-)}{\overset{F_{\mathbf{P}}}{\rightleftarrows}} \mathrm{Alg}_{\mathbf{P}}.$$

Here the right adjoint sends a $\mathbf{P}$-algebra A in $\mathbf{Sets}$ to the left fibration of dendroidal sets $N(\mathbf{P}, A) \to N\mathbf{P}$. A T-dendrex of $N(\mathbf{P}, A)$ consists of a map $\xi \colon \Omega[T] \to N\mathbf{P}$ together with a labelling of each leaf l of T by an element of the set $A_{\xi(l)}$. The left adjoint $F_{\mathbf{P}}$ is characterized by the fact that it sends a representable $\xi \colon \Omega[T] \to N\mathbf{P}$ to the free $\mathbf{P}$-algebra generated by the leaves of T with their labelling by a colour of $\mathbf{P}$ provided by ξ.

Now if $\mathbf{P}$ is a *simplicial* operad and $\mathrm{Alg}_{\mathbf{P}}$ the category of $\mathbf{P}$-algebras in the category of simplicial sets, we may apply the construction of the previous paragraph levelwise to obtain an adjunction

$$\mathbf{dSpaces}/N\mathbf{P} \underset{N(\mathbf{P},-)}{\overset{F_{\mathbf{P}}}{\rightleftarrows}} \mathrm{Alg}_{\mathbf{P}}.$$

Here $N\mathbf{P}$ is now the dendroidal space defined by $(N\mathbf{P})_n = N(\mathbf{P}_n)$. The aim of this section is to establish the following:

Theorem 13.37 *Suppose* $\mathbf{P}$ *is a* Σ*-free simplicial operad. Then the adjoint pair* $(F_{\mathbf{P}}, N(\mathbf{P}, -))$ *described above is a Quillen equivalence between the projective covariant model structure on the category* $\mathbf{dSpaces}/N\mathbf{P}$ *and the projective model structure on the category* $\mathrm{Alg}_{\mathbf{P}}$ *of simplicial* $\mathbf{P}$*-algebras.*

The proof of the theorem will be a straightforward combination of the following three lemmas.

Lemma 13.38 *The functor*

$$N(\mathbf{P}, -)\colon \mathrm{Alg}_{\mathbf{P}} \to (\mathbf{dSpaces}_P/N\mathbf{P})_{\mathrm{cov}}$$

is right Quillen.

Proof It is clear that $N(\mathbf{P}, -)$ sends fibrations and weak equivalences in $\mathrm{Alg}_{\mathbf{P}}$ to fibrations and weak equivalences in $\mathbf{dSpaces}_P/N\mathbf{P}$, respectively. Thus, by Proposition 11.24, it suffices to verify that $N(\mathbf{P}, -)$ sends fibrant objects of $\mathrm{Alg}_{\mathbf{P}}$ to local objects for the Bousfield localization $(\mathbf{dSpaces}_P/N\mathbf{P})_{\mathrm{cov}}$ of $\mathbf{dSpaces}_P$. But this is precisely the content of Example 13.4. □

Lemma 13.39 *The functor*

$$N(\mathbf{P}, -)\colon \mathrm{Alg}_{\mathbf{P}} \to (\mathbf{dSpaces}_P/N\mathbf{P})_{\mathrm{cov}}$$

preserves and detects arbitrary weak equivalences.

Proof It is clear that a map $f\colon A \to B$ of $\mathbf{P}$-algebras is a weak equivalence if and only if $N(\mathbf{P}, f)$ is a weak equivalence with respect to the projective model structure on the category $\mathbf{dSpaces}/N\mathbf{P}$. Now pick a square in $\mathrm{Alg}_{\mathbf{P}}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ \widehat{A} & \xrightarrow{g} & \widehat{B}, \end{array}$$

where the vertical maps are weak equivalences and $\widehat{A}$ and $\widehat{B}$ are fibrant. Then f is a weak equivalence if and only if g is a weak equivalence; moreover, our previous observation implies that this is the case if and only if $N(\mathbf{P}, g)$ is a weak equivalence with respect to the projective model structure. But since this is a map between fibrant objects in the covariant model structure, this is the case if and only if $N(\mathbf{P}, g)$ is a weak equivalence in the covariant model structure by virtue of Lemma 8.49. Applying $N(\mathbf{P}, -)$ to the square above, we see that this is the case if and only if $N(\mathbf{P}, f)$ is a weak equivalence in the covariant model structure, completing the proof. □

The previous two lemmas do not need the hypothesis that $\mathbf{P}$ is Σ-free. This is only necessary for the following statement:

Lemma 13.40 *Let $X \xrightarrow{p} N\mathbf{P}$ be a cofibrant object of $\mathbf{dSpaces}_P/N\mathbf{P}$. Then the unit $X \to N(\mathbf{P}, F_{\mathbf{P}}(X))$ is a covariant weak equivalence of dendroidal spaces over $N\mathbf{P}$.*

Proof By Example 13.36(a), any projectively cofibrant dendroidal space X is a retract of a dendroidal space Y with the property that Y_n is a coproduct of representables $\Omega[T]$ for every n. Thus it suffices to prove the lemma for such Y. Moreover, by Corollary 13.24 it suffices to prove that for every n, the morphism $Y_n \to N(\mathbf{P}_n, F_{\mathbf{P}_n}(Y_n))$ is a covariant weak equivalence of dendroidal sets over $N\mathbf{P}_n$. This follows immediately from Corollary 9.68. □

We are now in a position to prove the promised result:

Proof (of Theorem 13.37) Suppose $X \to N\mathbf{P}$ is a cofibrant object of $\mathbf{dSpaces}_P/N\mathbf{P}$ and choose a fibrant replacement $F_{\mathbf{P}}(p) \to A$ in $\mathrm{Alg}_{\mathbf{P}}$. Then Lemmas 13.39 and 13.40 imply that the composite

$$X \to N(\mathbf{P}, F_{\mathbf{P}}(X)) \to N(\mathbf{P}, A)$$

is a weak equivalence. As a consequence, the derived unit $\mathrm{id} \to \mathbf{R}N(\mathbf{P}, -) \circ \mathbf{L}F_{\mathbf{P}}$ is an isomorphism. By Lemma 13.39, the assertion that the derived counit

$$\mathbf{L}F_{\mathbf{P}} \circ \mathbf{R}N(\mathbf{P}, -) \to \mathrm{id}$$

is also an isomorphism can be verified after postcomposing with $\mathbf{R}N(\mathbf{P}, -)$. But then the assertion follows from the triangle identity

$$\begin{tikzcd}
\mathbf{R}N(\mathbf{P}, -) \arrow[r, "\cong"] \arrow[dr, equal] & \mathbf{R}N(\mathbf{P}, -) \circ \mathbf{L}F_{\mathbf{P}} \circ \mathbf{R}N(\mathbf{P}, -) \arrow[d] \\
& \mathbf{R}N(\mathbf{P}, -)
\end{tikzcd}$$

and two-out-of-three. □

We conclude this section with some first consequences of Theorem 13.37.

Corollary 13.41 *Let $\mathbf{E}_\infty$ be any Σ-free simplicial operad equivalent to* **Com** *(such as the Barratt–Eccles operad). Then there exist zigzags of Quillen equivalences between the categories* **dSpaces**, **dSets** *(both equipped with the absolute covariant model structure) and the category $\mathrm{Alg}_{\mathbf{E}_\infty}$ of $\mathbf{E}_\infty$-spaces equipped with the projective model structure.*

Proof The comparison between the covariant model structures on the categories of dendroidal spaces and dendroidal sets is a consequence of Proposition 13.7. Theorem 13.37 provides a Quillen equivalence between the (projective) covariant model structure on $\mathbf{dSpaces}/N\mathbf{E}_\infty$ and $\mathrm{Alg}_{\mathbf{E}_\infty}$. Finally, the forgetful functor

$$(\mathbf{dSpaces}/N\mathbf{E}_\infty)_{\mathrm{cov}} \to \mathbf{dSpaces}_{\mathrm{cov}}$$

is the left adjoint in a Quillen equivalence by Remark 13.27. □

The fibrant objects of the absolute covariant model structure on the category of dendroidal sets are precisely the left dendroidal Kan complexes. We saw in Section 9.6 that for a left dendroidal Kan complex X, the category $\tau(i^*X)$ is naturally a symmetric monoidal groupoid. In particular, the set of connected components $\pi_0(i^*X)$ inherits the structure of a commutative monoid by identifying it with the set of isomorphism classes of objects in $\tau(i^*X)$. In Section 9.7 we introduced the Picard model structure on the category of dendroidal sets. It is the Bousfield localization of the (absolute) covariant model structure characterized by the fact that its fibrant objects are the dendroidal Kan complexes, i.e., those dendroidal sets having the extension property with respect to *all* horn inclusions $\Lambda^x[T] \to T$ of trees. In Proposition 9.86 we saw a convenient alternative description of these fibrant objects; indeed, if X is already fibrant with respect to the covariant model structure (i.e., is a dendroidal left Kan complex), then it is fibrant for the Picard model structure if every object of $\tau(i^*X)$ admits an inverse with respect to the tensor product. In other words, a dendroidal left Kan complex X is fibrant in the Picard model structure if the monoid $\pi_0(i^*X)$ is in fact a group.

To state the following, recall that an $\mathbf{E}_\infty$-space Y is called *grouplike* if the commutative monoid $\pi_0 Y$ is a group. Write $\mathrm{Alg}^{\mathrm{gp}}_{\mathbf{E}_\infty}$ for the full subcategory of the category of $\mathbf{E}_\infty$-spaces on the grouplike objects.

Corollary 13.42 *The zigzag of Quillen equivalences of the previous corollary induces an equivalence of homotopy categories*

$$\mathrm{Ho}(\mathbf{dSets}_{\mathrm{Picard}}) \simeq \mathrm{Ho}(\mathrm{Alg}^{\mathrm{gp}}_{\mathbf{E}_\infty}).$$

Proof The homotopy category $\mathrm{Ho}(\mathbf{dSets}_{\mathrm{Picard}})$ can be identified with the full subcategory of $\mathrm{Ho}(\mathbf{dSets}_{\mathrm{cov}})$ spanned by the dendroidal left Kan complexes X satisfying the condition above, i.e., for which the commutative monoid $\pi_0(i^*X)$ is a group. For a dendroidal space Z that is covariantly fibrant, the underlying bisimplicial set i^*Z is homotopically constant in both simplicial directions: in the original simplicial direction by completeness (cf. Corollary 12.19) and in the other one by the fact that for a linear tree T and its unique leaf l, the restriction $Z(T) \to Z(\eta_l)$ is a weak homotopy equivalence. In particular, i^*Z_0 and $Z(\eta)$ are weakly equivalent as simplicial sets, hence have the same set of connected components.

If Y is an $\mathbf{E}_\infty$-space, then clearly $\pi_0 Y$ is $\pi_0(N(\mathbf{E}_\infty, Y)(\eta))$. By the preceding discussion, this agrees with $i^*N(\mathbf{E}_\infty, Y)_0$, which is the set of connected components of the corresponding dendroidal set. This gives the identification of the subcategories of ‘grouplike’ objects on both sides. □

Remark 13.43 It is a classical result that the homotopy theory of grouplike $\mathbf{E}_\infty$-spaces is equivalent to that of connective spectra [112]. Thus the previous corollary shows that the Picard model structure on the category of dendroidal sets provides another model for the homotopy theory of connective spectra.

Historical Notes

While the covariant model structure on dendroidal sets goes back to [77], the covariant model structure on dendroidal spaces was first studied in [23]. Both references prove versions of the ‘homotopy invariance’ of the covariant model structure (Theorem 13.26). Our proof of this result is new and uses the notion of totalization of a simplicial system of model categories, reminiscent of the category of sheaves on a simplicial space as described by Deligne [48]. The projective model structure for algebras over a (coloured) simplicial operad is discussed in many places in the literature. The case of simplicial algebras over an ordinary operad already follows from Quillen’s work [123], but his arguments apply to the more general case as well; this is made explicit by Rezk [128]. Model structures for algebras over operads in a differential graded context were studied by Hinich [83]; for algebras and operads in more general model categories we refer to [16]. A comprehensive recent account, including an extensive bibliography of references on the subject, is given by Pavlov–Scholbach [121]. The relation between the model category of $\mathbf{P}$-algebras and the covariant model structure on the category of dendroidal spaces over $N\mathbf{P}$ has many precursors; we will discuss the relation to the literature in more detail in the historical notes of the next chapter, after we have proved corresponding results about the covariant model structure on slice categories of dendroidal sets, rather than spaces. The results of Section 13.5 first appeared in [23].

Chapter 14
Simplicial Operads and ∞-Operads

In this final chapter we fulfil one of the main promises of this book, namely we prove that the homotopy theory of ∞-operads is equivalent to that of simplicial (or topological) operads. To prepare for this, Sections 14.1 and 14.2 establish some rather classical material on the homotopy theory of simplicial categories, most of it going back to the work of Dwyer and Kan. Then in Section 14.3 we establish a model structure on the category of simplicial operads in which the weak equivalences are the fully faithful and essentially surjective maps (in an appropriate interpretation of those terms). When restricting to simplicial categories, thought of as simplicial operads with only unary operations, this model structure specializes to the well-known Bergner model structure.

In Section 14.5 we establish the first form of an equivalence between the homotopy theory of dendroidal spaces and that of simplicial operads. This requires us to work with a model structure on the category of dendroidal spaces that is slightly different from the ones we have considered before, which we establish in Section 14.4. Finally, in Section 14.6, we deduce from these results that the homotopy-coherent nerve functor w^* provides a Quillen equivalence from the category of simplicial operads to the category of dendroidal sets, equipped with the operadic model structure. As a corollary, we reproduce the important result that the homotopy-coherent nerve provides a Quillen equivalence from the category of simplicial categories to the Joyal model structure on the category of simplicial sets.

Many applications of the theory of ∞-operads concern operads with only a single colour and therefore we devote Section 14.7 to this special case. Several arguments and results simplify considerably there; in particular, the notion of completion can be circumvented. As such, the reader only interested in the case of a single colour could start reading this section immediately and refer back to others as needed. We include an application of the theory of dendroidal spaces that has turned out to be important in the literature, namely that the space of maps between simplicial operads $\mathbf{P}$ and $\mathbf{Q}$ with a single colour can equivalently be computed as the mapping space (with respect to the projective model structure) between the dendroidal spaces $N\mathbf{P}$ and $N\mathbf{Q}$, cf. Corollary 14.42.

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In the concluding Section 14.8 we record some further consequences of our results in this chapter and the previous one. In particular, we will see that for an ∞-operad X, there is an equivalence of homotopy theories between the category of algebras for the simplicial operad $w_!X$ and the covariant model structure on the category $\mathbf{dSets}/X$. This specializes to a version of Lurie's 'straightening-unstraightening' equivalence when X is a simplicial set.

14.1 Simplicial Categories with Fixed Objects

In this section we review some of the classical homotopy theory of simplicial categories. Fix a set O. We write $\mathbf{sCat}_O$ for the category of (small) simplicial categories with O as their set of objects and functors between them that are the identity on objects. In Example 13.32(b) we observed that $\mathbf{sCat}_O$ admits a model structure in which a functor $\varphi\colon \mathbf{C} \to \mathbf{D}$ is a weak equivalence (resp. a fibration) if

$$\varphi_{x,y}\colon \mathbf{C}(x, y) \to \mathbf{D}(x, y)$$

is a weak homotopy equivalence (resp. a Kan fibration) of simplicial sets, for every pair of objects $x, y \in O$. Let us fix some terminology for these classes of maps.

Definition 14.1 A functor $\varphi\colon \mathbf{C} \to \mathbf{D}$ between simplicial categories (not necessarily with fixed set of objects) is *fully faithful* if

$$\varphi_{x,y}\colon \mathbf{C}(x, y) \to \mathbf{D}(\varphi(x), \varphi(y))$$

is a weak homotopy equivalence for each pair of objects x, y of $\mathbf{C}$. We call φ a *local fibration* if each $\varphi_{x,y}$ is a Kan fibration.

Recall from Example 13.36(b) that a simplicial category $\mathbf{C}$ is cofibrant precisely if it is a retract of an s-free simplicial category $\mathbf{D}$, i.e., a simplicial category for which each $\mathbf{D}_n$ is free on a set of arrows G_n and these generating sets can be chosen so that $\alpha^*(G_m) \subseteq G_n$ for each surjective map $\alpha\colon [n] \to [m]$ in $\mathbf{\Delta}$. For later use we observe that the property of being cofibrant is preserved by 'restriction of objects'. To be precise, if $f\colon M \to O$ is an injective function between finite sets, then there is a functor

$$f^*\colon \mathbf{sCat}_O \to \mathbf{sCat}_M$$

sending a simplicial category with objects O to the full subcategory on the objects M (regarding M as a subset of O via f).

Lemma 14.2 *For any injective function f as above, the restriction f^* preserves cofibrant simplicial categories.*

Proof By the characterization of cofibrant objects quoted above it suffices to check this for s-free simplicial categories. Suppose $\mathbf{C}$ is the free category $F(G)$ on some directed graph G with vertices O. Then one can form a new graph $f^!G$ with vertices

M and edges from m to m' given by the finite strings

$$m \to o_1 \to \cdots \to o_n \to m'$$

with $n \geq 0$ and $o_i \in O - M$. (Here the case $n = 0$ just refers to edges $m \longrightarrow m'$ in G.) Then clearly $f^*\mathbf{C}$ is free on the graph $f^!G$. Applying this observation levelwise to an s-free simplicial category with objects O, we see that f^* of it is another s-free category. □

Recall the functor

$$\pi_0 \colon \mathbf{sSets} \to \mathbf{Sets}$$

sending a simplicial set X to its set $\pi_0 X$ of connected components. It is left adjoint to the functor assigning to a set the corresponding discrete simplicial set. Explicitly, $\pi_0 X$ is the colimit over $\boldsymbol{\Delta}^{\mathrm{op}}$ of X or (more efficiently) just the coequalizer of the two maps $d_0, d_1 \colon X_1 \longrightarrow X_0$. Since the functor π_0 preserves products, it defines a corresponding functor

$$\pi_0 \colon \mathbf{sCat} \to \mathbf{Cat}$$

from the category of simplicial categories to that of categories by applying π_0 to 'spaces of morphisms'. More precisely, any simplicial category $\mathbf{C}$ defines a category $\pi_0\mathbf{C}$ having the same set of objects as $\mathbf{C}$ and with

$$(\pi_0\mathbf{C})(c, d) := \pi_0(\mathbf{C}(c, d)).$$

The remainder of this section will be devoted to the following result of Dwyer and Kan. It can be paraphrased as saying that if all morphisms in a cofibrant simplicial category $\mathbf{C}$ are invertible up to homotopy, then actually inverting them does not affect the homotopy type of $\mathbf{C}$. We write $\mathbf{C}[\mathbf{C}^{-1}]$ for the localization of $\mathbf{C}$ at all of its morphisms: it is the simplicial category which in degree n is obtained from the category $\mathbf{C}_n$ by formally inverting all of its morphisms.

Theorem 14.3 *Let $\mathbf{C}$ be a cofibrant simplicial category for which $\pi_0\mathbf{C}$ is a groupoid. Then the localization functor $\mathbf{C} \to \mathbf{C}[\mathbf{C}^{-1}]$ is fully faithful, hence a weak equivalence of simplicial categories.*

Remark 14.4 It is possible to establish a more general result, where one inverts only a certain subset W of the morphisms in $\mathbf{C}$ under the weaker assumption that only the morphisms in W go to isomorphisms in $\pi_0\mathbf{C}$. We will not need this added generality here.

We prove the theorem at the end of this section, after establishing some preliminary lemmas. Recall that we may take the nerve of a simplicial category $\mathbf{C}$ to obtain a bisimplicial set $N\mathbf{C}$:

$$N\mathbf{C}_{p,q} = N(\mathbf{C}_p)_q.$$

For the duration of this section we denote the diagonal $\delta^* N\mathbf{C}$ by $B\mathbf{C}$ and refer to it as the *classifying space* of $\mathbf{C}$. It follows immediately from the fundamental property of bisimplicial sets (Corollary 10.27) that a fully faithful functor of simplicial categories $\mathbf{C} \to \mathbf{D}$ induces a weak homotopy equivalence of classifying spaces $B\mathbf{C} \to B\mathbf{D}$.

Lemma 14.5 *If* $\mathbf{C}$ *is a cofibrant simplicial category, then the localization* $\mathbf{C} \to \mathbf{C}[\mathbf{C}^{-1}]$ *induces a weak homotopy equivalence of classifying spaces* $B\mathbf{C} \simeq B(\mathbf{C}[\mathbf{C}^{-1}])$.

Proof By Corollary 10.27 it suffices to show that for each p, the map $N(\mathbf{C}_p) \to N(\mathbf{C}_p[\mathbf{C}_p^{-1}])$ is a weak homotopy equivalence of simplicial sets. We will argue that this works for any free category $\mathbf{D}$, generated by some graph G. That graph may equivalently be thought of as a simplicial set with vertices those of G and one non-degenerate 1-simplex for every edge of G. Viewing G as a simplicial set in this way, we have $\mathbf{D} = \tau(G)$. There is an evident inclusion $G \to N\mathbf{D}$, which we claim to be a weak homotopy equivalence. The lemma then follows from this. Indeed, a graph has the homotopy type of a disjoint union of wedges of circles, and in particular has vanishing higher homotopy groups π_n for $n > 1$ (at an arbitrary basepoint). It is well known (and rather easy to see) that for any category $\mathbf{C}$, the localization map $N\mathbf{C} \to N(\mathbf{C}[\mathbf{C}^{-1}])$ gives an isomorphism on fundamental groupoids and that moreover $N(\mathbf{C}[\mathbf{C}^{-1}])$ has vanishing higher homotopy groups, being a disjoint union of classifying spaces of discrete groups. In particular, if $N\mathbf{C}$ itself has vanishing homotopy groups in dimensions $n > 1$, then the localization map is a weak homotopy equivalence.

It remains to prove our claim. First observe that this is clear in the case where G has no edges at all (so that $G \to N\mathbf{D}$ is even an isomorphism). Then one proves the general case by induction on the edges of G. Indeed, if H is built from G by adjoining a single edge f, there are squares as follows, the right obtained from the left by applying $N\tau$:

$$\begin{tikzcd} \partial\Delta[1] \arrow[r] \arrow[d] & G \arrow[d] \\ \Delta[1] \arrow[r] & H, \end{tikzcd} \qquad \begin{tikzcd} \partial\Delta[1] \arrow[r] \arrow[d] & N\mathbf{D} \arrow[d] \\ \Delta[1] \arrow[r] & N(\mathbf{D}[f]). \end{tikzcd}$$

The one on the left is a pushout, whereas Proposition 5.25 states that on the right the map from the pushout $\Delta[1] \cup_{\partial\Delta[1]} N\mathbf{D}$ to $N(\mathbf{D}[f])$ is inner anodyne, so in particular a weak homotopy equivalence. The cube lemma and the inductive hypothesis on $\mathbf{D}$ then show that $H \to N\tau(H) = N(\mathbf{D}[f])$ is a weak homotopy equivalence as well. □

Lemma 14.6 *If* $\mathbf{C}$ *is a simplicial category for which* $\pi_0\mathbf{C}$ *is a groupoid, then the simplicial set* $\mathbf{C}(c, d)$ *is a model for the path space of* $B\mathbf{C}$ *from* c *to* d*, i.e., it is naturally weakly equivalent to the homotopy pullback of the maps*

$$\Delta[0] \xrightarrow{c} B\mathbf{C} \xleftarrow{d} \Delta[0].$$

Proof Consider the simplicial functor

$$\mathbf{C}(c, -)\colon \mathbf{C} \to \mathbf{sSets}\colon d \mapsto \mathbf{C}(c, d).$$

We may form its homotopy colimit (as in Section 10.6)

$$\operatorname{hocolim}_{\mathbf{C}} \mathbf{C}(c,-) = \delta^*(\operatorname{hocolim}_{\mathbf{C}_p} \mathbf{C}_p(c,-))$$

and obtain a projection map

$$\operatorname{hocolim} \mathbf{C}(c,-) \to B\mathbf{C}.$$

Observe that its fibre over a vertex $d \in B\mathbf{C}$ is precisely the simplicial set $\mathbf{C}(c,d)$. On the other hand, we claim that its homotopy fibre is the homotopy pullback described in the lemma. To see this, we first observe that the inclusion of the 'initial vertex'

$$\mathrm{id}_c \colon \Delta[0] \to \operatorname{hocolim} \mathbf{C}(c,-)$$

is a weak homotopy equivalence. Indeed, it can be obtained by taking the diagonal of a map of bisimplicial sets which in degree p is the inclusion $\mathrm{id}_c \colon \Delta[0] \to N(c/\mathbf{C}_p)$. Since id_c is initial in $c/\mathbf{C}_p$, the nerve of the latter category is weakly contractible. We conclude that the homotopy fibre of $\operatorname{hocolim} \mathbf{C}(c,-) \to B\mathbf{C}$ at d is equivalent to the homotopy fibre at d of the map

$$\Delta[0] \xrightarrow{c} B\mathbf{C}$$

as desired.

It remains to argue that the fibre and homotopy fibre of the map $\operatorname{hocolim} \mathbf{C}(c,-) \to B\mathbf{C}$ agree up to weak homotopy equivalence. The assumption that $\pi_0\mathbf{C}$ is a groupoid implies that any morphism $f \colon d \to e$ in $\mathbf{C}_0$ induces a weak homotopy equivalence $f_* \colon \mathbf{C}(c,d) \to \mathbf{C}(c,e)$. Thus, the functor $\mathbf{C}(c,-)$ satisfies the hypothesis of Proposition 10.36, concluding the proof. □

Proof (of Theorem 14.3) Suppose $\mathbf{C}$ is a cofibrant simplicial category with $\pi_0\mathbf{C}$ is a groupoid. We should argue that for any objects c and d of $\mathbf{C}$, the map

$$\mathbf{C}(c,d) \to \mathbf{C}[\mathbf{C}^{-1}](c,d)$$

is a weak homotopy equivalence of simplicial sets. By Lemma 14.6 it suffices to show that the map of classifying spaces $B\mathbf{C} \to B(\mathbf{C}[\mathbf{C}^{-1}])$ is a weak homotopy equivalence, which is precisely the statement of Lemma 14.5. □

14.2 Equivalences in Simplicial Categories

In Corollary 5.52 we proved that an edge $f \colon \Delta[1] \to X$ of an ∞-category X is an equivalence if and only if f extends along the inclusion $\Delta[1] \to J$. In this section we establish an analog of this fact for simplicial categories, which is one of the key ingredients in establishing the Bergner model structure for simplicial categories (cf. Corollary 14.13). We will use this analog in the next section to provide a model structure on the category of simplicial operads. We begin by specifying the relevant version of the 'interval' J.

Definition 14.7 Let $O = \{0, 1\}$. A *categorical interval* is a simplicial category $\mathbf{E}$ with objects O such that:

(a) $\mathbf{E}$ is cofibrant in $\mathbf{sCat}_O$,
(b) the category $\pi_0\mathbf{E}$ is the free isomorphism $(0 \cong 1)$, and
(c) the functor $\mathbf{E} \to \pi_0\mathbf{E}$ is a local weak equivalence (so all mapping spaces in $\mathbf{E}$ are weakly contractible).

A morphism $f : c \to d$ in a simplicial category $\mathbf{C}$ is called an *equivalence* if the image of f in $\pi_0\mathbf{C}$ is an isomorphism. Writing $[1]$ for the 'free morphism' category $(0 \to 1)$, the main result of this section can be formulated as follows.

Theorem 14.8 *A morphism $f : c \to d$ in a simplicial category $\mathbf{C}$ is an equivalence if and only if the corresponding functor $[1] \to \mathbf{C}$ extends to a functor $\mathbf{E} \to \mathbf{C}$ from a categorical interval.*

Proof Without loss of generality we may assume that the set of objects of $\mathbf{C}$ is $O = \{0, 1\}$; indeed, we can pull back a general $\mathbf{C}$ along the map of objects $\{0, 1\} \to \mathrm{ob}(\mathbf{C})$ induced by f. Also, we may assume that $\pi_0\mathbf{C}$ is a groupoid, simply by restricting to those components of the mapping spaces $\mathbf{C}(c, d)$ corresponding to isomorphisms in $\pi_0\mathbf{C}$ and observing that f can only map into such components. Finally, we may replace $\mathbf{C}$ by a cofibrant simplicial category. Indeed, since the category $[1] = (0 \to 1)$ is cofibrant in $\mathbf{sCat}_O$, the map f will factor through such a replacement.

Write I for the free isomorphism $(0 \cong 1)$. Then the functor $f : [1] \to \mathbf{C}$ gives a functor $I \to \mathbf{C}[\mathbf{C}^{-1}]$, which we may factor as a local trivial cofibration followed by a local fibration:

$$I \overset{\sim}{\rightarrowtail} I_f \twoheadrightarrow \mathbf{C}[\mathbf{C}^{-1}].$$

Then f factors through the pullback $\mathbf{D}$ in the following square:

$$\begin{tikzcd} \mathbf{D} \arrow[r] \arrow[d] & I_f \arrow[d, two heads] \\ \mathbf{C} \arrow[r] & \mathbf{C}[\mathbf{C}^{-1}]. \end{tikzcd}$$

The bottom arrow is fully faithful by Theorem 14.3; since $\mathbf{sCat}_O$ is a right proper model category, the top arrow is fully faithful as well. It follows that $\mathbf{D}$ satisfies requirements (b) and (c) of Definition 14.7. Simply taking a cofibrant replacement $\widetilde{\mathbf{D}} \to \mathbf{D}$ now provides a categorical interval through which f factors. □

In the next section we will have to construct a certain *set* of generating trivial cofibrations using categorical intervals. For this purpose it will be useful to know that one can bound the cardinality of the intervals $\mathbf{E}$ appearing in Theorem 14.8:

Proposition 14.9 *In the statement of Theorem 14.8 the categorical interval $\mathbf{E}$ can be arranged to be countable, meaning a simplicial category with countably many morphisms in each simplicial degree.*

The proof of the proposition relies on the following observation:

Lemma 14.10 *Let $f\colon A \to X$ be a map of simplicial sets with A countable and X weakly contractible. Then there exists a factorization of f as in*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, tail, "i"'] & & X \\ & B \arrow[ur] & \end{tikzcd}$$

in which i is a monomorphism and B is a simplicial set that is both countable and weakly contractible.

Proof This is essentially a special case of Lemma 8.14, or alternatively of our arguments concerning the 'countable approximation property' in Section 11.5. Since those arguments simplify considerably in this case, let us describe a direct proof here for the reader's convenience. Factor f as a monomorphism $i\colon A \to Y$ followed by a trivial fibration $p\colon Y \to X$. Assume we have a functorial anodyne map $Z \to E(Z)$ for any simplicial set Z, such that E preserves monomorphisms and $E(Z)$ is countable whenever Z is countable. (Such an E can always be produced abstractly from the small object argument as in Lemma 8.25, or one takes Kan's Ex^∞-functor.) Then in particular we have the square

$$\begin{tikzcd} A \arrow[r, tail, "\sim"] \arrow[d, tail] & E(A) \arrow[d, tail] \\ Y \arrow[r, tail, "\sim"] & E(Y). \end{tikzcd}$$

Since $E(Y)$ is a contractible Kan complex there exists a homotopy

$$h\colon \Delta[1] \times E(Y) \to E(Y)$$

with $h_0 = \mathrm{id}_{E(Y)}$ and h_1 a constant map. We will inductively define countable simplicial subsets of Y, starting from $B_0 := i(A)$,

$$i(A) = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots .$$

Assuming B_i has been defined, we construct B_{i+1} in the following way. Consider the simplicial subset $C_0 := E(B_i)$ of $E(Y)$. Inductively define $C_j := h(\Delta[1] \times C_{j-1})$ for all $j \geq 1$. Then $C_\infty := \bigcup_j C_j$ is still countable and h restricts to a contracting homotopy

$$h\colon \Delta[1] \times C_\infty \to C_\infty.$$

Now let B_{i+1} be any countable simplicial subset of Y such that B_{i+1} contains B_i and $E(B_{i+1})$ contains C_∞ (cf. Lemma 8.25). Note that the inclusion $E(B_i) \to E(B_{i+1})$ is nullhomotopic, since it factors through C_∞. It follows that $\bigcup_i E(B_i)$ is contractible. Therefore $B := \bigcup_i B_i$ is weakly contractible (and still countable), proving the lemma. $\square$

Proof (of Proposition 14.9) Suppose that $\mathbf{E}$ is a categorical interval. We will show that the inclusion of objects $O = \{0, 1\} \to \mathbf{E}$ can be factored as a cofibration $O \to \mathbf{D}$ followed by a fully faithful functor $\mathbf{D} \to \mathbf{E}$ with $\mathbf{D}$ a countable categorical interval. To do this, we inductively construct cofibrant countable simplicial categories $\mathbf{D}^{(n)}$ with functors as in the following diagram:

$$\begin{tikzcd}
\mathbf{D}^{(-1)} := O \arrow[r, tail] \arrow[d] & \mathbf{D}^{(0)} \arrow[r, tail] \arrow[dl] & \mathbf{D}^{(1)} \arrow[r, tail] \arrow[dll] & \mathbf{D}^{(2)} \arrow[r, tail] \arrow[dlll] & \cdots \\
\mathbf{E}. & & & &
\end{tikzcd}$$

As indicated in the diagram, we start our construction by setting $\mathbf{D}^{(-1)} := O$. Now suppose $n \geq 0$ and $\mathbf{D}^{(n-1)}$ has been constructed. Let $i, j \in O$ and consider the corresponding mapping space $A_{ij} = \mathbf{D}^{(n-1)}(i, j)$. Applying Lemma 14.10 to the map $A_{ij} \to \mathbf{E}(i, j)$ yields a factorization $A_{ij} \to B_{ij} \to \mathbf{E}(i, j)$ with B_{ij} countable and weakly contractible. Now construct $\mathbf{D}^{(n)}$ by forming the pushout

$$\begin{tikzcd}
\coprod_{(i,j)} C_1[A_{ij}] \arrow[r] \arrow[d, tail] & \mathbf{D}^{(n-1)} \arrow[d, tail] \\
\coprod_{(i,j)} C_1[B_{ij}] \arrow[r] & \mathbf{D}^{(n)}.
\end{tikzcd}$$

Here $C_1[A_{ij}]$ denotes the simplicial category with objects i, j, and nonidentity morphisms A_{ij} from i to j. Finally, define $\mathbf{D} := \varinjlim_n \mathbf{D}^{(n)}$. Then $\mathbf{D}$ is clearly cofibrant and countable; it remains to show that its mapping spaces are weakly contractible. But observe that for each n, the map

$$\mathbf{D}^{(n-1)}(i, j) \to \mathbf{D}^{(n)}(i, j)$$

factors through the weakly contractible space B_{ij} by construction. Thus

$$\mathbf{D}(i, j) = \varinjlim_n \mathbf{D}^{(n)}(i, j)$$

is weakly contractible as well. □

We conclude this section with a useful observation on isofibrations of simplicial categories. Recall that an isofibration of categories is a functor $\varphi\colon \mathbf{C} \to \mathbf{D}$ such that for any object c of $\mathbf{C}$ and any isomorphism $g\colon \varphi(c) \to d$ in $\mathbf{D}$, there exists an isomorphism $f\colon c \to \overline{d}$ in $\mathbf{C}$ with $\varphi(f) = g$. We will say that a functor φ of *simplicial* categories is an isofibration if $\pi_0\varphi$ is an isofibration between ordinary categories.

Proposition 14.11 *A functor $\varphi\colon \mathbf{C} \to \mathbf{D}$ between simplicial categories is an isofibration whenever it has the right lifting property with respect to the maps $\{0\} \to \mathbf{E}$, where $\mathbf{E}$ ranges over the set of countable categorical intervals. The converse is also true if φ is assumed to be a local fibration.*

Proof Suppose that φ has the stated lifting property. To see that it is an isofibration, suppose that c is an object of $\mathbf{C}$ and $g\colon \varphi(c) \to d$ is an isomorphism in $\pi_0\mathbf{D}$. Then Theorem 14.8 and Proposition 14.9 imply that g can be extended to a functor from a countable categorical interval $\mathbf{E}$ to $\mathbf{D}$. Lifting this to a functor $\mathbf{E} \to \mathbf{C}$ in particular determines an isomorphism f in $\pi_0\mathbf{C}$ with $\varphi(f) = g$.

For the converse, assume that φ is an isofibration and a local fibration. Consider a lifting problem of the form

$$\begin{tikzcd} \{0\} \arrow[r, "c"] \arrow[d] & \mathbf{C} \arrow[d, "\varphi"] \\ \mathbf{E} \arrow[r] \arrow[ur, dashed] & \mathbf{D}. \end{tikzcd}$$

The bottom horizontal arrow in particular gives an equivalence $g\colon \varphi(c) \to d$ in $\mathbf{D}$. By assumption there exists a morphism $f\colon c \to \overline{d}$ in $\mathbf{C}$ such that g and $\varphi(f)$ have the same image in $\pi_0\mathbf{D}$. In other words, g and $\varphi(f)$ are in the same path component of the simplicial set $\mathbf{D}(\varphi(c), d)$. Since $\mathbf{C}(c, \overline{d}) \to \mathbf{D}(\varphi(c), d)$ is a Kan fibration, there must then also exist a vertex $f' \in \mathbf{C}(c, \overline{d})$ with $\varphi(f') = g$. Thus we find a commutative diagram

$$\begin{tikzcd} {[1]} \arrow[r, "f'"] \arrow[d] & \mathbf{C} \arrow[d, "\varphi"] \\ \mathbf{E} \arrow[r] & \mathbf{D} \end{tikzcd}$$

and it suffices to show that there exists a lift in this square. Without loss of generality we may assume that the sets of objects of $\mathbf{C}$ and $\mathbf{D}$ are $O = \{0, 1\}$ and φ is the identity on objects; indeed, if not, we can always pull back $\mathbf{C}$ and $\mathbf{D}$ to the set of objects of $\mathbf{E}$. Thus we may consider the square above as a lifting problem in $\mathbf{sCat}_O$. Writing $\mathbf{P}$ for the pullback in the square, the resulting map $[1] \to \mathbf{P}$ extends over a categorical interval $\mathbf{F}$ by Theorem 14.8. Now observe that the functor $\mathbf{F} \to \mathbf{E}$ is a weak equivalence (since both are categorical intervals) and factor it as a trivial cofibration $\mathbf{F} \to \mathbf{G}$ followed by a local trivial fibration $\mathbf{G} \to \mathbf{E}$. Lifting in the two squares

$$\begin{tikzcd} \mathbf{F} \arrow[r] \arrow[d, tail, "\wr"] & \mathbf{C} \arrow[d, two heads, "\varphi"] \\ \mathbf{G} \arrow[r] \arrow[ur, dashed, "k"] & \mathbf{D} \end{tikzcd} \qquad \begin{tikzcd} {[1]} \arrow[r] \arrow[d, tail] & \mathbf{G} \arrow[d, two heads, "\wr"] \\ \mathbf{E} \arrow[r, equal] \arrow[ur, dashed, "l"] & \mathbf{E} \end{tikzcd}$$

provides a functor $kl\colon \mathbf{E} \to \mathbf{C}$ solving our previous lifting problem. ◻

14.3 A Model Structure for Simplicial Operads

The aim of this section is to construct a model structure on the category **sOp** of simplicial operads in which the weak equivalences are the *fully faithful* and *essentially surjective* maps. Here a map $\varphi\colon \mathbf{P} \to \mathbf{Q}$ of simplicial operads is said to be fully faithful if for every tuple of colours $(c_1, \ldots, c_n, d)$ of $\mathbf{P}$, the map

$$\mathbf{P}(c_1, \ldots, c_n; d) \to \mathbf{Q}(\varphi(c_1), \ldots, \varphi(c_n); \varphi(d))$$

is a weak homotopy equivalence of simplicial sets. The map φ is called essentially surjective if the map $\pi_0\varphi$ is essentially surjective as a map between operads in sets. By definition, this means that the corresponding functor between underlying categories (obtained by restricting $\pi_0\mathbf{P}$ and $\pi_0\mathbf{Q}$ to unary operations only) is essentially surjective.

Before stating the main result of this section we introduce some more terminology. In analogy with the case of simplicial categories from the preceding sections, we say that a map $\varphi\colon \mathbf{P} \to \mathbf{Q}$ of simplicial operads is a *local fibration* if each of the maps

$$\mathbf{P}(c_1, \ldots, c_n; d) \to \mathbf{Q}(\varphi(c_1), \ldots, \varphi(c_n); \varphi(d))$$

is a Kan fibration of simplicial sets. We will say that φ is an *isofibration* if the corresponding functor of underlying simplicial categories is an isofibration. Recall that this means that for any object p of $\pi_0\mathbf{P}$ and isomorphism $g\colon \varphi(p) \to q$ in $\pi_0\mathbf{Q}$, there exists an isomorphism f in $\pi_0\mathbf{P}$ with $\varphi(f) = g$.

Theorem 14.12 *There exists a model structure on the category* **sOp** *of simplicial operads as follows:*

(1) *The weak equivalences are the fully faithful and essentially surjective maps of simplicial operads.*
(2) *The fibrations are the maps of simplicial operads that are both local fibrations and isofibrations.*

The cofibrations are those maps having the left lifting property with respect to the trivial fibrations.

The theorem in particular yields a model structure on the category **sCat** of simplicial categories by regarding it as the slice category of **sOp** over the trivial operad. This is known as the *Bergner model structure*:

Corollary 14.13 *There exists a model structure on the category* **sCat** *of simplicial categories in which the weak equivalences are the fully faithful and essentially surjective functors and the fibrations are those functors that are both local fibrations and isofibrations.*

For a simplicial set A and $n \geq 0$ we write $\tau(C_n)[A]$ for the simplicial operad with colours $\{0, 1, \ldots, n\}$ and

$$\tau(C_n)[A](1, \ldots, n; 0) = A$$

as its only nontrivial space of operations. This operad can be thought of as (τ applied to) the corolla C_n with leaves $1, \ldots, n$, root 0, and unique vertex labelled by A. By slight abuse of notation we will also write η for the trivial operad with one colour and no non-identity operations; it would be more precise to write $\tau(\eta)$, but it should always be clear whether we mean an operad or a dendroidal set.

To prove Theorem 14.12 we introduce two classes of maps. First, let $\mathcal{C}$ be the set consisting of the following two kinds of morphisms:

(C1) The morphisms $\tau(C_m)[\partial\Delta[n]] \to \tau(C_m)[\Delta[n]]$ for $m, n \geq 0$.
(C2) The morphism $\varnothing \to \eta$.

The elements of $\mathcal{C}$ will serve as generating cofibrations for the model structure of Theorem 14.12. Secondly, write $\mathcal{A}$ for the set consisting of the following morphisms:

(A1) The maps $\tau(C_m)[\Lambda^k[n]] \to \tau(C_m)[\Delta[n]]$ for $n \geq 1$, $m \geq 0$, and $0 \leq k \leq n$.
(A2) All the maps $\eta \to \mathbf{E}$, where $\mathbf{E}$ ranges over the countable categorical intervals (in the sense of Definition 14.7), thought of as simplicial operads with only unary operations.

These will play the role of generating trivial cofibrations. Indeed, we have the following characterizations:

Lemma 14.14 *A map of simplicial operads is a fibration (resp. a fibration and a weak equivalence) if and only if it has the right lifting property with respect to all the maps in $\mathcal{A}$ (resp. all the maps in $\mathcal{C}$).*

Proof Let $\varphi \colon \mathbf{P} \to \mathbf{Q}$ be a map of simplicial operads. Clearly φ is a local fibration if and only if it has the right lifting property with respect to the maps (A1). Proposition 14.11 then implies that it is also an isofibration if and only if it has the right lifting property with respect to (A2).

If φ has the right lifting property with respect to (C1) and (C2) then it is a fully faithful local fibration and surjective on objects. But clearly any fully faithful functor that is surjective on objects is an isofibration, so φ is a fibration and a weak equivalence. Conversely, if φ is a fibration and a weak equivalence then in particular it is a local trivial fibration, hence has the right lifting property with respect to (C1). An essentially surjective isofibration is actually surjective on objects, so that φ also has the right lifting property with respect to (C2). □

To prove Theorem 14.12 we will also need the following observation:

Lemma 14.15 *Any morphism in the saturation of the set $\mathcal{A}$ is a weak equivalence of simplicial operads.*

Proof Clearly it will suffice to show that any pushout of a map in (A1) or (A2) is a weak equivalence of simplicial operads. For (A1) we can again apply the path object argument used in the proof of Lemma 13.31. Indeed, any product-preserving fibrant replacement E of simplicial sets allows us to construct a natural map $\mathbf{P} \to E\mathbf{P}$, for any simplicial operad $\mathbf{P}$, which is a weak equivalence and so that $E\mathbf{P}$ is locally fibrant, meaning all of the simplicial sets $E\mathbf{P}(c_1, \ldots, c_n; d)$ are Kan complexes. Now if $\varphi \colon \mathbf{P} \to \mathbf{Q}$ is a pushout of a morphism in (A1), one forms a square

$$\begin{tikzcd}
\mathbf{P} \arrow[r, "\sim"] \arrow[d, "\varphi"] & E\mathbf{P} \times_{E\mathbf{Q}} E\mathbf{Q}^{\Delta[1]} \arrow[d] \\
\mathbf{Q} \arrow[r, "\sim"] & E\mathbf{Q}
\end{tikzcd}$$

just as in the proof of Lemma 13.31. The right vertical map is a local fibration (by the same argument as before) and hence a lift in the square exists. Then all maps in the square are weak equivalences by the two-out-of-six property (cf. Proposition 7.35).

Now consider a morphism $\eta \to \mathbf{E}$ in (A2). Without loss of generality we may assume that it picks out the object 0 of $\mathbf{E}$. Write $\mathbf{E}_0$ for the full subcategory of $\mathbf{E}$ on the object 0. Then for any pushout square of simplicial operads

$$\begin{tikzcd}
\eta \arrow[r] \arrow[d] & \mathbf{P} \arrow[d] \\
\mathbf{E} \arrow[r] & \mathbf{Q}
\end{tikzcd}$$

we may factor $\mathbf{P} \to \mathbf{Q}$ as a composition of morphisms $\mathbf{P} \to \mathbf{Q}_0 \to \mathbf{Q}$, where the first map is the pushout along $\eta \to \mathbf{P}$ of the morphism $\eta \to \mathbf{E}_0$. That morphism is a cofibration in $\mathbf{sCat}_{\{0\}}$ by Lemma 14.2; it is also fully faithful, hence a trivial cofibration. It is therefore in the saturation of the set of morphisms of type (A1). In particular, $\mathbf{P} \to \mathbf{Q}_0$ is a weak equivalence of simplicial operads by what we have already proved in the previous paragraph. It remains to deal with the morphism $\mathbf{Q}_0 \to \mathbf{Q}$. Observe that for any colours $c_1, \ldots, c_n, d$ of $\mathbf{Q}_0$, the map

$$\mathbf{Q}_0(c_1, \ldots, c_n; d) \to \mathbf{Q}(c_1, \ldots, c_n; d)$$

is an isomorphism. (This is a consequence of the fact that for operads in sets, the pushout of a fully faithful morphism along a map injective on colours is again fully faithful, as the reader may easily verify.) In particular, $\mathbf{Q}_0 \to \mathbf{Q}$ is fully faithful. It is also essentially surjective; indeed, the only colour not in the image corresponds to the object 1 of $\mathbf{E}$, but in $\pi_0\mathbf{Q}$ this object is connected to 0 by an isomorphism in the groupoid $\pi_0\mathbf{E}$. □

Establishing the promised model structure on the category of simplicial operads is now an easy task:

Proof (of Theorem 14.12) Axioms (M1–3) are easily checked. The factorization axiom (M5) follows by applying the small object argument to the classes $\mathcal{C}$ and $\mathcal{A}$. Indeed, Lemma 14.14 implies that with respect to $\mathcal{C}$ one obtains a factorization into a cofibration followed by a trivial fibration; with respect to $\mathcal{A}$ one obtains a factorization into a trivial cofibration (cf. Lemma 14.15) followed by a fibration. Finally there is the lifting axiom (M4). The cofibrations have the left lifting property with respect to trivial fibrations by definition. To see that trivial cofibrations have the left lifting property with respect to fibrations, one applies the retract argument again. Indeed, if $i\colon \mathbf{P} \to \mathbf{Q}$ is a trivial cofibration, it can be factored as a map $j\colon \mathbf{P} \to \mathbf{R}$ in the saturation of $\mathcal{A}$ (which is a weak equivalence by Lemma 14.15) followed by a fibration $p\colon \mathbf{R} \to \mathbf{Q}$. The latter is a weak equivalence by two-out-of-three, so a trivial fibration. Then there exists a lift in the square

$$\begin{tikzcd} \mathbf{P} \arrow[r, "j"] \arrow[d, tail, "i"] & \mathbf{R} \arrow[d, two heads, "p"', "\wr"] \\ \mathbf{Q} \arrow[r, equal] & \mathbf{Q} \end{tikzcd}$$

exhibiting i as a retract of j. But j has the left lifting property with respect to fibrations by Lemma 14.14, so that i has this lifting property as well. □

14.4 The Sparse Model Structure

Consider a simplicial operad $\mathbf{P}$. It can be thought of as a simplicial object $[n] \mapsto \mathbf{P}_n$ in the category of operads with the property that the sets of colours $\mathrm{col}(\mathbf{P}_n)$ form a constant simplicial set. Taking the nerve levelwise defines a dendroidal space with n-simplices $N(\mathbf{P}_n)$. As we have seen before this defines a functor

$$N\colon \mathbf{sOp} \to \mathbf{dSpaces}\colon \mathbf{P} \mapsto N\mathbf{P}.$$

Like the usual nerve functor, it admits a left adjoint τ. However, this does slightly more than just to apply τ levelwise; indeed, for a general dendroidal space X there is no reason for the simplicial set $\mathrm{col}(\tau(X_n))$ to be constant. Rather, one can first construct a dendroidal space $\widetilde{X}$ as the pushout in the following diagram:

$$\begin{tikzcd} X(\eta) \arrow[r] \arrow[d] & X \arrow[d] \\ \pi_0 X(\eta) \arrow[r] & \widetilde{X}. \end{tikzcd}$$

Here the set $\pi_0 X(\eta)$ is to be interpreted as a constant simplicial set. The object $\widetilde{X}$ is the universal way of replacing X by a dendroidal space with $\widetilde{X}(\eta)$ a constant simplicial set. One then easily deduces that $\tau(X)$ is the simplicial operad with n-simplices

$$\tau(X)_n = \tau(\widetilde{X}_n).$$

In the next section we will prove that the pair (τ, N) gives an equivalence of homotopy theories between simplicial operads and complete dendroidal Segal spaces. However, the functors τ and N do not form a Quillen adjunction with respect to any of the model structures we have considered thus far. Indeed, τ sends the projective cofibration

$$C_n \boxtimes \partial\Delta[1] \to C_n \boxtimes \Delta[1]$$

to the morphism

$$\tau(C_n) \amalg \tau(C_n) \to \tau(C_n)[\Delta[1]]$$

of simplicial operads. The latter is not injective on colours, so cannot be a cofibration. Therefore τ cannot be left Quillen with respect to the projective model structure, nor with respect to the Reedy one (which has more cofibrations). To circumvent this problem we will introduce another structure on the category of dendroidal spaces, called the *sparse model structure*. It has the same weak equivalences as the model category $\mathbf{dSpaces}_{RSC}$ for complete dendroidal Segal spaces, but fewer cofibrations. To state our result, write $\mathcal{C}$ for the set consisting of the following kinds of maps:

(C1) The maps $\varnothing \to \eta \boxtimes \Delta[n]$ for all $n \geq 0$.
(C2) The maps

$$T \boxtimes \partial\Delta[n] \cup_{E(T)\boxtimes\partial\Delta[n]} E(T) \boxtimes \Delta[n] \to T \boxtimes \Delta[n]$$

for all trees T and $n \geq 0$.

Here $E(T)$ denotes the set of edges of T, thought of as the discrete dendroidal set consisting as a coproduct of copies of η. We call a map of dendroidal spaces a *sparse cofibration* if it lies in the saturation of $\mathcal{C}$, i.e., if it can be obtained as a retract of a transfinite composition of pushouts of maps of the kinds (C1) and (C2). A more direct description is as follows:

Lemma 14.16 (i) *A map $X \to Y$ of dendroidal spaces has the right lifting property with respect to sparse cofibrations if and only if the map of simplicial sets $X(\eta) \to Y(\eta)$ is surjective in each simplicial degree and for each tree T the map*

$$X(T) \to Y(T) \times_{Y(E(T))} X(E(T))$$

is a trivial fibration of simplicial sets.
(ii) *Dually, a map of $A \to B$ of dendroidal spaces is a sparse cofibration if and only if the simplicial set $B(\eta) - A(\eta)$ is a disjoint union of representables and the map*

$$A \amalg_{\eta\boxtimes A(\eta)} \eta \boxtimes B(\eta) \to B$$

is a projective cofibration of dendroidal spaces.

Proof From the generating sets (C1) and (C2) above, the claim about $X \to Y$ is clear. The set of maps $A \to B$ described in the lemma is precisely the set of maps having the left lifting property with respect to those $X \to Y$ satisfying the two properties described in the lemma and is therefore the saturation of $\mathcal{C}$. □

The following is the main result of this section, establishing the existence of the sparse model structure (which we will denote by $\mathbf{dSpaces}_{sp}$):

Theorem 14.17 *There exists a model structure on the category of dendroidal spaces in which the cofibrations are the sparse cofibrations and the weak equivalences are the complete weak equivalences, i.e., the weak equivalences in the model category* $\mathbf{dSpaces}_{RSC}$ *for complete dendroidal Segal spaces. The identity functor defines a Quillen equivalence*

$$\mathbf{dSpaces}_{sp} \underset{\mathrm{id}^*}{\overset{\mathrm{id}_!}{\rightleftarrows}} \mathbf{dSpaces}_{RSC}.$$

We will deduce the theorem from the following two lemmas:

Lemma 14.18 *Let f be a map of dendroidal spaces having the right lifting property with respect to sparse cofibrations. Then f is a complete weak equivalence.*

We call a map between dendroidal spaces a *sparse trivial cofibration* if it is a sparse cofibration and a complete weak equivalence.

Lemma 14.19 *The class of sparse trivial cofibrations is saturated. Moreover, any sparse trivial cofibration is a retract of a transfinite composition of pushouts of sparse trivial cofibrations between countable dendroidal spaces.*

Let us first show how to deduce Theorem 14.17. The remainder of this section will then be devoted to proving the two lemmas.

Proof (of Theorem 14.17) The only nontrivial verifications are the factorization axiom (M5) and the lifting axiom (M4). Applying the small object argument to the set of maps $\mathcal{C}$ provides a factorization into a sparse cofibration followed by a map with the right lifting property with respect to sparse cofibrations, which is then a trivial fibration by Lemma 14.18. Lemma 14.19 guarantees that the trivial cofibrations are generated (as a saturated class) by a set, namely that of trivial cofibrations between countable dendroidal spaces (or rather a set of representatives of isomorphism classes of such). Applying the small object argument to this set then gives a factorization into a trivial cofibration followed by a fibration. For the lifting axiom (M4) we follow a familiar pattern: fibrations have the right lifting property with respect to trivial cofibrations by definition, whereas trivial fibrations have the right lifting property with respect to cofibrations by the familiar retract argument.

It remains to argue that the pair $(\mathrm{id}_!, \mathrm{id}^*)$ defines a Quillen equivalence between $\mathbf{dSpaces}_{sp}$ and $\mathbf{dSpaces}_{RSC}$. Observe that every sparse cofibration is in particular a Reedy cofibration and that the weak equivalences of these two model structures coincide (by definition), so that the identity functor indeed defines a left Quillen functor $\mathrm{id}_!$ from $\mathbf{dSpaces}_{sp}$ to $\mathbf{dSpaces}_{RSC}$. Since these model structures share the same weak equivalences, it is evident that $\mathrm{id}_!$ induces an equivalence of homotopy categories and is thus a Quillen equivalence. □

Proof (of Lemma 14.18) Suppose $f\colon X \to Y$ has the right lifting property with respect to sparse cofibrations. Write

$$f^*Y(T) := Y(T) \times_{Y(E(T))} X(E(T)),$$

so that f factors as a composition

$$X \to f^*Y \to Y.$$

The first map is a weak equivalence of dendroidal spaces (even in the Reedy model structure $\mathbf{dSpaces}_R$) by Lemma 14.16. To see that the second map is a complete weak equivalence it suffices to prove that for every n, the map $f^*Y_n \to Y_n$ is a trivial fibration of dendroidal sets (cf. Proposition 12.29). This map is a pullback of $E^*X_n \to E^*Y_n$, where

$$E^*X_n(T) := X(E(T))_n$$

and similarly for Y. Now a lifting problem as on the left is equivalent to one as on the right in the following diagram:

$$\begin{tikzcd} \partial T \arrow[r] \arrow[d] & E^*X_n \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & E^*Y_n, \end{tikzcd} \qquad \begin{tikzcd} E(\partial T) \arrow[r] \arrow[d] & X_n \arrow[d] \\ E(T) \arrow[r] \arrow[ur, dashed] & Y_n. \end{tikzcd}$$

The map $E(\partial T) \to E(T)$ is an isomorphism for any tree T that is not η, whereas for $T = \eta$ the lifting problem on the right admits a solution by the fact that $X(\eta)_n \to Y(\eta)_n$ is surjective (cf. Lemma 14.16). This completes the proof. □

To prepare for the proof of Lemma 14.19, we first observe that the sparse trivial cofibrations satisfy the following variation of the 'countable approximation property' of Definition 11.33:

Lemma 14.20 *For any commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

with f a sparse trivial cofibration and i a sparse cofibration between countable objects A and B, there exists an extension of this diagram to

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & A' \arrow[r] \arrow[d, "j"] & X \arrow[d, "f"] \\ B \arrow[r] & B' \arrow[r] & Y \end{tikzcd}$$

in which A' and B' are countable and j is a sparse trivial cofibration.

Proof We first exploit the countable approximation property of the trivial cofibrations in the model category $\mathbf{dSpaces}_{RSC}$. Indeed, that model category was constructed as a Bousfield localization of the Reedy model structure $\mathbf{dSpaces}_{R}$ and thus Proposition 11.35 implies that there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & A' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & Y \end{tikzcd}$$

in which $A' \to D$ is a trivial cofibration between countable objects in $\mathbf{dSpaces}_{RSC}$, i.e., a Reedy cofibration that is also a complete weak equivalence. Now form the pushout of i along the map $A \to A'$ to obtain maps

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & A' \arrow[d, "k"] \arrow[dr] & \\ B \arrow[r] & C \arrow[r] & D \end{tikzcd}$$

with k a sparse cofibration between countable objects. The map $C \to D$ may be factored as a sparse cofibration $C \to B'$ followed by a map $p\colon B' \to D$ having the right lifting property with respect to sparse cofibrations; moreover, p may be arranged to be a morphism with countable fibres (cf. Lemma 11.31), so that B' is also countable. By Lemma 14.18 the map p is a complete weak equivalence. By two-out-of-three we conclude that $A' \to B'$ is a complete weak equivalence as well. It is also a sparse cofibration, being the composition of $A' \to C$ and $C \to B'$. □

Proof (of Lemma 14.19) Checking that the class of sparse trivial cofibrations is saturated is entirely straightforward. Indeed, this class is the intersection of the class of sparse cofibrations with the class of trivial cofibrations in the model category $\mathbf{dSpaces}_{RSC}$. Both of those classes are closed under retracts, transfinite composition, and pushout, implying the same for the class of sparse trivial cofibrations.

Now suppose $f\colon A \to B$ is a sparse trivial cofibration. We claim that there exists a factorization of f as a composition

$$A \xrightarrow{j} X \xrightarrow{p} B$$

with j a transfinite composition of pushouts of sparse trivial cofibrations between countable dendroidal spaces and p a map having the right lifting property with respect to sparse cofibrations. Granted this factorization, there exists a lift in the square

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "f"'] & X \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dashed] & B \end{tikzcd}$$

demonstrating that f is a retract of j. In particular, f is contained in the saturated class generated by the sparse trivial cofibrations between countable dendroidal spaces, as desired.

To verify the claim of the previous paragraph, we use a variation of the small object argument. Inductively construct a sequence of dendroidal spaces and maps

$$\begin{tikzcd} A = A_0 \arrow[r] \arrow[d, "f"'] & A_1 \arrow[r] \arrow[dl] & A_2 \arrow[r] \arrow[dll] & \cdots \\ B & & & \end{tikzcd}$$

by constructing A_{k+1} out of A_k as follows. Consider the set I of commutative squares of the form

$$\begin{tikzcd} C_i \arrow[r] \arrow[d] & A_k \arrow[d] \\ D_i \arrow[r] & B, \end{tikzcd}$$

with $C_i \to D_i$ a generating sparse cofibration of the form (C1) or (C2), as defined earlier in this section. Apply Lemma 14.20 to each such square to find a larger diagram

$$\begin{tikzcd} C_i \arrow[r] \arrow[d] & C_i' \arrow[r] \arrow[d] & A_k \arrow[d] \\ D_i \arrow[r] & D_i' \arrow[r] & B, \end{tikzcd}$$

with the middle vertical map a sparse trivial cofibration between countable dendroidal spaces. Now define A_{k+1} to be the pushout in the square

$$\begin{tikzcd} \coprod_{i \in I} C_i' \arrow[r] \arrow[d] & A_k \arrow[d] \\ \coprod_{i \in I} D_i' \arrow[r] & A_{k+1} \end{tikzcd}$$

and define $X = \varinjlim_k A_k$. Then $j\colon A \to X$ is a transfinite composition of pushouts of sparse trivial cofibrations between countable dendroidal spaces and $p\colon X \to B$ has the right lifting property with respect to maps of type (C1) and (C2), so that it also has the right lifting property with respect to general sparse cofibrations. This concludes the proof. □

14.5 Simplicial Operads and Dendroidal Spaces

In the previous section we recalled the nerve construction for simplicial operads, giving rise to an adjoint pair

$$\mathbf{dSpaces} \underset{N}{\overset{\tau}{\rightleftarrows}} \mathbf{sOp}.$$

The aim of this section is to prove that these functors induce an equivalence of homotopy theories between dendroidal spaces and simplicial operads in a suitable sense. This result will be a key ingredient in the next section, where we finally establish a Quillen equivalence between the categories of dendroidal sets and of simplicial operads.

Theorem 14.21 *With respect to the sparse model structure on the category* **dSpaces** *and the model structure of Theorem 14.12 on the category* **sOp** *of simplicial operads, the functor τ admits a left derived functor $\mathbf{L}\tau$ and the nerve N admits a right derived functor $\mathbf{R}N$. These functors give an adjoint equivalence of homotopy categories*

$$\mathrm{Ho}(\mathbf{dSpaces}) \underset{\mathbf{R}N}{\overset{\mathbf{L}\tau}{\rightleftarrows}} \mathrm{Ho}(\mathbf{sOp}).$$

We will establish three preliminary lemmas and subsequently prove the theorem at the end of this section.

Lemma 14.22 *The functor τ sends sparse cofibrations between dendroidal spaces to cofibrations of simplicial operads.*

Proof It suffices to check that τ sends the generating sparse cofibrations of types (C1) and (C2) to cofibrations of simplicial operads. The functor τ was described explicitly at the start of Section 14.4. From that description it is clear that it sends the maps

$$\varnothing \longrightarrow \eta \boxtimes \Delta[n]$$

of type (C1) to the inclusion $\varnothing \longrightarrow \eta$, where now η is interpreted as the trivial operad. This is a cofibration of simplicial operads. To deal with the maps

$$T \boxtimes \partial\Delta[n] \cup_{E(T)\boxtimes\partial\Delta[n]} E(T) \boxtimes \Delta[n] \longrightarrow T \boxtimes \Delta[n]$$

of type (C2) we introduce a bit more notation. Recall that any tree T generates a free operad $\Omega(T) = \tau(T)$. For a simplicial set A, we write $\Omega(T)[A]$ for the simplicial operad generated by the vertices of T and simplices of A, in the following sense. If T is a corolla C_k, then $\Omega(C_k)[A]$ is the operad $\tau(C_k)[A]$ as before. For general T with more than one vertex, one can always decompose T as a grafting $T_1 \circ_e T_2$ of smaller trees along a common edge. Then inductively define the pushout

$$\Omega(T)[A] := \Omega(T_1)[A] \cup_e \Omega(T_2)[A].$$

Of course $\Omega(T)[A]$ could also be expressed directly as a colimit, indexed over vertices and edges of T, of operads of the forms $\Omega(C_k)[A]$ and η respectively. With this notation in place, applying τ to the map of type (C2) displayed above gives the map

$$\Omega(T)[\partial\Delta[n]] \to \Omega(T)[\Delta[n]].$$

This is a composition of pushouts (one for each vertex of T) of maps of the form $\tau(C_k)[\partial\Delta[n]] \to \tau(C_k)[\Delta[n]]$, each of which is a cofibration of simplicial operads.□

Lemma 14.23 *A morphism $\varphi\colon \mathbf{P} \to \mathbf{Q}$ of simplicial operads is a weak equivalence if and only if $N\varphi$ is a complete weak equivalence of dendroidal spaces. In other words, N preserves and detects weak equivalences of simplicial operads.*

Proof First observe that the dendroidal spaces $N\mathbf{P}$ and $N\mathbf{Q}$ satisfy the Segal property, cf. Example 12.11(i). They might not be dendroidal Segal spaces in the sense of Definition 12.9, but this can always be arranged by taking Reedy fibrant replacements of $N\mathbf{P}_f$ and $N\mathbf{Q}_f$ of $N\mathbf{P}$ and $N\mathbf{Q}$ respectively. Theorem 12.36 implies that $N\varphi$ is a complete weak equivalence if and only if $N\mathbf{P}_f \to N\mathbf{Q}_f$ is a fully faithful and essentially surjective map of dendroidal Segal spaces. Now note that for colours $c_1, \ldots, c_n, d$ of $\mathbf{P}$, the mapping space $(N\mathbf{P}_f)(c_1, \ldots, c_n; d)$ introduced right before Definition 12.12 is weakly equivalent to the simplicial set $\mathbf{P}(c_1, \ldots, c_n; d)$. Of course the same applies with $\mathbf{Q}$ in place of $\mathbf{P}$. Hence $N\mathbf{P}_f \to N\mathbf{Q}_f$ is fully faithful if and only if φ is fully faithful.

Similar comments apply to essential surjectivity. Indeed, it is straightforward to see that the operad $\pi_0\mathbf{P}$ agrees with the homotopy operad $\mathrm{ho}(N\mathbf{P}_f)$ of the dendroidal Segal space $N\mathbf{P}_f$. Therefore φ is essentially surjective if and only if $N\mathbf{P}_f \to N\mathbf{Q}_f$ is. We conclude that $N\varphi$ is a complete weak equivalence if and only if φ is a fully faithful and essentially surjective map of operads, which proves the lemma. □

Lemma 14.24 *Let X be a dendroidal space that is cofibrant in the sparse model structure. Then the unit map $X \to N\tau(X)$ is a complete weak equivalence.*

Proof Consider the class $\mathcal{D}$ of sparsely cofibrant dendroidal spaces X for which the unit $X \to N\tau(X)$ is a complete weak equivalence. It is clear that $\mathcal{D}$ is closed under retracts and directed colimits and contains the empty dendroidal space. Therefore it suffices to show that it satisfies the following additional closure property: if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

is a pushout square in which i is a generating sparse cofibration of type (C1) or (C2) and X is in $\mathcal{D}$, then also Y is in $\mathcal{D}$. For the maps (C1) of the form $\varnothing \to \eta \boxtimes \Delta[n]$ this is quite clear; Y is simply a coproduct of X with $\eta \boxtimes \Delta[n]$ and on the second term the unit is the map of dendroidal spaces

$$\eta \boxtimes \Delta[n] \to \eta,$$

which is even a Reedy weak equivalence. For maps (C2) of the form

$$T \boxtimes \partial\Delta[n] \cup_{E(T)\boxtimes\partial\Delta[n]} E(T) \boxtimes \Delta[n] \to T \boxtimes \Delta[n],$$

recall that applying τ gives the map of simplicial operads

$$\Omega(T)[\partial\Delta[n]] \to \Omega(T)[\Delta[n]].$$

Decomposing $\Omega(T)[\Delta[n]]$ as an iterated pushout of simplicial operads of the form $\tau(C_k)[\Delta[n]]$ as in the proof of Lemma 14.22, the map $\tau(X) \to \tau(Y)$ can be written as a composition of pushouts of maps of the form $\tau(C_k)[\partial\Delta[n]] \to \tau(C_k)[\Delta[n]]$. Hence we can reduce to the case where T is a corolla C_k.

To deduce that Y is in $\mathcal{D}$, the cube lemma (cf. Corollary 7.50) shows that it will suffice to prove that the square

$$\begin{array}{ccc} N\tau(C_k)[\partial\Delta[n]] & \longrightarrow & N\tau(X) \\ \downarrow & & \downarrow \\ N\tau(C_k)[\Delta[n]] & \longrightarrow & N\tau(Y) \end{array}$$

is a homotopy pushout. Since the left-hand map is a Reedy cofibration, this is equivalent to checking that the map

$$P \to N\tau(Y)$$

from the pushout P in the square to $N\tau(Y)$ is a complete weak equivalence of dendroidal spaces. This will follow from Proposition 12.29 if we check that for every $m \geq 0$, the morphism $P_m \to (NY)_m$ is an operadic equivalence of dendroidal sets. For $m < n$ it is an isomorphism and there is nothing to prove. For $m = n$ the pushout above adjoins a single operation f; in other words, the map we are considering is of the form

$$(N\tau(X))[f] \to N(\tau(X)[f])$$

described in Proposition 6.23 and hence an inner anodyne. (For the sake of completeness, observe that X is a normal dendroidal space, so that the simplicial operad $\tau(X)$ is Σ-free.) For $m > n$ it is a composition of pushouts of maps of this kind (one for each m-simplex of $\Delta[n]$ not contained in $\partial\Delta[n]$) and therefore again inner anodyne as a consequence of Proposition 6.23. This completes the proof. □

Corollary 14.25 *The functor τ preserves weak equivalences between sparsely cofibrant objects.*

Proof If $f : X \to Y$ is a complete weak equivalence of dendroidal spaces, then $\tau(f)$ is a weak equivalence if and only if $N\tau(f)$ is a weak equivalence as a consequence of Lemma 14.23. But the map $N\tau(f)$ is weakly equivalent to f itself by Lemma 14.24, at least if X and Y are sparsely cofibrant. □

Proof (of Theorem 14.21) Combining Corollary 14.25 with the fact that τ preserves cofibrations (Lemma 14.22) it follows that τ admits a left derived functor by Lemma 8.36. The right adjoint N preserves arbitrary weak equivalences and therefore induces a right derived functor $\mathbf{R}N$ on homotopy categories. It is clear that $\mathbf{L}\tau$ and $\mathbf{R}N$ form an adjoint pair.

We have seen in Lemma 14.24 that the unit $\mathrm{id} \to N\tau$ is a weak equivalence on sparsely cofibrant objects. This in particular shows that the derived unit $\mathrm{id} \to \mathbf{R}N \circ \mathbf{L}\tau$ is an isomorphism. The fact that $\mathbf{R}N$ detects isomorphisms then proves that the pair $(\mathbf{L}\tau, \mathbf{R}N)$ forms an adjoint equivalence. □

14.6 The Homotopy-Coherent Nerve

In this section we come to one of the main results in this book, namely Theorem 14.27, relating dendroidal sets and simplicial operads. We will use Theorem 14.21 to produce a direct Quillen equivalence between the categories of simplicial operads and of dendroidal sets (equipped with the operadic model structure). This equivalence will be implemented by the homotopy-coherent nerve w^*, as first introduced in Example 3.20(i), and its left adjoint $w_!$ defined in terms of the Boardman–Vogt resolution. The functor $w_!$ is characterized by the fact that for a tree T, the simplicial operad $w_!T$ is the Boardman–Vogt resolution $W\Omega(T)$ of the free operad $\Omega(T)$.

Proposition 14.26 *The adjunction*

$$\mathbf{dSets} \underset{w^*}{\overset{w_!}{\rightleftarrows}} \mathbf{sOp}$$

is a Quillen pair.

Proof We check that $w_!$ preserves cofibrations by considering its effect on generating cofibrations $\partial T \to T$. For a sequence of edges $c_1, \ldots, c_n, d$ of T, the map

$$w_!(\partial T)(c_1, \ldots, c_n; d) \to w_!T(c_1, \ldots, c_n; d)$$

is almost always an isomorphism of simplicial sets, except in the case where $c_1, \ldots, c_n$ is exactly the collection of leaves of T and d is its root. In that case

$$w_!T(c_1, \ldots, c_n; d) = \prod_{e \in I(T)} \Delta[1]$$

is the cube whose coordinates are indexed by the inner edges of T. We denote this cube by K. The simplicial set $w_!(\partial T)(c_1, \ldots, c_n; d)$ is the boundary ∂K of that cube. It follows that there is a pushout square

$$\begin{tikzcd} \tau(C_n)[\partial K] \arrow[r] \arrow[d] & w_!(\partial T) \arrow[d] \\ \tau(C_n)[K] \arrow[r] & w_! T. \end{tikzcd}$$

In particular, $w_!(\partial T) \to w_! T$ is a cofibration of simplicial operads.

It remains to be checked that $w_!$ sends J-anodyne maps (cf. Definition 9.35) of dendroidal sets to weak equivalences between simplicial operads. The morphism $w_!(\eta \to J)$ is one of the generating trivial cofibrations of simplicial operads of the kind (A2), since $w_! J$ is a categorical interval. For an inner edge x of a tree T, the map

$$w_!(\Lambda^x[T])(c_1, \ldots, c_n; d) \to w_! T(c_1, \ldots, c_n; d)$$

is an isomorphism in all cases, except (again) the one where $c_1, \ldots, c_n$ is the collection of leaves of T and d is the root. In this case the domain is the simplicial subset L of the cube $K = \prod_{e \in I(T)} \Delta[1]$ given by the union of the following subcubes:

(a) For inner edges f, the cubes $\{1\} \times \prod_{e \in I(T) - \{f\}} \Delta[1]$ where the edge f is labelled by the coordinate 1.
(b) For inner edges f not equal to x, the cubes $\{0\} \times \prod_{e \in I(T) - \{f\}} \Delta[1]$ where the edge f is labelled by the coordinate 0.

Thus, the map $L \to K$ is the pushout-product of the boundary inclusion of the cube

$$\prod_{e \in I(T) - \{x\}} \Delta[1]$$

with the inclusion $\{1\} \to \Delta[1]$ of the final vertex of the edge corresponding to x. The latter is an anodyne map of simplicial sets, hence so is the pushout-product $L \to K$. In particular, it is a weak homotopy equivalence. $\square$

The various constructions in this chapter can be summarized in the following diagram of left adjoint functors:

$$\begin{tikzcd} \mathbf{dSpaces}_{sp} \arrow[r, "\tau"] \arrow[d, "\mathrm{id}_!"] & \mathbf{sOp} \\ \mathbf{dSpaces}_{RSC} \arrow[r, "|\cdot|_J"] & \mathbf{dSets}. \arrow[u, "w_!"'] \end{tikzcd}$$

The reader should be warned that this diagram does *not* commute. However, we will show that it commutes up to natural isomorphism after passing to homotopy categories and derived functors. Since we have already proved that all but the right vertical arrow give equivalences on the level of homotopy categories, it will follow that the same is true for the functor $w_!$ and its adjoint w^*, the homotopy-coherent nerve. This will then imply the main result of this chapter:

Theorem 14.27 *The adjoint pair*

$$\mathbf{dSets} \underset{w^*}{\overset{w_!}{\rightleftarrows}} \mathbf{sOp}$$

is a Quillen equivalence.

We will prove the theorem after establishing two preparatory lemmas. The first concerns the behaviour of τ with respect to cosimplicial resolutions. Although τ is not quite a left Quillen functor, it does satisfy the following:

Lemma 14.28 *Let Y be a cofibrant object of* $\mathbf{dSpaces}_{\mathrm{sp}}$ *and $\check{Y}^\bullet$ a cosimplicial resolution of Y. Then $\tau(\check{Y}^\bullet)$ is a cosimplicial resolution of the simplicial operad $\tau(Y)$. In particular, this applies when Y is (the dendroidal space represented by) a tree T.*

Proof Since τ preserves colimits and cofibrations (Lemma 14.22), it sends Reedy cofibrant cosimplicial objects of $\mathbf{dSpaces}_{\mathrm{sp}}$ to Reedy cofibrant cosimplicial objects of $\mathbf{sOp}$. Furthermore, the fact that τ preserves weak equivalences between cofibrant objects (Corollary 14.25) implies that $\tau(\check{Y}^\bullet) \to \tau(Y)$ is a weak equivalence of cosimplicial objects. Thus, $\tau(\check{Y}^\bullet)$ is indeed a cosimplicial resolution of $\tau(Y)$. For the last sentence of the lemma, it remains to check that a tree T represents a sparsely cofibrant dendroidal space. The map $\varnothing \to E(T)$ is a sparse cofibration, being a coproduct of generating sparse cofibrations of type (C1), and $E(T) \to T$ is a generating sparse cofibration of type (C2). Hence their composition $\varnothing \to T$ is a sparse cofibration as desired. □

Lemma 14.29 *For* $\mathbf{P}$ *a simplicial operad, there is a natural complete weak equivalence between $N\mathbf{P}$ and the dendroidal space*

$$T \mapsto \mathrm{Map}_{\mathbf{sOp}}(\Omega(T), \mathbf{P}).$$

Proof As in Examples 12.11(ii) and 12.24 we will use the specific cosimplicial resolution $J[\bullet] \otimes T$ of a tree T. For X a dendroidal set and $m \geq 0$, consider the map of dendroidal sets

$$X(-) \to \mathbf{dSets}(J[m] \otimes -, X)$$

induced by the unique map $J[m] \to J[0] = \eta$. If X is an ∞-operad, then the map above is an operadic equivalence of dendroidal sets; indeed, any choice of section $\eta \to J[m]$ induces a map

$$\mathbf{dSets}(J[m] \otimes -, X) \to X(-)$$

that is a trivial fibration of dendroidal sets. This last claim follows from the fact that the pushout-product of the trivial cofibration $\eta \to J[m]$ with a boundary inclusion $\partial T \to T$ is again a trivial cofibration of dendroidal sets by Proposition 9.28.

Now suppose that X is a dendroidal *space* such that for every $n \geq 0$, the dendroidal set X_n is an ∞-operad. Then using the observation above, taking the diagonal with respect to m and n and applying Proposition 12.29, we see that the map

$$X_\bullet(-) \to \mathbf{dSets}(J[\bullet] \otimes -, X_\bullet)$$

is a complete weak equivalence of dendroidal spaces. In particular, this applies to $X = N\mathbf{P}$. Now observe that

$$\mathbf{dSets}(J[\bullet] \otimes -, N\mathbf{P}) \cong \mathbf{sOp}(\tau(J[\bullet] \otimes -), N\mathbf{P}).$$

By Lemma 14.28 above, the expression $\tau(J[\bullet] \otimes T)$ is a cosimplicial resolution of $\tau(T) = \Omega(T)$, so that the right-hand side can be interpreted as the dendroidal space

$$T \mapsto \mathrm{Map}_{\mathbf{sOp}}(\Omega(T), \mathbf{P}),$$

completing the proof. □

We can now prove the promised result:

Proof (of Theorem 14.27) Consider the following diagram of functors derived right adjoint to those in the previous square:

$$\begin{tikzcd}
\mathrm{Ho}(\mathbf{dSpaces}_{sp}) & \mathrm{Ho}(\mathbf{sOp}) \arrow[l, "\mathbf{R}N"'] \arrow[d, "\mathbf{R}w^*"] \\
\mathrm{Ho}(\mathbf{dSpaces}_{RSC}) \arrow[u, "\mathbf{R}\mathrm{id}^*"] & \mathrm{Ho}(\mathbf{dSets}). \arrow[l, "\mathbf{R}\mathrm{Sing}_J"']
\end{tikzcd}$$

It will suffice to show that $\mathbf{R}w^*$ is an equivalence of categories. We already know this for the other three functors: indeed, for $\mathbf{R}N$ this is Theorem 14.21, for $\mathbf{R}\mathrm{id}^*$ this is part of Theorem 14.17, and for $\mathbf{R}\mathrm{Sing}_J$ this is explained in Example 12.24. Thus the theorem will follow if we can show that the diagram commutes up to natural isomorphism.

To do this, consider a fibrant simplicial operad $\mathbf{P}$. By Lemma 14.29 above, there is a natural isomorphism

$$\mathbf{R}N(\mathbf{P}) \cong \mathrm{Map}_{\mathbf{sOp}}(\Omega(-), \mathbf{P})$$

in $\mathrm{Ho}(\mathbf{dSpaces}_{sp})$. On the other hand, by definition we have

$$\mathrm{id}^*\mathrm{Sing}_J w^*\mathbf{P}(T) = \mathbf{sOp}(w_!(J[\bullet] \otimes T), \mathbf{P}).$$

Since $w_!$ is left Quillen, the object $w_!(J[\bullet] \otimes T)$ is a cosimplicial resolution of $w_!T$, which in turn is weakly equivalent to $\Omega(T)$. Hence the right-hand side is another model for $\mathrm{Map}_{\mathbf{sOp}}(\Omega(T), \mathbf{P})$. In particular, in the category $\mathrm{Ho}(\mathbf{dSpaces}_{sp})$ the object $\mathbf{R}(\mathrm{id}^*\mathrm{Sing}_J w^*)(\mathbf{P})$ is naturally isomorphic to $\mathbf{R}N(\mathbf{P})$, concluding the proof. □

For future use, let us also record the following consequence of our arguments:

Corollary 14.30 *Let X be an ∞-operad X and let Y be a sparsely cofibrant replacement of the dendroidal space* $\mathrm{Sing}_J X$. *Then the objects $w_!X$ and $\tau(Y)$ are isomorphic in the homotopy category* $\mathrm{Ho}(\mathbf{sOp})$ *of simplicial operads.*

Proof Taking inverses of the functors $\mathbf{R}N$ and $\mathbf{R}w^*$ in the square above, we conclude that the following must also commute up to natural isomorphism:

$$\begin{tikzcd} \mathrm{Ho}(\mathbf{dSpaces}_{sp}) \arrow[r, "\mathbf{L}\tau"] & \mathrm{Ho}(\mathbf{sOp}) \\ \mathrm{Ho}(\mathbf{dSpaces}_{RSC}) \arrow[u, "\mathbf{R}\mathrm{id}^*"] & \mathrm{Ho}(\mathbf{dSets}). \arrow[l, "\mathbf{R}\mathrm{Sing}_J"'] \arrow[u, "\mathbf{L}w_!"'] \end{tikzcd}$$

Chasing X around both ways gives the conclusion. □

Theorem 14.27 allows us to compare spaces of operations in an ∞-operad (see Definition 9.42) with those in the corresponding simplicial operad. To be precise:

Proposition 14.31 *Let* $\mathbf{P}$ *be a fibrant simplicial operad and* $c_1, \ldots, c_n, d$ *a sequence of colours of* $\mathbf{P}$*. Then there is a natural weak homotopy equivalence between* $\mathbf{P}(c_1, \ldots, c_n; d)$ *and the space of operations* $(w^*\mathbf{P})(c_1, \ldots, c_n; d)$ *in the ∞-operad* $w^*\mathbf{P}$*. Similarly, if* X *is a normal ∞-operad and* $x_1, \ldots, x_n, y$ *a sequence of colours of* X*, then there is a natural weak homotopy equivalence between the space of operations* $X(x_1, \ldots, x_n; y)$ *and the simplicial set* $(w_!X)(x_1, \ldots, x_n; y)$*.*

Proof Example 11.18 shows that $(w^*\mathbf{P})(c_1, \ldots, c_n; d)$ is (a model for) the mapping space from C_n to $w^*\mathbf{P}$ in the model category $\partial C_n/\mathbf{dSets}$, where the reference map $\partial C_n \to w^*\mathbf{P}$ assigns the colours $c_1, \ldots, c_n$ to the leaves of C_n and d to its root. The fact that $(w_!, w^*)$ is a Quillen adjunction implies that this space is equivalent to the mapping space $\mathrm{Map}(w_!C_n, \mathbf{P})$ computed with respect to the model category $\tau(\partial C_n)/\mathbf{sOp}$. Note that $w_!C_n = \tau(C_n)$ and that the cosimplicial object $\tau(C_n)[\Delta[\bullet]]$ is a cosimplicial resolution of $\tau(C_n)$. Thus the space $\mathrm{Map}(\tau(C_n), \mathbf{P})$ is weakly equivalent to the simplicial set whose k-simplices are morphisms of simplicial operads $\tau(C_n)[\Delta[k]] \to \mathbf{P}$ sending the edges of C_n to the specified colours of $\mathbf{P}$. But that simplicial set is precisely $\mathbf{P}(c_1, \ldots, c_n; d)$.

For the second part, pick a fibrant replacement $w_!X \xrightarrow{\sim} \mathbf{P}$ of $w_!X$. Then the derived unit

$$X \to w^*\mathbf{P}$$

is a weak equivalence by Theorem 14.27. In particular, the map

$$X(x_1, \ldots, x_n; y) \to (w^*\mathbf{P})(x_1, \ldots, x_n; y)$$

is a weak homotopy equivalence. By what we proved above, the space on the right is weakly equivalent to $\mathbf{P}(x_1, \ldots, x_n; y)$, which in turn is weakly equivalent to $(w_!X)(x_1, \ldots, x_n; y)$ by construction. □

Specializing from dendroidal sets to simplicial sets, Theorem 14.27 in particular reproduces the well-established equivalence between the homotopy theories of simplicial categories and of ∞-categories. To be precise, slicing the category of dendroidal spaces over η and the category of simplicial operads over the trivial operad $\tau(\eta)$ gives the following:

Corollary 14.32 *The homotopy-coherent nerve functor w^* gives a Quillen equivalence*

$$\mathbf{sSets} \underset{w^*}{\overset{w_!}{\rightleftarrows}} \mathbf{sCat}$$

between the Joyal (or categorical) model structure on the category of simplicial sets and the Bergner model structure on the category of simplicial categories.

Another specialization of Theorem 14.27 concerns uncoloured dendroidal sets and simplicial operads. (We treat uncoloured dendroidal *spaces* in the next section.) Recall from Section 9.3 that the pair $(w_!, w^*)$ restricts to given an adjunction

$$\mathbf{udSets} \underset{w^*}{\overset{w_!}{\rightleftarrows}} \mathbf{usOp}.$$

According to Theorem 9.49, the category **udSets** of uncoloured dendroidal sets admits a model structure for which cofibrations and weak equivalences are 'the same' as in the operadic model structure on **dSets**, i.e., both are detected by the inclusion functor $\mathbf{udSets} \to \mathbf{dSets}$. For simplicial operads the situation is even better; the inclusion $\mathbf{usOp} \to \mathbf{sOp}$ of uncoloured simplicial operads into all simplicial operads detects weak equivalences, fibrations, and cofibrations.

Corollary 14.33 *The adjoint pair $(w_!, w^*)$ induces a Quillen equivalence between the categories* **udSets** *of uncoloured dendroidal sets and* **usOp** *of uncoloured simplicial operads.*

Proof The remarks above the corollary and the fact that $w_! \colon \mathbf{dSets} \to \mathbf{sOp}$ is left Quillen immediately imply that the restricted functor $w_! \colon \mathbf{udSets} \to \mathbf{usOp}$ is left Quillen as well. Now the derived unit and counit of the restricted pair $(w_!, w^*)$ are isomorphisms, simply because they are the restrictions of those featuring in the Quillen equivalence of Theorem 14.27. □

14.7 Operads with a Single Colour

Many applications of the theory of ∞-operads concern operads with only a single colour. There are various equivalent ways to describe such a theory, for example as dendroidal spaces X with $X(\eta)$ a one-point set (as at the end of the previous section), or as dendroidal spaces X with $X(\eta)$ a weakly contractible space (see Proposition 14.40 below). In this section we establish the equivalence between dendroidal spaces and simplicial operads in this restricted setting (Theorem 14.36). Many of our arguments simplify considerably in this case and, in particular, the notion of completion can be avoided in the relevant results (although it does feature in some of our proofs). As such, the reader only interested in operads with a single colour can read this section first and refer to the rest of this chapter as needed. We will conclude with an important and useful application (cf. Corollary 14.42), namely the fact that the space of maps between uncoloured simplicial operads **P** and **Q** is equivalent to the space of maps $\mathrm{Map}(N\mathbf{P}, N\mathbf{Q})$ between the corresponding dendroidal spaces $N\mathbf{P}$ and $N\mathbf{Q}$, computed with respect to the *projective* (or equivalently Reedy) model structure on the category **dSpaces**, rather than any of the more complicated model structures we have considered before.

Let us briefly discuss some terminology and notation concerning uncoloured dendroidal sets and spaces, which already featured in Section 3.5.6. We say a dendroidal space X is *uncoloured* if $X(\eta) \cong \Delta[0]$ and write **udSpaces** for the full subcategory of **dSpaces** on the uncoloured dendroidal spaces. Similarly, we will write **usOp** for the category of uncoloured simplicial operads, consisting of simplicial operads **P** having only a single colour. There is an adjoint pair of functors

$$\mathbf{dSpaces} \underset{r^*}{\overset{r_!}{\rightleftarrows}} \mathbf{udSpaces},$$

where the right adjoint r^* is simply the inclusion and the left adjoint $r_!$ collapses the space $X(\eta)$ to a single point. To be precise, for a dendroidal space X the uncoloured dendroidal space $r_!(X)$ fits in a pushout square

$$\begin{tikzcd} \eta \boxtimes X(\eta) \arrow[r] \arrow[d] & X \arrow[d] \\ \eta \arrow[r] & r_!(X). \end{tikzcd}$$

If **P** is an uncoloured simplicial operad, then $N\mathbf{P}$ is an uncoloured dendroidal space; similarly, an uncoloured dendroidal space produces an uncoloured simplicial operad $\tau(X)$. Thus, the functors τ and N restrict to give an adjoint pair

$$\mathbf{udSpaces} \underset{N}{\overset{\tau}{\rightleftarrows}} \mathbf{usOp}.$$

The main result of this section (Theorem 14.36) will be that this pair is a Quillen equivalence; moreover, the relevant model structure on **udSpaces** will just be a variant of the projective model structure localized for the Segal condition, rather than any of the more involved ones considered previously (such as those involving completion or sparseness). To prepare for this result we begin by observing that the projective model structure on **dSpaces** induces a corresponding model structure on the category **udSpaces**, to which we will also refer as the projective model structure:

Proposition 14.34 *There exists a cofibrantly generated left proper model structure on the category* **udSpaces** *such that a map is a weak equivalence (resp. a fibration) if and only if it is a weak equivalence (resp. a fibration) in the projective model structure on the category* **dSpaces**. *With respect to these projective model structures, the pair*

$$\mathbf{dSpaces} \underset{r^*}{\overset{r_!}{\rightleftarrows}} \mathbf{udSpaces}$$

is a Quillen adjunction.

Proof Observe that for a generating projective trivial cofibration

$$T \boxtimes \Lambda^k[n] \xrightarrow{j} T \boxtimes \Delta[n]$$

between dendroidal spaces, the morphism $r_!(j)$ fits into a pushout square of dendroidal spaces as follows:

$$\begin{tikzcd} T \boxtimes \Lambda^k[n] \cup_{E(T)\boxtimes\Lambda^k[n]} E(T) \boxtimes \Delta[n] \arrow[r] \arrow[d] & r_!(T \boxtimes \Lambda^k[n]) \arrow[d, "r_!(j)"] \\ T \boxtimes \Delta[n] \arrow[r] & r_!(T \boxtimes \Delta[n]). \end{tikzcd}$$

The left-hand vertical morphism is not quite a projective cofibration, but it is a trivial cofibration (and in particular a weak equivalence) in the Reedy model structure on dendroidal spaces. Hence the same is true for $r_!(j)$ and any pushout (or transfinite composition of pushouts) of such maps. We conclude that the conditions for transfer are satisfied; to be precise, Theorem 7.44 implies that a cofibrantly generated model structure on **udSpaces** with the desired fibrations and weak equivalences exists. Left properness follows from the corresponding fact for the Reedy model structure on **dSpaces**, combined with the observation that any cofibration in **udSpaces** is in particular a Reedy cofibration of dendroidal spaces (cf. also Remark 14.35 below). The fact that $(r_!, r^*)$ is a Quillen pair is immediate from the definitions of the fibrations and weak equivalences on **udSpaces**. □

Remark 14.35 It will be useful to observe that the transfer above also yields an explicit set of generating cofibrations for the projective model structure on **udSpaces**, namely the collection of maps $r_!(T \boxtimes \partial\Delta[n]) \to r_!(T \boxtimes \Delta[n])$ for T ranging over $\mathbf{\Omega}$ and $n \geq 0$. Note that these maps fit into pushout squares as follows:

$$\begin{tikzcd} T \boxtimes \partial\Delta[n] \cup_{E(T)\boxtimes\partial\Delta[n]} E(T) \boxtimes \Delta[n] \arrow[r] \arrow[d] & r_!(T \boxtimes \partial\Delta[n]) \arrow[d] \\ T \boxtimes \Delta[n] \arrow[r] & r_!(T \boxtimes \Delta[n]). \end{tikzcd}$$

The left-hand map is a sparse cofibration between dendroidal spaces. It follows that every cofibration in $\mathbf{udSpaces}_P$, when interpreted as a map in the category **dSpaces**, is in particular a sparse cofibration.

We will denote the model category of the preceding proposition by $\mathbf{udSpaces}_P$. As before, we can now consider the Bousfield localization of this model category with respect to the Segal condition. To be precise, localizing with respect to the maps $r_!(\mathrm{Sc}[T]) \to r_!(T)$, for T ranging over objects of $\mathbf{\Omega}$, gives a new model category that we denote $\mathbf{udSpaces}_{PS}$. The fibrant objects of this model category are precisely those uncoloured dendroidal spaces that are projectively fibrant (so $X(T)$ is a Kan complex for every T) and satisfy the Segal condition. Since $X(\eta)$ is a point, this reduces to the condition that for every T, the map

$$X(T) \to \prod_{v \in V(T)} X(C_v)$$

is a weak homotopy equivalence.

According to Example 13.32(a) the category **usOp** of uncoloured simplicial operads can be equipped with a model structure with weak equivalences and fibrations defined 'locally', i.e., by considering the effect of morphisms on spaces of operations. With respect to this model structure we can state the main result of this section:

Theorem 14.36 *The adjoint pair*

$$\mathbf{udSpaces}_{PS} \underset{N}{\overset{\tau}{\rightleftarrows}} \mathbf{usOp}$$

is a Quillen equivalence.

We will prove the theorem after establishing three lemmas.

Lemma 14.37 *Let X and Y be fibrant objects of* $\mathbf{udSpaces}_{PS}$*, i.e., projectively fibrant uncoloured dendroidal spaces satisfying the Segal condition. Then a map $f: X \to Y$ is a complete weak equivalence (when considered as a map in the category* **dSpaces***) if and only if it is a projective weak equivalence, i.e., $X(T) \to Y(T)$ is a weak equivalence of simplicial sets for each tree T.*

Proof Observe that any map between uncoloured dendroidal spaces is essentially surjective. Thus Theorem 12.36 implies that f is a complete weak equivalence if and only if it is fully faithful. Since X and Y are uncoloured, this is equivalent to $X(C_n) \to Y(C_n)$ being a weak equivalence for every $n \geq 0$. Invoking the Segal condition for X and Y, this is equivalent to $X(T) \to Y(T)$ being a weak equivalence for every tree T. □

Lemma 14.38 *The functor N:* **usOp** $\to$ $\mathbf{dSpaces}_P$ *preserves and detects weak equivalences.*

Proof A map $f: \mathbf{P} \to \mathbf{Q}$ of uncoloured simplicial operads is a weak equivalence if and only if the induced map $(N\mathbf{P})(C_n) \to (N\mathbf{Q})(C_n)$ is a weak equivalence for every corolla C_n. But since $N\mathbf{P}$ and $N\mathbf{Q}$ satisfy the Segal condition, this is the case if and only if $(N\mathbf{P})(T) \to (N\mathbf{Q})(T)$ is a weak equivalence for every tree T. □

Lemma 14.39 *For any cofibrant uncoloured dendroidal space X, the unit map $X \to N\tau(X)$ is a complete weak equivalence.*

Proof By Remark 14.35, every cofibrant uncoloured dendroidal space X is in particular sparsely cofibrant when considered as an object of **dSpaces**, so the conclusion follows immediately from Lemma 14.24. □

Proof (of Theorem 14.36) Clearly N sends fibrations between uncoloured simplicial operads to projective fibrations between dendroidal spaces. Combining this with Lemma 14.38 shows that (τ, N) defines a Quillen adjunction

$$\mathbf{udSpaces}_P \underset{N}{\overset{\tau}{\rightleftarrows}} \mathbf{usOp}.$$

To see that this is also a Quillen adjunction with respect to the localized model category $\mathbf{udSpaces}_{PS}$, it suffices to show that N sends fibrant simplicial operads to local objects. This is indeed the case; for any simplicial operad $\mathbf{P}$, the dendroidal space $N\mathbf{P}$ satisfies the Segal condition.

Let X be an object of **udSpaces** that is both fibrant and cofibrant; in particular, it satisfies the Segal condition. Then the unit $X \to N\tau(X)$ is a complete weak equivalence by Lemma 14.39 and hence also a projective weak equivalence, by Lemma 14.37 and the fact that $N\mathbf{P}$ satisfies the Segal condition for any simplicial operad $\mathbf{P}$. Thus, the unit of the adjoint pair of derived functors $(\mathbf{L}\tau, \mathbf{R}N)$ is an equivalence. The counit $\mathbf{L}\tau \circ \mathbf{R}N \to \mathrm{id}$ is then also a weak equivalence; indeed, by Lemma 14.38 this may be checked after applying $\mathbf{R}N$, when it follows from the triangle identity

$$\begin{tikzcd}
\mathbf{R}N \arrow[r, "\cong"] \arrow[dr, equal] & \mathbf{R}N \circ \mathbf{L}\tau \circ \mathbf{R}N \arrow[d] \\
& \mathbf{R}N.
\end{tikzcd}$$

□

We conclude this section with a discussion of the relation between the model categories **udSpaces** and **dSpaces**, equipped with either the projective model structures or their localizations with respect to the Segal condition. Let us write $\mathbf{dSpaces}_{P,*}$ for the left Bousfield localization of the projective model structure on the category of dendroidal spaces with respect to the morphism $\varnothing \to \eta$. Observe that η is a projectively cofibrant object and $\eta \boxtimes \Delta[\bullet]$ is a cosimplicial resolution of it with respect to the projective model structure. It follows from this that for any dendroidal space X, there is a weak equivalence

$$\mathrm{Map}(\eta, X) \simeq X(\eta),$$

where the mapping space is taken with respect to the projective model structure on the category **dSpaces**. It follows that the local objects of $\mathbf{dSpaces}_{P,*}$ are precisely those dendroidal spaces X for which $X(\eta)$ is weakly contractible.

Proposition 14.40 *The adjunction*

$$\mathbf{dSpaces}_{P,*} \underset{r^*}{\overset{r_!}{\rightleftarrows}} \mathbf{udSpaces}_P$$

is a Quillen equivalence. Similarly, it also gives a Quillen equivalence after localizing both model categories with respect to the Segal condition:

$$\mathbf{dSpaces}_{PS,*} \underset{r^*}{\overset{r_!}{\rightleftarrows}} \mathbf{udSpaces}_{PS}.$$

Proof To see that $(r_!, r^*)$ is still a Quillen pair with respect to the localized model structure $\mathbf{dSpaces}_{P,*}$, it suffices to observe that $r_!$ sends the localizing morphism $\varnothing \to \eta$ to the isomorphism $\eta = \eta$ in **udSpaces**. To prove that it is in fact a Quillen equivalence, we begin by observing that r^* preserves and detects arbitrary weak equivalences. Hence it suffices to show that for a dendroidal space X that is both fibrant and cofibrant in $\mathbf{dSpaces}_{P,*}$, the unit map $X \to r^* r_!(X)$ is a weak equivalence. By assumption, the simplicial set $X(\eta)$ is weakly contractible. Factor the evident inclusion $\eta \boxtimes X(\eta) \to X$ as a projective cofibration $\eta \boxtimes X(\eta) \to Y$ followed by a projective trivial fibration $p\colon Y \to X$. Since $\eta \boxtimes X(\eta)$ is projectively cofibrant, the same is true for Y, so that p is a projective weak equivalence between cofibrant objects. Hence the same is true of $r^* r_!(p)$; indeed, $r_!$ preserves projective weak equivalences between cofibrant objects by Brown's lemma, whereas r^* preserves arbitrary projective weak equivalences. Hence it suffices to examine the unit map $Y \to r^* r_!(Y)$. By definition of $r_!$, this map fits into a pushout square

$$\begin{tikzcd}
\eta \boxtimes Y(\eta) \arrow[r, "\simeq"] \arrow[d, tail] & \Delta[0] \arrow[d] \\
Y \arrow[r] & r^* r_!(Y).
\end{tikzcd}$$

By left properness (or the observation that this square is a homotopy pushout), it follows that the bottom horizontal map is a weak equivalence as desired. The final claim of the proposition now follows by observing that (by definition) $r_!$ sends the localizing maps for the Segal condition in **dSpaces** to the localizing maps for the Segal condition in **udSpaces**. □

Combining the proposition with Theorem 14.36 yields the following:

Corollary 14.41 *The adjoint pair (τ, N) gives a Quillen equivalence*

$$\mathbf{dSpaces}_{PS,*} \underset{N}{\overset{\tau}{\rightleftarrows}} \mathbf{usOp}.$$

Finally, the previous results also imply the following very useful observation about mapping spaces between uncoloured simplicial operads.

Corollary 14.42 *Let* **P** *and* **Q** *be uncoloured simplicial operads. Then the functor N induces a natural weak equivalence of mapping spaces*

$$\mathrm{Map}(\mathbf{P}, \mathbf{Q}) \xrightarrow{\sim} \mathrm{Map}(N\mathbf{P}, N\mathbf{Q}),$$

where the mapping space on the right is computed in the model category $\mathbf{dSpaces}_P$.

Proof For emphasis write $\mathrm{Map}_{PS,*}$ (resp. Map_P) for the space of maps between dendroidal spaces with respect to the model structure $\mathbf{dSpaces}_{PS,*}$ (resp. $\mathbf{dSpaces}_P$). Then the previous corollary, combined with Corollary 11.12(1), in particular implies a weak equivalence

$$\mathrm{Map}(\mathbf{P}, \mathbf{Q}) \xrightarrow{\sim} \mathrm{Map}_{PS,*}(N\mathbf{P}, N\mathbf{Q}).$$

(Note that there is no need to distinguish between N and $\mathbf{R}N$, since N preserves arbitrary weak equivalences.) Then the observation that $\mathbf{dSpaces}_{PS,*}$ is a left Bousfield localization of $\mathbf{dSpaces}_P$ gives a further weak equivalence (cf. Corollary 11.12(2))

$$\mathrm{Map}_{PS,*}(N\mathbf{P}, N\mathbf{Q}) \xrightarrow{\sim} \mathrm{Map}_P(N\mathbf{P}, N\mathbf{Q}),$$

completing the proof. □

14.8 Algebras for ∞-Operads and for Simplicial Operads

In this short concluding section we record some further consequences of our results for the theory of algebras over operads. Specifically, we show that for an ∞-operad X the covariant model structure on the category $\mathbf{dSets}/X$ is a model for the homotopy theory of algebras for the simplicial operad $w_!X$ (cf. Theorem 14.44). This specializes to a version of Lurie's 'straightening-unstraightening' equivalence for left fibrations over a simplicial set (Corollary 14.46). Before we do this, let us observe that our results imply 'homotopy invariance' of the model category $\mathrm{Alg}_{\mathbf{P}}$ of $\mathbf{P}$-algebras:

Theorem 14.43 *Suppose that $f: \mathbf{P} \to \mathbf{Q}$ is a weak equivalence of Σ-free simplicial operads. Then the adjoint pair*

$$\mathrm{Alg}_{\mathbf{P}} \underset{\varphi^*}{\overset{\varphi_!}{\rightleftarrows}} \mathrm{Alg}_{\mathbf{Q}}$$

is a Quillen equivalence.

Proof It is evident that φ^* preserves fibrations and weak equivalences, so that the pair above is a Quillen adjunction. To see that it is a Quillen equivalence, consider the following commutative square of right Quillen functors:

$$\begin{tikzcd}
\mathrm{Alg}_{\mathbf{P}} \arrow[d, "N(\mathbf{P},-)"] & & \mathrm{Alg}_{\mathbf{Q}} \arrow[ll, "\varphi^*"'] \arrow[d, "N(\mathbf{Q},-)"] \\
(\mathbf{dSpaces}_P/N\mathbf{P})_{\mathrm{cov}} & & (\mathbf{dSpaces}_P/N\mathbf{Q})_{\mathrm{cov}}. \arrow[ll, "N\varphi^*"']
\end{tikzcd}$$

The vertical functors are part of Quillen equivalences by Theorem 13.37. The bottom horizontal arrow is part of a Quillen equivalence by Corollary 13.28 and the fact that $N\varphi: N\mathbf{P} \to N\mathbf{Q}$ is a complete weak equivalence between dendroidal spaces that satisfy the Segal property (cf. Lemma 14.23). Therefore the top horizontal arrow in the square is also part of a Quillen equivalence. □

Combining the various Quillen equivalences we have established, we can now finally relate the covariant model structure on $\mathbf{dSets}/X$ to the homotopy theory of $w_!X$-algebras.

Theorem 14.44 *Let X be a normal ∞-operad. There exists a natural zig-zag of Quillen equivalences between the covariant model structure on $\mathbf{dSets}/X$ and the projective model structure on the category $\mathrm{Alg}_{w_!X}$ of simplicial $w_!X$-algebras. Under these equivalences, a map $x\colon \eta \to X$ corresponds (up to weak equivalence) to the free $w_!X$-algebra generated by x.*

Proof According to Example 12.24, the dendroidal space $\mathrm{Sing}_J(X)$ is a complete dendroidal Segal space with $\mathrm{Sing}_J(X)_0 = X$. Hence Proposition 13.7 provides a Quillen equivalence

$$(\mathbf{dSets}/X)_{\mathrm{cov}} \underset{\mathrm{dis}^*}{\overset{\mathrm{dis}_!}{\rightleftarrows}} (\mathbf{dSpaces}_R/\mathrm{Sing}_J(X))_{\mathrm{cov}}.$$

Now pick a sparsely cofibrant replacement Y of $\mathrm{Sing}_J(X)$. In detail, fix a map $Y \to \mathrm{Sing}_J(X)$ that has the right lifting property with respect to sparse cofibrations and such that Y is sparsely cofibrant. (If desired, this map can be constructed functorially from the small object argument, of course.) Using Lemma 14.16(i) it is easily seen that Y still satisfies the Segal property. Hence Corollary 13.28 provides a Quillen equivalence

$$(\mathbf{dSpaces}_P/Y)_{\mathrm{cov}} \rightleftarrows (\mathbf{dSpaces}_P/\mathrm{Sing}_J(X))_{\mathrm{cov}}.$$

Note that we have switched from the Reedy to the projective model structure on the category of dendroidal spaces here, but of course the two are Quillen equivalent. To proceed, note that Lemma 14.24 states that the unit map $Y \to N\tau(Y)$ is a complete weak equivalence of dendroidal spaces. Since both satisfy the Segal property, we may apply Corollary 13.28 again to find a Quillen equivalence

$$(\mathbf{dSpaces}_P/Y)_{\mathrm{cov}} \rightleftarrows (\mathbf{dSpaces}_P/N\tau(Y))_{\mathrm{cov}}.$$

Now Theorem 13.37 provides a further Quillen equivalence

$$(\mathbf{dSpaces}_P/N\tau(Y))_{\mathrm{cov}} \rightleftarrows \mathrm{Alg}_{\tau(Y)}.$$

Corollary 14.30 implies that the simplicial operad $\tau(Y)$ is weakly equivalent to $w_!X$. Finally, combining this with Theorem 14.43 gives a zigzag of Quillen equivalences between $\mathrm{Alg}_{\tau(Y)}$ and $\mathrm{Alg}_{w_!X}$. □

Remark 14.45 Of course the chain of Quillen equivalences used to prove Theorem 14.44 is rather long and somewhat indirect. It is possible to write down a direct Quillen pair relating $\mathbf{dSpaces}/X$ to $\mathrm{Alg}_{w_!X}$, but since the extra payoff is marginal we refrain from doing so here.

Specializing to simplicial sets, Theorem 14.44 gives a version of the ‘straightening-unstraightening’ correspondence (for left fibrations) first established by Lurie:

Corollary 14.46 *Let X be an ∞-category. There exists a natural zig-zag of Quillen equivalences between the covariant model structure on* $\mathbf{sSets}/X$ *and the projective model structure on the category* $\mathbf{sSets}^{w_!X}$ *of simplicial diagrams on the simplicial category* $w_!X$*. Under these equivalences, a map* $x\colon \Delta[0] \to X$ *corresponds (up to weak equivalence) to the functor corepresented by* x*.*

Historical Notes

Localizations of simplicial categories were introduced by Dwyer–Kan [51] and all of the results in Section 14.1 are due to them. The characterization of equivalences in simplicial categories given in Section 14.2 is due to Bergner [19] and is the crucial input in establishing what is now called the Bergner model structure on the category of simplicial categories. The generalization to a model structure for simplicial operads in Section 14.3 first appears in [42] and in work of Robertson [130].

The equivalence of homotopy theories between simplicial operads and dendroidal sets was established in [41, 42] (including the case of uncoloured operads); our proof in Sections 14.5 and 14.6 is different, relying on the sparse model structure. In the proof of Lemma 14.19 we used a modification of the usual small object argument originating with Jeff Smith; an account of this appears right before Lemma 1.8 of [15].

As we have observed, our results in particular imply equivalences between the homotopy theories of simplicial categories, ∞-categories, and complete Segal spaces. The equivalence between simplicial categories and complete Segal spaces was established by Bergner [20], the equivalence between complete Segal spaces and the Joyal model structure on simplicial sets by Joyal–Tierney [94]. Bergner’s proof uses the Segal categories of Hirschowitz–Simpson [85] as an intermediate device; the proof presented in this chapter sidesteps this by the use of the sparse model structure. A direct proof that the homotopy-coherent nerve gives a Quillen equivalence between the Joyal model structure on simplicial sets and the Bergner model structure on simplicial categories was given by Lurie [105].

Section 14.7 on the relation between uncoloured dendroidal spaces and simplicial operads contains several proofs and results that have not appeared in the literature before. The application of dendroidal spaces to mapping spaces between simplicial operads (Corollary 14.42) has been important in recent years, for example in the work of Boavida–Weiss [25] and Göppl [70] (see also the appendix of [79]).

In the final Section 14.8 we started by discussing the homotopy invariance of the category of simplicial **P**-algebras; this result was first demonstrated (with a different proof) in [16]. The straightening-unstraightening equivalence between left fibrations over a simplicial set X and simplicial diagrams on the simplicial category $w_!X$ was established by Lurie [105]; see also [81] for a different approach. The generalization to left fibrations over a dendroidal set X and $w_!X$-algebras appears in [77].

Epilogue

In this book we have developed the theory of dendroidal sets and spaces. We have not attempted to cover the many variations and extensions of this theory that have appeared in the recent literature. In this epilogue we would like to direct the reader to some of these, as well as highlight some applications. Also, we point out alternative (but equivalent) models for the theory of ∞-operads.

Different Models for the Theory of ∞-Operads

In this book we have described the theory of dendroidal sets and spaces as an approach to the homotopy theory of ∞-operads and shown that it is equivalent to the theory of simplicial (or topological) operads. These are certainly not the only two available models for a theory of operads 'up to coherent homotopy'. Let us briefly review the various other perspectives that have been developed in the literature and point the reader to the available comparisons between them.

Perhaps the most well-known model is that of Lurie [106]. His starting point is the *category of operators* associated to an operad $\mathbf{O}$. This category has as its objects the finite lists $(c_1, \ldots, c_n)$ of colours of $\mathbf{O}$, with a morphism to another such list $(d_1, \ldots, d_m)$ consisting of a partial map $f: \{1, \ldots, n\} \to \{1, \ldots, m\}$ of sets and for each $1 \leq i \leq m$ an operation of $\mathbf{O}$ with output d_i and set of inputs $f^{-1}(d_i)$. This category of operators is equipped with an evident forgetful functor to the category $\mathbf{F}_{\mathrm{part}}$ of finite sets and partial maps. The properties of such categories can be captured in a short list of axioms, which can then be generalized to the setting of ∞-categories: Lurie defines an ∞-operad to be a map of simplicial sets $X \to N\mathbf{F}_{\mathrm{part}}$ satisfying these axioms. This approach can be seen as a generalization of Segal's approach to infinite loop spaces as particular kinds of simplicial diagrams indexed by the category $\mathbf{F}_{\mathrm{part}}$.

The relation between simplicial sets over $N\mathbf{F}_{\mathrm{part}}$ and dendroidal objects can be visualized very concretely. Roughly speaking, a simplex $S_0 \to S_1 \to \cdots \to S_n$ of $N\mathbf{F}_{\mathrm{part}}$ can be pictured as a forest (meaning a disjoint union of trees), with the elements of the various sets S_i serving as edges and the partial maps going from

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S_i to S_{i+1} expressing how edges are attached to each other via vertices. Assigning to such a simplex the corresponding disjoint union of representable dendroidal sets provides a functor relating Lurie's framework to the one of this book. This idea was made precise in [80], where it is shown that this gives an equivalence of homotopy categories between the two formalisms, at least in the case of operads without nullary operations. (A full comparison is also available, as will be explained shortly.)

Another approach was proposed by Barwick [11]. He introduces a category $\mathbf{\Delta}_{\mathrm{Fin}}$ of which the objects are strings

$$S_0 \xrightarrow{f_1} S_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} S_m$$

of functions between finite sets, i.e., simplices $f\colon [m] \to \mathbf{F}$ of the nerve of the category of finite sets. The morphisms between such (m, f) and (n, g) are pairs of a morphism $\alpha\colon [m] \to [n]$ in $\mathbf{\Delta}$ and an injective natural transformation $f \to \alpha^* g$ with the property that all naturality squares are pullbacks. One can then consider presheaves (of sets or of spaces) on this category and impose various Segal conditions on them. Barwick proves that the homotopy theory obtained in this way is equivalent to that of Lurie's ∞-operads. On the other hand, it is proved in [39] that this model is equivalent to that of dendroidal spaces, thus in particular proving that Lurie's model and the dendroidal one of this book are equivalent without any restrictions on nullary operations.

To conclude we mention two further alternative approaches. First of all, any (uncoloured) operad $\mathbf{O}$ in simplicial sets induces a corresponding free algebra monad

$$X \mapsto \coprod_{n \geq 0} \mathbf{O}(n) \times_{\Sigma_n} X^n$$

on the category of simplicial sets. Working in the ∞-category of spaces, Gepner–Haugseng–Kock [63] characterize the monads arising in this way from an ∞-operad as the *analytic* ones, meaning those which are cartesian and preserve sifted colimits and weakly contractible limits. The ∞-category of such monads (and a particular kind of morphisms between them) can then serve as a theory of (uncoloured) ∞-operads. The three authors also include a version for operads with colours and prove a comparison result to the theory of dendroidal spaces. Another approach by Haugseng [76] is perhaps closest in spirit to the traditional treatment of operads. A *symmetric sequence* of simplicial sets is a collection $\{S(n)\}_{n\geq 0}$ of simplicial sets with a specified action of the symmetric group Σ_n on the term $S(n)$. There is a monoidal structure on this category, called the *composition product*, such that algebras with respect to this product are precisely (uncoloured) operads in simplicial sets. Haugseng develops an analogue of this notion in the ∞-categorical setting and shows that the theory of ∞-operads constructed in this way is equivalent to any of the other models described above.

One appealing feature of the problem of comparing different models for ∞-operads is that once an equivalence has been established, it is essentially unique; indeed, Ara, Groth, and Gutierrez [4] prove that the space of automorphisms of the theory of ∞-operads is contractible.

Spaces of Maps Between Little Disks Operads

As we have seen in Section 14.7, the space of maps $\mathrm{Map}(\mathbf{P}, \mathbf{Q})$ between uncoloured simplicial operads $\mathbf{P}$ and $\mathbf{Q}$ may be computed as the space of maps $\mathrm{Map}(N\mathbf{P}, N\mathbf{Q})$ between the associated dendroidal spaces $N\mathbf{P}$ and $N\mathbf{Q}$. This space is of particular interest for the operads $\mathbf{E}_n$ of little n-cubes, for varying n, since in this case these mapping spaces feature in the study of differential topology. For example, for $n - m \geq 3$, Boavida and Weiss [25] show that the space $\mathrm{emb}_\partial(D^m, D^n)$ of embeddings between disks which agree with a standard embedding near the boundary is equivalent to the homotopy fibre of a certain map from the space of immersions $\mathrm{imm}_\partial(D^m, D^n)$ to the m-fold loop space of the mapping space $\mathrm{Map}(\mathbf{E}_m, \mathbf{E}_n)$. For $m = 1$, this embedding space is a space of 'long knots' and has received much attention in the literature. In fact, Boavida and Weiss prove results of this kind for general manifolds M and N by introducing their 'configuration categories'. These are a type of Segal space capturing the spaces of configurations of n points in these manifolds (for $n \geq 1$), as well as the relations between them as points collide.

One approach to analyzing mapping spaces of the kind $\mathrm{Map}(\mathbf{P}, \mathbf{Q})$ is to approximate them by a tower

$$\cdots \to \mathrm{Map}_{\leq k}(\mathbf{P}, \mathbf{Q}) \to \mathrm{Map}_{\leq k-1}(\mathbf{P}, \mathbf{Q}) \to \cdots,$$

where $\mathrm{Map}_{\leq k}(\mathbf{P}, \mathbf{Q})$ is the mapping space between the 'k-truncations' of the operads $\mathbf{P}$ and $\mathbf{Q}$. This idea features explicitly in the paper of Boavida and Weiss [25] as well as that of Arone and Turchin [5], both in the context of the embedding problem described above. The computation of these mapping spaces between truncated operads can be carried over the setting of dendroidal spaces as well, by considering diagrams indexed by the subcategory $\mathbf{\Omega}_{\leq k}$ of $\mathbf{\Omega}$ spanned by trees that have at most k input edges for every vertex. This translation is explained by Weiss in [141] and in the appendix of [79] in the setting of operads without constants. In the latter case one has the additional option of restricting attention to the subcategory of $\mathbf{\Omega}$ of open trees with at most k leaves. Göppl [70] uses these truncations in the case $\mathbf{P} = \mathbf{E}_m$ and $\mathbf{Q} = \mathbf{E}_n$ to prove that the mapping space $\mathrm{Map}(\mathbf{E}_m, \mathbf{E}_n)$ is $(n - m - 1)$-connected if $n - m \geq 2$.

Equivariant Operads

In equivariant homotopy theory, one replaces the basic categories of spaces and spectra by those of G-spaces and G-spectra, meaning all objects are equipped with an action by some fixed group G. Corresponding algebraic notions, such as $\mathbf{E}_\infty$-ring spectra, are upgraded accordingly to G-spectra equipped with an action by a suitable 'G-equivariant operad'. The theory of equivariant homotopy-coherent algebraic structures has seen much development by May and his collaborators and has recently gained renewed interest because of the solution of the Kervaire invariant one problem by Hill–Hopkins–Ravenel [82], relying heavily on equivariant stable homotopy theory.

The extension of the theory of simplicial operads to that of equivariant simplicial operads admits a parallel in the context of dendroidal sets developed by Bonventre and Pereira. The paper [122] introduces the notion of *equivariant dendroidal sets*, which are based on an extension $\mathbf{\Omega}_G$ of the category $\mathbf{\Omega}$. The objects of $\mathbf{\Omega}_G$ are 'G-trees', but the reader should be warned that these are bit more than simply a tree T with an action of a group G. There is a notion of 'G-∞-operad', defined as a presheaf on the category $\mathbf{\Omega}_G$ satisfying an inner horn filling condition. Bonventre and Pereira establish a theory of equivariant dendroidal Segal spaces in [27] and prove a comparison result with the theory of G-equivariant simplicial operads in [26].

Enriched ∞-Operads

This book has focused on operads in the category of simplicial sets. One of the great virtues of the concept of an operad (and its algebras) is that it makes sense in any symmetric monoidal category, and that operads and their algebras can be transported along suitable functors between such categories. For example, taking the homology of the little 2-cubes operad yields an operad in graded vector spaces whose algebras are precisely Gerstenhaber algebras [65], explaining why the homology of a double loop space has a Gerstenhaber algebra structure.

However, if $\mathbf{P}$ is an operad in a symmetric monoidal category $\mathbf{M}$, we cannot simply construct its nerve $N\mathbf{P}$ as a presheaf on $\mathbf{\Omega}$ as we did in the case where $\mathbf{M}$ is the category of (simplicial) sets. Indeed, the action by external face maps in $\mathbf{\Omega}$ would require $\mathbf{M}$ to have projections from a tensor product to its factors. For many purposes it is enough to consider $N\mathbf{P}$ in such a situation as a functor on the smaller category $\mathbf{\Omega}' \subseteq \mathbf{\Omega}$ of inner faces and isomorphisms only, i.e., the subcategory of morphisms which send leaves to leaves and preserve the root. For example, this applies to treatments of bar-cobar duality that we will briefly touch upon below.

A more elaborate solution to this problem that applies generally has been proposed by Chu and Haugseng [38], extending a similar treatment of ∞-categories in a symmetric monoidal ∞-category by Gepner and Haugseng [62]. To explain the basic idea, recall that a morphism $\alpha\colon S \to T$ in $\mathbf{\Omega}$ defines a partial map $V(\alpha)\colon V(T) \to$

$V(S)$ between sets of vertices. Now define a new category $\mathbf{\Omega}[\mathbf{M}]$ whose objects are pairs (S, m) where S is an object of $\mathbf{\Omega}$ and $m = (m_v)_{v \in V(S)}$ is a labelling of the vertices of S by objects of $\mathbf{M}$. Morphisms $(S, m) \to (T, n)$ are pairs consisting of a map $\alpha \colon S \to T$ in $\mathbf{\Omega}$ and a family of morphisms

$$\alpha_v \colon m_v \to \bigotimes_{V(\alpha)(w)=v} n_w$$

in $\mathbf{M}$, one for each vertex v of S. An operad $\mathbf{P}$ (uncoloured, for simplicity) defines a presheaf $N\mathbf{P}$ on $\mathbf{\Omega}[\mathbf{M}]$ by

$$N\mathbf{P}(S, m) = \prod_{v \in V(S)} \mathbf{M}(m_v, \mathbf{P}(\mathrm{in}(v))),$$

where $\mathrm{in}(v)$ is the collection of inputs of a vertex v, as usual. Carrying this idea over the context where $\mathbf{M}$ is an ∞-category (rather than an ordinary category), Chu and Haugseng now define a theory of ∞-operads enriched in $\mathbf{M}$ by considering such simplicial presheaves X on $\mathbf{\Omega}[\mathbf{M}]$ which satisfy a Segal property: the map induced by the corollas $C_v \to S$ for vertices v in S,

$$X(S, m) \to \prod_v X(C_v, m_v)$$

is an equivalence in **sSets**. Moreover, they require that for each corolla C_n with n leaves, the map

$$X(C_n, -) \colon \mathbf{M}^{\mathrm{op}} \to \mathbf{sSets}/X(\eta)^{n+1}$$

is fibrewise representable. This then defines for any vertices $x_1, \ldots, x_n$, y in $X(\eta)$ a fibre $X(x_1, \ldots, x_n; y)$, an object in $\mathbf{M}$ of 'operations from $x_1, \ldots, x_n$ to y'.

They go on to formulate a completeness condition in this context and prove general comparison theorems analogous to our comparison between dendroidal spaces and simplicial operads, as well as the comparison results mentioned in the first section of this epilogue.

Bar Constructions

An important application of the theory of operads is to what is often referred to as 'Koszul duality'. Briefly said, for an operad $\mathbf{P}$, say of spaces or of chain complexes, one can associate a dual cooperad $B\mathbf{P}$ by performing a certain bar construction. Then there exists an adjoint pair of functors relating $\mathbf{P}$-algebras to $B\mathbf{P}$-coalgebras with a host of interesting properties. Examples of this include Moore's bar-cobar duality for associative differential graded algebras and coassociative differential graded coalgebras, as well as Quillen's duality between differential graded Lie algebras and cocommutative differential graded coalgebras. Another example is the relation between $\mathbf{E}_n$-algebras and $\mathbf{E}_n$-coalgebras (and the resulting 'self-duality' of the theory of $\mathbf{E}_n$-algebras). The bar construction of an operad $\mathbf{P}$ in chain complexes was first defined by Ginzburg–Kapranov [68] and around the same time (in slightly different terms) by Getzler–Jones [65]. However, the construction is not limited to the differential graded context; both Salvatore [131] and Ching [36] define the bar construction for an operad $\mathbf{P}$ of spaces and Ching also for operads of spectra [37].

All of these constructions can be formulated in terms of colimits over certain categories of trees closely related to the category $\mathbf{\Omega}$ we have considered in this book. Let us single out the case where $\mathbf{P}$ is an operad in spaces for concreteness. Consider a finite set A with at least two elements. Then we write $\mathbf{\Omega}_A$ for the subcategory of $\mathbf{\Omega}$ on open trees with set of leaves precisely A and maps between them the injective morphisms of trees that preserve the root and are the identity on leaves. Any topological operad $\mathbf{P}$ defines a presheaf $N\mathbf{P}$ of spaces on $\mathbf{\Omega}$ and hence by restriction also on this category $\mathbf{\Omega}_A$. Following Ching and Salvatore, associate to an object T of $\mathbf{\Omega}_A$ a cube

$$w(T) := I \times \Big(\prod_{e \in I(T)} I \Big).$$

Here the product is over the inner edges $I(T)$ of T; note that this is almost identical to the space used to define the Boardman–Vogt W-construction, except that now we have an extra factor of I in front, which we think of as a label on the root of T. Write $w_0(T)$ for the subspace consisting of points where at least one edge is labelled 1 or the root edge is labelled 0 and define $\overline{w}(T) := w(T)/w_0(T)$. These spaces are functorial in T as in the case of the Boardman–Vogt construction; for an inner face map $T \longrightarrow S$ one assigns the label 0 to any newly arising inner edges. Then the bar construction $B\mathbf{P}$ is defined by a coend over the category $\mathbf{\Omega}_A$:

$$B\mathbf{P}(A) := \overline{w}(-) \otimes_{\mathbf{\Omega}_A} N\mathbf{P}(-).$$

It is proved in [131, 36, 37] that this construction produces a cooperad out of an operad. Also, as is already clear from this description, the bar construction is closely related to the W-construction. To be precise, if one defines the space $\mathrm{Indec}(W\mathbf{P}(A))$ of indecomposables to be the quotient of $W\mathbf{P}(A)$ by the subspace of operations that can be obtained as compositions of (non-identity) operations, then there is a homeomorphism $B\mathbf{P}(A) \cong \Sigma\mathrm{Indec}(W\mathbf{P}(A))$, with Σ denoting the reduced suspension. The description of the bar construction for differential graded operads, as in the work of Ginzburg–Kapranov and Getzler–Jones, is described in very similar terms in [87]. There it is also explained how Koszul duality for such operads may be understood in terms of combinatorial features of the category of trees and inner face maps between them.

Cyclic Operads, Modular Operads, Properads

There are several important variations of the notion of operad occurring in the literature. One of these is that of a *cyclic operad* introduced by Getzler and Kapranov [66]. In a cyclic operad, one can not only permute the input colours (as for any symmetric operad), but also 'rotate' the colours by an isomorphism

$$\mathbf{P}(c_1, \ldots, c_n; c_0) \cong \mathbf{P}(c_0, c_1, \ldots, c_{n-1}; c_n).$$

This typically happens when the **P**-algebras are objects in a category with duality such as that of finite-dimensional vector spaces. In the tree picture that we have advertised in this book, this corresponds to rotating a tree so that the root becomes a leaf and one of the leaves becomes a root. Getzler and Kapranov [67] also introduced the notion of a *modular operad*, which is a cyclic operad that moreover has 'contraction operations'

$$\mathbf{P}(c_1, \ldots, c_n; c_0) \to \mathbf{P}(c_1, \ldots, c_{n-2}; c_0).$$

A typical (uncoloured) example is the one where $\mathbf{P}(n)$ is the space of Riemann surfaces with $n + 1$ numbered boundary circles, with n of them serving as inputs and one as output. One can glue one such surface to another by identifying the output circle of the first with one of the input circles of the second to get an operadic composition, but one could also glue two of the boundary circles of a given surface to each other to obtain the stated 'contraction', which produces a surface with genus increased by one.

Yet another variation of the notion of operad is that of a so-called PROP, where operations can have multiple inputs as for operads, but also multiple outputs. Such PROPs are relevant for describing algebraic structures such as bialgebras and Hopf algebras in monoidal categories in which the tensor product is not the cartesian product. There is also a slightly restricted version of the notion of PROP, called a properad, enjoying better Koszul duality properties [138], and there are 'wheeled' variations of these notions allowing contractions as for modular operads.

In this book, we have observed that the nerve functor enables one to view operads (in **Sets**) as presheaves on the category $\mathbf{\Omega}$ satisfying the strict inner Kan condition and we have shown that the more general (not necessarily strict) inner Kan complexes, or ∞-operads, form the fibrant objects in a model structure on the category of presheaves on $\mathbf{\Omega}$. We have also shown that there is an equivalent model structure for complete Segal spaces on the category of simplicial presheaves on $\mathbf{\Omega}$. It is natural to ask to what extent such results can be proved for the generalizations of operads described above, by replacing $\mathbf{\Omega}$ by some other suitable category of graphs. There are a number of results in this direction of which we would like to mention a few here. Hackney, Robertson, and Yau [73, 74] study modular ∞-operads by considering presheaves on a category U of graphs, closely related to the Feynman graphs introduced by Joyal–Kock [93]. They prove that the category of modular operads in **Sets** embeds fully faithfully into the category of presheaves on **U** and study a localization of the Reedy model structure on the category of simplicial presheaves on **U** of which the fibrant objects can be viewed as modular ∞-operads. In [72], the same three authors define a variant $\boldsymbol{\Xi}$ of $\boldsymbol{\Omega}$, which plays a similar role, but now for cyclic operads. In [75] they extend these methods and results to (wheeled) properads. Another approach to cyclic operads, better suited to the coloured version, is taken by Walde in [140]. He also shows that dendroidal Segal spaces that are 'invertible', in the sense that inner face maps act by weak equivalences, are closely related to the 2-Segal spaces of Dyckerhoff and Kapranov [53].

Brane Actions and Gromov–Witten Invariants

In work by Toën [137] and Mann–Robalo [109], ∞-operads are used to give an interpretation of Gromov–Witten invariants. In order to explain some of the main ideas, let us start with operads in the category of sets for the moment. The 2-category of *spans* has sets as objects, while morphisms from A to B are diagrams of the form $A \leftarrow S \rightarrow B$ and 2-cells between two such spans are morphisms $S \rightarrow S'$ over $A \times B$. Composition is given by pullback of spans. Dually, there is a 2-category of *cospans* $A \rightarrow S \leftarrow B$ where composition is given by pushout. For a fixed object X, the contravariant functor $A \mapsto \mathrm{Hom}(A, X)$ sends cospans to spans. These span and cospan categories have a symmetric monoidal structure given by product and coproduct of sets, respectively. If $\mathbf{P}$ is a operad, then a $\mathbf{P}$-algebra in cospans is given by a set A and for each operation $p \in \mathbf{P}(n)$ a cospan

$$\amalg_{i=1}^{n} A \rightarrow S_p \leftarrow A$$

or, in other words, a single cospan

$$\amalg_{i=1}^{n} (\mathbf{P}(n) \times A) \rightarrow S \leftarrow A \times \mathbf{P}(n)$$

in the category of sets over $\mathbf{P}(n)$. These data then satisfy unit and associativity conditions with respect to the composition of operations in $\mathbf{P}$. A collection of maps as above is called a *lax* algebra if these conditions for a $\mathbf{P}$-action only hold up to specified 2-cells between cospans. If A is a lax $\mathbf{P}$-algebra in cospans, then $\mathrm{Hom}(A, X)$ is a lax $\mathbf{P}$-algebra in spans.

Toën observes that if $\mathbf{P}$ is any operad with $\mathbf{P}(0) = \mathbf{P}(1) = *$, then the set $\mathbf{P}(2)$ is a lax $\mathbf{P}$-algebra in cospans, with action given by

$$\begin{tikzcd} \amalg_i \mathbf{P}(n) \times \mathbf{P}(2) \arrow[r] \arrow[dr, "\pi_1"'] & \mathbf{P}(n+1) \arrow[d] & \mathbf{P}(2) \times \mathbf{P}(n) \arrow[l, "\circ_1"'] \arrow[dl, "\pi_2"] \\ & \mathbf{P}(n). & \end{tikzcd}$$

Here the middle arrow substitutes the constant of $\mathbf{P}(0)$, say in the first variable, and the left horizontal map is given by the $\circ_i$-operations and symmetries of $\mathbf{P}$ so as to make the diagram commute. If $\mathbf{P}$ satisfies certain conditions (is *of configuration* *type* in Toën's terminology, or *coherent* in Lurie's), then this lax action is in fact a true action, meaning the specified 2-cells are invertible. So for such a **P**, the set Hom(**P**(2), X) is a **P**-algebra in spans. Much of the work of [137] goes into showing that the same applies to an ∞-operad **P** and in the much wider context where one replaces the sets by objects with a geometric structure, such as spaces or objects in a topos or an ∞-topos. For the derived category of quasi-coherent sheaves $\mathbf{D}A$ (or some suitable variant of it) of an object A in such a category, a span

$$A \xleftarrow{p} S \xrightarrow{q} B$$

yields functors $q_* p^* \colon \mathbf{D}A \to \mathbf{D}B$ and these respect the composition of spans if a base change formula for direct images holds. Thus, under favourable circumstances, an operad **P** of configuration type acts on $\mathbf{D}\mathrm{Hom}(\mathbf{P}(2), X)$, the derived category of the 'spaces' of maps from $\mathbf{P}(2)$ into X. For example, if **P** is the $\mathbf{E}_{n+1}$-operad, then $\mathbf{E}_{n+1}(2)$ is homotopy equivalent to the n-sphere S^n and one concludes that $\mathbf{D}\mathrm{Hom}(S^n, X)$ is an $\mathbf{E}_{n+1}$-algebra for suitable X. Toën subsequently applies this to higher formality problems.

Mann–Robalo [109] apply these ideas to the construction of Gromov–Witten invariants roughly as follows. Consider the operad **P** with arity n operations given by $\overline{\mathcal{M}}_{0,n+1}$, the moduli space of genus zero stable curves with $n+1$ marked points. (Really, Mann–Robalo use a derived enhancement and consider it in a suitable ∞-category of derived stacks.) The marked points on such a curve C can be labelled $x_0, x_1, \ldots, x_n$, with x_0 considered the output and the other x_i's the inputs. Composition of operations is given by gluing curves. In this case, the operad **P** acts on $\mathbf{P}(2)$ by cospans as in the diagram above.

Now fix a smooth projective variety X and apply the internal hom $\mathrm{Map}_{\mathbf{P}(n)}(-, \mathbf{P}(n) \times X)$ in the category of spaces (or derived stacks) over $\mathbf{P}(n)$ to the earlier diagram expressing the **P**-action by cospans to get a span

$$\begin{tikzcd}
\overline{\mathcal{M}}_{0,n+1} \times X^n \arrow[dr] & \overline{\mathcal{M}}_{0,n+1}(X) \arrow[l] \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{0,n+1} \times X \arrow[dl] \\
& \overline{\mathcal{M}}_{0,n+1} &
\end{tikzcd}$$

over $\mathbf{P}(n)$. In this diagram, we have used that $\mathbf{P}(2) = \overline{\mathcal{M}}_{0,3} = *$ to identify $\mathrm{Map}(\mathbf{P}(2), X)$ with X itself. Also, the middle term $\overline{\mathcal{M}}_{0,n+1}(X)$ denotes the moduli space of stable curves C with $n+1$ marked points $x_0, \ldots, x_n$, equipped with a map $f \colon C \to X$. The arrow to the right evaluates f at x_0, whereas the arrow to the left evaluates at the remaining points $x_1, \ldots, x_n$.

The diagram above describes an action by spans of the operad $\overline{\mathcal{M}}_{0,\bullet+1}$ on X. As explained above, this also induces an action on (a version of) the derived category of X. In particular, one obtains an action of the homology of the operad **P** of moduli spaces of stable curves on the cohomology of X, which is the more usual perspective on Gromov–Witten invariants. The work of Mann–Robalo is an enhancement of this action on cohomology to a ‘geometric’ action by spans on X itself.

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