On some aspects of the dynamics of a ball in a rotating surface of revolution and of the kasamawashi art

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Dedicated to memory of Alexey V. Borisov

Abstract

We study some aspects of the dynamics of the nonholonomic system formed by a heavy homogeneous ball constrained to roll without sliding on a steadily rotating surface of revolution. First, in the case in which the figure axis of the surface is vertical (and hence the system is $SO(3) \times SO(2)$-symmetric) and the surface has a (nondegenerate) maximum at its vertex, we show the existence of motions asymptotic to the vertex and rule out the possibility of blow up. This is done passing to the 5-dimensional SO(3)-reduced system. The SO(3)-symmetry persists when the figure axis of the surface is inclined with respect to the vertical—and the system can be viewed as a simple model for the Japanese kasamawashi (turning umbrella) performance art—and in that case we study the (stability of the) equilibria of the 5-dimensional reduced system.

Keywords: Nonholonomic mechanics, Nonholonomic mechanical systems with symmetry, Rolling rigid bodies, Kasamawashi.

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1 Introduction

The system formed by a heavy homogeneous ball that rolls without sliding on a surface of revolution, which either is at rest or steadily rotates around its vertical figure axis with constant angular velocity Ω, is a classical system of nonholonomic mechanics. Its first studies go back to Routh, and there has recently been a renew of interest, see e.g. [19, 24, 9, 14, 13, 16, 8, 5, 6, 7, 2, 1, 11]. A rather general study of the dynamics of the system has been the object of the very recent article [11], which is the basis for the present study.

The system has an 8-dimensional phase space and an $SO(3) \times SO(2)$-symmetry (rotate the ball about its center and the center about the surface's figure axis). Reduction can be done in stages, obtaining first a 5-dimensional SO(3)-reduced system and then a 4-dimensional $SO(3) \times SO(2)$-reduced system. Most of the above analyses have been performed in either reduced system. The

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5-dimensional reduced system looses the information on the attitude of the ball and describes the motion of the center (or of the contact point) of the ball along the surface and of the ball's angular velocity. Specifically, a possible choice of the five coordinates in the SO(3)-reduced space are the horizontal coordinates and velocities of the center of the ball and the vertical component of the angular velocity vector (the other two components of the angular velocity vector are then determined by the rolling constraint). The 4-dimensional reduced system neglects also the rotation of the center of the ball around the surface's figure axis and describes only the radial motion of the center of the ball and, again, the angular velocity.

The unreduced system has three independent $SO(3) \times SO(2)$-invariant first integrals, which are inherited by both reduced systems. One is the energy if $\Omega = 0$ and a generalization of it called 'moving energy' if $\Omega \neq 0$ [16, 11]. The existence of the other two was proven by Routh if $\Omega = 0$ [21] and by Borisov, Kilin and Mamaev [9] if $\Omega \neq 0$. Therefore, the 4-dimensional $SO(3) \times SO(2)$-reduced system has three independent integrals of motion, and this has made it possible to prove a number of results on its dynamics. In particular, if the surface on which the ball rolls goes to $+\infty$ at infinity (sufficiently fast, if $\Omega \neq 0$), then the common level sets of these three integrals in the 4-dimensional reduced phase space are compact and the dynamics of the 4-dimensional reduced system is generically periodic; correspondingly, reconstruction results for relative periodic orbits of symmetric systems with compact symmetry groups (which date back to the 1980's and are due to Krupa and Field [17, 20], see also [19, 13, 10]) ensure that the dynamics of the 5-dimensional reduced system is generically almost-periodic on tori of dimension 2 and that of the unreduced system is generically almost-periodic on tori of dimension 3. This result was proven in the 1990s by Hermans [19] and Zenkov [24] in the case $\Omega = 0$, but its extension to the case of a rotating surface [16, 11] had to wait for the discovery of the conservation of the moving energy because the energy is (except in special situations [15]) not conserved for a nonholonomic system with nonhomogeneous constraints.

The study of the 4-dimensional reduced system benefits of the fact that, thanks to the existence of a rank-two Poisson structure that makes the system Hamiltonian ([9] for $\Omega = 0$, [11] for $\Omega \neq 0$) its phase space is foliated by two-dimensional invariant submanifolds on which the dynamics is Hamiltonian (and even Lagrangian). This allowed to study and classify, for instance, its equilibria [11]. Numerical investigations of the reduced dynamics in the particular case of a rotating conical surface are given in [7].

1.1 The dynamics near the vertex. Even though very successful, the analysis in the 4-dimensional $SO(3) \times SO(2)$-reduced space has a limitation due to the fact that the SO(2)-action is not free (the rotation about the figure axis keeps fixed all kinematical states in which the center of the ball is at the vertex of the surface with zero velocity—and the ball has any vertical angular velocity) and the SO(2)-reduced space is singular. This complicates the study of motions in which the ball passes through the vertex, which to our knowledge has never been undertaken so far.

Of course, it is intuitively clear that, whichever the geometry of the surface[1] and its rotational velocity Ω, the 4-dimensional reduced system has equilibria that correspond to the ball sitting at the vertex and spinning with any vertical angular velocity. However, their stability has not been investigated so far. In particular, it is not known if there are motions asymptotic to such equilibria at the vertex. Reference [11] points out that, particularly when $\Omega \neq 0$, it is not even ruled out the possibility of 'blow up' at the vertex, namely, of motions in which the center of the ball approaches (or even reaches in finite time) the vertex with the angular velocity of the ball that goes to infinity.

The main objective of the present article is to give some answers to these questions. Following an indication in [11], we will do it by studying the 5-dimensional SO(3)-reduced system, whose phase space is regular at the vertex. We will first of all prove that there is no possibility of blow up at the vertex. Next, we will investigate the reduced equilibria of the 5-dimensional reduced system

[1] As long as it is regular at the vertex, thus excluding e.g. the case of a conical surface

that correspond to the ball sitting at the vertex. Quite clearly, there is a one-parameter family of them (parametrized by the vertical component of the ball's angular velocity) and this implies that their Lyapunov stability may be elusive. Nevertheless, the study of the linearization at these equilibria gives important information, because the presence of eigenvalues with negative (positive) real part implies the existence of a stable (unstable) manifold and hence of motions asymptotic to the vertex for $t \to +\infty$ ($t \to -\infty$). We will show that, if the surface has a (local or global) nondegenerate maximum at the vertex, then motions of this type do exist. In addition, we will study some aspects of the stability of the reduced equilibria at the vertex.

1.2 Kasamawashi, or the ball on a rotating umbrella. We take the opportunity of approaching this study from a more general perspective and consider the more general case in which the figure axis of the surface of revolution on which the ball rolls may also be inclined of a certain angle α with respect to the vertical. For $\alpha = 0$ we have the system described above. The system with $\alpha \neq 0$ does not appear to have been investigated so far, except in the case in which the surface is a plane [5].

If $\alpha \neq 0$ the system looses the SO(2)-symmetry (except for special geometries of the surface, such as that of a sphere) but retains its SO(3)-symmetry. It is therefore possible to consider the 5-dimensional SO(3)-reduced system. We do not undertake here a systematic study of the dynamics of this reduced system, which if $\alpha \neq 0$ can be expected to be nonintegrable. However, as a slight extension of our study of the case $\alpha = 0$ we will investigate the equilibria of the SO(3)-reduced system and their stability. We shall show that the only equilibria of such reduced system correspond to motion of the unreduced system in which the center of the ball stays fixed in space, touching the surface at a point at which the tangent plane is horizontal (due to the rotational symmetry of the surface, the contact takes place at a point that changes in the surface but stays fixed in space), and spins with any vertical angular velocity. We shall analyze the spectral stability of these reduced equilibria.

It is tempting, if not even natural, to relate this analysis to the Japanese *kasamawashi* ("turning umbrella") art, in which a ball (or a disk or ring) is posed on a tilted conic umbrella, that the performer rotates so as to keep the ball at the same spatial position. The art is very fascinating and its modelling, of course, is a matter of control (the realization of a robot that performs kasamawashi through a PID controller has been reported in [22], without however any mathematical or modelling detail). Nevertheless, this purely dynamical approach seems capable of giving some information.

2 The system

2.1 The nonholonomic system We follow the description of the system given in [11], which however considers only the case $\alpha = 0$ (and, less important, the case in which the surface is a graph over $\mathbb{R}^2$, namely $D = \mathbb{R}^2$ in the notation below). We begin considering the holonomic system formed by a homogeneous ball of mass m and radius a whose center C is constrained to a smooth surface of revolution Σ which is embedded in $\mathbb{R}^3 \ni (\mathrm{X}, \mathrm{Y}, \mathrm{Z})$ with its vertex at the origin and its figure axis that forms an angle α, $0 \le \alpha < \frac{\pi}{2}$, with the Z-axis, which is the ascending vertical. We describe the system with respect to a (spatial) reference frame $\{O; \mathrm{x}, \mathrm{y}, \mathrm{z}\}$ with the origin O at the vertex of Σ, the z-axis aligned with the figure axis of Σ, the y-axis horizontal and the x-axis tangent to Σ at the vertex, see Fig. 1. We parametrize the system with the rescaled coordinates $x = (x_1, x_2) = (\frac{\mathrm{x}}{a}, \frac{\mathrm{y}}{a})$ and describe the surface Σ, in the frame $\{O; \mathrm{x}, \mathrm{y}, \mathrm{z}\}$, via the parametrization

$$D \ni x = (x_1, x_2) \mapsto \big(ax_1, ax_2, af(|x|)\big)$$

where $D = \{x \in \mathbb{R}^2 \,:\, |x| < R\}$ for some $R > 0$ or $R = +\infty$ and $f : I \to \mathbb{R}$ with $I = (-R, R)$ is an even smooth function that we call the profile function ($|\ |$ denotes the Euclidean norm in $\mathbb{R}^2$). Obviously, $f'(0) = 0$.

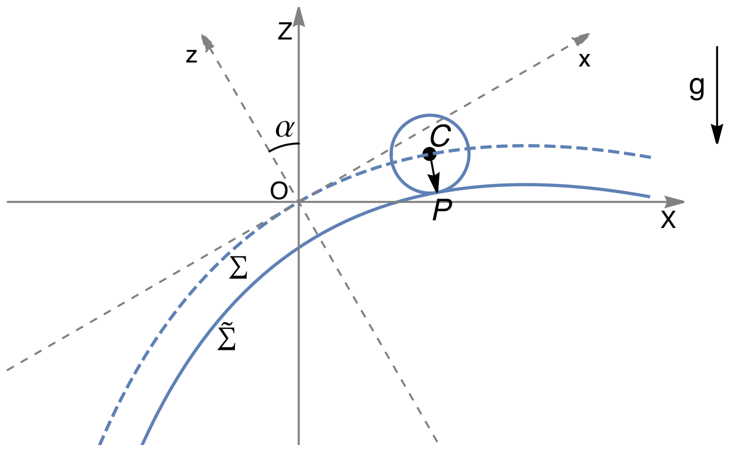

Figure 1: The generatrices Γ and $\tilde\Gamma$ of the surfaces Σ and $\tilde\Sigma$.

Since smoothness at $x = 0$ of the function $x \mapsto f(|x|)$ is not manifest, and we are specifically interested to the dynamics near that set, following [14, 11] we will use instead a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$f(r) = \psi\big(\tfrac12 r^2\big) \qquad \forall r \in I\,.$$

The existence of such a function is granted by a result of Whitney [23] (see also [18], pages 103 and 108, and [14]) on account of the fact that f is even. Note that

$$\begin{aligned} \psi'\big(\tfrac12 r^2\big) &= \frac{f'(r)}{r} \qquad \text{for } r>0\,, & \psi'(0) &= f''(0)\,, \\ \psi''\big(\tfrac12 r^2\big) &= \frac{rf''(r)-f'(r)}{r^3} \qquad \text{for } r>0\,, & \psi''(0) &= \frac13 f^{(\mathrm{iv})}(0)\,. \end{aligned} \tag{1}$$

In stating our results, we will however use as much as possible the profile function f and its derivatives, whose interpretation is more direct.

The configuration manifold of this holonomic system can be identified with $D\times \mathrm{SO}(3) \ni (x,\mathcal{R})$, where $\mathcal{R}$ is the matrix that turns the spatial frame $\{O;\mathrm{x},\mathrm{y},\mathrm{z}\}$ into some chosen body frame, and, after (right) trivialization of the tangent bundle of $SO(3)$, its phase space can be identified with the 10-dimensional manifold

$$M_{10} = D\times \mathrm{SO}(3)\times\mathbb{R}^2\times\mathbb{R}^3 \ni (x,\mathcal{R},v,\omega)$$

where $v = (v_1,v_2) = \dot x$ and $\omega = (\omega_{\mathrm x},\omega_{\mathrm y},\omega_{\mathrm z})$ is the angular velocity of the ball relative to, and written in, the frame $\{O;\mathrm{x},\mathrm{y},\mathrm{z}\}$.

We assume that the only active force that acts on the system is weight, directed as the downward vertical Z-axis, and denote by $a\hat g$ the gravity acceleration and by mka^2, with some $0<k<1$, the moment of inertia of the ball with respect to its center. Thus, since $Z = \mathrm{x}\sin\alpha + a\psi\big(\tfrac12|x|^2\big)\cos\alpha$ in the points of Σ and the velocity V_C of C equals $a\big(v_1, v_2,\, x\cdot v\,\psi'(\tfrac12|x|^2)\big)$, the Lagrangian of the holonomic system is

$$\mathcal{L}(x,\mathcal{R},v,\omega) = \frac12|v|^2 + \frac12\big(x\cdot v\,\psi'\big(\tfrac12|x|^2\big)\big)^2 + \frac12 k|\omega|^2 - \hat g\big(x_1\sin\alpha + \psi\big(\tfrac12|x|^2\big)\cos\alpha\big)\,. \tag{2}$$

We now add to the system the nonholonomic constraint that the ball rolls without sliding on a surface $\widetilde\Sigma$ which lies *below* Σ, is parallel to it, and rotates with constant angular velocity $\Omega e_{\mathrm z}$ about its figure axis, namely, the z-axis. We assume that

$$f''(r) > -(1+f'(r)^2)^{3/2} \qquad \forall r \in I\,, \tag{3}$$

which ensures that $\tilde\Sigma$ is regular and that, in any configuration, the ball touches $\tilde\Sigma$ in a single point, see [11].

Since the point P of the ball in contact with $\tilde\Sigma$ has velocity $V_C + \omega \times CP$ and the point of $\tilde\Sigma$ with which P is in contact has velocity $\Omega e_z \times OP$, the nonholonomic constraint is given by

$$V_C + \omega \times CP = \Omega e_{\rm z} \times OP\,. \tag{4}$$

Here, $OP = OC + CP = \big(ax_1, ax_2, a\psi\big(\tfrac12|x|^2\big)\big) + an(x)$ with $n(x)$ the (downward) normal unit vector to Σ at its point C, namely $n(x) = \frac{1}{F(|x|)}\big(x_1\psi'(\tfrac12|x|^2), x_2\psi'(\tfrac12|x|^2), -1\big)$ with the function $F: I \to \mathbb{R}$ defined as

$$F(r) \;:=\; \sqrt{1+f'(r)^2} = \sqrt{1+r^2\psi'(\tfrac12 r^2)^2}\,. \tag{5}$$

Thus, the first two entries of (4) can be written[2]

$$\omega_{\rm x} = \;-Fv_2 - x_1\psi'\omega_{\rm z} + \Omega x_1(1+\psi')\,, \quad \omega_{\rm y} = \;Fv_1 - x_2\psi'\omega_{\rm z} + \Omega x_2(1+\psi') \tag{6}$$

(the third entry is not independent) and define an 8-dimensional submanifold of M_{10} which is diffeomorphic to

$$M_8 = D \times \mathrm{SO}(3) \times \mathbb{R}^2 \times \mathbb{R} \ni (x, \mathcal{R}, \dot x, \omega_{\rm z})$$

and is the phase space of the nonholonomic system.

The equations of motion of the nonholonomic system in M_8 can be obtained with various standard techniques, and are the five equations

$$\begin{aligned}
\dot x_1 =& v_1\\
\dot x_2 =& v_2\\
\dot v_1 =& -\frac{1}{F^2}\Big[\gamma\big(x_1\psi'\cos\alpha + (1+x_2^2\psi'^2)\sin\alpha\big) - \mu\big(v_2\psi' + x_2\, x\!\cdot\! v\,\psi''\big)\omega_{\rm z}F\\
&\quad + \mu v_1\, x\!\cdot\! v\,\big(\psi' + |x|^2\psi''\big)\psi' + \frac{x_1}{1+k}\big(|v|^2\psi' + (x\!\cdot\! v)^2\psi''\big)\psi'\\
&\quad - \Omega\mu\big[v_2F(F+\psi') + x_2\, x\!\cdot\! v\,\big(\psi'^2 + |x|^2\psi'\psi'' + F\psi''\big)\big]\Big]\\
\dot v_2 =& -\frac{1}{F^2}\Big[\gamma x_2\psi'(\cos\alpha - x_1\psi'\sin\alpha) + \mu\big(v_1\psi' + x_1\, x\!\cdot\! v\,\psi''\big)\omega_{\rm z}F\\
&\quad + \mu v_2\, x\!\cdot\! v\,\big(\psi' + |x|^2\psi''\big)\psi' + \frac{x_2}{1+k}\big(|v|^2\psi' + (\,x\!\cdot\! v\,)^2\psi''\big)\psi'\\
&\quad + \Omega\mu\big[v_1F(F+\psi') + x_1\, x\!\cdot\! v\,\big(\psi'^2 + |x|^2\psi'\psi'' + F\psi''\big)\big]\Big]\\
\dot\omega_{\rm z} =& -\frac{1}{F}\gamma x_2\psi'\sin\alpha - \frac{x\!\cdot\! v\,\psi'}{(1+k)F^3}\Big[\big(\omega_{\rm z}F + (x_1v_2 - x_2v_1)\psi'\big)\big(\psi' + |x|^2\psi''\big)\\
&\quad - \Omega\big[F^2 + (F+|x|^2\psi')\big(\psi' + |x|^2\psi''\big)\big]\Big]
\end{aligned} \tag{7}$$

where $\gamma = \frac{\hat g}{1+k}$ and $\mu = \frac{k}{1+k}$, completed with the restriction to M_8 of the equation $\dot{\mathcal{R}} = \hat\omega\mathcal{R}$ with $\hat\omega$ the antisymmetric matrix associated to the vector $(\omega_{\rm x}, \omega_{\rm y}, \omega_{\rm z}) \in \mathbb{R}^3$ (with $\omega_{\rm x}$ and $\omega_{\rm y}$ as in (6)). Some indications on how to obtain these equations are given in the Appendix.

[2]From now it is understood that, unless differently specified, ψ and its derivatives are evaluated at $\frac12|x|^2$ and F at $|x|$.

Remark. *This formulation assumes smoothness of the surface Σ. In certain cases—such as that of a cone—the surface is not smooth at the vertex. In such cases, Eqs.* (7) *describe the motions outside a neighbourhood of the vertex. Thus, they can be used to study the equilibria of the system at locations different from the vertex, which is what we will do for an inclined conic surface in section 5.*

2.2 The SO(3)-reduced system Consider now the right action Ξ of SO(3) on M_{10} on the SO(3)-factor: $\Xi_S(x, \mathcal{R}, \dot{x}, \omega) = (x, \mathcal{R}S, \dot{x}, \omega)$. From (6) it follows that the constraint manifold M_8 is invariant under the action Ξ and thus Ξ restricts to an action on M_8. Since the Lagrangian (2) as well is invariant under Ξ, the equations of motion of the nonholonomic system in M_8 can be reduced to $M_8/\mathrm{SO}(3)$ [3, 4]. Since the Lagrangian and the constraint are independent of the attitude $\mathcal{R}$ of the ball, the SO(3)-reduction consists in simply cutting off the factor SO(3) of M_8. Thus, the SO(3)-reduced space is the five-dimensional manifold

$$M_5 = D \times \mathbb{R}^2 \times \mathbb{R} \ni (x, v, \omega_{\mathrm{z}})$$

and the equations of motion of the reduced system are Eqs. (7). These equations define a vector field on M_5.

Note that the motion $t \mapsto (x(t), v(t), \omega_{\mathrm{z}}(t))$ of the SO(3)-reduced system determines the motion $t \mapsto (x(t), \mathcal{R}(t), v(t), \omega_{\mathrm{z}}(t))$ of the unreduced system except for the attitude $t \mapsto \mathcal{R}(t)$ of the ball, which can in principle be determined via the "reconstruction equation" $\dot{\mathcal{R}}(t) = \hat{\omega}(t)\mathcal{R}(t)$, where $t \mapsto \omega(t) = (\omega_{\mathrm{x}}(t), \omega_{\mathrm{y}}(t), \omega_{\mathrm{z}}(t))$ with the first two components determined by the constraint Eq. (6).

3 The equilibria of the SO(3)-reduced system

3.1 The SO(3)-reduced equilibria. We determine now the equilibria of the SO(3)-reduced system.

Proposition 1. *The equilibria of the SO(3)-reduced system are the points $(x, 0, \omega_{\mathrm{z}}) \in M_5$ with any $\omega_{\mathrm{z}} \in \mathbb{R}$ and any x such that the normal to the surface Σ at the point of coordinate x has horizontal tangent plane, namely:*

i. If $\alpha = 0$, x such that $f'(|x|) = 0$.

ii. If $\alpha \neq 0$, $x_2 = 0$ and x_1 such that $f'(|x_1|) = -\mathrm{sign}(x_1)\tan(\alpha)$.

Proof. At an equilibrium, $v_1 = v_2 = 0$ and the vanishing of $\dot{v}_1$, $\dot{v}_2$ and $\dot{\omega}_{\mathrm{z}}$ in (7) gives the three conditions

$$\begin{aligned} & x_1\psi'\big(\tfrac{1}{2}|x|^2\big)\cos\alpha + \big(1 + x_2^2\psi'(\tfrac{1}{2}|x|^2)^2\big)\sin\alpha = 0 \\ & x_2\psi'\big(\tfrac{1}{2}|x|^2\big)\big(x_1\psi'\big(\tfrac{1}{2}|x|^2\big)\sin\alpha - \cos\alpha\big) = 0 \\ & x_2\psi'\big(\tfrac{1}{2}|x|^2\big)\sin\alpha = 0 \end{aligned} \tag{8}$$

on $x = (x_1, x_2)$. Since ω_{z} does not enter them, it is arbitrary at the equilibria.

If $\alpha = 0$, then the last condition (8) is satisfied for all x while the first two give $x_1\psi'\big(\frac{1}{2}|x|^2\big) = x_2\psi'\big(\frac{1}{2}|x|^2\big) = 0$. These two conditions are satisfied at all points at which $x = 0$ and/or $\psi'\big(\frac{1}{2}|x|^2\big) = 0$, namely, as follows from (1), all points at which $f'(|x|) = 0$.

If $\alpha \neq 0$, then the last condition (8) is satisfied if $x_2 = 0$ and/or if $\psi'\big(\frac{1}{2}|x|^2\big) = 0$. But in the latter case the first condition (8) is never satisfied because $\sin\alpha \neq 0$. If $x_2 = 0$ then the second condition (8) is satisfied by all x_1 and the first one reduces to $x_1\psi'\big(\frac{1}{2}x_1^2\big)\cos\alpha + \sin\alpha = 0$. Since $\sin\alpha \neq 0$, necessarily $x_1 \neq 0$ and $f'(|x_1|)\mathrm{sign}(x_1) = x_1\psi'\big(\frac{1}{2}x_1^2\big) = -\tan\alpha$.

The normal to Σ at the point C, in the frame $\{O; X, Y, Z\}$, is $\begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} n(x)$ and the vanishing of its first two components is equivalent to $x_2 = 0$, $\sin\alpha + x_1\psi'\big(\frac{1}{2}x_1^2\big)\cos\alpha = 0$. $\square$

The SO(3)-reduced equilibria reconstruct to (SO(3)-families of) motions of the unreduced system in which the ball 'sits' at a point in space and either spins around its center or stays still. These families of motions form the so-called relative equilibria of the unreduced system. It follows from the already mentioned reconstruction theory of Krupa and Field that, since SO(3) is compact and has rank one, all motions of the ball in a relative equilibrium are periodic (or, as a particular case, equilibria, which happens if $\omega_{\rm z} = \Omega = 0$).

Since $f'(0) = 0$, when $\alpha = 0$ there is always a family of reduced equilibria with $x = 0$ and any $\omega_{\rm z}$, that we call "reduced equilibria at the vertex".

In addition, when $\alpha = 0$, there are families of reduced equilibria with any $\omega_{\rm z} \in \mathbb{R}$ and any x in a 'critical parallel' of the surface Σ, namely the parallels on which $f' = 0$. We note that the existence of these reduced equilibria was already proven in [11]. Specifically, the equilibria of "type RE2" of the SO(3) $\times$ SO(2)-reduced system found in [11] reconstruct exactly to these equilibria of the SO(3)-reduced system (see particularly section 5.2 of [11]). Since their (spectral) stability properties have already been investigated in [11], we will not consider them here anymore.

When $\alpha \neq 0$, instead, the reduced equilibria reconstruct to periodic orbits (equilibria) of the unreduced system in which the ball spins around the vertical (stays still) and touches the surface at a point at which the tangent plane to the surface is horizontal and stays fixed in space. Note that, if $\alpha \neq 0$, the contact point at such reduced equilibrium is never at the vertex of $\tilde{\Sigma}$.

Remark. *It follows from the reconstruction of the equilibria of the SO(3)* $\times$ *SO(2)-reduced system in [11] that, for* $\alpha = 0$*, the SO(3)-reduced system possesses periodic orbits in which the center of the ball moves steadily on any parallel of the surface.*

3.2 Linearization. Since in the SO(3)-reduced equations of motion (7) the coordinate $\omega_{\rm z}$ is always multiplied by either v_1 or v_2, the last column of the Jacobian matrix of the SO(3)-reduced vector field vanishes at the equilibria. Therefore, the linearization at a reduced equilibrium has always an eigenvalue 0. Its presence is related to the fact that the reduced equilibria all come in families, parametrized by $\omega_{\rm z} \in \mathbb{R}$. The remaining four eigenvalues are determined by the first 4×4 block of the linearization matrix.

As already said, when $\alpha = 0$ we exclude from our consideration the reduced equilibria with $x \neq 0$.[3] In the remaining equilibria $x_2 = 0$ and the first 4×4 block of the linearization matrix at the equilibrium $(x_1, 0, 0, 0, \omega_{\rm z})$ has the form

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_{31} & 0 & 0 & a_{34} \\ 0 & a_{42} & a_{43} & 0 \end{pmatrix} \qquad (9)$$

[3] They form two-parameter families and therefore there at least two zero eigenvalues of the linearization. But in fact, there are always three zero eigenvalues; this can be explained through the already mentioned fact that the SO(3) $\times$ SO(2)-reduced system has a Hamiltonian structure.

with

$$\begin{aligned}
a_{31} &= \frac{\gamma}{F^4}\big(\psi' + x_1^2\psi''\big)\big(2x_1\psi'\sin\alpha + (x_1^2\psi'^2 - 1)\cos\alpha\big)\\
a_{34} &= \frac{\mu}{F}\psi'\omega_{\rm z} - \Omega\frac{\mu}{F^2}\big(1 + F\psi' + x_1^2\psi'^2\big)\\
a_{42} &= \frac{\gamma}{F^2}\psi'\big(x_1\psi'\sin\alpha - \cos\alpha\big)\\
a_{43} &= -\frac{\mu}{F}\big(\psi' + x_1^2\psi''\big)\omega_{\rm z} + \Omega\frac{\mu}{F^2}\big(F^2 + (x_1^2\psi' + F)\psi' + x_1^2(F + x_1^2\psi')\psi''\big)\,.
\end{aligned} \tag{10}$$

where ψ' and ψ'' are evaluated at $\frac{1}{2}x_1^2$ and F at x_1. The characteristic polynomial of this matrix is the biquadratic polynomial

$$\lambda^4 - (a_{31} + a_{42} + a_{34}a_{43})\lambda^2 + a_{31}a_{42}\,. \tag{11}$$

4 The dynamics near the vertex in the case $\alpha = 0$

In this section we consider the system formed by the ball nonholonomically constrained to the surface with $\alpha = 0$. The main question is whether there exist motions in which, asymptotically, the ball tends to the vertex.

4.1 No blow up at the vertex. First, we show that no such motions exists in which the angular velocity $\omega_{\rm z}$ explodes. This answers a question raised in [11]. This question is not completely trivial because, when $\Omega \neq 0$, the energy is not conserved. Nevertheless, when $\alpha = 0$ the unreduced system has the first integral

$$\begin{aligned}
E(x, v, \omega_{\rm z}) =& \frac{1}{2}|v|^2 + \frac{1}{2}\Big(x\cdot v\,\psi'\big(\tfrac{1}{2}|x|^2\big)\Big)^2 + \frac{k}{2}\omega_{\rm z}^2 - \Omega(x_1v_2 - x_2v_1) + k\Omega\omega_{\rm z}\\
&+ \frac{k}{2}\Big((v_1 + \Omega x_2)F(|x|) + x_2(\Omega - \omega_{\rm z})\psi'\big(\tfrac{1}{2}|x|^2\big)\Big)^2\\
&+ \frac{k}{2}\Big((v_2 - \Omega x_1)F(|x|) - x_1(\Omega - \omega_{\rm z})\psi'\big(\tfrac{1}{2}|x|^2\big)\Big)^2 + \hat{g}\psi\big(\tfrac{|x|^2}{2}\big)
\end{aligned}$$

which coincides with the energy for $\Omega = 0$ and, for $\Omega \neq 0$, is called a 'moving energy'. The existence of this integral for $\Omega \neq 0$ was proven in [16] and its expression was computed in [8]. This function coincides with the function variously called "energy", "total energy", "Jacobi integral" in Lagrangian mechanics but the fact that—under certain conditions—it is a first integral for nonholonomic systems with constraints which are affine (linear nonhomogenous) in the velocities was proven only very recently. We refer to [16, 8, 12] for the theory of moving energies in nonholonomic mechanics.

The impossibility of blow ups is certainly ensured by the compactness of the level sets of the moving energy, which intuitively prevents $\omega_{\rm z}$ to "go to infinity" and, more precisely, ensures the completeness of the dynamical vector field. The compactness of *all* the level sets of E in the $\mathrm{SO}(3) \times \mathrm{SO}(2)$-reduced system was proven in [11], Proposition 7, in the case $D = \mathbb{R}^2$, under the hypothesis that the profile function goes to $+\infty$ at infinity, and does it sufficiently fast if $\Omega \neq 0$. Due to the compactness of SO(2) and SO(3), this result extends to the SO(3)-reduced system and to the unreduced one. However if, at infinity, the profile function goes to $-\infty$ or is bounded, then certainly there are level sets of the moving energy which reach infinity in the factor $\mathbb{R}^2 \ni x$ of M_8 and are not compact.

Nevertheless, as we show here, there cannot be blow ups at the vertex. This is due to the fact that, on each level set of E, the coordinates v and $\omega_{\rm z}$ cannot go to infinity near $x = 0$:

Proposition 2. *Assume $\alpha = 0$. Then, for any $\Omega \in \mathbb{R}$ and any $L > 0$ the level sets of E have compact intersection with the subset of M_5 where $|x| \leq L$.*

Proof. Consider $E_0 \in \mathbb{R}$ such that the set $S_{E_0} = \{(x, v, \omega_z) \in M_5 : E(x, v, \omega_z) = E_0 \ \ |x| \le L\}$ is not empty. Since E is continuous, S_{E_0} is closed and we need to prove that it is bounded. Note that

$$\begin{aligned} E_0 = E(x, v, \omega_z) \ &\ge \ \frac{1}{2}|v|^2 + \frac{k}{2}\omega_z^2 - |\Omega|\,|x|\,|v| - k|\Omega|\,|\omega_z| - \hat{g}\psi\big(\tfrac{1}{2}|x|^2\big) \\ &= \ \frac{1}{2}\big(|v| - |\Omega|\,|x|\big)^2 + \frac{k}{2}\big(|\omega_z| - |\Omega|\big)^2 - \frac{1}{2}\Omega^2|x|^2 - \frac{1}{2}k\Omega^2 - \hat{g}\psi\big(\tfrac{|x|^2}{2}\big)\,. \end{aligned}$$

Hence, for $|x| \le L$,

$$E_0 \ge \frac{1}{2}\big(|v| - |\Omega|\,|x|\big)^2 + \frac{k}{2}\big(|\omega_z| - |\Omega|\big)^2 - C$$

with $C = \frac{1}{2}(k + L^2)\Omega^2 + \max_{0\le r\le L}|f(r)|$. Thus, $\big(|v| - |\Omega|\,|x|\big)^2 + k\big(|\omega_z| - |\Omega|\big)^2\big) \le 2(E_0 + C)$ and so $|v| \le L|\Omega| + \sqrt{2(E_0 + C)}$ and $|\omega_z| \le |\Omega| + \sqrt{\frac{2}{k}(E_0 + C)}$. □

4.2 Linearization at the vertex. We study now the possibility that motions tend asymptotically to the vertex. To simplify the exposition we say that an eigenvalue of the linearization is of type Z if it is zero, of type C if it is purely imaginary and nonzero, of type R_+ (R_-) if it is real and positive (negative) and of type F_+ (F_-) if it has nonzero imaginary part and positive (negative) real part.

As is well known, the presence of only eigenvalues with zero real part, hence of types Z and C, is a necessary condition for Lyapunov stability called "spectral stability". The presence of some eigenvalue with positive real part, namely of types R_+ and F_+, implies Lyapunov instability.

But foremost, we are interested in the existence of motions which are asymptotic, in the future or in the past, to the equilibria at the vertex, which are related to the presence of eigenvalues of types R_-, F_- and R_+, F_+, respectively.

We may limit our analysis to the 4×4 block (9) of the linearization. Obviously, its complex eigenvalues come in conjugate pairs, but further limitations come from the biquadratic structure of the characteristic polynomial (11).

Proposition 3. *Assume $\alpha = 0$ and define the function*

$$B(\omega_z, \Omega) = \big(1 + f''(0)\big)\Omega - f''(0)\omega_z\,.$$

Then, for any $\Omega \in \mathbb{R}$, the four eigenvalues of the 4×4 block (13) of the linearization at the reduced equilibrium $(0, 0, \omega_z)$ are of the following types:

i. If $f''(0) = 0$: $ZZZZ$ if $\Omega = 0$ and $ZZCC$ if $\Omega \ne 0$.

ii. If $f''(0) > 0$: $CCCC$.

iii. If $f''(0) < 0$: $CCCC$ if $B(\omega_z, \Omega)^2 \ge 4\gamma\mu^{-2}|f''(0)|$, $F_+F_+F_-F_-$ if $0 < B(\omega_z, \Omega)^2 < 4\gamma\mu^{-2}|f''(0)|$, and $R_+R_+R_-R_-$ if $B(\omega_z, \Omega) = 0$.

Proof. Preliminarily note that, if $c \ge 0$, then the four roots of the biquadratic equations $\lambda^4 + 2b\lambda^2 + c = 0$ are of the following types. If $c = 0$: $ZZZZ$ if $b = 0$, $ZZCC$ if $b > 0$, ZZR_+R_- if $b < 0$. If $c > 0$: $F_+F_+F_-F_-$ if $b^2 < c$, $R_+R_+R_-R_-$ if $b^2 \ge c$ and $b < 0$, $CCCC$ if $b^2 \ge c$ and $b > 0$.

When $\alpha = 0$, the four coefficients (10) evaluated at the equilibrium $(0, 0, \omega_z)$ are $a_{31} = a_{42} = -\gamma f''(0)$ and $a_{34} = -a_{43} = -\mu B(\omega_z, \Omega)$ (use $\psi'(0) = f''(0)$, $F(0) = 1$). Therefore, the characteristic polynomial (11) is $\lambda^4 + 2b\lambda^2 + c$ with

$$b = \gamma f''(0) + \frac{1}{2}\mu^2 B(\omega_z, \Omega)^2\,, \qquad c = \big(\gamma f''(0)\big)^2\,.$$

(i.) If $f''(0) = 0$ then $B(\omega_z, \Omega) = \Omega$ and so $b = \frac{1}{2}(\mu\Omega)^2$ and $c = 0$. If $\Omega = 0$ then $b = 0$ and the eigenvalues type is $ZZZZ$. If $\Omega \neq 0$ then $b > 0$ and the eigenvalues type is $ZZCC$.

(ii.) If $f''(0) > 0$ then $c > 0$ and, since $B(\omega_z, \Omega)^2 \geq 0$, $b \geq \gamma f''(0) > 0$ and $b^2 \geq \big(\gamma f''(0)\big)^2 = c$. Thus, the eigenvalues type is $CCCC$.

(iii.) Assume $f''(0) < 0$ and write B for $B(\omega_z, \Omega)$. Thus $b = \frac{1}{2}\mu^2 B^2 - \gamma|f''(0)|$, $c = (\gamma|f''(0)|)^2 > 0$ and

$$b^2 - c = \big(b + \gamma|f''(0)|\big)\big(b - \gamma|f''(0)|\big) = \frac{1}{4}\mu^2 B^2\big(\mu^2 B^2 - 4\gamma|f''(0)|\big)\,.$$

We now distinguish two cases. (1) If $\mu^2 B^2 - 4\gamma|f''(0)| \geq 0$ then $b^2 - c \geq 0$ and, since $b = \frac{1}{2}\mu^2 B^2 - \gamma|f''(0)| \geq \gamma|f''(0)| > 0$, the eigenvalues type is $CCCC$. (2) If $\mu^2 B^2 - 4\gamma|f''(0)| < 0$ and $B \neq 0$ then $b^2 - c < 0$ and the eigenvalues type is $F_+F_+F_-F_-$. If instead $B = 0$ then $b^2 - c = 0$, $b = -\gamma|f''(0)| < 0$ and the eigenvalues type is $R_+R_+R_-R_-$. □

Proposition 3 implies that when $f''(0) \geq 0$ all reduced equilibria at the vertex are spectrally stable.

Instead, when $f''(0) < 0$, namely the surface has a nondegenerate maximum at the vertex, the situation is richer. In such a case $B(\omega_z, \Omega) = (1 - |f''(0)|)\Omega + |f''(0)|\omega_z$, with $1 - |f''(0)| > 0$ because of (3), the loci $B(\omega_z, \Omega) = \text{const}$ in the (ω_z, Ω)-plane are straight lines, and the regions of different eigenvalues types are as in Fig. 2. Therefore, for fixed Ω, the spectrally stable reduced equilibria $(0, 0, \omega_z)$ are those with ω_z outside an open bounded interval (which depends on Ω, and may include $\omega_z = 0$). In particular, when $\Omega = 0$, the spectrally stable reduced equilibria are those with $|\omega_z| \geq \frac{2}{\mu}\sqrt{\frac{\gamma}{|f''(0)|}}$. Interestingly, each reduced equilibrium $(0, 0, \omega_z)$ becomes eventually spectrally stable for $|\Omega|$ large enough. In this sense, the rotation of the surface has a "stabilizing" effect—a phenomenon of which some instances had already been pointed out in [11].

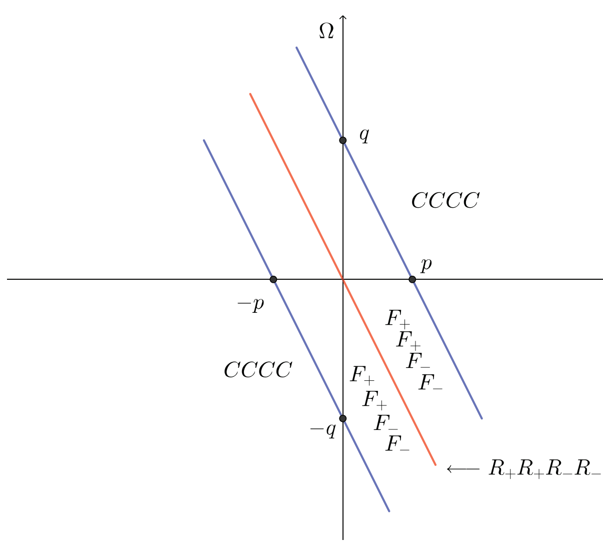

Figure 2: The types of the four eigenvalues of the block (13) when $f''(0) < 0$ as functions of ω_z and Ω. The marked points are $p = \Big(\frac{2}{\mu}\sqrt{\frac{\gamma}{|f''(0)|}}, 0\Big)$ and $q = \Big(0, \frac{2\sqrt{\gamma|f''(0)|}}{\mu(1-|f''(0)|)}\Big)$.

But moreover, when $f''(0) < 0$, for (ω_z, Ω) in the instability region

$$-2\mu^{-1}\sqrt{\gamma|f''(0)|} < \big(1 + f''(0)\big)\Omega - f''(0)\omega_z < 2\mu^{-1}\sqrt{\gamma|f''(0)|} \tag{12}$$

the reduced equilibrium $(0, 0, \omega_z)$ at the vertex has a two-dimensional stable manifold and a two-dimensional unstable manifold on which all motions tend to the equilibrium for, respectively, $t \to +\infty$ and $t \to -\infty$. (The existence of these invariant manifolds is often stated for hyperbolic equilibria, but it is granted also in the present case because the eigenvalues with negative (positive) real parts are separated by a "spectral gap" from all the others, including zero; see section 4.1 of [**?**]). Thus, in all motions in these submanifolds, for either $t \to +\infty$ or $t \to -\infty$ the center of the ball tends asymptotically to the vertex, with the z-component of the angular velocity of the ball approaching a finite value. Note that, in region ((12)), the eigenvalues of the 4×4 block ((9)) of the linearization have generically nonzero imaginary parts. Therefore, in that region, generically motions will tend to the vertex with some kind of spiraling. Motions that tend to the equilibrium without spiraling are exceptional ($B(\omega_z, \Omega) = 0$).

4.3 Lyapunov stability. Going beyond the linearized analysis, it would be interesting to study the Lyapunov stability of the spectrally stable reduced equilibria at the vertex. The natural candidate for a Lyapunov function is the moving energy. However, $dE(0, 0, \omega_z) = (0, 0, k(\omega_z - \Omega))$ and the moving energy has a critical point only at those reduced equilibria $(0, 0, \omega_z)$ with $\omega_z = \Omega$ (the ball stands still relative to the rotating surface, but spins in space). We restricts our considerations to this case.

Proposition 4. *Assume $\alpha = 0$ and $\Omega \in \mathbb{R}$. If $f''(0) > 0$ and $\Omega^2 < \hat{g}f''(0)$ then the reduced equilibrium $(0, 0, \omega_z = \Omega)$ is Lyapunov stable.*

Proof. Lyapunov stability of $(0, 0, \omega_z = \Omega)$ is granted if the Hessian

$$\begin{pmatrix} k\Omega^2 + \hat{g}f''(0) & 0 & 0 & -(1+k)\Omega & 0 \\ 0 & k\Omega^2 + \hat{g}f''(0) & (1+k)\Omega & 0 & 0 \\ 0 & (1+k)\Omega & 1+k & 0 & 0 \\ -(1+k)\Omega & 0 & 0 & 1+k & 0 \\ 0 & 0 & 0 & 0 & k \end{pmatrix} \tag{13}$$

of the moving energy E at that point is positive definite. Clearly, its last three principal minors are all positive. The first two minors equal $k(1+k)(\hat{g}f''(0) - \omega_z^2)^2$ and $k(1+k)(\hat{g}f''(0) - \omega_z^2)$, respectively, and are both positive if $\hat{g}f''(0) > \omega_z^2$. □

This result is somewhat poor, because it applies only to cases in which the vertex is a point of nondegenerate minimum of the surface, and only to the equilibria with $\omega_z = \Omega$. It does not allow to say anything about Lyapunov stability in all other cases. But also in the considered case, it detects Lyapunov stability only for $|\omega_z| = |\Omega|$ not too large ($< \sqrt{\hat{g}f''(0)}$), while in that situation there is spectral stability for all $\omega_z = \Omega \in \mathbb{R}$: it would be interesting to establish if Lyapunov stability of this class of equilibria is retained for all $|\Omega|$ or if it is actually lost at large $|\Omega|$ (a sort of gyrostatic de-stabilization?). Perhaps, a study of Lyapunov stability beyond the result of Proposition 4 could be based on trying to build a Lyapunov function out of the moving energy and of the two "Routhian" integrals.

5 The Kasamawashi case ($\alpha \neq 0$, $f'' \leq 0$)

We consider now the case in which the surface Σ is inclined of an angle α, $0 < \alpha < \frac{\pi}{2}$. Imagining a ball that rolls on the surface of an umbrella we assume that f is concave, $f''(r) \leq 0$ for all r. Thus, $f'(r) \leq 0$ for all $r > 0$ as well. In such a situation, an equilibrium $(x_1, 0, 0, 0, \omega_z)$ has necessarily $x_1 > 0$ and $f'(x_1) = -\tan\alpha < 0$.

Proposition 5. *Under the stated hypotheses, let* $\mathcal{E} = (x_1, 0, 0, 0, \omega_z)$ *be an equilibrium of the system, with* $x_1 > 0$.

i. If $f''(x_1) = 0$*, then* $\mathcal{E}$ *is spectrally stable if and only if*

$$\Big(\frac{x_1}{\sin\alpha} - 1\Big)\Omega^2 + \Omega\omega_z \geq \frac{\gamma}{\mu^2}\,. \tag{14}$$

ii. If $f''(x_1) < 0$*, define* $h := \frac{f''(x_1)}{f'(x_1)} = -f''(x_1)\frac{\cos\alpha}{\sin\alpha} > 0$*. Then,* $\mathcal{E}$ *is spectrally stable if and only if*

$$a_{11}\Omega^2 + 2a_{12}\Omega\omega_z + a_{22}\omega_z^2 \geq a_0 \tag{15}$$

with $a_{11} = (x_1 - \sin\alpha)\big(1 + h\sin\alpha - hx_1\sin^2\alpha\big)$, $a_{12} = \frac{1}{2}\big(1 + 2h\sin\alpha - hx_1 - hx_1\sin^2\alpha\big)\sin\alpha$, $a_{22} = -h\sin^2\alpha$, $a_0 = \frac{\gamma}{\mu^2}\big(1 + 2\sqrt{hx_1}\cos\alpha + hx_1\cos^2\alpha\big)\sin\alpha$.

Proof. Since $x_1 > 0$ and $0 < \alpha < \frac{\pi}{2}$, $\sin\alpha$ and $\cos\alpha$ are both positive, and $f'(x_1) = -\tan\alpha$. Thus $F(x_1) = \frac{1}{\cos\alpha}$, $\psi'(\frac{1}{2}x_1^2) = -\frac{\tan\alpha}{x_1}$, $\psi''(\frac{1}{2}x_1^2) = f''(x_1) - \frac{\tan\alpha}{x_1^2}$ and the entries (10) of the 4×4 block (9) of the linearization can be written as

$$a_{31} = -\gamma f''(x_1)\cos^3(\alpha)\,, \qquad a_{34} = -\mu\Omega + \mu(\Omega - \omega_z)\frac{\sin\alpha}{x_1}\,,$$
$$a_{42} = \gamma\frac{\sin\alpha}{x_1}\,, \qquad a_{43} = \mu\Omega + \mu(\Omega - \omega_z - x_1\Omega\sin\alpha)f''(x_1)\cos\alpha\,.$$

Spectral stability of $\mathcal{E}$ is equivalent to the fact that all the roots of the characteristic polynomial (11), namely $\lambda^4 + 2b\lambda^2 + c$ with $2b = -(a_{31} + a_{42} + a_{34}a_{43})$ and $c = a_{31}a_{42}$, have nonpositive real part.

(i.) If $f''(0) = 0$ then $c = 0$ and, as noticed in the proof of Proposition (4), the roots of the characteristic polynomial have all nonpositive real part if and only if $b \geq 0$. For $f''(0) = 0$, $2b = \mu^2\big(1 - \frac{\sin\alpha}{x_1}\big)\Omega^2 + \mu^2\frac{\sin\alpha}{x_1}\Omega\omega_z - \gamma\frac{\sin\alpha}{x_1}$. Since $x_1 > 0$, $\sin\alpha > 0$ and $\mu > 0$, condition $b \geq 0$ is equivalent to (14).

(ii.) If $f''(0) < 0$ then $c > 0$ and (see again the proof of Proposition (4)) the roots of the characteristic polynomial have all nonpositive real part if and only if $b^2 \geq c$ and $b > 0$, namely $b \geq \sqrt{c}$. Since $x_1 > 0$ and, as noticed above, $f'(x_1) = -\tan\alpha < 0$, $h := \frac{f''(x_1)}{f'(x_1)} > 0$. Writing $f''(x_1) = -h\tan\alpha$, condition $b \geq \sqrt{c}$ becomes $\frac{\mu^2}{2x_1}(a_{11}\Omega^2 + 2a_{12}\Omega\omega_z + a_{22}\omega_z^2 - a_{00}) \geq 0$. □

We now analyze the conditions given by Proposition 5.

Given x_1, when $f''(x_1) = 0$ the condition of spectral stability (14) is satisfied in a region of the (ω_z, Ω)-plane which is bounded by the two branches of a hyperbola and is shown in Fig. 3. One asymptote of the hyperbola is the ω_z-axis, and the equilibrium is never spectrally stable (and hence is always unstable) if $\Omega = 0$. The rotation of the surface has a stabilizing effect, in the sense that if $\Omega \neq 0$ then spectral stability of the equilibrium becomes possible for certain ω_z, but this effect depends on the distance of the equilibrium position from the rotation axis. Indeed, the other asymptote of the hyperbola is the line $\Omega = (1 - \frac{\sin\alpha}{x_1})\omega_z$ and counterclockwise rotates from the diagonal to the horizontal axis as x_1 grows from 0 to $+\infty$.

Thus, for equilibria near the rotation axis ($x_1 < \sin\alpha$) spectral stability is achieved for ω_z of the same sign as Ω and in an unbounded interval which does not contain 0, and whose size first decreases and then increases with $|\Omega|$.

Instead, for equilibria far from the rotation axis ($x_1 > \sin\alpha$), spectral stability is achieved for ω_z in an interval that contains 0 and whose size steadily increases as $|\Omega|$ increases.

As already mentioned in the Introduction, the case $f''(x_1) = 0$ is that of the kasamawashi, which uses an umbrella with conic profile. The umbrella is inclined so that the upper generatrix of the cone is horizontal, and there are reduced equilibria at all points of this horizontal line. Inspection of movies showing actual kasamawashi performances[4] suggests that the performer manages to have $\omega_z = 0$ and that, consistently with the above remarks, $x_1 > \sin\alpha$.[5] Of course, these conclusions should be taken for what they are because—besides the fact that, as already pointed out, kasamawashi involves control—not only spectral stability does non guarantees stability but, moreover, the presence of zero eigenvalues might be an indication of unstable behaviours. Some further study of the dynamics might be interesting.

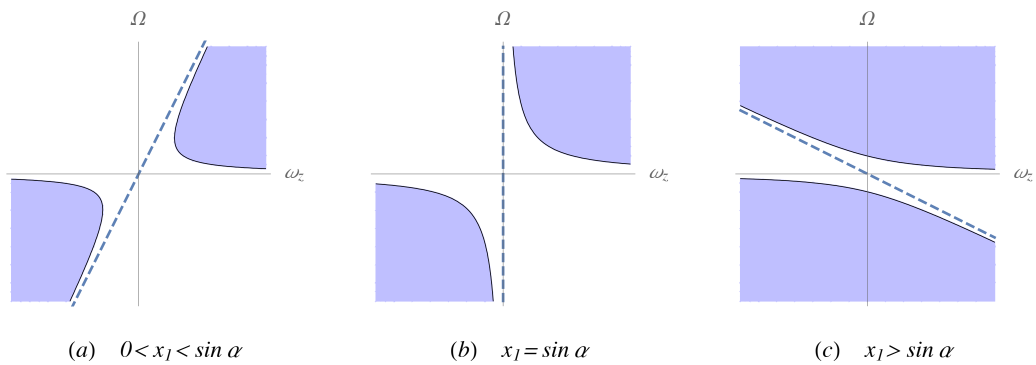

Figure 3: The region of spectral stability of the equilibrium $(x_1, 0, 0, \omega_z)$ in the plane (ω_z, Ω) when $f''(x_1) = 0$. The dashed line is the asymptote $\Omega = \frac{\sin\alpha}{\sin\alpha - x_1}\omega_z$

When $f''(x_1) < 0$ the situation is similar, though more complex to analyze. First, when $\Omega = 0$ condition (15) reduces to

$$\omega_z^2 \geq \frac{\gamma}{\mu^2}\Big(\frac{1}{h} + \sqrt{\frac{x_1}{h}}\cos\alpha + x_1\cos^2\alpha\Big) . \tag{16}$$

Therefore, at variance from the case $f''(x_1) = 0$, for $\Omega = 0$ there is spectral stability for $|\omega_z|$ not too small (with a threshold which however increases with x_1). For all Ω,

$$a_{11}a_{22} - a_{12}^2 = -\frac{1}{4}\mu^2\big(2 + hx_1\cos(2\alpha)\big)^2\sin^2\alpha$$

is negative (unless $hx_1\cos(2\alpha) = -2$, which could only happen if $\alpha \geq \frac{\pi}{4}$) and region (15) is again bounded by the two branches of a hyperbola. These curves intersect the ω_z-axis in the two points where (16) is satisfied with the = sign. From this it follows that the region where (15) is satisfied is the one outside the two branches of the hyperbola—very much as in Fig. 3.

Remark. *If $f''(r) < 0$ for all r then for any $\alpha \in (0, \frac{\pi}{2})$ there is a unique ω_z-family of equilibria $(x_1, 0, 0, 0, \omega_z)$. For $\alpha \to 0$, these equilibria tend to the equilibria $(0, 0, 0, 0, \omega_z)$ at the vertex. It is not difficult to check that, for small α, at first order in α the condition for spectral stability (15) coincides with the condition $B(\omega_z, \Omega) \geq 4\gamma|f''(0)|$ which, in item iii. of Proposition 3, ensures the spectral stability of the equilibria at the vertex. (Since $\sin\alpha \sim \alpha$ etc, $f'(x_1) \sim \alpha$ and $f'(x_1) \sim f''(0)x_1$ which give $x_1 \sim \frac{\alpha}{|f''(x_1)|}$ and $h \sim \frac{1}{\alpha}$).*

[4]Such as the one available at `https://www.youtube.com/watch?v=FeDyMdh1JLQ`

[5]In the movie, the angle α is small and the ball sits at a distance from the rotation axis which is approximately two-to-three times its radius, hence $x_1 > 1$.

6 Conclusions

We have studied two new problems in the dynamics of a heavy homogeneous ball that rolls without sliding on a surface of revolution which rotates with constant angular velocity $\Omega \in \mathbb{R}$ about its figure axis. The system has an SO(3)-invariance which allows reduction to 5-dimensions.

First, assuming that the figure axis of the surface is vertical, we have studied those equilibria of the reduced system which correspond to periodic orbits of the unreduced system in which the ball sits at the vertex of the surface and rotates steadily about its center with vertical angular velocity $\omega_{\mathrm{z}} \in \mathbb{R}$. We have shown that no blow up is possible at these reduced equilibria and we have studied their spectral stability as a function of the parameters, in particular of ω_z, Ω and the curvature of the surface's profile at the vertex. We have shown that they are all spectrally stable unless the profile of the surface has a nondegenerate maximum at the vertex, in which case spectral stability is attained for (ω_z, Ω) outside of a strip in $\mathbb{R}^2$. For (ω_z, Ω) inside that strip the reduced equilibrium is spectrally unstable, and this implies the existence of motions which are asymptotic (in the past or in the future) to the reduced equilibrium. Finally, we have proven the nonlinear stability of a special subclass of the spectrally stable reduced equilibria: in the case in which the surface has a nondegenerate minimum at the vertex, those with $\omega_z = \Omega$ and $|\Omega|$ not too large. It is likely that the class of nonlinearly stable reduced equilibria at the vertex is larger, but this question remains open and deserves to be studied.

Second, we have considered the case in which the figure axis is tilted with respect to the vertical. The reduced equilibria correspond to periodic motions of the unreduced system in which the ball steadily rotates with vertical angular velocity ω_z about its center, which stands still in space over a point in which the surface has horizontal tangent plane. We have limited the study of the spectral stability of these reduced equilibria to the case of a non-convex profile, a particular case of which is that of the conic umbrella used in the kasamawashi performances, remarking in particular its dependence on the distance from the vertex. A study of the nonlinear stability of these reduced equilibria, and even more so of the dynamics near them, is left open and is worth further investigation.

Appendix: the equations of motion of the system

The equations of motion of the system can be determined in various routine ways which however, as often happens with nonholonomic systems, involve some tedious computations. Here we follow the approach of [11].

Reference [11] employs a known form of the equations of motion of mechanical nonholonomic systems as the restriction to the constraint manifold of Lagrange equations with the nonholonomic reaction forces, writing however them in a way that allows for the use of quasi-velocities (Proposition 16 in the Appendix of [11]). Of course, one might just specialize those formula to the present case, and this would indeed be the straightest—though somewhat laborious—approach. However, since the computations are there already made for the case $\alpha = 0$, in order to keep the length of this article to a minimum we prefer here to indicate how to modify that deduction to allow for $\alpha \neq 0$. There are in fact three other minor differences. One is technically irrelevant: reference [11] assumes that the domain $I = (-R, R)$ of the profile function is the entire real axis, so that $D = \mathbb{R}^2$. In addition, the derivation of the equations of motion in the Appendix of [11] uses the profile function f, not ψ, and a different parametrization of M_8, which excludes the vertex and uses polar coordinates, namely $(r, \theta, v_r, v_\theta, \mathcal{R}, \omega_{\mathrm{z}}) \in \mathbb{R}_+ \times S^1 \times \mathbb{R} \times \mathbb{R} \times \mathrm{SO}(3) \times \mathbb{R} =: M_8^{\mathrm{pol}}$ with $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $v_r = \dot{r}$, $v_\theta = \dot{\theta}$. We thus indicate how to modify such a derivation.

First, the inclination of the surface has the only effect of changing the potential energy of the weight force: instead of $g\mathrm{z}|_{M_8^{\mathrm{pol}}} = a\hat{g}f(r)$, it becomes $g(\mathrm{z}\cos\alpha + \mathrm{x}\sin\alpha)|_{M_8^{\mathrm{pol}}} = a\hat{g}(f(r)\cos\alpha + r\sin\alpha\cos\theta)$. This has the consequence that the nonholonomic reaction force R, given in formula

(46) within the proof of Proposition 17 of [11], gets the following changes: in its $\dot r$-component the term $\mu\hat g f'$ has to be replaced with $\mu\hat g(f'\cos\alpha+\sin\alpha\,\cos\theta)$, its $\dot\theta$-component acquires a term $-\mu\hat g r^{-1}\sin\alpha\,\sin\theta$ and its $\omega_{\rm z}$-component acquires a term $-\mu\hat g F^{-1}f'\sin\alpha\,\sin\theta$. These changes propagate to the equations for $\dot v_r$, $\dot v_\theta$ and $\dot\omega_{\rm z}$ as given in Proposition 17 of [11] after multiplication by the appropriate entries of the inverse of the kinetic matrix (namely F^{-2}, r^{-2} and k^{-1} respectively).

Second, the equations for $\dot v_r$ and $\dot v_\theta$ can be transformed into equations for $\dot v_1$ and $\dot v_2$ using the kinematical identities $\dot v_1=\big(\frac{\dot v_r}{r}-v_\theta^2\big)x_1-\big(\dot v_\theta+2\frac{v_rv_\theta}{r}\big)x_2$ and $\dot v_2=\big(\frac{\dot v_r}{r}-v_\theta^2\big)x_2+\big(\dot v_\theta+2\frac{v_rv_\theta}{r}\big)x_1$ and making the obvious substitutions $r\to|x|$, $\sin\theta\to\frac{x_2}{r}$, $\cos\theta\to\frac{x_1}{r}$, $v_r\to x\cdot v$, $v_\theta\to\frac{x_1v_2-x_2v_1}{r}$. This leads to the equations $\dot x_1=v_1$, $\dot x_2=v_2$, $\dot{\mathcal R}=\mathcal R^T\omega$ and

$$
\begin{aligned}
\dot v_1 =& -\frac{\gamma}{F^2}\Big(\frac{x_1}{|x|}f'\cos\alpha+\Big(1+\frac{x_2^2}{|x|^2}f'^2\Big)\sin\alpha\Big)+\frac{\mu}{F}\Big(\frac{x_1}{|x|^3}\,x\cdot Jv\,f'+\frac{x_2}{|x|^2}\,x\cdot v\,f''\Big)\omega_z\\
&-\frac{\mu}{F^2}\frac{v_1}{|x|}\,x\cdot v\,f'f''-\frac{f'}{(1+k)F^2}\frac{x_1}{|x|^4}\Big((x\cdot Jv)^2f'+|x|(x\cdot v)^2f''\Big)\\
&-\Omega\mu\Big(v_2+\frac{1}{F}\frac{x_1}{|x|^3}\,x\cdot Jv\,f'+\frac{x_2}{|x|^2}\frac{x\cdot v}{F^2}f''\big(F+|x|f'\big)\Big)\\
\dot v_2 =& -\frac{\gamma}{F^2}\frac{x_2}{|x|}f'\Big(\cos\alpha-\frac{x_1}{|x|}f'\sin\alpha\Big)+\frac{\mu}{F}\Big(\frac{x_2}{|x|^3}\,x\cdot Jv\,f'-\frac{x_1}{|x|^2}\,x\cdot v\,f''\Big)\omega_z\\
&-\frac{\mu}{F^2}\frac{v_2}{|x|}\,x\cdot v\,f'f''-\frac{f'}{(1+k)F^2}\frac{x_2}{|x|^4}\Big((x\cdot Jv)^2f'+|x|(x\cdot v)^2f''\Big)\\
&+\Omega\mu\Big(v_1-\frac{1}{F}\frac{x_2}{|x|^3}\,x\cdot Jv\,f'+\frac{x_1}{|x|^2}\frac{x\cdot v}{F^2}f''\big(F+|x|f'\big)\Big)\\
\dot\omega_{\rm z} =& -\frac{\gamma}{F}\frac{x_2}{|x|}f'\sin\alpha-\frac{f'f''}{(1+k)F^3}\frac{x\cdot v}{|x|^2}\big(|x|F\omega_{\rm z}-\,x\cdot Jv\,f'\big)\\
&+\Omega\frac{f'}{(1+k)F}\frac{x\cdot v}{|x|}\Big(1+\frac{f''}{F}+|x|\frac{f'f''}{F^2}\Big)
\end{aligned}
$$

with $J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. After replacing f' with $|x|\psi'$ and f'' with $\psi'+|x|^2\psi''$, see (1), these equations take the form (7).

In this way we have proven that (7) are the equations of motion of the system in the subset of the phase space M_8 where $x\neq0$. Therefore, their right hand side defines a vector field Y in $M_8\setminus\{x=0\}$ which coincides with the restriction to such a set of the dynamical vector field of the system. But since the latter is known (from the general theory) to exists in all of M_8, $M_8\setminus\{x=0\}$ is dense in M_8 and Y has a continuous extension to M_8, the extension of Y is the dynamical vector field of the system in all of M_8.

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