Black-Karasinski Short Rate Tree Model

The Black-Karasinski model is a short rate model that assumes the short-term interest rates to be log-normally distributed. We implement the one factor Black-Karasinski model as a binomial or trinomial tree.

Assume that short term interest rate process, $\{r_t \mid t \geq 0\}$, satisfies, under the risk neutral probability measure, a SDE of Black-Karasinski form,

$$d \log r_t = (\theta(t) - a \log r_t) dt + \sigma_r dW_t^r, \quad t \geq 0,$$

where

- $\{W_t^r \mid t \geq 0\}$ denotes standard Brownian motion,
- σ_r is the volatility,
- a, with $a > 0$, is the mean reversion,
- $\theta(t)$ is chosen to match the initial term structure of zero coupon bond prices.

Our approach towards building a tree for the short-term interest rate process, $\{r_t \mid t \in [0,T]\}$, is based on the single-factor tree construction technique. Specifically let

$$\log r_t = (\log r_t - \tilde{r}_t) + \tilde{r}_t,$$

where the process $\{\tilde{r}_t \mid t \in [0,T]\}$ satisfies the SDE

$$
\begin{cases} d\tilde{r}_t = -a\tilde{r}_t + \sigma_r dW_t^r, \\ \tilde{r}_0 = 0. \end{cases} \tag{2.1}
$$

Then

$$
\begin{cases} d\alpha_t = (\theta_t - a\alpha_t)dt, \\ \alpha_0 = \log r_0, \end{cases}
$$

where $\alpha_t = \log r_t - \tilde{r}_t$. We first build a tree for the process $\{\tilde{r}_t \mid t \in [0,T]\}$ as described below.

Let $x_t = e^{at}\tilde{r}_t$. From Ito's Lemma,

$$
dx_t = \sigma_r e^{at} dW_t^r,
$$

$$
x_0 = \tilde{r}_0.
$$

Then

$$
\begin{aligned}
& x_{t+\Delta t} = x_t + \sigma_r \int_t^{t+\Delta t} e^{as} dW_s^r, \\
& \Rightarrow e^{a(t+\Delta t)}\tilde{r}_{t+\Delta t} = e^{at}\tilde{r}_t + \sigma_r \int_t^{t+\Delta t} e^{as} dW_s^r, \\
& \Rightarrow \tilde{r}_{t+\Delta t} = e^{-a\Delta t}\tilde{r}_t + \sigma_r e^{-a(t+\Delta t)} \int_t^{t+\Delta t} e^{as} dW_s^r,
\end{aligned}
$$

where $\Delta t > 0$.

Next let

$$\prod = \{t_i\}_{i=0}^{N},$$

where $N > 0$ and $0 = t_0 < \cdots < t_N = T$, be a partition of the interval $[0,T]$; furthermore, let $t_{N+1} = t_N + (t_N - t_{N-1})$ be an additional time slice. We can view our tree for the process $\{\tilde{r}_t \mid t \in [0,T]\}$ as a directed graph, which is defined by a set of vertices and directed edges. Let i_j and $\tilde{r}_{i_j}$, for $j = 1,\ldots,n_i$, respectively denote a tree node at time slice t_i and the associated value for $\tilde{r}_{t_j}$; here n_i, for $i = 0,\ldots,N$, denotes the total number of nodes on the i^{th} time slice. Furthermore let

$$\left(\tilde{r}_{t_{i+1}} \middle| \tilde{r}_{i_j}\right)$$

denote the random variable

$$e^{-a\Delta t_i}\tilde{r}_{i_j} + \sigma_r e^{-at_{i+1}} \int_{t_i}^{t_{i+1}} e^{as} dW_s^r,$$

where $\Delta t_i = t_{i+1} - t_i$. Then

$$E\left(\left(\tilde{r}_{t_{i+1}} \middle| \tilde{r}_{i_j}\right)\right) = e^{-a\Delta t_i}\tilde{r}_{i_j}.$$

Since $\int_{t_i}^{t_{i+1}} e^{as} dW_s^r = e^{at_{i+1}} W_{t_{i+1}}^r - e^{at_i} W_{t_i}^r - a\int_{t_i}^{t_{i+1}} e^{as} W_s^r ds,$

$$Var\left(\int_{t_i}^{t_{i+1}} e^{as} dW_s^r\right)$$

$$= Var\left(e^{at_{i+1}}W_{t_{i+1}}^r - e^{at_i}W_{t_i}^r - a\int_{t_i}^{t_{i+1}} e^{as}W_s^r ds\right),$$

$$= E\Bigg(e^{2at_{i+1}}\left[W_{t_{i+1}}^r\right]^2 + e^{2at_i}\left[W_{t_i}^r\right]^2 + a^2\int_{s=t_i}^{s=t_{i+1}}\int_{u=t_i}^{u=t_{i+1}} e^{a(s+u)}W_s^r W_u^r du\, ds$$

$$- 2e^{a(t_i+t_{i+1})}W_{t_{i+1}}^r W_{t_i}^r - 2ae^{at_{i+1}}W_{t_{i+1}}^r\int_{t_i}^{t_{i+1}} e^{as}W_s^r ds + 2ae^{at_i}W_{t_i}^r\int_{t_i}^{t_{i+1}} e^{as}W_s^r ds\Bigg),$$

$$= e^{2at_{i+1}}t_{i+1} + e^{2at_i}t_i - 2e^{a(t_i+t_{i+1})}t_i - 2ae^{at_{i+1}}\int_{t_i}^{t_{i+1}} e^{as}s\, ds + 2ae^{at_i}t_i\int_{t_i}^{t_{i+1}} e^{as} ds$$

$$+ a^2\int_{s=t_i}^{s=t_{i+1}}\int_{u=t_i}^{u=t_{i+1}} e^{a(s+u)}\min(s,u)du\, ds,$$

$$= e^{2at_{i+1}}t_{i+1} + e^{2at_i}t_i - 2e^{a(t_i+t_{i+1})}t_i - 2ae^{at_{i+1}}\int_{t_i}^{t_{i+1}} e^{as}s\, ds + 2ae^{at_i}t_i\int_{t_i}^{t_{i+1}} e^{as} ds$$

$$+ a^2\int_{s=t_i}^{s=t_{i+1}} e^{as}\left(\int_{u=t_i}^{u=s} e^{au}u\, du + s\int_{u=s}^{u=t_{i+1}} e^{au}\, du\right)ds,$$

Then

$$Var\left(\left(\tilde{r}_{t_{i+1}} \middle| \tilde{r}_{i_j}\right)\right) = \frac{\sigma_r^2}{2a}e^{-2at_{i+1}}\left(e^{2at_{i+1}} - e^{2at_i}\right),$$

$$= \frac{\sigma_r^2}{2a}\left(1 - e^{-2a\Delta t_i}\right).$$

We build a tree for $\left\{\tilde{r}_t \middle| t \in [0,T]\right\}$, based on Myint’s equity tree construction technique, using the expressions above for $E\left(\left(\tilde{r}_{t_{i+1}} \middle| \tilde{r}_{i_j}\right)\right)$ and $Var\left(\left(\tilde{r}_{t_{i+1}} \middle| \tilde{r}_{i_j}\right)\right)$. Here we employ a partition with spacing of

$$\omega_r \sqrt{Var\left(\left(\tilde{r}_{t_{i+1}} \middle| \tilde{r}_{i_j}\right)\right)}$$

at time slice t_{i+1}, for $i = 0,\ldots,N-1$, where $\omega_r = 1.95$.

Let $e_k\left(i_j\right)$, for $k = 1,\ldots,m_{i_j}$ and $i = 0,\ldots,N$, denote a child, at the $(\mathrm{i}+1)^{st}$ time slice, of the tree node i_j; here m_{i_j} denotes the number of children emanating from the parent node, i_j (e.g., $m_{i_j} = 3$ for a trinomial tree). Let A_{i_j}, for $i = 0,\ldots,N$ and $j = 1,\ldots,n_i$, denote the price at time zero of an Arrow-Debreu security at the node i_j, that is, a security that pays 1 currency unit if the node i_j is reached at time t_i and zero otherwise.

Black-Karasinski short rate tree approach can be used to price convertible bond. Convertible bond is not only a coupon paying bond but also can be converted at the discretion of the holder within the periods of time specified by the conversion schedule. Typically, the issuer has the option to buy the bond back at a predetermined strike price(s) during the callable period(s). Also, there are provisions that allow the holder to return the bond to the issuer in exchange for a predetermined cash price during certain period(s). Find convertible bond valuation details at
https://finpricing.com/lib/EqConvertible.html

Let $P(0,t)$ denote the price at time zero of a zero coupon bond with maturity of t and face value of 1 currency unit. We determine $\alpha_t = \log r_t - \tilde{r}_t$ at each time slice by matching the initial term structure of zero coupon bond prices. We first solve

$$e^{-e^{\alpha_0} t_1} = P(0,t_1)$$

for α_0, that is, $\alpha_0 = \log\left(-\frac{\log P(0,t_1)}{t_1}\right)$. We then set the Arrow-Debreu security values at the time slice t_1 to

$$A_{e_k(0_1)} = e^{-e^{\alpha_0} t_1} prob(0_1, e_k(0_1)),$$

for $k = 1,\ldots,n_{0_1}$, where 0_1 denotes the tree root node.

For $i = 1,\ldots,N$, let

$$f_i(\alpha_i) = \sum_{j=1}^{n_i} A_{i_j} e^{-e^{\tilde{r}_{i_j}+\alpha_i}\Delta t_i} - P(0,t_{i+1}).$$

Sequentially, for $i = 1,\ldots,N$, we then numerically solve

$$f_i(\alpha_i) = 0 \qquad (1)$$

for the unknown α_i. Here we employ the Newton iteration scheme

$$\alpha_i^{(j+1)} = \alpha_i^{(j)} - \left[\frac{d\ f_i\left(\alpha_i^{(j)}\right)}{d\alpha_i}\right]^{-1} f_i\left(\alpha_i^{(j)}\right),$$

for $j \geq 1$. Observe that

$$f_i(\alpha_i) \approx \sum_{j=1}^{n_i} A_{i_j} e^{-\left(1+\tilde{r}_{i_j}+\alpha_i\right)\Delta t_i} - P(0,t_{i+1}),$$

$$= e^{-(1+\alpha_i)\Delta t_i} \sum_{j=1}^{n_i} A_{i_j} e^{-\tilde{r}_{i_j}\Delta t_i} - P(0,t_{i+1}),$$

which we denote by $g_i(\alpha_i)$. An initial guess to the Newton iteration scheme above, $\alpha_i^{(0)}$, is then obtained by solving

$$g_i\left(\alpha_i^{(0)}\right) = 0$$

for the unknown $\alpha_i^{(0)}$; that is,

$$\alpha_i^{(0)} = -1 - \frac{\log\left(\dfrac{P(0,t_{i+1})}{\sum_{j=1}^{n_i} A_{i_j} e^{-\tilde{r}_{i_j}\Delta t_i}}\right)}{\Delta t_i}.$$