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EVOLUTIONARY MATHEMATICS AND SCIENCE FOR

TWO ASPECTS OF BINOMIAL COEFFICIENTS: POLYNOMIAL C(n, k) AND COMBINATION $\binom{n}{k}$

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ABSTRACT

We make distinction of two important roles for binomial coefficients to play in Number Theory and Trigonometry, respectively. The polynomial aspect of C(n, k) enables the derivation of both explicit and implicit formulas for sums of powers of arithmetic progressions, whereas the combinatorial aspect of $\binom{n}{k}$ helps the derivation of multiple angle formulas for Sine, Cosine and Tangent functions.

KEYWORDS: Polynomial expression, Sorting, Commutative ring, Stirling number, Eulerian number, Natural sequence, Powered sum, Binomial coefficient, Pascal triangle, Bernoulli coefficient, Arithmetically progressive sequence, Recursion, Iteration, Trigonometry, Multi angle formula Sine, Cosine, Tangent, De Moivre's Theorem.

NOMENCLATURE

C(n, k), $\binom{n}{k}$	binomial coefficient, combination
Σ	sum
$b(k, j)$	Bernoulli coefficient
$(i)_1^\infty$	the natural sequence
$(a+(i-1)d)_1^\infty$	arithmetically progressive sequence
$S_n^{(k)}$	the sum of the first kth powers of the natural sequence
P(n, k) , $\left(\binom{n}{k}\right)$	the permutation of n elements taken k at a time
k!	k factorial
$\begin{bmatrix} n \\ k \end{bmatrix}$	Stirling number of the first kind
$\begin{Bmatrix} n \\ k \end{Bmatrix}$	Stirling number of the second kind
$\begin{Bmatrix} n \\ k \end{Bmatrix}_{a;d}$	Stirling number of the second kind for $(a + (i-1)d)_1^\infty$

1. POLYNOMIAL ASPECT

According to the binomial theorem, we are familiar with the following expansion

$$(x + y)^n = \sum_{k=0}^{n} C(n,k)x^{n-k}y^k \ , \qquad \text{Eq. 1}$$

where $C(n,k)$ is known to be Binomial coefficient as displayed in Table 1.

n\k	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Table 1. Pascal Triangle

Since

$$C(n,k) = n(n-1)(n-2)\ldots(n-k+1), \qquad \text{Eq. 2}$$

we see that C(n, k) is a polynomial in n of degree k. This feature of binomial coefficients enable us to derive both explicit and implicit formulas for sums of powers of arithmetic progressions. Despite of the fact that the closed form for the polynomial expression for a powered sum of the arithmetically progressive sequence $(a+(i-1)d)_1^{\infty}$ has been obtained in (3), we feel obliged to present two more efficient ways for computer calculation.

One way is to use recursion for finding the Bernoulli number b(k, j) in

$$\sum_{i=1}^{n} i^k = \sum_{j=1}^{n+1} b(k,j) n^{k+1-j}, \qquad \text{Eq. 3}$$

which is displayed in Table 2.

k\j	1	2	3	4	5	6	7	8	9	10	11
1	$\frac{1}{2}$	$\frac{1}{2}$									
2	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$								
3	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$							
4	$-\frac{1}{30}$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{5}$						
5	0	$-\frac{1}{12}$	0	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{6}$					
6	$\frac{1}{42}$	0	$-\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{7}$				
7	0	$\frac{1}{12}$	0	$-\frac{7}{24}$	0	$\frac{7}{12}$	$\frac{1}{2}$	$\frac{1}{8}$			
8	$-\frac{1}{30}$	0	$\frac{2}{9}$	0	$-\frac{7}{15}$	0	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{9}$		
9	0	$-\frac{3}{20}$	0	$\frac{1}{2}$	0	$-\frac{7}{10}$	0	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{10}$	
10	$\frac{5}{66}$	0	$-\frac{1}{2}$	0	1	0	-1	0	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{11}$

Table 2. Bernoulli triangle

By taking n = k, x = 1 and y = n in Eq. 1, we can obtain

$$\sum_{i=1}^{n+1} i^k - \sum_{i=1}^{n} i^k = (1+n)^k = \sum_{j=0}^{k} C(k,j) n^j \qquad \text{Eq. 4}$$

and equate the coefficients of the like terms in the expansions of which for $j = 0,1,2,\ldots,k$ to

to come up with

$$C(k,i) = \sum_{j=i}^{k+1} C(j,i) b(k,j)\ . \qquad \text{Eq. 5}$$

Take $k = 3$ in Eq. 5 for instance, by equating the coefficients of the like terms of

$$C(3,0) + C(3,1)n + C(3,2)n^2 + C(3,3)n^3$$

$$= \sum_{j=1}^{4} b(3, j)(n+1)^j - \sum_{j=1}^{4} b(3, j)n^j$$

$$= b(3,1)C(1,0) + b(3,2)[C(2,0) + C(2,1)n] + b(3,3)[C(3,0) + C(3.1)n + C(3,2)n^2]$$

$$+ b(3,4)[C(4,0) + C(4,1)n + C(4,2)n^2 + C(4,3)n^3]$$

$$= [C(1,1)b(3,1) + C(2,2)b(3,2) + C(3,3)b(3,3) + C(4,4)b(3,4)]$$

$$+ [C(2,1)b(3,2) + C(3,2)b(3,3) + C(4,3)b(3,4)]n$$

$$+ [C(4,2)b(3,3) + C(3,1)b(3,4)]n^2 + [C(4,1)b(3,4)]n^3$$

we can obtain

$$C(4,1)b(3,4) = C(3,0) ,$$

$$C(4,2)b(3,3) + C(3,1)b(3,4) = C(3,1) ,$$

$$C(2,1)b(3,2) + C(3,2)b(3,3) + C(4,3)b(3,4) = C(3,2)$$

and

$$C(1,1)b(3,1) + C(2,2)b(3,2) + C(3,3)b(3,3) + C(4,4)b(3,4) = C(3,3) .$$

Moreover, let us generalize Eq. 3 to

$$\sum_{i=1}^{n} [a + (i-1)d]^k = \sum_{j=1}^{k+1} b_{a;d}(k, j)n^{k+1-j} \qquad \text{Eq. 6}$$

for an arithmetically progressive sequence $(a + (i-1)d)_1^\infty$ with $b_{1;1}(k, j) = b(k, j)$.

Likewise, we can equate the coefficients of the like terms for $j = 0,1,2,,...,k$ in the expansions of both sides of the identity

$$(dn+a)^k = \sum_{i=1}^{n+1}[a+(i-1)d]^k - \sum_{i=1}^{n}[a+(i-1)d]^k \qquad \text{Eq. 7}$$

to obtain the following generalization of Eq. 4:

$$a^i d^{k-i} C(k,i) = \sum_{j=i}^{k+1} C(j,i) b_{a,d}(k,j)\,. \qquad \text{Eq. 8}$$

Putting $i = k+1, k, k-1$ in Eq. 8, we see that

$$d^k C(k,0) = C(k+1,1) b_{a;d}(k,k+1)$$

gives

$$b_{a;d}(k,k+1) = d^k \frac{1}{k+1}\,;$$

$$ad^{k-1}C(k,1) = C(k+1,2) b_{a;d}(k,k+1) + C(k,1) b_{a;d}(k.k)$$

gives

$$b_{a;d}(k,k) = d^{k-1}\left(a - \frac{d}{2}\right)$$

and

$$a^2 d^{k-2} C(k,2) = C(k+1,3) b_{a;d}(k,k+1) + C(k,2) b_{a;d}(k.k) + C(k-1,1) b_{a;d}(k,k-1)$$

gives

$$b_{a;d}(k,k-1) = d^{k-2}\left(a^2 - ad + \frac{d^2}{6}\right)\frac{C(k,1)}{2}\,.$$

In this manner, we can successively obtain

$$b_{a;d}(k,k-2)=d^{k-3}\left(a-\frac{d}{2}\right)(a^2-ad)\frac{C(k,2)}{3}\ ;$$

$$b_{a;d}(k,k-3)=d^{k-4}\left[(a^2=ad)^2-\frac{d^4}{30}\right]\frac{C(k,3)}{4}\ ;$$

$$b_{a;d}(k,k-4)=d^{k-5}\left(a-\frac{d}{2}\right)\left[(a^2=ad)^2-\frac{d^2}{3}(a^2-ad)\right]\frac{C(k,4)}{5}\ ;$$

$$b_{a;d}(k,k-5)=d^{k-6}\left[(a^2=ad)^3-\frac{d^2}{2}(a^2-ad)^2+\frac{d^6}{42}\right]\frac{C(k,5)}{6}\ ;$$

$$b_{a;d}(k,k-6)=d^{k-7}\left(a-\frac{d}{2}\right)\left[(a^2=ad)^2-d^2(a^2-ad)^2+\frac{d^4}{3}(a^2-ad)\right]\frac{C(k,6)}{7}\ ;$$

$$b_{a;d}(k,k-7)=d^{k-8}\left[(a^2-ad)^4-\frac{4d^2}{3}(a^2=ad)^3+\frac{2d^4}{3}(a^2-ad)^2-\frac{d^8}{30}\right]\frac{C(k,7)}{8}\ ;$$

$$b_{a;d}(k,k-8)$$

$$=d^{k-9}\left(a-\frac{d}{2}\right)\left[(a^2-ad)^4-2d^2(a^2=ad)^2+\frac{9d^4}{4}(a^2-ad)^2-\frac{3d^6}{5}(a^2-ad)\right]\frac{C(k,8)}{9}\ ;$$

$$b_{a;d}(k,k-9)$$

$$=d^{k-10}\left[(a^2-ad)^5-\frac{5d^2}{2}(a^2-ad)^4+3d^4(a^2=ad)^3-\frac{3d^6}{2}(a^2-ad)^2+\frac{5d^{10}}{66}\right]\frac{C(k,9)}{10}\,.$$

2. COMBINATORIAL ASPECT

The number $\binom{n}{n_1, n_2, \ldots, n_m}$ of ways of sorting the first n terms of the natural sequence $(i)_1^\infty$ into m subsets with n_j elements in the jth subset is $\frac{n!}{\prod_{j=1}^{m} n_j!}$, where $n = \sum_{j=1}^{m} n_j$. For example, $\binom{11}{2,4,5} = \frac{11!}{2!4!5!}$ and $\binom{14}{2,3,3,6} = \frac{14!}{2!3!3!6!}$.

In particular, the number of ways of sorting the first n terms of $(i)_1^\infty$ into 2 subsets with k elements in one and $n-k$ elements in another is the combination $\binom{n}{k, n-k} = \frac{n!}{k!(n-k)!}$, which will be further abbreviated as the binomial coefficient $\binom{n}{k}$ or $C(n,k)$; while the number of ways of sorting the first n terms of $(i)_1^\infty$ into k singletons and a subset of $n-k$ elements is the permutation $\binom{n}{1,1,\ldots,1,k} = \frac{n!}{(n-k)!}$, which will be abbreviated as $\left(\binom{n}{k}\right)$ or $P(n,k)$. Hence we write

$$\left(\binom{n}{k}\right) = n(n-1)(n-2)\ldots[n-(k-1)] \qquad \text{Eq. 9}$$

and

$$\binom{n}{k} = \frac{\left(\binom{n}{k}\right)}{\left(\binom{k}{k}\right)}, \qquad \text{Eq. 10}$$

where $\left(\!\binom{k}{k}\!\right) = k!$. Since this first level of sortation can be expressed as the product of a

combination and one or more permutations such as $\binom{9}{2,3,4} = \binom{5}{2}\left(\!\binom{9}{4}\!\right)$ and

$\binom{14}{2,3,4,5} = \binom{5}{2}\left(\!\binom{9}{4}\!\right)\left(\!\binom{14}{5}\!\right)$, the familiar recursive formulas

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad \text{Eq. 11}$$

and

$$\left(\!\binom{n}{k}\!\right) = k\left(\!\binom{n-1}{k-1}\!\right). \qquad \text{Eq. 12}$$

We use Eq. 11 to generate the first-order Pascal triangle, same as Table 1, in Table 3 and Eq. 12 to generate the second-order Pascal triangle as in Table 4.

$\binom{n}{k}\Delta$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Table 3. Table for the first-order Pascal triangle

$\left(\binom{n}{k}\right)$ Δ	0	1	2	3	4	5	6	7	8	9	10
1	1	1									
2	1	2	2								
3	1	3	6	6							
4	1	4	12	24	24						
5	1	5	20	60	120	120					
6	1	6	30	120	360	720	720				
7	1	7	42	210	840	2520	5040	5040			
8	1	8	56	336	1680	6720	20160	40320	40320		
9	1	9	72	504	3024	15120	60480	181440	362880	362880	
10	1	10	90	720	5040	30240	151200	604800	1814400	3628800	3628800

Table 4. Table for the second-order Pascal triangle

The number of ways of sorting the first n terms of $(i)_1^\infty$ into k cycles is the Stirling number of the first kind $\left[{n \atop k}\right]$. We can obtain Table 5 via the recursive formula

$$\left[{n \atop k}\right] = \left[{n-1 \atop k-1}\right] + (n-1)\left[{n-1 \atop k}\right]. \qquad \text{Eq. 13}$$

$\left[{n \atop k}\right]$ Δ	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	2	3	1							
4	6	11	6	1						
5	24	50	35	10	1					
6	120	274	225	85	15	1				
7	720	1764	1624	735	175	21	1			
8	5040	13068	13132	6769	1960	322	28	1		
9	40320	109584	118124	67284	22449	4536	546	36	1	
10	362880	1026576	1172700	723680	269325	63273	9450	870	45	1

Table 5. Table for Stirling numbers of the first kind

We further write Eq. 9 into

$$\left(\!\binom{n}{k}\!\right) = \sum_{j=0}^{k-1}(-1)^{k-j}\begin{bmatrix} k \\ k-j \end{bmatrix} \ . \qquad \text{Eq. 14}$$

For example,

$$\left(\!\binom{n}{4}\!\right) = n(n-1)(n-2)(n-3) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}\mathrm{n}^4 - \begin{bmatrix} 4 \\ 3 \end{bmatrix}\mathrm{n}^3 + \begin{bmatrix} 4 \\ 2 \end{bmatrix}\mathrm{n}^2 - \begin{bmatrix} 4 \\ 4 \end{bmatrix}\mathrm{n} \ .$$

On the other hand, the number of ways of sorting the first n terms of $(i)_1^\infty$ into k sets is the Stirling number of the second kind $\left\{ n \atop k \right\}$. We can obtain Table 6 via the recursive formula

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k\left\{ n-1 \atop k \right\}, \qquad \text{Eq. 15}$$

n\k	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

Table 6. Table for Stirling triangle of the second kind

Alternatively, the Stirling triangle $\left\{ n \atop k \right\}\Delta$ of the second kind can be constructed based on $\left\{ n \atop 1 \right\} = 1$, $\left\{ n \atop n-1 \right\} = \left[n \atop n-1 \right] = \binom{n}{2}$ and $\left\{ n \atop n \right\} = 1$ via the inversion formula

$$\left\{ n \atop k \right\} = \sum_{j=1}^{n-k}(-1)^{j-1}\left[k+j \atop k \right]\left\{ n \atop k+j \right\} \ . \qquad \text{Eq. 16}$$

$$\left\{ 4 \atop 2 \right\} = \left[3 \atop 2 \right]\left\{ 4 \atop 3 \right\} - \left[4 \atop 2 \right]\left\{ 4 \atop 4 \right\} = 7, \ \left\{ 5 \atop 3 \right\} = \left[4 \atop 3 \right]\left\{ 5 \atop 4 \right\} - \left[5 \atop 3 \right]\left\{ 5 \atop 5 \right\} = 25,$$

$$\left\{ 5 \atop 2 \right\} = \left[3 \atop 2 \right]\left\{ 5 \atop 3 \right\} - \left[4 \atop 2 \right]\left\{ 5 \atop 4 \right\} + \left[5 \atop 2 \right]\left\{ 5 \atop 5 \right\} = 15, \ldots$$

3. EXPLICIT FORMULAS FOR POWERED SUMS

To attain our goal, we first derive the following identity

$$\binom{n}{k} = \sum_{j=k}^{n}(-1)^{j-k}\begin{bmatrix} j \\ k \end{bmatrix}\begin{Bmatrix} n+1 \\ j+1 \end{Bmatrix}. \qquad \text{Eq. 17}$$

We shall only look at the case for $n=5$ and $k=3$, since the general case is similar. Let us first verify Eq. 17 with the following examples:

$$\binom{6}{3} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}\begin{Bmatrix} 7 \\ 4 \end{Bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{Bmatrix} 7 \\ 5 \end{Bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{Bmatrix} 7 \\ 6 \end{Bmatrix} - \begin{bmatrix} 6 \\ 3 \end{bmatrix}\begin{Bmatrix} 7 \\ 7 \end{Bmatrix} = 1\times 350 - 6\times 140 + 35\times 21 - 225\times 1 = 20,$$

$$\binom{5}{3} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 4 \end{Bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 5 \end{Bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 6 \end{Bmatrix} = 1\times 65 - 6\times 15 + 35\times 1 = 10,$$

$$\binom{5}{2} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{Bmatrix} 6 \\ 3 \end{Bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix}\begin{Bmatrix} 6 \\ 4 \end{Bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}\begin{Bmatrix} 6 \\ 5 \end{Bmatrix} - \begin{bmatrix} 5 \\ 2 \end{bmatrix}\begin{Bmatrix} 6 \\ 6 \end{Bmatrix} = 1\times 90 - 3\times 65 + 11\times 15 - 50\times 1 = 10.$$

Next we use Eqs. 13 and 15 to show the inductive step:

$$\binom{6}{3} = \left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 4 \end{Bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 5 \end{Bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 6 \end{Bmatrix}\right) + \left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 3 \end{Bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 4 \end{Bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 5 \end{Bmatrix} - \begin{bmatrix} 6 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 6 \end{Bmatrix}\right)$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{Bmatrix} 6 \\ 3 \end{Bmatrix} - \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} - 4\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right)\begin{Bmatrix} 6 \\ 4 \end{Bmatrix} + \left(\begin{bmatrix} 5 \\ 3 \end{bmatrix} - 5\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right)\begin{Bmatrix} 6 \\ 5 \end{Bmatrix} - \left(\begin{bmatrix} 6 \\ 3 \end{bmatrix} - 6\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right)\begin{Bmatrix} 6 \\ 6 \end{Bmatrix}$$

$$= \left(4\begin{Bmatrix} 6 \\ 4 \end{Bmatrix} + \begin{Bmatrix} 6 \\ 3 \end{Bmatrix}\right) - \begin{bmatrix} 4 \\ 3 \end{bmatrix}\left(5\begin{Bmatrix} 6 \\ 5 \end{Bmatrix} + \begin{Bmatrix} 6 \\ 4 \end{Bmatrix}\right) + \begin{bmatrix} 5 \\ 3 \end{bmatrix}\left(6\begin{Bmatrix} 6 \\ 6 \end{Bmatrix} + \begin{Bmatrix} 6 \\ 5 \end{Bmatrix}\right) - \begin{bmatrix} 6 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 6 \end{Bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix}\begin{Bmatrix} 7 \\ 4 \end{Bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{Bmatrix} 7 \\ 5 \end{Bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{Bmatrix} 7 \\ 6 \end{Bmatrix} - \begin{bmatrix} 6 \\ 3 \end{bmatrix}\begin{Bmatrix} 7 \\ 7 \end{Bmatrix}.$$

We then derive the following identity

$$(1+n)^k = \sum_{j=1}^{k+1} \left(\!\binom{n}{j-1}\!\right) \begin{Bmatrix} k+1 \\ j \end{Bmatrix}. \qquad \text{Eq. 18}$$

We only look at the case where $k=4$. From Eq. 4, we can use Eqs. 16 and 17 to write

$$(1+n)^4 = \binom{4}{0} + \binom{4}{1} n + \binom{4}{2} n^2 + \binom{4}{3} n^3 + \binom{4}{4} n^4$$

$$= 1 + \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{Bmatrix} 5 \\ 2 \end{Bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{Bmatrix} 5 \\ 3 \end{Bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{Bmatrix} 5 \\ 4 \end{Bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix} \begin{Bmatrix} 5 \\ 5 \end{Bmatrix} \right) n$$

$$+ \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{Bmatrix} 5 \\ 3 \end{Bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{Bmatrix} 5 \\ 4 \end{Bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{Bmatrix} 5 \\ 5 \end{Bmatrix} \right) n^2 + \left(\begin{bmatrix} 3 \\ 3 \end{bmatrix} \begin{Bmatrix} 5 \\ 4 \end{Bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix} \begin{Bmatrix} 5 \\ 5 \end{Bmatrix} \right) n^3 + n^4$$

$$= 1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} n \begin{Bmatrix} 5 \\ 2 \end{Bmatrix} + \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} n^2 - \begin{bmatrix} 2 \\ 1 \end{bmatrix} n \right) \begin{Bmatrix} 5 \\ 3 \end{Bmatrix} + \left(\begin{bmatrix} 3 \\ 3 \end{bmatrix} n^3 - \begin{bmatrix} 3 \\ 2 \end{bmatrix} n^2 + \begin{bmatrix} 3 \\ 1 \end{bmatrix} n \right) \begin{Bmatrix} 5 \\ 4 \end{Bmatrix}$$

$$+ \left(\begin{bmatrix} 4 \\ 4 \end{bmatrix} n^4 - \begin{bmatrix} 4 \\ 3 \end{bmatrix} n^3 + \begin{bmatrix} 4 \\ 2 \end{bmatrix} n^2 - \begin{bmatrix} 4 \\ 1 \end{bmatrix} n \right) \begin{Bmatrix} 5 \\ 5 \end{Bmatrix}$$

$$= \left(\!\binom{n}{0}\!\right) \begin{Bmatrix} 5 \\ 1 \end{Bmatrix} + \left(\!\binom{n}{1}\!\right) \begin{Bmatrix} 5 \\ 2 \end{Bmatrix} + \left(\!\binom{n}{2}\!\right) \begin{Bmatrix} 5 \\ 3 \end{Bmatrix} + \left(\!\binom{n}{3}\!\right) \begin{Bmatrix} 5 \\ 4 \end{Bmatrix} + \left(\!\binom{n}{4}\!\right) \begin{Bmatrix} 5 \\ 5 \end{Bmatrix}.$$

Next, we use the mathematical induction to prove

$$\sum_{i=1}^{n} i^k = \sum_{j=1}^{k+1} \frac{1}{j} \left(\!\binom{n}{j}\!\right) \begin{Bmatrix} k+1 \\ j \end{Bmatrix}, \qquad \text{Eq. 19}$$

with Eq. 18 being used in the inductive step:

$$\sum_{i=1}^{n+1} i^k = \sum_{j=1}^{k+1} \frac{1}{j} \left(\!\!\binom{n}{j}\!\!\right) \left\{ {k+1 \atop j} \right\} + \sum_{j=1}^{k+1} \left(\!\!\binom{n}{j-1}\!\!\right) \left\{ {k+1 \atop j} \right\}$$

$$= \sum_{j=1}^{k+1} \left[1 + \frac{j}{n-j+1} \right] \frac{1}{j} \left(\!\!\binom{n}{j}\!\!\right) \left\{ {k+1 \atop j} \right\}$$

$$= \sum_{j=1}^{k+1} \left[\frac{n+1}{n-j+1} \left(\!\!\binom{n}{j}\!\!\right) \right] \frac{1}{j} \left\{ {k+1 \atop j} \right\}$$

$$= \sum_{j=1}^{k+1} \frac{1}{j} \left(\!\!\binom{n+1}{j}\!\!\right) \left\{ {k+1 \atop j} \right\}. \qquad \text{Eq. 20}$$

Finally, we can obtain

$$\sum_{i=1}^{n} i^k = \sum_{r=0}^{k} \sum_{j=k+1-r}^{k+1} (-1)^{j-k-1+r} \frac{1}{j} \left[{j \atop k+1-r} \right] \left\{ {k+1 \atop j} \right\} n^{k+1-r} \qquad \text{Eq. 21}$$

by regrouping the following display of Eq. 20:

$$\left[{1 \atop 1} \right] \left\{ {k+1 \atop 1} \right\} n + \frac{1}{2} \left\{ {k+1 \atop 2} \right\} n^2 - \frac{1}{2} \left[{2 \atop 1} \right] \left\{ {k+1 \atop 2} \right\} n$$

$$+ \frac{1}{3} \left\{ {k+1 \atop 3} \right\} n^3 - \frac{1}{3} \left[{3 \atop 2} \right] \left\{ {k+1 \atop 3} \right\} n^2 + \frac{1}{3} \left[{3 \atop 1} \right] \left\{ {k+1 \atop 3} \right\} n \quad + ,,,$$

$$+ \frac{1}{k+1} \left[{k+1 \atop k+1} \right] \left\{ {k+1 \atop k+1} \right\} n^{k+1} - \frac{1}{k+1} \left[{k+1 \atop k} \right] \left\{ {k+1 \atop k+1} \right\} n^k + \ldots + (-1)^k \frac{1}{k+1} \left[{k+1 \atop 1} \right] \left\{ {k+1 \atop k+1} \right\} n.$$

4. IMPLICIT FORMULAS FOR POWERED SUMS

For simplicity, let us write

$$S_{a;d}^{(k)}(n) \quad = \quad \sum_{i=1}^{n}[a+(i-1)d]^{k} \qquad \text{Eq. 22}$$

and derive as follows.

$$S_{a;d}^{(k)}(n)$$

$$=\sum_{i=1}^{n}[a+(i-1)d]^{k-1}[a+(i-1)d]$$

$$=\sum_{i=1}^{n}[a+(i-1)d]^{k-1}d\left[\left(n+\frac{a}{d}\right)-(n+1-i)\right]$$

$$=d\left\{\left(n+\frac{a}{d}\right)\sum_{i=1}^{n}[a+d(i-1)]^{k-1}-\sum_{i=1}^{n}(n+1-i)[a+d(i-1)]^{k-1}\right\}$$

$$=d\left\{\left(n+\frac{a}{d}\right)S_{a;d}^{k-1}(n)-\sum_{k=1}^{n}(n+1-i)[a+d(i-1)]^{k-1}\right\},$$

the last term of which can further be derived as

$$\sum_{i=1}^{n}(n+1-i)[a+d(i-1)]^{k-1}$$

$$=S_{a;d}^{(k-1)}(n)+\sum_{i=1}^{n-1}[(n-1)+1-i][a+d(i-1)]^{k-1}$$

and in turn

$$\sum_{i=1}^{n-1}[(n-1)+1-i][a+d(i-1)]^{k-1}$$

$$= S_{a;d}^{(k-1)}(n-1) + \sum_{i=1}^{n-2}[(n-2)+1-i][a+d(i-1)]^{k-1}\ , \ \ldots$$

$$\sum_{i=1}^{n-(n-2)}\{[n-(n-2)]+1-i][a+d(i-1)]^{k-1} = S_{a;d}^{(k-1)}(2) + \sum_{i=1}^{n-(n-1)}\{[n-(n-1)]+1-i\}[a+d(i-1)]^{k-1}$$

so that

$$S_{a;d}^{(k)}(n) = d\left[\left(n+\frac{a}{d}\right)S_{a;d}^{(k-1)}(n) - \sum_{j=1}^{n} S_{a;d}^{(k-1)}(j)\right]. \qquad \text{Eq. 23}$$

From

$$S_{a;d}^{(1)}(n) = \sum_{i=1}^{n}[a+(i-1)d] = an + d\,\frac{n(n-1)}{2} = \frac{d}{2}n^2 + \left(a-\frac{d}{2}\right)n\,,$$

we use Eq. 6 to obtain

$$S_{a;d}^{(2)}(n)$$

$$= d\left\{\left(n+\frac{a}{d}\right)\left[\frac{d}{2}n^2 + \left(a-\frac{d}{2}\right)n\right] - \sum_{j=1}^{n}\left[\frac{d}{2}j^2 + \left(a-\frac{d}{2}\right)j\right]\right\}$$

$$= d\left\{\left[\frac{d}{2}n^3 + \left(\frac{3}{2}a-\frac{1}{2}d\right)n^2 + \left(\frac{a^2}{d}-\frac{a}{2}\right)n\right] - \left[\frac{d}{2}\left(\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n\right) + \left(a-\frac{d}{2}\right)\left(\frac{1}{2}n^2+\frac{1}{2}n\right)\right]\right\}$$

$$= \frac{d^2}{3}n^3 + d\left(a-\frac{d}{2}\right)n^2 + \left[a(a-d)+\frac{d^2}{6}\right]n\,.$$

We further write

$$S_{a;d}^{(1)}(n) = \frac{d}{2}n^2 + \frac{1}{2}[a + (a-d)]n \, ,$$

$$S_{a;d}^{(2)}(n) \; = \frac{d^2}{3}n^3 + \frac{d}{2}[a + (a-d)]n^2 + \frac{1}{6}[a^2 + 4a(a-d) + (a-d)^2]n \, ,$$

$$\ldots$$

$$S_{a;d}^{(k)}(n) = (dn + a)S_{a;d}^{(k-1)}(n) - d\sum_{j=1}^{n} S_{a;d}^{(k-1)}(j) \; .$$

We shall use $<10>$ for a, $<1-1>$ for d, $<11>$ for $a+(a-d)$, $<141>$ for $a^2 + 4a(a-d) + (a-d)^2$. Accordingly, we can derive

$$S_{a;d}^{(3)}(n)$$

$$= (dn + <10>)S_{a;d}^{(2)}(n) - d\sum_{j=1}^{n} S_{a;d}^{(2)}(j)$$

$$= (dn + <10>) \left(\frac{d^2}{3}n^3 + \frac{d}{2} <11> n^2 + \frac{1}{6} <141> n \right)$$

$$- d\sum_{j=1}^{n} \left(\frac{d^2}{3} j^3 + \frac{d}{2} <11> j^2 + \frac{1}{6} <141> j \right)$$

$$- d\left[\frac{d^2}{3}\left(\frac{1}{4}n^4 + \frac{1}{2}n^3 \frac{1}{4}n^2 \right) + \frac{d}{2} <11> \left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \right) + \frac{1}{6} <141> \left(\frac{1}{2}n^2 + \frac{1}{2}n \right) \right]$$

$$= \frac{d^3}{3} n^4 + \left(\frac{d^2}{2} < 11 > + \frac{d^2}{3} < 10 > \right) n^3 + \left(\frac{d}{6} < 141 > + \frac{d}{2} < 10 >< 11 > \right) n^2 + \frac{1}{6} < 10 >< 141 > n$$

$$= \left(\frac{d^3}{3} - \frac{d^3}{12} \right) n^4 + \left(\frac{d^2}{2} < 11 > + \frac{d^2}{3} < 10 > - \frac{d^2}{6} < 1-1 > - \frac{d^2}{6} < 11 > \right) n^3$$

$$+ \left(\frac{d}{6} < 141 > + \frac{d}{2} < 10 >< 11 > - \frac{d}{12} < 1-21 > - \frac{d}{6} < 141 > \right) n^2$$

$$+ \left(\frac{1}{6} < 10 >< 141 > - \frac{1}{12} < 1-21 >< 11 > - \frac{1}{12} < 1-1 >< 141 > \right) n$$

$$= \frac{d^3}{4} n^4 + \frac{d^2}{6} < 33 > n^3 + \frac{d^2}{12} < 282 >) n^2 + \frac{1}{12} < 0660 > n$$

$$= \frac{d^3}{4} n^4 + \frac{d^2}{2} < 11 > n^3 + \frac{d^2}{6} < 141 >) n^2 + \frac{1}{2} < 0110 > n \, .$$

Defining

$$b^*(k, k-2t+1) = b(k, k-2t+1)$$

and

$$b^*(k, k-2t) = (k-2t+1) b(k, k-2t+1) \, ,$$

we can write

$$S_{a;d}^{(k)}(n) = \sum_{j=0}^{k} b^*(k, k+1-j) d^{k-j} p_j(a,d) n^{k+1-j} \, ,$$

where $p_j(a, a-d)$ is a homogeneous polynomial in a and $a-d$ such as

$$p_0(a,d) = 1,$$

$$p_1(a,d) = a + (a-d) = <1,>,$$

$$p_2(a,d) = a^2 + 4a(a-d) + (a-d)^2 = <1,4>,$$

$$p_3(a,d) = a^2(a-d) + a(a-d)^2 = <0,1,>,$$

$$p_4(a,d) = a^4 - 4a^3(a-d) + 24a^2(a-d)^2 - 4a(a-d)^3 + (a-d)^4 = <1,-4,24>,$$

$$p_5(a,d) = a^4(a-d) - 4a^3(a-d)^2 - 4a^2(a-d)^3 + a(a-d)^4 = <0,1,-4,>, \ldots \quad \text{Eq. 24}$$

with $<1,>$ abbreviating $<11>$, $<1,4>$ abbreviating $<141>$, $<0,1,>$ abbreviating $<0110>$, $<1,-4,24>$ abbreviating $<1(-4)(24)(-4)1>$, $<1,-4,24>$ abbreviating $<1(-4)(24)(-4)1>$, $<0,1,-4,>$ abbreviating $<01(-4)(-4)10>$ … due to the symmetry of $p_j(a,a-d)$. Thus we can derive

$$S_{a;d}^{(4)}(n) = \frac{d^4}{5}n^5 + \frac{d^3}{2}<1,>n^4 + \frac{d^2}{3}<1,4>n^3 + d<0,1,>n^2 - \frac{1}{30}<1,-4,-24>n;$$

$$S_{a;d}^{(5)}(n) = \frac{d^5}{6}n^6 + \frac{d^4}{2}<1,>n^5 + \frac{5d^3}{12}<1,4>n^4 + \frac{5d^2}{3}<0,1,>n^3 - \frac{d}{12}<1,-4,-24>n^2$$

$$-\frac{1}{6}<0,1,-4,>n;$$

$$S_{a;d}^{(6)}(n) = \frac{d^6}{7}n^7 + \frac{d^5}{2}<1,>n^6 + \frac{d^4}{2}<1,4>n^5 + \frac{5d^3}{2}<0,1,>n^4 - \frac{d^2}{6}<1,-4,-24>n^3$$

$$-\frac{d}{2}<0,1,-4,>n^2 + \frac{1}{42}<1,-6,-6,64>n;$$

$$S_{a;d}^{(7)}(n) = \frac{d^7}{8}n^8 + \frac{d^6}{2} < 1, > n^7 + \frac{7d^5}{12} < 1,4 > n^6 + \frac{7d^4}{2} < 0,1, > n^5 - \frac{7d^3}{24} < 1,-4,-24 > n^4$$

$$- \frac{7d^2}{6} < 0,1,-4, > n^3 + \frac{d}{42} < 1,-6,-6,64 > n^2 + \frac{1}{12} < 0,1,-6,8, > n \ ;$$

$$S_{a;d}^{(8)}(n) = \frac{d^8}{9}n^9 + \frac{d^7}{2} < 1, > n^8 + \frac{2d^6}{3} < 1,4 > n^7 + \frac{14d^5}{3} < 0,1, > n^6 - \frac{7d^4}{15} < 1,-4,-24 > n^5$$

$$- \frac{7d^3}{3} < 0,1,-4, > n^4 + \frac{2d^2}{9} < 1,-6,-6,64 > n^3 + \frac{2d}{3} < 0,1,-6,8, > n^2$$

$$- \frac{1}{30} < 1,-8,8,64,-160 > n \ ;$$

$$S_{a;d}^{(9)}(n) = \frac{d^9}{10}n^{10} + \frac{d^8}{2} < 1, > n^9 + \frac{3d^7}{4} < 1,4 > n^8 + 6d^6 < 0,1, > n^7 - \frac{7d^5}{10} < 1,-4,-24 > n^6$$

$$- \frac{21d^4}{5} < 0,1,-4, > n^5 + \frac{d^3}{2} < 1,-6,-6,64 > n^4 + 2d^2 < 0,1,-6,8, > n^3$$

$$- \frac{3d}{20} < 1,-8,8,64,-160 > n^2 - \frac{3}{10} < 0,3,-24,64,-48, > n \ ;$$

$$S_{a;d}^{(10)}(n) = \frac{d^{10}}{11}n^{11} + \frac{d^9}{2} < 1, > n^{10} + \frac{5d^8}{6} < 1,4 > n^9 + \frac{15d^7}{2} < 0,1, > n^8 - d^6 < 1,-4,-24 > n^7$$

$$- 7d^5 < 0,1,-4, > n^6 + d^4 < 1,-6,-6,64 > n^5 + 5d^3 < 0,1,-6,8, > n^4$$

$$- \frac{d^2}{2} < 1,-8,8,64,-160 > n^3 - \frac{3d}{2} < 0,3,-24,64,-48, > n^2$$

$$+ \frac{5}{66} < 5,-50,126,192,-1392,2304 > n \ .$$

Now we can use Eq. 8 to further come up with

$$S_{a;d}^{(10)}(n)=\frac{d^{10}}{11}n^{11}+d^9(a-\frac{d}{2})n^{10}+5d^8\left[a(a-d)+\frac{d^2}{6}\right]n^9+15d^7(a-\frac{d}{2})a(a-d)n^8$$

$$+30d^6\left[a^2(a-d)^2-\frac{d^4}{30}\right]n^7+42d^5\left(a-\frac{d}{2}\right)\left[a^2(a-d)^2-\frac{d^2}{3}a(a-d)\right]n^6$$

$$+42d^4\left[a^3(a-d)^3-\frac{d^2}{2}a^2(a-d)^2+\frac{d^6}{42}\right]n^5$$

$$+30d^3\left(a-\frac{d}{2}\right)\left[a^3(a-d)^3-d^2(a-d)^2+\frac{d^4}{3}a(a-d)\right]n^4$$

$$+15d^2\left[a^4(a-d)^4-\frac{4d^2}{3}a^3(a-d)^3+\frac{2d^4}{3}a^2(a-d)^2-\frac{d^8}{30}\right]n^3$$

$$+5d\left(a-\frac{d}{2}\right)\left[a^4(a-d)^4-2d^2a^3(a-d)^3+\frac{9d^4}{4}a^2(a-d)^2-\frac{3d^6}{5}a(a-d)\right]n^2$$

In order to see the trend, use $A(j,r)$ to denote the coefficients of the terms in $p_j(a,a-d)$.

For example,

$A(1,1)=1$, $A(1,2)=1$;

$A(2,1)=1$, $A(2,2)=4$, $A(2,3)=1$;

$A(3,1)=0$, $A(3,2)=1$, $A(3,3)=1$, $A(3,4)=0$;

$A(4,1)=1$, $A(4,2)=-4$, $A(4,3)=-24$, $A(4,4)=-4$, $A(4,5)=1$;

$A(5,1)=0$, $A(5,2)=1$, $A(5,3)=-4$, $A(5,4)=-4$, $A(5,5)=1$, $A(5,6)=0$.

In fact, we can iteratively obtain more values of $A(j,r)$ as in the triangle $A\Delta$ of Table 7.

j\r	1	2	3	4	5	6	7	8	9	10	11
1	1	1									
2	1	4	1								
3	0	1	1	0							
4	1	-4	-24	-4	1						
5	0	1	-4	-4	1	0					
6	1	-6	-6	64	-6	-6	1				
7	0	1	-6	8	8	-6	1	0			
8	1	-8	8	64	160	64	8	-8	1		
9	0	1	-8	64/3	-16	-16	64/3	-8	1	0	
10	1	-10	126/5	192/5	-1392/5	2304/5	-1392/5	192/5	126/5	-10	1

Table 7. Table for the coefficients $A(j,r)$ of the terms in $p_j(a,a-d)$

Besides $A(2t,1)=A(2t,2t+1)=1$ and $A(2t+1,1)=A(2t+1,2t+2)=0$, by further observing

$$A(4,2)=A(4,4)=A(5,3)=A(5,4)=-4\,,$$

$$A(6,2)=A(6,6)=A(7,3)=A(7,6)=-6$$

$$A(8,2)=A(8,8)=A(9,3)=A(9,8)=-8\,,$$

we see the trend that, for $t>1$,

$$A(2t,2)=A(2t,2t)=A(2t+1,3)=A(2t+1,2t)=-2t\,,$$

$$A(2t,t+1)=(2t+2)A(2t+1,t+1)\,,$$

$$\sum_{j=1}^{2t+1}A(2t,j)=(2t+1)\sum_{j=1}^{2t+2}A(2t+1,j)\,,$$

which can be verified in Table 4. Due to the symmetry, only the first half of each row of $A(j,r)$ will be shown in Table 8.

1								
1	4							
0	1							
1	- 4	- 24						
0	1	-4						
1	- 6	- 6	64					
0	1	- 6	8					
1	- 8	8	64	-160				
0	1	- 8	$\frac{64}{3}$	- 16				
1	- 10	$\frac{126}{5}$	$\frac{192}{5}$	$-\frac{1392}{5}$	$\frac{2304}{5}$			
0	1	- 10	$\frac{192}{5}$	$-\frac{336}{5}$	$\frac{192}{5}$			
1	– 12	$\frac{31956}{691}$	$-\frac{15520}{691}$	$-\frac{227160}{691}$	$\frac{730368}{691}$	$-\frac{1026816}{691}$		
0	1	- 12	$\frac{41056}{691}$	$-\frac{106520}{691}$	$\frac{146304}{691}$	$-\frac{73344}{691}$		
1	– 14	$\frac{2494}{35}$	$-\frac{4448}{35}$	$-\frac{8296}{35}$	$\frac{11840}{7}$	$-\frac{137568}{35}$	$\frac{178176}{35}$	
0	1	-14	$\frac{8864}{105}$	$-\frac{29928}{105}$	$\frac{60864}{105}$	$-\frac{14368}{21}$	$\frac{33408}{105}$	
1	– 16	$\frac{362640}{3617}$	$-\frac{1025920}{3617}$	$\frac{320480}{3617}$	$\frac{7371264}{3617}$	$-\frac{273098}{3617}$	$\frac{52971520}{3617}$	$-\frac{65272320}{3617}$

Table 8. Table for the first half of each row of $A(j,r)$

As can be seen from Table 8, it is peculiar that

$$A(2t,t+1)=(2t+2)A(2t+1,t+1)$$

for $t<8$, but not for $t=8$.

For further investigation, let us continue the list of Eq. 10 below.

$p_6(a,a-d) =< 1,-6,-6,64 >$,

$p_7(a,a-d) =< 0,1,-6,8,>$,

$p_8(a,a-d) =< 1,-8,8,64,-160 >$,

$p_9(a,a-d) =< 0,3,-24,64,-48,>$,

$p_{10}(a,a-d) =< 5,-50,126,192,-1392,2304 >$,

$P_{11}(a, a - d) =<0, 5, -50, 192, -336, 192>$,

$p_{12}(a,a-d) =< 691,-8292,31956,15520,-227160,730368,-1026816 >$,

$p_{13}(a,a-d) =< 0,691,-8292,41056,-106520,146304,-73344,>$,

$p_{14}(a,a-d) =< 35,-490,2494,-4448,-8296,59200,-137568,178176 >$,

$p_{15}(a,a-d) =< 0,35,-490,8864,-29928,60864,-14368,33408,>$,

so that

$$S_{a;d}^{(11)}(n)=\frac{d^{11}}{12}n^{12}+\frac{d^{10}}{2}p_1(a,a-d)n^{11}+\frac{11d^9}{12}p_2(a,a-d)n^{10}+\frac{55d^8}{6}p_3(a,a-d)n^9$$

$$-\frac{11d^7}{8}p_4(a,a-d)n^8-11d^6p_5(a,a-d)n^7+\frac{11d^5}{6}p_6(a,a-d)n^6$$

$$+11d^4p_7(a,a-d)n^5-\frac{11d^3}{8}p_8(a,a-d)n^4-\frac{11d^2}{2}p_9(a,a-d)n^3$$

$$+\frac{d}{12}p_{10}(a,a-d)n^2+\frac{1}{6}p_{11}(a,a-d)n\,,$$

$$S_{a;d}^{(12)}(n)=\frac{d^{12}}{13}n^{13}+\frac{d^{11}}{2}p_1(a,a-d)n^{12}+d^{10}p_2(a,a-d)n^{11}+11d^9p_3(a,a-d)n^{10}$$

$$-\frac{11d^8}{6}p_4(a,a-d)n^9-\frac{33d^7}{2}p_5(a,a-d)n^8+\frac{22d^6}{7}p_6(a,a-d)n^7$$

$$+22d^5p_7(a,a-d)n^6-\frac{33d^4}{10}p_8(a,a-d)n^5-\frac{11d^3}{2}p_9(a,a-d)n^4$$

$$+\frac{d^2}{3}p_{10}(a,a-d)n^3+dp_{11}(a,a-d)n^2-\frac{1}{2730}p_{12}(a,a-d)\,,$$

$$S_{a;d}^{(13)}(n)=\frac{d^{13}}{14}n^{14}+\frac{d^{12}}{2}p_1(a,a-d)n^{13}+\frac{13d^{11}}{12}p_2(a,a-d)n^{12}+13d^{10}p_3(a,a-d)n^{11}$$

$$-\frac{143d^9}{60}p_4(a,a-d)n^{10}-\frac{143d^8}{6}p_5(a,a-d)n^9+\frac{143d^7}{28}p_6(a,a-d)n^8$$

$$+\frac{286d^6}{7}p_7(a,a-d)n^7-\frac{143d^5}{20}p_8(a,a-d)n^6-\frac{143d^4}{10}p_9(a,a-d)n^5$$

$$+\frac{13d^3}{12}p_{10}(a,a-d)n^4+\frac{13d^2}{3}p_{11}(a,a-d)n^3-\frac{d}{420}p_{12}(a,a-d)n^2$$

$$-\frac{1}{210}p_{13}(a,a-d)n\,,$$

$$S_{a;d}^{(14)}(n) = \frac{d^{14}}{15} n^{15} + \frac{d^{13}}{2} p_1(a, a-d)n^{14} + \frac{7d^{12}}{6} p_2(a, a-d)n^{13} + \frac{91d^{11}}{6} p_3(a, a-d)n^{12}$$

$$-\frac{91d^{10}}{30} p_4(a, a-d)n^{11} - \frac{1001d^9}{30} p_5(a, a-d)n^{10} + \frac{143d^8}{18} p_6(a, a-d)n^9$$

$$+\frac{143d^7}{2} p_7(a, a-d)n^8 - \frac{143d^6}{10} p_8(a, a-d)n^7 - \frac{1001d^5}{30} p_9(a, a-d)n^6$$

$$+\frac{91d^4}{30} p_{10}(a, a-d)n^5 + \frac{91d^3}{6} p_{11}(a, a-d)n^4 - \frac{d^2}{90} p_{12}(a, a-d)n^3$$

$$-\frac{d}{30} p_{13}(a, a-d)n^2 + \frac{1}{30} p_{14}(a, a-d)n,$$

$$S_{a;d}^{(15)}(n) = \frac{d^{15}}{16} n^{16} + \frac{d^{14}}{2} p_1(a, a-d)n^{15} + \frac{5d^{13}}{4} p_2(a, a-d)n^{14} + \frac{35d^{12}}{2} p_3(a, a-d)n^{13}$$

$$-\frac{91d^{11}}{24} p_4(a, a-d)n^{12} - \frac{91d^{10}}{2} p_5(a, a-d)n^{11} + \frac{143d^9}{12} p_6(a, a-d)n^{10}$$

$$+\frac{715d^8}{6} p_7(a, a-d)n^9 - \frac{429d^7}{16} p_8(a, a-d)n^8 - \frac{143d^6}{2} p_9(a, a-d)n^7$$

$$+\frac{91d^5}{12} p_{10}(a, a-d)n^6 + \frac{91d^4}{2} p_{11}(a, a-d)n^5 - \frac{d^3}{24} p_{12}(a, a-d)n^4$$

$$-\frac{d^2}{6} p_{13}(a, a-d)n^3 + \frac{d}{4} p_{14}(a, a-d)n^2 + \frac{1}{6} p_{15}(a, a-d)n.$$

We have, however, yet to find a better way to come up with satisfactory patterns!

we can derive

$$\tan(2n-1)\theta = \frac{\sum_{k=1}^{n}(-1)^{k-1}\binom{2n-1}{2k-1}\tan^{2k-1}\theta}{\sum_{k=1}^{n}(-1)^{k-1}\binom{2n-1}{2k-2}\tan^{2k-2}\theta} \qquad \text{Eq. 25}$$

and

$$\tan 2n\theta = \frac{\sum_{k=1}^{n}(-1)^{k-1}\binom{2n}{2k-1}\tan^{2k-1}\theta}{\sum_{k=0}^{n}(-1)^{k}\binom{2n}{2k}\tan^{2k}\theta} \qquad \text{Eq. 26}$$

by using

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}\,. \qquad \text{Eq. 27}$$

Although the general formulas for $\sin m\theta$ and $\cos m\theta$ have been known due to De Moivre’s Theorem, we can use Eq. 25 to derive

$$\cos(2n-1)\theta = \sum_{k=0}^{n-1}\sum_{i=k+1}^{n}(-1)^{k}\binom{i-1}{k}\binom{2n-1}{2i-2}\cos^{2n-2k-1}\theta \qquad \text{Eq. 28}$$

and

$$\sin(2n-1)\theta = \sum_{k=0}^{n-1}\sum_{i=k+1}^{n}(-1)^{k}\binom{i-1}{k}\binom{2n-1}{2i-2}\sin^{2n-2k-1}\theta\,; \qquad \text{Eq. 29}$$

and use Eq. 26 to derive

$$\cos 2n\theta = \sum_{k=0}^{n}\sum_{i=k+1}^{n+1}(-1)^{k}\binom{i-1}{k}\binom{2n}{2i-2}\cos^{2n-2k}\theta \qquad \text{Eq. 30}$$

and

$$\sin 2n\theta = \cos\theta\sum_{k=0}^{n-1}\sum_{i=k+1}^{n}(-1)^{k}\binom{i-1}{k}\binom{2n}{2i-1}\sin^{2n-2k-1}\theta \qquad \text{Eq. 31}$$

via combinatorial method.

Since $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$, we can use Eq. 27 to derive

$$\tan 3\theta = \frac{2\tan\theta + (1-\tan^2\theta)\tan\theta}{(1-\tan^2\theta) - 2\tan\theta} = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = \frac{\binom{3}{1}\tan\theta - \binom{3}{3}\tan^3\theta}{\binom{3}{0} - \binom{3}{2}\tan^2\theta}$$

and in turn

$$\tan 5\theta = \frac{2\tan\theta\left[\binom{3}{0} - \binom{3}{2}\tan^2\theta\right] + (1-\tan^2\theta)\left[\binom{3}{1}\tan\theta - \binom{3}{3}\tan^3\theta\right]}{(1-\tan^2\theta)\left[\binom{3}{0} - \binom{3}{2}\tan^2\theta\right] - 2\tan\theta\left[\binom{3}{1}\tan\theta - \binom{3}{3}\tan^3\theta\right]}$$

$$= \frac{5\tan\theta - \left[2\binom{3}{2} + \binom{3}{3} + \binom{3}{1}\right]\tan^3\theta + \tan^5\theta}{1 - \left[2\binom{3}{1} + \binom{3}{2} + \binom{3}{0}\right]\tan^2\theta + 5\tan^4\theta}$$

$$= \frac{\binom{5}{1}\tan\theta - \binom{5}{3}\tan^3\theta + \binom{5}{5}\tan^5\theta}{\binom{5}{0} - \binom{5}{2}\tan^2\theta + \binom{5}{4}\tan^4\theta},$$

due to $2\binom{l}{r} + \binom{l}{r+1} + \binom{l}{r-1} = \binom{l+2}{r+1}$.

In general, we can derive

$$\tan(2n-1)\theta$$

$$= \frac{2\tan\theta\sum_{k=1}^{n-1}(-1)^{k-1}\binom{2n-3}{2k-2}\tan^{2k-2}\theta + (1-\tan^2\theta)\sum_{k=1}^{n-1}(-1)^{k-1}\binom{2n-3}{2k-1}\tan^{2k-1}\theta}{(1-\tan^2\theta)\sum_{k=1}^{n-1}(-1)^{k-1}\binom{2n-3}{2k-2}\tan^{2k-2}\theta - 2\tan\theta\sum_{k=1}^{n-1}(-1)^{k-1}\binom{2n-3}{2k-1}\tan^{2k-1}\theta}$$

$$= \frac{(2n-1)\tan\theta + \sum_{k=2}^{n-1}(-1)^{k-1}\left[2\binom{2n-3}{2k-2} + \binom{2n-3}{2k-1} + \binom{2n-3}{2k-3}\right]\tan^{2k-1}\theta + (-1)^{n-1}\tan^{2n-1}\theta}{1 + \sum_{k=2}^{n-1}(-1)^{k-1}\left[2\binom{2n-3}{2k-3} + \binom{2n-3}{2k-2} + \binom{2n-3}{2k-4}\right]\tan^{2k-2}\theta + (-1)^{n-1}(2n-1)\tan^{2n-2}\theta},$$

which leads to Eq. 25.

Similarly, we can derive

$$\tan 4\theta = \frac{\binom{4}{1}\tan\theta - \binom{4}{3}\tan^3\theta}{\binom{4}{0} - \binom{4}{2}\tan^2\theta + \binom{4}{4}\tan^4\theta}$$

and in turn

$$\tan 6\theta = \frac{2\tan\theta\left[\binom{4}{0} - \binom{4}{2}\tan^2\theta + \binom{4}{4}\tan^4\theta\right] + (1-\tan^2\theta)\left[\binom{4}{1}\tan\theta - \binom{4}{3}\tan^3\theta\right]}{(1-\tan^2\theta)\left[\binom{4}{0} - \binom{4}{2}\tan^2\theta + \binom{4}{4}\tan^4\theta\right] - 2\tan\theta\left[\binom{4}{1}\tan\theta - \binom{4}{3}\tan^3\theta\right]}$$

Eq. 30

$$= \frac{6\tan\theta - \left[2\binom{4}{2} + \binom{4}{3} + \binom{4}{1}\right]\tan^3\theta + 6\tan^5\theta}{1 - \left[2\binom{4}{1} + \binom{4}{2} + \binom{4}{0}\right]\tan^2\theta + \left[2\binom{4}{3} + \binom{4}{4} + \binom{4}{2}\right]\tan^4\theta - \tan^6\theta}$$

$$= \frac{\binom{6}{1}\tan\theta - \binom{6}{3}\tan^3\theta + \binom{6}{5}\tan^5\theta}{\binom{6}{0} - \binom{6}{2}\tan^2\theta + \binom{6}{4}\tan^4\theta - \binom{6}{6}\tan^6\theta}. \qquad \text{Eq. 32}$$

In general, we can derive

$\tan 2n\theta$

$$= \frac{2\tan\theta\sum_{k=0}^{n-1}(-1)^k\binom{2n-2}{2k}\tan^{2k}\theta + (1-\tan^2\theta)\sum_{k=1}^{n-1}(-1)^{k-1}\binom{2n-2}{2k-1}\tan^{2k-1}\theta}{(1-\tan^2\theta)\sum_{k=0}^{n-1}(-1)^k\binom{2n-2}{2k}\tan^{2k}\theta - 2\tan\theta\sum_{k=1}^{n-1}(-1)^{k-1}\binom{2n-2}{2k-1}\tan^{2k-1}\theta}$$

$$= \frac{2n\tan\theta + \sum_{k=1}^{n-2}(-1)^k\left[2\binom{2n-2}{2k} + \binom{2n-2}{2k+1} + \binom{2n-2}{2k-1}\right]\tan^{2k-1}\theta + (-1)^{n-1}2n\tan^{2n-1}\theta}{1 + \sum_{k=2}^{n-1}(-1)^{k-1}\left[2\binom{2n-3}{2k-3} + \binom{2n-3}{2k-2} + \binom{2n-3}{2k-4}\right]\tan^{2k-2}\theta + (-1)^{n-1}(2n-1)\tan^{2n-2}\theta},$$

which leads to Eq. 26.

Before deriving Eq. 28, we first look at the case when $n=4$. From Eq. 25, we can derive

$$\tan 7\theta = \frac{\binom{7}{1}\tan\theta - \binom{7}{3}\tan^3\theta + \binom{7}{5}\tan^5\theta - \binom{7}{7}\tan^7\theta}{\binom{7}{0} - \binom{7}{2}\tan^2\theta + \binom{7}{4}\tan^4\theta - \binom{7}{6}\tan^6\theta}$$

$$= \tan\theta \frac{\binom{7}{1} - \binom{7}{3}(\sec^2\theta - 1) + \binom{7}{5}(\sec^2\theta-1)^2 - \binom{7}{7}(\sec^2\theta-1)^3}{\binom{7}{0} - \binom{7}{2}(\sec^2\theta - 1) + \binom{7}{4}(\sec^2\theta-1)^2 - \binom{7}{6}(\sec^2\theta-1)^3}$$

$$= \frac{\sin\theta \sum_{k=0}^{3}\sum_{i=k+1}^{4}(-1)^k \binom{i-1}{k}\binom{7}{2i-1}\sec^{2k}\theta}{\cos\theta \sum_{k=0}^{3}\sum_{i=k+1}^{4}(-1)^k \binom{i-1}{k}\binom{7}{2i-2}\sec^{2k}\theta}$$

$$= \frac{\sin\theta \sum_{k=0}^{3}\sum_{i=k+1}^{4}(-1)^k \binom{i-1}{k}\binom{7}{2i-1}(1-\sin^2\theta)^{3-k}}{\cos\theta \sum_{k=0}^{3}\sum_{i=k+1}^{4}(-1)^k \binom{i-1}{k}\binom{7}{2i-2}\cos^{2(3-k)}\theta}. \qquad \text{Eq. 33}$$

Obviously, the denominator of Eq. 33 equals the right-hand side of Eq. 28 for $n=4$. Instead of showing that the numerator of Eq. 33 equals the right-hand side of Eq. 29 for $n=4$, we can likewise derive

$$\cot 7\theta = \frac{\cos\theta \sum_{k=0}^{3}\sum_{i=k+1}^{4}(-1)^k \binom{i-1}{k}\binom{7}{2i-1}(1-\cos^2\theta)^{3-k}}{\sin\theta \sum_{k=0}^{3}\sum_{i=k+1}^{4}(-1)^k \binom{i-1}{k}\binom{7}{2i-2}\sin^{2(3-k)}\theta}$$

so that the denominator of which equals the right-hand side of Eq. 29.

In general, we can similarly derive

$$\tan(2n-1)\theta = \frac{\sin\theta \sum_{k=0}^{n-1}\sum_{i=k+1}^{n}(-1)^k \binom{i-1}{k}\binom{2n-1}{2i-1}(1-\sin^2\theta)^{n-k-1}}{\cos\theta \sum_{k=0}^{n-1}\sum_{i=k+1}^{n}(-1)^k \binom{i-1}{k}\binom{2n-1}{2i-2}\cos^{2(n-k-1)}\theta}$$

and

$$\cot(2n-1)\theta = \frac{\cos\theta \sum_{k=0}^{n-1}\sum_{i=k+1}^{n}(-1)^k \binom{i-1}{k}\binom{2n-1}{2i-1}(1-\cos^2\theta)^{n-k-1}}{\sin\theta \sum_{k=0}^{n-1}\sum_{i=k+1}^{n}(-1)^k \binom{i-1}{k}\binom{2n-1}{2i-2}\sin^{2(n-k-1)}\theta}$$

so that Eqs. 28 and 29 are true.

Finally, we note that the numerator and the denominator of Eq. 32 equal to $\dfrac{\sin 6\theta}{\cos^6\theta}$ and $\dfrac{\cos 6\theta}{\cos^6\theta}$, respectively. Accordingly, Eqs. 31 and 32 can similarly be derived from Eq. 26. All the multiple angle formulae derived above can be proved by mathematical induction. We shall only prove Eq. 25, which is obviously true for $n=1$. So we are left to prove

$$\tan 2(n+1)\theta = \frac{\sum_{k=1}^{n+1}(-1)^{k-1}\binom{2(n+1)}{2k-1}\tan^{2k-1}\theta}{\sum_{k=0}^{n+1}(-1)^{k}\binom{2(n+1)}{2k}\tan^{2k}\theta}. \qquad \text{Eq. 34}$$

Using Eq. 27, we first obtain

$$\tan 2(n+1)\theta = \frac{\tan 2\theta+\tan 2n\theta}{1-\tan 2\theta\tan 2n\theta}$$

$$= \frac{2\tan\theta\sum_{k=0}^{n}(-1)^{k}\binom{2n}{2k}\tan^{2k}\theta+(1-\tan^2\theta)\sum_{k=1}^{n}(-1)^{k-1}\binom{2n}{2k-1}\tan^{2k-1}\theta}{(1-\tan^2\theta)\sum_{k=0}^{n}(-1)^{k}\binom{2n}{2k}\tan^{2k}\theta-2\tan\theta\sum_{k=1}^{n}(-1)^{k-1}\binom{2n}{2k-1}\tan^{2k-1}\theta}. \qquad \text{Eq. 35}$$

Then use Eq. 27 to regroup the numerator of Eq. 35 successively as

$$2\tan\theta+\sum_{k=1}^{n}(-1)^{k}\binom{2n}{2k}\tan^{2k+1}\theta+\sum_{k=1}^{n}(-1)^{k-1}\binom{2n}{2k-1}\tan^{2k-1}\theta$$
$$+\sum_{k=1}^{n}\left[(-1)^{k}\binom{2n}{2k}\tan^{2k+1}\theta-(-1)^{k-1}\binom{2n}{2k-1}\tan^{2k+1}\theta\right]$$
$$=2\tan\theta+\sum_{k=1}^{n-1}(-1)^{k}\binom{2n}{2k}\tan^{2k+1}\theta+(-1)^{n}\tan^{2n+1}\theta$$
$$+2n\tan\theta+\sum_{k=2}^{n}(-1)^{k-1}\binom{2n}{2k-1}\tan^{2k-1}\theta+\sum_{k=1}^{n}(-1)^{k}\binom{2n+1}{2k}\tan^{2k+1}\theta$$
$$=2(n+1)\tan\theta+\sum_{k=1}^{n-1}(-1)^{k}\left[\binom{2n}{2k}+\binom{2n}{2k+1}\right]\tan^{2k+1}\theta+(-1)^{n}\tan^{2n+1}\theta$$
$$+\sum_{k=1}^{n-1}(-1)^{k}\binom{2n+1}{2k+1}\tan^{2k+1}\theta+(-1)^{n}(2n+1)\tan^{2n+1}\theta$$
$$=2(n+1)\tan\theta+\sum_{k=1}^{n}(-1)^{k}\binom{2n+1}{2k+1}\tan^{2k+1}\theta+\sum_{k=1}^{n-1}(-1)^{k}\binom{2n+1}{2k}\tan^{2k+1}\theta$$
$$+(-1)^{n}2(n+1)\tan^{2n+1}\theta$$
$$=2(n+1)\tan\theta+\sum_{k=1}^{n-1}(-1)^{k}\binom{2n+2}{2k+1}\tan^{2k+1}\theta+(-1)^{n}\tan^{2n+1}\theta$$
$$=\sum_{k=0}^{n}(-1)^{k}\binom{2(n+1)}{2k+1}\tan^{2k+1}\theta=\sum_{k=1}^{n+1}(-1)^{k}\binom{2(n+1)}{2k-1}\tan^{2k-1}\theta,$$

which is the numerator of Eq. 34. We can similarly prove both denominators to be equal.

GLOSSARY

Polynomial: A mathematical expression such as ax^3+bx^2-cx, where x is a variable and a, b, c are called coefficients.

Binomial expansion: According to the binomial theorem, it is possible to expand the polynomial $(x + y)^n$ into a sum involving terms of the form $ax^b y^c$, where the exponents b and c are nonnegative integers with $b + c = n$, and the coefficient a of each term is a specific positive integer depending on n and b.

Combinatorics: The branch of mathematics dealing with combinations of objects belonging to a finite set in accordance with certain constraints.

Integration of a polynomial: The polynomial rule of integration via term-wise integration can be justified by the fact that the volume i^{k+1} of a $k+1$ dimensional cube with side i is the integral of the surface area $(k+1)i^k$. To explain why it works, let r_i be the inradius of a k dimensional cube with side i. Then the volume $(2r_i)^k$ is geometrically the integral of the surface area $2k(2r_i)^{k-1}$, i.e. $(2r_i)^k=\int 2k(2r_i)^{k-1}\mathrm{d}r_i$ Since $i = 2r_i$, we have $i^k = k\int i^{k-1}di$. It follows that

$$\sum_{i=1}^{n} i^k = \sum_{i=1}^{n} k\int \sum_{i=1}^{n} i^{k-1}di = k\int \sum_{i=1}^{n} i^{k-1}dn.$$

Mathematical induction: To prove a statement S(n) is true for any natural number n, it suffices first to establish the inductive basis [to prove S(1) is true] and then to provide the inductive step [to prove S(m+1) is true by assuming S(m) is true].

REFERENCES

https://hcommons.org/deposits/?facets%5Bauthor_facet%5D%5B%5D=Hung-ping+Tsao

APPENDIX: LIST OF TABLES

Editors of *"EVOLUTIONARY PROGRESS IN SCIENCE, TECHNOLOGY, ENGINEERING, ARTS AND MATHEMATICS (STEAM)"*

1. Dr. Lawrence K. Wang (王抗曝)

Lawrence K. Wang has over 30+ years of professional experience in facility design, environmental sustainability, natural resources, STEAM education, global pollution control, construction, plant operation, and management. He has expertise in water supply, air pollution control, solid waste disposal, drinking water treatment, waste treatment, and hazardous waste management.

He was the Director/Acting President of the Lenox Institute of Water Technology, Engineering Director of Krofta Engineering Corporation and Zorex Corporation, and a Professor of RPI/SIT/UIUC, in the USA.

He was also a Senior Advisor of the United Nations Industrial and Development Organization (UNIDO) in Austria.

Dr. Wang is the author of over 700 technical papers and 45+ books, and is credited with 24 US patents and 5 foreign patents.

He earned his two HS diplomas from the High School of National Taiwan Normal University and the State University of New York. He also earned his BS degree from National Cheng-Kung University, Taiwan, ROC, his two MS degrees from the University of Missouri and the University of Rhode Island, USA, and his PhD degree from Rutgers University, USA.

Currently he is the Chief Series Editor of the Handbook of Environmental Engineering series (Springer); Chief Series Editor of the Advances in Industrial and Hazardous Wastes Treatment series, (CRC Press, Taylor & Francis); co-author of the Water and Wastewater Engineering series (John Wiley & Sons); and Co-Series Editor of the Handbook of Environment and Waste Management series (World Scientific). Dr. Wang is active in professional activities of AWWA, WEF, NEWWA, NEWEA, AIChE, ACS, OCEESA, etc.

2. Dr. Hung-ping Tsao (曹恆平)

Hung-ping Tsao has been a mathematician, a university professor, and an assistant actuary, serving private firms and universities in the United States and Taiwan for 30+ years. He used to be an Associate Member of the Society of Actuaries and a Member of the American Mathematical Society. His research have been in the areas of college mathematics, actuarial mathematics, management mathematics, classic number theory and Sudoku puzzle solving.

In particular, bikini and open top problems are presented to share some intuitive insights and some type of optimization problems can be solved more efficiently and categorically by using the idea of the boundary being the marginal change of a well-rounded region with respect to its inradius; theory of interest, life contingency functions and pension funding are presented in more simplified and generalized fashions; the new way of the simplex method using cross-multiplication substantially simplified the process of finding the solutions of optimization problems; the generalization of triangular arrays of numbers from the natural sequence based to arithmetically progressive sequences based opens up the dimension of explorations; the introduction of step-by-step attempts to solve Sudoku puzzles makes everybody's life so much easier and other STEAM project development.

Dr. Tsao is the author of 3 books and over 30 academic publications. Among all of the above accomplishments, he is most proud of solving manually in the total of ten hours the hardest Sudoku posted online by Arto Inkala in early July of 2012.

He earned his high school diploma from the High School of National Taiwan Normal University, his BS and MS degrees from National Taiwan Normal University, Taipei, Taiwan, his second MS degree from the UWM in USA, and a PhD degree from the University of Illinois, USA.

Editors of the eBOOK Series of the *"EVOLUTIONARY PROGRESS IN SCIENCE, TECHNOLOGY, ENGINEERING, ARTS AND MATHEMATICS (STEAM)"*

Dr. Lawrence K. Wang (王抗曝) - - left

Dr. Hung-ping Tsao (曹恆平) -- right

E-BOOK SERIES AND CHAPTER INTRODUCTON

Introduction to the E-BOOK Series of the *"EVOLUTIONARY PROGRESS IN SCIENCE, TECHNOLOGY, ENGINEERING, ARTS AND MATHEMATICS (STEAM)"* and This Chapter *"*EVOLUTIONARY MATHEMATICS AND SCIENCE FOR TWO ASPECTS OF BINOMIAL COEFFICIENTS: POLYNOMIAL C(n, k) AND COMBINATION $\binom{n}{k}$*"*

The acronym STEM stands for "science, technology, engineering and mathematics". In accordance with the National Science Teachers Association (NSTA), "A common definition of STEM education is an interdisciplinary approach to learning where rigorous academic concepts are coupled with real-world lessons as students apply science, technology, engineering, and mathematics in contexts that make connections between school, community, work, and the global enterprise enabling the development of STEM literacy and with it the ability to compete in the new economy". The problem of this country has been pointed out by the US Department of Education that "All young people should be prepared to think deeply and to think well so that they have the chance to become the innovators, educators, researchers, and leaders who can solve the most pressing challenges facing our nation and our world, both today and tomorrow. But, right now, not enough of our youth have access to quality STEM learning opportunities and too few students see these disciplines as springboards for their careers." STEM learning and applications are very popular topics at present, and STEM related careers are in great demand. According to the US Department of Education reports that the number of STEM jobs in the United States will grow by 14% from 2010 to 2020, which is much faster than the national average of 5-8 % across all job sectors. Computer programming and IT jobs top the list of the hardest to fill jobs.

Despite this, the most popular college majors are business, law, etc., not STEM related. For this reason, the US government has just extended a provision allowing foreign students that are earning degrees in STEM fields a seven month visa extension, now allowing them to stay for up to three years of “on the job training”. So, at present STEM is a legal term. The acronym STEAM stands for “science, technology, engineering, arts and mathematics”. As one can see, STEAM (adds “arts”) is simply a variation of STEM. The word of “arts” means application, creation, ingenuity, and integration, for enhancing STEM inside, or exploring of STEM outside.   It may also mean that the word of “arts” connects all of the humanities through an idea that a person is looking for a solution to a very specific problem which comes out of the original inquiry process. STEAM is an academic term in the field of education.

The University of San Diego and Concordia University offer a college degree with a STEAM focus. Basically STEAM is a framework for teaching or R&D, which is customizable and functional, thence the “fun” in functional. As a typical example, if STEM represents a normal cell phone communication tower looking like a steel truss or concrete column, STEAM will be an artificial green tree with all devices hided, but still with all cell phone communication functions. This e-book series presents the recent evolutionary progress in STEAM with many innovative chapters contributed by academic and professional experts.

This e-book chapter, *“*EVOLUTIONARY MATHEMATICS AND SCIENCE FOR TWO

ASPECTS OF BINOMIAL COEFFICIENTS: POLYNOMIAL C(n, k) AND

COMBINATION $\binom{n}{k}$*”* is Dr. Hung-ping Tsao’s collection of thoughts, works and articles

about various ways of coming up with formulas for sums of powers throughout his retired period for seventeen years now. Three years prior to the publication of “*EXPLICIT POLYNOMIAL EXPRESSIONS FOR SUMS OF POWERS OF AN ARITHMETIC PROGRESSION*”, he gave a few talks among universities in Taiwan and a class of gifted students of his Alma Mater (High School of National Taiwan Normal University). He was then invited to present “General Triangular Arrays of Numbers” by “22nd Asian Technology Conference in Mathematics” (Chung Yuan Christian University, December 19, 2017). He is also grateful that Professor Ronald Graham [author of “*CONCRETE MATHEMATICS*”] replied promptly to my e-mails with two separate attachments of his manuscripts that he generalized most of the special functions in Chapter 6 of “*CONCRETE MATHEMATICS*”. He is presenting here a systemic but rather long account of his personal excursion into the realm of numbers initiated by Blaise Pascal, James Stirling, Leonhard Euler and Jacob Bernoulli, which is therefore not meant to be a categorical survey of the topic.