How to Construct Cap Volatility Surfaces

FinPricing

An implied volatility is the volatility implied by the market price of an option based on the Black-Scholes option pricing model. In cap market, a cap/floor is quoted by implied volatilities but not prices. An interest rate cap volatility surface is a three-dimensional plot of the implied volatility of a cap as a function of strike and maturity.

The term structures of implied volatilities which provide indications of the market's near- and long-term uncertainty about future short- and long-term forwar interest rates. A crucial property of the implied volatility surface is the absence of arbitrage.

Summary

- Cap Volatility Surface Introduction

- The Summary of Volatility Surface Construction Approaches

- Arbitrage Free Conditions

- The SABR Model

- Cap Volatility Surface Construction via The SABR Model

Cap Volatility Surface Introduction

- An implied volatility is the volatility implied by the market price of an option based on the Black-Scholes option pricing model.

- In market, a cap/floor is quoted by implied volatilities rather than prices.

- An interest rate cap volatility surface is a three-dimensional plot of the implied volatility of a cap as a function of strike and maturity.

- The term structures of implied volatilities provide indications of the market's near- and long-term uncertainty about future short- and long-term forward rates.

- Vol skew or smile pattern is directly related to the conditional non-nomality of the underlying returns. In particular, a smile reflects fat tails in the return distribution whereas a skew indicates asymmetry.

- A crucial property of the implied volatility surface is the absence of arbitrage.

The Summary of Volatility Surface Construction Approaches

▪ To construct a reliable volatility surface, it is necessarily to apply robust interpolation methods to a set of discrete volatility data. Arbitrage free conditions may be implicitly or explicitly embedded in the procedure. Typical approaches are

  ▪ Local Volatility Model: a generalisation of the Blaack-Scholes model.

  ▪ Stochastic Volatility Models: such as SABR, Heston, Levy

  ▪ Parametric or Semi-Parametric Models: such as SVI, Omega

  ▪ Market Volatility Model: directly modeling the implied volatility dynamics

  ▪ Interpolation/Extrapolation Model: interpolating or extrapolating volatility data using specific function forms

Arbitrage Free Conditions

- Any volatility models must meet arbitrage free conditions.

- Typical arbitrage free conditions

  - Static arbitrage free condition: Static arbitrage free condition makes it impossible to invest nothing today and receive positive return tomorrow.

  - Calendar arbitrage free condition: The cost of a calendar spread should be positive.

  - Vertical (spread) arbitrage free condition: The cost of a vertical spread should be positive.

  - Horizontal (butterfly) arbitrage free condition: The cost of a butterfly spread should be positive.

Arbitrage Free Conditions (Cont)

- Vertical arbitrage free and horizontal arbitrage free conditions for cap volatility surfaces are based on different strikes

- There is no calendar arbitrage in cap volatility surfaces as caps with different maturities have different cash flows and are associated with different indices. In other words, they can be treated independently.

- At FinPricing, we use the SABR model to construct cap volatility surfaces following the best market practice.

The SABR Model

- SABR stands for "stochastic alpha, beta, rho" referring to the parameters of the model.

- The SABR model is a stochastic volatility model for the evolution of the forward price of an asset, which attempts to capture the volatility smile/skew in derivative markets.

- There is a closed-form approximation of the implied volatility of the SABR model.

- In the cap volatility case, the underlying asset is the forward interest rate.

The SABR Model (Cont)

- The dynamics of the SABR model

$$d\hat{F} = \hat{\alpha}\hat{F}^{\beta}dW_1$$

$$d\hat{\alpha} = v\hat{\alpha}dW_2$$

$$dW_1 dW_2 = \rho dt$$

$$\hat{\alpha}(0) = \alpha$$

where

$\hat{F}$ the forward rate

$\hat{\alpha}$ the forward volatility

W_1, W_2 the standard Brownian motions

ρ the instantaneous correlation between W_1 and W_2

The SABR Model (Cont)

- The four model parameters (a , b , r, n) above have the following intuitive meaning:

  - α the volatility that is tied closely to the at-the-money volatility value.

  - β the exponent that is related to the backbone that the ATM volatility traces when the ATM forward rate varies.

  - ρ the correlation that describes the "skew" or average slope of the volatility curve across strike.

  - v the volatility of volatility that describes the "smile" or curvature/convexity of the volatility curve across the strike for a given term and tenor.

Constructing Cap Volatility Surface

via The SABR Model

- For each maturity of a cap/floor, conduct the following calibration procedure.

- The β parameter is estimated first and typically chosen a priri according to how the market prices are to be observed.

- Alternatively β can be estimated by a linear regression on a time series of ATM volatilities and of forward rates.

- After β is set, we can obtain α by using σ_{ATM} to solve the following equition

$$\sigma_{ATM} = \frac{\alpha}{f^{1-\beta}}\left\{1 + \left[\frac{\alpha^2(1-\beta)^2}{24f^{2(1-\beta)}} + \frac{\rho\beta v\alpha}{4f^{1-\beta}} + \frac{(2-3\rho^2)v^2}{24}\right]T\right\}$$

The Viete method is used to solve this equation

Constructing Cap Volatility Surface
via The SABR Model

- Given the α and β solved above, we can find the optimized value of (ρ, v) by minimizing the distance between the SABR model output volatilities and market volatilities across all strikes for each term and tenor.

$$min \sum_{i=1}^{n} \left[\sigma_i^{SABR}(\alpha, \beta, \rho, v) - \sigma_i^{Market} \right]^2$$

  The Levenberg-Marquardt least-squares optimization routine is used for optimization.

- After α, β, ρ, v calibrated, one can generate SABR volatility (cap volatility) for any moneyness.

- Repeat the above process for each term and tenor.

Thank You

You can find more details at

https://finpricing.com/lib/EqWarrant.html