Quanto Total Return LIBOR Swap Model

A quanto total return Libor Swap is a swap where one leg is a regular floating leg paying LIBOR less a constant spread and the other leg makes a single payment at the swap's maturity equal to a leveraged non-negative return on USD-for-EURO exchange rate paid in CAD. The main focus of the valuation model is the quantoed total return on the FX rate.

The payoff of the quanto-return leg is max(1.3 x (FX at maturity - initialFX)/initialFx), 0). In other words, The payoff of the leg based on the return of the foreign exchange rate is a payoff of a European call option. Its present value is given by Black's formula for futures with the discounting factor equal to the Canadian zero-coupon bond and the future price. You can find bond valuation details at https://finpricing.com/lib/EqConvertible.html

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$$F = X \frac{B_E}{B_U} \exp(\rho \, \sigma_{\mathrm{EU}} \sigma_{\mathrm{CU}} T), \qquad (1)$$

where

- X is the spot USD-for-EURO exchange rate,
- B_U is the price of the US T-maturity zero coupon bond,
- B_E is the price of the EURO T-maturity zero coupon bond,
- ρ is the correlation between the USD-for-EURO and the USD-for-CAD exchange rates,

- σ_{EU} is the volatility of the USD-for-EURO exchange rate,
- σ_{CU} is the volatility of the USD-for-CAD exchange rate,
- T is the swap's maturity.

The important point is that the option price depends on the volatilities and correlation of the exchange rates as well as on the interest rates of all the three currencies.

Let $X_{UE}(t)$ be the USD-for-EURO exchange rate at time t, which gives the amount of USD funds exchanged for one EURO. Let $X_{UC}(t)$ be a similar USD-for-CAD exchange rate. Assume further that both rates follow geometric Brownian motions with constant volatilities under the natural probability measure:

$$dX_{EU} = X_{EU}\left(\mu_{EU}(t)dt + \boldsymbol{\sigma}_{EU}\, d\mathbf{w}_t\right) \qquad (2)$$

$$dX_{CU} = X_{CU}\left(\mu_{CU}(t)dt + \boldsymbol{\sigma}_{CU}\, d\mathbf{w}_t\right) \qquad (3)$$

Here $\mathbf{w}_t$ is a standard n-dimensional Brownian motion, while $\boldsymbol{\sigma}_{EU}$ and $\boldsymbol{\sigma}_{CU}$ are n-dimensional volatility vectors, such that the volatility of $X_{EU}(t)$ is $\sigma_{EU} = \sqrt{\boldsymbol{\sigma}_{EU}^2}$, the volatility of $X_{CU}(t)$ is $\sigma_{CU} = \sqrt{\boldsymbol{\sigma}_{CU}^2}$, and their correlation coefficient is $\rho = \dfrac{\boldsymbol{\sigma}_{EU}\boldsymbol{\sigma}_{CU}}{\sigma_{EU}\sigma_{CU}}$.

The CAD-for-EURO exchange rate is not independent and is expressed through $X_{EU}(t)$ and $X_{CU}(t)$ as

$$X_{EC}(t) = \frac{X_{EU}(t)}{X_{CU}(t)} \tag{4}$$

Assume the Canadian, US, and EURO short time interest rates r_C, r_U, and r_E, to be deterministic functions of time. Then the savings accounts of the respective currencies evolve as follows:

$$b_C(t) = \exp\left(\int_o^t r_C(s)ds\right) \tag{5}$$

$$b_U(t) = \exp\left(\int_o^t r_U(s)ds\right) \tag{6}$$

$$b_E(t) = \exp\left(\int_o^t r_E(s)ds\right) \tag{7}$$

We now form the values of the US and EURO saving accounts converted into the Canadian currency with the Canadian saving account used as a numeraire:

$$C_U = \frac{B_U}{X_{CU} B_C} \tag{8}$$

$$C_E = \frac{X_{EU} B_E}{X_{CU} B_C} \tag{9}$$

The evolution of C_U and C_E is then governed by the following SDE:

$$dC_U = C_U\left[(-\mu_{CU} + r_U - r_C + \sigma_{CU}^2)dt - \boldsymbol{\sigma}_{CU} d\mathbf{w}\right] \tag{10}$$

$$dC_E = C_E\left[(\mu_{EU} - \mu_{CU} + r_E - r_C - \boldsymbol{\sigma}_{CU}\boldsymbol{\sigma}_{EU} + \sigma_{CU}^2)dt + (\boldsymbol{\sigma}_{EU} - \boldsymbol{\sigma}_{CU})d\mathbf{w}\right] \tag{11}$$

There must exist a measure under which all the tradeables on the Canadian market, discounted by the Canadian saving account, including C_U and C_E, are martingales. A standard Brownian motion under this measure $\mathbf{w}^*$ is coupled with the original Brownian motion:

$$d\mathbf{w} = d\mathbf{w}^* - \boldsymbol{\gamma}\, dt \tag{12}$$

For C_U and C_E to be martingales under the new measure, the following expressions should hold:

$$\boldsymbol{\sigma}_{CU}\boldsymbol{\gamma} = \mu_{CU} - r_U + r_C - \sigma_{CU}^2 \tag{13}$$

$$(\boldsymbol{\sigma}_{CU} - \boldsymbol{\sigma}_{EU})\boldsymbol{\gamma} = -\mu_{EU} + \mu_{CU} - r_E + r_C + \rho\,\sigma_{CU}\sigma_{EU} - \sigma_{CU}^2 \tag{14}$$

The underlying of our option is the USD-for-EURO exchange rate, $X_{EU}(t)$, given by eq. (1). Using eqs. (12) – (14) one finds that under the Canadian martingale measure eq. (1) becomes

$$dX_{EU} = X_{EU}\left[(r_U - r_E + \rho\,\sigma_{CU}\sigma_{EU})dt + \boldsymbol{\sigma}_{EU}\; d\mathbf{w}^*\right] \tag{15}$$

Defining a standard one dimensional w^* by

$$\sigma_{EU}\; dw^* = \boldsymbol{\sigma}_{EU}\; d\mathbf{w}^*$$

one can cast eq. (15) into the form

$$dX_{EU} = X_{EU}\left[(r_U - r_E + \rho\,\sigma_{CU}\sigma_{EU})dt + \sigma_{EU}\; dw^*\right] \tag{16}$$

whose solution is

$$X_{EU}(t) = X_{EU}(0)\exp\left(\int_0^t r_U(s)ds - \int_0^t r_E(s)ds + (\rho\,\sigma_{CU}\sigma_{EU} - \frac{\sigma_{EU}^2}{2})t + \sigma_{EU} w^*\right) \tag{17}$$

Since the interest rates are assumed to be deterministic, the zero-coupon bonds in each currency can be represented as

$$B_C(t) = \frac{1}{b_C(t)} = \exp\left(-\int_o^t r_C(s)ds\right) \tag{18}$$

$$B_U(t) = \frac{1}{b_U(t)} = \exp\left(-\int_o^t r_U(s)ds\right) \tag{19}$$

$$B_E(t) = \frac{1}{b_{EC}(t)} = \exp\left(-\int_o^t r_E(s)ds\right) \tag{20}$$

and eq. (17) takes the form

$$X_{EU}(t) = X_{EU}(0)\frac{B_E(t)}{B_U(t)}\exp\left((\rho\,\sigma_{CU}\sigma_{EU} - \frac{\sigma_{EU}^2}{2})t\right)\exp\left(\sigma_{EU}w^*\right) \equiv F(t)\exp\left(-\frac{\sigma_{EU}^2}{2}t + \sigma_{EU}w^*\right) \tag{21}$$

where *F(t)* is defined in section 3 above.

The price of the call option is

$$V_{FX} = E\left(\frac{(X_{EU}(T)-K)^+}{Kb_C(T)} \mid \Im_0\right) = \frac{B_C(T)}{K}E\left((X_{EU}(T)-K)^+ \mid \Im_0\right),$$

where *K* is the strike, and *T* is the swap's maturity. Explicit calculation of the expectation yields

$$V_{FX} = \frac{B_C(T)}{K}\left[FN\left(\frac{\ln \frac{F}{K} + \frac{\sigma_{EU}{}^2}{2} T}{\sigma_{EU}\sqrt{T}} \right) - KN\left(\frac{\ln \frac{F}{K} - \frac{\sigma_{EU}{}^2}{2} T}{\sigma_{EU}\sqrt{T}} \right) \right] \qquad (22)$$

where

- $F = X_{EU}^0 \frac{B_E(T)}{B_U(T)} \exp(\rho\, \sigma_{\mathrm{EU}} \sigma_{\mathrm{CU}} T)$,
- X_{EU}^0 is the spot USD-for-EURO exchange rate,
- $B_C(T)$ is the price of the Canadian T-maturity zero coupon bond,
- $B_U(T)$ is the price of the US T-maturity zero coupon bond,
- $B_E(T)$ is the price of the EURO T-maturity zero coupon bond,
- σ_{EU} is the volatility of the USD-for-EURO exchange rate,
- σ_{CU} is the volatility of the USD-for-CAD exchange rate,
- $N(\cdot)$ is the standard normal cumulative distribution function.