**Lenox Institute Press**

**Newtonville, NY, USA; Auburndale, MA, USA**

**Evolutionary Progress in Science, Technology, Engineering, Arts and Mathematics (STEAM) Series**


# MATHEMATICS OF HUNG-PING TSAO


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Tsao, Hung-ping (2020). Mathematics of Hung-ping Tsao. In: "*Evolutionary Progress in Science, Technology, Engineering, Arts, and Mathematics (STEAM)"*, Wang, Lawrence K. and Tsao, Hung-ping (editors). Volume 2, Number 11, November 2020; 336 pages. Lenox Institute Press, Newtonville, NY, 12128-0405, USA. No. STEAM-VOL2-NUM11-NOV2020; ISBN 978-0-9890870-3-2. US Department of Commerce, National Technical Information Service, 5301 Shawnee Road, Alexandria, VA 22312, USA.

# REPORT DOCUMENTATION PAGE

*Form Approved*
*OMB No. 0704-0188*



**1. REPORT DATE** *(DD-MM-YYYY)* 11-15-2020

**2. REPORT TYPE** NEW REPORT

**3. DATES COVERED** *(From - To)* JAN 2003-JUN 2020

**4. TITLE AND SUBTITLE** MATHEMATICS OF HUNG-PING TSAO

**5a. CONTRACT NUMBER** N/A

**5b. GRANT NUMBER** N/A

**5c. PROGRAM ELEMENT NUMBER** N/A

**5d. PROJECT NUMBER** STEAM-VOL2-NUM11-NOV2020

**5e. TASK NUMBER** N/A

**5f. WORK UNIT NUMBER** N/A

**6. AUTHOR(S)** **Tsao, Hung-ping**

**7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)**

TSAO RESIDENCE, 1151 Highland Drive, Novato, CA 94949, USA

**8. PERFORMING ORGANIZATION REPORT NUMBER** STEAM-VOL2-NUM11-NOV2020

**9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)**

Wang, Lawrence K. and Tsao, Hung-ping (editors). "*Evolutionary Progress in Science, Technology, Engineering, Arts, and Mathematics (STEAM)*", Volume 2, Number 11, November 2020; Lenox Institute Press, PO Box 405, Newtonville, NY, 12128-0405, USA

**10. SPONSOR/MONITOR'S ACRONYM(S)** LENOX

**11. SPONSOR/MONITOR'S REPORT NUMBER(S)** STEAM-VOL2-NUM11-NOV2020

**12. DISTRIBUTION / AVAILABILITY STATEMENT** NO RESTRICTION

**13. SUPPLEMENTARY NOTES** REGISTRATION ISBN 978-0-9890870-3-2

**14. ABSTRACT**

I would like to share some of my ideas in Number Theory, Actuarial Mathematics,

Sudoku Solving and Optimization Teaching with college students and colleagues.

**15. SUBJECT TERMS** (Keywords) Polynomial expression, Natural sequence, Power-sum, Binomial coefficient, Product-sum, Stirling number, Pascal triangle, Alhazen's Problem, Bernoulli coefficient, Arithmetically progressive sequence, Linear operator, Commutative ring, Recursive formula, n-layer integration, Sorting, Permutation, Combination, Cycle, Subset, Eulerian number, Stirling polynomial, Worpitzky's identity, q-Gaussian coefficient, Bell number, Ordered Bell polynomial, Eulerian Bell polynomial,, Life insurance and annuity, Mortality rate, Survivorship function, Mortality, Interest rate, Premium, Reserve, Sudoku, Puzzle, Row, Column, Box.

**16. SECURITY CLASSIFICATION OF:** UNCLASSIFIED UNLIMITED (UU)

| a. REPORT | b. ABSTRACT | c. THIS PAGE |
|---|---|---|
| UU | UU | UU |

**17. LIMITATION OF ABSTRACT** UU

**18. NUMBER OF PAGES** 336

**19a NAME OF RESPONSIBLE PERSON** Wang, Lawrence K.

**19a. TELEPHONE NUMBER** *(include area code)* (518) 250-0012

**Standard Form 298 (Rev. 8-98)**
**Prescribed by ANSI Std.**

# MATHEMATICS OF HUNG-PING TSAO

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# MATHEMATICS OF HUNG-PING TSAO

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ABSTRACT

I would like to share some of my ideas in Number Theory, Actuarial Mathematics, Sudoku Solving and Optimization Teaching with college students and colleagues.

**Keywords:** Natural sequence, AP-sequence, Power-sum, Product-sum, Sorting, Combination, Permutation, Cycle, Subset, Binomial coefficient, Stirling number, Pascal triangle, Bernoulli coefficient, Eulerian number, Bell number, Ordered Bell polynomial, Eulerian Bell polynomial, Recursive formula, q-Gaussian coefficient, Life insurance, Life annuity, Interest, Mortality, Contingency, Premium, Reserve, Sudoku, Puzzle, Row, Column, Box, Unique solution, Flipflops chain, Residue.

NOMENCLATURE

| | |
|---|---|
| C(n, k), $\binom{n}{k}$ | combination |
| $\Sigma$ | sum |
| $b(k, j)$ | Bernoulli coefficient |
| $(i)_1^\infty$ | the natural sequence |
| $S_n^{(k)}$ | the sum of the first kth powers of the natural sequence |
| $\int$ | integral |
| P(n, k) , $_nP_k$ | the permutation of n elements taken k at a time |
| k! | k factorial |
| $s(n,k)$ | small Stirling number |
| $S(k+1, j)$ | large Stirling number |
| $e(k, j)$ | small Euler number |
| $E(k, j)$ | large Euler number |
| $\left\langle {n \atop k} \right\rangle$ | first-order Eulerian number |
| $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ | second-order Eulerian number |
| $\left(\!\left( {n \atop k} \right)\!\right)$ | the permutation of n elements taken k at a time |
| $\left[ {n \atop k} \right]$ | Stirling number of the first kind |
| $\left\{ {n \atop k} \right\}$ | Stirling number of the second kind |

$(a+(i-1)d)_1^\infty$ arithmetically progressive sequence

$\begin{bmatrix} n \\ k \end{bmatrix}_{a;d}$ Stirling triangle of the first kind for $(a+(i-1)d)_1^\infty$

$\begin{Bmatrix} n \\ k \end{Bmatrix}_{a;d}$ Stirling triangle of the second kind for $(a+(i-1)d)_1^\infty$

$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{a;d}$ first-order Eulerian number for $(a+(i-1)d)_0^\infty$

$\left\langle\left\langle \begin{matrix} k \\ j \end{matrix} \right\rangle\right\rangle_{a;d}$ second-order Eulerian number for $(a+(i-1)d)_0^\infty$

$T^{r,s}{}_{(a_i)_0^\infty}$ general triangular array for $(a_i)_0^\infty$

$\prod$ product

$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q$ $q$ - Gaussian coefficient

$B_n$ Bell number

$\delta$ the force of interest

$\mu_x$ the force of mortality

a(x) the accumulation function

i the nominal rate of interest

$i^{(m)}$ the nominal rate of interest payable m times a year

$d^{(m)}$ the nominal rate of discount payable m times a year

$\ddot{a}_n^{(m)}$ the present value of an annuity due which pays $m^{-1}$ at the beginning of each mth of a year for n years

$a_n^{(m)}$ the present value of an annuity immediate which pays $m^{-1}$ at the end of each mth of a year for n years

$\bar{a}_n$ the present value of a continuous annuity payable continuously for n years with the total of 1 paid during each year

$\ddot{s}_n^{(m)}$ the future value of an annuity due which pays $m^{-1}$ at the beginning of each mth of a year for n years

$s_n^{(m)}$ the future value of an annuity immediate which pays $m^{-1}$ at the end of each mth of a year for n years

$\bar{s}_n$ the future value of a continuous annuity payable continuously for n years, with the total of 1 paid during each year

X the random variable of a newborn's age-at-death

$\omega$ the terminal age

F(x) the distribution function (d. f.) of X

S(x) the survival function

(x) the life aged x

${}_{t|u}q_x$ the probability that a life (x) aged x will die between ages x + t and x + t + u

${}_tq_x$ the probability that (x) will die within t years

${}_tp_x$ the probability that (x) will survive for t years

$q_x$ the probability that (x) will die within a year

$p_x$ the probability that (x) will survive for a year

L(0) the cohort of newborns

$l_0$ the number of newborns in L(0)

L(x) those in L(0) who survive to age x

$l_x$ the number of lives in L(x)

${}_nd_x$ the number of those in L(x) who will die within n years

| | |
|---|---|
| $d_x$ | those in L(x) who will die within a year |
| $P_x^z$ | the number of persons aged between x and x + 1 at the beginning of the calendar year z |
| $E_x^z$ | the number of persons attained age x during the calendar year z |
| ${}_\alpha D_x^z$ | the number of deaths among $E_x^z$ during the calendar year z |
| ${}_\delta D_x^z$ | the number of deaths among $P_x^z$ before the attainment of age x + 1 |
| ${}_\alpha m_x^z$ | the number of migrants in addition to $E_x^z$ during the calendar year z |
| ${}_\delta m_x^z$ | the number of migrants in addition to $P_x^z$ before the attainment of age x + 1 |
| $A_{x:n}^1$ | n-year term insurance of 1 payable at the end of the year of death |
| $A_{x:n}^{\;\;1}$ | n-year pure endowment of 1 payable at the end of the nth year when (x) lives |
| $A_{x:n}$ | n-year endowment insurance of 1 payable either at the end of the year of death or at the end of the nth year when (x) survives |
| $a_{x:n}$ | n-year annuity of 1 payable at the end of each year while (x) survives |
| $\ddot{a}_{x:n}$ | n-year annuity of 1 payable at the beginning of each year while (x) survives |
| $\alpha_{x:n}^1$ | n-year term life contingency function with the death benefit $\alpha_k$ payable at the end of the year of death |
| $\alpha_{x:n}^{\;\;1}$ | n-year pure endowment of $\alpha_n$ at the date of maturity |
| $A_{x:n}^1$ | n-year term insurance of 1 payable at the end of the year of death |
| $A_{x:n}^{\;\;1}$ | n-year pure endowment of 1 payable at the end of the nth year when (x) lives |

$A_{x:n}$ n-year endowment insurance of 1 payable either at the end of the year of death or at the end of the nth year when (x) lives

$d_\delta(x,t)$ the discount function of interest

$d_\mu(x,t)$ the discount function of mortality

${}_{h|}\overline{\alpha}_{x:n}$ an h-year deferred n-year continuous life contingency function

${}_{h|}\overline{\alpha}_{x}$ an h-year deferred whole life continuous contingency function

$\overline{\alpha}_{x:n}$ an n-year continuous life contingency function

$\overline{\alpha}_{x}$ a whole life continuous contingency function

${}_r\overline{P}({}_{h|}\overline{\alpha}_{x:n})$ the continuously paid net level premium of ${}_{h|}\overline{\alpha}_{x:n}$ , with payments for r years

$(I\overline{\alpha})_{x:n}$ an n-year continuous contingency function providing the present value of the death benefit $(\underline{t+1})\overline{\alpha}_t$ at time t and the maturity benefit $n\overline{\alpha}_n$.

$(I\overline{\alpha})^1_{x:n}$ an n-year continuous contingency function providing the present value of the death benefit $(\underline{t+1})\overline{\alpha}_t$ at time t and the maturity benefit 0

$(D\overline{\alpha})^1_{x:n}$ an n-year continuous contingency function providing the present value of the death benefit $(n-\underline{t})\overline{\alpha}_t$ at time t

$(D^{(m)}\overline{\alpha})^1_{x:n}$ , an mthly decreasing life contingency function

$(D_h^{(m)}\overline{\alpha})^1_{x:n}$ an mthly decreasing life contingency function with only h years death benefit decrease

$6_{10}$(86)b6:6(75): u56c3b3/c9b9 The tenth step with the law of unique solution to avoid the dilemma of double choices between 5, 6 in column 3 of box 3 and 6, 5 in column 9 of box 9 so as to place 6 at (86) box 6

$3_7$r3: fcr-56(32)(33)&49(36)(39) The seventh step with row move to place 3 in row 3 since 3 is the residue of the flipflops chain 56(32)(33)&49(36)(39) in row 3.

## PROLOGUE

I am an amateur mathematician, who has been working rather diligently all my life. After receiving Ph.D. from UIUC in the area of combinatorics and teaching for two years in college, I pursued my actuarial career for eight years before returning to teach. I have published (22) in Chinese, an English excerpt of which was published in (20). During the 17 years of teaching at SFSU, I used the textbook “College Mathematics” tailor-made for my own students in College of Business. Since these books contain fruitful of innovative ideas, I am eager to benefit undergraduate students worldwide.

Ever since my retirement in the year of 2002, I have been working on the classic Number Theory. Although I only used elementary methods, I was able to get a breakthrough and recently published a “EVOLUTIONARY MATHEMATICS AND SCIENCE FOR NUMBER INTRICACY INVESTIGATION” (in addition to “EVOLUTIONARY MATHEMATICS AND SCIENCE FOR LIFE CONTINGENCY INVESTIGATION”) by Lenox Institute Press. I hope my efforts and results could evolve into an undergraduate textbook in the area of Number Theoretical Combinatorics.

I also published “Cracking Sudoku Completely” in Chinese. It contains the detailed solution of the hardest Sudoku puzzle posted online which I solved manually in 2013. Besides being referenced by a course of Computer Games in Taiwan, its excerpt was published in (17). I believe the inclusion of the English version of it should benefit the vast majority of Sudoku lovers worldwide. As a matter of fact, four “EVOLUTIONARY MATHEMATICS AND ART FOR SUDOKU” e-books have recently published.

1. Number Theory

Dated back to the Eighteenth Century, James Sterling, Leonhard Euler and Jacob Bernoulli have collectively set up a solid foundation for the number theoretically combinatorics. You might have heard of numbers bearing their names, but I bet most people are not familiar with their work.

Although my doctoral dissertation included some discussion of certain properties of Stirling numbers, I should admit that I knew very little about them. Late until around the turn of this Century, I came across a book titled Concrete Mathematics in a Stanford library which was used as a textbook for the first year graduate students of Computer Science major.

At that time, I was about to retire from teaching in the College of Business, San Francisco State University and got interested in pursuing the explicit polynomial expression for the sum of powers of the natural numbers. Through my tireless efforts, I came up with two unusual arrays of numbers to be used for my purpose. No sooner I flipped through that textbook than I realized that those unusual numbers were indeed Stirling numbers of the first and second kind in disguise!

The authors of the book did suggest some plausible approach for expressions of the sum of powers of the natural numbers without reaching the goal, as an example of a failed attempt. I would like to share with you my successful attempt. I further generalize the related numbers based on the natural sequence to those that are arithmetically progressive sequence-based. As a result, various structures of triangular arrays can be built on top of different underlying bases.

2. Actuarial Mathematics

We shall start with the accumulation function and use the geometric point of view to generalize and simplify the theory of interest. Then survey the laws of mortality from both points of view of stochastic theory and traditional actuaries. There is a thorough discussion and simple visualization of Balducci and uniform distribution of deaths assumptions of mortality rates of fractional ages.

Two least square-fit cubic survivorship functions for fitting the Male Table of 1958 CSO are presented. Various complicated exposure formulas for a mortality study are obtained by a simple inspection of the valuation schedule in demography.

Life insurance and annuities are first introduced in three different points of view: deterministic, stochastic and dynamic. Then a uniform representation of a general life contingency function and its derivative is defined in such a fashion that deferred, term, endowment, life insurance and life annuity with level or varying benefit and premium can be treated all in one shot.

3. Sudoku Solving

We provide a unique step-by-step Sudoku solving procedure by using subscripts and annotations so that the entire solving process can be recorded. We train the beginners into champion players with enabling "kung fu skills" coupled with surprisingly easy measures. In the end, we demonstrate how to crack down the hardest Sudoku ever!

4. Optimization Teaching

We introduce more efficient methods in Differential Calculus and Linear Programming.

# 1. TALK ELEGANCY

## 1.1. TOWER OF HANOI

We are given a tower of eight disks, initially stacked in decreasing size on one of three pegs. The objective is to transfer the entire tower to one of the other pegs, moving only one disk at a time and never moving a larger one onto a smaller.

The problem of the Tower of Hanoi was invented by E. Lucas in 1883 and had been discussed extensively without touching on the number system in (6). Let us first give an optimal solution to this problem with four disks by imposing subscripts to each number to keep track of the number of moves hereunto taken for the corresponding disk as follows.

| | | | |
|---|---|---|---|
| Step 0 | $1_0$ | | |
| | $2_0$ | | |
| | $3_0$ | | |
| | $4_0$ | | |
| | peg 1 | peg 2 | peg 3 |
| Step 1 | $2_0$ | | |
| | $3_0$ | | |
| | $4_0$ | $1_1$ | |
| | peg 1 | peg 2 | peg 3 |
| Step 2 | $3_0$ | | |
| | $4_0$ | $1_1$ | $2_1$ |
| | peg 1 | peg 2 | peg 3 |
| Step 3 | $3_0$ | | $1_2$ |
| | $4_0$ | | $2_1$ |
| | peg 1 | peg 2 | peg 3 |
| Step 4 | | | $1_2$ |
| | $4_0$ | $3_1$ | $2_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 5 | $1_3$ | | |
| | $4_0$ | $3_1$ | $2_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 6 | $1_3$ | $2_2$ | |
| | $4_0$ | $3_1$ | |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 7 | | $1_4$ | |
| | | $2_2$ | |
| | $4_0$ | $3_1$ | |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 8 | | $1_4$ | |
| | | $2_2$ | |
| | | $3_1$ | $4_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 9 | | $2_2$ | $1_5$ |
| | | $3_1$ | $4_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 10 | | | $1_5$ |
| | $2_3$ | $3_1$ | $4_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 11 | $1_6$ | | |
| | $2_3$ | $3_1$ | $4_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 12 | $1_6$ | | $3_2$ |
| | $2_3$ | | $4_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 13 | | | $3_2$ |
| | $2_3$ | $1_7$ | $4_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 14 | | | $2_4$ |
| | | | $3_2$ |
| | | $1_7$ | $4_1$ |
| | peg 1 | peg 2 | peg 3 |

Step 15 | | |
| --- | --- | --- |
| | | $1_8$<br>$2_4$<br>$3_2$<br>$4_1$ |
| peg 1 | peg 2 | peg 3 |

The general problem with n disks can be solved the same way. Let $T_n$ denote the total number of moves taken to accomplish the objective in the general case. Then we see from the above illustration that

$$T_3 = 8 + 4 + 2 + 1 = 15$$

and in general

$$T_n = \sum_{i=0}^{n-1} 2^i = 2^n - 1.$$

Now with a modification we can convert our problem into the binary number system. Give some of the disks the weight 0 and the others the weight 1. Then the weighted sum $T_{n;2}$ of all the moves would represent a unique binary number such as

$$T_{5;2}(1,1,0,1,0) = 2^4 + 2^3 + 2^1 = (1,1,0,1,0)_2,$$

where (1,1,0,1,0) represents 5 disks with weights listed in the ascending order of sizes.

If we further restrict that all moves can only be made between the adjacent pegs and consider each move as a half count, then the direct solution of the modified problem with three disks can be demonstrated as follows.

Step 0

| | | |
| --- | --- | --- |
| $1_0$<br>$2_0$<br>$3_0$ | | |
| peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 1 | $2_0$ | | |
| | $3_0$ | $1_{0.5}$ | |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 2 | $2_0$ | | |
| | $3_0$ | | $1_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 3 | $3_0$ | $2_{0.5}$ | $1_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 4 | | $1_{1.5}$ | |
| | $3_0$ | $2_{0.5}$ | |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 5 | $1_2$ | | |
| | $3_0$ | $2_{0.5}$ | |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 6 | $1_2$ | | |
| | $3_0$ | | $2_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 7 | $3_0$ | $1_{2.5}$ | $2_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 8 | | | $1_3$ |
| | $3_0$ | | $2_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 9 | | | $1_3$ |
| | | $3_{0.5}$ | $2_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 10 | | $1_{3.5}$ | |
| | | $3_{0.5}$ | $2_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 11 | $1_4$ | $3_{0.5}$ | $2_1$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 12 | | $2_{1.5}$ | |
| | $1_4$ | $3_{0.5}$ | |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 13 | | $1_{4.5}$ | |
| | | $2_{1.5}$ | |
| | | $3_{0.5}$ | |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 14 | | $2_{1.5}$ | |
| | | $3_{0.5}$ | $1_{5}$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 15 | $2_{2}$ | $3_{0.5}$ | $1_{5}$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 16 | | $1_{5.5}$ | |
| | $2_{2}$ | $3_{0.5}$ | |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 17 | $1_{6}$ | | |
| | $2_{2}$ | $3_{0.5}$ | |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 18 | $1_{6}$ | | |
| | $2_{2}$ | | $3_{1}$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 19 | $2_{2}$ | $1_{6.5}$ | $3_{1}$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 20 | | | $1_{7}$ |
| | $2_{2}$ | | $3_{1}$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 21 | | | $1_{7}$ |
| | | $2_{2.5}$ | $3_{1}$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 22 | | $1_{7.5}$ | |
| | | $2_{2.5}$ | $3_{1}$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 23 | $1_{8}$ | $2_{2.5}$ | $3_{1}$ |
| | peg 1 | peg 2 | peg 3 |

| | | | |
|---|---|---|---|
| Step 24 | | | $2_{3}$ |
| | $1_{8}$ | | $3_{1}$ |
| | peg 1 | peg 2 | peg 3 |

| Step 25 | | | |
|---|---|---|---|
| | | | $2_3$ |
| | | $1_{8.5}$ | $3_1$ |
| | peg 1 | peg 2 | peg 3 |

| Step 26 | | | |
|---|---|---|---|
| | | | $1_9$ |
| | | | $2_3$ |
| | | | $3_1$ |
| | peg 1 | peg 2 | peg 3 |

The above detailed process can be recapped as follows.

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $1_0$ | | | | | | | | | $1_{4.5}$ | | | | | | | | | $1_9$ |
| $2_0$ | $1_2$ | | $1_6$ | | | $1_{1.5}$ | | $1_{3.5}$ | $2_{1.5}$ | $1_{5.5}$ | | $1_{7.5}$ | | | $1_3$ | | $1_7$ | $2_3$ |
| $3_0$ | $3_0$ | $1_4$ | $2_2$ | $1_8$ | $1_{0.5}$ | $2_{0.5}$ | $1_{2.5}$ | $3_{0.5}$ | $3_{0.5}$ | $3_{0.5}$ | $1_{6.5}$ | $2_{2.5}$ | $1_{8.5}$ | $1_1$ | $2_1$ | $1_5$ | $3_1$ | $3_1$ |
| | | peg 1 | | | | | | | peg 2 | | | | | | | peg 3 | | |

As before, with a modification we can convert our problem into the tri-nary system. Give some of the disks the weight 0, some others the weight 1 and the rest the weight 2. Then the weighted sum $T_{n;3}$ of all the moves would represent a unique tri-nary number. For example,

$$T_{5;3}(2,0,1,0,2) = 2(3^4) + 3^2 + 2 = (2,0,1,0,2)_3,$$

where (2,0,1,0,2) represents 5 disks with weights listed in the ascending order of sizes.

The initial problem can be generalized as follows.

<u>General Tower of Hanoi with k pegs</u>. We are given a tower of n disks, initially stacked in decreasing size on the first of k ordered pegs. The objective is to transfer the entire tower to the last peg, moving only one disk at a time and never moving a larger one onto a smaller. We further restrict that all moves can only be made between the adjacent pegs.

Can we convert the problem of General Tower of Hanoi into the k-ary number system?

## 1.2. EINSTEIN RIDDLE

The question is: Based on the following hints, who owns the fish?

1) There are five houses in five different colors.

2) In each house lives a person with a different nationality.

3) These five owners drink a certain type of beverages, smoke a certain brand of cigar and keep a certain pet.

4) No owners have the same pet, smoke the same brand of cigar or drink the same beverage.

5) The Brit lives in the red house.

6) The Swede keeps dogs as pets.

7) The Dane drinks tea.

8) The green house is on the left of the white house.

9) The green house's owner drinks coffee.

10) The person who smokes Pall Mall rears birds.

11) The owner of the yellow house smokes Dunhill.

12) The man living in the center house drinks milk.

13) The Norwegian lives in the first house.

14) The man who smokes blends lives next to the one who keeps cats.

15) The man who keeps horses lives next to the man who smokes Dunhill.

16) The owner who smokes Blue Master drinks beer.

17) The German smokes Prince.

18) The Norwegian lives next to the blue house.

19) The man who smokes blends has a neighbor who drinks water.

The author's motivation came from the notion that only two percent of people in the whole world could solve Einstein Riddle. He would try to prove it otherwise by using an illustrative method incorporate with subscripted annotations.

Many logical reasoning problems can be solved this way, especially Sudoku puzzles. He was invited to present the following talk at "The 2017 International Conference in Management Sciences and Decision Making" (Tamkang University), which is an example of a talk that could appeal to general audience with no math background.

### 1.2.1 Talk Topic: Illustrative Problem Solving

First of all, I would like to thank Professor Ruey-Chyn Tsaur for inviting me cordially here at the Department of Management Sciences, which is by no means a stranger to me. It has to trace back to its origin the Institute of Management Sciences, where I was invited by Professor Horng-Jinh Chang to be a visiting professor for three months in the year of 1984.

As you might have known, Professor Chang was a student of Professor Wen-tao Huang, who has been my dearest friend since our graduate student years together in Tsinghua University, where Professor Tsaur received his PhD from much later.

I still remember quite vividly the founding year of the Institute of Mathematics there nearly fifty five years ago. Wen-tao was the first student to register, but to his dismay was soon called back to fulfill his obliged teaching duty in Tainan as required for every student graduated from the National Taiwan Normal University.

Among the seven students registered in the Spring semester of 1962, six (one female) graduated from the National Taiwan Normal University and one from the National Taiwan University. Despite of the wide range of age disparity, we got along pretty well in the male student dorm. After the first day of orientation, we had dinner together. To make the story short, let me propose the following three problems that are most suitable for our theme: the illustrative problem solving.

Problem 1**.** From the following requirements, can you figure out the order of the age seniority and the gift-exchanging arrangement?

#1. The sitting is three on each side of the table.

#2. Each gives gift in a non-reciprocate fashion.

#3. HT gives gift to the one sitting opposite him.

#4. WH sitting opposite to the second oldest gives gift to the second youngest.

#5. HL sitting by the side of the second youngest gives gift to the youngest.

#6. CH not giving gift to the second oldest sits between the youngest and the third youngest.

#7. FH and the third oldest do not give gift to each other.

#8. WL being not the oldest sits in a corner and gives gift to the third youngest.

#9. The third youngest sits opposite the oldest.

Solution

According to #1, divide the table surface into six parts as shown in Figure 1, each of which is to place one’s number of the age seniority with 1 being the youngest.

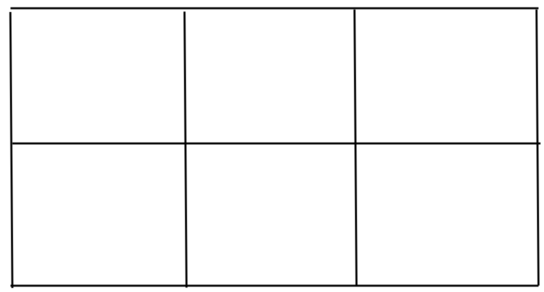


Figure 1. The table surface for placing

According to #6, sit CH first with the subscript indicating the order of occurrence and co-place 1 and 3 as shown in Figure 2. According to #9, place 6 opposite to 3.

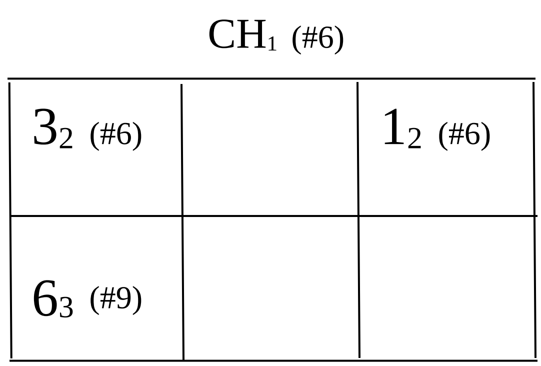



Figure 2. The first stage of placing

According to #4 and #6, place 5 in the remaining corner and sit WH opposite to the second oldest. According to #8, sit WL (being neither the oldest nor the third youngest) in the remaining right corner.

|  | $CH_1$ | $WH_5$ (#4) |
|---|---|---|
| $3_2$ |  | $1_2$ |
| $6_3$ |  | $5_4$ (#4.6) |
|  |  | $WL_6$ (#8) |

Figure 3. The second stage of placing

Therefore, CH must be the third oldest. Otherwise, co-place 2 and 4 as shown below. According to #5, sit HL by the side of CH. According to #7, sit FH (being not the third oldest) in the remaining corner. According to #3, HT gives gift to CH contradicting #4 (WH gives gift to CH).

| $HL_2$ (#5) | CH | WH |
|---|---|---|
| 3 | $2_1$ | 1 |
| 6 | $4_1$ | 5 |
| $FH_3$ (#7) | $HT_4$ (#3) | WL |

Figure 4. The third stage of placing

Therefore, the only two possible cases are displayed below.

Case 1.

| HT | CH | WH |
|---|---|---|
| 3 | 4 | 1 |
| 6 | 2 | 5 |
| HL | FH | WL |

Figure 5. The fourth stage of placing

The gift-giving arrangement is displayed with the requirement number as follows.

| | WH (1) | CH (4) | HT (3) | WL (5) | FH (2) | HL (6) |
|---|---|---|---|---|---|---|
| WH (1) | | | #4 | | | |
| CH (4) | | | | #6 | | |
| HT (3) | | #3 | | | | |
| WL (5) | | | | | #8 | |
| FH (2) | | | | | | #2 |
| HL (6) | #5 | | | | | |

Table 1. The first stage of gift-giving arrangement

The requirement #6 prevents FH from giving gift to CH, so nobody gives gift to CH.

Case 2.

| FH | CH | WH |
|---|---|---|
| 3 | 4 | 1 |
| 6 | 2 | 5 |
| HL | HT | WL |

Figure 6. The fifth stage of placing

The gift-giving arrangement is displayed with the requirement number and the age seniority order is WH, HT, FH, CH, WL, HL as illustrated below.

| | WH (1) | CH (4) | HT (2) | WL (5) | FH (3) | HL (6) |
|---|---|---|---|---|---|---|
| WH (1) | | | #4 | | | |
| CH (4) | | | | #6 | | |
| HT (2) | | #3 | | | | |
| WL (5) | | | | | #8 | |
| FH (3) | | | | | | #2 |
| HL (6) | #5 | | | | | |

Table 2. The second stage of gift-giving arrangement

Problem 2. On the second day, we three young ones decided to elect courses among Statistics, Analysis, Algebra and Topology according to the following agreements. From the following agreements, can you figure out who does not elect Topology?

#1. Each elects exactly three courses.

#2. Each course is elected by exactly two.

#3. If WH elects Statistics, so does he Topology.

#4. If HT elects Algebra, so does he Analysis.

#5. If FH elects Algebra, so does he Analysis.

#6. If WH elects Topology, so does he Analysis.

#7. If FH elects Topology, so does he Analysis.

Solution

Assumption 1. WH does not elect Topology.

According to #2, TH and FH elect Topology, abbreviated as (HT, TP) and (FH,TP) in Figure 7.

| | ST | AN | AL | TP | |
|---|---|---|---|---|---|
| WH | 1.1.1 | | 1.1 | 1.1.1 (#3) | < 4 |
| HT | | 1.1 (#4) | 1.1 | 1 | < 4 |
| FH | | 1 (#7) | | 1 | < 4 |
| | = 2 | = 2 | = 2 | = 2 | |

Figure 7. Assumption 1 of course electing

According to #7, (FH, AN).

Assumption 1.1. (WH, AL), (HT, AL).

According to #4, (HT, AN).

Assumption 1.1.1. (WH, ST).

According to #3, (WH, TP), contradicting #2, since Topology is elected by three.

Assumption 1.2. (WH, AL), (FH, AL).

| | ST | AN | AL | TP |
|---|---|---|---|---|
| WH | | | 1.2 | |
| HT | | | | 1 |
| FH | | 1 | 1.2 | 1 |

Figure 8. Assumption1.2 of course electing

According to #2, (WH,ST), (HT,ST). Or else, FH would elect four courses, contradicting #1. According to #3, (WH, TP), contradicting Assumption 1.

Assumption 1.3. (HT, AL), (FH, AL).

According to #4, (HT, AN).

| | ST | AN | AL | TP |
|---|---|---|---|---|
| WH | | | | |
| HT | | | 1.3 | 1 |
| FH | | 1 | 1.3 | 1 |

Figure 9. Assumption1.3 of course electing

Same as before, (WH, TP), contradicting Assumption 1. So instead of Figure 8 and Figure 9, let us look at Figure 10 under Assumption 2.

Assumption 2. HT does not elect Topology. According to #2, (WH,TP), (FH,TP). According to #6, (WH, AN). According to #7, (FH, AN).

| | ST | AN | AL | TP |
|---|---|---|---|---|
| WH | | 2 (#6) | | 2 |
| HT | | 2.1 (#4) | 2.1 | |
| FH | | 2 (#7) | | 2 |

Figure 10. Assumption 2 of course electing

Assumption 2.1. (HT, AL).

According to #4, (HT, AN), contradicting #2, since Analysis is elected by three.

Assumption 2.2. (WH, AL), (FH, AL).

| | ST | AN | AL | TP |
|---|---|---|---|---|
| WH | | 2 (#6) | 2.2 | 2 |
| HT | | | | |
| FH | | 2 (#7) | 2.2 | 2 |

Figure 11. Assumption 2.2 of course electing

So instead of Figure 11, let us look at Figure 12 under Assumption 3. According to #1, HT can only elect Topology, contradicting #2.

Assumption 3. FH does not elect Topology.

According to #2, (WH,TP), (HT,TP). According to #6, (WH, AN).

| | ST | AN | AL | TP |
|---|---|---|---|---|
| WH | | 3 (#6) | 3.1 | 3 |
| HT | | 3.1 (#4) | 3.1 | 3 |
| FH | | | | |

Figure 12. Assumption 3 of course electing

Assumption 3.1. (WH, AL), (HT, AL). According to #4, (HT, AN).

According to #1, FH can only elect Statistics, contradicting #2.

Assumption 3.2. (HT, AL), (FH, AL). According to #4, (HT, AN).

| | ST | AN | AL | TP |
|---|---|---|---|---|
| WH | | 3 (#6) | | 3 |
| HT | | 3.2 (#4) | 3.2 | 3 |
| FH | | 3.2 (#5) | 3.2 | |

Figure 13. Assumption 3.2 of course electing

According to #5, (FH, AN), contradicting #2, since Analysis is elected by three. The false attempt shown in Figure 13 leads us to Assumption 3.3 for the last resort.

Assumption 3.3. (WH, AL), (FH, AL). According to #7, (FH, AN).

Assumption 3.3.1. (HT, ST), (FH, ST).

| | ST | AN | AL | TP |
|---|---|---|---|---|
| WH | | 3 (#6) | 3.3 | 3 |
| HT | 3.3.1 | | | 3 |
| FH | 3.3.1 | 3.3 (#7) | 3.3 | |

Figure 14. Assumption 3.3.1 of course electing

Agreements #1 and #2 are satisfied in Figure 14. Therefore, FH does not elect Topology.

Problem 3**.** Oddly enough, we all came from different places (P), later majored in Different fields (F), with advisors of different nationalities (N) and now each of us lives in different cities (C) . Based on the hints below, can you figure out who resides in San Francisco?

#1. CH came from Danshuei.

#2. HL lives in Hsinchu.

#3. FH advised by American.

#4. The individual who came from Taipei registered before that from Chiayi.

#5. The individual who came from Taipei majored in Number Theory.

#6. The individual who advised by Canadian lives in Taipei.

#7. The individual who came from Tainan advised by Indian.

#8. The individual who registered third majored in Geometry.

#9. WH registered first.

#10. The individual who advised by American registered either right before or right after that lives in Danshuei.

#11. The individual who lives in Los Angeles registered either right before or right after that advised by Indian.

#12. The individual who advised by Japanese majored in Game Theory.

#13. HT advised by Chinese.

#14. WH registered right before the one came from Changhua.

#15. The individual who advised by American registered either right before or right after that majored in Geometry.

Solution. (This is, in fact, the Einstein Riddle in disguise.)

By rearranging the hints in the new order: 9, 14, 8, 4, 5, 1, 7 and 11, we can first come up with Table 3 of the first stage information.

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| I | WH (#9) | | CH (#1) | | |
| P | TN (#1) | CH (#14) | DS (#1) | TP (#4,#5)* | CY (#4) |
| F | | | GM (#8) | NT (#5) | |
| N | IN (#7) | | | | |
| C | | LA (#11) | | | |

*. Had TP been placed in (P, 3), NT would have to be placed in (F, 3) according to #5, contradicting GM (F, 3).

Table 3. Table of the first stage of information

According to #15, we have the following two cases to consider.

Case 1. AM (N, 4)

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| I | WH (#7) | | CH (#1) | FH (#3) | HL (#2) |
| P | TN (#1) | CH (#14) | DS (#1) | TP (#4,#5) | CY (#4) |
| F | | | GM (#8) | NT (#5) | |
| N | IN (#7) | | | AM (1) | |
| C | | LA (#11) | | | HC (#2) |

Table 4. Table of the second stage of information in Case 1

According to #3, we place FH in (I, 4). According to #2, we place HL in (I, 5), HC in (C, 5) and HT in (I, 2), contradicting #13.

Case 2. AM (N, 2)

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| I | WH (#7) | FH (#3) | CH (#1) | HT (#13) | |
| P | TN (#1) | CH (#14) | DS (#1) | TP (#4,#5) | CY (#4) |
| F | | | GM (#8) | NT (#5) | GT (#12) |
| N | IN (#7) | AM (2) | | CH (#13) | JA (#12) |
| C | | LA (#11) | | | |

Table 5. Table of the second stage of information in Case 2

According to #3, we place FH in (I, 2). According to #12, we place JA in (N, 5) and GT in (F, 5). According to #13, we place CH in (N, 4) and HT in (I, 4).

We can now use the rest of the hints to complete Table 6 for the answer.

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| I | WH | FH | CH | HT | HL |
| P | TN | CH | DS | TP | CY |
| F | ST | AN | GM | NT | GT |
| N | IN | AM | CN | CH | JA |
| C | DS | LA | TP | SF | HC |

Table 6. The final table of the complete information

Therefore, HT lives in San Francisco.

Now you know the rest of the story. Not quite! Let me finish with the following episode.

<u>Fifty four years ago,</u>

HT: "WH, **W**here is your **H**ome?"

WH: "HT, **H**ome is in **T**ainan."

<u>Fifty four years later,</u>

Rock watching HT

White waves splash upon shore rock slate,

One dashes another lest getting there late;

An idle by-passer simply sits still watching,

Poetic rhythms well up during long gazing.

Watching waves   WH

White waves seize the rocks,

Waves went yet turning back;

If only for sentimental blocks,

Nag forever not holding back.

Wen-tao and I have been teasing each other (first by mail, then through e-mail) ever since we graduated from Tsinghua University. In conclusion, I would like to challenge Wen-tao with the following poem in hope that our exchange of teasing would continue.

I have almost left my heart in San Francisco,

Come over here to talk about fifty years ago;

Along the bank of the same old Damshueiho*,

Hope for your applaud and a long loud acho.

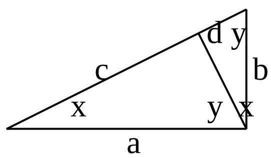
* Damshueiho means Damshuei river.

## 1.3. GEOMETRY VIA TRIGONOMETRY

In high school mathematics, Geometry and Trigonometry are taught separately in that order.

Here are some the other way around examples

### 1.3.1 Pythagorean Theorem




$(c + d) \cos x = a$ $\quad$ $(c + d) \cos^2 x = c$

$(c + d) \sin x = b$ $\quad$ $(c + d) \sin^2 x = d$

$\cos^2 x + \sin^2 x = 1$ $\quad$ $a^2 + b^2 = (c + d)^2$

Figure 15. Figure for P. T.

### 1.3.2 Tsao's Theorem I

Let ADEK be a rectangle. If OB = 3 OA and OC = 4 OA, then BK = HG.

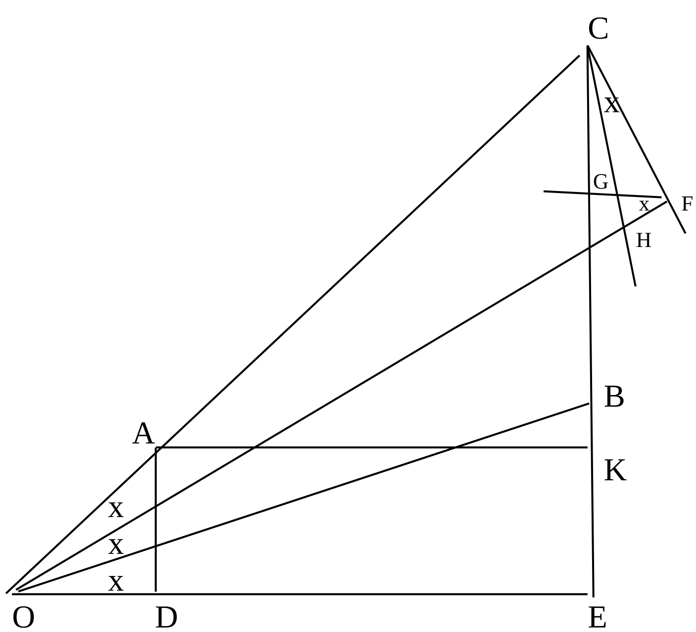



Figure 16. Figure for T. T. 1

Proof via the trigonometrical identity $\sin 3x = 3 \sin x - 4 \sin^3 x$

Let OA = 1. Then OB = 3 and OC = 4. It follows that

$$4 \sin x = CF,\ 4 \sin^2 x = FG,\ 4 \sin^3 x = HG,\ \sin 3x = AD \text{ and } 3 \sin x = BE.$$

Therefore,

$$HG = 4 \sin^3 x = 3 \sin x - \sin 3x = BE – AD = BK.$$

1.3.3 Tsao's Theorem II

If BF and OE are perpendicular, then BC = DF.

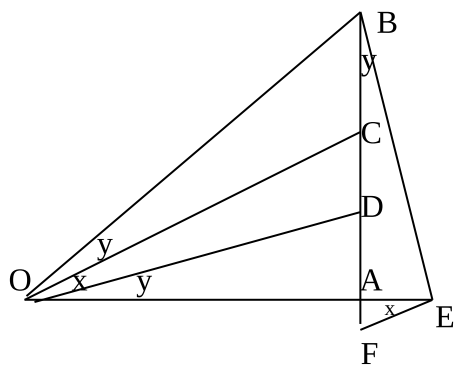



Figure 17. Figure for T. T. 2

Proof via the formula $\tan (x + y) = (\tan x + \tan y)/(1 – \tan x \tan y)$

Let OA = 1. Then $\tan (x + y) = BA$, $\tan x = CA$, $\tan y = DA$, $\tan (x + y) \tan y = EA$ and $\tan (x + y) \tan y \tan x = FA$. It follows that

$$BC = \tan (x + y) – \tan x = \tan y + \tan (x + y) \tan y \tan x = DA + FA = DF.$$

## 1.4. PROOFS WITHOUT WORDS: PYTHAGOREAN THEOREM

### 1.4.1 $a^2 + b^2 = c^2$

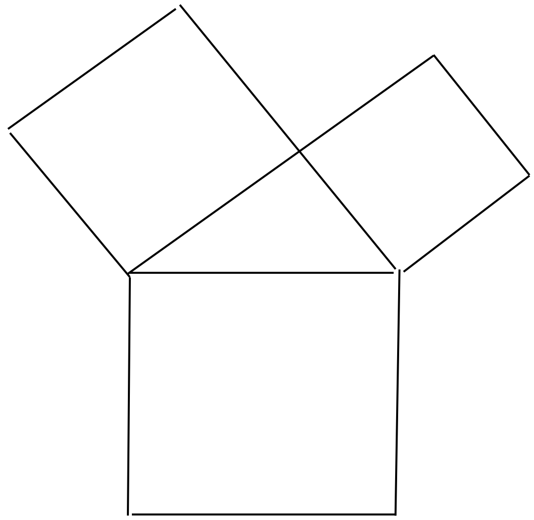


Figure 18.　　　Figure with three squares

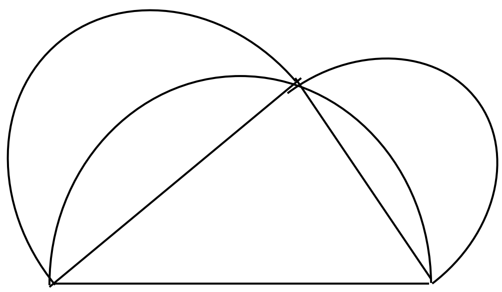


Figure 19.　　　Figure with three half circles

1.4.2　$a^2 + b^2 = 2ab + (b-a)^2$

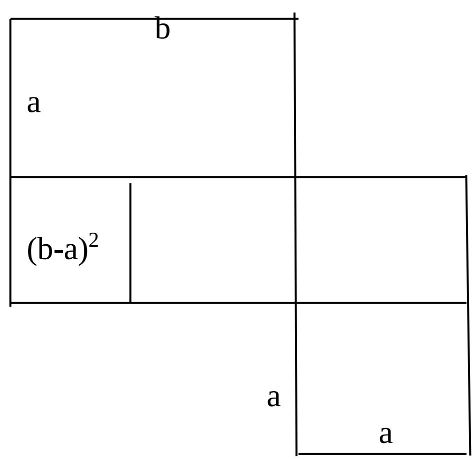



Figure 20.　　　Figure with $(b-a)^2$ off center

1.4.3 $2ab + (b-a)^2$

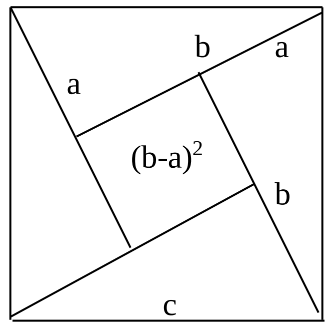



Figure 21. Figure with $(b-a)^2$ centered

2.4.4 $c^2 = (b - a \cos C)^2 + a^2 \sin^2 C = a^2 + b^2 - 2ab \cos C$

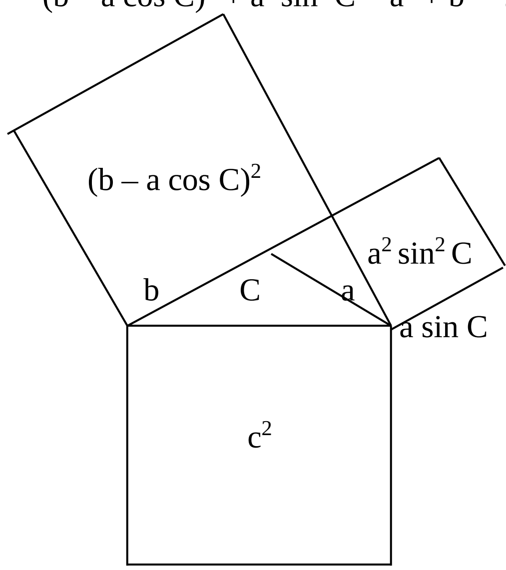



Figure 22. Figure for obtuse triangle

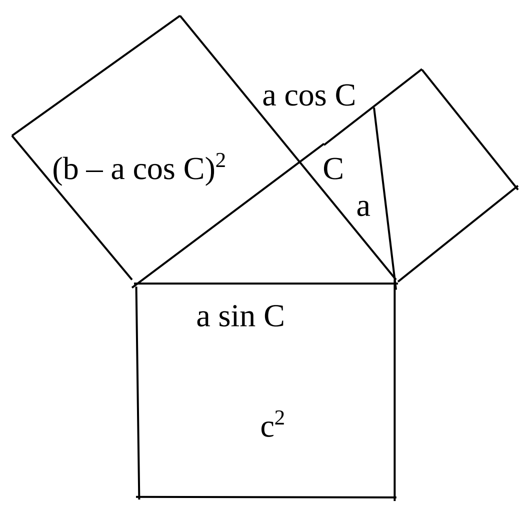



Figure 23. Figure for acute triangle

## 1.5. GEOMETRY PLUS TRIGONOMETRY

Let ABCD be a square such that

$$BE = CE$$

and

$$BF = 2\ AF.$$

If FG is perpendicular to DE and DH is perpendicular to EF, then

$$DG = FG$$

and

$$HB \mathbin{//} DE.$$

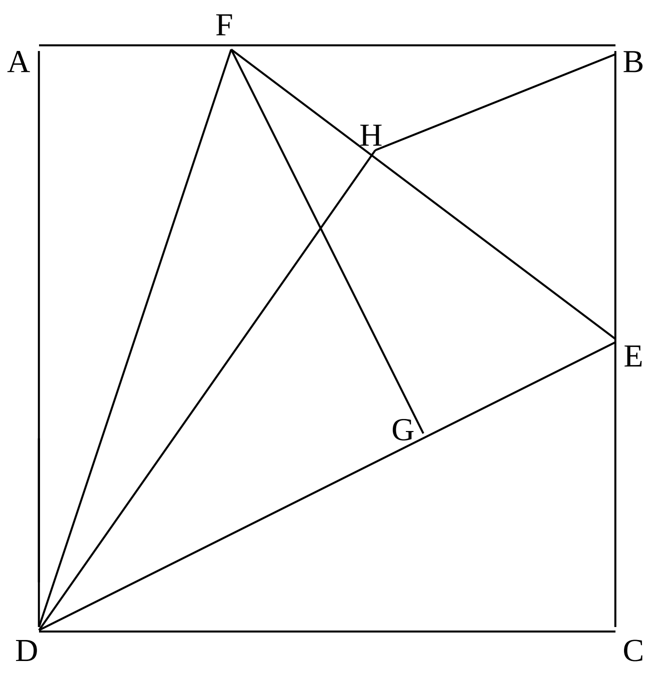



Figure 24. Figure for DG = FG and HB // DE

Let AB = 1. Then Pythagorean Theorem gives

$$DE = \sqrt{5}/2,$$

$$DF = \sqrt{10}/3,$$

$$EF = 5/6$$

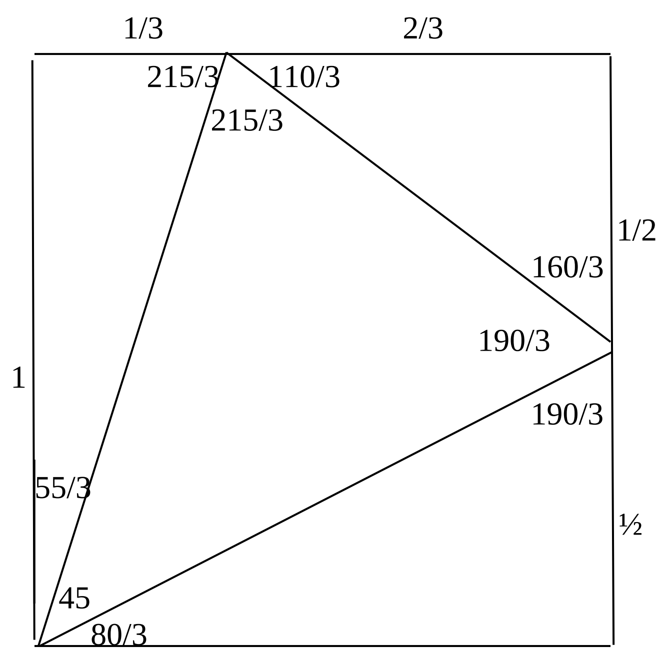
and Law of Cosines gives

$$\cos x = (DE^2 + DF^2 - EF^2)/2(DE)(DF) = \sqrt{2}/2,$$

where x = angle EDF.

It follows that $DG = \sqrt{5}/3 = FG$ and the rest is clear.

The following figure and table can further be obtained.




| x ( ◦ ) | tan x |
|---|---|
| 55/3 | 1/3 |
| 80/3 | 1/2 |
| 110/3 | 3/4 |
| 135/3 | 1 |
| 160/3 | 4/3 |
| 190/3 | 2 |
| 215/3 | 3 |

Figure 25. Figure supplemented by a convenient table

## 1.6. COUNTING TIDBITS

The author shall touch on miscellaneous combinatorial problems that can be pondered in a jail house.

He gave two talks of the same nature at Sonoma State University (1988) and National Taiwan Normal University (2017). Since the time span is nearly thirty years, the contents had been modified a great deal.

The following is a combined excerpt of both talks that could appeal to general audience with some math background.

### 1.6.1. Talk Topic: Jail House Mathematics

I was once caught speeding and put in jail overnight. At first, I was eager to get out of the jail-cell. I could not help but to stare at the grid gate.

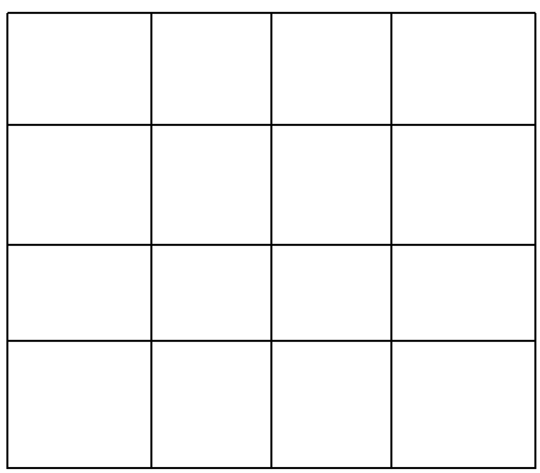


Figure 26. Figure for the grid gate

After some inner struggle, I calmed myself down and began to wonder how many squares were there in the grid.

The answer was $\sum_{r=1}^{n} r^2$.

I continued to indulge in my wonderland and became a combinatorialist. I'll tell you what happened inside and out. Unlike many long-term inmates (who later became philosophers, writers or politicians), I was fortunate just being in jail one night to become a mathematician.

First, let's get back to the grid. What came to my mind was, in fact, "how many different (shortest) paths connecting A and B?" (See Figure 27)

In jail, people usually use the brute force approach. There are 1 (via 5) plus 4 (via 4) plus 10(via 3) plus 20 (via 2) plus 35 (via 1),i.e. 70 paths.

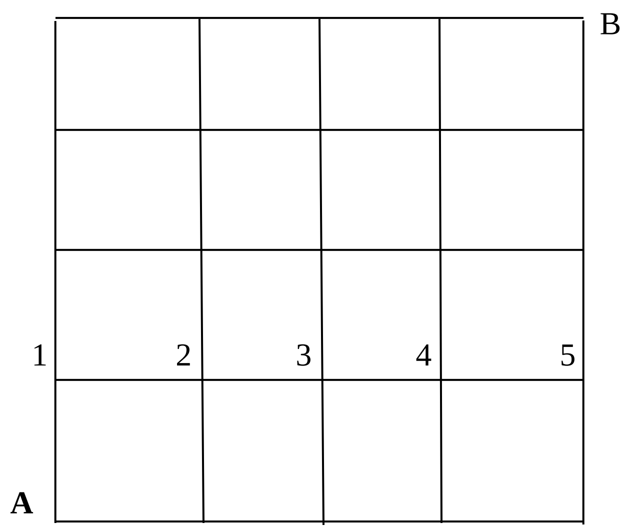



Figure 27. Figure for the grid gate with two corners marked A and B

A combinatorial approach is to pick all possible 4 horizontal (or vertical**)** moves out of 8 moves needed to go from A to B. There are, therefore, $\binom{8}{4} = 70$ ways.

After such mind boggling, I rested myself in the restroom. Then I saw the graded window.

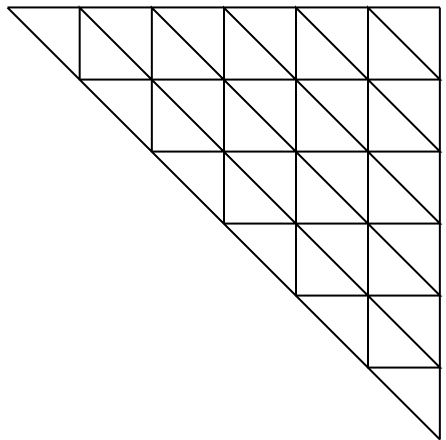


Figure 28. Figure for the graded window

I wondered "how many of triangles of all sizes in a subdivided triangle of n layers?" To come up with the answer in a rather academic way.

Theorem 1

Let T(n) be the number in question and S(n) = $\sum_{r=1}^{n} r$. Then

$$T(n-1) + T(n) = 2\sum_{r=1}^{n-1} S(r) + \sum_{r=1}^{n} S(r).$$

Proof

As can be seen from the figure, when extending from n-1 layers to n+1 layers S(n) + S(n+1) new forward triangles and S(n) new backward triangles are added. Hence

$$T(n+1) = T(n-1) + 2S(n) + S(n+1).$$

We shall only show the inductive step of the mathematical induction:

$$T(n) + T(n+1) = T(n) + T(n-1) + 2S(n) + S(n+1)$$

$$= 2\sum_{r=1}^{n-1} S(r) + \sum_{r=1}^{n} S(r) + 2S(n) + S(n+1)$$

$$= 2\sum_{r=1}^{n} S(r) + \sum_{r=1}^{n+1} S(r).$$

<u>Theorem 2</u>

$$S^{(2)}(n) = \sum_{r=1}^{n} r^2 = C(n+1, 3) + C(n+2, 3).$$

<u>Proof</u>

Since $S(r-1) + S(r) = r^2$ and since $\sum_{r=2}^{n} C(r,2) = C(n+1, 3)$, we have

$$S^{(2)}(n) = \sum_{r=1}^{n-1} S(r) + \sum_{r=1}^{n} S(r) = \sum_{r=1}^{n-1} C(r+1,2) + \sum_{r=1}^{n} C(r+1,2) = C(n+1, 3) + C(n+2, 3).$$

<u>Corollary</u>

$$T(n) = C(n+1, 3) + S^{(2)}(n) - T(n-1).$$

We can use Theorem 2 and Corollary to come up with $S^{(2)}(n)$ and $T(n)$ recursively.

| n | C(n+1, 3) | C(n+2, 3) | $S^{(2)}(n)$ | T(n-1) | T(n) |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 1 |
| 2 | 1 | 4 | 5 | 1 | 5 |
| 3 | 4 | 10 | 14 | 5 | 13 |
| 4 | 10 | 20 | 30 | 13 | 27 |
| 5 | 20 | 35 | 55 | 27 | 48 |
| 6 | 35 | 56 | 91 | 48 | 78 |
| 7 | 56 | 84 | 140 | 78 | 118 |
| 8 | 84 | 120 | 204 | 118 | 170 |
| 9 | 120 | 165 | 285 | 170 | 235 |
| 10 | 165 | 220 | 385 | 235 | 315 |

Table 7. Table for recursive calculations of $S^{(2)}(n)$ and T(n)

Theorem 3

$$T(n) = [\ n(n+2)(2n+1)/8\ ],$$

where [ x ] is the integral part of x.

Proof

Due to Theorem 1, we can write

$$T(n) + T(n-1) = 2C(n+1, 3) + C(n+2, 3) = (n+1)n(n-1)/3 + (n+2)(n+1)n/6 = n^3/2 + n^2/2$$

$$= (2n^3 + 5n^2 + 2n)/8 + [(2(n-1)^3 + 5(n-1)^2 + 2(n-1)]/8 - 1/8,$$

from which we see that

$$T(n) = (2n^3 + 5n^2 + 2n)/8 + [-1 + (-1)^n]/16 = [\ n(n+2)(2n+1)/8\ ].$$

When leaving the jail, I left behind the following graffiti on the wall.

1) $\sum_{r=1}^{n} r = C(n+1,2)$

| | | | | | |
|---|---|---|---|---|---|
| O | O | O | O | O | O |
| X | O | O | O | O | O |
| X | X | O | O | O | O |
| X | X | X | O | O | O |
| X | X | X | X | O | O |
| X | X | X | X | X | O |
| X | X | X | X | X | X |

Figure 29. Figure for graffiti picture 1

2) $\sum_{r=1}^{n} r^2 = C(n+2,3) + C(n+1,3)$

1 + (1+2) + (1+2+3) + . . . + (1+2+3+ … +n) $= C(n+2,3)$

O   O O   O O O   O O O O   O O O O O<br>
  X O   X O O   X O O O   X O O O O<br>
  X X O   X X O O   X X O O O<br>
  X X X O   X X X O O<br>
  X X X X O

1 + (1+2) + . . . + [1+2+. . .+(n-1)] $= C(n+1,3)$

Figure 30. Figure for graffiti picture 2

3) $\sum_{r=1}^{n} r^3 = [C(n+1,2)]^2$

$1^3 + 2^3 + 3^3 + \; . \; . \; . + n^3$

```
X O O X X X O O O O X X X X X
O O O X X X O O O O X X X X X
O O O X X X O O O O X X X X X
X X X X X X O O O O X X X X X
X X X X X X O O O O X X X X X
X X X X X X O O O O X X X X X
O O O O O O O O O O X X X X X
O O O O O O O O O O X X X X X
O O O O O O O O O O X X X X X
O O O O O O O O O O X X X X X
X X X X X X X X X X X X X X X
X X X X X X X X X X X X X X X
X X X X X X X X X X X X X X X
X X X X X X X X X X X X X X X
X X X X X X X X X X X X X X X
```

$(1 + 2 + 3 + \; . . . \; + n)^2$

Figure 31. Figure for graffiti picture 3

The moment that I got out, I was able to see a better picture below and wrote a couple of articles (19) and (18) about $\sum_{r=1}^{n} r^k$.

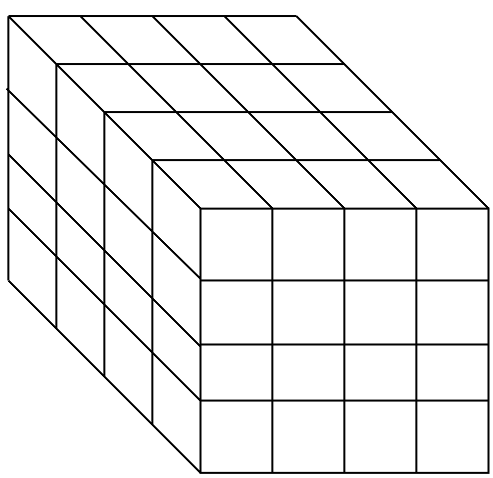


Figure 32. Figure for the jail house

## 1.7. GLOBAL APPROACH TO PROBABILITY PROBLEMS

For the instructional purpose, both global and local approaches are equally important to solve probability problems. There are excellent demonstrations of the latter in (13), from which the author shall select four problems, namely twin knights, the ballot box, ties in matching pennies and the theater row for the global approach. The author shall also provide the combinatorial realization of the probability for "Squares among rectangles" and the graphical visualization of the odds for "Same birthday among classmates".

### 1.7.1. Talk Topic: Probability Problems and Concepts Made Simple

Let us start with the following four problems selected from (13).

1) Twin knight

Suppose King Arthur holds a jousting tournament where the jousts are in pairs as in a tennis tournament. The 8 knights in the tournament are evenly matched, and they include the twin knights Balin and Balan. What is the chance that the twins meet in a match during the tournament?

If the knights were not evenly matched, the calculations of each probability for all possible locations that the twins meet as in the book would have been necessary. In our case, all we need is to divide the total number of matches by the total number of pairs. Therefore, in the case of $2^n$ knights, the answer is

$$\frac{2^n - 1}{\binom{2^n}{2}} = \frac{1}{2^{n-1}},$$

which was proved by induction in (13).

2) The ballot box

In an election, two candidates, Albert and Benjamin, have in a ballot box $a$ and $b$ votes respectively, $a > b$. If ballots are randomly drawn and tallied, what is the chance that at least once after the first tally the candidates have the same number of tallies?

If we approach this problem by considering the last tie in the tallying, then the detail discussions of the first tie as in (13) can be avoided. Out of $a+b$ positions in a tallying sequence, the last tie can occur with A or B being tallied at each of the $2b$ even positions so that the answer is $\dfrac{2b}{a+b}$.

3) Ties in matching pennies

Players A and B match pennies N times. They keep a tally of their gains and losses. After the first toss, what is the chance that at no time during the game will they be even?

The solution in (13) made use of its previous problem and $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$.

In fact, when $N = 2n$ or $N = 2n+1$, the answer is $\dfrac{\binom{2n}{n}}{2^{2n}}$.

(To achieve no tie, the first $n$ tallies out of the first $2n$ need to be of the same kind.)

<u>Combinatorial solution</u>

To achieve no tie, the first n tallies but not the second n tallies out of the first 2n tallies need to be of the same kind. For the case of n=2, no tie tallies:

N=4:

AAAA, AAAB, AABB, BBAB, BBBA, BBBB.

N=5:

AAAAA, AAAAB, AAABA, AABBA, AAABB, AABAB,

BBABA, BBBAA, BBABB, BBBAB, BBBBA, BBBBB

4) The theater row

With $b$ elements of one kind and $m$ of another, randomly arranged in a line, what is the expected number of unlike adjacent elements?

Instead of being caught up with "unlike adjacent elements", we shall consider the matching of unlike pair. A match will produce two adjacent cases. Since each of the first kind has the chance of $\frac{1}{b}$ to match with the second kind and each of the second kind has the chance of $\frac{1}{m}$ to match with the first kind, the answer is $\frac{2}{\frac{1}{b}+\frac{1}{m}}=\frac{2mb}{m+b}$.

5) Squares among rectangles

What is the chance $P(n)$ that a randomly selected rectangle from a gridded square of size $n$ is a square? (See Figure 33 for $n = 4$)

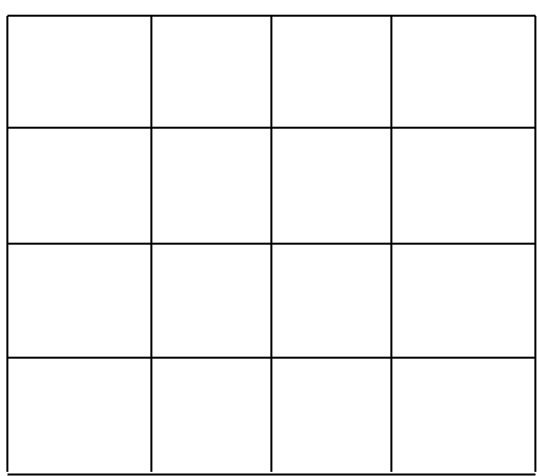


Figure 33. Figure for the gridded square

We can certainly use the local approach to find that

$$P(1) = 1,$$

$$P(2) = \frac{5}{9},$$

$$P(3) = \frac{7}{18},$$

$$P(4) = \frac{3}{10}$$

and

$$P(5) = \frac{11}{45}.$$

Globally, we need to realize that a pair of identical lengths and a pair of identical widths determine a unique rectangle so that the total number of rectangles is

$$\binom{n+1}{2}^2.$$

On the other hand, the total number of squares is $\sum_{i=1}^{n} i^2$. Therefore, we have

$$P(n) = \frac{\frac{n^2(n+1)^2}{4}}{\frac{n(n+1)(2n+1)}{6}} = \frac{1}{n} + \frac{n-1}{3n(n+1)} \quad .$$

Two observations are in order.

First, $P(n) \approx \frac{1}{n}$. Second, $P(n) = \frac{\sum_{i=1}^{n} i^2}{\sum_{i=1}^{n} i^3}$.

6) Same birthday among classmates

What is the probability that two students in a classroom have the same birthday?

During my teaching at San Francisco State University, I did the experiment for each of my classes. This is how the experiment went. Each student was asked to submit his/her birthday written in a piece of paper. Then I collected them according to the birth month, from January to December. I still remember vividly the very experiment the match of birthdays late until December papers were collected. Then came a loud laughter, when two Korean twin students walked all the way from the last row to submit their papers.

Let us first consider the following figure, in which 25 points are uniformly spread over the uniform sample space of 441 points as in Figure 34.

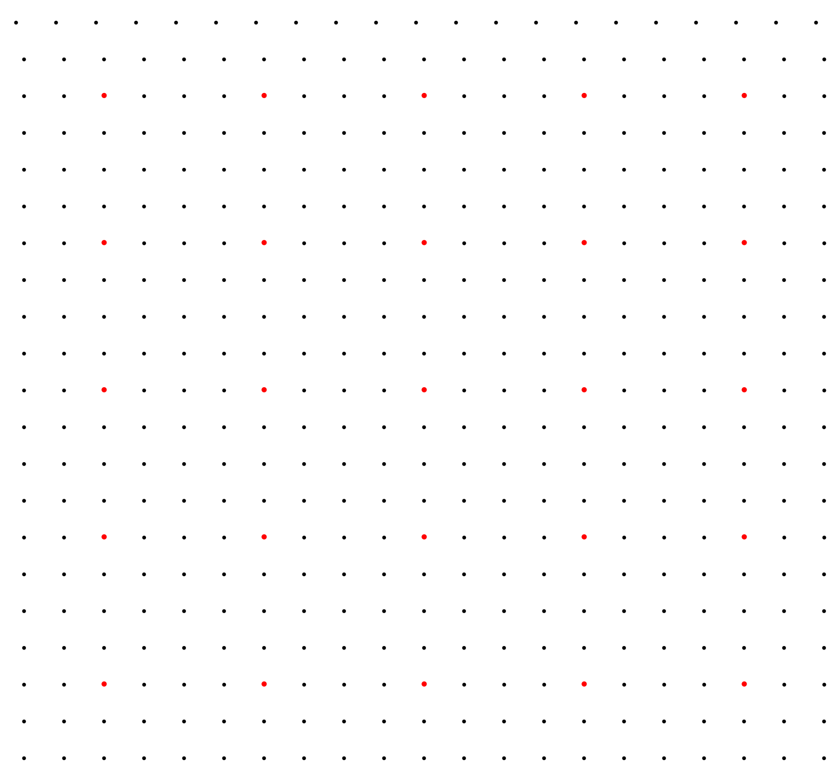


Figure 34. Figure for the uniform sample space of 441 points

This is equivalent of saying that if there were 441 days in a year, then the maximum number of people to have different 'birthdays' spread out uniformly would be 25 so that among 25 people the probability of at least two having the same 'birthday' would be

$$1 - {}_{441}P_{25} \,/\, 441^{25} = 0.5.$$

Similarly, Figure 35 shows that if there were 368 days in a year, then among 23 people the probably of at least two having the same 'birthday' would be

$$1 - {}_{368}P_{23} \,/\, 368^{23} = 0.5.$$

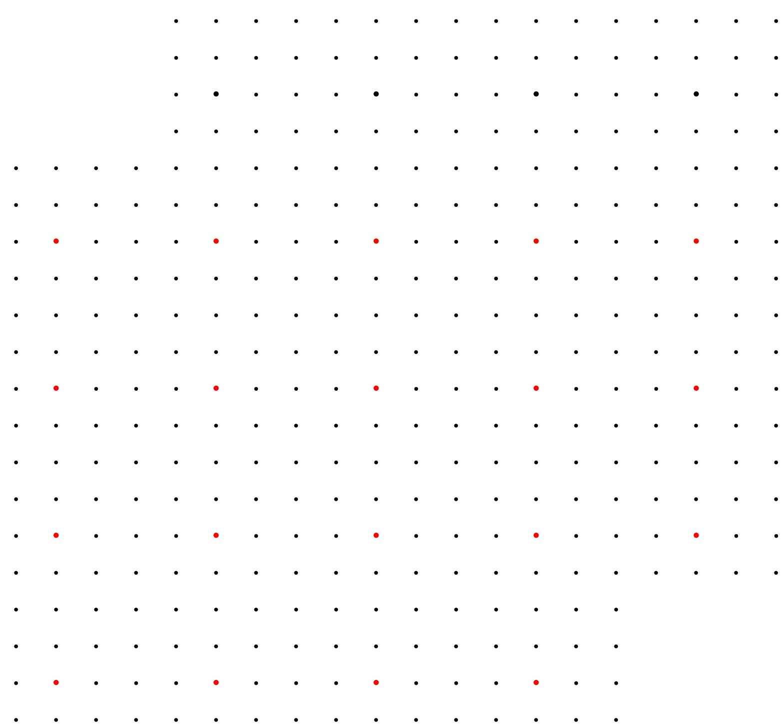


Figure 35. Figure for the uniform sample space of 368 points

Since there are actually 365 days in a year, it follows that among 23 people the probability of at least two having the same birthday is slightly exceeding 50%.

## 1.8. MUTUAL INDEPENDANCY VERSUS MUTAL EXCLUSIVENESS

It was pointed out in (12) that the concepts of mutual exclusivity and probabilistic independency are difficult for students to grasp. However, the concept of the former is self-explanatory. Two events that are likely to occur are said to be mutually exclusive if the occurrence of one prevents the other from occurring. This concept does not involve the probability.

So the problem comes from the definition of the latter: two events are said to be independent if the occurrence of one does not affect the probability of the other to occur. Since this concept involves the probability, there shouldn't be any confusion with the previous concept. Rather, the difficulty lies on the judgment of the "affection".

One way of solving this problem is to introduce a measure of evaluating the degree of dependency.

Let $A$ and $B$ be events. The conditional probability $P(B \mid A)$ of $B$ given $A$ is the probability of $B$ given that $A$ has already occurred.

Thus two events $A$ and $B$ are independent if and only if $P(B \mid A) = P(B)$ and/or $P(A \mid B) = P(A)$.

Since $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)}$, we can also say that $A$ and $B$ are independent if and only if

$$P(B \cap A) = P(B)P(A). \qquad \text{Eq. 1}$$

To help judging of the "affection", we define the discrepancy of independency $DI(B \mid A)$ in the probability of $B$ given $A$ to be the percentage change in the probability of $B$ affected by the occurrence of $A$, namely

$$DI(B \mid A) = \frac{P(B \mid A) - P(B)}{P(B)} = \frac{P(B \cap A)}{P(B)P(A)} - 1 \qquad \text{Eq. 2}$$

which is 0 if $A$ and $B$ are independent due to Eq. 1.

In an experiment of picking 6 distinct months randomly from the calendar year, construct 2 events that are mutually exclusive.

Apparently, the event $A$ of picking the odd months and the event $B$ of picking the even months are mutually exclusive. Are they independent? Certainly not, since the occurrence of $A$ does affect the probability of $B$ to occur.

To be more specific,

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)} = 0,$$

$$P(B) = \frac{1}{2},$$

$$DI(B \mid A) = \frac{P(B \mid A) - P(B)}{P(B)} = \frac{0 - \frac{1}{2}}{\frac{1}{2}} = -1.$$

For convenience, let the sample space be $S = \{1,2,3,4,5,6,7,8,9,10,11,12\}$.

By setting $E_i = \{i, i+1, i+2, i+3, i+4, i+5\}$, we can use Eq. 74 to find

$$DI(E_1 \mid E_1) = 1,$$

$$DI(E_1 \mid E_2) = \frac{2}{3},$$

$$DI(E_1 \mid E_3) = \frac{1}{3},$$

$$DI(E_1 \mid E_4) = 0,$$

$$DI(E_1 \mid E_5) = -\frac{1}{3},$$

$$DI(E_1 \mid E_6) = -\frac{2}{3},$$

$$DI(E_1 \mid E_7) = -1.$$

We see from the above $E_1$ is 100% dependent to itself, $E_1$ and $E_7$ are mutually exclusive (-100% dependent to each other), whereas $E_1$ and $E_4$ are independent (0% ).

The following graphical views might further help readers to envision the matter.

Case 1. A and B are mutually exclusive.

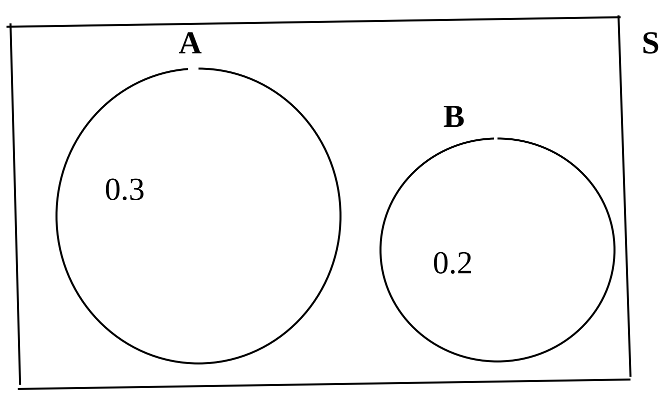



Figure 36. Figure for the mutually exclusive case

In this case, $DI(B \mid A) = \dfrac{P(B \cap A)}{P(B)P(A)} - 1 = \dfrac{0}{0.2 X 0.3} - 1 = -1$

<u>Case 2</u>. A and B are independent.

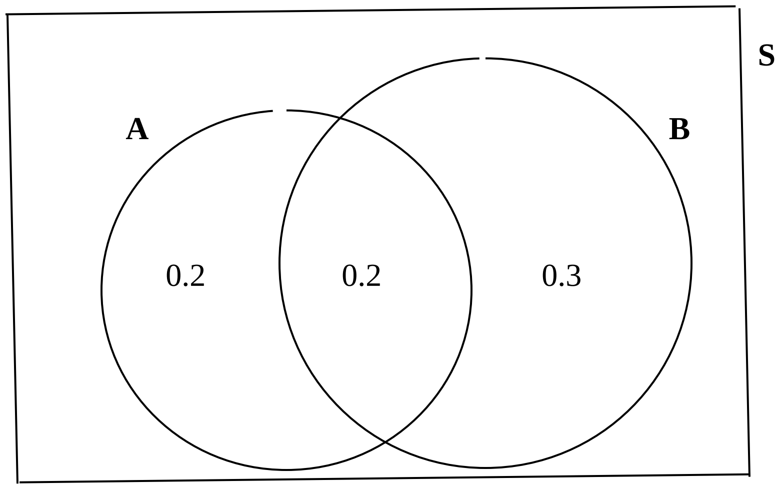



Figure 37. Figure for the independent case

In this case, $DI(B|A) = \dfrac{P(B \cap A)}{P(B)P(A)} - 1 = \dfrac{0.2}{0.5 X 0.4} - 1 = 0$

<u>Case 3</u>. A and B are nearly independent.

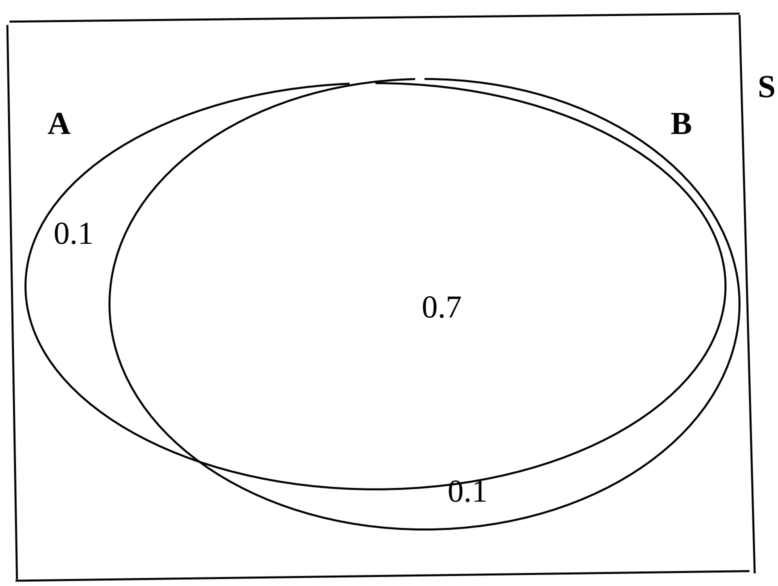



Figure 38. Figure for the nearly independent case

In this case, $DI(B|A) = \dfrac{P(B \cap A)}{P(B)P(A)} - 1 = \dfrac{0.7}{0.8 X 0.8} - 1 = 0.09375$

## 1.9. THE SOLID GROUND IN A BIG PICTURE OF LIFE INSURANCE

The main theme of this speech is to demonstrate how not to be lost in a big picture by way of diligently laying down a solid foundation, especially for actuaries. I am very pleased to have this opportunity to inform you about two explosive well-kept secrets.

The life actuarial theory had been well developed to a near perfection throughout the twentieth century. The whole hundred years of development is like the entire life span of an ideal individual in the insurance industry. For a human life, the mid-age is the juncture of two distinguishing stages: growing and maturing. Therefore, it is not coincidental that the formation of the SOA organization in U. S. A. and the construction of the CSO life table in 1958 came about all in the mid-century. By constructing two cubic models for 1958 CSO male life table, I discovered the first secret: the live curve is symmetrical with respect to the mid-age!

Analogous to the apparent deterioration at the very end of a human life, near the turning point of the last century the life contingency theory suffered a severe setback and in despair bizarrely resorted to some fuzzy model for savage!

Through the tireless effort of unifying the insurance and annuity functions from both deterministic and stochastic points of view, I discovered the second secret: the dynamic model could very well be an important tool to cope with the drastic change of the financial environment in this new century!

You might have been confused by these two secrets, especially the first one. What is exactly the mid-age? If it means 50, then what I just told you shouldn't be true at all.

In my studies, the mid-age is actually 65, the age of retirement!

As I have shown in (20), my models fit well with 1958 CSO male life table up to age 75. This is good enough for practical use, isn't it?

My first secret would have been true, if the terminal age of a human life were 130. Who knows? Some day we might reach that goal. Although actuaries should not develop theories without looking at the reality, they won't prosper without relying on theories either. Most theories are based on two important factors: mortality rate and interest rate. Life actuaries in the last century collectively built up a gigantic mansion by laying down a solid foundation.

After giving you my perspectives ranging from mortality models construction to unification of life contingencies in an orderly manner, I'll then elaborate on my second secret. In there, I'll point out that the mansion we have built is now precarious, not because of the mortality pillar, rather of the interest pillar. In the big picture, there is an urgent need for a revolutionary change in the concept of interest rate. Otherwise, we would run into the same unrealistic dilemma as I mentioned in the first secret.

The key word in the actuarial profession is fairness. In the past experience, actuaries have not been able to predict correctly about the interest rate. For that matter, nobody could have.

Therefore, the dynamic approach using discounting functions of interest and mortality retrospectively should be the way to go in this new century.

## 1.10. SUDOKU PREVIEW

The following is an excerpt of the introduction in (17).

> Maze with clues has been built in every foreseeable place,
>
> All barriers could be removed without any frustrating face
>
> ;
>
> In idle time please come to visit the three treasures palace,
>
> Relax your mood and nerves and indulge in Sudoku space.

The inventor of Sudoku games was Tetsuya Nishio, who first came across a game named Number Place in Dell Magazine in early 1980's while visiting U.S. and then developed it into a more complicated puzzle to be played in Japan. Its name was immediately changed to Sudoku by Nikoli Magazine in Japan and prevailed there for a while. Now, people all over the world are indulging in this game thanks to Wayne Gould, a retired Hong Kong judge from New Zealand. Not until 1997 while touring Tokyo, he encountered this gadget. After six years of study, he came up with the computer software named Pappocom which enabled him to massively produce fiendish Sudoku puzzles. In 2004, this wonderful workmanship game frantically hit the entire England and subsequently the whole Europe. Soon after that, it returned to U.S. and Japan, further extended to Taiwan in 2005. Surging from the outset of this century, "Sudoku" is indeed self-entertaining, time-killing, loneliness-removing, solitude-exempting and senile-preventing.

The purpose of the Sudoku game is using logical inference, starting from the puzzle form of Figure 39, to uncover those un-starred numbers in Figure 40 step by step according to the order of subscripts. The rule of Sudoku game is to require each row, each column and each box to have each of all numbers ranging from 1 through 9.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 5* | 7* | 1* | | 9* | 4* | 3* | 2* |
| 9* | 1* | 3* | 4* | 5* | 2* | 7* | 6* | 8* |
| 4* | | 2* | 3* | | 7* | 9* | 5* | 1* |
| | | 9* | 2* | | 4* | 5* | | 7* |
| | | 1* | | | 5* | 6* | | 4* |
| 5* | | 4* | 6* | | 8* | 3* | 1* | 9* |
| 2* | 4* | 6* | 5* | | 1* | 8* | | 3* |
| 1* | 3* | 8* | | 4* | 6* | 2* | | 5* |
| 7* | 9* | 5* | 8* | 2* | 3* | 1* | 4* | 6* |

Figure 39. The first figure of Sudoku preview

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $6_{14}$ | 5* | 7* | 1* | $8_{18}$ | 9* | 4* | 3* | 2* |
| 9* | 1* | 3* | 4* | 5* | 2* | 7* | 6* | 8* |
| 4* | $8_{15}$ | 2* | 3* | $6_{19}$ | 7* | 9* | 5* | 1* |
| $3_{12}$ | $6_{13}$ | 9* | 2* | $1_{1}$ | 4* | 5* | $8_{3}$ | 7* |
| $8_{16}$ | $7_{17}$ | 1* | $9_{7}$ | $3_{6}$ | 5* | 6* | $2_{2}$ | 4* |
| 5* | $2_{4}$ | 4* | 6* | $7_{5}$ | 8* | 3* | 1* | 9* |
| 2* | 4* | 6* | 5* | $9_{9}$ | 1* | 8* | $7_{10}$ | 3* |
| 1* | 3* | 8* | $7_{8}$ | 4* | 6* | 2* | $9_{11}$ | 5* |
| 7* | 9* | 5* | 8* | 2* | 3* | 1* | 4* | 6* |

Figure 40. The second figure of Sudoku preview

Freshly retired from the teaching post of San Francisco State Business School, I started to play this game sporadically. No sooner than 2005, the returning year of my son Michael from medical training, I began to indulge myself in this fascinating game, thanks to his thoughtful choices of all sorts of challenging Sudoku books as birthday, father day and Chistmas presents for the subsequent three years.

Those books include 1001 SUDOKU (Thunder’s Mouth Press, copy right to Nicoli) and SUDOKU GENIUS (Tom Scheldon, 144 of the Most Friendish Puzzles Ever Devised) of 2005; Su Doku (Wayne Gould, Challenging Sudoku 4), HIGHER SUDOKU (Tetsuya Nishio, New Variations from Japan) and Sudoku Puzzles (Aline Ribeiro de Almeida,TOP 100 HARDEST) of 2006; Extreme Sudoku (Dell, Sudoku puzzles with an X factor!) of 2007.

Therefore, I literally ate and drank Sudoku during the entire period of those three years. However, unlike most speed-oriented players, I took my time to enjoy the logical reasoning provided by each puzzle and kept the detailed record of the whole solving process. The joy of life is to share. With this belief, I had prepared a draft of my book “Completely Cracking Sudoku“ way back in 2007 blending the most inspiring ideas of puzzle structures enlightened by the afore-mentioned books in order to introduce the unique step by step method. The key is to take and record each step in accordance with a logical reasoning instead of hasty trials and errors, so that everyone can enjoy and refresh one’s memorable moments.

That draft was then sent to my youngest brother Yung-Shyeng who never played a single game of Sudoku. He made lots of valuable suggestions from a beginner’s point of view. He also added a finishing touch, liking of the secrete codes in kung-fu practice, on this originally scrupulous and methodical manuscript of knowhow. This has revived the spirit of my book as if bringing the painted dragon to life by putting in the pupils of its eyes. Soon after that, I was sidetracked by my breakthrough in the classic number theory.

Coincidentally, the afore-mentioned Euler was a famous classic number theorist, who along with Gauss, Bernoulli and Stirling had almost simultaneously discovered various formulas for expressing the sum of powers of the natural sequence. Imaging that, had he had spare time to spend on Latin squares, Sudoku games could have come about some three hundred years ago! As to my breakthrough, I generalized most of those formulas from the natural sequence to arithmetically progressive sequences and obtained their polynomial expressions.

Just around the conclusion of my breakthrough, I was informed by Mr. Ray Leo in early July of 2012 that the hardest Sudoku was newly posted online. After being able to crack down this hardest Sudoku in a couple of days using my Sudoku-solving techniques, I have revived the desire of publishing my book. During this five years of “idling period“, I have actually perfected the method of explaining how puzzles can be solved step by step using various techniques with the aid of shorthand annotations to be introduced in my book. In fact, most of so called challenging puzzles turned out to be so so under the scrutiny of my examination. Nevertheless, they more or less reflected those authors’ special view points and therefore should not be categorically denied.

Interestingly, in 2008 I picked up and studied “Cracking Sudoku“ (in Chinese, by Wang Tung Chiao) while strolling the “Bookstore Street“ in Taipei. The following year, I have pointed out an erroneous puzzle of (16) and received three of his new books in return. So it is fair to say that I have not given up on Sudoku completely. Thus in the final section of this article, we shall let readers take part in solving the hardest Sudoku to manifest what they are about to learn is by no means a “flowery boxing“.

Furthermore, we might as well let veterans peek at a few puzzles from the above two books now so that they can foresee what would unfold in the later sections, for fear that they might give up on this article due to the unchallenging nature of the first few sections.

Although most puzzles we shall encounter were labeled as rank 5, they could be solved rather easily with patience and perseverance; even the beginners could follow the step by step guidance and enjoy the wonderful feeling.

Otherwise, they can skip this foreplay and come back to visit these puzzles after learning the basic skills. To begin with, let us try the most challenging puzzle claimed by Wang Tung Chiao in Cracking Sudoku.

First star all given numbers in Figure 41 and then start with the smallest number ready to be filled, according to the prescribed order of up-down and left-right.

After failing with 1, 2 and 3 for all boxes, you could try 4 in box 1. The junction of row 1 & column 2,Grid (12), is the only place for 4, abbreviated as 4(12). So the first step is $4_1$(12).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | | | | 9* | | | | 3* |
| | 7* | | | 5* | | | 6* | |
| | | 2* | 8* | | 1* | 4* | | |
| 4* | | | | | | | | 5* |
| | | 6* | | | | 7* | | |
| 9* | | | | | | | | 8* |
| | | 4* | 5* | | 9* | 2* | | |
| | 3* | | | 6* | | | 5* | |
| 2* | | | | 4* | | | | 6* |

Figure 41. The third figure of Sudoku preview

The second step is to enter 4 into the grid of row 8 and column 9 in box 9, abbreviated as $4_2$(89) and the third step is to enter 5 into the grid of row 1 and column 7 in box 7, abbreviated as $5_3$(17).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | $4_1$ | $8_4$ | | 9* | | $5_3$ | $2_7$ | 3* |
| | 7* | $9_9$ | | 5* | | $8_5$ | 6* | $1_6$ |
| | | 2* | 8* | | 1* | 4* | $7_{14}$ | $9_{13}$ |
| 4* | | | | | | | | 5* |
| | | 6* | | | | 7* | | $2_8$ |
| 9* | | | | | | | | 8* |
| | | 4* | 5* | | 9* | 2* | | $7_{15}$ |
| | 3* | | | 6* | | $9_{12}$ | 5* | $4_2$ |
| 2* | $9_{10}$ | $5_{11}$ | | 4* | | | | 6* |

Figure 42. The fourth figure of Sudoku preview

Now the first obstacle is encountered. With patience and perseverance, readers might find the grid in row 1 and column 3, but what number to fill in? Please scan in Figure 42 from left to right, row 1 has 1, 4, 9, 5, 3 and column 3 has 2, 6, 4, hence only 7 and 8 are left to be filled. But, wait! 7 can not be filled here either, due to the fact that box 1 where the grid in question is situated has 7. Hence for the fourth step, we can take $8_4$(13) as shown in Figure 42. This is called a grid move (g), abbreviated as $8_4$(13)g, since this move is determined by the surroundings (row, column & box) intersecting with this grid. After $8_5$(27) and $1_6$(29), you can look at box 7. The 2 can only be entered into (18), abbreviated as $2_7$(18)b7. This is called a box move (b), since this move is determined by the surroundings (all rows & columns) intersecting with this box. After $2_8$(59), you can look at row 2. The 9 can only be entered into (23), abbreviated as $9_9$(23)r2. This is called a row move (r), since this move is determined by the surroundings (all columns & boxes) intersecting with this row.

After $9_{10}$(92), $5_{11}$(93) and $9_{12}$(87), you can look at column 9. The 9 can only be entered into (39), abbreviated as $9_{13}$(39)c9. This is called a column move (c), since this move is determined by the surroundings (all rows & boxes) intersecting with this column. After $7_{14}$(38) and $7_{15}$(79), once again a stalemate, is encountered. By scanning three unfilled grids in box 1, readers can easily know to fill 3 into (21), abbreviated as $3_{16}$(21)g as shown in figure 43.

Readers can then move rather smoothly by taking $3_{17}$(35), $3_{18}$(78)r7, $8_{19}$(98) , $1_{20}$(97) and $7_{21}$(81)c1 as shown. The rest is easy with the following annotations.

$1_{22}$(83)g

$8_{24}$(55)g

$2_{32}$(62)c2

$2_{34}$(45)c5

$6_{38}$(67)g

$6_{40}$(46)g

$9_{46}$(58)r5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | $4_{1}$ | $8_{4}$ | $6_{41}$ | 9* | $7_{42}$ | $5_{3}$ | $2_{7}$ | 3* |
| $3_{16}$ | 7* | $9_{9}$ | $4_{48}$ | 5* | $2_{49}$ | $8_{5}$ | 6* | $1_{6}$ |
| $6_{28}$ | $5_{29}$ | 2* | 8* | $3_{17}$ | 1* | 4* | $7_{14}$ | $9_{13}$ |
| 4* | $8_{25}$ | $7_{36}$ | $9_{54}$ | $2_{34}$ | $6_{40}$ | $3_{39}$ | $1_{55}$ | 5* |
| $5_{30}$ | $1_{33}$ | 6* | $3_{45}$ | $8_{24}$ | $4_{47}$ | 7* | $9_{46}$ | $2_{8}$ |
| 9* | $2_{32}$ | $3_{37}$ | $1_{53}$ | $7_{35}$ | $5_{31}$ | $6_{38}$ | $4_{52}$ | 8* |
| $8_{26}$ | $6_{27}$ | 4* | 5* | $1_{23}$ | 9* | 2* | $3_{18}$ | $7_{15}$ |
| $7_{21}$ | 3* | $1_{22}$ | $2_{51}$ | 6* | $8_{50}$ | $9_{12}$ | 5* | $4_{2}$ |
| 2* | $9_{10}$ | $5_{11}$ | $7_{43}$ | 4* | $3_{44}$ | $1_{20}$ | $8_{19}$ | 6* |

Figure 43. The fifth figure of Sudoku preview

## 1.11. TEACHING EFFICIENCY

In Chapter 5, among other things, we shall use the idea of the boundary being the marginal change of a well-rounded region (a region possessing an inscribed circle) with respect to the inradius (the radius of the inscribed circle) to solve optimization problems more efficiently and categorically.

# 2. NUMBERS INTRICACY

## 2.1. INTRODUCTION

I have a unique experience of linking the following famous mathematicians together.

Pascal-Bernoulli-Stirling-Euler-Bell-Gauss

Frankly speaking, I was not familiar with their works when I first started the process of transforming product-sums to power-sums! Prior to all this, I have submitted an article to Mathematical Gazette using a simpler approach which will be presented in the end. All these endeavors had been undertaken two years after I retired from teaching at College of Business, San Francisco State University in 2002.

We first define the linear factorization of the “polynomial” in $\Theta = S$ or $O$:

$$P(\Theta) = b_k \Theta^{(k)} + b_{k-1}\Theta^{(k-1)} + b_{k-2}\Theta^{(k-2)} + ... + b_0 \Theta^{(0)},$$

by way of factorization of ordinary polynomials.

Let P(n, k) be the permutation of n elements taken k at a time. It is well-known that

P(n, k) = n(n-1)(n-2)...(n-k+1).

We shall use $P(S+2,2)$, which is $(S+2)(S+1)$, to denote

$$S^{(2)} + 3S^{(1)} + 2S^{(0)}$$

and use $Q(O+1,3)$ to denote

$$O^{(3)} - 3O^{(2)} - O^{(1)} + 4O^{(0)},$$

where $Q(n,k) = n(n-2)(n-4)...(n-2k+2)$ .

Lemma 1

$$n^k = S^{(k)} - (S-1)^k = \sum_{j=1}^{k} (-1)^{j-1} C(k, j) S^{(k-j)}.$$

Proof.

We shall only show the inductive step of mathematical induction on $n$:

$$(n+1)^k = (n+1)^k + S^{(k)}(n) - [S(n)-1]^k - [(n+1)-1]^k = S^{(k)}(n+1) - [S(n+1)-1]^k.$$

Therefore, any polynomial in $n$ can be converted into a polynomial in $S$. For example,

$$n^6 = 6S^{(5)} - 15S^{(4)} + 20S^{(3)} - 15S^{(2)} + 6S^{(1)} - S^{(0)}.$$

Lemma 2

$$Q(O) = Q(2S-1) \text{ or } P(S) = P(\frac{O+1}{2}).$$

Proof.

We shall only show the inductive step of mathematical induction on $n$:

$$Q(O(n+1)) = Q(O(n)) + Q(2n+1) = Q(2S(n)-1) + Q(2(n+1)-1) = Q(2S(n+1)-1).$$

Therefore, any even (respectively, odd) polynomial in $n$ can be converted into an odd (respectively, even) polynomial in $O$, since

$$n^{2k+1} = \frac{1}{2^{2k}}[C(2k+1,1)O^{(2k)} + C(2k+1,3)O^{(2k-2)} + ... + C(2k+1,2k-1)O^{(2)} + O^{(0)}];$$

$$n^{2k} = \frac{1}{2^{2k-1}}[C(2k,1)O^{(2k-1)} + C(2k,3)O^{(2k-3)} + ... + C(2k,2k-3)O^{(3)} + C(2k,2k-1)O^{(1)}].$$

For example, $n^9 = \frac{1}{256}[9O^{(8)} + 84O^{(6)} + 126O^{(4)} + 36O^{(2)} + O^{(0)}]$ and

$$n^{10} = \frac{1}{512}[10O^{(9)} + 120O^{(7)} + 252O^{(5)} + 120O^{(3)} + 10O^{(1)}].$$

Lemma 3

$$P(n,r) = rP(S-1,r-1) = \frac{r}{2^{r-1}} Q(O-1,r-1) .$$

Proof.

We shall only show the inductive step of mathematical induction on $n$ :

$$\begin{aligned}
& P(n+1,r-1) \\
&= P(n,r) + rP(n,r-1) \\
&= rP(S(n)-1,r-1) + rP((n+1)-1,r-1) \\
&= rP(S(n+1)-1,r-1)
\end{aligned}$$

and

$$\begin{aligned}
& rP(S-1,r-1) \\
&= rP\left(\frac{O+1}{2}-1,r-1\right) \\
&= rP\left(\frac{O-1}{2},r-1\right) \\
&= \frac{r}{2^{r-1}} Q(O-1,r-1) .
\end{aligned}$$

In addition to

$$P(n,1) = S^{(0)} = O^{(0)},$$

we can use Lemma 3 to derive

$$P(n,2) = 2\left(S^{(1)} - S^{(0)}\right) = O^{(1)} - O^{(0)},$$

$$P(n,3) = 3\left(S^{(2)} - 3S^{(1)} + 2S^{(0)}\right) = \frac{3}{4}\left(O^{(2)} - 4O^{(1)} + 3O^{(0)}\right)$$

and in general

$P(n,r)$

$$= r\left(S^{(r-1)} - S(r-1,1)S^{(r-2)} + \ldots + (-1)^j S(r-1,j)S^{(r-j-1)} + \ldots + (-1)^{r-1} S(r-1,r-1)S^{(0)}\right)$$

$$= \frac{r}{2^{r-1}}\left(O^{(r-1)} - O(s-1,1)O^{(r-2)} + \ldots + (-1)^j O(s-1,j)O^{(r-j-1)} + \ldots + (-1)^{r-1} O(s-1,r-1)O^{(0)}\right)$$

,
where $s = 2(r-1)$, $S(m, j)$ and $O(m, j)$ denote the sums of all products of j elements of the sets $\{1,2,3,\ldots,m\}$ and $\{1,3,5,\ldots,2m-1\}$, respectively.

We can use Lemma 3 to derive the summation formulas for each $k$. For example,

$$S^{(2)} = (S-1)(S-2) + 3(S-1) + 1$$

$$= P(S-1,2) + 3P(S-1,1) + P(S-1,0)$$

$$= \frac{P(n,3)}{3} + 3\left(\frac{P(n,2)}{2}\right) + P(n,1) \; = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

and

$$O^{(2)} = (O-1)(O-3) + 4(O-1) + 1$$

$$= Q(O-1,2) + 4Q(O-1,1) + Q(O-1,0)$$

$$= \frac{2^{3-1}}{3}P(n,3) + 4\left(\frac{2^{2-1}}{2}P(n,2)\right) + P(n,1) \; = \frac{4n^3}{3} - \frac{n}{3} \; .$$

Soon after that, I received a notice of passing of the referee from Mathematical Gazette and the request of the new referee for some final revisions of my pending manuscript. Having already generalized my findings to power-sums of arithmetic progressions based on the above three lemmas, I submitted the new version of my article with an essentially different approach, which is quoted as follows.

We define the general permutation notation, with ${}_nP_{r;1} = {}_nP_r$ ,

$${}_nP_{r;d} = n(n-d)(n-2d)...[n-(r-1)d] \ ;$$

$${}_{S_n-a}P_{r;d} = (S_n - a)(S_n - a - d)...[S_n - a - (r-1)d]$$

with

$${}_{S_n-a}P_{3;d} = S_n^{(3)} - (3a+3d)S_n^{(2)} + (3a^2 + 6ad + 2d^2)S_n^{(1)} - (a^3 + 3a^2d + 2ad^2)S_n^{(0)} .$$

In such “polynomials”, $S_n$ is a linear operator over any commutative ring; in particular, if $S_n = S_n' + S_n''$, then ${}_{S_n-a}P_{r;d} = {}_{S_n'-a}P_{r;d} + {}_{S_n''-a}P_{r;d}$ since in three “polynomial” expansions all the coefficients of the same “power” are equal. Our method is based on the following.

Theorem 1.

$${}_{dn}P_{r;d} = rd\,{}_{S_n-a}P_{r-1;d} .$$

Proof.

$${}_{d(n+1)}P_{r;d} = {}_{dn}P_{r;d} + rd\,{}_{dn}P_{r-1;d} = rd\,{}_{S_n-a}P_{r-1;d} + rd\,{}_{(a+nd)-a}P_{r-1;d} = rd\,{}_{S_{n+1}-a}P_{r-1;d} .$$

It then follows from ${}_{dn}p_{r;d} = d^r\,{}_np_r$ that

$${}_{S-a}p_{r-1;d} = \frac{d^{r-1}}{r}\,{}_np_r .$$

Next we can first obtain

$$\sum_{j=1}^{r}\binom{r}{j}n^{r-j} = \frac{1}{d^r}\sum_{j=1}^{r}(-1)^{j-1}\binom{r}{j}[a^j - (a-d)^j](a+nd)^{r-j}$$

via expanding $[(a+nd)-a]^r$ and $[(a+nd)-(a-d)]^r$, then use mathematical induction on $n$ to prove

$$n^r = \frac{1}{d^r}\sum_{j=1}^{r}(-1)^{j-1}\binom{r}{j}[a^j - (a-d)^j]S^{(r-j)} \ :$$

$$(n+1)^r$$

$$= n^r + \sum_{j=1}^{r}\binom{r}{j}n^{r-j}$$

$$= \frac{1}{d^r}\sum_{j=1}^{r}(-1)^{j-1}\binom{r}{j}[a^j - (a-d)^j]\sum_{i=1}^{n}[a+(i-1)d]^{r-j}$$

$$+ \frac{1}{d^r}\sum_{j=1}^{r}(-1)^{j-1}\binom{r}{j}[a^j - (a-d)^j](a+nd)^{r-j}$$

$$= \frac{1}{d^r}\sum_{j=1}^{r}(-1)^{j-1}\binom{r}{j}[a^j - (a-d)^j]\sum_{i=1}^{n+1}[a+(i-1)d]^{r-j} \ ,$$

with $\binom{r}{j}$ and $C(r, j)$ being interchangeable. Since ${}_{dn}P_{r;d} = d^r\,{}_nP_r$, it follows from Theorem 1 that

$${}_{S_n-a}P_{r-1;d} = \frac{d^{r-1}}{r}\,{}_nP_r \ ,$$

which can be used to derive the polynomial expression in $n$ for $S_n^{(k)}$ as follows.

$$S_n^{(1)} = (S_n - a) + aS_n^{(0)} = {}_{S_n-a}P_{1;d} + a\,{}_{S_n-a}P_{0;d} = \frac{d\,{}_nP_2}{2} + a\,{}_nP_1 = \frac{d}{2}n^2 + (a - \frac{d}{2})n \ ;$$

$$S_n^{(2)} = (S_n - a)(S_n - a - d) + (2a+d)(S_n - a) + a^2 S_n^{(0)}$$

$$= {}_{S_n-a}P_{2;d} + (2a+d)\,{}_{S_n-a}P_{1;d} + a^2\,{}_{S_n-a}P_{0;d} = \frac{d^2\,{}_nP_3}{3} + \frac{(2a+d)d\,{}_nP_2}{2} + a^2\,{}_nP_1$$

$$= \frac{d^2}{3}n^3 + d(a - \frac{d}{2})n^2 + (a^2 - ad + \frac{d^2}{6})n \ ;$$

$$S_n^{(3)} = (S_n - a)(S_n - a - d)(S_n - a - 2d) + (3a + 3d)(S_n - a)(S_n - a - d)$$

$$+ (3a^2 + 3ad + d^2)(S_n - a) + a^3 S_n^{(0)}$$

$$= {}_{S_n - a}P_{3;d} + (3a + 3d)\,{}_{S_n - a}P_{2;d} + (3a^2 + 3ad + d^2)\,{}_{S_n - a}P_{1;d} + a^3\,{}_{S_n - a}P_{0;d}$$

$$= \frac{d^3\,{}_nP_4}{4} + \frac{(3a + 3d)d^2\,{}_nP_3}{3} + \frac{(3a^2 + 3ad + d^2)d\,{}_nP_2}{2} + a^3\,{}_nP_1$$

$$= \frac{d^3}{4}n^4 + d^2(a - \frac{d}{2})n^3 + \frac{3d}{2}(a^2 - ad + \frac{d^2}{6})n^2 + a(a - d)(a - \frac{d}{2})n\,.$$

During the new pending period, I received a letter from the referee to recommend reading (1). Accordingly, I incorporated the integration method into my new article (18) evolved from the following.

Lemma 4.

Let $S^{(k)}(n) = a_{k+1}n^{k+1} + a_k n^k + a_{k-1}n^{k-1} + ... + a_2 n^2 + a_1 n$. Then

$$S^{(k+1)}(n) = (k+1)\left\lfloor \int S^{(k)}(n)dn + cn \right\rfloor, \qquad \text{Eq. 3}$$

where

$$c = \frac{1}{k+1} - \left( \frac{1}{k+2}a_{k+1} + \frac{1}{k+1}a_k + \frac{1}{k}a_{k-1} + ... + \frac{1}{3}a_2 + \frac{1}{2}a_1 \right).$$

Proof.

We shall only show the inductive step of mathematical induction on $n$ :

$$S^{(k+1)}(n+1) = S^{(k+1)}(n) + (n+1)^{k+1}$$

$$= (k+1)\left\lfloor \int S^{(k)}(n)dn + cn \right\rfloor + (k+1)\left\lfloor \int (n+1)^k d(n+1) + c \right\rfloor$$

$$= (k+1)\left\lfloor \int S^{(k)}(n+1)d(n+1) + c(n+1) \right\rfloor.$$

By using Lemma 4, we can successively obtain

$$S^{(0)}(n) = n\,;$$

$$S^{(1)}(n) = \frac{1}{2}n^2 + \frac{1}{2}n\,;$$

$$S^{(2)}(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\,;$$

$$S^{(3)}(n) = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2\,;$$

$$S^{(4)}(n) = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n\,;$$

$$S^{(5)}(n) = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2\,;$$

$$S^{(6)}(n) = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n\,;$$

$$S^{(7)}(n) = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2\,;$$

$$S^{(8)}(n) = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n\,;$$

$$S^{(9)}(n) = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{3}{20}n^2\,;$$

$$S^{(10)}(n) = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n\,;$$

$$S^{(11)}(n) = \frac{1}{12}n^{12} + \frac{1}{2}n^{11} + \frac{11}{12}n^{10} - \frac{11}{8}n^8 + \frac{11}{6}n^6 - \frac{11}{8}n^4 + \frac{5}{12}n^2\,;$$

$$S^{(13)}(n) = \frac{1}{14}n^{13} + \frac{1}{2}n^{12} + \frac{13}{12}n^{11} - \frac{143}{60}n^{10} + \frac{143}{28}n^8 - \frac{143}{20}n^6 + \frac{65}{12}n^4 - \frac{691}{420}n^2\,.$$

Lemma 5.

Let $O^{(k)}(n) = b_{k+1}n^{k+1} + b_k n^k + b_{k-1}n^{k-1} + ... + b_2 n^2 + b_1 n$. Then

$$O^{(k+1)}(n) = (2k+2)\left\lfloor \int O^{(k)}(n)dn + cn \right\rfloor,$$

where

$$c = \frac{1}{2k+2} - \left( \frac{1}{k+2}b_{k+1} + \frac{1}{k+1}b_k + \frac{1}{k}b_{k-1} + ... + \frac{1}{3}b_2 + \frac{1}{2}b_1 \right).$$

Proof

We shall only show the inductive step of mathematical induction on $n$ :

$$\begin{aligned} O^{(k+1)}(n+1) &= O^{(k+1)}(n) + (2n+1)^{k+1} \\ &= (2k+2)\left\lfloor \int O^{(k)}(n)dn + cn \right\rfloor + (2k+2)\left\lfloor \int (2n+1)^k d(n+1) + c \right\rfloor \\ &= (2k+2)\left\lfloor \int O^{(k)}(n+1)d(n+1) + c(n+1) \right\rfloor \end{aligned}$$

The following list can be obtained by successive use of Lemma 5.

$O^{(0)}(n) = n$;

$O^{(1)}(n) = n^2$;

$O^{(2)}(n) = \frac{4}{3}n^3 - \frac{1}{3}n$;

$O^{(3)}(n) = 2n^4 - n^2$;

$O^{(4)}(n) = \frac{16}{5}n^5 - \frac{8}{3}n^3 + \frac{7}{15}n$;

$O^{(5)}(n) = \frac{16}{3}n^6 - \frac{20}{3}n^4 + \frac{7}{3}n^2$;

$$O^{(6)}(n)=\frac{64}{7}n^7-\frac{16}{3}n^5+\frac{28}{3}n^3-\frac{31}{21}n\,;$$

$$O^{(7)}(n)=16n^8-\frac{112}{3}n^6+\frac{98}{3}n^4-\frac{31}{3}n^2\,;$$

$$O^{(8)}(n)=\frac{256}{9}n^9-\frac{256}{3}n^7+\frac{1568}{15}n^5-\frac{496}{9}n^3+\frac{127}{15}n\,;$$

$$O^{(9)}(n)=\frac{512}{10}n^{10}-192n^8+\frac{3136}{10}n^6-248n^4+\frac{762}{10}n^2\,;$$

$$O^{(10)}(n)=\frac{1024}{11}n^{11}-\frac{1280}{3}n^9+896n^7-992n^5+508n^3-\frac{2555}{33}n$$

Furthermore, by letting $N=2n^2$, we can obtain

$$O^{(1)}(n)=\frac{1}{2}N\,,$$

$$O^{(2)}(n)=\frac{n}{3}(2N-1)\,,$$

$$O^{(3)}(n)=\frac{1}{2}\left(N^2-N\right),$$

$$O^{(4)}(n)=\frac{n}{15}(2N-1)(6N-7)\,,$$

$$O^{(5)}(n)=\frac{1}{6}\left(4N^3-10N^2+7N\right),$$

$$O^{(6)}(n)=\frac{n}{21}(2N-1)\left(12N^2-36N+31\right),$$

$$O^{(7)}(n)=\frac{1}{6}\left(6N^4-28N^3+49N^2-31N\right),$$

$$O^{(8)}(n)=\frac{n}{45}(2N-1)\left(40N^3-220N^2+478N-381\right),$$

$$O^{(9)}(n) = \frac{1}{10}\left(16N^5 - 120N^4 + 392N^3 - 620N^2 + 381N\right),$$

$$O^{(10)}(n) = \frac{n}{33}(2N-1)\left(48N^4 - 416N^3 + 1640N^2 - 3272N + 2555\right),$$

where the second factor of $O^{(2k)}(n)$ is the derivative of that of $O^{(2k+1)}(n)$.

The above two lemmas led to the following.

Theorem 2.

Let $S_n^{(k)} = \sum_{j=1}^{k} a_{k+1-j}^{(k)} n^{k+1-j}$, where $a_{k+1-j}^{(k)}$ is a polynomial in $a$ and $d$. Then

$$S_n^{(k+1)} = d(k+1)\int S_n^{(k)} dn + c_{k+1} n, \qquad \text{Eq. 4}$$

where $c_{k+1}$ is a polynomial in $a$ and $d$ that can be determined by $S_1^{(k+1)} = a^{k+1}$.

Proof.

We use mathematical induction on $n$:

$$S_{n+1}^{(k+1)} = S_n^{(k+1)} + (a+nd)^{k+1}$$

$$= d(k+1)\left[\int S_n^{(k)}(n)dn + c_{k+1}n\right] + d(k+1)\left[\int (a+nd)^k dn + a^{k+1}\right]$$

$$= d(k+1)\int S_{n+1}^{(k)} d(n+1) + c_{k+1}n + a^{k+1}$$

$$= d(k+1)\int S_{n+1}^{(k)} d(n+1) + c_{k+1}(n+1),$$

where the last step is true, since $a^{k+1} = S_1^{(k+1)} = c_{k+1}$ when $n = 0$.

By abbreviating $S^{(k)}(n)$ to $S^k$, I came up with the following interesting approximation

$$\frac{S^{k+2} - S^k}{S^k - S^{k-2}} \approx \frac{k+1}{k+3}(n^2+n) - \frac{(k^2+3k+8)(k-3)}{6(k-1)(k+3)} \qquad \text{Eq. 5}$$

by taking $m = k$ and $m = k-2$ in

$$S^{m+2} - S^{m} \approx \frac{1}{m+3} n^{m+3} + \frac{1}{2} n^{m+2} + \frac{m+2}{12} n^{m+1} - \frac{1}{m+1} n^{m} - \frac{1}{2} n^{m}$$

respectively and then via long division. Note that Eq. 5 is exact for $k = 3,4,5$ :

$$\frac{S^5 - S^3}{S^3 - S^1} = \frac{4n^2 + 4n}{6} \ ;$$
$$\frac{S^6 - S^4}{S^4 - S^2} = \frac{5n^2 + 5n - 2}{7} ;$$
$$\frac{S^7 - S^5}{S^5 - S^3} = \frac{6n^2 + 6n - 4}{8} .$$

Although Eq. 5 is not exact for $k = 6$, it only underestimates the exact value of

$$\frac{S^8 - S^6}{S^6 - S^4} = \frac{7n^2 + 7n - 6}{9} - \frac{n^2 + n - 6}{9(5n^2 + 5n - 2)}$$

with the discrepancy approximately 2/3. In the similar manner, we can also derive the following approximation formulas

$$\frac{S^{k+2}}{S^k} \approx \frac{k+1}{k+3}(n^2 + n) - \frac{k(k+1)}{6(k+3)} \ ;$$

$$\frac{S^{k+1}S^{k-1} - S^k S^k}{S^{2k+1}} \approx 3C(k+2,3) .$$

Three years prior to the publication of (18), I gave a few talks among universities in Taiwan and a class of gifted students of my Alma Mater (High School of National Taiwan Normal University). I was then invited to present “General Triangular Arrays of Numbers” by “22nd Asian Technology Conference in Mathematics” (Chung Yuan Christian University, December 19, 2017). I am also grateful that Professor Ronald Graham [author of (6)] replied promptly to my e-mails with two separate attachments of my manuscripts that I generalized most of the special functions in Chapter 6 of (6).

## 2.2. PASCAL-BERNOULLI-STIRLING-EULER-BELL-GAUSS

I am presenting here a systemic but rather long account of my personal excursion into the realm of numbers initiated by Blaise Pascal, James Stirling, Leonhard Euler and Jacob Bernoulli, which is therefore not meant to be a categorical survey of the topic.

### 2.2.1 Pascal-Bernoulli

Nothing is more impressive than the Pascal triangle,
It displays those numbers ever so natural and simple;
I have long dreamed of writing a prospective article,
To show the inner beauty of numbers from my angle.

Binomial coefficients $C(n,k)$ can be displayed as Pascal triangle (see Table 8), which was discovered about one thousand years ago by Al-Karaji. In fact, it could trace back to the second century B.C. by Pingala and for the subsequent thousand years there had been documentary evidences that Pascal triangle had been mentioned independently in India, Greece, China and Persia.

| $C(n,k)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | |
| 1 | 1 | 1 | | | | | | | | | |
| 2 | 1 | 2 | 1 | | | | | | | | |
| 3 | 1 | 3 | 3 | 1 | | | | | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | | | | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | | | | | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | | |
| 7 | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | |
| 8 | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | |
| 9 | 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | |
| 10 | 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 |

Table 8. Pascal Triangle

As a matter of fact, $C(n,k)$ and $\sum_{i=1}^{n} i^k$ got intertwined in the Eighteen Century by Blaise Pascal, James Stirling, Leonhard Euler and Jacob Bernoulli.

My goal had been to use $C(k,j)$ in Table 8 to find the general Bernoulli coefficient $b(k,j)$, with $b(k,1)$ denoting Bernoulli numbers, in the following expression

$$\sum_{i=1}^{n} i^k = \sum_{j=1}^{k+1} b(k,j) n^j \ , \qquad \text{Eq.6}$$

which is also denoted as $S_n^{(k)}$ , displayed in the Bernoulli triangle in Table 9.

| $b\Delta$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | | | | | | | | | |
| 2 | $\frac{1}{6}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | | | | | | | | |
| 3 | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ | | | | | | | |
| 4 | $-\frac{1}{30}$ | 0 | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{1}{5}$ | | | | | | |
| 5 | 0 | $-\frac{1}{12}$ | 0 | $\frac{5}{12}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | | | | | |
| 6 | $\frac{1}{42}$ | 0 | $-\frac{1}{6}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{7}$ | | | | |
| 7 | 0 | $\frac{1}{12}$ | 0 | $-\frac{7}{24}$ | 0 | $\frac{7}{12}$ | $\frac{1}{2}$ | $\frac{1}{8}$ | | | |
| 8 | $-\frac{1}{30}$ | 0 | $\frac{2}{9}$ | 0 | $-\frac{7}{15}$ | 0 | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{1}{9}$ | | |
| 9 | 0 | $-\frac{3}{20}$ | 0 | $\frac{1}{2}$ | 0 | $-\frac{7}{10}$ | 0 | $\frac{3}{4}$ | $\frac{1}{2}$ | $\frac{1}{10}$ | |
| 10 | $\frac{5}{66}$ | 0 | $-\frac{1}{2}$ | 0 | 1 | 0 | $-1$ | 0 | $\frac{5}{6}$ | $\frac{1}{2}$ | $\frac{1}{11}$ |

Table 9. Bernoulli triangle

The Intuitive approach is to equate the coefficients of the like terms in the expansions of

$$(n+1)^k = \sum_{i=1}^{n+1} i^k - \sum_{i=1}^{n} i^k \qquad \text{Eq. 7}$$

for $j = 0,1,2,...,k$, then use the identity

$$(1+n)^k = \sum_{j=0}^{k} C(k,j)n^j \qquad \text{Eq. 8}$$

to obtain

$$C(k,i) = \sum_{j=i}^{k+1} C(j,i)b(k,j) \,. \qquad \text{Eq. 9}$$

Take $k = 3$ in Eq. 7 for instance, by equating the coefficients of the like terms of

$$C(3,0) + C(3,1)n + C(3,2)n^2 + C(3,3)n^3$$

$$= \sum_{j=1}^{4} b(3,j)(n+1)^j - \sum_{j=1}^{4} b(3,j)n^j$$

$$= b(3,1)C(1,0) + b(3,2)[C(2,0) + C(2,1)n] + b(3,3)[C(3,0) + C(3.1)n + C(3,2)n^2]$$

$$+ b(3,4)[C(4,0) + C(4,1)n + C(4,2)n^2 + C(4,3)n^3]$$

$$= [C(1,1)b(3,1) + C(2,2)b(3,2) + C(3,3)b(3,3) + C(4,4)b(3,4)]$$

$$+ [C(2,1)b(3,2) + C(3,2)b(3,3) + C(4,3)b(3,4)]n$$

$$+ [C(4,2)b(3,3) + C(3,1)b(3,4)]n^2 + [C(4,1)b(3,4)]n^3$$

we can obtain

$$C(4,1)b(3,4) = C(3,0) \;;\; C(4,2)b(3,3) + C(3,1)b(3,4) = C(3,1)$$

$$C(2,1)b(3,2) + C(3,2)b(3,3) + C(4,3)b(3,4) = C(3,2)$$

and

$$C(1,1)b(3,1) + C(2,2)b(3,2) + C(3,3)b(3,3) + C(4,4)b(3,4) = C(3,3) \,.$$

Moreover, let us generalize Eq. 6 to

$$\sum_{i=1}^{n}[a+(i-1)d]^k = \sum_{j=1}^{k+1} b_{a;d}(k,j)n^{k+1-j} \qquad \text{Eq. 10}$$

for an arithmetically progressive sequence $(a+(i-1)d)_1^{\infty}$ with $b_{1;1}(k,j)=b(k,j)$.

Likewise, we can equate the coefficients of the like terms for $j=0,1,2,,...,k$ in the expansions of both sides of the identity $(dn+a)^k = \sum_{i=1}^{n+1}[a+(i-1)d]^k - \sum_{i=1}^{n}[a+(i-1)d]^k$ to obtain the following generalization of Eq. 7:

$$a^i d^{k-i} C(k,i) = \sum_{j=i}^{k+1} C(j,i) b_{a,d}(k,j). \qquad \text{Eq. 11}$$

When $i=k+1,k,k-1$ in Eq. 4,

$$d^k C(k,0) = C(k+1,1) b_{a;d}(k,k+1)$$

gives

$$b_{a;d}(k,k+1) = d^k \frac{1}{k+1};$$

$$ad^{k-1}C(k,1) = C(k+1,2) b_{a;d}(k,k+1) + C(k,1) b_{a;d}(k.k)$$

gives

$$b_{a;d}(k,k) = d^{k-1}\left(a - \frac{d}{2}\right)$$

and

$$a^2 d^{k-2} C(k,2) = C(k+1,3) b_{a;d}(k,k+1) + C(k,2) b_{a;d}(k.k) + C(k-1,1) b_{a;d}(k,k-1)$$

gives

$$b_{a;d}(k,k-1) = d^{k-2}\left(a^2 - ad + \frac{d^2}{6}\right)\frac{C(k,1)}{2}.$$

In this manner, we can successively obtain

$$b_{a;d}(k,k-2) = d^{k-3}\left(a - \frac{d}{2}\right)(a^2 - ad)\frac{C(k,2)}{3},$$

$$b_{a;d}(k,k-3) = d^{k-4}\left[(a^2 = ad)^2 - \frac{d^4}{30}\right]\frac{C(k,3)}{4},$$

$$b_{a;d}(k,k-4) = d^{k-5}\left(a - \frac{d}{2}\right)\left[(a^2 = ad)^2 - \frac{d^2}{3}(a^2 - ad)\right]\frac{C(k,4)}{5},$$

$$b_{a;d}(k,k-5) = d^{k-6}\left[(a^2 = ad)^3 - \frac{d^2}{2}(a^2 - ad)^2 + \frac{d^6}{42}\right]\frac{C(k,5)}{6},$$

$$b_{a;d}(k,k-6) = d^{k-7}\left(a - \frac{d}{2}\right)\left[(a^2 = ad)^2 - d^2(a^2 - ad)^2 + \frac{d^4}{3}(a^2 - ad)\right]\frac{C(k,6)}{7},$$

which certainly would not lead to Eq. 11. So it is time to introduce my approach.

### 2.2.2 Stirling

Let $p(n,k)$ be the product-sum (sum of products) of all $k$ numbers of row $n$ of the unity triangle $U\Delta$, where $U(n,k) = 1$ .By further assuming $p(n,0) = C(n,0)$ , we see that $p(n,k) = C(n,k)$ as displayed in Table 8. For example, $p(1,1) = 1 = C(1,1)$, $p(2,1) = 1 + 1 = 2 = C(2,1)$, $p(2,2) = 1\times 1 = 1 = C(2,2)$, $p(3,1) = 1 + 1 + 1 = 3 = C(3,1)$, $p(3,2) = 1\times 1 + 1\times 1 + 1\times 1 = 3 = C(3,2)$ and $p(3,3) = 1\times 1\times 1 = 1 = C(3,3)$.

Likewise, we define the small Stirling numbers $s(n,k)$ with $s(n,0) = 1$ and $s(n+1,k)$ being the product-sum of all $k$ numbers in row $n$ of the natural triangle $N\Delta$, where $N(n,k) = k$. For example, as in Table 10, $s(2,1) = 1$, $s(3,1) = 1 + 2 = 3$, $s(3,2) = 1\times 2 = 2$, $s(4,1) = 1 + 2 + 3 = 6$, $s(4,2) = 1\times 2 + 1\times 3 + 2\times 3 = 11$, $s(4,3) = 1\times 2\times 3 = 6$.

| $s\Delta$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | |
| 2 | 1 | 1 | | | | | | | | |
| 3 | 1 | 3 | 2 | | | | | | | |
| 4 | 1 | 6 | 11 | 6 | | | | | | |
| 5 | 1 | 10 | 35 | 50 | 24 | | | | | |
| 6 | 1 | 15 | 85 | 225 | 274 | 120 | | | | |
| 7 | 1 | 21 | 175 | 735 | 1624 | 1764 | 720 | | | |
| 8 | 1 | 28 | 322 | 1960 | 6769 | 13132 | 13068 | 5040 | | |
| 9 | 1 | 36 | 546 | 4536 | 22449 | 67284 | 118124 | 109584 | 40320 | |
| 10 | 1 | 45 | 870 | 9450 | 63273 | 269325 | 723680 | 1172700 | 1026576 | 362880 |

Table 10. The small Stirling triangle

We further notice that $s(n,n-1)=(n-1)!$ and

$$s(n,k)=(n-1)s(n-1,k-1)+s(n-1,k)\,,\ k\le n-2\,. \qquad \text{Eq. 12}$$

Next, we define the large Stirling numbers $S(k+1,\,j)$ by way of

$$\sum_{i=0}^{n} i^k = \sum_{i=1}^{k+1} S(k+1,\,j)C(n,\,j)\,. \qquad \text{Eq. 13}$$

Since $\sum_{i=1}^{n} i^0 = n = C(n,1)$ and $\sum_{i=1}^{n} i^1 = C(n+1,2) = C(n,1)+C(n,2)$, we have

$S(1,1)=1$, $S(2,1)=1$ and $S(2,2)=1$.

Likewise, since

$$\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6} = \frac{2n^3+3n^2+n}{6} + \left[2C(n,3) - \frac{2n^3-6n^2+4n}{6}\right]$$

$$= 2C(n,3) + \frac{9n^2-3n}{6} + \left[3C(n,2) - \frac{9n^2-9n}{6}\right]$$

$$= 2C(n,3) + 3C(n,2) + C(n,1)\,,$$

we have $S(3,1)=1$, $S(3,2)=3$ and $S(3,3)=2$;

and since

$$\sum_{i=1}^{n} i^3 = \frac{(n+1)^2 n^2}{4}$$

$$= \frac{n^4 + 2n^3 + n^2}{4} + \left[ 6C(n,4) - \frac{n^4 - 6n^3 + 11n^2 - 6n}{4} \right]$$

$$= 6C(n,4) + \frac{8n^3 - 10n^2 + 6n}{4} + \left[ 12C(n,3) - \frac{8n^3 - 24n^2 + 16n}{4} \right]$$

$$= 6C(n,4) + 12C(n,3) + \frac{14n^2 - 10n}{4} + \left[ 7C(n,2) - \frac{14n^2 - 14n}{4} \right]$$

$$= 6C(n,4) + 12C(n,3) + 7C(n,2) + C(n,1),$$

we have $S(4,1) = 1$, $S(4,2) = 7$, $S(4,3) = 12$ and $S(4,4) = 6$.

We can further find that $S(k,1) = 1$, $S(k,k) = (k-1)!$, $\sum_{j=1}^{k} (-1)^j S(k,j) = 0$ and

$$S(k,j) = (j-1)S(k-1,j-1) + jS(k-1,j) \quad 1 \le j \le k, \qquad \text{Eq. 14}$$

via which we can obtain the large Stirling triangle as in Table 11.

| $S\Delta$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | |
| 2 | 1 | 1 | | | | | | | | |
| 3 | 1 | 3 | 2 | | | | | | | |
| 4 | 1 | 7 | 12 | 6 | | | | | | |
| 5 | 1 | 15 | 50 | 60 | 24 | | | | | |
| 6 | 1 | 31 | 180 | 390 | 360 | 120 | | | | |
| 7 | 1 | 63 | 602 | 2100 | 3360 | 2520 | 720 | | | |
| 8 | 1 | 127 | 1932 | 10206 | 25200 | 31920 | 20160 | 5040 | | |
| 9 | 1 | 255 | 6050 | 46620 | 166284 | 317520 | 332640 | 181440 | 40320 | |
| 10 | 1 | 511 | 188660 | 204630 | 1017900 | 2736540 | 4233600 | 3780000 | 1814400 | 362880 |

Table 11. The large Stirling triangle

Next we shall show how to come up with Bernoulli coefficients $b(k,j)$ by observing both Tables 10 and 11 simultaneously. For example,

$$\frac{s(4,0)S(4,4)}{4!} = \frac{1}{4} = b(3,4), \frac{3(3,0)S(4,3)}{3!} - \frac{s(4,1)S(4,4)}{4!} = \frac{1}{2} = b(3,3)$$

$$\frac{s(2,0)S(4,2)}{2!} - \frac{s(3,1)S(4,3)}{3!} + \frac{s(4,2)S(4,4)}{4!} = \frac{1}{4} = b(3,2)$$

$$\frac{s(1,0)S(4,1)}{1!} - \frac{s(2,1)S(4,2)}{2!} + \frac{s(3,2)S(4,3)}{3!} - \frac{s(4,3)S(4,4)}{4!} = 0 = b(3,1).$$

In general, we have

$$b(k,j) = \sum_{t=0}^{k+1-j} \frac{(-1)^t s(j+t,t)S(k+1,j+t)}{(j+t)!}, \qquad \text{Eq. 15}$$

which can be substituted in Eq. 6 to yield

$$\sum_{i=1}^{n} i^k = \sum_{j=0}^{k}\left[\sum_{t=0}^{k+1-j} \frac{(-1)^t s(j+t,t)S(k+1,j+t)}{(j+t)!}\right] n^{j+1}. \qquad \text{Eq. 16}$$

### 2.2.3 Euler

Let us define the small Euler numbers $e(k,j)$ by $e(k,1) = e(k,k) = 1$ and

$$e(k,j) = \sum_{t=0}^{j-1} (-1)^t C(k+1-j+t,t)S(k+1,j-t). \qquad \text{Eq. 17}$$

For example,

$$e(3,2) = C(2,0)S(4,2) - C(3,1)S(4,1) = 4,$$

$$e(4,2) = C(3,0)S(5,2) - C(3,1)S(5,1) = 11,$$

$$e(4,3) = C(2,0)S(5,3) - C(3,1)S(5,2) + C(4,2)S(5,1) = 11,$$

$$e(5,2) = C(4,0)S(6,2) - C(5,1)S(6,1) = 26,$$

$$e(5,3) = C(3,0)S(6,3) - C(4,1)S(6,2) + C(5,2)S(6,1) = 66,$$

$$e(5,4) = C(2,0)S(6,4) - C(3,1)S(6,3) + C(4,2)S(6,2) - C(5,3)S(6,1) = 26.$$

In addition to $e(k, j) = e(k, k+1-j)$, we can further observe

$$e(3,2) = (3+1-2)e(2,1) + 2e(2,2)\ ,\ e(4,2) = (4+1-2)e(3,1) + 2e(3,2)\ ,$$

$$e(4,3) = (4+1-3)e(3,2) + 3e(3,3)\ ,\ e(5,2) = (5+1-2)e(4,1) + 2e(4,2)\ ,$$

$$e(5,3) = (5+1-3)e(4,2) + 3e(4,3)\ ,\ e(5,4) = (5+1-4)e(4,3) + 4e(4,4)\ ,$$

to come up with

$$e(k, j) = (k+1-j)e(k-1, j-1) + je(k-1, j) \qquad \text{Eq. 18}$$

and the small Euler triangle as shown in Table 12.

| $e\Delta$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | |
| 2 | 1 | 1 | | | | | | | | |
| 3 | 1 | 4 | 1 | | | | | | | |
| 4 | 1 | 11 | 11 | 1 | | | | | | |
| 5 | 1 | 26 | 66 | 26 | 1 | | | | | |
| 6 | 1 | 57 | 302 | 302 | 57 | 1 | | | | |
| 7 | 1 | 120 | 1191 | 2416 | 1191 | 120 | 1 | | | |
| 8 | 1 | 247 | 4293 | 15619 | 15619 | 4293 | 247 | 1 | | |
| 9 | 1 | 502 | 14608 | 88234 | 156189 | 88234 | 14608 | 502 | 1 | |
| 10 | 1 | 1013 | 47840 | 455192 | 1310354 | 1310354 | 455192 | 47840 | 1013 | 1 |

Table 12. The small Euler triangle

Since $(n+1)^1 = C(n+1,1)$, we can also obtain $e(k, j)$ via

$$(n+1)^k = \sum_{j=1}^{k} e(k, j)C(n+j, k): \qquad \text{Eq. 19}$$

$$(n+1)^2 = C(n+1,2) + C(n+2,2)\,,$$

$$(n+1)^3 = C(n+1,3) + 4C(n+2,3) + C(n+3,3)\ ,$$

$$(n+1)^4 = C(n+1,4) + 11C(n+2,4) + 11C(n+3,4) + C(n+4,4)\,,$$

$$(n+1)^5 = C(n+1,5) + 26C(n+2,5) + 66C(n+3,5) + 26C(n+4,5) + C(n+5,5)\ ,\ldots$$

By virtue of Eq. 14, we can use mathematical induction to establish

$$\sum_{i=1}^{n} i^k = \sum_{j=1}^{k} e(k,j)C(n+j,k+1): \qquad \text{Eq. 20}$$

$$\sum_{i=1}^{n+1} i^k = \sum_{j=1}^{k} e(k,j)C(n+j,k+1) + \sum_{j=1}^{k} e(k,j)C(n+j,k)$$

$$= \sum_{j=1}^{k} e(k,j)C(n++1+j,k+1).$$

For $k=3$, we can write

$$\sum_{i=1}^{n} i^3 = e(3,1)C(n+1,4) + e(3,2)C(n+2,4) + e(3,3)C(n+3,4)$$

$$= e(3,1)[C(1,1)C(n,3) + C(1,0)C(n,4)]$$

$$+ e(3,2)[C(2,2)C(n,2) + C(2,1)C(n,3) + C(2,0)C(n,4)]$$

$$+ e(3,3)[C(3,3)C(n,1) + C(3,2)C(n,2) + C(3,1)C(n,3) + C(3,0)C(n,4)]$$

$$= [e(3,1)C(1,0) + e(3,2)C(2,0) + e(3,3)C(3,0)]\sum_{j+0}^{3} \frac{(-1)^j s(4,j)n^{4-j}}{4!}$$

$$+ [e(3,1)C(1,1) + e(3,2)C(2,1) + e(3,3)C(3,1)]\sum_{j=0}^{2} \frac{(-1)^j s(3,j)n^{3-j}}{3!}$$

$$+ [e(3,2)C(2,2) + e(3,3)C(3,2)]\sum_{j+0}^{1} \frac{(-1)^j s(2,j)n^{2-j}}{2!} + e(3,3)C(3,3)\frac{s(1,0)}{1!}.$$

Hence we have

$$b(3,4) = [e(3,1)C(1,0) + e(3,2)C(2,0) + e(3,3)C(3,0)]\frac{s(4,0)}{4!} = \frac{1}{4},$$

$$b(3,3) = -[e(3,1)C(1,0) + e(3,2)C(2,0) + e(3,3)C(3,0)]\frac{s(4,1)}{4!}$$

$$+ [e(3,1)C(1,1) + e(3,2)C(2,1) + e(3,3)C(3,1)]\frac{s(3,0)}{3!} = \frac{1}{2},$$

$$b(3,2)=[e(3,1)C(1,0)+e(3,2)C(2,0)+e(3,3)C(3,0)]\frac{s(4,2)}{4!}$$

$$-[e(3,1)C(1,1)+e(3,2)C(2,1)+e(3,3)C(3,1)]\frac{s(3,1)}{3!}$$

$$+[e(3,2)C(2,2)+e(3,3)C(3,2)]\frac{s(2,0)}{2!}=\frac{1}{4},$$

$$b(3,1)=-[e(3,1)C(1,0)+e(3,2)C(2,0)+e(3,3)C(3,0)]\frac{s(4,3)}{4!}$$

$$+[e(3,1)C(1,1)+e(3,2)C(2,1)+e(3,3)C(3,1)]\frac{s(3,2)}{3!}$$

$$-[e(3,2)C(2,2)+e(3,3)C(3,2)]\frac{s(2,1)}{2!}$$

$$+e(3,3)C(3,3)\frac{s(1,0)}{1!}=0\,.$$

Since $S(k,j)=\sum_{t=k-j}^{k-1}e(k-1,t)C(t,k-j)$ and $S(k,k)=\sum_{t=1}^{k-1}e(k-1,t)$, we can write Eq. 16 as

$$\sum_{i=1}^{n}i^k=\sum_{j=1}^{k}\frac{e(k,j)}{(k+1)!}\left\{\sum_{t=0}^{k}\left[\sum_{r=0}^{t}(-1)^r s(k,r)C(k+1-r,t-r)i^{t-r}\right]n^{k+1-t}\right\}. \qquad \text{Eq. 21}$$

Next, let us observe Table 11 diagonally. We can recognize that the second rightmost diagonal entries, in fact, give $\frac{S(n,n-1)}{(n-2)!}=C(n,2)$. We further dictate the trend:

$$\frac{S(n,n-2)}{(n-3)!}=C(n+1,4)+2C(n,4)\,,$$

$$\frac{S(n,n-3)}{(n-4)!}=C(n+2,6)+8C(n+1,6)+6C(n,6)\,,$$

$$\frac{S(n,n-4)}{(n-5)!} = C(n+3,8) + 22C(n+2,8) + 58C(n+1,8) + 24C(n,8) \ ,\ldots$$

In such manner, we can define the large Euler number $E(k, j)$ by way of

$$S(n,n-k) = (n-k-1)!\sum_{j=1}^{k} E(k, j)C(n+k-j,2k) \,. \qquad \text{Eq. 22}$$

and come up with $E(k,1) = E(k-1,1)$, $E(k,k) = kE(k-1,k-1)$ and in general

$$E(k, j) = (2k-j)E(k-1, j-1) + kF(k-1, j) \,, \qquad \text{Eq. 23}$$

via which we can generate the large Euler triangle in Table 13
.

| $E\Delta$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | |
| 2 | 1 | 2 | | | | | | | |
| 3 | 1 | 8 | 6 | | | | | | |
| 4 | 1 | 22 | 58 | 24 | | | | | |
| 5 | 1 | 52 | 328 | 444 | 120 | | | | |
| 6 | 1 | 114 | 1452 | 4400 | 3780 | 720 | | | |
| 7 | 1 | 240 | 5610 | 32129 | 58140 | 33984 | 5040 | | |
| 8 | 1 | 494 | 19950 | 198580 | 644020 | 785304 | 341136 | 40320 | |
| 9 | 1 | 1004 | 67260 | 1062500 | 5765500 | 12440064 | 11026296 | 3733920 | 362880 |

Table 13. The large Euler triangle

On the other hand, we can also obtain

$$s(n,k) = \sum_{j=1}^{k} E(k, j)C(n+j-1,2k) \qquad \text{Eq. 24}$$

Since Eq. 24 is true for $k=1$:

$$s(n,1) = \sum_{j=1}^{1} E(1, j)C(n+1-1,2\cdot 1) \,,$$

all we need to show is that

$$s(n,m+1)=\sum_{j=1}^{m+1}E(m+1,j)C(n+j-1,2m+2) \qquad \text{Eq. 25}$$

by assuming Eq. 24 is true for $k=m$. Prior to proving Eq. 24 by mathematical induction, let us do it in the case of $m=4$. Using Eqs. 12 and 23, we can write

$$s(n,5)=(n-1))s(n-1,4)+s(n-1,5)$$

$$=(n-1)[E(4,1)C(n-1,8)+E(4,2)C(n,8)+E(4,3)C(n+1,8)+E(4,4)C(n+2,8)]$$

$$+E(5,1)C(n-1,10)+E(5,2)C(n,10)+E(5,3)C(n+1,10)+E(5,4)C(n+2,10)+E(5,5)C(n+3,10)$$

$$=\sum_{j=1}^{4}[(n-1)C(n+j-2,8)+jC(n+j-2,10)+(9-j)C(n+j-1,10)]E(4,j)$$

and

$$\sum_{j=1}^{5}E(5,j)C(n+j-1,10)$$

$$=E(4,1)C(n,10)+[8E(4,1)+2E(4,2)]C(n+1,10)+[7E(4,2)+3E(4,3)]C(n+2,10)$$

$$+[6E(4,3)+4E(4,4)]C(n+3,10)+5E(4,4)C(n+4,10)$$

$$=\sum_{j=1}^{4}[jC(n+j-1,10)+(9-j)C(n+j,10)]E(4,j).$$

The coefficients of the like term $E(4,j)$ are equal, since

$$[(n-1)C(n+j-2,8)+jC(n+j-2,10)+(9-j)C(n+j-1,10)]$$

$$-[jC(n+j-1,10)+(9-j)C(n+j,10)]$$

$$=(n+j-1)C(n+j-2,8)-[jC(n+j-2,8)+jC(n+j-2,9)+(9-j)C(n+j-1,9)]$$

$$=9C(n+j-1,9)-[jC(n+j-2,8)+jC(n+j-2,9)+(9-j)C(n+j-1,9)]$$

$$=0.$$

Thus we have proved that $s(n,5) = \sum_{j=1}^{5} E(5,j)C(n+j-1,10)$ . In general, we can write

$$s(n,m+1) - \sum_{j=1}^{m+1} E(m+1,j)C(n+j-1,2m+2)$$

$$= \sum_{j=1}^{m} [(n-1)C(n+j-2,2m) + jC(n+j-2,2m+2) + (2m+1-j)C(n+j-1,2m+2)]E(m,j)$$

$$- \sum_{j=1}^{m} [jC(n+j-1,2m+2) + (9-j)C(n+j,2m+2)]E(m,j) .$$

The coefficients of the like term $E(m,j)$ are equal, since

$$[(n-1)C(n+j-2,2m) + jC(n+j-2,2m+2) + (2m+1-j)C(n+j-1,2m+2)]$$

$$-[jC(n+j-1,2m+2) + (2m+1-j)C(n+j,2m+2)]$$

$$= (n+j-1)C(n+j-2,2m)$$

$$-[jC(n+j-2,2m) + jC(n+j-2,2m+2) + (2m+1-j)C(n+j-1,2m+1)]$$

$$= (2m+1)C(n+j-1,2m+1)$$

$$-[jC(n+j-2,2m) + jC(n+j-2,2m+1) + (2m+1-j)C(n+j-1,2m+1)] = 0 .$$

We have completed the proof of Eq. 24 by the mathematical induction. Therefore, by virtue of Eqs. 15, 22 and 24, $b(k,j)$ can be expressed in terms of the large Euler numbers. Note that the small and large Euler numbers are in essence the same as the first-order and second-order Eulerian numbers $\left\langle {n \atop k} \right\rangle$ and $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ (which will be introduced next), since $\left\langle {n \atop k} \right\rangle = e(n,k-1)$ and $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle = E(n,k-1)$ .

2.2.4 Sorting

The number $\binom{n}{n_1,n_2,\ldots n_m}$ of ways of sorting the first $n$ terms of the natural sequence $(i)_1^\infty$ into $m$ subsets with $n_j$ elements in the $jth$ subset is $\frac{n!}{\prod_{j=1}^{m} n_j!}$, where $n=\sum_{j=1}^{m} n_j$ .

In particular, the number of ways of sorting the first $n$ terms of $(i)_1^\infty$ into $2$ subsets with $k$ elements in one and $n-k$ elements in another is the combination $\binom{n}{k,n-k}=\frac{n!}{k!(n-k)!}$, which will be further abbreviated as the binomial coefficient $\binom{n}{k}$ or $C(n,k)$ ; while the number of ways of sorting the first $n$ terms of $(i)_1^\infty$ into $k$ singletons and a subset of $n-k$ elements is the permutation $\binom{n}{1,1,\ldots 1,k}=\frac{n!}{(n-k)!}$, which will be abbreviated as $\left(\binom{n}{k}\right)$ or $P(n,k)$ . Hence we write

$$\left(\binom{n}{k}\right)=n(n-1)(n-2)\ldots[n-(k-1)] \qquad \text{Eq. 26}$$

and

$$\binom{n}{k}=\frac{\left(\binom{n}{k}\right)}{\left(\binom{k}{k}\right)}, \qquad \text{Eq. 27}$$

where $\left(\binom{k}{k}\right)=k!$ .

Since this first level of sortation can be expressed as the product of a combination and one or more permutations such as $\binom{9}{2,3,4} = \binom{5}{2}\left(\binom{9}{4}\right)$ and $\binom{14}{2,3,4,5} = \binom{5}{2}\left(\binom{9}{4}\right)\left(\binom{14}{5}\right)$, the familiar recursive formulas

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad \text{Eq. 28}$$

and

$$\left(\binom{n}{k}\right) = k\left(\binom{n-1}{k-1}\right). \qquad \text{Eq. 29}$$

We can use Eq. 28 to generate the first-order Pascal triangle, same as Table 8, in Table 14.

| $\binom{n}{k}\Delta$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | |
| 1 | 1 | 1 | | | | | | | | | |
| 2 | 1 | 2 | 1 | | | | | | | | |
| 3 | 1 | 3 | 3 | 1 | | | | | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | | | | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | | | | | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | | |
| 7 | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | |
| 8 | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | |
| 9 | 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | |
| 10 | 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 |

Table 14. The first-order Pascal triangle

Likewise, we can use Eq. 29 to generate the second-order Pascal triangle as in Table 15.

| $\left(\!\binom{n}{k}\!\right)$ Δ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | | | | | | | | | |
| 2 | 1 | 2 | 2 | | | | | | | | |
| 3 | 1 | 3 | 6 | 6 | | | | | | | |
| 4 | 1 | 4 | 12 | 24 | 24 | | | | | | |
| 5 | 1 | 5 | 20 | 60 | 120 | 120 | | | | | |
| 6 | 1 | 6 | 30 | 120 | 360 | 720 | 720 | | | | |
| 7 | 1 | 7 | 42 | 210 | 840 | 2520 | 5040 | 5040 | | | |
| 8 | 1 | 8 | 56 | 336 | 1680 | 6720 | 20160 | 40320 | 40320 | | |
| 9 | 1 | 9 | 72 | 504 | 3024 | 15120 | 60480 | 181440 | 362880 | 362880 | |
| 10 | 1 | 10 | 90 | 720 | 5040 | 30240 | 151200 | 604800 | 1814400 | 3628800 | 3628800 |

Table 15. The second-order Pascal triangle

We further write Eq. 26 into

$$\left(\!\binom{n}{k}\!\right)=\sum_{j=1}^{k}(-1)^{j-1}s(k,j)n^{k-j+1}\ , \qquad \text{Eq. 30}$$

where $s(k,j)$ is the small Stirling number as in Table 10. For example,

$$\left(\!\binom{n}{4}\!\right)=n(n-1)(n-2)(n-3)\ = s(4,0)n^4-s(4,1)n^3+s(4,3)n^2-s(4,4)n\ .$$

Next, let us proceed to sorting of the second level: Stirling numbers.

The number of ways of sorting the first $n$ terms of $(i)_1^\infty$ into $k$ cycles is the Stirling number of the first kind $\left[{n \atop k}\right]$. Clearly, $\left[{1 \atop 1}\right]=1$. In general, sorting the first $n$ terms of $(i)_1^\infty$ into $k$ cycles, there are $\left[{n-1 \atop k-1}\right]$ ways including the one-cycle $[n]$, since the number of ways of sorting the first $n-1$ terms into $k-1$ cycles is $\left[{n-1 \atop k-1}\right]$.

For any one of the $\begin{bmatrix} n-1 \\ k \end{bmatrix}$ ways of sorting, not including $n$, we need to insert $n$ into 1 (say, a $j_i$-cycle) of the $k$ cycles. Since there are $j_i$ ways of doing such insertion, the total possible ways of inserting $n$ into any of those $k$ cycles is $\sum_{i=1}^{k} j_i = n-1$.

Thus we have

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}, \qquad \text{Eq. 31}$$

which is the same as Eq. 12, since $\begin{bmatrix} n \\ k \end{bmatrix} = s(n, n-k)$.

Also, we can write Eq. 30 as

$$\left(\!\!\binom{n}{k}\!\!\right) = \sum_{j=1}^{k} (-1)^{j-1} \begin{bmatrix} k \\ k-j+1 \end{bmatrix} n^{k-j+1} . \qquad \text{Eq. 32}$$

We can use Eq. 31 to generate the Stirling triangle of the first kind as shown in Table 16.

| $\begin{bmatrix} n \\ k \end{bmatrix}\Delta$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | |
| 2 | 1 | 1 | | | | | | | | |
| 3 | 2 | 3 | 1 | | | | | | | |
| 4 | 6 | 11 | 6 | 1 | | | | | | |
| 5 | 24 | 50 | 35 | 10 | 1 | | | | | |
| 6 | 120 | 274 | 225 | 85 | 15 | 1 | | | | |
| 7 | 720 | 1764 | 1624 | 735 | 175 | 21 | 1 | | | |
| 8 | 5040 | 13068 | 13132 | 6769 | 1960 | 322 | 28 | 1 | | |
| 9 | 40320 | 109584 | 118124 | 67284 | 22449 | 4536 | 546 | 36 | 1 | |
| 10 | 362880 | 1026576 | 1172700 | 723680 | 269325 | 63273 | 9450 | 870 | 45 | 1 |

Table 16. Table for Stirling numbers of the first kind

On the other hand, $\begin{bmatrix} n \\ k \end{bmatrix}\Delta$ can be built from $\binom{n}{k}\Delta$ via $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$, $\begin{bmatrix} n \\ n \end{bmatrix} = 1$ and

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \binom{k}{0}\begin{bmatrix} n \\ k \end{bmatrix} + \binom{k+1}{1}\begin{bmatrix} n \\ k+1 \end{bmatrix} + \binom{k+2}{2}\begin{bmatrix} n \\ k+2 \end{bmatrix} + ... + \binom{n}{n-k}\begin{bmatrix} n \\ n \end{bmatrix} \qquad \text{Eq. 33}$$

as shown in Table 17.

| $n \backslash k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | | | | |
| 2 | 1 | 1 | | | |
| 3 | 2 | $\binom{1}{1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \binom{2}{1}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ | 1 | | |
| 4 | 6 | $\binom{1}{1}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \binom{2}{1}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \binom{3}{1}\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ | $\binom{2}{2}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \binom{3}{2}\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ | 1 | |
| 5 | 24 | $\binom{1}{1}\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \binom{2}{1}\begin{bmatrix} 4 \\ 2 \end{bmatrix} + \binom{3}{1}\begin{bmatrix} 4 \\ 3 \end{bmatrix} + \binom{4}{1}\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ | $\binom{2}{2}\begin{bmatrix} 4 \\ 2 \end{bmatrix} + \binom{3}{2}\begin{bmatrix} 4 \\ 3 \end{bmatrix} + \binom{4}{2}\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ | $\binom{3}{3}\begin{bmatrix} 4 \\ 3 \end{bmatrix} + \binom{4}{3}\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ | 1 |

Table 17. Stirling triangle of the first kind via the recursive formula

By using Eq. 26 to expand both sides of

$$\left(\binom{n}{k}\right) = n(n-1)(n-2)...[n-(k-1)] = n\left(\binom{n-1}{k-1}\right),$$

we can obtain Eq. 28 by equating the like terms of

$$\sum_{j=0}^{k}(-1)^j \begin{bmatrix} k+1 \\ k+1-j \end{bmatrix} n^{k+1-j} = n\sum_{j=0}^{k-1}(-1)^j \begin{bmatrix} k \\ k-j \end{bmatrix}(n-1)^{k-j} .$$

Let us continue our excursion of this second level of sortation. The number of ways of sorting the first $n$ terms of $(i)_1^\infty$ into $k$ sets is the Stirling number of the second kind $\begin{Bmatrix} n \\ k \end{Bmatrix}$.

Clearly, $\left\{ {1 \atop 1} \right\} = 1$. In general, sorting the first $n$ terms of $(i)_1^\infty$ into $k$ sets, there are two cases to consider. First, there are $\left\{ {n-1 \atop k-1} \right\}$ ways if the singleton $\{n\}$ is included in the sorted arrangements, since the number of ways of sorting the first $n-1$ terms into $k-1$ sets is $\left\{ {n-1 \atop k-1} \right\}$**.** Second, there are $k\left\{ {n-1 \atop k} \right\}$ ways if the singleton $\{n\}$ is not included in the sorted arrangements, since for any one of $\left\{ {n-1 \atop k} \right\}$ ways of sorting the term $n$ can be inserted into any one of those $k$ sets. Thus we have proved the recursive formula

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}, \qquad \text{Eq. 34}$$

which is equivalent to Eq. 14, since $S(n,k) = (k-1)!\left\{ {n \atop k} \right\}$.

The Stirling triangle $\left\{ {n \atop k} \right\}\Delta$ of the second kind can be generated via Eq. 34 as in Table 18.

| $\left\{ {n \atop k} \right\}\Delta$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | |
| 2 | 1 | 1 | | | | | | | | |
| 3 | 1 | 3 | 1 | | | | | | | |
| 4 | 1 | 7 | 6 | 1 | | | | | | |
| 5 | 1 | 15 | 25 | 10 | 1 | | | | | |
| 6 | 1 | 31 | 90 | 65 | 15 | 1 | | | | |
| 7 | 1 | 63 | 301 | 350 | 140 | 21 | 1 | | | |
| 8 | 1 | 127 | 966 | 1701 | 1050 | 266 | 28 | 1 | | |
| 9 | 1 | 255 | 3025 | 7770 | 6951 | 2646 | 462 | 36 | 1 | |
| 10 | 1 | 511 | 9330 | 34105 | 42525 | 22827 | 5880 | 750 | 45 | 1 |

Table 18. Stirling triangle of the second kind via the recursive formula

To attain our goal, we first derive the following identity

$$C(n,k)=\sum_{j=k}^{n}(-1)^{j-k}\left[{j \atop k}\right]\left\{{n+1 \atop j+1}\right\}. \qquad \text{Eq. 35}$$

We shall only look at the case for $n=5$ and $k=3$, since the general case is similar. So we use Eqs. 31 and 34 to show the inductive step:

$$\begin{aligned}
\binom{6}{3} &= \left(\left[{3 \atop 3}\right]\left\{{6 \atop 4}\right\}-\left[{4 \atop 3}\right]\left\{{6 \atop 5}\right\}+\left[{5 \atop 3}\right]\left\{{6 \atop 6}\right\}\right)+\left(\left[{3 \atop 3}\right]\left\{{6 \atop 3}\right\}-\left[{4 \atop 3}\right]\left\{{6 \atop 4}\right\}+\left[{5 \atop 3}\right]\left\{{6 \atop 5}\right\}-\left[{6 \atop 3}\right]\left\{{6 \atop 6}\right\}\right)\\
&= \left[{2 \atop 2}\right]\left\{{6 \atop 3}\right\}-\left(\left[{4 \atop 3}\right]-4\left[{3 \atop 3}\right]\right)\left\{{6 \atop 4}\right\}+\left(\left[{5 \atop 3}\right]-5\left[{4 \atop 3}\right]\right)\left\{{6 \atop 5}\right\}-\left(\left[{6 \atop 3}\right]-6\left[{5 \atop 3}\right]\right)\left\{{6 \atop 6}\right\}\\
&= \left(4\left\{{6 \atop 4}\right\}+\left\{{6 \atop 3}\right\}\right)-\left[{4 \atop 3}\right]\left(5\left\{{6 \atop 5}\right\}+\left\{{6 \atop 4}\right\}\right)+\left[{5 \atop 3}\right]\left(6\left\{{6 \atop 6}\right\}+\left\{{6 \atop 5}\right\}\right)-\left[{6 \atop 3}\right]\left\{{6 \atop 6}\right\}\\
&= \left[{3 \atop 3}\right]\left\{{7 \atop 4}\right\}-\left[{4 \atop 3}\right]\left\{{7 \atop 5}\right\}+\left[{5 \atop 3}\right]\left\{{7 \atop 6}\right\}-\left[{6 \atop 3}\right]\left\{{7 \atop 7}\right\}.
\end{aligned}$$

Alternatively, the Stirling triangle $\left\{{n \atop k}\right\}\Delta$ of the second kind can be constructed based on

$\left\{{n \atop 1}\right\}=1$, $\left\{{n \atop n-1}\right\}=\left[{n \atop n-1}\right]=\binom{n}{2}$ and $\left\{{n \atop n}\right\}=1$ via the inversion formula

$$\left\{{n \atop k}\right\}=\sum_{j=1}^{n-k}(-1)^{j-1}\left[{k+j \atop k}\right]\left\{{n \atop k+j}\right\} \qquad \text{Eq. 36}$$

as follows.

$$\left\{{4 \atop 2}\right\}=\left[{3 \atop 2}\right]\left\{{4 \atop 3}\right\}-\left[{4 \atop 2}\right]\left\{{4 \atop 4}\right\}=7,\ \left\{{5 \atop 3}\right\}=\left[{4 \atop 3}\right]\left\{{5 \atop 4}\right\}-\left[{5 \atop 3}\right]\left\{{5 \atop 5}\right\}=25,$$

$$\left\{{5 \atop 2}\right\}=\left[{3 \atop 2}\right]\left\{{5 \atop 3}\right\}-\left[{4 \atop 2}\right]\left\{{5 \atop 4}\right\}+\left[{5 \atop 2}\right]\left\{{5 \atop 5}\right\}=15, \ldots$$

We then derive the following identity

$$(1+n)^k = \sum_{j=1}^{k+1}\left(\binom{n}{j-1}\right)\left\{{k+1 \atop j}\right\}. \qquad \text{Eq. 37}$$

We only look at the case where $k=4$. From Eq. 8, we can use Eqs. 35 and 36 to write

$$(1+n)^4 = \binom{4}{0}+\binom{4}{1}n+\binom{4}{2}n^2+\binom{4}{3}n^3+\binom{4}{4}n^4$$

$$=1+\left(\left[{1 \atop 1}\right]\left\{{5 \atop 2}\right\}-\left[{2 \atop 1}\right]\left\{{5 \atop 3}\right\}+\left[{3 \atop 1}\right]\left\{{5 \atop 4}\right\}-\left[{4 \atop 1}\right]\left\{{5 \atop 5}\right\}\right)n$$

$$+\left(\left[{2 \atop 2}\right]\left\{{5 \atop 3}\right\}-\left[{3 \atop 2}\right]\left\{{5 \atop 4}\right\}+\left[{4 \atop 2}\right]\left\{{5 \atop 5}\right\}\right)n^2+\left(\left[{3 \atop 3}\right]\left\{{5 \atop 4}\right\}-\left[{4 \atop 3}\right]\left\{{5 \atop 5}\right\}\right)n^3+n^4$$

$$=1+\left[{1 \atop 1}\right]n\left\{{5 \atop 2}\right\}+\left(\left[{2 \atop 2}\right]n^2-\left[{2 \atop 1}\right]n\right)\left\{{5 \atop 3}\right\}+\left(\left[{3 \atop 3}\right]n^3-\left[{3 \atop 2}\right]n^2+\left[{3 \atop 1}\right]n\right)\left\{{5 \atop 4}\right\}$$

$$+\left(\left[{4 \atop 4}\right]n^4-\left[{4 \atop 3}\right]n^3+\left[{4 \atop 2}\right]n^2-\left[{4 \atop 1}\right]n\right)\left\{{5 \atop 5}\right\}$$

$$=\left(\binom{n}{0}\right)\left\{{5 \atop 1}\right\}+\left(\binom{n}{1}\right)\left\{{5 \atop 2}\right\}+\left(\binom{n}{2}\right)\left\{{5 \atop 3}\right\}+\left(\binom{n}{3}\right)\left\{{5 \atop 4}\right\}+\left(\binom{n}{4}\right)\left\{{5 \atop 5}\right\}.$$

Next, we use the mathematical induction to prove

$$\sum_{i=1}^{n} i^k = \sum_{j=1}^{k+1}\frac{1}{j}\left(\binom{n}{j}\right)\left\{{k+1 \atop j}\right\}, \qquad \text{Eq. 38}$$

with Eq. 37 being used in the inductive step:

$$\sum_{i=1}^{n+1} i^k = \sum_{j=1}^{k+1}\frac{1}{j}\left(\binom{n}{j}\right)\left\{{k+1 \atop j}\right\}+\sum_{j=1}^{k+1}\left(\binom{n}{j-1}\right)\left\{{k+1 \atop j}\right\}=\sum_{j=1}^{k+1}\left[1+\frac{j}{n-j+1}\right]\frac{1}{j}\left(\binom{n}{j}\right)\left\{{k+1 \atop j}\right\}$$

$$=\sum_{j=1}^{k+1}\left[\frac{n+1}{n-j+1}\left(\binom{n}{j}\right)\right]\frac{1}{j}\left\{{k+1 \atop j}\right\}=\sum_{j=1}^{k+1}\frac{1}{j}\left(\binom{n+1}{j}\right)\left\{{k+1 \atop j}\right\}. \qquad \text{Eq. 39}$$

Finally, we can obtain

$$\sum_{i=1}^{n} i^k = \sum_{r=0}^{k} \sum_{j=k+1-r}^{k+1} (-1)^{j-k-1+r} \frac{1}{j} \left[ {j \atop k+1-r} \right] \left\{ {k+1 \atop j} \right\} n^{k+1-r} \qquad \text{Eq. 40}$$

by regrouping the following display of Eq. 39:

$$\left[ {1 \atop 1} \right] \left\{ {k+1 \atop 1} \right\} n + \frac{1}{2} \left\{ {k+1 \atop 2} \right\} n^2 - \frac{1}{2} \left[ {2 \atop 1} \right] \left\{ {k+1 \atop 2} \right\} n$$

$$+ \frac{1}{3} \left\{ {k+1 \atop 3} \right\} n^3 - \frac{1}{3} \left[ {3 \atop 2} \right] \left\{ {k+1 \atop 3} \right\} n^2 + \frac{1}{3} \left[ {3 \atop 1} \right] \left\{ {k+1 \atop 3} \right\} n \ + ,,,$$

$$+ \frac{1}{k+1} \left[ {k+1 \atop k+1} \right] \left\{ {k+1 \atop k+1} \right\} n^{k+1} - \frac{1}{k+1} \left[ {k+1 \atop k} \right] \left\{ {k+1 \atop k+1} \right\} n^k + \ldots + (-1)^k \frac{1}{k+1} \left[ {k+1 \atop 1} \right] \left\{ {k+1 \atop k+1} \right\} n \, .$$

We finally come to the third level of sortation.

The first-order Eulerian number $\left\langle {n \atop k} \right\rangle$ is the number of permutations $p_1 p_2 ... p_n$ of the set $\{1,2,...n\}$ that have $k$ ascents, i.e. $k$ places where $p_j < p_{j+1}$. Let us first look at simple examples: $1$ gives $\left\langle {1 \atop 0} \right\rangle = 1$ and $\left\langle {1 \atop 1} \right\rangle = 0$; $21$ gives $\left\langle {2 \atop 0} \right\rangle = 1$; $12$ gives $\left\langle {2 \atop 1} \right\rangle = 1$ and $\left\langle {2 \atop 2} \right\rangle = 0$; $321$ gives $\left\langle {3 \atop 0} \right\rangle = 1$; $132, 213, 231, 312$ gives $\left\langle {3 \atop 1} \right\rangle = 4$; $123$ gives $\left\langle {3 \atop 2} \right\rangle = 1$ and $\left\langle {3 \atop 3} \right\rangle = 0$; $4321$ gives $\left\langle {4 \atop 0} \right\rangle = 1$; $1234$ gives $\left\langle {4 \atop 3} \right\rangle = 1$ and

$\left\langle {4 \atop 4} \right\rangle = 0$; $1432, 2143, 2431, 3142, 3214, 3412, 3241, 4132, 4213, 4231, 4312$ gives

$\left\langle {4 \atop 1} \right\rangle = 11$

and $1243, 1324, 1342, 1423, 2134, 2314, 2341, 2413, 3124, 3412, 4123$ gives $\left\langle {4 \atop 2} \right\rangle = 11$.

In general, for a permutation $p_1 p_2 ... p_{n-1}$ of $\{1,2,...n-1\}$ with $k-1$ ascents, we have two cases to consider.

<u>Case 1</u>. We can insert $n$ into $p_1 p_2 ... p_{n-1}$ either after $p_{n-1}$ or between $p_{j-1}$ and $p_j$ whenever $P_{j-1} > P_j$ to form a permutation of $\{1,2,...n\}$ that increases the number of ascents by 1 so that the total number of permutations of $\{1,2,...n\}$ that have $k$ ascents in this case is $(n-k)\left\langle {n-1 \atop k-1} \right\rangle$.

<u>Case 2</u>. For a permutation $p_1 p_2 ... p_{n-1}$ of $\{1,2,...n-1\}$ with $k$ ascents, we can insert $n$ into $p_1 p_2 ... p_{n-1}$ either before $p_1$ or between $p_{j-1}$ and $p_j$ whenever $p_{j-1} < p_j$ to form a permutation of $\{1,2,...n\}$ that maintains the same number of ascents so that the total number of permutations of $\{1,2,...n\}$ that have $k$ ascents in this case is $(k+1)\left\langle {n-1 \atop k} \right\rangle$.

Therefore, we have proved that

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle, \qquad \text{Eq. 41}$$

Comparing Eqs. 41 and 18, we see that $\left\langle {n \atop k} \right\rangle = e(n, k-1)$ and Eq. 41 can be written as

$$\sum_{i=1}^{n} i^k = \sum_{j=0}^{k-1} \left\langle {k \atop j} \right\rangle \binom{n+j+1}{k+1}. \qquad \text{Eq. 42}$$

Now, by virtue of Eq. 35, we can derive

$$\sum_{i=1}^{n} i^k = \frac{1}{(k+1)!}\sum_{j=0}^{k-1}\left\langle {k \atop j} \right\rangle \sum_{r=1}^{k+1}(-1)^{r-1}\left[ {k+1 \atop k+2-r} \right](n+j+1)^{k+2-r}$$

$$= \frac{1}{(k+1)!}\sum_{j=0}^{k-1}\left\langle {k \atop j} \right\rangle \sum_{r=1}^{k+1}(-1)^{r-1}\left[ {k+1 \atop k+2-r} \right]\sum_{t=0}^{k+2-r}\binom{k+2-r}{k+2-r-t} n^{k+2=r=t}(j+1)^t$$

$$= \frac{1}{(k+1)!}\sum_{j=0}^{k+1}\left\{ \sum_{t=0}^{j}(-1)^r \left[ {k+1 \atop k+1-r} \right]\binom{k+1-r}{k+1-j}\sum_{t=0}^{k-1}(t+1)^{j-r}\left\langle {k \atop t} \right\rangle \right\} n^{k+1-j} \text{ .} \qquad \text{Eq. 43}$$

The second-order Eulerian number $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ is the number of permutations $p_1 p_2 ... p_n$ of the multiset $\{1,1,2,2,...n,n\}$ that have $k$ ascents, i.e. $k$ places where $p_j < p_{j+1}$, provided that all numbers between the two occurrences of $m$ are greater than m for $1 \le m \le n$.

Here are some simple cases: $11$ gives $\left\langle\!\left\langle {1 \atop 0} \right\rangle\!\right\rangle = 1$ and $\left\langle\!\left\langle {1 \atop 1} \right\rangle\!\right\rangle = 0$; $2211$ gives $\left\langle\!\left\langle {2 \atop 0} \right\rangle\!\right\rangle = 1$;

$1122, 12211$ gives $\left\langle\!\left\langle {2 \atop 1} \right\rangle\!\right\rangle = 2$; $332211$ gives $\left\langle\!\left\langle {3 \atop 0} \right\rangle\!\right\rangle = 1$; $112233$ gives $\left\langle\!\left\langle {3 \atop 3} \right\rangle\!\right\rangle = 0$;

$113322, 133221, 221133, 221331, 223311, 233211, 331122, 331221$ gives $\left\langle\!\left\langle {3 \atop 1} \right\rangle\!\right\rangle = 8$ and

$112233, 112332, 122133, 122331, 123321, 133122$ gives $\left\langle\!\left\langle {3 \atop 2} \right\rangle\!\right\rangle = 6$. For a permutation

$p_1 p_2 ... p_{2n-2}$ of $\{1,1,2,2,...n-1,n-1\}$ with $k-1$ ascents, we can insert $n,n$ into

$p_1 p_2 ... p_{2n-2}$ either after $p_{n-1}$ or between $p_{j-1}$ and $p_j$ whenever $P_{j-1} \ge P_j$ to form a permutation of $\{1,1,2,2,...n,n\}$ that increases the number of ascents by 1 so that the total number of permutations of $\{1,1,2,2,...n,n\}$ that have $k$ ascents is $(2n-1-k)\left\langle\!\left\langle {n-1 \atop k} \right\rangle\!\right\rangle$;

whereas for a permutation $p_1 p_2 ... p_{2n-2}$ of $\{1,1,2,2,...n-1,n-1\}$ with $k$ ascents, we can insert $n,n$ into $p_1 p_2 ... p_{2n-2}$ either before $p_1$ or between $p_{j-1}$ and $p_j$ whenever $p_{j-1} < p_j$ to form a permutation of $\{1,1,2,2,...n,n\}$ that maintains the same number of ascents so that the total number of permutations of $\{1,1,2,2,...n,n\}$ that have $k$ ascents is $(k+1)\left\langle\left\langle {n-1 \atop k} \right\rangle\right\rangle$.

Therefore, we have proved that

$$\left\langle\left\langle {n \atop k} \right\rangle\right\rangle = (2n-1-k)\left\langle\left\langle {n-1 \atop k-1} \right\rangle\right\rangle + (k+1)\left\langle\left\langle {n-1 \atop k} \right\rangle\right\rangle,$$

which is equivalent to Eq. 18, since $\left\langle\left\langle {n \atop k} \right\rangle\right\rangle = E(n,k-1)$ .

Furthermore, there is a curious link between Stirling numbers of the second kind and the first-order Eulerian numbers , that is,

$$\sum_{k=1}^{n} k! \left\{ {n \atop k} \right\} = \sum_{k=0}^{n-1} 2^k \left\langle {n \atop k} \right\rangle, \qquad \text{Eq.44}$$

as can be verified via Tables 18 and 12.

For $(a+(i-1)d)_1^\infty$, the Stirling triangle of the first kind $\left[ {n \atop k} \right]_{a;d}$ can be constructed via

$$\left[ {n \atop k} \right]_{a;d} = \left[ {n-1 \atop k-1} \right]_{a;d} + [a+(n-2)d])\left[ {n-1 \atop k} \right]_{a;d} \qquad \text{Eq.45}$$

with $\left[ {n \atop n} \right]_{a;d} = 1$ and the Stirling triangle of the second kind $\left\{ {n \atop k} \right\}_{a;d}$ can be constructed via

$$\left\{\begin{matrix} n \\ k \end{matrix}\right\}_{a;d} = \left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\}_{a;d} + [a+(k-1)d]\left\{\begin{matrix} n-1 \\ k \end{matrix}\right\}_{a;d} \qquad \text{Eq.46}$$

with $\left\{\begin{matrix} n \\ n \end{matrix}\right\}_{a;d} = 1$. On the other hand, Eq.36 can be generalized to

$$\left\{\begin{matrix} n \\ k \end{matrix}\right\}_{a;d} = \sum_{j=1}^{n-k} (-1)^{j-1} \left[\begin{matrix} k+j \\ k \end{matrix}\right]_{a;d} \left\{\begin{matrix} n \\ k+j \end{matrix}\right\}_{a;d} . \qquad \text{Eq. 47}$$

Next, we shall prove

$$\sum_{i=1}^{n} [a+(i-1)d]^k = \sum_{r=0}^{k} \sum_{j=k+1-r}^{k+1} (-1)^{j-k-1+r} \frac{d^{j-1}}{j} \left[\begin{matrix} j \\ k+1-r \end{matrix}\right] \left\{\begin{matrix} k+1 \\ j \end{matrix}\right\}_{a;d} n^{k+1-r} , \qquad \text{Eq. 48}$$

which is the generalization of Eq. 37. By virtue of $(n-j+1)\left(\!\!\left(\begin{matrix} n \\ j-1 \end{matrix}\right)\!\!\right) = \left(\!\!\left(\begin{matrix} n \\ j \end{matrix}\right)\!\!\right)$ and Eq. 46,

we first use mathematical induction to prove

$$\sum_{j=1}^{k+1} d^{j-1} \left\{\begin{matrix} k+1 \\ j \end{matrix}\right\}_{a;d} \left(\!\!\left(\begin{matrix} n \\ j-1 \end{matrix}\right)\!\!\right) = (a+nd)^k \qquad \text{Eq. 49}$$

as follows. Since the inductive basis is trivially true, we only show the inductive step.

$$\begin{aligned}
&\sum_{j=1}^{k+2} d^{j-1} \left\{\begin{matrix} k+2 \\ j \end{matrix}\right\}_{a;d} \left(\!\!\left(\begin{matrix} n \\ j-1 \end{matrix}\right)\!\!\right) \\
&= \sum_{j=1}^{k+1} d^{j-1} \left([a+(j-1)d]\left\{\begin{matrix} k+1 \\ j \end{matrix}\right\}_{a;d} + \left\{\begin{matrix} k+1 \\ j-1 \end{matrix}\right\}_{a;d}\right) \left(\!\!\left(\begin{matrix} n \\ j-1 \end{matrix}\right)\!\!\right) + d^{k+1} \left(\!\!\left(\begin{matrix} n \\ k+1 \end{matrix}\right)\!\!\right) \\
&= \sum_{j=1}^{k+1} a d^{j-1} \left\{\begin{matrix} k+1 \\ j \end{matrix}\right\}_{a;d} \left(\!\!\left(\begin{matrix} n \\ j-1 \end{matrix}\right)\!\!\right) + \sum_{j=1}^{k+1} d^{j}(j-1) \left\{\begin{matrix} k+1 \\ j \end{matrix}\right\}_{a;d} \left(\!\!\left(\begin{matrix} n \\ j-1 \end{matrix}\right)\!\!\right) + \sum_{j=1}^{k+1} d^{j-1} \left\{\begin{matrix} k+1 \\ j-1 \end{matrix}\right\}_{a;d} \left(\!\!\left(\begin{matrix} n \\ j-1 \end{matrix}\right)\!\!\right) + d^{k+1} \left(\!\!\left(\begin{matrix} n \\ k+1 \end{matrix}\right)\!\!\right) \\
&= \sum_{j=1}^{k+1} (a+nd) d^{j-1} \left\{\begin{matrix} k+1 \\ j \end{matrix}\right\}_{a;d} \left(\!\!\left(\begin{matrix} n \\ j-1 \end{matrix}\right)\!\!\right) - \sum_{j=1}^{k+1} d^{j} \left\{\begin{matrix} k+1 \\ j \end{matrix}\right\}_{a;d} \left(\!\!\left(\begin{matrix} n \\ j \end{matrix}\right)\!\!\right) + \sum_{j=1}^{k+1} d^{j-1} \left\{\begin{matrix} k+1 \\ j-1 \end{matrix}\right\}_{a;d} \left(\!\!\left(\begin{matrix} n \\ j-1 \end{matrix}\right)\!\!\right) + d^{k+1} \left(\!\!\left(\begin{matrix} n \\ k+1 \end{matrix}\right)\!\!\right) \\
&= (a+nd) \sum_{j=1}^{k+1} d^{j-1} \left\{\begin{matrix} k+1 \\ j \end{matrix}\right\}_{a;d} \left(\!\!\left(\begin{matrix} n \\ j-1 \end{matrix}\right)\!\!\right) - \sum_{j=1}^{k+1} d^{j} \left\{\begin{matrix} k+1 \\ j \end{matrix}\right\}_{a;d} \left(\!\!\left(\begin{matrix} n \\ j \end{matrix}\right)\!\!\right) + \sum_{j=1}^{k} d^{j} \left\{\begin{matrix} k+1 \\ j \end{matrix}\right\}_{a;d} \left(\!\!\left(\begin{matrix} n \\ j \end{matrix}\right)\!\!\right) + d^{k+1} \left(\!\!\left(\begin{matrix} n \\ k+1 \end{matrix}\right)\!\!\right) \\
&= (a+nd)^{k+1} .
\end{aligned}$$

We can now use mathematical induction to prove Eq. 48 via Eq. 49:

$$\begin{aligned}
&\sum_{j=1}^{k+1}\frac{d^{\,j-1}}{j}\left\{{k+1 \atop j}\right\}_{a;d}\left(\!\!\left({n+1 \atop j}\right)\!\!\right)\\
&=\sum_{j=1}^{k+1}\frac{d^{\,j-1}}{j}\left\{{k+1 \atop j}\right\}_{a;d}\left[\left(\!\!\left({n \atop j}\right)\!\!\right)+j\left(\!\!\left({n \atop j-1}\right)\!\!\right)\right]\\
&=\sum_{j=1}^{k+1}\frac{d^{\,j-1}}{j}\left\{{k+1 \atop j}\right\}_{a;d}\left(\!\!\left({n \atop j}\right)\!\!\right)+\sum_{j=1}^{k+1}d^{\,j-1}\left\{{k+1 \atop j}\right\}_{a;d}\left(\!\!\left({n \atop j-1}\right)\!\!\right)\\
&=\sum_{i=1}^{n}[a+(i-1)d]^k+(a+nd)^k\\
&=\sum_{i=1}^{n+1}[a+(i-1)d]^k\ .
\end{aligned}$$

Finally, we can obtain Eq. 49 by regrouping the following display of Eq. 48.

$$\begin{aligned}
&\left[{1 \atop 1}\right]\left\{{k+1 \atop 1}\right\}_{a;d} n+\frac{d}{2}\left[{2 \atop 2}\right]\left\{{k+1 \atop 2}\right\}_{a;d} n^2-\frac{d}{2}\left[{2 \atop 1}\right]\left\{{k+1 \atop 2}\right\}_{a;d} n\\
&+\frac{d^2}{3}\left[{3 \atop 3}\right]\left\{{k+1 \atop 3}\right\}_{a;d} n^3-\frac{d^2}{3}\left[{3 \atop 2}\right]\left\{{k+1 \atop 3}\right\}_{a;d} n^2+\frac{d^2}{3}\left[{3 \atop 1}\right]\left\{{k+1 \atop 3}\right\}_{a;d} n+\dots\\
&+\frac{d^k}{k+1}\left[{k+1 \atop k+1}\right]\left\{{k+1 \atop k+1}\right\}_{a;d} n^{k+1}-\frac{d^k}{k+1}\left[{k+1 \atop k}\right]\left\{{k+1 \atop k+1}\right\}_{a;d} n^k+\dots+(-1)^k\frac{d^k}{k+1}\left[{k+1 \atop 1}\right]\left\{{k+1 \atop k+1}\right\}_{a;d} n.
\end{aligned}$$

Based on $\left[{1 \atop 0}\right]_{a;d}=0$ and $\left[{1 \atop 1}\right]_{a;d}=1$, we can use Eq. 45 to tabulate $\left[{n \atop k}\right]_{a;d}$ in Table 19.

| $n/k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | | | | |
| 2 | $a$ | 1 | | | |
| 3 | $a(a+d)$ | $2a+d$ | 1 | | |
| 4 | $a(a+d)(a+2d)$ | $3a^2+6ad+2d^2$ | $3a+3d$ | 1 | |
| 5 | $a(a+d)(a+2d)(a+3d)$ | $4a^3+18a^2d+22ad^2+6d^3$ | $6a^2+18ad+11d^2$ | $4a+6d$ | 1 |

Table 19. Table for general Stirling numbers of the first kind $\left[{n \atop k}\right]_{a;d}$

Next, we shall come up with the second-order Stirling numbers of the second kind in the same manner. Based on $\left\{ {1 \atop 0} \right\}_{a;d} = 0$ and $\left\{ {1 \atop 1} \right\}_{a;d} = 1$, we can use Eq. 46 to tabulate $\left\{ {n \atop k} \right\}_{a;d}$ in Table 20.

| $n/k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | | | | |
| 2 | $a$ | 1 | | | |
| 3 | $a^2$ | $2a+d$ | 1 | | |
| 4 | $a^3$ | $3a^2+3ad+d^2$ | $3a+3d$ | 1 | |
| 5 | $a^4$ | $4a^3+6a^2d+4ad^2+d^3$ | $6a^2+12ad+7d^2$ | $4a+6d$ | 1 |

Table 20. Table for general Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_{a;d}$

Lastly, we shall generalize Eulerian numbers $\left\langle {n \atop k} \right\rangle$ and $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ for $(a+(n-1)d)_{n=1}^{\infty}$. It is quite easy to derive

$$\sum_{i=1}^{n}[a+(i-1)d] = (d-a)\binom{n}{2} + a\binom{n+1}{2}, \qquad \text{Eq. 50}$$

$$\sum_{i=1}^{n}[a+(i-1)d]^2 = (d-a)^2\binom{n}{3} + (-2a^2+2ad+d^2)\binom{n+1}{3} + a^2\binom{n+2}{3}, \qquad \text{Eq. 51}$$

and

$$\sum_{i=1}^{n}[a+(i-1)d]^3 = (d-a)^3\binom{n}{4} + (3a^3-6a^2d+4d^3)\binom{n+1}{4}$$
$$+(-3a^3+3a^2d+3ad^2+d^3)\binom{n+2}{4} + a^3\binom{n+3}{4}. \qquad \text{Eq. 52}$$

Now, we can define $\genfrac{\langle}{\rangle}{0pt}{}{n}{k}_{a;d}$ according to Eqs. 50-52 analogous to $\genfrac{\langle}{\rangle}{0pt}{}{n}{k}$. By virtue of Eq. 50, we define $\genfrac{\langle}{\rangle}{0pt}{}{1}{-1}_{a;d} = d - a$ and $\genfrac{\langle}{\rangle}{0pt}{}{1}{0}_{a;d} = a$ so that $\genfrac{\langle}{\rangle}{0pt}{}{1}{-1} = \genfrac{\langle}{\rangle}{0pt}{}{1}{-1}_{1;1} = 0$. Unlike $\genfrac{\langle}{\rangle}{0pt}{}{n}{k}$, we start with $k = -1$ for $\genfrac{\langle}{\rangle}{0pt}{}{n}{k}_{a;d}$. Due to Eqs. 51 and 52, we can define $\genfrac{\langle}{\rangle}{0pt}{}{2}{-1}_{a;d} = (d - a)^2$, $\genfrac{\langle}{\rangle}{0pt}{}{2}{0}_{a;d} = -2a^2 + 2ad + d^2$, $\genfrac{\langle}{\rangle}{0pt}{}{2}{1}_{a;d} = a^2$, $\genfrac{\langle}{\rangle}{0pt}{}{3}{-1}_{a;d} = (d - a)^3$, $\genfrac{\langle}{\rangle}{0pt}{}{3}{0}_{a;d} = 3a^3 - 6a^2 d + 4d^3$, $\genfrac{\langle}{\rangle}{0pt}{}{3}{1}_{a;d} = -3a^3 + 3a^2 d + 3ad^2 + d^3$ and $\genfrac{\langle}{\rangle}{0pt}{}{3}{2}_{a;d} = a^3$. Thus we have generalized Eq. 42 up to $k = 3$, which is sufficient for us to generalize Eq. 41.

For our purpose, let us first define $\genfrac{\langle}{\rangle}{0pt}{}{0}{-1}_{a;d} = 0$. Then we write $\genfrac{\langle}{\rangle}{0pt}{}{2}{-1}_{a;d} = (-a + d)\genfrac{\langle}{\rangle}{0pt}{}{1}{-1}_{a;d}$, $\genfrac{\langle}{\rangle}{0pt}{}{2}{0}_{a;d} = (a + d)\genfrac{\langle}{\rangle}{0pt}{}{1}{-1}_{a;d} + (-a + 2d)\genfrac{\langle}{\rangle}{0pt}{}{1}{0}_{a;d}$, $\genfrac{\langle}{\rangle}{0pt}{}{2}{1}_{a;d} = a\genfrac{\langle}{\rangle}{0pt}{}{1}{0}_{a;d}$, $\genfrac{\langle}{\rangle}{0pt}{}{3}{-1}_{a;d} = (-a + d)\genfrac{\langle}{\rangle}{0pt}{}{2}{-1}_{a;d}$, $\genfrac{\langle}{\rangle}{0pt}{}{3}{0}_{a;d} = (a + 2d)\genfrac{\langle}{\rangle}{0pt}{}{2}{-1}_{a;d} + (-a + 2d)\genfrac{\langle}{\rangle}{0pt}{}{2}{0}_{a;d}$, $\genfrac{\langle}{\rangle}{0pt}{}{3}{1}_{a;d} = (a + d)\genfrac{\langle}{\rangle}{0pt}{}{2}{0}_{a;d} + (-a + 3d)\genfrac{\langle}{\rangle}{0pt}{}{2}{1}_{a;d}$ and $\genfrac{\langle}{\rangle}{0pt}{}{3}{2}_{a;d} = a\genfrac{\langle}{\rangle}{0pt}{}{2}{1}_{a;d}$.

As we can check, the above are the special cases of

$$\genfrac{\langle}{\rangle}{0pt}{}{n}{k}_{a;d} = [a + (n - k - 1)d]\genfrac{\langle}{\rangle}{0pt}{}{n-1}{k-1}_{a;d} + [-a + (k + 2)d]\genfrac{\langle}{\rangle}{0pt}{}{n-1}{k}_{a;d}, \qquad \text{Eq. 53}$$

which is the generalization of Eq. 36 and can be used to tabulate $\genfrac{\langle}{\rangle}{0pt}{}{n}{k}_{a;d}$ in Table 21.

| $n \backslash k$ | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| 0 | 0 | | | |
| 1 | $d-a$ | $a$ | | |
| 2 | $(d-a)^2$ | $-2a^2+2ad+d^2$ | $a^2$ | |
| 3 | $(d-a)^3$ | $3a^3-6a^2d+4d^3$ | $-3a^3+3a^2d+3ad^2+d^3$ | $a^3$ |

Table 21. Table for general first order Eulerian numbers $\left\langle {n \atop k} \right\rangle_{a;d}$

Moreover, we can write

$$[a+(n-1)d] = (d-a)\binom{n-1}{1} + a\binom{n}{1},$$

$$[a+(n-1)d]^2 = (d-a)^2\binom{n-1}{2} + (-2a^2+2ad+d^2)\binom{n}{2} + a^2\binom{n+1}{2},$$

$$[a+(n-1)d]^3 = (d-a)^3\binom{n-1}{3} + (3a^3-6a^2d+4d^3)\binom{n}{3}$$

$$+(-3a^3+3a^2d+3ad^2+d^3)\binom{n+1}{3} + a^3\binom{n+2}{3}$$

and in general

$$[a+(n-1)d]^k = \sum_{j=-1}^{k-1} \left\langle {k \atop j} \right\rangle_{a;d} \binom{n+j}{k},$$

which the generalization of Eq. 19 since $\left\langle {n \atop k} \right\rangle = e(n,k-1)$ . Therefore, the proof of Eq. 20

can be generalized to prove

$$\sum_{i=1}^{n}[a+(i-1)d]^k = \sum_{j=-1}^{k-1} \left\langle {k \atop j} \right\rangle_{a;d} \binom{n+j+1}{k+1}.$$

By recalling $\begin{bmatrix} n \\ k \end{bmatrix} = s(n, n-k)$ and $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle = E(n, k-1)$, we generalize Eq. 51 as follows.

Since $\begin{bmatrix} n \\ n-1 \end{bmatrix}_{a;d} = \sum_{i=1}^{n-1} [a + (i-1)d] = (d-a)\binom{n-1}{2} + a\binom{n}{2}$, to generalize Eq. 51 we assume

$\begin{bmatrix} n \\ n-2 \end{bmatrix}_{a;d} = x\binom{n-1}{4} + y\binom{n}{4} + z\binom{n+1}{4}$. Taking $n = 1,2,3$, we have

$z = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{a;d} = a^2 + ad$,

$y + 5z = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{a;d} = 3a^2 + 6ad + 2d^2$ and $x + 5y + 15z = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{a;d} = 6a^2 + 18ad + 11d^2$ so that

$x = (d-a)^2$, $y = -2a^2 + ad + 2d^2$ and $z = a(a+d)$. Thus, we have arrived at

$$\begin{bmatrix} n \\ n-2 \end{bmatrix}_{a;d} = (d-a)^2\binom{n-1}{4} + (-2a^2 + ad + 2d^2)\binom{n}{4} + a(a+d)\binom{n+1}{4}.$$

Likewise, we can obtain

$$\begin{bmatrix} n \\ n-3 \end{bmatrix}_{a;d} = (d-a)^3\binom{n-1}{6} + (3a^3 - 3a^2d - 7ad^2 + 8d^3)\binom{n}{6}$$

$$+ (-3a^3 - 3a^2d + 8ad^2 + 6d^3)\binom{n+1}{6} + a(a+d)(a+2d)\binom{n+2}{6}.$$

By defining $\left\langle\!\left\langle {0 \atop -1} \right\rangle\!\right\rangle_{a;d} = 1$, $\left\langle\!\left\langle {1 \atop -1} \right\rangle\!\right\rangle_{a;d} = d - a$ and $\left\langle\!\left\langle {1 \atop 0} \right\rangle\!\right\rangle_{a;d} = a$, we can use

$$\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle_{a;d} = [a + (2n-2-k)d]\left\langle\!\left\langle {n-1 \atop k-1} \right\rangle\!\right\rangle_{a;d} + [-a + (k+2)d]\left\langle\!\left\langle {n-1 \atop k} \right\rangle\!\right\rangle_{a;d} \quad \text{Eq. 54}$$

to tabulate $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle_{a;d}$ in Table 22.

| $n \backslash k$ | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| 0 | 1 | | | |
| 1 | $d-a$ | $a$ | | |
| 2 | $(d-a)^2$ | $-2a^2+ad+2d^2$ | $a(a+d)$ | |
| 3 | $(d-a)^3$ | $3a^3-3a^2d-7ad^2+8d^3$ | $-3a^3-3a^2d+8ad^2+6d^3$ | $a(a+d)(a+2d)$ |

Table 22. Table for general second order Eulerian numbers $\left\langle\left\langle {n \atop k} \right\rangle\right\rangle_{a;d}$

Accordingly, we can derive $\begin{bmatrix} n \\ n-k \end{bmatrix}_{a;d} = \sum_{j=-1}^{k-1} \left\langle\left\langle {k \atop j} \right\rangle\right\rangle_{a;d} \binom{n+j}{2k}$ so that $\left\langle\left\langle {k \atop j} \right\rangle\right\rangle_{a;d}$ is the second-order Eulerian number for $(a+(i-1)d)_0^\infty$.

### 2.2.5 Bell

Let us now consider the ordered Bell polynomial

$$F_n(a,d) = \sum_{k=1}^{n} d^k k! \left\{ {n \atop k} \right\}_{a;d} \quad \text{for } n > 1 \qquad \text{Eq. 55}$$

and the Eulerian Bell polynomial

$$E_n(a,d) = \sum_{k=0}^{n-1} 2^k \left\langle {n \atop k} \right\rangle_{a;d} \quad \text{for } n > 1. \qquad \text{Eq. 56}$$

Can we generalize Eq. 44 for $n > 1$? We shall see that $F_n(a,d) = E_n(a,d)$ only when $a = d$. For our purpose, let us define the difference Bell polynomial $D_n(a,d)$ to be

$$D_n(a,d) = F_n(a,d) - E_n(a,d). \qquad \text{Eq. 57}$$

The Bell number $B_n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}$ satisfying $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$ can be generalized to

$$B_n(a,d) = \sum_{k=1}^{n} d^{k-1} \left\{ {n \atop k} \right\}_{a;d}$$

as follows.

$$B_1(a,d) = 1,$$

$$B_2(a,d) = a + d,$$

$$B_3(a,d) = a^2 + 2ad + 2d^2,$$

$$B_4(a,d) = a^3 + 3a^2d + 6ad^2 + 5d^3$$

so that

$$B_1(a,d) = 1 = B_0,$$

$$B_2(a,d) = aB_0 + dB_1,$$

$$B_3(a,d) = a^2B_0 + 2adB_1 + d^2B_2,$$

$$B_4(a,d) = a^3B_0 + 3a^2dB_1 + 3ad^2B_2 + d^3B_3$$

and in general,

$$B_{n+1}(a,d) = \sum_{k=0}^{n} a^{n-k} d^k \binom{n}{k} B_k .$$

Finally, we use Eq. 55 to find

$$F_2(a,d) = ad + 2d^2,$$

$$F_3(a,d) = a^2d + 4ad^2 + 8d^3,$$

$$F_4(a,d) = a^3d + 6a^2d^2 + 24ad^3 + 44d^4$$

and use Eq. 56 to find

$$E_2(a,d) = 2ad + d^2,$$

$$E_3(a,d) = a^3 + 6ad^2 + 6d^3,$$

$$E_4(a,d) = 4a^3d + 6a^2d^2 + 28ad^3 + 37d^4$$

In the same fashion, we can use Eq. 57 to obtain

$$D_2(a,d) = (d-a)d \ ,$$

$$D_3(a,d) = (d-a)(a^2 + 2d^2)$$

$$D_4(a,d) = (d-a)(3a^2d + 3ad^2 + 7d^3) \ .$$

$$D_5(a,d) = (d-a)(a^4 + 12a^2d^2 + 24ad^3 + 38d^4) \ ,$$

$$D_6(a,d) = (d-a)(5a^4d + 10a^3d^2 + 70a^2d^3 + 185ad^4 + 271d^5) \ ,$$

$$D_7(a,d) = (d-a)(a^6 + 30a^4d^2 + 120a^3d^3 + 570a^2d^4 + 1620ad^5 + 2342d^6)$$

and in general, we propose the following

**CONJECTURE** $D_n(a,d) = (d-a)[(d-a)^{n-1} + E_{n-1}(a,d)]$. Eq. 58

2.2.6 General triangular arrays

We shall further rewriting Eqs. 45, 46, 53 and 54 as

$$\left[\begin{matrix} n \\ k \end{matrix}\right]_{(a_i)_1^\infty} = [a + (n-2)d]\left[\begin{matrix} n-1 \\ k \end{matrix}\right]_{(a_i)_1^\infty} + \left[\begin{matrix} n-1 \\ k \end{matrix}\right]_{(a_i)_1^\infty} = a_{n-1}\left[\begin{matrix} n-1 \\ k \end{matrix}\right]_{(a_i)_1^\infty} + \left[\begin{matrix} n-1 \\ k \end{matrix}\right]_{(a_i)_1^\infty} , \qquad \text{Eq. 59}$$

$$\left\{\begin{matrix} n \\ k \end{matrix}\right\}_{(a_i)_1^\infty} = \left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\}_{(a_i)_1^\infty} + [a + (k-1)d]\left\{\begin{matrix} n-1 \\ k \end{matrix}\right\}_{(a_i)_1^\infty} = \left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\}_{(a_i)_1^\infty} + a_k\left\{\begin{matrix} n-1 \\ k \end{matrix}\right\}_{(a_i)_1^\infty} , \qquad \text{Eq. 60}$$

$$\left\langle\begin{matrix} n \\ k \end{matrix}\right\rangle_{(a_i)_0^\infty} = a_{n-k}\left\langle\begin{matrix} n-1 \\ k-1 \end{matrix}\right\rangle_{(a_i)_0^\infty} + [(n+k)a_n - (n+1+k)a_{n-1}]\left\langle\begin{matrix} n-1 \\ k \end{matrix}\right\rangle_{(a_i)_0^\infty}$$

$$+ [(n+k)a_n - (n+1+k)a_{n-1}]\left\langle\!\!\left\langle\begin{matrix} n-1 \\ k \end{matrix}\right\rangle\!\!\right\rangle_{(a_i)_0^\infty} , \qquad \text{Eq. 61}$$

$$\left\langle\!\!\left\langle\begin{matrix} n \\ k \end{matrix}\right\rangle\!\!\right\rangle_{(a_i)_0^\infty} = [(n-k)a_n - (n-1-k)a_{n-1}]\left\langle\!\!\left\langle\begin{matrix} n-1 \\ k-1 \end{matrix}\right\rangle\!\!\right\rangle_{(a_i)_0^\infty}$$

$$+ [(n+k)a_n - (n+1+k)a_{n-1}]\left\langle\!\!\left\langle\begin{matrix} n-1 \\ k \end{matrix}\right\rangle\!\!\right\rangle_{(a_i)_0^\infty} , \qquad \text{Eq. 62}$$

where $a_i = a + (i-1)d$ . In the same token, Eqs. 28 and 29 can be written as

$$\binom{n}{k}_{(1)_0^\infty} = \binom{n-1}{k}_{(1)_0^\infty} + \binom{n-1}{k-1}_{(1)_0^\infty}; \qquad \text{Eq. 63}$$

$$\left(\binom{n}{k}\right)_{(i)_0^\infty} = 0\left(\binom{n-1}{k}\right)_{(i)_0^\infty} + k\left(\binom{n-1}{k-1}\right)_{(i)_0^\infty}. \qquad \text{Eq. 64}$$

More generally, a triangular array $T^{r,s}{}_{(a_i)_0^\infty}$ for $(a_i)_0^\infty$ can be defined as $T^{r,s}{}_{(a_i)_0^\infty} = 0$ for $k \le -2$ and $k \ge n$, $T^{r,s}{}_{(a_i)_0^\infty}(1,-1) = -a_0$, $T^{r,s}{}_{(a_i)_0^\infty}(1,0) = a_1$ and

$$T^{r,s}{}_{(a_i)_0^\infty}(n,k) = M(r)T^{r,s}{}_{(a_i)_0^\infty}(n-1,k-1) + M(s)T^{r,s}{}_{(a_i)_0^\infty}(n-1,k), \qquad \text{Eq. 65}$$

where $M(r)$ and $M(s)$ can be taken the following model list.

$M(0) = 0$, $M(1) = 1$, $M(2) = a_{n-1}$, $M(3) = a_k$, $M(4) = a_{n-k}$,

$M(5) = (n+k)a_n - (n+1+k) - (n+1+k)a_{n-1}$,

$$M(6) = (n-k)a_n - (n-1-k)a_{n-1} \qquad \text{Eq. 66}$$

So, Eqs. 63 and 64 can be rewritten as $\binom{n}{k} = T^{1,1}{}_{(1)_1^\infty}(n,k)$ and $\left(\binom{n}{k}\right) = T^{0,2}{}_{(i)_1^\infty}(n,k)$,

where $(1)_1^\infty$ is the unity sequence and $(i)_1^\infty$ is the natural sequence. Moreover, Eqs. 59-62 can be rewritten as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(a_i)_1^\infty} = T^{1,2}_{(a_i)_1^\infty}(n, k-1),$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{(a_i)_1^\infty} = T^{1,3}_{(a_i)_1^\infty}(n, k-1),$$

$$\left\langle {n \atop k} \right\rangle_{(a_i)_1^\infty} = T^{3,4}_{(a_i)_0^\infty}(n,k),$$

$$\left\langle\left\langle {n \atop k} \right\rangle\right\rangle_{(a_i)_0^\infty} = T^{5,4}_{(a_i)_0^\infty}(n,k).$$

2.2.7 Gauss

To close out, let us consider Stirling numbers based on the sequence $\left(q^{i-1}\right)_1^{\infty}$ with $q \neq 0$.

First, we look at $\begin{bmatrix} n \\ k \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}}$. Based on $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = 0$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = 1$, we can derive from

Eq. 59 for $\left(q^{i-1}\right)_1^{\infty}$, namely

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = \begin{bmatrix} n-1 \\ k \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} + q^{n-2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}},$$

the following:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = q^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = 1, \begin{bmatrix} 2 \\ 2 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = 1,$$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = q^1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = q, \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} + q^1 \begin{bmatrix} 2 \\ 2 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = 1 + q, \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = 1,$$

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = q^2, \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = q + q^2 + q^3, \begin{bmatrix} 4 \\ 3 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = 1 + q + q^2, \begin{bmatrix} 4 \\ 4 \end{bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = 1, \ldots$$

and in general

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(q^{i-1})_1^{\infty}} = q^{\binom{n-k}{2}} \prod_{i=1}^{k-1} \frac{1-q^{n-k+i}}{1-q^i} = q^{\binom{n-k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q,$$

where $\begin{bmatrix} n-1 \\ k \end{bmatrix}_q$ is known to be a $q$ - Gaussian coefficient. Likewise, we can use Eq. 63 for

$\left(q^{i-1}\right)_1^{\infty}$, namely $\begin{Bmatrix} n \\ k \end{Bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} = \binom{k+1}{2} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}_{\left(q^{i-1}\right)_1^{\infty}} + \begin{Bmatrix} n-1 \\ k \end{Bmatrix}_{\left(q^{i-1}\right)_1^{\infty}}$

to arrive at

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{(q^{i-1})_1^{\infty}} = \prod_{i=1}^{k-1} \frac{1-q^{n-k+i}}{1-q^i} = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.$$

In conclusion, the properties of unimodal and log concave in (36) can be generalized.

## 2.3. MULTIPLE ANGLE FORMULAS IN TRIGONOMETRY

We shall derive the general formula for $\tan m\theta$, which is hitherto not known.

We can derive

$$\tan(2n-1)\theta = \frac{\sum_{k=1}^{n}(-1)^{k-1}\binom{2n-1}{2k-1}\tan^{2k-1}\theta}{\sum_{k=1}^{n}(-1)^{k-1}\binom{2n-1}{2k-2}\tan^{2k-2}\theta} \qquad \text{Eq. 67}$$

and

$$\tan 2n\theta = \frac{\sum_{k=1}^{n}(-1)^{k-1}\binom{2n}{2k-1}\tan^{2k-1}\theta}{\sum_{k=0}^{n}(-1)^{k}\binom{2n}{2k}\tan^{2k}\theta} \qquad \text{Eq. 68}$$

by using

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}. \qquad \text{Eq. 69}$$

Although the general formulas for $\sin m\theta$ and $\cos m\theta$ have been known due to De Moivre's Theorem, we can use Eq. 67 to derive

$$\cos(2n-1)\theta = \sum_{k=0}^{n-1}\sum_{i=k+1}^{n}(-1)^{k}\binom{i-1}{k}\binom{2n-1}{2i-2}\cos^{2n-2k-1}\theta \qquad \text{Eq. 70}$$

and

$$\sin(2n-1)\theta = \sum_{k=0}^{n-1}\sum_{i=k+1}^{n}(-1)^{k}\binom{i-1}{k}\binom{2n-1}{2i-2}\sin^{2n-2k-1}\theta; \qquad \text{Eq. 71}$$

and use Eq. 68 to derive, via combinatorial method,

$$\cos 2n\theta = \sum_{k=0}^{n}\sum_{i=k+1}^{n+1}(-1)^{k}\binom{i-1}{k}\binom{2n}{2i-2}\cos^{2n-2k}\theta \qquad \text{Eq. 72}$$

and

$$\sin 2n\theta = \cos\theta \sum_{k=0}^{n-1} \sum_{i=k+1}^{n} (-1)^k \binom{i-1}{k} \binom{2n}{2i-1} \sin^{2n-2k-1}\theta \qquad \text{Eq. 73}$$

Since

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta},$$

we can use Eq. 69 to derive

$$\tan 3\theta = \frac{2\tan\theta + (1-\tan^2\theta)\tan\theta}{(1-\tan^2\theta) - 2\tan\theta} = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = \frac{\binom{3}{1}\tan\theta - \binom{3}{3}\tan^3\theta}{\binom{3}{0} - \binom{3}{2}\tan^2\theta}$$

and in turn

$$\tan 5\theta = \frac{2\tan\theta\left[\binom{3}{0} - \binom{3}{2}\tan^2\theta\right] + (1-\tan^2\theta)\left[\binom{3}{1}\tan\theta - \binom{3}{3}\tan^3\theta\right]}{(1-\tan^2\theta)\left[\binom{3}{0} - \binom{3}{2}\tan^2\theta\right] - 2\tan\theta\left[\binom{3}{1}\tan\theta - \binom{3}{3}\tan^3\theta\right]}$$

$$= \frac{5\tan\theta - \left[2\binom{3}{2} + \binom{3}{3} + \binom{3}{1}\right]\tan^3\theta + \tan^5\theta}{1 - \left[2\binom{3}{1} + \binom{3}{2} + \binom{3}{0}\right]\tan^2\theta + 5\tan^4\theta}$$

$$= \frac{\binom{5}{1}\tan\theta - \binom{5}{3}\tan^3\theta + \binom{5}{5}\tan^5\theta}{\binom{5}{0} - \binom{5}{2}\tan^2\theta + \binom{5}{4}\tan^4\theta},$$

due to $2\binom{l}{r} + \binom{l}{r+1} + \binom{l}{r-1} = \binom{l+2}{r+1}$.

In general, we can derive

$\tan(2n-1)\theta$

$$= \frac{2\tan\theta\sum_{k=1}^{n-1}(-1)^{k-1}\binom{2n-3}{2k-2}\tan^{2k-2}\theta + (1-\tan^2\theta)\sum_{k=1}^{n-1}(-1)^{k-1}\binom{2n-3}{2k-1}\tan^{2k-1}\theta}{(1-\tan^2\theta)\sum_{k=1}^{n-1}(-1)^{k-1}\binom{2n-3}{2k-2}\tan^{2k-2}\theta - 2\tan\theta\sum_{k=1}^{n-1}(-1)^{k-1}\binom{2n-3}{2k-1}\tan^{2k-1}\theta}$$

$$= \frac{(2n-1)\tan\theta + \sum_{k=2}^{n-1}(-1)^{k-1}\left[2\binom{2n-3}{2k-2}+\binom{2n-3}{2k-1}+\binom{2n-3}{2k-3}\right]\tan^{2k-1}\theta + (-1)^{n-1}\tan^{2n-1}\theta}{1+\sum_{k=2}^{n-1}(-1)^{k-1}\left[2\binom{2n-3}{2k-3}+\binom{2n-3}{2k-2}+\binom{2n-3}{2k-4}\right]\tan^{2k-2}\theta + (-1)^{n-1}(2n-1)\tan^{2n-2}\theta},$$

which leads to Eq. 67.

Similarly, we can derive

$$\tan 4\theta = \frac{\binom{4}{1}\tan\theta - \binom{4}{3}\tan^3\theta}{\binom{4}{0} - \binom{4}{2}\tan^2\theta + \binom{4}{4}\tan^4\theta}$$

and in turn

$$\tan 6\theta = \frac{2\tan\theta\left[\binom{4}{0}-\binom{4}{2}\tan^2\theta+\binom{4}{4}\tan^4\theta\right] + (1-\tan^2\theta)\left[\binom{4}{1}\tan\theta - \binom{4}{3}\tan^3\theta\right]}{(1-\tan^2\theta)\left[\binom{4}{0}-\binom{4}{2}\tan^2\theta+\binom{4}{4}\tan^4\theta\right] - 2\tan\theta\left[\binom{4}{1}\tan\theta - \binom{4}{3}\tan^3\theta\right]}$$

$$= \frac{6\tan\theta - \left[2\binom{4}{2}+\binom{4}{3}+\binom{4}{1}\right]\tan^3\theta + 6\tan^5\theta}{1-\left[2\binom{4}{1}+\binom{4}{2}+\binom{4}{0}\right]\tan^2\theta + \left[2\binom{4}{3}+\binom{4}{4}+\binom{4}{2}\right]\tan^4\theta - \tan^6\theta}$$

$$= \frac{\binom{6}{1}\tan\theta - \binom{6}{3}\tan^3\theta + \binom{6}{5}\tan^5\theta}{\binom{6}{0} - \binom{6}{2}\tan^2\theta + \binom{6}{4}\tan^4\theta - \binom{6}{6}\tan^6\theta}. \qquad \text{Eq. 74}$$

In general, we can derive

$$\tan 2n\theta$$

$$= \frac{2\tan\theta\sum_{k=0}^{n-1}(-1)^k\binom{2n-2}{2k}\tan^{2k}\theta+(1-\tan^2\theta)\sum_{k=1}^{n-1}(-1)^{k-1}\binom{2n-2}{2k-1}\tan^{2k-1}\theta}{(1-\tan^2\theta)\sum_{k=0}^{n-1}(-1)^k\binom{2n-2}{2k}\tan^{2k}\theta-2\tan\theta\sum_{k=1}^{n-1}(-1)^{k-1}\binom{2n-2}{2k-1}\tan^{2k-1}\theta}$$

$$= \frac{2n\tan\theta+\sum_{k=1}^{n-2}(-1)^k\left[2\binom{2n-2}{2k}+\binom{2n-2}{2k+1}+\binom{2n-2}{2k-1}\right]\tan^{2k-1}\theta+(-1)^{n-1}2n\tan^{2n-1}\theta}{1+\sum_{k=2}^{n-1}(-1)^{k-1}\left[2\binom{2n-3}{2k-3}+\binom{2n-3}{2k-2}+\binom{2n-3}{2k-4}\right]\tan^{2k-2}\theta+(-1)^{n-1}(2n-1)\tan^{2n-2}\theta},$$

which leads to Eq. 68. Before deriving Eq. 70, we first look at the case when $n=4$.

From Eq. 67, we can derive

$$\tan 7\theta = \frac{\binom{7}{1}\tan\theta-\binom{7}{3}\tan^3\theta+\binom{7}{5}\tan^5\theta-\binom{7}{7}\tan^7\theta}{\binom{7}{0}-\binom{7}{2}\tan^2\theta+\binom{7}{4}\tan^4\theta-\binom{7}{6}\tan^6\theta}$$

$$= \tan\theta\frac{\binom{7}{1}-\binom{7}{3}(\sec^2\theta-1)+\binom{7}{5}(\sec^2\theta-1)^2-\binom{7}{7}(\sec^2\theta-1)^3}{\binom{7}{0}-\binom{7}{2}(\sec^2\theta-1)+\binom{7}{4}(\sec^2\theta-1)^2-\binom{7}{6}(\sec^2\theta-1)^3}$$

$$= \frac{\sin\theta\sum_{k=0}^{3}\sum_{i=k+1}^{4}(-1)^k\binom{i-1}{k}\binom{7}{2i-1}\sec^{2k}\theta}{\cos\theta\sum_{k=0}^{3}\sum_{i=k+1}^{4}(-1)^k\binom{i-1}{k}\binom{7}{2i-2}\sec^{2k}\theta}$$

$$= \frac{\sin\theta\sum_{k=0}^{3}\sum_{i=k+1}^{4}(-1)^k\binom{i-1}{k}\binom{7}{2i-1}(1-\sin^2\theta)^{3-k}}{\cos\theta\sum_{k=0}^{3}\sum_{i=k+1}^{4}(-1)^k\binom{i-1}{k}\binom{7}{2i-2}\cos^{2(3-k)}\theta}. \qquad \text{Eq. 75}$$

Obviously, the denominator of Eq. 75 equals the right-hand side of Eq. 70 for $n=4$.

Instead of showing that the numerator of Eq. 75 equals the right-hand side of Eq. 71 for $n = 4$, we can likewise derive

$$\cot 7\theta = \frac{\cos\theta \sum_{k=0}^{3} \sum_{i=k+1}^{4} (-1)^k \binom{i-1}{k} \binom{7}{2i-1} (1-\cos^2\theta)^{3-k}}{\sin\theta \sum_{k=0}^{3} \sum_{i=k+1}^{4} (-1)^k \binom{i-1}{k} \binom{7}{2i-2} \sin^{2(3-k)}\theta}$$

so that the denominator of which equals the right-hand side of Eq. 71.

In general, we get

$$\tan(2n-1)\theta = \frac{\sin\theta \sum_{k=0}^{n-1} \sum_{i=k+1}^{n} (-1)^k \binom{i-1}{k} \binom{2n-1}{2i-1} (1-\sin^2\theta)^{n-k-1}}{\cos\theta \sum_{k=0}^{n-1} \sum_{i=k+1}^{n} (-1)^k \binom{i-1}{k} \binom{2n-1}{2i-2} \cos^{2(n-k-1)}\theta}$$

and

$$\cot(2n-1)\theta = \frac{\cos\theta \sum_{k=0}^{n-1} \sum_{i=k+1}^{n} (-1)^k \binom{i-1}{k} \binom{2n-1}{2i-1} (1-\cos^2\theta)^{n-k-1}}{\sin\theta \sum_{k=0}^{n-1} \sum_{i=k+1}^{n} (-1)^k \binom{i-1}{k} \binom{2n-1}{2i-2} \sin^{2(n-k-1)}\theta}$$

so that Eqs. 67 and 70 are true. Finally, we note that the numerator and the denominator of Eq. 74 equal to $\frac{\sin 6\theta}{\cos^6\theta}$ and $\frac{\cos 6\theta}{\cos^6\theta}$, respectively. Similarly, Eqs. 72 and 73 can be derived from Eq. 68.

All the multiple angle formulas derived above can be proved by mathematical induction. We shall only prove Eq. 67, which is obviously true for $n = 1$. So we are left to prove

$$\tan 2(n+1)\theta = \frac{\sum_{k=1}^{n+1} (-1)^{k-1} \binom{2(n+1)}{2k-1} \tan^{2k-1}\theta}{\sum_{k=0}^{n+1} (-1)^k \binom{2(n+1)}{2k} \tan^{2k}\theta}. \qquad \text{Eq. 76}$$

Using Eq. 64, we first obtain $\tan 2(n+1)\theta = \dfrac{\tan 2\theta + \tan 2n\theta}{1-\tan 2\theta \tan 2n\theta}$

$$= \frac{2\tan\theta\sum_{k=0}^{n}(-1)^k\binom{2n}{2k}\tan^{2k}\theta + (1-\tan^2\theta)\sum_{k=1}^{n}(-1)^{k-1}\binom{2n}{2k-1}\tan^{2k-1}\theta}{(1-\tan^2\theta)\sum_{k=0}^{n}(-1)^k\binom{2n}{2k}\tan^{2k}\theta - 2\tan\theta\sum_{k=1}^{n}(-1)^{k-1}\binom{2n}{2k-1}\tan^{2k-1}\theta} . \qquad \text{Eq. 77}$$

Then use Eq. 28 to regroup the numerator of Eq. 77 successively as

$$2\tan\theta + \sum_{k=1}^{n}(-1)^k\binom{2n}{2k}\tan^{2k+1}\theta + \sum_{k=1}^{n}(-1)^{k-1}\binom{2n}{2k-1}\tan^{2k-1}\theta$$

$$+\sum_{k=1}^{n}\left[(-1)^k\binom{2n}{2k}\tan^{2k+1}\theta - (-1)^{k-1}\binom{2n}{2k-1}\tan^{2k+1}\theta\right]$$

$$= 2\tan\theta + \sum_{k=1}^{n-1}(-1)^k\binom{2n}{2k}\tan^{2k+1}\theta + (-1)^n\tan^{2n+1}\theta$$

$$+ 2n\tan\theta + \sum_{k=2}^{n}(-1)^{k-1}\binom{2n}{2k-1}\tan^{2k-1}\theta + \sum_{k=1}^{n}(-1)^k\binom{2n+1}{2k}\tan^{2k+1}\theta$$

$$= 2(n+1)\tan\theta + \sum_{k=1}^{n-1}(-1)^k\left[\binom{2n}{2k}+\binom{2n}{2k+1}\right]\tan^{2k+1}\theta + (-1)^n\tan^{2n+1}\theta$$

$$+\sum_{k=1}^{n-1}(-1)^k\binom{2n+1}{2k+1}\tan^{2k+1}\theta + (-1)^n(2n+1)\tan^{2n+1}\theta$$

$$= 2(n+1)\tan\theta + \sum_{k=1}^{n}(-1)^k\binom{2n+1}{2k+1}\tan^{2k+1}\theta + \sum_{k=1}^{n-1}(-1)^k\binom{2n+1}{2k}\tan^{2k+1}\theta$$

$$+(-1)^n 2(n+1)\tan^{2n+1}\theta$$

$$= 2(n+1)\tan\theta + \sum_{k=1}^{n-1}(-1)^k\binom{2n+2}{2k+1}\tan^{2k+1}\theta + (-1)^n\tan^{2n+1}\theta$$

$$= \sum_{k=0}^{n}(-1)^k\binom{2(n+1)}{2k+1}\tan^{2k+1}\theta = \sum_{k=1}^{n+1}(-1)^k\binom{2(n+1)}{2k-1}\tan^{2k-1}\theta ,$$

which is the numerator of Eq. 71. The proof for denominator case is similar.

## 3. LIFE CONTINGENCY

### 3.1. INTRODUCTION

Some forty years ago, I started out my eight years of actuarial career as an actuarial student and was provided by all my employers with study time and materials to prepare for Actuarial Exams sponsored by SOA. By studying the only one textbook (9) inside out, I personally invented many short cuts for solving various problems. I later published thirteen papers (23)-(35), a lecture note (20) and a textbook (22) in Chinese, which I did consult (2). Recently, I found out that my innovative ideas such as the uniform representation of a general life contingency function and its derivative were not even mentioned in (15). Thus, I feel obliged to write this chapter for the benefit of readers.

A life actuarial model is based on three major factors: interest, mortality and expense. We first give a general way of constructing it in terms of accumulation functions.

| Force of Interest | Force of Mortality | Expense Percentage |
|---|---|---|
| $\delta$ | $\mu_x$ | $\varepsilon$ |

Interest Related Accumulation Function: $a_\delta(t) = e^{\delta t}$

Mortality Related Accumulation Function: $a_{\mu_x}(t) = e^{\int_0^t \mu_{x+u} du}$

Annuity

$$\bar{a}_{x:n} = \int_0^n [a_\delta(t) a_{\mu_x}(t)]^{-1} dt$$

Endowment Insurance

$$\bar{A}_{x:n} = \int_0^n [a_\delta(t)]^{-1} \frac{d}{dt}\{-[a_{\mu_x}(t)]^{-1}\} dt + [a_\delta(n) a_{\mu_x}(n)]^{-1}$$

Net Level Premium

$$\bar{P}_{x:n} = \bar{A}_{x:n} / \bar{a}_{x:n}$$

Gross Level Premium

$$\bar{G}_{x:n} = \bar{P}_{x:n}(1+\varepsilon)$$

Net Level Premium Reserve at Year $t$

$${}_t\bar{V}_{x:n} = \bar{A}_{x+t:n-t} - \bar{P}_{x:n}\bar{a}_{x+t:n-t}$$

Cash Value at Year $t$

$${}_r\bar{C}_{x:n} = \bar{A}_{x+t:n-t} - \bar{G}_{x:n}\bar{a}_{x+t:n-t}$$

Figure 44. The structure of an $n$ year continuous life actuarial model

In the model outlined in Figure 45, let $\overline{a}(t) = \int_0^t [a_\delta(u)]^{-1} du$. Then by using the integration by parts, we can write

$$\overline{a}_{x:n} = \int_0^n \overline{a}(t) \frac{d}{dt} \{-[a_{\mu_x}(t)]^{-1}\} dt + \overline{a}(n)[a_{\mu_x}(n)]^{-1}$$

so that all life contingency functions can be written as

$$\overline{\alpha}_{x:n} = \int_0^n \overline{\alpha}(t) \frac{d}{dt} \{-[a_{\mu_x}(t)]^{-1}\} dt + \overline{\alpha}(n)[a_{\mu_x}(n)]^{-1},$$

where $\overline{\alpha}(t)$ is the present value of the benefit at time $t$ so that the first term $\overline{\alpha}^1_{x:n}$ is the death benefit and the second term $\overline{\alpha}_{x:\overset{1}{n}}$ is the maturity benefit. This is generally true in any model.

## 3.2. THEORY OF COMPOUND INTEREST

### 3.2.1 Functions of compound interest

We shall start with the accumulation function and use the geometric point of view to generalize and simplify the theory of interest.

Let a(x) be an increasing positive function satisfying

$$a(-x) = [a(x)]^{-1}. \qquad \text{Eq. 78}$$

From Eq. 78, it follows that a(0) = 1 and that

$$[a(0) – a(-x)]^{-1} – [a(x) – a(0)]^{-1} = 1, \quad x > 0. \qquad \text{Eq. 79}$$

A continuous a(x) further requires the existence of a’(0) (denoted by δ). For example, $a(x) = (1+i)^x$, where i is the nominal rate of interest.

We first list the notations and definitions of major functions with the illustrative Figures 45-50 as follows:

$i^{(m)}$ = the nominal rate of interest payable m times a year

$d^{(m)}$ = the nominal rate of discount payable m times a year

$\delta$ = the force of interest

$\ddot{a}_n^{(m)}$ = the present value of an annuity due which pays $m^{-1}$ at the beginning of each mth of a year for n years

$m^{-1}$ $m^{-1}$ $m^{-1}$ . . . . . . . . . . . . . . $m^{-1}$

0 n

$\ddot{a}_n^{(m)}$

Figure 45. The present value of $\ddot{a}_n^{(m)}$

$a_n^{(m)}$ = the present value of an annuity immediate which pays $m^{-1}$ at the end of each mth of a year for n years

$m^{-1}$ $m^{-1}$ $m^{-1}$ . . . . . . . . . . . . $m^{-1}$

0 n

$a_n^{(m)}$

Figure 46. The present value of $a_n^{(m)}$

$\bar{a}_n$ = the present value of a continuous annuity payable continuously for n years, with the total of 1 paid during each year

---1--- ---1--- ---1--- . . . . . . . . . . ---1---

0 1 2 3 n-1 n

$\bar{a}_n$

Figure 47. The present value of $\bar{a}_n$

$\ddot{s}_n^{(m)}$ = the future value of $\ddot{a}_n^{(m)}$

$m^{-1}$ $m^{-1}$ $m^{-1}$ . . . . . . . . . . . . . $m^{-1}$

0 n

$\ddot{s}_n^{(m)}$

Figure 48. The future value of $\ddot{a}_n^{(m)}$

$s_n^{(m)}$ = the future value of $a_n^{(m)}$

$m^{-1}$ $m^{-1}$ $m^{-1}$ . . . . . . . . . . . . $m^{-1}$

0 n

$s_n^{(m)}$

Figure 49. The future value of $a_n^{(m)}$

$\bar{s}_n$ = the future value of $\bar{a}_n$

---1--- ---1--- ---1--- . . . . . . . . . . . ---1---

0 1 2 3 n-1 n

$\bar{s}_n$

Figure 50. The present value of $\bar{a}_n$

When m = 1, the superscript (m) of the above notations is simply dropped.

On the other hand, we can generalize the definition of the force of interest at time x to be

$$\delta(x) = \lim_{t \to 0} \{[a(x+t) - a(x)]/a(x)\}/t \, .$$

Then

$$\delta(x) = a'(x)/a(x)$$

and

$$\delta = \delta(0).$$

In the case that

$$a(x) = (1+i)^x,$$

we have

$$\delta(x) = \lim_{t\to 0} \{[a(x+t) - a(x)]/t\}/a(x) = \lim_{t\to 0} [(1+i)^t - 1]/t = \delta$$

for all x and

$$\delta = \ln (1+i)$$

or

$$i = e^{\delta} - 1.$$

### 3.2.2 Geometric point of view

Let a(x) be an accumulation function. Define $i^{(m)}$ to be the slope of the line joining (0, 1) and $(m^{-1}, a(m^{-1}))$. Let $d^{(p)} = i^{(-p)}$. Then $d^{(p)}$ is the slope of the line joining (0, 1) and $(-p^{-1}, a(-p^{-1}))$.

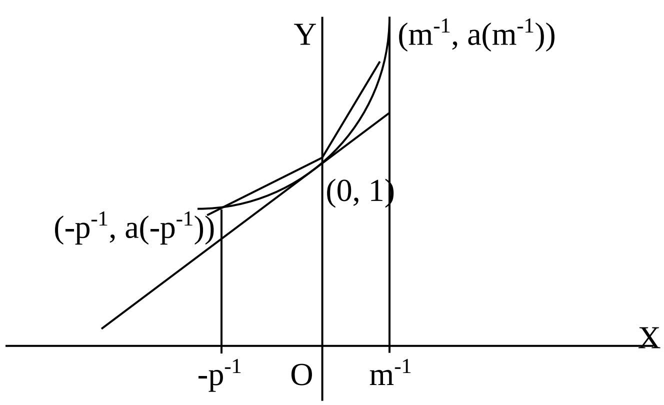



Figure 51. Geometric visualization of the force of interest

We can visualize from Figure 8 that

$$d^{(p)} < \delta < i^{(m)}$$

and

$$\lim_{p\to\infty} d^{(p)} = \delta = \lim_{m\to\infty} i^{(m)}.$$

The latter can also be derived as follows according to the definitions of $d^{(p)}$, $\delta$ and $i^{(m)}$:

$$\lim_{p\to\infty} d^{(p)} = \lim_{-p^{-1}\to 0} [a(-p^{-1}) - a(0)]/(-p^{-1}) = a'(0) = \delta = a'(0) = \lim_{m^{-1}\to 0} [a(m^{-1}) - a(0)]/(m^{-1}) = \lim_{m\to\infty} i^{(m)}.$$

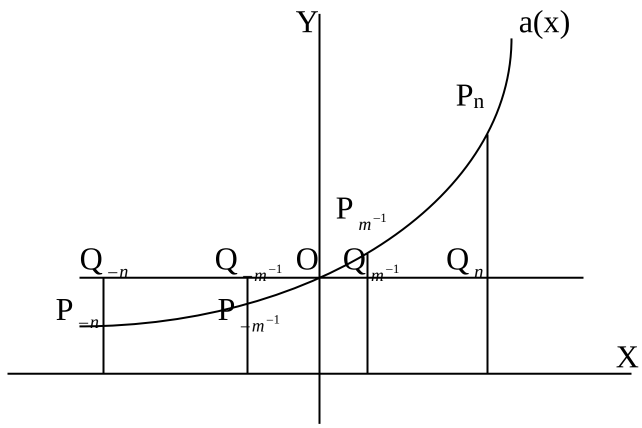



Figure 52. Geometric visualization of annuity functions

Referring to Figure 52, we define the following annuity functions:

$\ddot{a}_n^{(m)} =$ the ratio of the length of $P_{-n}Q_{-n}$ to the slope of $OP_{-m^{-1}}$

$a_n^{(m)} =$ the ratio of the length of $P_{-n}Q_{-n}$ to the slope of $OP_{m^{-1}}$

$\bar{a}_n =$ the ratio of the length of $P_{-n}Q_{-n}$ to the slope of the tangent at O

$\ddot{s}_n^{(m)} =$ the ratio of the length of $P_nQ_n$ to the slope of $OP_{-m^{-1}}$

$s_n^{(m)} =$ the ratio of the length of $P_nQ_n$ to the slope of $OP_{m^{-1}}$

$\bar{s}_n =$ the ratio of the length of $P_nQ_n$ to the slope of the tangent at O

With the exception of continuous functions, the above can also be defined as follows:

$\ddot{a}_n^{(m)} =$ the ratio of the length of $P_{-n}Q_{-n}$ to m times that of $P_{-m^{-1}}Q_{-m^{-1}}$

$a_n^{(m)}$ = the ratio of the length of $P_{-n}Q_{-n}$ to m times that of $P_{m^{-1}}$ $Q_{m^{-1}}$

$\ddot{s}_n^{(m)}$ = the ratio of the length of $P_nQ_n$ to m times that of $P_{-m^{-1}}$ $Q_{-m^{-1}}$

$s_n^{(m)}$ = the ratio of the length of $P_nQ_n$ to m times that of $P_{m^{-1}}$ $Q_{m^{-1}}$

Hence

$\ddot{a}_n^{(m)} = [1 - a(-nm)]/d^{(m)} = [1 - a(-nm)]/\{m[1 - a(-m^{-1})]\}$,

$a_n^{(m)} = [1 - a(-nm)]/i^{(m)} = [1 - a(-nm)]/\{m[a(m^{-1}) - 1]\}$,

$\bar{a}_n = [1 - a(-n)]/a'(0) = [1 - a(-n)]/\delta$,

$\ddot{s}_n^{(m)} = [a(nm) - 1]/d^{(m)} = [a(nm) - 1]/\{m[1 - a(-m^{-1})]\}$,

$s_n^{(m)} = [a(nm) - 1]/i^{(m)} = [a(nm) - 1]/\{m[a(m^{-1}) - 1]\}$,

$\bar{s}_n = [a(n) - 1]/a'(0) = [a(n) - 1]/\delta$.

In conclusion, we shall derive the following important formulas from Eqs. 78 and 79.

$i^{(m)} = a(m^{-1})d^{(m)}$; Eq. 80

$[d^{(m)}]^{-1} - [i^{(m)}]^{-1} = m^{-1}$ ; Eq. 81

$[\ddot{a}_n^{(m)}]^{-1} - [\ddot{s}_n^{(m)}]^{-1} = d^{(m)}$; Eq. 82

$[a_n^{(m)}]^{-1} - [s_n^{(m)}]^{-1} = i^{(m)}$; Eq. 83

$[[\bar{a}_n]^{-1} - [\bar{s}_n]^{-1} = \delta$. Eq. 84

Since a(0) = 1, we can derive Eq. 80 from Eq. 78 as follows.

$$i^{(m)} = \frac{a(m^{-1}) - a(0)}{m^{-1} - 0} = a(m^{-1})\{\frac{1 - [a(m^{-1})]^{-1}}{0 - (-m^{-1})}\} = a(m^{-1})[\frac{a(0) - a(-m^{-1})}{0 - (-m^{-1})}] = a(m^{-1})d^{(m)}.$$

Eq. 81 can be derived from Eq. 80 as follows.

$$\frac{1}{d^{(m)}}-\frac{1}{i^{(m)}}=a(\frac{1}{m})[\frac{1}{i^{(m)}}]-\frac{1}{i^{(m)}}=\frac{a(m^{-1})-a(0)}{i^{(m)}}=\frac{1}{m}\ .$$

From Eq. 79, we can derive Eq. 82 [similar for Eqs. 83 and 84] as follows.

$$\frac{1}{\ddot{a}_n^{(m)}}-\frac{1}{\ddot{s}_n^{(m)}}=\frac{d^{(m)}}{a(0)-a(-nm)}-\frac{d^{(m)}}{a(nm)-a(0)}=d^{(m)}[\frac{1}{a(0)-a(-nm)}-\frac{1}{a(nm)-a(0)}]=d^{(m)}.$$

In the case that

$$a(x) = (1+i)^x,$$

we have

$$\ddot{a}_n=[1-(1+i)^{-n}]/d=\ddot{s}_n(1+i)^{-n};$$

$$a_n=[1-(1+i)^{-n}]/i=s_n(1+i)^{-n};$$

$$\bar{a}_n=[1-(1+i)^{-n}]/\delta=\bar{s}_n(1+i)^{-n}.$$

From these formulas we can readily derive Eqs. 84-81 for the case that m = 1, which can also be visualized from Figures 53-55.

| | n-year payments of annuity due | | | | | future value |
|---|---|---|---|---|---|---|
| | $1/\ddot{s}_n$ | $1/\ddot{s}_n$ | $1/\ddot{s}_n$ | . . . . . . . . | $1/\ddot{s}_n$ | 1 |
| + | d | d | d | . . . . . . . . . | d | $d\,\ddot{s}_n$ |
| = | $1/\ddot{a}_n$ | $1/\ddot{a}_n$ | $1/\ddot{a}_n$ | . . . . . . . . | $1/\ddot{a}_n$ | $(1+i)^n$ |

$$(1/\ddot{a}_n)\,\ddot{s}_n = (1+i)^n = 1 + d\,\ddot{s}_n = (1/\ddot{s}_n)\,\ddot{s}_n + d\,\ddot{s}_n$$

Figure 53. Future value of n-year payments of annuity due

| n-year payments of annuity immediate | | | | future value |
|---|---|---|---|---|
| $1/s_n$ | $1/s_n$ | $1/s_n$ . . . . . . . . | $1/s_n$ | 1 + i |
| + i | i | i . . . . . . . . . | i | $i\ddot{s}_n$ |
| = $1/a_n$ | $1/a_n$ | $1/a_n$ . . . . . . . . | $1/a_n$ | $(1+i)^{n+1}$ |

$$(1/a_n)\,\ddot{s}_n = (1+i)^{n+1} = (1 + i) + [\,(1+i)^{n+1} - (1 + i)] = (1/s_n)\,\ddot{s}_n + i\,\ddot{s}_n$$

Figure 54. Future value of n-year payments of annuity immediate

| n-year payments of continuous annuity | | | | future value |
|---|---|---|---|---|
| $1/\bar{s}_n$ | $1/\bar{s}_n$ | $1/\bar{s}_n$ . . . . . . . . | $1/\bar{s}_n$ | δ/d |
| + δ | δ | δ . . . . . . . . . | δ | $\delta\,\ddot{s}_n$ |
| = $1/\bar{a}_n$ | $1/\bar{a}_n$ | $1/\bar{a}_n$ . . . . . . . . | $1/\bar{a}_n$ | $(\delta/d)\,(1+i)^n$ |

$$(1/\bar{a}_n)\,\ddot{s}_n = (\delta/d)\,(1+i)^n = \delta/d + (\delta/d)[\,(1+i)^n - 1] = (1/\bar{s}_n)\,\ddot{s}_n + \delta\,\ddot{s}_n$$

Figure 55. Future value of n-year payments of continuous annuity

## 3.3. LAWS OF MORTALITY

### 3.3.1 Point of view of stochastic theory

Let us first introduce the conventional notations as follows.

X : The random variable of a newborn's age-at-death.

$\omega$ : The terminal age.

F(x) : The distribution function (d. f.) of X.

S(x) = 1 – F(x) : The survival function.

Let ${}_{t|u}q_x$ be the probability that a life (x) aged x will die between ages x + t and x + t + u.

Then

$${}_{t|u}q_x = \Pr[x+t < X \le x+t+u \mid x < X \le \omega] = \frac{F(x+t+u)-F(x+t)}{F(\omega)-F(x)} = \frac{S(x+t)-S(x+t+u)}{S(x)-S(\omega)},$$

where F( $\omega$ ) = 1 and S( $\omega$ ) = 0. The above relationships can be visualized from Figure 57.

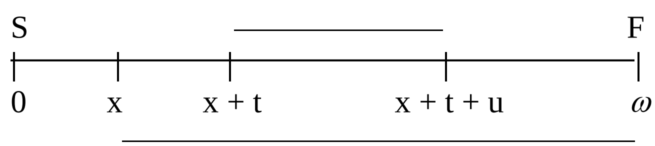



Figure 56. Linear visualization of the death rate ${}_{t|u}q_x$

When u = 1, we have (by omitting 1)

$${}_{t|}q_x = \Pr[x+t < X \le x+t+1 \mid x < X \le \omega] = \frac{F(x+t+1)-F(x+t)}{1-F(x)} = \frac{S(x+t)-S(x+t+1)}{S(x)}$$

and when t = 0, we have (by omitting 0 and replacing u by t)

$${}_{t}q_x = \Pr[x < X \le x+t \mid x < X \le \omega] = \frac{F(x+t)-F(x)}{1-F(x)} = \frac{S(x)-S(x+t)}{S(x)}$$

Let ${}_tp_x$ = 1 - ${}_tq_x$ be the probability that (x) will survive for t years. Then

$${}_{t|u}q_x = {}_{t+u}q_x - {}_tq_x = {}_tp_x - {}_{t+u}p_x = {}_tp_x - {}_tp_x\,{}_up_{x+t} = {}_tp_x\,{}_uq_{x+t}.$$

### 3.3.2 Point of view of traditional actuaries

In a life table, we can always find $q_x$, x = 0, 1, 2, 3, . . . $\omega$, which is the probability that (x) will die within a year, namely ${}_1q_x$ as introduced in the previous section; while $p_x = {}_1p_x$ is the probability that (x) will survive in a year.

Let S(x) be a survival function. Then

$$q_x = \frac{S(x) - S(x+1)}{S(x)}$$

and

$$p_x = \frac{S(x+1)}{S(x)}.$$

Let L(0) be a cohort of $l_0$ newborns. Then the survivorship function $l_x = l_0 S(x)$ is the number of those in L(0) who survive to age x. Let L(x) denote such a set.

In this manner, the number of survivors can be tracked down as follows:

$$l_1 = l_0 S(1) = l_0 p_0,$$

$$l_2 = l_0 S(2) = l_0\, {}_2p_0 = l_0 p_0 p_1 = l_1 p_1,$$

$$l_3 = l_0 S(3) = l_0\, {}_3p_0 = l_0 p_0 p_1 p_2 = l_2 p_2, \ldots$$

$$l_\omega = l_0 S(\omega) = l_0\, {}_\omega p_0 = l_0 p_0 p_1 p_2 p_3 \ldots p_{\omega-1} = l_{\omega-1} p_{\omega-1} = 0.$$

Let ${}_nd_x$ be the number of those in L(x) who will die within n years and let $d_x$ be those in L(x) who will die within a year. Then

$${}_nd_x = l_x - l_{x+n},$$

$$d_x = l_x - l_{x+1},$$

$${}_nq_x = {}_nd_x / l_x,$$

$$q_x = d_x / l_x,$$

$${}_np_x = (l_x + {}_nd_x) / l_x,$$

$$p_x = (l_x + d_x)/l_x ,$$

$$_{t+u}q_x - {}_t q_x = {}_{t|u}q_x = {}_u d_{x+t} / l_x = (l_{x+t}/l_x)({}_u d_{x+t}/l_{x+t}) = {}_t p_x \, {}_u q_{x+t} .$$

The above relationships can be visualized in Figure 57.

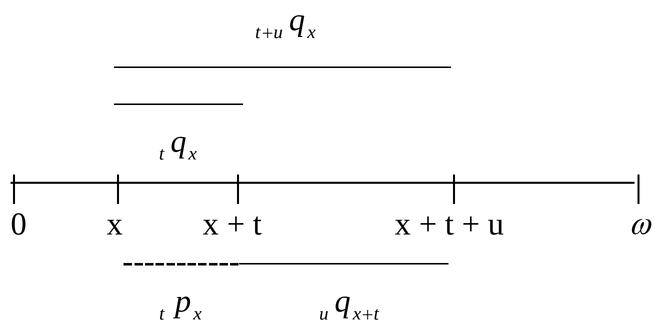



Figure 57. Linear visualization of various death rates

Now let us look at the instantaneous rate of mortality

$$\mu_x = \lim_{t \to 0} ({}_t q_x / t) , \qquad \text{Eq. 85}$$

called the force of mortality. Since

$$_t q_x = \frac{S(x) - S(x+t)}{S(x)} ,$$

we have

$$\mu_x = -\frac{S'(x)}{S(x)} = \frac{F'(x)}{S(x)} = \frac{F'(x)}{{}_x p_0} . \qquad \text{Eq. 86}$$

Hence ${}_x p_0 \mu_x$ is the p. d. f. (probability density function) of X.

## 3.4. MORTALITY RATES OF FRACTIONAL AGES

When $0 < t < 1$, ${}_t q_x$ can not be found in a life table.

The following two methods are commonly used to solve this problem.

1) Linear interpolation:

$$l_{x+t} = (1-t)l_x + tl_{x+1}.$$

2) Reciprocal interpolation:

$$l_{x+t}^{-1} = (1-t)l_x^{-1} + tl_{x+1}^{-1}.$$

The first method assumes the uniform distribution of deaths throughout a year, called U-Assumption; while the second method is due to Balducci, called B-Assumption.

Dual Theorem. $l_x$ imposes B-Assumption if and only if $l_{\omega-x}^{-1}$ imposes U-Assumption.

Proof. We shall assume that $l_\omega$ is close to 0 but not 0 to avoid $l_\omega^{-1}$ being undefined. Let

$$l_x^* = l_{\omega-x}^{-1}.$$

Then the theorem can be proved as follows:

$$l_{x+t}^* = l_{\omega-(x+t)}^{-1} = l_{[\omega-(x+1)]+(1-t)}^{-1} = [1-(1-t)]l_{\omega-(x+1)}^{-1} + (1-t)l_{[\omega-(x+1)]+1}^{-1} = tl_{x+1}^* + (1-t)l_x^*.$$

Now, we shall derive the formula of ${}_hq_{x+t}$ for the following two cases.

Case 1. $l_x$ imposes U-Assumption.

The following formula can also be visualized from Figure 58.

$${}_hq_{x+t} = \frac{l_{x+t} - l_{x+t+h}}{l_{x+t}} = \frac{(1-t)l_x + tl_{x+1} - (1-t-h)l_x - (t+h)l_{x+1}}{(1-t)l_x + tl_{x+1}} = \frac{[h(l_x - l_{x+1})]/l_x}{[l_x - t(l_x - l_{x+1})]/l_x} = \frac{hq_x}{1-tq_x}$$

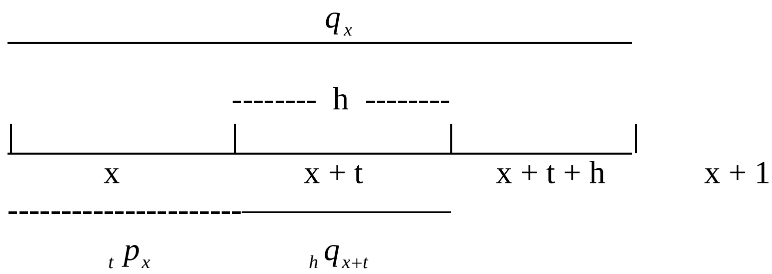



Figure 58. Linear visualization of the death rate in Case 1

It follows from

$$ {}_t p_x \, {}_h q_{x+t} = h q_x $$

that

$$ {}_h q_{x+t} = \frac{hq_x}{1 - {}_t q_x} = \frac{hq_x}{1 - tq_x} \qquad \text{Eq. 87} $$

and that

$$ \mu_{x+t} = \lim_{h \to 0} ({}_h q_{x+t} / h) = q_x / (1 - tq_x). $$

By taking t = 0 in Eq. 87, we have

$$ {}_h q_x = hq_x $$

and by taking h = 1 – t , we have

$$ {}_{1-t} q_{x+t} = \frac{(1-t)q_x}{1 - tq_x}. $$

<u>Case 2</u>. $l_x$ imposes B-Assumption.

According to Dual Theorem, $l_x^*$ imposes U-Assumption. From Case 1, we have

$$ {}_h q^*_{[\omega-(x+1)]+(1-t-h)} = \frac{hq^*_{\omega-(x+1)}}{1 - (1-t-h)q^*_{\omega-(x+1)}}. $$

Since

$$q^*_{\omega-(x+1)} = \frac{l^*_{\omega-(x+1)} - l^*_{\omega-x}}{l^*_{\omega-(x+1)}} = \frac{l^{-1}_{x+1} - l^{-1}_{x}}{l^{-1}_{x+1}} = \frac{l_x - l_{x+1}}{l_x} = q_x ,$$

it follows that

$${}_h q_{x+t} = \frac{l_{x+t} - l_{x+t+h}}{l_{x+t}} = \frac{l^{-1}_{x+t+h} - l^{-1}_{x+t}}{l^{-1}_{x+t+h}} = \frac{l^*_{\omega-(x+t+h)} - l^*_{\omega-(x+t)}}{l^*_{\omega-(x+t+h)}} = {}_h q^*_{\omega-(x+t+h)} = {}_h q^*_{[\omega-(x+1)]+(1-t-h)} = \frac{hq_x}{1-(1-t-h)q_x} .$$

Eq. 88

Eq. 88 can be visualized from Figures 58 and 59.

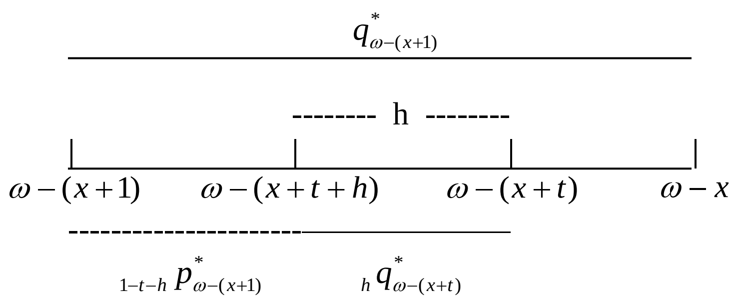



Figure 59. Linear visualization of the death rate in Case 2

To simplify the matter, we can further combine Figures 58 and 59 into Figure 60 as though the time is running from x + 1 to x (having in mine that the time is actually running from $\omega-(x+1)$ to $\omega - x$ .)

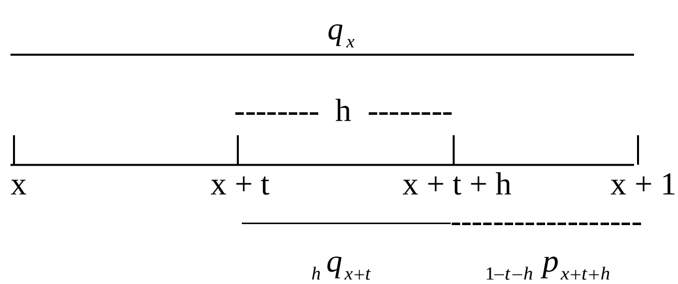



Figure 60. Linear visualization of the death rate in both cases

By taking t = 0 in Eq. 88, we have

$$ {}_h q_x = \frac{hq_x}{1-(1-h)q_x} $$

and by taking h = 1 – t , we have

$$ {}_{1-t} q_{x+t} = (1-t)q_x . \qquad \text{Eq. 89} $$

Similar to Case 1, we can also derive

$$ \mu_{x+t} = \lim_{h \to 0}({}_h q_{x+t} / h) = q_x /[1-(1-t)q_x]. $$

Because of the simplicity of Eq. 89, B-Assumption is often used as the basis of calculating the mortality rates.

## 3.5. MODELS OF THE SURVIVALSHIP FUNCTION

Mathematicians have long been searching for appropriate models for the survivorship function $l_x$. From Eq. 86, we can derive

$$ l_x = l_0 S(x) = l_0 \frac{S(x)}{S(0)} = l_0 e^{\ln\frac{S(x)}{S(0)}} = l_0 e^{\int_0^x \frac{s'(y)}{S(y)}dy} = l_0 e^{-\int_0^x \mu_y dy} . \qquad \text{Eq. 90} $$

In 1724 de Moivre first introduced, as the basis of the model,

$$ \mu_x = \frac{1}{\omega - x} $$

the survivorship function of which can be calculated from Eq. 90 as

$$ l_x = l_0 (1 - \frac{x}{\omega}) , $$

where $\omega = 105$. This was used in those days for simplifying the calculation of life annuities primarily only for the range of ages from 12 to 86, which was generalized to

$$ \mu_x = \frac{\alpha}{\omega - x} , \qquad \alpha > 0 , $$

the survivorship function of which being

$$l_x = l_0 (1 - \frac{x}{\omega})^{\alpha} .$$

In 1825, Gompertz believed that the force of mortality was increasing in geometric progression and introduced

$$\mu_x = Bc^x , \qquad \text{Eq. 91}$$

the survivorship function of which being

$$l_x = l_0 e^{-\frac{B}{\ln c}(c^x - 1)} .$$

By suitably adjusting B and c, this model could fit the range of ages from 10 to 55. Therefore, it was used to construct the 1937 Standard Annuity Table.

In 1860, Makeham further generalize the model to

$$\mu_x = A + Bc^x \qquad \text{Eq. 92}$$

and later to

$$\mu_x = A + Hx + Bc^x ,$$

the survivorship function of which being

$$l_x = l_0 e^{-\frac{B}{\ln c}(c^x - 1) - Ax - \frac{Hx^2}{2} - \frac{Bc^x}{\ln c}} .$$

By suitably adjusting A, B and c, this model could fit any age over 20 and was used to construct the Commissioners 1941 Standard Ordinary Mortality Table and also the Annuity Table for 1949. Furthermore, both Eqs. 91 and 92 are often used nowadays to simplify compound probability problems involving multiple life insurance.

Later, the model based on

$$\mu_x = \frac{Ac^x}{1+Bc^x},$$

the survivorship function of which being

$$l_x = l_0 (\frac{1+B}{1+Bc^x})^{\frac{A}{B \ln c}}.$$

In 1939, Weibull introduced

$$\mu_x = kx^n$$

the survivorship function of which being

$$l_x = l_0 e^{-\frac{k}{n+1} x^{n+1}}.$$

In 1997, the author obtained the following two least-square-fit cubic survivorship functions:

$$l_x^O = l_0[-\frac{1}{365}(\frac{x-46}{10})^3 - \frac{1}{45}(\frac{x-46}{10})^2 - \frac{1}{15}(\frac{x-46}{10}) + \frac{1}{1.115}]; \qquad \text{Eq. 93}$$

$$l_x^E = l_0[-\frac{1}{1970}(\frac{x-41}{5})^3 - \frac{1}{205}(\frac{x-41}{5})^2 - \frac{1}{55}(\frac{x-41}{5}) + \frac{1}{1.075}]. \qquad \text{Eq. 94}$$

The function Eq. 93 was derived based on $l_6$, $l_{16}$, $l_{26}$, . . . $l_{86}$ of 1958 CSO Male Life Table and fit well the range of ages from 0 to 70. The function Eq. 94 was derived based on $l_6$, $l_{16}$, $l_{26}$, . . . $l_{76}$ of the same table and fit well the range of ages from 3 to 79. The error for each of these models is within 1% for most of the ages described above and about 2% for few ages as can be seen in the following comparison chart (Tables 23 and 24 combined).

| x | 1958 CSO $l_x$ | Tsao's $l_x^O$ | % Error | Tsao's $l_x^E$ | % Error |
|---|---|---|---|---|---|
| 0 | 10,000,000 | 9,999,794 | .00 | 10,312,057 | 3.12 |
| 1 | 9,929,200 | 9,965,185 | .36 | 10,233,904 | 3.07 |
| 2 | 9,911,725 | 9,933,529 | .22 | 10,161,596 | 2.52 |
| 3 | 9,896,659 | 9,904,662 | .08 | 10,094,887 | 2.00 |
| 4 | 9,882,210 | 9,878,418 | - .04 | 10,033,535 | 1.53 |
| 5 | 9,868,375 | 9,854,634 | - .14 | 9,977,296 | 1.10 |
| 6 | 9,855,053 | 9,833,146 | - .22 | 9,925,926 | .72 |
| 7 | 9,842,241 | 9,813,788 | - .29 | 9,879,181 | .38 |
| 8 | 9,829,840 | 9,796,397 | - .34 | 9,836,818 | .07 |
| 9 | 9,817,749 | 9,780,808 | - .38 | 9,798,593 | - .20 |
| 10 | 9,805,870 | 9,766,856 | - .40 | 9,764,263 | - .42 |
| 11 | 9,794,005 | 9,754,378 | - .40 | 9,733,584 | - .62 |
| 12 | 9,781,958 | 9,743,210 | - .40 | 9,706,312 | - .77 |
| 13 | 9,769,633 | 9,733,185 | - .37 | 9,682,203 | - .89 |
| 14 | 9,756,737 | 9,724,141 | - .33 | 9,661,014 | - .98 |
| 15 | 9,743,175 | 9,715,913 | - .28 | 9,642,502 | - 1.03 |
| 16 | 9,728,950 | 9,708,336 | - .21 | 9,626,422 | - 1.05 |
| 17 | 9,713,967 | 9,701,246 | - .13 | 9,612,531 | - 1.04 |
| 18 | 9,698,230 | 9,694,479 | - .04 | 9,600,585 | - 1.01 |
| 19 | 9,681,840 | 9,687,870 | .06 | 9,590,341 | - .95 |
| 20 | 9,664,994 | 9,681,255 | .17 | 9,581,555 | - .86 |
| 21 | 9,647,694 | 9,674,470 | .28 | 9,573,984 | - .76 |
| 22 | 9,630,039 | 9,667,350 | .39 | 9,567,382 | - .65 |
| 23 | 9,612,127 | 9,659,730 | .50 | 9,561,508 | - .53 |
| 24 | 9,593,960 | 9,651,447 | .60 | 9,556,118 | - .39 |
| 25 | 9,575,636 | 9,642,336 | .70 | 9,550,966 | - .26 |
| 26 | 9,557,155 | 9,632,232 | .79 | 9,545,812 | - .12 |
| 27 | 9,538,423 | 9,620,972 | .87 | 9,540,409 | .02 |
| 28 | 9,519,442 | 9,608,391 | .93 | 9,534,515 | .16 |
| 29 | 9,500,118 | 9,594,324 | .99 | 9,527,886 | .29 |
| 30 | 9,480,358 | 9,578,607 | 1.04 | 9,520,279 | .42 |
| 31 | 9,460,165 | 9,561,076 | 1.07 | 9,511,449 | .54 |
| 32 | 9,439,447 | 9,541,566 | 1.08 | 9,501,154 | .65 |
| 33 | 9,418,208 | 9,519,913 | 1.08 | 9,489,148 | .75 |
| 34 | 9,396,358 | 9,495,952 | 1.06 | 9,475,190 | .84 |
| 35 | 9,373,807 | 9,469,520 | 1.02 | 9,459,035 | .91 |
| 36 | 9,350,279 | 9,440,450 | .96 | 9,440,439 | .96 |
| 37 | 9,325,594 | 9,408,582 | .89 | 9,419,160 | 1.00 |
| 38 | 9,299,482 | 9,373,748 | .80 | 9,394,952 | 1.03 |
| 39 | 9,271,491 | 9,335,785 | .69 | 9,367,573 | 1.04 |

Table 23. First half of the comparison chart

| | | | | | |
|---|---|---|---|---|---|
| 40 | 9,241,359 | 9,294,527 | .56 | 9,336,779 | 1.03 |
| 41 | 9,208,737 | 9,249,812 | .45 | 9,302,326 | 1.02 |
| 42 | 9,173,375 | 9,201,474 | .31 | 9,263,970 | .99 |
| 43 | 9,135,122 | 9,149,350 | .16 | 9,221,468 | .95 |
| 44 | 9,093,740 | 9,093,273 | .01 | 9,174,577 | .89 |
| 45 | 9,048,999 | 9,033,082 | - .18 | 9,123,052 | .82 |
| 46 | 9,000,587 | 8,968,610 | - .36 | 9,066,651 | .73 |
| 47 | 8,948,114 | 8,899,694 | - .54 | 9,005,128 | .64 |
| 48 | 8,891,204 | 8,826,168 | - .73 | 8,938,241 | .53 |
| 49 | 8,829,410 | 8,747,870 | - .92 | 8,865,746 | .41 |
| 50 | 8,762,306 | 8,664,634 | - 1.11 | 8,787,400 | .29 |
| 51 | 8,689,404 | 8,576,296 | - 1.30 | 8,702,958 | .16 |
| 52 | 8,610,244 | 8,482,692 | - 1.48 | 8,612,177 | .02 |
| 53 | 8,524,486 | 8,383,657 | - 1.65 | 8,514,814 | .11 |
| 54 | 8,431,654 | 8,279,027 | - 1.81 | 8,410,624 | .25 |
| 55 | 8,331,317 | 8,168,637 | - 1.95 | 8,299,364 | .38 |
| 56 | 8,223,010 | 8,052,324 | - 2.08 | 8,180,791 | .51 |
| 57 | 8,106,161 | 7,929,922 | - 2.17 | 8,054,660 | .64 |
| 58 | 7,980,191 | 7,801,267 | - 2.24 | 7,920,729 | .75 |
| 59 | 7,844,528 | 7,666,196 | - 2.27 | 7,778,752 | .84 |
| 60 | 7,698,698 | 7,524,543 | - 2.26 | 7,628,488 | .91 |
| 61 | 7,542,106 | 7,376,144 | - 2.20 | 7,469,692 | .96 |
| 62 | 7,374,370 | 7,220,835 | - 2.08 | 7,302,120 | .98 |
| 63 | 7,195,099 | 7,058,452 | - 1.90 | 7,125,529 | .97 |
| 64 | 7,003,925 | 6,888,829 | - 1.64 | 6,939,675 | .92 |
| 65 | 6,800,531 | 6,711,803 | - 1.30 | 6,744,315 | .83 |
| 66 | 6,584,614 | 6,527,210 | - .87 | 6,539,205 | .69 |
| 67 | 6,355,865 | 6,334,884 | - .33 | 6,324,100 | .50 |
| 68 | 6,114,088 | 6,134,662 | .34 | 6,098,759 | .25 |
| 69 | 5,859,253 | 5,926,379 | 1.15 | 5,862,936 | .06 |
| 70 | 5,592,012 | 5,709,870 | 2.11 | 5,616,388 | .44 |
| 71 | 5,313,586 | 5,484,972 | 3.23 | 5,358,872 | .85 |
| 72 | 5,025,855 | 5,251,520 | 4.49 | 5,090,144 | 1.28 |
| 73 | 4,731,089 | 5,009,350 | 5.88 | 4,809,960 | 1.67 |
| 74 | 4,431,800 | 4,758,296 | 7.37 | 4,518,077 | 1.95 |
| 75 | 4,129,906 | 4,498,196 | 8.92 | 4,214,251 | 2.04 |
| 76 | 3,826,895 | 4,228,884 | 10.50 | 3,898,238 | 1.86 |
| 77 | 3,523,881 | 3,950,196 | 12.10 | 3,569,794 | 1.30 |
| 78 | 3,221,884 | 3,661,968 | 13.66 | 3,228,677 | .21 |
| 79 | 2,922,055 | 3,364,034 | 15.13 | 2,874,642 | - .62 |

Table 24. Second half of the comparison chart

## 3.6. SIMPLE VISUALIZATIONS FOR SCHEDULE EXPOSURE FORMULAS

We shall introduce the method of valuation schedule in demography to be used to calculate the mortality rates for any observed group in the insurance industry such as the insured of a life insurance company, the annuitants of an annuity contract or the participants of a pension plan. To undertake a mortality study for such a group, we need to specify the observation period and the mechanism of calculating the exposure and deaths. These calculations involve with starters, new entrants, withdrawers, deaths and enders. For a large group, the valuation schedule exposure formulas are often considered rather than the individual record exposure formulas because of the obvious reason. These formulas are based only on the observed deaths and the periodic numeration of the individuals in the observed group, which are readily available from the data for the valuation purpose just as in the population study in demography.

We first adopt pertinent notations from the demography.

$P_x^z =$ the number of persons aged between x and x + 1 at the beginning of the calendar year z;

$E_x^z =$ the number of persons attained age x during the calendar year z;

${}_\alpha D_x^z =$ the number of deaths among $E_x^z$ during the calendar year z;

${}_\delta D_x^z =$ the number of deaths among $P_x^z$ before the attainment of age x + 1;

$D_x^z =$ the number of deaths at age x last birthday during the calendar year z;

$D_{x\backslash}^z = {}_\delta D_{x-1}^z + {}_\alpha D_x^z$;

$D_x^{z\backslash z+1} = {}_\alpha D_x^z + {}_\delta D_x^{z+1}$;

${}_{\alpha}m_x^z =$ the number of migrants in addition to $E_x^z$ during the calendar year z;

${}_{\delta}m_x^z =$ the number of migrants in addition to $P_x^z$ before the attainment of age x + 1.

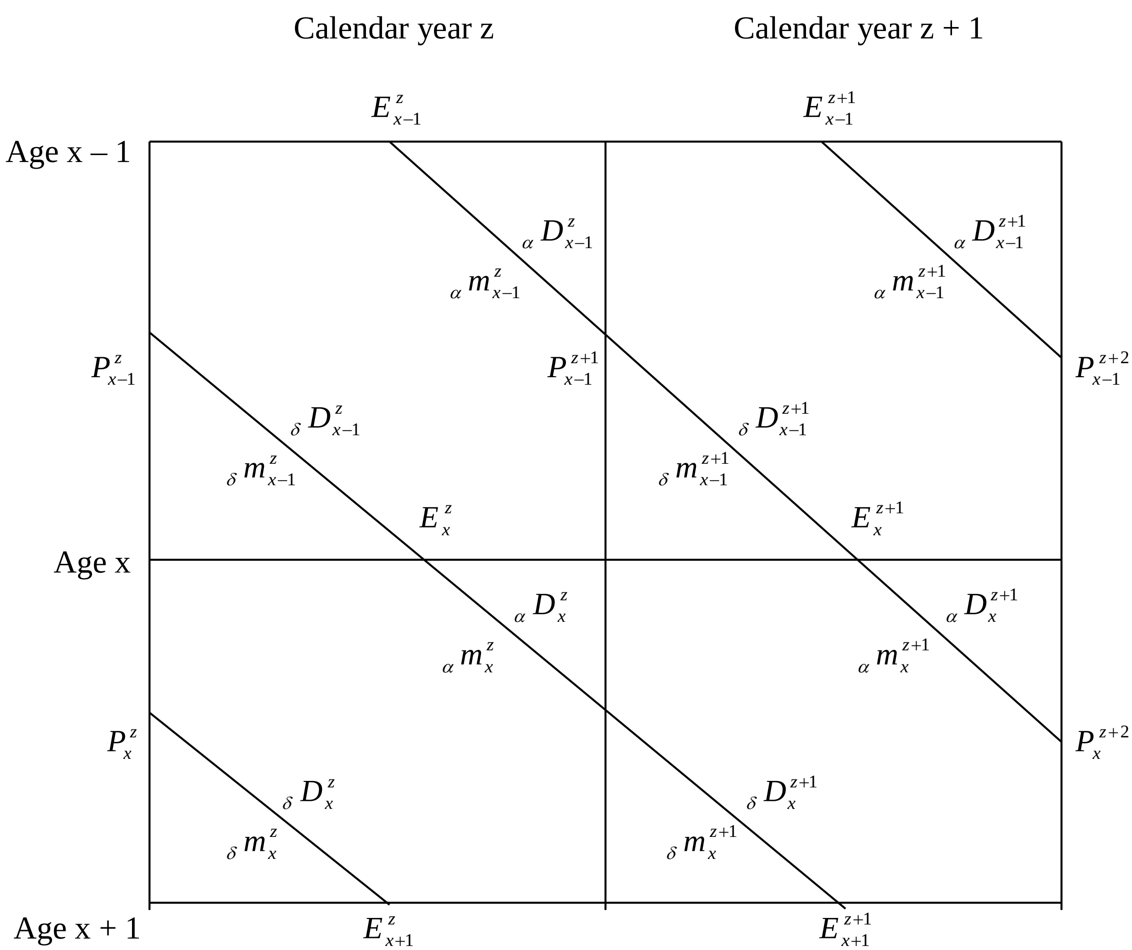



Figure 61. The observed deaths and the periodic numeration of the individuals

From the above figure, we have

$$ {}_{\alpha}m_x^z = E_x^z - P_x^{z+1} + {}_{\alpha}D_x^z \qquad \text{Eq. 95} $$

and

$$ {}_{\delta}m_x^z = P_x^z - E_{x+1}^z + {}_{\delta}D_x^z . \qquad \text{Eq. 96} $$

The number of migrants is the number of new entrants minus the number of withdrawers. In the insurance industry, the migration can be assumed to occur either at the insured's birthday or at the end of calendar year. Different migration and mortality assumptions will lead to different exposure formulas. The mortality rate is calculated as the ratio of the number of deaths over the total exposure. The treatment of deaths plays the major role in the calculation of different exposure formulas as discussed below. Let k be the number of months after January 1 for the average birthday of an observed group. For a large group, k is usually assumed to be 6. If the observation period is the calendar year z, we can group the deaths by age last birthday or by calendar age. If the observation period is from birthday in z to the birthday in z + 1, then the grouping is always by age last birthday.

<u>Case 1</u>. Calendar year study, deaths by age last birthday.

In the following figure, we assume that ${}_{\alpha}m_x^z$ occurs m months after January 1 and ${}_{\delta}m_x^z$ occurs n months after January 1.

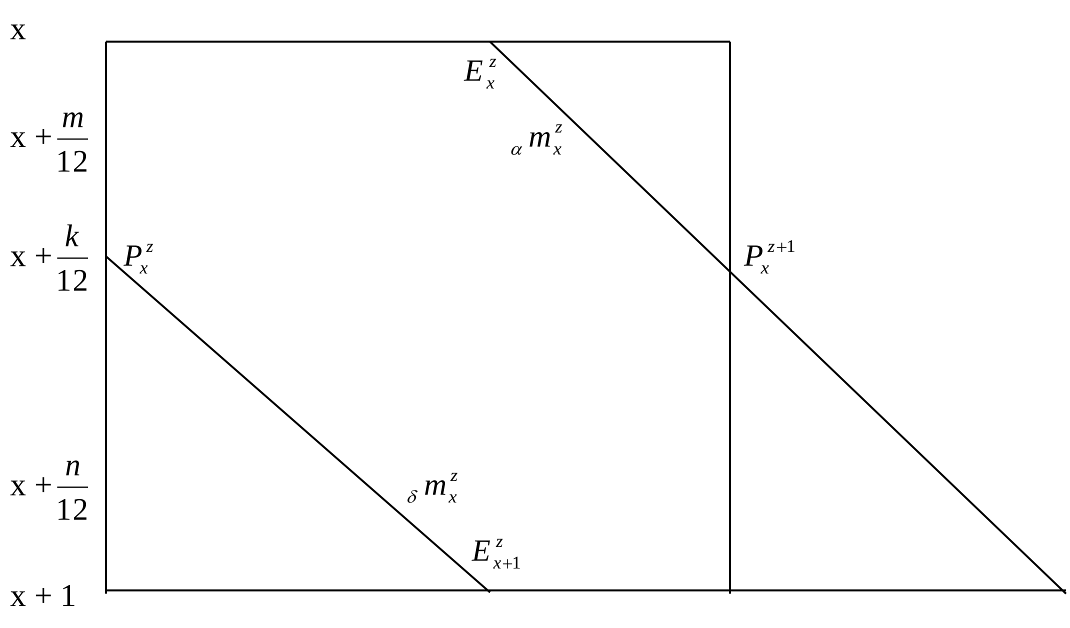



Figure 62. Visualization of Case 1

1) B-Assumption on deaths

Using the idea of potential and cancelled exposure, we can obtain

$$E_x^z q_x + {}_\alpha m_x^z \, {}_{\frac{12-m}{12}} q_{x+\frac{m}{12}} - P_x^{z+1} \, {}_{\frac{12-k}{12}} q_{x+\frac{k}{12}} + P_x^z \, {}_{\frac{12-k}{12}} q_{x+\frac{k}{12}} + {}_\delta m_x^z \, {}_{\frac{12-n}{12}} q_{x+\frac{n}{12}} = D_x^z .$$

It follows from Eqs. 89, 95 and 96 that

$$[E_x^z + (P_x^{z+1} + {}_\alpha D_x^z - E_x^z)\frac{12-m}{12} + (P_x^z - P_x^{z+1})\frac{12-k}{12} + (E_{x+1}^z + {}_\delta D_x^z - P_x^z)\frac{12-n}{12}]q_x = D_x^z .$$

Hence

$$q_x = \frac{D_x^z}{E},$$

where

$$E = \frac{m}{12}E_x^z + \frac{n-k}{12}P_x^z + \frac{12-n}{12}E_x^{z+1} + \frac{k-m}{12}P_x^{z+1} + \frac{12-m}{12}{}_\alpha D_x^z + \frac{12-n}{12}{}_\delta D_x^z .$$

If the migration occurs on birthdays (m = 0 and n = 12), then

$$E = \frac{12-k}{12}P_x^z + \frac{k}{12}P_x^{z+1} + \frac{k}{12}{}_\alpha D_x^z ,$$

which can be visualized directly from the diagram below.

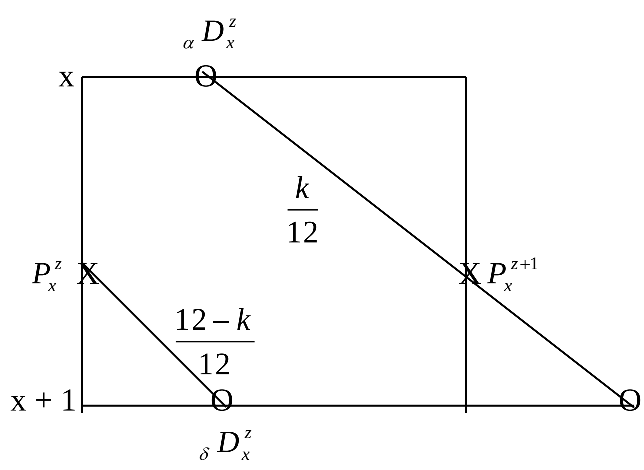



Figure 63. The migration occurs on birthdays under B-Assumption

The coefficient (exposure) of $P_x^z$ is the length of the line segment X---O, the coefficient of $P_x^{z+1}$ is the length of the line segment O---X and the coefficient of ${}_{\alpha}D_x^z$ is the length of the line segment O---O.

If the migration occurs at year-ends (m = n = k), then

$$E = \frac{k}{12}E_x^z + \frac{12-k}{12}E_{x+1}^z + \frac{12-k}{12}D_x^z,$$

which can be visualized directly from the diagram below.

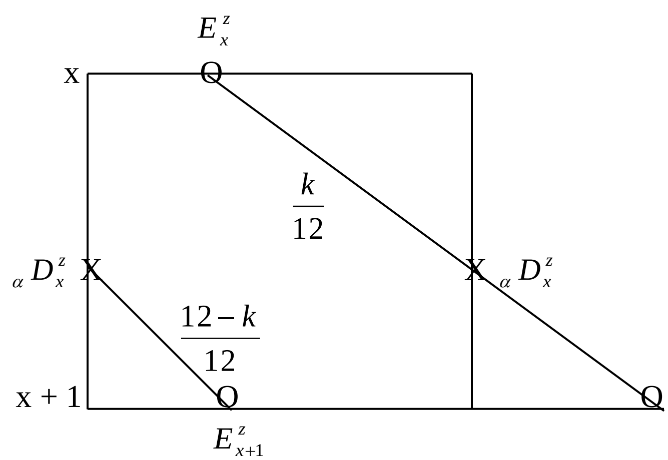



Figure 64. The migration occurs at year-ends under B-Assumption

The coefficient of $E_{x+1}^z$ is the length of the line segment X---O, the coefficient of $E_x^z$ is the length of the line segment O---X, the coefficient of ${}_{\delta}D_x^z$ is the length of the line segment X---O and the coefficient of ${}_{\alpha}D_x^z$ is the length of the line segment X---O.

2) U-Assumption on deaths

Since the equivalent formula to Eq. 87 is far more complicated under U-Assumption, we shall use the direct approach by tracing down the deaths from segment to segment in the original diagram to obtain

$$E_x^z \, {}_{\frac{k}{12}}q_x + {}_\alpha m_x^z \, {}_{\frac{k-m}{12}}q_{x+\frac{m}{12}} + P_x^z \, {}_{\frac{12-k}{12}}q_{x+\frac{k}{12}} + {}_\delta m_x^z \, {}_{\frac{12-n}{12}}q_{x+\frac{n}{12}} = D_x^z. \qquad \text{Eq. 97}$$

Due to the fact that U-Assumption is usually accompanied with the migration assumption either on birthdays or at year-ends, we shall only discuss these two special cases. If the migration occurs on birthdays (m = 0 and n = 12), then

$$(E_x^z + {}_\alpha m_x^z)\, {}_{\frac{k}{12}}p_x = P_x^{z+1}.$$

We shall make use of the following identity

$$ {}_{\frac{k}{12}}p_x \, {}_{\frac{12-k}{12}}q_{x+\frac{k}{12}} = {}_{\frac{k}{12}}p_x (1 - {}_{\frac{12-k}{12}}p_{x+\frac{k}{12}}) = {}_{\frac{k}{12}}p_x - p_x = q_x - {}_{\frac{k}{12}}q_x. \qquad \text{Eq. 98}$$

By multiplying ${}_{\frac{k}{12}}p_x$ to Eq. 97 and making use of the above, we can obtain

$$P_x^{z+1} \, {}_{\frac{k}{12}}q_x + P_x^z (q_x - {}_{\frac{k}{12}}q_x) + {}_\delta m_x^z \, {}_{\frac{k}{12}}p_x \, {}_0q_{x+1} = D_x^z (1 - {}_{\frac{k}{12}}q_x).$$

It then follows from U-Assumption, as can be visualized from the figure below, that

$$E = \frac{12-k}{12} P_x^z + \frac{k}{12} P_x^{z+1} + \frac{k}{12} {}_\alpha D_x^z,$$

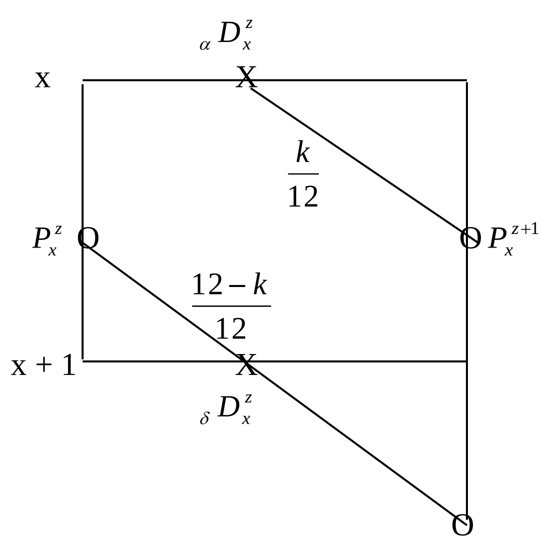



Figure 65. The migration occurs at year-ends under U-Assumption

The coefficient of $P_x^z$ is the length of the line segment O---X, the coefficient of $P_x^{z+1}$ is the length of the line segment X---O, the coefficient of ${}_{\alpha}D_x^z$ is the length of the line segment X---O and the coefficient of ${}_{\delta}D_x^z$ is the length of the line segment X---O. If the migration occurs at year-ends (m = n = k), by multiplying ${}_{\frac{k}{12}}p_x$ to Eq. 97 and making use of Eq. 96, we can obtain

$$(E_x^z\ {}_{\frac{k}{12}}p_x)\ {}_{\frac{k}{12}}q_x + {}_{\alpha}m_x^z\ {}_{\frac{k}{12}}p_x\ {}_0q_{x+\frac{k}{12}} + (P_x^z + {}_{\delta}m_x^z - {}_{\delta}D_x^z)(q_x - {}_{\frac{k}{12}}q_x) + {}_{\delta}D_x^z q_x = D_x^z.$$

Since $E_x^z\ {}_{\frac{k}{12}}p_x = P_x^{z+1}$, by applying Eq. 96 and U-Assumption to the above we can obtain

$$E = \frac{k}{12}E_x^z + \frac{12-k}{12}E_{x+1}^z + {}_{\delta}D_x^z,$$

which can be visualized directly from the figure below.

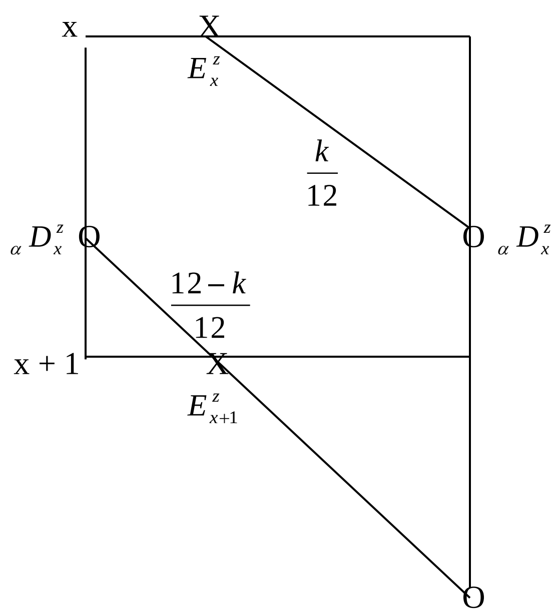



Figure 66. The migration occurs at year-ends under U-Assumption

The coefficient of $E_{x+1}^{z}$ is the length of the line segment O---X, the coefficient of $E_{x}^{z}$ is the length of the line segment X---O, and the coefficient of ${}_{\alpha}D_{x}^{z}$ is the length of the line segment O---O. The derivation of exposure formulas for the last two of the following cases is similar to the first and therefore will be omitted. However, we shall summarize the formulas of the case with accompanying figures and follow suit.

Case 1. Calendar year study, deaths by age last birthday.

Case 2. Calendar year study, deaths by calendar year.

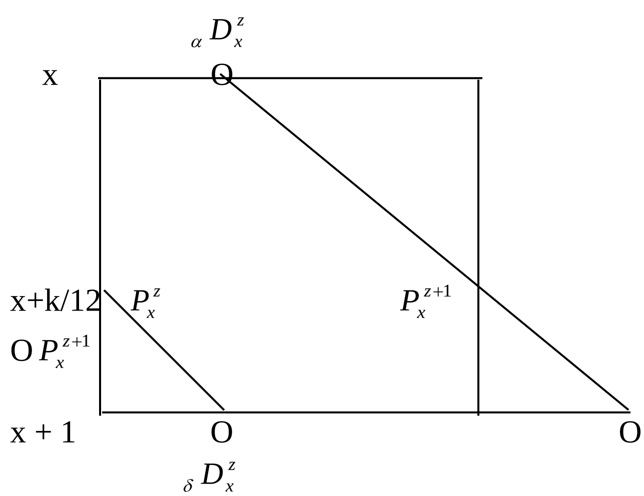
Case 3. Birthday to birthday study, deaths by age last birthday.

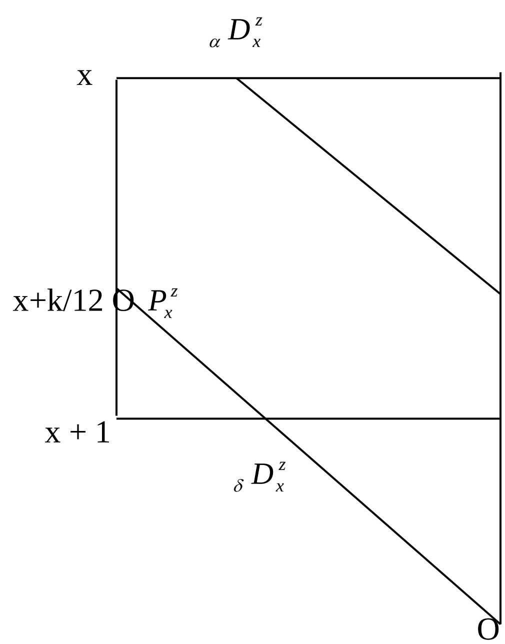
Case 1. Calendar year study, deaths by age last birthday.

Exposure formulas

B-Assumption on deaths

U-Assumption on deaths

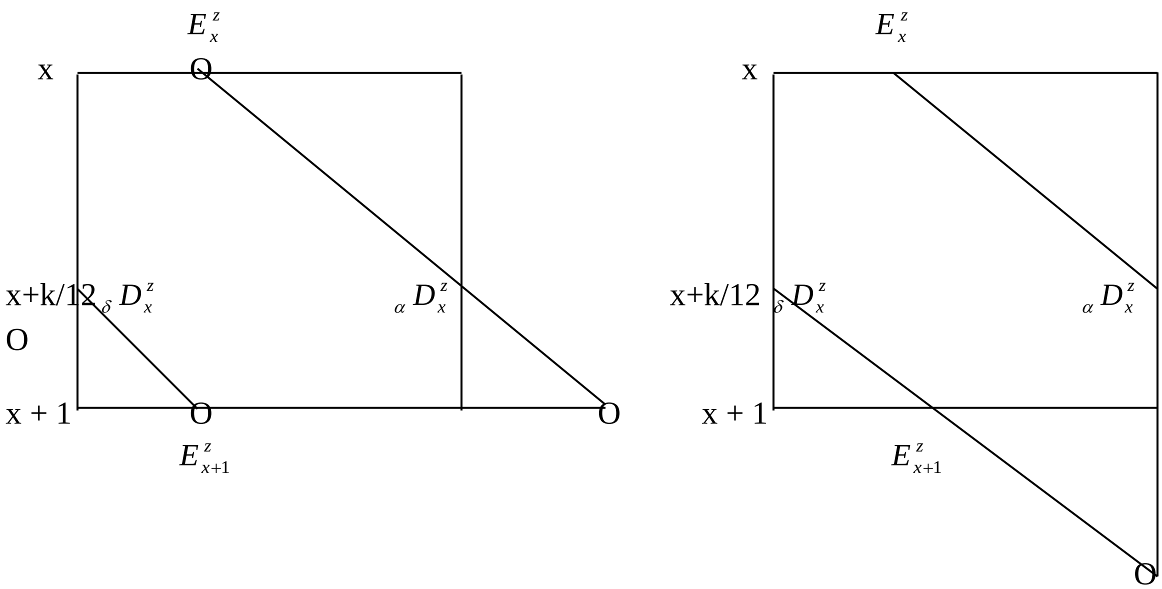



$$E = \frac{12-k}{12} P_{x}^{z} + \frac{k}{12} P_{x}^{z+1} + \frac{k}{12} {}_{\alpha}D_{x}^{z}$$

$$E = \frac{12-k}{12} P_{x}^{z} + \frac{k}{12} P_{x}^{z+1} + \frac{k}{12} D_{x}^{z}$$

Figure 67. Calendar year study, deaths by age last birthday, migration on birthdays



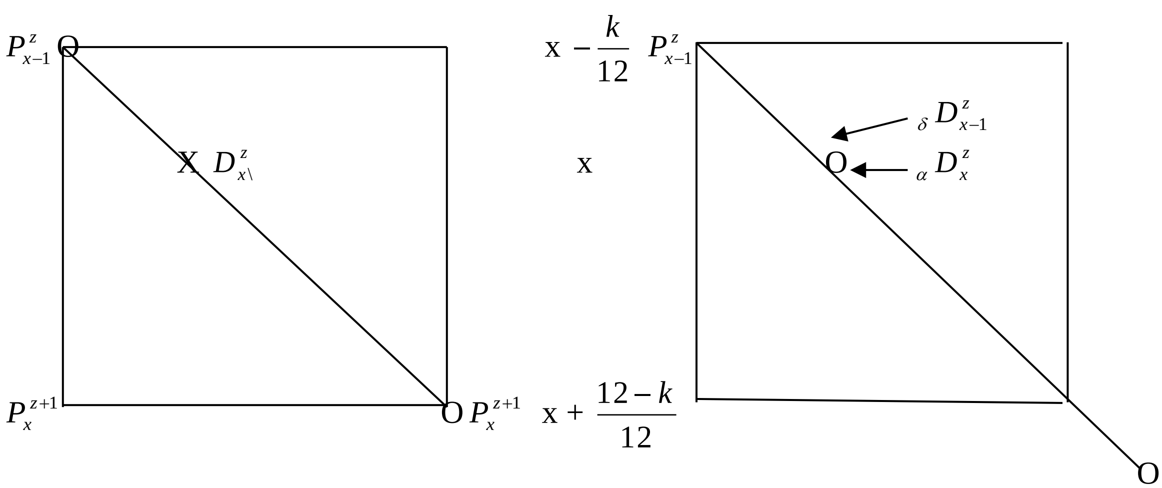

$$E = \frac{k}{12}E_x^z + \frac{12-k}{12}E_{x+1}^z + \frac{12-k}{12}D_x^z \qquad E = \frac{k}{12}E_x^z + \frac{12-k}{12}E_{x+1}^z + {}_{\delta}D_x^z$$

Figure 68. Calendar year study, deaths by age last birthday, migration at year-ends

Case 2. Calendar year study, deaths by calendar year.

Exposure formulas

B-Assumption on deaths U-Assumption on deaths




$$E = \frac{12-k}{12}P_{x-1}^z + \frac{k}{12}P_x^{z+1} + \frac{k}{12}D_{x\backslash}^z \qquad E = \frac{12-k}{12}P_{x-1}^z + \frac{k}{12}P_x^{z+1} + {}_{\alpha}D_x^z$$

Figure 69. Calendar year study, deaths by calendar year, migration on birthdays

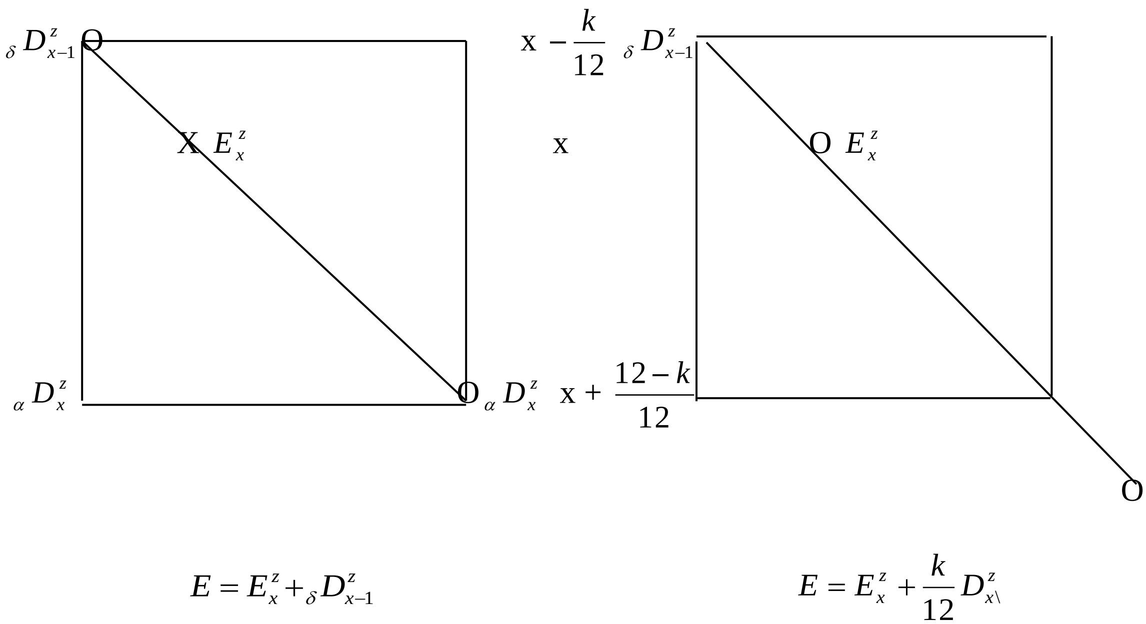



Figure 70. Calendar year study, deaths by calendar year, migration at year-ends

## 3.7. LIFE INSURANCE AND ANNUITIES

### 3.7.1 Deterministic point of view

Let $_{k-1|}q_x$ be the probability that a life (x) aged x will die between ages x + k –1 and x + k. Let $_k p_x$ be the probability that (x) will survive to age x + k. Let i be the nominal interest rate and let v = 1/(1 + i).

Let $A^{1}_{x:n}$ denote an n-year term insurance of 1 payable at the end of the year of death. Then

$$A^{1}_{x:n} = \sum_{k=0}^{n-1} v^{k+1} {}_{k|}q_x \ . \qquad \text{Eq. 99}$$

Let $A_{x:\overset{1}{n}}$ denote an n-year pure endowment of 1 payable at the end of the nth year when (x) survives. Then

$$A_{x:\overset{1}{n}} = v^n {}_n p_x \ . \qquad \text{Eq. 100}$$

Let $A_{x:n}$ denote an n-year endowment insurance of 1 payable either at the end of the year of death or at the end of the nth year when (x) survives. Then

$$A_{x:n} = A^{1}_{x:n} + A_{x:\overset{1}{n}}. \qquad \text{Eq. 101}$$

Let $a_{x:n}$ denote an n-year annuity of 1 payable at the end of each year while(x) survives. Then it is called an annuity immediate and

$$a_{x:n} = \sum_{k=1}^{n} v^k \, {}_k p_x \ .$$

Let $\ddot{a}_{x:n}$ denote an n-year annuity of 1 payable at the beginning of each year while (x) survives. Then it is called an annuity due and

$$\ddot{a}_{x:n} = \sum_{k=0}^{n-1} v^k \, {}_k p_x = 1 + a_{x:n-1} \ .$$

An annuity due can also be interpreted as an endowment insurance with $\ddot{a}_k$ payable at the year of death and $\ddot{a}_n$ payable at the date of maturity. Hence

$$\ddot{a}_{x:n} = \sum_{k=0}^{n-1} \ddot{a}_{k+1} \, {}_{k|}q_x + \ddot{a}_n \, {}_n p_x \ . \qquad \text{Eq. 102}$$

Therefore, we can consider an n-year term life contingency function $\alpha^{1}_{x:n}$ with the death benefit $\alpha_k$ payable at the end of the year of death. Then

$$\alpha^{1}_{x:n} = \sum_{k=0}^{n-1} \alpha_{k+1} \, {}_{k|}q_x \qquad \text{Eq. 103}$$

and an n-year endowment contingency function with $\alpha_n$ payable at the date of maturity is

$$\alpha_{x:n} = \alpha^{1}_{x:n} + \alpha_{x:\overset{1}{n}} \ ,$$

where $\alpha_{x:n}^{\ \ 1}$ is an n-year pure endowment of $\alpha_n$ at the date of maturity, namely

$$\alpha_{x:n}^{\ \ 1} = \alpha_{n\ n} p_x . \qquad \text{Eq. 104}$$

By taking $\alpha_k = v^k$ in Eq. 103, we can obtain Eq. 99. In this case, Eq. 100 follows from Eq. 104. These are the formulas for life insurance.

By taking $\alpha_k = \ddot{a}_k$ in Eq. 103, we can obtain

$$\ddot{a}_{x:n}^{1} = \sum_{k=0}^{n-1} \ddot{a}_{k+1\ k|} q_x \ ;$$

$$\ddot{a}_{x:n} = \sum_{k=0}^{n-1} \ddot{a}_{k+1\ k|} q_x + \ddot{a}_{n\ n} p_x \ .$$

In this case, Eq. 102 follows from Eq. 141 and the above. These are the formulas for annuity due.

By taking $\alpha_k = a_k$ in Eq. 103, we can obtain

$$a_{x:n}^{1} = \sum_{k=0}^{n-1} a_{k+1\ k|} q_x \ ;$$

$$a_{x:n} = \sum_{k=0}^{n-1} a_{k+1\ k|} q_x + a_{n\ n} p_x \ .$$

Finally, from Eqs. 100 and 101, we can derive

$$A_{x:n} + d\ddot{a}_{x:n} = \sum_{k=0}^{n-1} (v^{k+1} + d\ddot{a}_{k+1})_{k|} q_x + (v^n + d\ddot{a}_n)_n p_x = 1 .$$

Likewise, we can obtain

$$A_{x:n}^{1} + d\ddot{a}_{x:n}^{1} = \sum_{k=0}^{n-1} {}_{k|} q_x = {}_n q_x \ .$$

### 3.7.2 Stochastic point of view

Let K be the random variable of the integral future-life-time of (x). Then its p.d.f.is ${}_{k|}q_x, 0 \le k \le \omega - x - 1$, where $\omega$ is the terminal age.

Let ${}_{h|}\alpha_{x:n}$ be an h-year deferred n-year life contingency function, with the random variable of the present value of the benefit being

$$Z_{{}_{h|}\alpha_{x:n}} = \begin{cases} \alpha_K & K = h, h+1, h+2, \ldots, h+n-1 \\ \alpha_{h+n} & K = h+n, h+n+1, h+n+2, \ldots \end{cases}$$

Then

$$ {}_{h|}\alpha_{x:n} = E[Z_{{}_{h|}\alpha_{x:n}}] = \sum_{k=h}^{h+n-1} \alpha_{k+1}\, {}_{k|}q_x + \alpha_{h+n}\, {}_{h+n}p_x \,. \qquad \text{Eq. 105}$$

As in Eqs. 103 and 104, we write

$$ {}_{h|}\alpha_{x:\overset{1}{n}} = \alpha_{h+n}\, {}_{h+n}p_x $$

and

$$ {}_{h|}\alpha^{1}_{x:n} = \sum_{k=h}^{h+n-1} \alpha_{k+1}\, {}_{k|}q_x \,. $$

Let ${}_{h|}A_{x:n}$ be an h-year deferred n-year endowment payable at the end of the year. Then

$$\alpha_k = v^k, \qquad h+1 \le k \le h+n.$$

From Eq. 105, we have

$$ {}_{h|}A_{x:n} = \sum_{k=h}^{h+n-1} v^{k+1}\, {}_{k|}q_x + v^{h+n}\, {}_{h+n}p_x \,. \qquad \text{Eq. 106}$$

Let ${}_{h|}A^1_{x:n}$ be an h-year deferred n-year term insurance payable at the end of the year.

Then from Eq. 105, we have

$$ {}_{h|}A^1_{x:n} = \sum_{k=h}^{h+n-1} v^{k+1}\,{}_{k|}q_x \,, \qquad \text{Eq. 107} $$

since $\alpha_{h+n} = 0$. From Eqs. 106 and 107, we have

$$ {}_{h|}A_{x:n} = {}_{h|}A^1_{x:n} + v^{h+n}\,{}_{h+n}p_x = {}_{h|}A^1_{x:n} + {}_{h+n}E_x = {}_{h|}A^1_{x:n} + {}_{h|}A_{x:\overset{1}{n}} \,. $$

Let ${}_{h|}a_{x:n}$ be an h-year deferred n-year annuity payable at the end of the year. By taking

$\alpha_k = a_k - a_h$ in Eq. 105, we have

$$ {}_{h|}a_{x:n} = \sum_{k=h+1}^{h+n} (a_k - a_h)\,{}_{k|}q_x + (a_{h+n} - a_h)\,{}_{h+n}p_x = \sum_{k=h+1}^{h+n} a_k\,{}_{k|}q_x - a_h\,{}_{h+1}p_x + a_{h+n}\,{}_{h+n}p_x \qquad \text{Eq. 108} $$

and

$$ {}_{h|}a^1_{x:n} = \sum_{k=h+1}^{h+n} a_k\,{}_{k|}q_x \,. $$

Since $a_k = v + v^2 + v^3 + \ldots + v^k$ and ${}_{k|}q_x = {}_k p_x - {}_{k+1}p_x$, from Eq. 108 we have

$$ {}_{h|}a_{x:n} = \sum_{k=h+1}^{h+n} v^k\,{}_k p_x \,. $$

Let ${}_{h|}\ddot{a}_{x:n}$ be an h-year deferred n-year annuity payable at the beginning of the year. By

taking $\alpha_k = \ddot{a}_{k+1} - \ddot{a}_h$, we can, likewise, obtain

$$ {}_{h|}\ddot{a}_{x:n} = \sum_{k=h}^{h+n-1} \ddot{a}_{k+1}\,{}_{k|}q_x - \ddot{a}_h\,{}_h p_x + \ddot{a}_{h+n+1}\,{}_{h+n}p_x = \sum_{k=h}^{h+n+1} v^k\,{}_k p_x \,; $$

$$ {}_{h|}\ddot{a}^1_{x:n} = \sum_{k=h}^{h+n-1} \ddot{a}_{k+1}\,{}_{k|}q_x \,. $$

Let T be the random variable of the life-until-death of (x). Then its p.d.f. is ${}_t p_x \mu_{x+t}$,

where $\mu_{x+t}$ is the force of mortality at age x+t.

Let ${}_{h|}\overline{\alpha}_{x:n}$ be an h-year deferred n-year continuous life contingency function with the present value of the death benefit at time t being $\overline{\alpha}_t$ and that of the maturity benefit being $\overline{\alpha}_{h+n}$.

When n = $\omega - x$, ${}_{h|}\overline{\alpha}_{x:n}$ becomes ${}_{h|}\overline{\alpha}_x$, called an h-year deferred whole life continuous contingency function. When h = 0, they are denoted as $\overline{\alpha}_{x:n}$ and $\overline{\alpha}_x$, respectively.

Since the random variable of the present value of the benefit is

$$Z_{{}_{h|}\overline{\alpha}_{x:n}} = \begin{cases} \overline{\alpha}_T & h \le T \le h+n \\ \overline{\alpha}_{h+n} & T > h+n , \end{cases}$$

it follows that

$${}_{h|}\overline{\alpha}_{x:n} = E[Z_{{}_{h|}\overline{\alpha}_{x:n}}] = \int_h^{h+n} \overline{\alpha}_t \, {}_t p_x \mu_{x+t} dt + \overline{\alpha}_{h+n} \, {}_{h+n} p_x . \quad \text{Eq. 109}$$

By taking $\overline{\alpha}_t = v^t$ in Eq. 106, we obtain

$${}_{h|}\overline{A}_{x:n} = \int_h^{h+n} v^t \, {}_t p_x \mu_{x+t} dt + v^{h+n} \, {}_{h+n} p_x \qquad \text{Eq. 110}$$

and

$${}_{h|}\overline{A}^1_{x:n} = \int_h^{h+n} v^t \, {}_t p_x \mu_{x+t} dt ,$$

where ${}_{h|}\overline{A}_{x:n} (respectivdy, {}_{h|}\overline{A}^1_{x:n})$ is an h-year deferred n-year endowment (respectively, term)insurance of 1 payable at the moment of death or at the date of maturity.

Let ${}_{h|}\bar{a}_{x:n}$ denote an h-year deferred n-year continuous annuity. Then

$$Z = Z({}_{h|}\bar{a}_{x:n}) = \begin{cases} 0 & 0 \le T \le h \\ \bar{a}_T - \bar{a}_n & h \le T \le h+n \\ \bar{a}_{h+n} - \bar{a}_h & T > h+n. \end{cases}$$

It follows from Eq. 106 that

$${}_{h|}\bar{a}_{x:n} = E[Z_{{}_{h|}\bar{a}_{x:n}}] = \int_h^{h+n} (\bar{a}_t - \bar{a}_h)\,{}_t p_x \mu_{x+t} dt + (\bar{a}_{h+n} - \bar{a}_h)\,{}_{h+n}p_x \,. \qquad \text{Eq. 111}$$

Since

$$\bar{a}_t = \frac{1-v^t}{\delta} \ ,$$

we can obtain from Eqs. 110 and 111 that

$${}_{h|}\bar{a}_{x:n} = ({}_hE_x - {}_{h|}\bar{A}_{x:n})/\delta = {}_hE_x(1-\bar{A}_{x+h:n})/\delta\,,$$

where ${}_hE_x = v^h\,{}_hp_x$ and $\delta$ is the force of interest.

### 3.7.3 Dynamic point of view

Let $d_\delta(x,t)$ and $d_\mu(x,t)$ be the discount function of interest and mortality, respectively. Define an h-year deferred n-year continuous annuity as

$${}_{h|}\bar{a}_{x:n} = \int_h^{h+n} d_\delta(x,t) d_\mu(x,t) dt$$

and an h-year deferred n-year continuous term insurance as

$${}_{h|}\bar{A}^1_{x:n} = \int_h^{h+n} d_\delta(x,t) \frac{d}{dt}[-d_\mu(x,t)]dt.$$

Then an h-year deferred n-year continuous endowment insurance is defined to be

$$ {}_{h|}\bar{A}_{x:n} = {}_{h|}\bar{A}^{1}_{x:n} + {}_{h|}A_{x:\overset{1}{n}} , $$

where

$$ {}_{h+n}E_x = {}_{h|}A_{x:\overset{1}{n}} = d_{\delta}(x,h+n)d_{\mu}(x,h+n) $$

is an h-year deferred n-year pure endowment. This is the model of life contingency functions based on discount functions. In particular, if we let

$$ d_{\delta}(x,t) = e^{-\delta t} $$

and

$$ d_{\mu}(x,t) = e^{-\int_0^t \mu_{x+s} ds} , $$

then we can obtain the familiar (traditional) expressions for life contingency functions.

Another alternative is to let

$$ d_{\delta}(x,t) = e^{-\delta_x t} , $$

where $\delta_x$ is the force of interest at the issue age x. By integration by parts, we can obtain

$$ {}_{h|}\bar{A}^{1}_{x:n} = {}_{h}E_x - \delta_x \, {}_{h|}\bar{a}_{x:n} - d_{\delta}(x,h+n)d_{\mu}(x,h+n) $$

and

$$ {}_{h|}\bar{A}_{x:n} = {}_{h}E_x - \delta_x \, {}_{h|}\bar{a}_{x:n} . $$

Note that $\delta_x$ could be updated according to a certain national index. It can also include the expense factor for the calculation of the gross premiums, while $d_{\mu}(x,t)$ could be updated according to the national life table. On the other hand, $\bar{a}(x,n)$ can always be approximated. As for the discrete case, the conventional approximations are handy.

## 3.8. NET ANNUAL PREMIUMS AND RESERVES

### 3.8.1 Net annual premiums

Let ${}_r\overline{P}({}_{h|}\overline{\alpha}_{x:n})$ be the continuously paid net level premium, or simply net premium of ${}_{h|}\overline{\alpha}_{x:n}$, with payments for r years.

In the annuity case, it only makes sense that r < h + n. The reason is as follows. The insured pays r years of premiums when financially able, then starts receiving payments after h years for n years when financially needy. When h = 0, then the paying period should be less than the receiving period. There is no such restriction in the insurance case. In fact, when h = 0, r is usually equal to n. In this case, ${}_n\overline{P}(\overline{A}_{x:n})$ is abbreviated as $\overline{P}_{x:n}$, ${}_nP(\overline{A}^1_{x:n})$ as $\overline{P}^1_{x:n}$ and ${}_n\overline{P}(A_{x:\overset{1}{n}})$ as $\overline{P}_{x:\overset{1}{n}}$.

Let L be the random variable of the present value of the insurer's loss. Then

$$L = Z({}_{h|}\overline{\alpha}_{x:n}) - {}_r\overline{P}({}_{h|}\overline{\alpha}_{x:n})Z(\overline{a}_{x:n}).$$

If E[L] = 0, then

$${}_r\overline{P}({}_{h|}\overline{\alpha}_{x:n}) = {}_{h|}\overline{\alpha}_{x:n} / \overline{a}_{x:r}.$$

Hence we have

$${}_r\overline{P}({}_{h|}\overline{a}_{x:n}) = {}_{h|}\overline{a}_{x:n} / \overline{a}_{x:r},$$

$${}_r\overline{P}({}_{h|}\overline{a}^1_{x:n}) = {}_{h|}\overline{a}^1_{x:n} / \overline{a}_{x:r},$$

$${}_r\overline{P}({}_{h|}\overline{A}_{x:n}) = {}_{h|}\overline{A}_{x:n} / \overline{a}_{x:r},$$

$${}_r\overline{P}({}_{h|}\overline{A}^1_{x:n}) = {}_{h|}\overline{A}^1_{x:n} / \overline{a}_{x:r}$$

and

$$ {}_{r}\overline{P}({}_{h|}A^{\;1}_{x:n})={}_{h|}A^{\;1}_{x:n}\,/\,\overline{a}_{x:r}\,. $$

For the special cases, we have

$$ \overline{P}_{x:n} = \overline{A}_{x:n}\,/\,\overline{a}_{x:n}\,, $$

$$ \overline{P}^{1}_{x:n} = \overline{A}^{1}_{x:n}\,/\,\overline{a}_{x:n} $$

and

$$ \overline{P}_{x} = \overline{A}_{x}\,/\,\overline{a}_{x}\,. $$

For the discrete case, similar formulas can be derived.

### 3.8.2 Net premium reserves

We shall discuss the reserves based on the net level premium ${}^{r}\overline{P}({}_{h|}\overline{\alpha}_{x:n})$.

Define

${}^{r}_{t}\overline{V}({}_{h|}\overline{\alpha}_{x:n}) =$ the reserve needs to be provided for (x) by the insurer at the end of the t-th year, abbreviated as the reserve for (x) at the end of the t-th year or simply the reserve for (x + t).

Let U be the random variable of the future-life-time of (x + t). Then its p.d.f. is

$$ {}_{u}p_{x+t}\,\mu_{x+t+u}\,. $$

Let ${}_{t}L$ be the random variable of the loss of the insurer at the end of the t-th year. Then

$$ {}^{r}_{t}\overline{V}({}_{h|}\overline{\alpha}_{x:n}) = E[{}_{t}L], $$

the value of which is as follows.

i) If r < h, then

$$ {}_t^r\overline{V}({}_{h|}\overline{\alpha}_{x:n}) = \begin{cases} {}_{h-t|}\overline{\alpha}_{x+t:n} - {}^r\overline{P}({}_{h|}\overline{\alpha}_{x:n})\overline{a}_{x+t:r-t} & t < r \\ {}_{h-t|}\overline{\alpha}_{x+t:n} & r \le t < h \\ \overline{\alpha}_{x+t:n+h-t} & h \le t \le h+n \end{cases} $$

ii) If r = h, then

$$ {}_t^r\overline{V}({}_{h|}\overline{\alpha}_{x:n}) = \begin{cases} {}_{h-t|}\overline{\alpha}_{x+t:n} - {}^r\overline{P}({}_{h|}\overline{\alpha}_{x:n})\overline{a}_{x+t:r-t} & t < h \\ \overline{\alpha}_{x+t:n+h-t} & h \le t \le h+n \end{cases} $$

iii) If r > h, then

$$ {}_t^r\overline{V}({}_{h|}\overline{\alpha}_{x:n}) = \begin{cases} {}_{h-t|}\overline{\alpha}_{x+t:n} - {}^r\overline{P}({}_{h|}\overline{\alpha}_{x:n})\overline{a}_{x+t:r-t} & t < h \\ \overline{\alpha}_{x+t:n+h-t} - {}^r\overline{P}({}_{h|}\overline{\alpha}_{x:n})\overline{a}_{x+t:r-t} & h \le t < r \\ \overline{\alpha}_{x+t:n+h-t} & r \le t \le h+n \end{cases} $$

The above formulas also hold for ${}_{h|}\overline{\alpha}^1_{x:n}$, with ${}_{h+n}^{r}\overline{V}({}_{h|}\overline{\alpha}^1_{x:n}) = 0$.

If h = 0, then

$$ {}_t^r\overline{V}(\overline{\alpha}_{x:n}) = \begin{cases} \overline{\alpha}_{x+t:n-t} - {}^r\overline{P}(\overline{\alpha}_{x:n})\overline{a}_{x+t:r-t} & t < r \\ \overline{\alpha}_{x+t:n-t} & r \le t \le n \end{cases} $$

and

$$ {}_t^r\overline{V}(\overline{\alpha}^1_{x:n}) = \begin{cases} \overline{\alpha}^1_{x+t:n-t} - {}^r\overline{P}(\overline{\alpha}^1_{x:n})\overline{a}_{x+t:r-t} & t < r \\ \overline{\alpha}^1_{x+t:n-t} & r \le t \le n \end{cases} $$

In the case of r = n, we have

$$
{}_t\bar{V}(\bar{\alpha}_{x:n}) = \begin{cases} \bar{\alpha}_{x+t:n-t} - {}^r\bar{P}(\bar{\alpha}_{x:n})\bar{a}_{x+t:n-t} & t < n \\ \bar{\alpha}_{x+n:0} & t = n \end{cases}
$$

and

$$
{}_t\bar{V}(\bar{\alpha}^1_{x:n}) = \bar{\alpha}^1_{x+t:n-t} - \bar{P}(\bar{\alpha}^1_{x:n})\bar{a}_{x+t:n-t} \qquad t < n
$$

We write ${}_t\bar{V}(\bar{A}_{x:n})$ and ${}_t\bar{V}(\bar{A}^1_{x:n})$, respectively as ${}_t\bar{V}_{x:n}$ and ${}_t\bar{V}^1_{x:n}$. Thus

$$
{}_t\bar{V}^1_{x:n} = {}_t\bar{V}(A^1_{x:n}) = \begin{cases} A^1_{x+t:n-t} - \bar{P}^1_{x:n}\bar{a}_{x+t:n-t} & t < n \\ 1 & t = n \end{cases}
$$

## 3.9. VARYING LIFE CONTINGENCY FUNCTIONS

### 3.9.1 Increasing life contingency functions

Let $(I\bar{\alpha})_{x:n}$ be an n-year continuous contingency function providing the present value of the death benefit $(\underline{t+1})\bar{\alpha}_t$ at time t and the maturity benefit $n\bar{\alpha}_n$, where $\underline{x}$ is the floor function of x (the greatest integer less than x). If the maturity benefit is 0, then the function is denoted by $(I\bar{\alpha})^1_{x:n}$. Thus

$$
(I\bar{\alpha})_{x:n} = (I\bar{\alpha})^1_{x:n} + n\bar{\alpha}_n\, {}_n p_x.
$$

It follows from Eq. 108 that

$$
(I\bar{\alpha})^1_{x:n} = \int_0^n (\underline{t+1})\bar{\alpha}_t\, {}_t p_x \mu_{x+t} dt.
$$

Since the death benefit of both functions increases by 1 each year, they are called annually increasing life contingency functions with the difference only in the maturity benefit.

If the present value of the death benefit at time t is

$$\frac{(tm+1)}{m}\overline{\alpha}_t,$$

then the above functions are denoted by $(I^{(m)}\overline{\alpha})_{x:n}$ and, called mthly increasing life contingency functions.

If the death benefit increases only for h years, then the pertinent functions are written as $(I_h{}^{(m)}\overline{\alpha})_{x:n}$ and $(I_h{}^{(m)}\overline{\alpha})^1_{x:n}$.

3.9.2 Decreasing life contingency functions

Let $(D\overline{\alpha})^1_{x:n}$ be an n-year continuous contingency function providing the present value of the death benefit $(n-\underline{t})\overline{\alpha}_t$ at time t. Then it follows from Eq. 109 that

$$(D\overline{\alpha})^1_{x:n} = \int_0^n (n-\underline{t})\overline{\alpha}_t\, {}_t p_x \mu_{x+t} dt.$$

The death benefit decreases by 1 annually from n to 1. Thus such a function is called an annually decreasing life contingency function. Since the maturity benefit is 0, the notation $(D\overline{\alpha})_{x:n}$ is redundant.

If the present value of the death benefit at time t is

$$(n-\frac{(tm)}{m})\overline{\alpha}_t,$$

then the pertinent function is denoted by $(D^{(m)}\overline{\alpha})^1_{x:n}$, called an mthly decreasing life contingency function. If the death benefit decreases only for h years, then the pertinent functions is written as $(D_h{}^{(m)}\overline{\alpha})^1_{x:n}$.

### 3.9.3 The supplementary relationships

Since

$$\frac{(tm+1)}{m} + (n - \frac{(tm)}{m}) = n + \frac{1}{m},$$

we have

$$(I^{(m)}\bar{\alpha})^1_{x:n} + (D^{(m)}\bar{\alpha})^1_{x:n} = (n + \frac{1}{m})\bar{\alpha}^1_{x:n}. \qquad \text{Eq. 112}$$

This supplementary relationship can also be seen from the following figure.

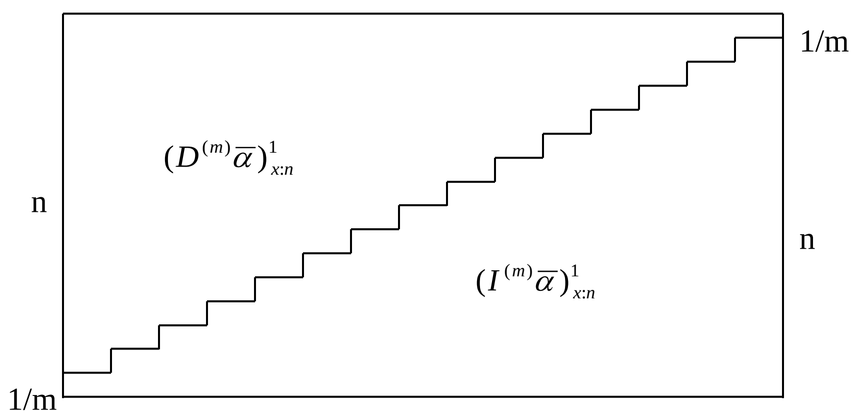



Figure 71. The supplementary relationship of $(D^{(m)}\bar{\alpha})^1_{x:n}$ and $(I^{(m)}\bar{\alpha})^1_{x:n}$

If $m \to \infty$, then from Eq. 112 we have

$$(\bar{I}\bar{\alpha})^1_{x:n} + (\bar{D}\bar{\alpha})^1_{x:n} = n\bar{\alpha}^1_{x:n}, \qquad \text{Eq. 113}$$

where $(\bar{I}\bar{\alpha})^1_{x:n}$ is an n-year continually increasing life contingency function and $(\bar{D}\bar{\alpha})^1_{x:n}$ is an n-year continually decreasing life contingency function.

If the present value of the maturity benefit is $n\bar{\alpha}_n$, then the pertinent function is

$$(\bar{I}\bar{\alpha})_{x:n} = \lim_{m\to\infty}(I^{(m)}\bar{\alpha})_{x:n} = (\bar{I}\bar{\alpha})^1_{x:n} + n\bar{\alpha}_{n}\,{}_{n}p_x. \qquad \text{Eq. 114}$$

The supplementary relationship in Eq. 113 can be seen from the following figure.

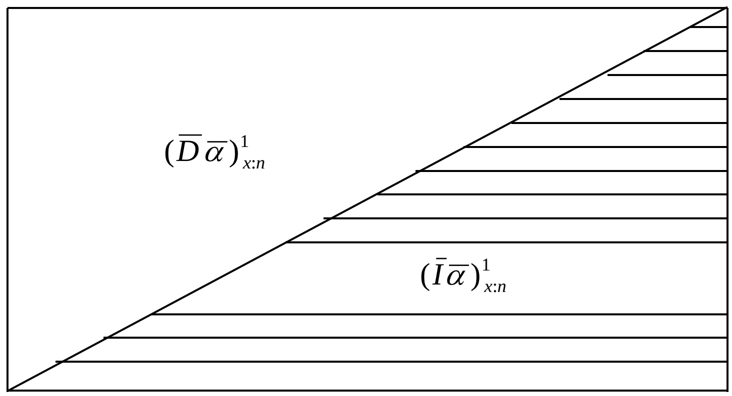



Figure 72. The supplementary relationship of $(\bar{D}\bar{\alpha})^1_{x:n}$ and $(\bar{I}\bar{\alpha})^1_{x:n}$

The triangle representing $(\bar{I}\bar{\alpha})^1_{x:n}$ consists of all the horizontal line segments, each representing ${}_{s|}\bar{\alpha}^1_{x:n-s}$, $0 \le s \le n.$ This relationship can also be proved as follows.

$$(\bar{I}\bar{\alpha})^1_{x:n} = \int_0^n t\bar{\alpha}_t\,{}_tp_x\mu_{x+t}dt = \int_0^n\int_0^t ds\bar{\alpha}_t\,{}_tp_x\mu_{x+t}dt = \int_0^n\int_s^n \bar{\alpha}_t\,{}_tp_x\mu_{x+t}dtds = \int_0^n {}_{s|}\bar{\alpha}^1_{x:n-s}ds. \quad \text{Eq. 115}$$

Similarly, we have

$$(\bar{D}\bar{\alpha})^1_{x:n} = \int_0^n (n-t)\bar{\alpha}_t\,{}_tp_x\mu_{x+t}dt = \int_0^n\int_t^n ds\bar{\alpha}_t\,{}_tp_x\mu_{x+t}dt = \int_0^n \bar{\alpha}^1_{x:s}ds. \quad \text{Eq. 116}$$

Combining Eqs. 115 and 116, we have

$$(\bar{I}\bar{\alpha})_{x:n} = \int_0^n ({}_{s|}\bar{\alpha}^1_{x:n-s} + \bar{\alpha}_n\,{}_np_x)ds = \int_0^n {}_{s|}\bar{\alpha}_{x:n-s}ds. \quad \text{Eq. 117}$$

On the other hand, we can use the integration by parts to obtain

$$(\bar{I}\bar{\alpha})_{x:n} = \int_0^n \bar{\alpha}_t\,{}_tp_xdt + \int_0^n t\bar{\alpha}_t'\,{}_tp_xdt; \quad \text{Eq. 118}$$

$$(\bar{D}\bar{\alpha})_{x:n} = n\bar{\alpha}_0 - \int_0^n \bar{\alpha}_t\,{}_tp_xdt + \int_0^n (n-t)\bar{\alpha}_t'\,{}_tp_xdt. \quad \text{Eq. 119}$$

3.9.4 Varying life insurance and annuities

By taking $\overline{\alpha}_t = v^t$ in the hitherto derived functions, we can obtain the formulas for

$(I\overline{A})_{x:n}$, $(I\overline{A})^1_{x:n}$, $(D\overline{A})^1_{x:n}$, $(I^{(m)}\overline{A})_{x:n}$, $(I^{(m)}\overline{A})^1_{x:n}$, $(D^{(m)}\overline{A})^1_{x:n}$, $(I_h{}^{(m)}\overline{A})_{x:n}$, $(I_h{}^{(m)}\overline{A})^1_{x:n}$,

$(D_h{}^{(m)}\overline{A})^1_{x:n}$, $(\overline{I}\overline{A})_{x:n}$, $(\overline{I}\overline{A})^1_{x:n}$ and $(\overline{D}\overline{A})^1_{x:n}$.

By taking $\overline{\alpha}_t{}' = v^t$, we can obtain the formulas for

$(I\overline{a})_{x:n}$, $(I\overline{a})^1_{x:n}$, $(D\overline{a})^1_{x:n}$, $(I^{(m)}\overline{a})_{x:n}$, $(I^{(m)}\overline{a})^1_{x:n}$, $(D^{(m)}\overline{a})^1_{x:n}$, $(I_h{}^{(m)}\overline{a})_{x:n}$, $(I_h{}^{(m)}a)^1_{x:n}$,

$(D_h{}^{(m)}\overline{a})^1_{x:n}$, $(\overline{I}\overline{a})_{x:n}$, $(\overline{I}\overline{a})^1_{x:n}$ and $(\overline{D}\overline{a})^1_{x:n}$.

Note that, in the case of annuities, $\overline{\alpha}_t = \overline{a}_t$ except for the last three types and for the last three types,

$$\overline{\alpha}_t = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} v^s ds = \lim_{\Delta t \to 0} (\overline{a}_{t+\Delta t} - \overline{a}_t) = 0.$$

Hence, from Eq. 141, we have

$$(\overline{I}\overline{A})_{x:n} = (\overline{I}\overline{A})^1_{x:n} + \overline{a}_{x:n}$$

and

$$(\overline{I}\overline{a})_{x:n} = (\overline{I}\overline{a})^1_{x:n}.$$

From Eq. 113, we have

$$(\overline{I}\overline{A})^1_{x:n} + (\overline{D}\overline{A})^1_{x:n} = n\overline{A}^1_{x:n}$$

and

$$(\overline{I}\overline{a})^1_{x:n} + (\overline{D}\overline{a})^1_{x:n} = n\overline{a}_{x:n}. \qquad \text{Eq. 120}$$

Furthermore, from Eqs. 115 and 116, we have

$$(\bar{I}\bar{a})_{x:n} = (\bar{I}\bar{a})^{1}_{x:n} = \int_0^n {}_{s|}\bar{a}^{1}_{x:n-s}ds$$

and

$$(\overline{D}\bar{a})^{1}_{x:n} = \int_0^n \bar{a}^{1}_{x:s}ds\,.$$

Let $\overline{\alpha}_t\,' = v^t$. Since $\overline{\alpha}_t = 0$, from Eq. 118 we have

$$(\bar{I}\bar{a})_{x:n} = \int_0^n tv^t\,{}_t\,p_x dt. \qquad \text{Eq. 121}$$

As a special case, we have

$$\bar{a}_{x:n} = \int_0^n v^t\,{}_t\,p_x dt. \qquad \text{Eq. 122}$$

From Eq. 119, we can also derive Eq. 120 as follows:

$$(\overline{D}\bar{a})^{1}_{x:n} = \int_0^n (n-t)v^t\,{}_t\,p_x dt = n\bar{a}_{x:n} - (\bar{I}\bar{a})_{x:n}\,.$$

For insurance, from Eqs. 115-118, we can obtain

$$(\bar{I}\overline{A})^{1}_{x:n} = \int_0^n {}_{s|}\overline{A}^{1}_{x:n-s}ds\,,$$

$$(\overline{D}\overline{A})^{1}_{x:n} = \int_0^n \overline{A}^{1}_{x:s}ds$$

and

$$(\bar{I}\overline{A})_{x:n} = \int_0^n {}_{s|}\overline{A}_{x:n-s}ds\,. \qquad \text{Eq. 123}$$

Let $\overline{\alpha}_t = v^t$. From Eqs. 118-120, we have

$$(\bar{I}\overline{A})_{x:n} = \int_0^n v^t\,{}_t\,p_x dt - \int_0^n t\delta v^t\,{}_t\,p_x dt = \bar{a}_{x:n} - \delta(\bar{I}\bar{a})_{x:n}\,.$$

From Eqs. 119, 122 and 123 we have

$$(\overline{D}\overline{A})^{1}_{x:n} = n - \int_0^n (n-t)v^t\,{}_t p_x dt - \int_0^n v^t\,{}_t p_x dt = n - \delta(\overline{D}\overline{a})_{x:n} - \overline{a}_{x:n} = n - \overline{a}_{x:n} - \delta n\overline{a}_{x:n} + \delta(\bar{I}\overline{a})_{x:n}.$$

Similar to Eq. 120, we can obtain

$$(I\overline{\alpha})_{x:n} = \sum_{k=0}^{n-1} {}_{k|}\overline{\alpha}_{x:n-k}$$

and then

$$(I\overline{A})_{x:n} + \delta(I\overline{a})_{x:n} = \sum_{k=0}^{n-1}({}_{k|}\overline{A}_{x:n-k} + \delta\,{}_{k|}\overline{a}_{x:n-k}) = \sum_{k=0}^{n-1} {}_k E_x = \ddot{a}_{x:n}.$$

## 3.10. DERIVATIVES OF LIFE CONTINGENCY FUNCTIONS

### 3.10.1 Derivatives of continuous life insurance and annuities

Let $l_x$ be the survivorship function. Since

$$\frac{dl_x}{dx} = -l_x\mu_x,$$

we have

$$\frac{d\,{}_t p_x}{dx} = \frac{d\left(l_{x+t}/l_x\right)}{dx} = -\frac{l_{x+t}\mu_{x+t}}{l_x} - \frac{l_{x+t}(-l_x\mu_x)}{l_x^{\;2}} = {}_t p_x(\mu_x - \mu_{x+t}). \qquad \text{Eq. 124}$$

It follows that

$$\frac{d\,{}_t E_x}{dx} = \frac{d(v^t\,{}_t p_x)}{dx} = v^t\,{}_t p_x(\mu_x - \mu_{x+t}) = {}_t E_x(\mu_x - \mu_{x+t}). \qquad \text{Eq. 125}$$

Using Eq. 125, we can differentiate Eq. 122 to obtain

$$\frac{d\,{}_{h|}\overline{\alpha}_{x:n}}{dx} = \int_h^{h+n} \overline{\alpha}_t \frac{d({}_t p_x \mu_{x+t})}{dx} dt + \overline{\alpha}_{h+n}\,{}_{h+n} p_x(\mu_x - \mu_{x+h+n}).$$

Since

$$\frac{d({}_tp_x\mu_{x+t})}{dx}=\frac{d(\frac{l_{x+t}\mu_{x+t}}{l_x})}{dx}=(\frac{1}{l_x})\frac{d(l_{x+t}\mu_{x+t})}{dx}+{}_tp_x\mu_{x+t}\mu_x,$$

it follows that

$$\frac{d\,{}_{h|}\overline{\alpha}_{x:n}}{dx}=\int_h^{h+n}(\frac{\overline{\alpha}_{tt}}{l_x})\frac{d(\frac{l_{x+t}\mu_{x+t}}{l_x})}{dx}dt+\mu_x\int_h^{h+n}\overline{\alpha}_{t\ t}\,p_x\mu_{x+t}dt+\overline{\alpha}_{h+n\ h+n}\,p_x(\mu_x-\mu_{x+h+n}).$$

Using the integration by parts, we have

$$\frac{d\,{}_{h|}\overline{\alpha}_{x:n}}{dx}=\mu_{x\ h|}\overline{\alpha}_{x:n}-\int_h^{h+n}\overline{\alpha}_t{}'\,{}_tp_x\mu_{x+t}dt-\overline{\alpha}_{h\ h}\,p_x\mu_{x+h}. \qquad \text{Eq. 126}$$

3.10.2 Derivatives of discrete life insurance and annuities

From Eqs. 105 and 124, we can derive

$$\frac{d\,{}_{h|}\alpha_{x:n}}{dx}$$

$$=\sum_{k=h}^{h+n-1}\alpha_{k+1}[{}_kp_x(\mu_x-\mu_{x+k})-{}_{k+1}p_x(\mu_x-\mu_{x+k+1})]+\alpha_{n+h\ n+h}\,p_x(\mu_x-\mu_{x+h+n})$$

$$=\sum_{k=h}^{h+n-1}[\alpha_{k+1\ k|}q_x\mu_x+\alpha_{k+1}({}_{k+1}p_x\mu_{x+k+1}-{}_kp_x\mu_{x+k})]+\alpha_{n+h\ n+h}\,p_x(\mu_x-\mu_{x+h+n})$$

$$=\mu_{x\ h|}\alpha_{x:n}+\sum_{k=h}^{h+n-1}(\alpha_k-\alpha_{k+1})_k\,p_x\mu_{x+k}. \qquad \text{Eq. 127}$$

Let $\alpha_k=v^k$. Then

$$\frac{d\,{}_{h|}A_{x:n}}{dx}=\mu_{x\ h|}A_{x:n}+\sum_{k=h}^{h+n-1}(v^k-v^{k+1})_k\,p_x\mu_{x+k}=\mu_{x\ h|}A_{x:n}+d\sum_{k=h}^{h+n-1}{}_kE_x\mu_{x+k},$$

where $d = \dfrac{i}{1+i}$.

Let $\alpha_k = \ddot{a}_k$. Then

$$\frac{d\ {}_{h|}\ddot{a}_{x:n}}{dx}$$

$$= \mu_x\ {}_{h|}\ddot{a}_{x:n} + \sum_{k=h}^{h+n-1} (\ddot{a}_k - \ddot{a}_{k+1})\ {}_k p_x \mu_{x+k}$$

$$= \mu_x\ {}_{h|}\ddot{a}_{x:n} - \sum_{k=h}^{h+n-1} {}_k E_x \mu_{x+k}.$$

Let $\alpha_k = a_k$. Then

$$\frac{d\ {}_{h|}a_{x:n}}{dx}$$

$$= \mu_x\ {}_{h|}a_{x:n} + \sum_{k=h}^{h+n-1} (a_k - a_{k+1})\ {}_k p_x \mu_{x+k}$$

$$= \mu_x\ {}_{h|}a_{x:n} - v \sum_{k=h}^{h+n-1} {}_k E_x \mu_{x+k}.$$

3.10.3 Derivatives of varying life insurance and annuities

From Eqs. 117 and 126, we can derive

$$\frac{d(\bar{I}\bar{\alpha})_{x:n}}{dx} = \int_0^n \frac{d\ {}_{s|}\bar{\alpha}_{x:n-s}}{dx} ds$$

$$= \int_0^n \bar{\alpha}_{x:n-s} \mu_x ds - \int_0^n \int_s^n \bar{\alpha}_t{}'\ {}_t p_x \mu_{x+t} dt ds - \int_0^n \bar{\alpha}_t\ {}_s p_x \mu_{x+s} ds$$

$$= \mu_x (\bar{I}\bar{\alpha})_{x:n} - \int_0^n t\bar{\alpha}_t{}'\ {}_t p_x \mu_{x+t} dt - \int_0^n \bar{\alpha}_s\ {}_s p_x \mu_{x+s} ds. \qquad \text{Eq. 128}$$

## 3.11. SOME USEFUL THEOREMS IN ACTUARIAL MATHEMATICS

Theorem A. Let a, c, d and e be positive numbers. Then the function

$$f(x)=\frac{dx+e}{\sqrt{ax^2+c}}$$

attains its maximum value

$$\sqrt{\frac{d^2}{a}+\frac{e^2}{c}} \text{ at } x=\frac{cd}{ae}.$$

Proof. We first derive

$$f'(x)=\frac{d\sqrt{ax^2+c}-\frac{ax(dx+e)}{\sqrt{ax^2+c}}}{\sqrt{ax^2+c}}=\frac{cd-aex}{\sqrt{(ax^2+c)^3}};$$

$$f''(x)=\frac{-ae\sqrt{(ax^2+c)}-\frac{3a(cd-aex)}{\sqrt{(ax^2+c)^3}}}{\sqrt{(ax^2+c)^3}}=-\frac{ae(ax^2+c)^3+3a(cd-aex)}{\sqrt{(ax^2+c)^5}}$$

.

Since the value of $f''(x)$ at the critical point $x=\frac{cd}{ae}$ is

$$-\frac{ae\left[a\left(\frac{cd}{ae}\right)^2+c\right]^3}{e\sqrt{\left[a\left(\frac{cd}{ae}\right)^2+c\right]^5}},$$

the maximum value is

$$f\left(\frac{cd}{ae}\right)=\frac{a\left(\frac{cd}{ae}\right)+e}{\sqrt{a\left(\frac{cd}{ae}\right)^2+c}}=\frac{\frac{c}{e}\left(\frac{d^2}{a}+\frac{e^2}{c}\right)}{\sqrt{\frac{c^2}{e^2}\left(\frac{d^2}{a}+\frac{e^2}{c}\right)}}=\sqrt{\frac{d^2}{a}+\frac{e^2}{c}}.$$

Corollary A. For an insurance organization, let S denote the random loss on a segment of its risks and let x be the retention limit the minimizes the probability

$$\Pr\left(\frac{S-E[s]}{\sqrt{Var[S]}} > f(x)\right),$$

where f(x) is the ratio of the security loading g(x) = dx + e and the standard deviation

$$h(x)=\sqrt{Var[S]}=\sqrt{ax^2+c}\ .$$

Then $x=\frac{cd}{ae}$ and

$$f\left(\frac{cd}{ae}\right)=\sqrt{\frac{d^2}{a}+\frac{e^2}{c}}\ .$$

Corollary B. Let a, b, c, d and e be positive numbers such that 4ac > $b^2$ and 2ae > bd. Then

$$f(x)=\frac{dx+e}{\sqrt{ax^2+bx+c}}$$

attains its maximum value

$$\sqrt{\frac{\frac{d^2}{a}+(2ae-bd)^2}{a(4ac-b^2)}}$$

at $x=\frac{2cd-be}{2ae-bd}$.

Proof. Write

$$f(x)=\frac{d\left(x+\frac{b}{2a}\right)+\frac{2ae-bd}{2a}}{\sqrt{a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}}}$$

and use Theorem A.

Theorem B. Let $f(x) = qb[\exp(-bx)]$ and $g(x) = -\exp(-ax)$. Then

$$h(d;c) = \int_0^\infty f(x)g(d-cx)dx = -\frac{qb[\exp(-ad)]}{b-ca}.$$

Corollary C. Let p be the probability that a property will not be damaged in the next period and let f(x) in Theorem B be the probability density function of a positive random variable X with q = 1 – p. If the owner of the property with wealth w has a utility function g(x) in Theorem B and is offered an insurance policy that will pay1 – c portion of any loss during the next period, then the maximum premium G that the property owner will pay for this insurance is

$$G = \frac{1}{a}\ln\frac{p+\frac{qb}{b-a}}{p+\frac{qb}{b-ca}}.$$

Proof. Equating the utilities with and without insurance, we have

$$pg(w-G) + h(w-G;c) = pg(w) + h(w;1).$$

It follows from Theorem B that

$$-p\{\exp[-a(w-G)]\} - \frac{qb}{b-a}\exp[-a(w-G)]$$

$$= -p[\exp(-aw)] - \frac{qb}{b-a}\exp(-aw)$$

so that

$$\left(p+\frac{qb}{b-ca}\right)\exp(aG) = p+\frac{qb}{b-a}.$$

The corollary follows.

Theorem C.   Let

$$f(x) = \frac{2}{a}\left(1 - \frac{x}{a}\right),\ 0 \le x \le a,$$

be the probability density function of a random variable X. Then

$$E[X^n] = \frac{a^n}{\binom{n+2}{2}}.$$

Corollary D.   The mean and variance of the random variable X in Theorem C are $\frac{a}{3}$ and $\frac{a^2}{18}$, respectively.

Theorem D.   A decision maker has wealth w, has a utility function

$$u(x) = x^r,\ 0 < x < 1$$

and faces a random loss X with a uniform distribution on [0,w]. Then the maximum amount this decision maker will pay for the complete insurance against the random loss is

$$G = \left[1 - \left(\frac{1}{r+1}\right)^{\frac{1}{r}}\right] w.$$

Proof.   Equating the utilities with and without insurance, we have

$$(w - G)^r = \int_0^w \frac{1}{w}(w - x)^r\, dx.$$

It follows that

$$(w - G)^r = \frac{w^r}{r+1}.$$

The theorem follows.

Theorem E. Assume that a decision maker will retain wealth w with probability p and will suffer a loss c with probability q = 1 – p. Based on the utility function

$$u(x) = x - ax^2,\ 0 < x < \frac{1}{2a}\ \ (\text{ a} > 0 \text{ }),$$

the maximum insurance premium that the decision maker will pay for the complete insurance is

$$G = w - \frac{1}{2a}\{1 - [1 - 4apw(1 - aw) + 4aq(w - c) - 4a^2 q(w - c)^2]^{\frac{1}{2}}\}.$$

Proof. Equating the utilities with and without insurance, we have

$$(w - G) - a(w - G)^2 = pw(1 - aw) + q(w - c)[1 - a(w - c)].$$

It follows that

$$w - G = \frac{1}{2a}\{1 - [4apw(1 - aw) + 4aq(w - c) - 4a^2 q(w - c)^2]^{\frac{1}{2}}\}.$$

The theorem follows.

Theorem F. Let $X_i$, i = 1, 2, 3, …, n, be nonnegative mutually independent random variable with the probability density function $f_i(t)$. If the moment generating function $M_{X_i}(t)$ of each $X_i$ is finite on some interval, then the convolution $f_1 * f_2(x)$ is the unique probability density function of $S = \sum_1^n X_i$.

Proof. We shall only prove the case with n = 2. For any t in the given interval, we have

$$M_S(t) = \int_0^\infty e^{tx} \int_0^x f_1(x - y) f_2(y) dy dx = \int_0^\infty e^{ty} f_2(y) \int_0^\infty e^{tz} f_1(z) dz dy,$$

where $z = x - y$.

Hence $M_S(t) = M_{X_1}(t) M_{X_2}(t)$ and hence the theorem follows.

# 4. SUDOKU ECSTACY

## 4.1. ORIENTATIONAL MOVES

On top of all sorts of skills, we have also developed shorthand annotations for keeping track of the order, location and type of each move! At the same time, the nomenclature of types of moves is very easy to remember and describe. For example, the move developed within a single box is called a “box move“ and the way of scanning among boxes and blocks is called the “scanning method“. As shown in Figure 73, the playground of Sudoku is divided into 81 grids, to be combined horizontally as nine rows top-down, vertically as nine columns left-right, and 3x3 squares as nine boxes. We follow the prescribed order of up-down and left-right, so the referral of each grid will be row first column next; for instance column next; for instance grid (32) stands for the grid located at the intersection of row 3 (r3) and column 2 (c2).

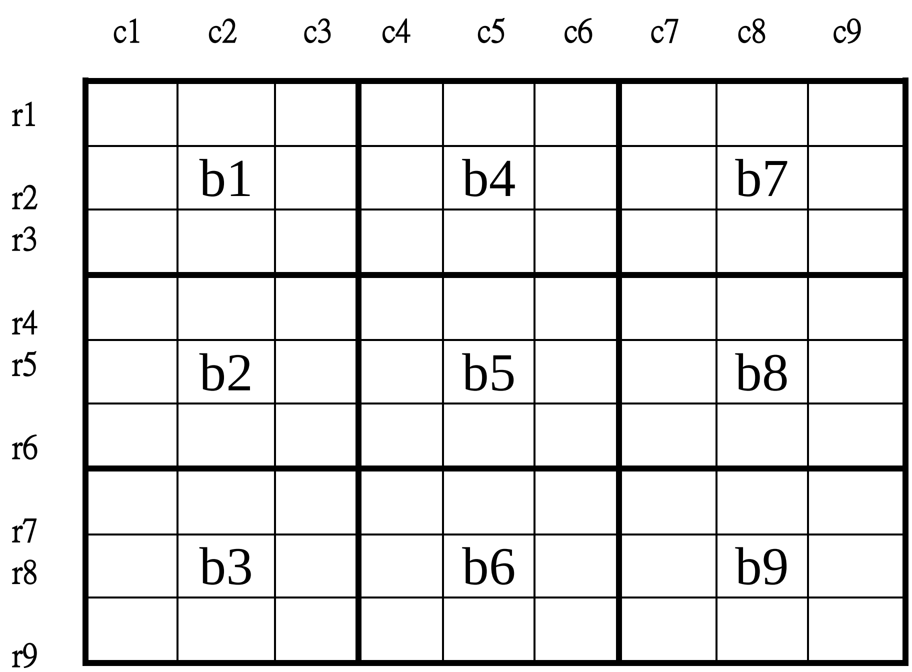



Figure 73. The playground of Sudoku

Similarly, the order of boxes is the same: box 1, box 2, box 3, box 4, box 5, box 6, box 7, box 8 and box 9 are called top-down and then left-right respectively as b1, b2, b3, b4, b5, b6, b7, b8 and b9. We shall later use the same prescribed order to place numbers at grids in rows, columns or boxes. The reason for doing so is simply to facilitate our explanations and mutual understanding with readers, but by no means to limit your flexibility in manipulation! Our unique invention is to combine three consecutive boxes as blocks: b1b2b3 as Left Block (abbreviated as LB), b4b5b6 as Middle Block (abbreviated as MB), b7b8b9 as Right Block (abbreviated as RB), b1b4b7 as Up Block (abbreviated as UB), b2b5b8 as Central Block (abbreviated as CB) and b3b6b9 as Down Block (abbreviated as DB).

A move that can be determined by scanning a single block is called a "single block move", while a move that requires the cross reference of two blocks is called a "double block move" such as left block move, up block move, up left block move, center middle block move, down right block move, etc.

## 4.2. FUNDAMENTAL MOVES

1) Single block move

Here we find the 1 in question can only be entered into grid (45), so as to avoid two 1's appearing either in the same row or column of central block. This is not only because both rows 5 and 6 have 1, but also the 1 in row 4 can neither be placed at grid (41) or grid (42) (which would contradict with the 1 in box 2) nor at any other grids already filled; just so we can gradually recognize the relationships of numbers among rows, columns and grids in the block, which is the skill of a block move.

The above first step was using central block move to place 1 at grid (45) , abbreviated as $1_1$(45)CB, where the subscript 1 indicates that this is the “first step“.

2) Double block move

Since each box is located in the intersection of two perpendicular blocks, we can simultaneously use both horizontal and vertical block moves to gradually find out the fillable number. There are altogether nine double block moves: up-left (UL), central-left (CL), down-left (DL), up-middle (UM), central-middle (CM), down-middle (DM), up-right (UR), central-right (CR) and down-right (DR).

Here we find 2 can only be entered into box 8 (grid (48) or grid (58)), because in right block both box 7 and box 9 have 8; had we entered 2 into grid (48), it would contradict with the 2 in row 4 of central block. Hence by double scanning of central and right blocks, we can only entered 2 into grid (58), abbreviated as $2_2$(58)CR.

3) Terminating move (t)

After each move, we should scan each related row, column and box to see if there is only one grid remained to be filled; if so, we should terminate it right away by filling in the very last number so that more easy target could reveal.

4) Row move (r)

Scanning unfilled grids of each row to gradually find among unfilled numbers a potential one to fill that won’t cause any conflict with the existing numbers in each column and each box.

5) Column move (c)

Scanning unfilled grids of each column to gradually find among unfilled numbers a potential one to fill that won't cause any conflict with the existing numbers in each row and each box.

6) Box move (b)

Scanning unfilled grids of each box to gradually find among unfilled numbers a potential one to fill that won't cause any conflict with the existing numbers in each row and each column.

7) Grid move (g)

Scanning unfilled grids of each row, column and box to gradually find among unfilled numbers a potential one to fill that won't cause any conflict with the existing numbers in its situated row, column and box. Unlike the above row column combo, after locating a grid we still need to preclude more than one potential unfilled number. Therefore, sometimes even the veterans would find the grid move "nowhere to set foot in". We first introduce the traditional solving methods, and later introduce our "unique secret skills".

8) Law of unique solution (u)

Suppose that a number has only two potential grids to fit in a row, column or box in attempting row column combo. If the choice of one of them would cause multiple solutions of the puzzle, then the number in question needs to be filled into the other grid. Sudoku puzzles do not allow multiple solutions to screw up the logical reasoning needed in the solving process.

**Puzzle 1**

When facing a y-junction, we don't simply take one road. Rather, be prepared for some easy way to pop up.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | | | 6* | 2* | | | | |
| 1* | | | | | | | 4* | |
| | | | | | | | | 1* |
| | | | 1* | | 4* | | 7* | |
| 2* | | | | | | | | |
| | | | | | 3* | | | |
| | 3* | | | | | 6* | | |
| | | | | 8* | | 2* | | |
| | 4* | 7* | | | | | | |

Figure 74. Figure 1 for Puzzle 1

For the puzzle in Figure 74, we can take to the thirteenth step as shown in Figure 75.

$2_2$b3 $2_3$c6

$4_8$r1 $4_{10}$c7

$6_{11}$b7 $7_{12}3_{13}$c1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | | $4_8$ | 6* | 2* | $1_1$ | | | |
| 1* | $2_7$ | | | | | | 4* | $6_{11}$ |
| $7_{12}$ | | | | | | | $2_6$ | 1* |
| $3_{13}$ | | | 1* | | 4* | | 7* | $2_5$ |
| 2* | | | | | | $4_{10}$ | | |
| $4_9$ | | | $2_4$ | | 3* | | | |
| | 3* | $2_2$ | | | | 6* | | |
| | | | | 8* | | 2* | | |
| | 4* | 7* | | | $2_3$ | | | |

Figure 75. Figure 2 for Puzzle 1

The situation shown in Figure 76 allows us to take

$7_{14}$r1: 7c9b9.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | | $4_8$ | 6* | 2* | $1_1$ | $7_{14}$ | | |
| 1* | $2_7$ | | | | | | 4* | $6_{11}$ |
| $7_{12}$ | | | | | | | $2_6$ | 1* |
| $3_{13}$ | | | 1* | | 4* | | 7* | $2_5$ |
| 2* | | | | | | $4_{10}$ | | |
| $4_9$ | | | $2_4$ | | 3* | | | |
| | 3* | $2_2$ | | | | 6* | | 7? |
| | | | | 8* | | 2* | | 7? |
| | 4* | 7* | | | $2_3$ | | | |

Figure 76. Figure 3 for Puzzle 1

The situation shown in Figure 77 allows us to take the next twenty one steps in Figure 78.

$3_{15}$c7: 3(18)(19)

$4_{17}7_{18}6_{19}9_{20}$r8

$1_{32}$b9 $8_{34}$b5

$7_{35}$c4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | | $4_{8}$ | 6* | 2* | $1_{1}$ | $7_{14}$ | 3? | 3? |
| 1* | $2_{7}$ | | | | | | 4* | $6_{11}$ |
| $7_{12}$ | | | | | | | $2_{6}$ | 1* |
| $3_{13}$ | | | 1* | | 4* | | 7* | $2_{5}$ |
| 2* | | | | | | $4_{10}$ | | |
| $4_{9}$ | | | $2_{4}$ | | 3* | | | |
| | 3* | $2_{2}$ | | | | 6* | | |
| | | | | 8* | | 2* | | |
| | 4* | 7* | | | $2_{3}$ | | | |

Figure 77. Figure 4 for Puzzle 1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | | $4_{8}$ | 6* | 2* | $1_{1}$ | $7_{14}$ | | |
| 1* | $2_{7}$ | | $7_{35}$ | $3_{25}$ | | | 4* | $6_{11}$ |
| $7_{12}$ | $6_{27}$ | $3_{26}$ | | $4_{24}$ | | | $2_{6}$ | 1* |
| $3_{13}$ | | $6_{30}$ | 1* | | 4* | | 7* | $2_{5}$ |
| 2* | | | $8_{34}$ | | $6_{28}$ | $4_{10}$ | | |
| $4_{9}$ | | | $2_{4}$ | | 3* | $1_{33}$ | $6_{29}$ | |
| | 3* | $2_{2}$ | $4_{23}$ | $1_{22}$ | | 6* | | $7_{31}$ |
| $6_{19}$ | | | $3_{16}$ | 8* | $7_{18}$ | 2* | $3_{20}$ | $4_{17}$ |
| | 4* | 7* | | $6_{21}$ | $2_{3}$ | $3_{15}$ | $1_{32}$ | |

Figure 78. Figure 5 for Puzzle 1

Now we are facing a y- junction to choose between 8(42) and 8(47) as shown in Figure 79.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | | $4_{8}$ | 6* | 2* | $1_{1}$ | $7_{14}$ | | |
| 1* | $2_{7}$ | | $7_{35}$ | $3_{25}$ | | | 4* | $6_{11}$ |
| $7_{12}$ | $6_{27}$ | $3_{26}$ | | $4_{24}$ | | | $2_{6}$ | 1* |
| $3_{13}$ | 8? | $6_{30}$ | 1* | | 4* | 8? | 7* | $2_{5}$ |
| 2* | | | $8_{34}$ | | $6_{28}$ | $4_{10}$ | | |
| $4_{9}$ | | | $2_{4}$ | | 3* | $1_{33}$ | $6_{29}$ | |
| | 3* | $2_{2}$ | $4_{23}$ | $1_{22}$ | | 6* | | $7_{31}$ |
| $6_{19}$ | | | $3_{16}$ | 8* | $7_{18}$ | 2* | $3_{20}$ | $4_{17}$ |
| | 4* | 7* | | $6_{21}$ | $2_{3}$ | $3_{15}$ | $1_{32}$ | |

Figure 79. Figure 6 for Puzzle 1

As displayed in Figure 80, taking the road of 8(42), we would come to a dead end.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | No8 | $4_{8}$ | 6* | 2* | $1_{1}$ | $7_{14}$ | No8 | No8 |
| 1* | $2_{7}$ | | $7_{35}$ | $3_{25}$ | | | 4* | $6_{11}$ |
| $7_{12}$ | $6_{27}$ | $3_{26}$ | | $4_{24}$ | | | $2_{6}$ | 1* |
| $3_{13}$ | $8_{1}$ | $6_{30}$ | 1* | | 4* | | 7* | $2_{5}$ |
| 2* | | | $8_{34}$ | | $6_{28}$ | $4_{10}$ | | |
| $4_{9}$ | | | $2_{4}$ | | 3* | $1_{33}$ | $6_{29}$ | $8_{2}$ |
| | 3* | $2_{2}$ | $4_{23}$ | $1_{22}$ | | 6* | $8_{3}$ | $7_{31}$ |
| $6_{19}$ | | | $3_{16}$ | 8* | $7_{18}$ | 2* | $3_{20}$ | $4_{17}$ |
| | 4* | 7* | | $6_{21}$ | $2_{3}$ | $3_{15}$ | $1_{32}$ | |

Figure 80. Figure 7 for Puzzle 1

Therefore, we should take the road not taken, namely

$8_{36}$r4: 8(42)
→8(69)
→8(78)
→No8r1

in Figure 81.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | | $4_{8}$ | 6* | 2* | $1_{1}$ | $7_{14}$ | | |
| 1* | $2_{7}$ | | $7_{35}$ | $3_{25}$ | | | 4* | $6_{11}$ |
| $7_{12}$ | $6_{27}$ | $3_{26}$ | | $4_{24}$ | | | $2_{6}$ | 1* |
| $3_{13}$ | | $6_{30}$ | 1* | | 4* | $8_{36}$ | 7* | $2_{5}$ |
| 2* | | | $8_{34}$ | | $6_{28}$ | $4_{10}$ | | |
| $4_{9}$ | | | $2_{4}$ | | 3* | $1_{33}$ | $6_{29}$ | |
| | 3* | $2_{2}$ | $4_{23}$ | $1_{22}$ | | 6* | | $7_{31}$ |
| $6_{19}$ | | | $3_{16}$ | 8* | $7_{18}$ | 2* | $3_{20}$ | $4_{17}$ |
| | 4* | 7* | | $6_{21}$ | $2_{3}$ | $3_{15}$ | $1_{32}$ | |

Figure 81. Figure 8 for Puzzle 1

We thus complete the puzzle in Figure 82.

$9_{37}$r1: 9c7b7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | $9_{37}$ | $4_{8}$ | 6* | 2* | $1_{1}$ | $7_{14}$ | $8_{52}$ | $3_{51}$ |
| 1* | $2_{7}$ | $8_{38}$ | $7_{35}$ | $3_{25}$ | $5_{62}$ | $9_{61}$ | 4* | $6_{11}$ |
| $7_{12}$ | $6_{27}$ | $3_{26}$ | $9_{59}$ | $4_{24}$ | $8_{63}$ | $5_{64}$ | $2_{6}$ | 1* |
| $3_{13}$ | $5_{43}$ | $6_{30}$ | 1* | $9_{45}$ | 4* | $8_{36}$ | 7* | $2_{5}$ |
| 2* | $7_{40}$ | $1_{41}$ | $8_{34}$ | $5_{47}$ | $6_{28}$ | $4_{10}$ | $3_{50}$ | $9_{49}$ |
| $4_{9}$ | $8_{39}$ | $9_{44}$ | $2_{4}$ | $7_{46}$ | 3* | $1_{33}$ | $6_{29}$ | $5_{48}$ |
| $8_{55}$ | 3* | $2_{2}$ | $4_{23}$ | $1_{22}$ | $9_{60}$ | 6* | $5_{53}$ | $7_{31}$ |
| $6_{19}$ | $1_{42}$ | $5_{57}$ | $3_{16}$ | 8* | $7_{18}$ | 2* | $3_{20}$ | $4_{17}$ |
| $9_{56}$ | 4* | 7* | $5_{58}$ | $6_{21}$ | $2_{3}$ | $3_{15}$ | $1_{32}$ | $8_{54}$ |

Figure 82. Figure 9 for Puzzle 1

**Puzzle 2**

After taking

$5_1$b1 $6_2$b9 $8_3$b5

in Figure 84 of the puzzle in Figure 83, we spot two sets of flipflops.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | 3* | | | | | | | |
| | | | | 8* | | 5* | | |
| 7* | | | | | | | | |
| | | | 3* | | 1* | | | 6* |
| | 8* | 5* | | | | | | |
| | | | 2* | | | | | |
| 1* | | | | | | | 2* | |
| | | 6* | | 4* | | | | |
| | | | | | | 7* | 3* | |

Figure 83. Figure 1 for Puzzle 2

As we can see the flipflop 23(26)(29) leads us nowhere, while 58(81)(91) would help! We can take

$3_4$b3: 58(81)(91).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | 3* | | | | | | | |
| | | | | 8* | 2/3 | 5* | | 3/2 |
| 7* | $5_{1}$ | | | | | | | |
| | | | 3* | | 1* | | | 6* |
| | 8* | 5* | | | | | | |
| | | | 2* | | $8_{3}$ | | | |
| 1* | | $3_{4}$ | | | | $6_{2}$ | 2* | |
| 5/8 | | 6* | | 4* | | | | |
| 8/5 | | | | | | 7* | 3* | |

Figure 84. Figure 2 for Puzzle 2

We can now complete the puzzle as in Figure 85.

$7_{12}$b3 $7_{15}$r2

$2_{19}1_{20}3_{21}$r5

$5_{25}$g $1_{28}6_{29}$c5

$1_{39}$b7 $4_{42}$b8

$4_{52}$b1 $5_{25}$g

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | 3* | $9_{53}$ | $5_{32}$ | $1_{28}$ | $7_{18}$ | $4_{44}$ | $6_{58}$ | $8_{49}$ |
| $4_{52}$ | $6_{38}$ | $1_{40}$ | $9_{59}$ | 8* | $2_{6}$ | 5* | $7_{15}$ | $3_{8}$ |
| 7* | $5_{1}$ | $8_{50}$ | $6_{61}$ | $3_{7}$ | $4_{62}$ | $1_{39}$ | $9_{60}$ | $2_{23}$ |
| $9_{54}$ | $4_{55}$ | $2_{11}$ | 3* | $7_{17}$ | 1* | $8_{27}$ | $5_{26}$ | 6* |
| $3_{21}$ | 8* | 5* | $4_{63}$ | $6_{29}$ | $9_{64}$ | $2_{19}$ | $1_{20}$ | $7_{16}$ |
| $6_{37}$ | $1_{22}$ | $7_{13}$ | 2* | $5_{25}$ | $8_{3}$ | $3_{24}$ | $4_{42}$ | $9_{43}$ |
| 1* | $7_{12}$ | $3_{4}$ | $8_{35}$ | $9_{30}$ | $5_{34}$ | $6_{2}$ | 2* | $4_{36}$ |
| $5_{48}$ | $2_{10}$ | 6* | $7_{14}$ | 4* | $3_{5}$ | $9_{45}$ | $8_{47}$ | $1_{41}$ |
| $8_{51}$ | $9_{56}$ | $4_{57}$ | $1_{31}$ | $2_{9}$ | $6_{33}$ | 7* | 3* | $5_{46}$ |

Figure 85. Figure 3 for Puzzle 2

**Puzzle 3**

For the puzzle in Figure 86, we can take to the sixteenth step as shown in Figure 87.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 9* | | | | | | | 1* |
| | | | | 2* | | 4* | | |
| | | | 3* | | | | | |
| 2* | | 8* | | 6* | | | | |
| | | | | | | | 9* | 5* |
| | | | | | 1* | | | |
| | | | 4* | | | 8* | 2* | |
| 1* | 5* | | | | | | | |
| 7* | | | | | | | | |

Figure 86. Figure 1 for Puzzle 3

$1_1 5_2$r7

$8_4$b3

$2_5$c2

$5_9 9_{10}$r4

$7_{11}$r7

$8_{12}$b8

$9_{13}$b4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 9* | | | $5_{14}$ | | $2_7$ | | 1* |
| | | | $1_3$ | 2* | | 4* | | $9_{15}$ |
| | $2_5$ | $1_6$ | 3* | $9_{13}$ | | | | $8_{16}$ |
| 2* | | 8* | $5_9$ | 6* | $9_{10}$ | | | |
| | | | | 3? | 3? | | 9* | 5* |
| | | | | 3? | 1* | | $8_{12}$ | $2_8$ |
| | | | 4* | $1_1$ | $5_2$ | 8* | 2* | $7_{11}$ |
| 1* | 5* | | | | | | | |
| 7* | $8_4$ | | | | | | | |

Figure 87. Figure 2 for Puzzle 3

As in Figure 87, we need to choose from column 5 or 6 for 3 to be in box 5, but the latter would run into the dilemma in Figure 88.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 9* | | | $5_{14}$ | | $2_7$ | | 1* |
| | | | $1_3$ | 2* | | 4* | | $9_{15}$ |
| | $2_5$ | $1_6$ | 3* | $9_{13}$ | | | | $8_{16}$ |
| 2* | | 8* | $5_9$ | 6* | $9_{10}$ | | | |
| | | | | 3? | | | 9* | 5* |
| | | | | 3? | 1* | | $8_{12}$ | $2_8$ |
| | | | 4* | $1_1$ | $5_2$ | 8* | 2* | $7_{11}$ |
| 1* | 5* | | | | | | | |
| 7* | $8_4$ | | | No# | | | | |

Figure 88. Figure 3 for Puzzle 3

Therefore, we can

Take in Figure 89

$3_{17}$b5:
3c5b5→No#(95),

but with the question

mark left behind.

What number to fill in

(95)? (See Figure 90)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 9* | | | $5_{14}$ | | $2_{7}$ | | 1* |
| | | | $1_{3}$ | 2* | | 4* | | $9_{15}$ |
| | $2_{5}$ | $1_{6}$ | 3* | $9_{13}$ | | | | $8_{16}$ |
| 2* | | 8* | $5_{9}$ | 6* | $9_{10}$ | | | |
| | | | | | $3_{17}$ | | 9* | 5* |
| | | | | | 1* | | $8_{12}$ | $2_{8}$ |
| | | | 4* | $1_{1}$ | $5_{2}$ | 8* | 2* | $7_{11}$ |
| 1* | 5* | | | | | | | |
| 7* | $8_{4}$ | | | ? | | | | |

Figure 89. Figure 4 for Puzzle 3

You guessed it, it is 3!

That's the exactly

place for 3! What we

are playing is Sudoku,

the Japanese meaning

of which is "Unique

Number Placement".

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 9* | | | $5_{14}$ | | $2_{7}$ | | 1* |
| | | | $1_{3}$ | 2* | | 4* | | $9_{15}$ |
| | $2_{5}$ | $1_{6}$ | 3* | $9_{13}$ | | | | $8_{16}$ |
| 2* | | 8* | $5_{9}$ | 6* | $9_{10}$ | | | |
| | | | 2 | | $3_{17}$ | | 9* | 5* |
| | | | | | 1* | | $8_{12}$ | $2_{8}$ |
| | | | 4* | $1_{1}$ | $5_{2}$ | 8* | 2* | $7_{11}$ |
| 1* | 5* | | | | | | | |
| 7* | $8_{4}$ | | | 3 | | | | |

Figure 90. Figure 5 for Puzzle 3

Now, as in Figure 91,

let us take a step back,

as in Figure 87, prior

to the seventeenth step

and ask: What number

to fill in (95)?

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 9* | | | $5_{14}$ | | $2_{7}$ | | 1* |
| | | | $1_{3}$ | 2* | | 4* | | $9_{15}$ |
| | $2_{5}$ | $1_{6}$ | 3* | $9_{13}$ | | | | $8_{16}$ |
| 2* | | 8* | $5_{9}$ | 6* | $9_{10}$ | | | |
| | | | | | | | 9* | 5* |
| | | | | | 1* | | $8_{12}$ | $2_{8}$ |
| | | | 4* | $1_{1}$ | $5_{2}$ | 8* | 2* | $7_{11}$ |
| 1* | 5* | | | | | | | |
| 7* | $8_{4}$ | | | ? | | | | |

Figure 91. Figure 6 for Puzzle 3

Let us now look back to those two choices between columns 5 and 6 in box 5 for the number 3. Because of 3(95) as in Figure 92, the right choice has to be column 6.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 9* | | | $5_{14}$ | | $2_{7}$ | | 1* |
| | | | $1_{3}$ | 2* | | 4* | | $9_{15}$ |
| | $2_{5}$ | $1_{6}$ | 3* | $9_{13}$ | | | | $8_{16}$ |
| 2* | | 8* | $5_{9}$ | 6* | $9_{10}$ | | | |
| | | | | | $3_{2}$ | | 9* | 5* |
| | | | | | 1* | | $8_{12}$ | $2_{8}$ |
| | | | 4* | $1_{1}$ | $5_{2}$ | 8* | 2* | $7_{11}$ |
| 1* | 5* | | | | | | | |
| 7* | $8_{4}$ | | | $3_{1}$ | | | | |

Figure 92. Figure 7 for Puzzle 3

Since things are straighten out, let us go back to Figure 90. We can continue to the twenty-third step as shown in Figure 93.

$4_{21}$b5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 9* | | | $5_{14}$ | | $2_{7}$ | | 1* |
| | | | $1_{3}$ | 2* | | 4* | | $9_{15}$ |
| | $2_{5}$ | $1_{6}$ | 3* | $9_{13}$ | | | | $8_{16}$ |
| 2* | | 8* | $5_{9}$ | 6* | $9_{10}$ | | | |
| | | | $2_{18}$ | $8_{19}$ | $3_{17}$ | | 9* | 5* |
| | | | $7_{22}$ | $4_{21}$ | 1* | | $8_{12}$ | $2_{8}$ |
| | | | 4* | $1_{1}$ | $5_{2}$ | 8* | 2* | $7_{11}$ |
| 1* | 5* | | | $7_{23}$ | | | | |
| 7* | $8_{4}$ | | | $3_{20}$ | | | | |

Figure 93. Figure 8 for Puzzle 3

From the inferences displayed in Figure 94, we can take

$6_{24}$g: 59(61)(63)
→3c2b2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 9* | | | $5_{14}$ | | $2_{7}$ | | 1* |
| | | | $1_{3}$ | 2* | | 4* | | $9_{15}$ |
| | $2_{5}$ | $1_{6}$ | 3* | $9_{13}$ | | | | $8_{16}$ |
| 2* | 3? | 8* | $5_{9}$ | 6* | $9_{10}$ | | | |
| | | | $2_{18}$ | $8_{19}$ | $3_{17}$ | | 9* | 5* |
| 5/9 | 3? | 9/5 | $7_{22}$ | $4_{21}$ | 1* | | $8_{12}$ | $2_{8}$ |
| | ? | | 4* | $1_{1}$ | $5_{2}$ | 8* | 2* | $7_{11}$ |
| 1* | 5* | | | $7_{23}$ | | | | |
| 7* | $8_{4}$ | | | $3_{20}$ | | | | |

Figure 94. Figure 9 for Puzzle 3

From the situation shown in Figure 95, we can take

$6_{25}3_{26}$r6: 59(61)(63)

in Figure 96.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 9* | | | $5_{14}$ | | $2_{7}$ | | 1* |
| | | | $1_{3}$ | 2* | | 4* | | $9_{15}$ |
| | $2_{5}$ | $1_{6}$ | 3* | $9_{13}$ | | | | $8_{16}$ |
| 2* | | 8* | $5_{9}$ | 6* | $9_{10}$ | | | |
| | | | $2_{18}$ | $8_{19}$ | $3_{17}$ | | 9* | 5* |
| 5/9 | | 9/5 | $7_{22}$ | $4_{21}$ | 1* | | $8_{12}$ | $2_{8}$ |
| | $6_{24}$ | | 4* | $1_{1}$ | $5_{2}$ | 8* | 2* | $7_{11}$ |
| 1* | 5* | | | $7_{23}$ | | | | |
| 7* | $8_{4}$ | | | $3_{20}$ | | | | |

Figure 95. Figure 10 for Puzzle 3

What number to fill in (22)? The grid move again! Accordingly, we take

$7_{27}$g

in Figure 96.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 9* | | | $5_{14}$ | | $2_{7}$ | | 1* |
| | ? | | $1_{3}$ | 2* | | 4* | | $9_{15}$ |
| | $2_{5}$ | $1_{6}$ | 3* | $9_{13}$ | | | | $8_{16}$ |
| 2* | | 8* | $5_{9}$ | 6* | $9_{10}$ | | | |
| | | | $2_{18}$ | $8_{19}$ | $3_{17}$ | | 9* | 5* |
| | $3_{26}$ | | $7_{22}$ | $4_{21}$ | 1* | $6_{25}$ | $8_{12}$ | $2_{8}$ |
| | $6_{24}$ | | 4* | $1_{1}$ | $5_{2}$ | 8* | 2* | $7_{11}$ |
| 1* | 5* | | | $7_{23}$ | | | | |
| 7* | $8_{4}$ | | | $3_{20}$ | | | | |

Figure 96. Figure 11 for Puzzle 3

Finally, we can complete the puzzle in Figure 97.

$4_{30}$r5

$3_{36}$c7

$3_{47}$c9

$6_{50}$r3

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $4_{46}$ | 9* | $6_{60}$ | $8_{59}$ | $5_{14}$ | $7_{38}$ | $2_{7}$ | $3_{55}$ | 1* |
| $8_{56}$ | $7_{27}$ | $3_{61}$ | $1_{3}$ | 2* | $6_{62}$ | 4* | $5_{54}$ | $9_{15}$ |
| $5_{51}$ | $2_{5}$ | $1_{6}$ | 3* | $9_{13}$ | $4_{39}$ | $7_{37}$ | $6_{50}$ | $8_{16}$ |
| 2* | $1_{32}$ | 8* | $5_{9}$ | 6* | $9_{10}$ | $3_{36}$ | $7_{40}$ | $4_{41}$ |
| $6_{29}$ | $4_{30}$ | $7_{28}$ | $2_{18}$ | $8_{19}$ | $3_{17}$ | $1_{31}$ | 9* | 5* |
| $9_{53}$ | $3_{26}$ | $5_{52}$ | $7_{22}$ | $4_{21}$ | 1* | $6_{25}$ | $8_{12}$ | $2_{8}$ |
| $3_{57}$ | $6_{24}$ | $9_{58}$ | 4* | $1_{1}$ | $5_{2}$ | 8* | 2* | $7_{11}$ |
| 1* | 5* | $2_{44}$ | $6_{63}$ | $7_{23}$ | $8_{64}$ | $9_{35}$ | $4_{42}$ | $3_{47}$ |
| 7* | $8_{4}$ | $4_{43}$ | $9_{49}$ | $3_{20}$ | $2_{45}$ | $5_{34}$ | $1_{33}$ | $6_{48}$ |

Figure 97. Figure 12 for Puzzle 3

**Puzzle 4**

For the puzzle in Figure 98, we can take to the twentyninth step as shown in Figure 99.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 6* | | | | | | 1* | |
| | 7* | 2* | | | | | | |
| | | | 3* | | | | | |
| 1* | | 5* | | 3* | | | | |
| | | | | 2* | | 8* | | |
| 4* | | | | | | | | |
| | | | | | | 7* | | 2* |
| 5* | | | 1* | | | | | |
| | | | 4* | | | 3* | | |

Figure 98. Figure 1 for Puzzle 4

$1_{2}4_{3}$b1

$2_{11}7_{12}6_{13}$c1

$4_{20}$r5

$5_{24}$b9

$5_{25}$r5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 6* | $4_{3}$ | $2_{15}$ | | | $5_{27}$ | 1* | |
| | 7* | 2* | | $1_{8}$ | $5_{28}$ | $4_{23}$ | | |
| | $5_{1}$ | $1_{2}$ | 3* | $4_{22}$ | | | | |
| 1* | | 5* | | 3* | $4_{21}$ | | | |
| $7_{12}$ | | | $5_{25}$ | 2* | $1_{7}$ | 8* | | $4_{20}$ |
| 4* | | | | | | $1_{6}$ | | $5_{26}$ |
| $6_{13}$ | $1_{4}$ | | | $5_{29}$ | $3_{16}$ | 7* | $4_{10}$ | 2* |
| 5* | $4_{9}$ | $3_{17}$ | 1* | $7_{19}$ | $2_{14}$ | | | |
| $2_{11}$ | | $7_{18}$ | 4* | | | 3* | $5_{24}$ | $1_{5}$ |

Figure 99. Figure 2 for Puzzle 4

Now, we come to a y-junction as shown in Figure 100: 8(42) or 8(44).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 6* | $4_{3}$ | $2_{15}$ | | | $5_{27}$ | 1* | |
| | 7* | 2* | | $1_{8}$ | $5_{28}$ | $4_{23}$ | | |
| | $5_{1}$ | $1_{2}$ | 3* | $4_{22}$ | | | | |
| 1* | 8? | 5* | 8? | 3* | $4_{21}$ | | | |
| $7_{12}$ | | | $5_{25}$ | 2* | $1_{7}$ | 8* | | $4_{20}$ |
| 4* | | | | | | $1_{6}$ | | $5_{26}$ |
| $6_{13}$ | $1_{4}$ | | | $5_{29}$ | $3_{16}$ | 7* | $4_{10}$ | 2* |
| 5* | $4_{9}$ | $3_{17}$ | 1* | $7_{19}$ | $2_{14}$ | | | |
| $2_{11}$ | | $7_{18}$ | 4* | | | 3* | $5_{24}$ | $1_{5}$ |

Figure 100. Figure 3 for Puzzle 4

From Figure 100, we can see that

$8_{30}$r4: 8(44)
→8(73)&679r6b5
→No#(63),

to be taken in Figure 101.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 6* | $4_{3}$ | $2_{15}$ | | | $5_{27}$ | 1* | |
| | 7* | 2* | | $1_{8}$ | $5_{28}$ | $4_{23}$ | | |
| | $5_{1}$ | $1_{2}$ | 3* | $4_{22}$ | | | | |
| 1* | | 5* | $8_{1}$ | 3* | $4_{21}$ | | | |
| $7_{12}$ | | | $5_{25}$ | 2* | $1_{7}$ | 8* | | $4_{20}$ |
| 4* | | No# | 7? | 6? | 9? | $1_{6}$ | | $5_{26}$ |
| $6_{13}$ | $1_{4}$ | $8_{2}$ | | $5_{29}$ | $3_{16}$ | 7* | $4_{10}$ | 2* |
| 5* | $4_{9}$ | $3_{17}$ | 1* | $7_{19}$ | $2_{14}$ | | | |
| $2_{11}$ | | $7_{18}$ | 4* | | | 3* | $5_{24}$ | $1_{5}$ |

Figure 101. Figure 4 for Puzzle 4

After the thirtysixth step, from the situation shown in Figure 102, we can take

$9_{37}$c3:9r6b5

as in Figure 103.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 6* | $4_{3}$ | $2_{15}$ | | | $5_{27}$ | 1* | |
| | 7* | 2* | | $1_{8}$ | $5_{28}$ | $4_{23}$ | | |
| | $5_{1}$ | $1_{2}$ | 3* | $4_{22}$ | | | | |
| 1* | $8_{30}$ | 5* | | 3* | $4_{21}$ | | | |
| $7_{12}$ | $3_{32}$ | | $5_{25}$ | 2* | $1_{7}$ | 8* | | $4_{20}$ |
| 4* | $2_{31}$ | | | 9? | 9? | $1_{6}$ | $3_{36}$ | $5_{26}$ |
| $6_{13}$ | $1_{4}$ | $8_{34}$ | $9_{35}$ | $5_{29}$ | $3_{16}$ | 7* | $4_{10}$ | 2* |
| 5* | $4_{9}$ | $3_{17}$ | 1* | $7_{19}$ | $2_{14}$ | | | |
| $2_{11}$ | $9_{33}$ | $7_{18}$ | 4* | | | 3* | $5_{24}$ | $1_{5}$ |

Figure 102. Figure 5 for Puzzle 4

Finally, we can complete the puzzle in Figure 103.

$7_{41}$c4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $8_{52}$ | 6* | $4_{3}$ | $2_{15}$ | $9_{47}$ | $7_{48}$ | $5_{27}$ | 1* | $3_{51}$ |
| $3_{53}$ | 7* | 2* | $8_{42}$ | $1_{8}$ | $5_{28}$ | $4_{23}$ | $9_{55}$ | $6_{50}$ |
| $9_{54}$ | $5_{1}$ | $1_{2}$ | 3* | $4_{22}$ | $6_{43}$ | $2_{62}$ | $7_{61}$ | $8_{59}$ |
| 1* | $8_{30}$ | 5* | $6_{40}$ | 3* | $4_{21}$ | $9_{63}$ | $2_{64}$ | $7_{60}$ |
| $7_{12}$ | $3_{32}$ | $9_{37}$ | $5_{25}$ | 2* | $1_{7}$ | 8* | $6_{39}$ | $4_{20}$ |
| 4* | $2_{31}$ | $6_{38}$ | $7_{41}$ | $8_{46}$ | $9_{49}$ | $1_{6}$ | $3_{36}$ | $5_{26}$ |
| $6_{13}$ | $1_{4}$ | $8_{34}$ | $9_{35}$ | $5_{29}$ | $3_{16}$ | 7* | $4_{10}$ | 2* |
| 5* | $4_{9}$ | $3_{17}$ | 1* | $7_{19}$ | $2_{14}$ | $6_{56}$ | $8_{58}$ | $9_{57}$ |
| $2_{11}$ | $9_{33}$ | $7_{18}$ | 4* | $6_{44}$ | $8_{45}$ | 3* | $5_{24}$ | $1_{5}$ |

Figure 103. Figure 6 for Puzzle 4

**Puzzle 5**

We can take to the thirtyfifth step in Figure 105 of the puzzle in Figure 104.

$5_{2}$c1 $7_{8}2_{9}$c7 $2_{15}$b1

$3_{17}$c6 $3_{19}$c7 $9_{20}$b4

$9_{22}$b9 $2_{29}$r4 $4_{34}$b2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | | | | 7* | | | | |
| | 3* | | | | | 9* | | |
| | | | | | | 5* | | |
| 1* | | | 5* | | | 4* | | |
| 7* | | 2* | | | 8* | | | |
| | | | | | | | | |
| | | | | 3* | | | 7* | |
| | | 3* | | | | | 2* | |
| | 5* | | 9* | | | | | |

Figure 104. Figure 1 for Puzzle 5

It's time for a chain of flipflops. Let us start from box 8 and then expand the chain to cover the entire right block as displayed in Figure 106.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | | $9_{31}$ | | 7* | $5_{4}$ | $2_{9}$ | $3_{29}$ | |
| $5_{2}$ | 3* | | $2_{28}$ | | | 9* | | $7_{14}$ |
| $2_{15}$ | $7_{13}$ | | | $9_{20}$ | $3_{17}$ | 5* | | |
| 1* | | | 5* | $2_{26}$ | $7_{10}$ | 4* | $9_{23}$ | $3_{24}$ |
| 7* | $9_{30}$ | 2* | $3_{18}$ | $4_{34}$ | 8* | | $5_{7}$ | |
| $3_{1}$ | $4_{33}$ | $5_{3}$ | | | $9_{21}$ | $7_{8}$ | $8_{35}$ | $2_{25}$ |
| $9_{32}$ | $2_{16}$ | | | 3* | | | 7* | $5_{6}$ |
| | | 3* | $7_{11}$ | $5_{5}$ | | | 2* | $9_{22}$ |
| | 5* | $7_{12}$ | 9* | | $2_{27}$ | $3_{19}$ | | |

Figure 105. Figure 2 for Puzzle 5

We are going to introduce here a brand new move, called "the residue of flpflops chain move".

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | | $9_{31}$ | | 7* | $5_{4}$ | $2_{9}$ | $3_{29}$ | 8/4 |
| $5_{2}$ | 3* | | $2_{28}$ | | | 9* | 1/6 | $7_{14}$ |
| $2_{15}$ | $7_{13}$ | | | $9_{20}$ | $3_{17}$ | 5* | 6/1 | 4/8 |
| 1* | | | 5* | $2_{26}$ | $7_{10}$ | 4* | $9_{23}$ | $3_{24}$ |
| 7* | $9_{30}$ | 2* | $3_{18}$ | $4_{34}$ | 8* | 1/6 | $5_{7}$ | 6/1 |
| $3_{1}$ | $4_{33}$ | $5_{3}$ | 1/6 | 6/1 | $9_{21}$ | $7_{8}$ | $8_{35}$ | $2_{25}$ |
| $9_{32}$ | $2_{16}$ | | | 3* | | | 7* | $5_{6}$ |
| | | 3* | $7_{11}$ | $5_{5}$ | | | 2* | $9_{22}$ |
| | 5* | $7_{12}$ | 9* | | $2_{27}$ | $3_{19}$ | | 1/6 |

Figure 106. Figure 3 for Puzzle 5

In Figure 107, we can find the residue 4(98) in column 8. So, we can take

$4_{36}$c8: rcf-16(28)(38)

in Figure 108.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | | $9_{31}$ | | 7* | $5_4$ | $2_9$ | $3_{29}$ | 8/4 |
| $5_2$ | 3* | | $2_{28}$ | | | 9* | 1/6 | $7_{14}$ |
| $2_{15}$ | $7_{13}$ | | | $9_{20}$ | $3_{17}$ | 5* | 6/1 | 4/8 |
| 1* | | | 5* | $2_{26}$ | $7_{10}$ | 4* | $9_{23}$ | $3_{24}$ |
| 7* | $9_{30}$ | 2* | $3_{18}$ | $4_{34}$ | 8* | 1/6 | $5_7$ | 6/1 |
| $3_1$ | $4_{33}$ | $5_3$ | | | $9_{21}$ | $7_8$ | $8_{35}$ | $2_{25}$ |
| $9_{32}$ | $2_{16}$ | | | 3* | | | 7* | $5_6$ |
| | | 3* | $7_{11}$ | $5_5$ | | | 2* | $9_{22}$ |
| | 5* | $7_{12}$ | 9* | | $2_{27}$ | $3_{19}$ | 4 | 1/6 |

Figure 107. Figure 4 for Puzzle 5

As a result, we can take to the fortythird step as shown in Figure 109.

$4_{43}$r2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | | $9_{31}$ | | 7* | $5_4$ | $2_9$ | $3_{29}$ | 8/4 |
| $5_2$ | 3* | | $2_{28}$ | | | 9* | 1/6 | $7_{14}$ |
| $2_{15}$ | $7_{13}$ | | | $9_{20}$ | $3_{17}$ | 5* | 6/1 | 4/8 |
| 1* | | | 5* | $2_{26}$ | $7_{10}$ | 4* | $9_{23}$ | $3_{24}$ |
| 7* | $9_{30}$ | 2* | $3_{18}$ | $4_{34}$ | 8* | 1/6 | $5_7$ | 6/1 |
| $3_1$ | $4_{33}$ | $5_3$ | | | $9_{21}$ | $7_8$ | $8_{35}$ | $2_{25}$ |
| $9_{32}$ | $2_{16}$ | | | 3* | | | 7* | $5_6$ |
| | | 3* | $7_{11}$ | $5_5$ | | | 2* | $9_{22}$ |
| | 5* | $7_{12}$ | 9* | | $2_{27}$ | $3_{19}$ | $4_{36}$ | 1/6 |

Figure 108. Figure 5 for Puzzle 5

In order to have a better read, we expand the chain further as displayed in Figure 110.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | | $9_{31}$ | | 7* | $5_4$ | $2_9$ | $3_{29}$ | 8/4 |
| $5_2$ | 3* | $4_{43}$ | $2_{28}$ | $8_{41}$ | | 9* | 1/6 | $7_{14}$ |
| $2_{15}$ | $7_{13}$ | | | $9_{20}$ | $3_{17}$ | 5* | 6/1 | 4/8 |
| 1* | | | 5* | $2_{26}$ | $7_{10}$ | 4* | $9_{23}$ | $3_{24}$ |
| 7* | $9_{30}$ | 2* | $3_{18}$ | $4_{34}$ | 8* | 1/6 | $5_7$ | 6/1 |
| $3_1$ | $4_{33}$ | $5_3$ | | | $9_{21}$ | $7_8$ | $8_{35}$ | $2_{25}$ |
| $9_{32}$ | $2_{16}$ | | $8_{38}$ | 3* | $4_{40}$ | 6/1 | 7* | $5_6$ |
| $4_{37}$ | | 3* | $7_{11}$ | $5_5$ | | $8_{42}$ | 2* | $9_{22}$ |
| $8_{39}$ | 5* | $7_{12}$ | 9* | | $2_{27}$ | $3_{19}$ | $4_{36}$ | 1/6 |

Figure 109. Figure 6 for Puzzle 5

We can easily spot 4 as the residue of box 4. Hence, we take

$4_{44}$b4: rcf-16(26)(34)

as shown in Figure 111.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | | $9_{31}$ | 4 | 7* | $5_{4}$ | $2_{9}$ | $3_{29}$ | 8/4 |
| $5_{2}$ | 3* | $4_{43}$ | $2_{28}$ | $8_{41}$ | 6/1 | 9* | 1/6 | $7_{14}$ |
| $2_{15}$ | $7_{13}$ | | 1/6 | $9_{20}$ | $3_{17}$ | 5* | 6/1 | 4/8 |
| 1* | | | 5* | $2_{26}$ | $7_{10}$ | 4* | $9_{23}$ | $3_{24}$ |
| 7* | $9_{30}$ | 2* | $3_{18}$ | $4_{34}$ | 8* | 1/6 | $5_{7}$ | 6/1 |
| $3_{1}$ | $4_{33}$ | $5_{3}$ | 6/1 | 1/6 | $9_{21}$ | $7_{8}$ | $8_{35}$ | $2_{25}$ |
| $9_{32}$ | $2_{16}$ | 1/6 | $8_{38}$ | 3* | $4_{40}$ | 6/1 | 7* | $5_{6}$ |
| $4_{37}$ | 6/1 | 3* | $7_{11}$ | $5_{5}$ | 1/6 | $8_{42}$ | 2* | $9_{22}$ |
| $8_{39}$ | 5* | $7_{12}$ | 9* | 6/1 | $2_{27}$ | $3_{19}$ | $4_{36}$ | 1/6 |

Figure 110. Figure 7 for Puzzle 5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | | $9_{31}$ | $4_{44}$ | 7* | $5_{4}$ | $2_{9}$ | $3_{29}$ | 8/4 |
| $5_{2}$ | 3* | $4_{43}$ | $2_{28}$ | $8_{41}$ | 6/1 | 9* | 1/6 | $7_{14}$ |
| $2_{15}$ | $7_{13}$ | | 1/6 | $9_{20}$ | $3_{17}$ | 5* | 6/1 | 4/8 |
| 1* | | | 5* | $2_{26}$ | $7_{10}$ | 4* | $9_{23}$ | $3_{24}$ |
| 7* | $9_{30}$ | 2* | $3_{18}$ | $4_{34}$ | 8* | 1/6 | $5_{7}$ | 6/1 |
| $3_{1}$ | $4_{33}$ | $5_{3}$ | 6/1 | 1/6 | $9_{21}$ | $7_{8}$ | $8_{35}$ | $2_{25}$ |
| $9_{32}$ | $2_{16}$ | 1/6 | $8_{38}$ | 3* | $4_{40}$ | 6/1 | 7* | $5_{6}$ |
| $4_{37}$ | 6/1 | 3* | $7_{11}$ | $5_{5}$ | 1/6 | $8_{42}$ | 2* | $9_{22}$ |
| $8_{39}$ | 5* | $7_{12}$ | 9* | 6/1 | $2_{27}$ | $3_{19}$ | $4_{36}$ | 1/6 |

Figure 111. Figure 8 for Puzzle 5

Now, we are ready to break the whole thing up in Figure 112.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | $1_{46}$ | $9_{31}$ | $4_{44}$ | 7* | $5_{4}$ | $2_{9}$ | $3_{29}$ | $8_{45}$ |
| $5_{2}$ | 3* | $4_{43}$ | $2_{28}$ | $8_{41}$ | $6_{60}$ | 9* | $1_{62}$ | $7_{14}$ |
| $2_{15}$ | $7_{13}$ | $8_{47}$ | $1_{59}$ | $9_{20}$ | $3_{17}$ | 5* | $6_{63}$ | $4_{64}$ |
| 1* | $8_{49}$ | $6_{50}$ | 5* | $2_{26}$ | $7_{10}$ | 4* | $9_{23}$ | $3_{24}$ |
| 7* | $9_{30}$ | 2* | $3_{18}$ | $4_{34}$ | 8* | $1_{53}$ | $5_{7}$ | $6_{54}$ |
| $3_{1}$ | $4_{33}$ | $5_{3}$ | $6_{58}$ | $1_{57}$ | $9_{21}$ | $7_{8}$ | $8_{35}$ | $2_{25}$ |
| $9_{32}$ | $2_{16}$ | $1_{51}$ | $8_{38}$ | 3* | $4_{40}$ | $6_{52}$ | 7* | $5_{6}$ |
| $4_{37}$ | $6_{48}$ | 3* | $7_{11}$ | $5_{5}$ | $1_{61}$ | $8_{42}$ | 2* | $9_{22}$ |
| $8_{39}$ | 5* | $7_{12}$ | 9* | $6_{56}$ | $2_{27}$ | $3_{19}$ | $4_{36}$ | $1_{55}$ |

Figure 112. Figure 9 for Puzzle 5

**Puzzle 6**

We can take to the seventh step in Figure 114 of the puzzle in Figure 113.

$3_1$c4

$5_7$b2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 2* | 3* | 6* | | |
| | 1* | | | | | | 5* | |
| | | | | | | | | |
| 3* | | 2* | | 7* | | | | |
| | | | 5* | | | | 4* | |
| 6* | | | | | | | | |
| | | | 1* | | | | | 8* |
| | 5* | | 4* | | | | | |
| 7* | | | | | | 3* | | |

Figure 113. Figure 1 for Puzzle 6

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_8$ | | | | 2* | 3* | 6* | | |
| | 1* | $3_4$ | | | | | 5* | |
| | | | | | | | $3_3$ | |
| 3* | | 2* | | 7* | | | | |
| | | | 5* | | | | 4* | $3_2$ |
| 6* | | $5_7$ | $3_1$ | | | | | |
| | $3_5$ | | 1* | | | | | 8* |
| | 5* | | 4* | $3_6$ | | | | |
| 7* | | | | | | 3* | | |

Figure 114. Figure 2 for Puzzle 6

Normally, it's too early to take the "two roads" approach. But let us try it for row 9 in Figure 115 to take a choice between 4(93) and 4(99).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_8$ | | | | 2* | 3* | 6* | | |
| | 1* | $3_4$ | | | | | 5* | |
| | | | | | | | $3_3$ | |
| 3* | 4? | 2* | | 7* | | | | |
| | | | 5* | | | | 4* | $3_2$ |
| 6* | 4? | $5_7$ | $3_1$ | | | | | |
| | $3_5$ | | 1* | | | | | 8* |
| | 5* | | 4* | $3_6$ | | | | |
| 7* | | 4? | | | | 3* | | 4? |

Figure 115. Figure 3 for Puzzle 6

The choice of 4(93) would lead us to the dilemma as demonstrated in Figure 116.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_8$ | | | | 2* | 3* | 6* | $1_4$ | $4_2$ |
| | 1* | $3_4$ | | | | | 5* | |
| | | | | | | | $3_3$ | |
| 3* | | 2* | | 7* | | | | |
| | | | 5* | | | | 4* | $3_2$ |
| 6* | | $5_7$ | $3_1$ | | | | | |
| | $3_5$ | | 1* | | | $4_3$ | No5 | 8* |
| | 5* | | 4* | $3_6$ | | No5 | No5 | No5 |
| 7* | | $4_1$ | | | | 3* | No5 | $1_5$ |

Figure 116. Figure 4 for Puzzle 6

So in Figure 117, we take

$4_9$r9: 4(93)
→4(19)
→4(77)
→1(18)
→1(99)
→No5b9.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_8$ | | $4_{10}$ | | 2* | 3* | 6* | | |
| | 1* | $3_4$ | | | | | 5* | |
| | | | | | | | $3_3$ | |
| 3* | 4? | 2* | | 7* | | | | $5_{13}$ |
| | | | 5* | | | | 4* | $3_2$ |
| 6* | 4? | $5_7$ | $3_1$ | | | | | |
| $4_{11}$ | $3_5$ | | 1* | | | $5_{12}$ | | 8* |
| | 5* | | 4* | $3_6$ | | | | |
| 7* | | | | | | 3* | | $4_9$ |

Figure 117. Figure 5 for Puzzle 6

We can take to the thirteenth step in Figure 117

$4_{10}$r1: 4c2b2

$5_{12}$b9

and in Figure 118 take $2_{14}7_{15}$r7&$2_{16}$r8:2c6b5.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_8$ | | $4_{10}$ | | 2* | 3* | 6* | | |
| | 1* | $3_4$ | | | | | 5* | |
| | | | | | | | $3_3$ | |
| 3* | | 2* | | 7* | | | | $5_{13}$ |
| | | | 5* | | 2? | | 4* | $3_2$ |
| 6* | | $5_7$ | $3_1$ | | 2? | | | |
| $4_{11}$ | $3_5$ | | 1* | | $7_{15}$ | $5_{12}$ | $2_{14}$ | 8* |
| $2_{16}$ | 5* | | 4* | $3_6$ | | | | |
| 7* | | | | | | 3* | | $4_9$ |

Figure 118. Figure 6 for Puzzle 6

We then take to the fortyseventh step in Figure 119.

.

$6_{18}7_{19}$b1 $1_{22}$b2

$6_{23}$r7 $6_{25}$c9 $2_{29}$r5

$8_{38}$b7 $8_{39}$c4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_{8}$ | $7_{19}$ | $4_{10}$ | $9_{40}$ | 2* | 3* | 6* | $8_{38}$ | $1_{41}$ |
| $9_{46}$ | 1* | $3_{4}$ | $6_{28}$ | | | $7_{34}$ | 5* | $2_{32}$ |
| $8_{47}$ | $2_{17}$ | $6_{18}$ | $7_{33}$ | | | $4_{35}$ | $3_{3}$ | $9_{42}$ |
| 3* | | 2* | $8_{39}$ | 7* | | | $6_{26}$ | $5_{13}$ |
| $1_{22}$ | $8_{44}$ | $7_{21}$ | 5* | $9_{45}$ | $6_{27}$ | $2_{29}$ | 4* | $3_{2}$ |
| 6* | | $5_{7}$ | $3_{1}$ | | $2_{30}$ | $8_{43}$ | | $7_{37}$ |
| $4_{11}$ | $3_{5}$ | $9_{24}$ | 1* | $6_{23}$ | $7_{15}$ | $5_{12}$ | $2_{14}$ | 8* |
| $2_{16}$ | 5* | | 4* | $3_{6}$ | | | $7_{36}$ | $6_{25}$ |
| 7* | $6_{20}$ | | $2_{31}$ | | | 3* | | $4_{9}$ |

Figure 119. Figure 7 for Puzzle 6

In Figure 120, the equilibrium of the chain of flipflops is maintained, i.e. all flipflops involved in each row, column or box are cancellable.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_{8}$ | $7_{19}$ | $4_{10}$ | $9_{40}$ | 2* | 3* | 6* | $8_{38}$ | $1_{41}$ |
| $9_{46}$ | 1* | $3_{4}$ | $6_{28}$ | 4/8 | 8/4 | $7_{34}$ | 5* | $2_{32}$ |
| $8_{47}$ | $2_{17}$ | $6_{18}$ | $7_{33}$ | 5/1 | 1/5 | $4_{35}$ | $3_{3}$ | $9_{42}$ |
| 3* | 9/4 | 2* | $8_{39}$ | 7* | 4/1 | 1/9 | $6_{26}$ | $5_{13}$ |
| $1_{22}$ | $8_{44}$ | $7_{21}$ | 5* | $9_{45}$ | $6_{27}$ | $2_{29}$ | 4* | $3_{2}$ |
| 6* | 4/9 | $5_{7}$ | $3_{1}$ | 1/4 | $2_{30}$ | $8_{43}$ | 9/1 | $7_{37}$ |
| $4_{11}$ | $3_{5}$ | $9_{24}$ | 1* | $6_{23}$ | $7_{15}$ | $5_{12}$ | $2_{14}$ | 8* |
| $2_{16}$ | 5* | | 4* | $3_{6}$ | 9 | 9/1 | $7_{36}$ | $6_{25}$ |
| 7* | $6_{20}$ | | $2_{31}$ | 8/5 | 5/8 | 3* | 1/9 | $4_{9}$ |

Figure 120. Figure 8 for Puzzle 6

In Figure 120, we can find the residue 9(86) in box 6 and take $9_{48}$b6: rcf-58(95)(96), which forces us to take all the flops in Figure 121.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_{8}$ | $7_{19}$ | $4_{10}$ | $9_{40}$ | 2* | 3* | 6* | $8_{38}$ | $1_{41}$ |
| $9_{46}$ | 1* | $3_{4}$ | $6_{28}$ | $8_{57}$ | $4_{58}$ | $7_{34}$ | 5* | $2_{32}$ |
| $8_{47}$ | $2_{17}$ | $6_{18}$ | $7_{33}$ | $3_{60}$ | $3_{59}$ | $4_{35}$ | $3_{3}$ | $9_{42}$ |
| 3* | $4_{53}$ | 2* | $8_{39}$ | 7* | $1_{55}$ | $9_{50}$ | $6_{26}$ | $5_{13}$ |
| $1_{22}$ | $8_{44}$ | $7_{21}$ | 5* | $9_{45}$ | $6_{27}$ | $2_{29}$ | 4* | $3_{2}$ |
| 6* | $9_{54}$ | $5_{7}$ | $3_{1}$ | $4_{56}$ | $2_{30}$ | $8_{43}$ | $1_{51}$ | $7_{37}$ |
| $4_{11}$ | $3_{5}$ | $9_{24}$ | 1* | $6_{23}$ | $7_{15}$ | $5_{12}$ | $2_{14}$ | 8* |
| $2_{16}$ | 5* | $8_{63}$ | 4* | $3_{6}$ | $9_{48}$ | $1_{49}$ | $7_{36}$ | $6_{25}$ |
| 7* | $6_{20}$ | $1_{64}$ | $2_{31}$ | $8_{61}$ | $5_{62}$ | 3* | $9_{52}$ | $4_{9}$ |

Figure 121. Figure 9 for Puzzle 6

**Puzzle 7**

We can readily take the first thirty-five steps of the puzzle in Figure 122 as shown in Figure 123.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9* | | | | | | 6* | 8* | |
| 4* | | | | 7* | | | | |
| | | | | | | 2* | | |
| | 8* | | | | | | 1* | 7* |
| | | | 4* | 2* | | | | |
| | | | | | | | | |
| | 7* | | 8* | | 1* | | | |
| | | | 9* | | | 4* | | |
| 3* | | | | | | | | |

Figure 122. Figure 1 for Puzzle 7

$7_1$r1 $4_7$c8

$1_{11}$b5 $2_{12}$b8

$3_{14}$b6 $8_{15}9_{16}6_{17}$c5

$1_{22}$c1 $1_{23}$r9

$9_{28}$b3 $9_{33}$r5:

9c9b7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9* | | $7_1$ | $1_{27}$ | $5_{18}$ | | 6* | 8* | |
| 4* | | $8_{19}$ | | 7* | | $1_{26}$ | | |
| $1_{22}$ | | | | $8_{15}$ | | 2* | $7_2$ | |
| $2_{13}$ | 8* | $4_8$ | | $9_{16}$ | | | 1* | 7* |
| $7_6$ | $1_{24}$ | | 4* | 2* | $8_{35}$ | $9_{33}$ | | |
| | $9_{29}$ | | $7_5$ | $1_{11}$ | | $8_{34}$ | $4_7$ | $2_{12}$ |
| | 7* | $9_{28}$ | 8* | $4_{10}$ | 1* | | $2_{32}$ | |
| $8_{20}$ | | $2_{31}$ | 9* | $3_{14}$ | $7_4$ | 4* | | $1_{25}$ |
| 3* | $4_9$ | $1_{23}$ | | $6_{17}$ | | $7_3$ | $9_{30}$ | $8_{21}$ |

Figure 123. Figure 2 for Puzzle 7

Although we spot a blank grid in box 9 of Figure 124, we are not sure what number can fit there.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9* | | $7_1$ | $1_{27}$ | $5_{18}$ | | 6* | 8* | |
| 4* | | $8_{19}$ | | 7* | | $1_{26}$ | | |
| $1_{22}$ | | | | $8_{15}$ | 4/9 | 2* | $7_2$ | 9/4 |
| $2_{13}$ | 8* | $4_8$ | | $9_{16}$ | | 3/5 | 1* | 7* |
| $7_6$ | $1_{24}$ | | 4* | 2* | $8_{35}$ | $9_{33}$ | | |
| 5/6 | $9_{29}$ | | $7_5$ | $1_{11}$ | | $8_{34}$ | $4_7$ | $2_{12}$ |
| 6/5 | 7* | $9_{28}$ | 8* | $4_{10}$ | 1* | 5/3 | $2_{32}$ | |
| $8_{20}$ | 5/6 | $2_{31}$ | 9* | $3_{14}$ | $7_4$ | 4* | 6/5 | $1_{25}$ |
| 3* | $4_9$ | $1_{23}$ | | $6_{17}$ | | $7_3$ | $9_{30}$ | $8_{21}$ |

Figure 124. Figure 3 for Puzzle 7

We can expand the chain of flipflops in Figure 124 as far as we can so that the equilibrium is mantained as in Figure 125.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9* | | $7_{1}$ | $1_{27}$ | $5_{18}$ | | 6* | 8* | |
| 4* | | $8_{19}$ | | 7* | | $1_{26}$ | | 5/3 |
| $1_{22}$ | 6/5 | 5/6 | | $8_{15}$ | 4/9 | 2* | $7_{2}$ | 9/4 |
| $2_{13}$ | 8* | $4_{8}$ | 5/3 | $9_{16}$ | | 3/5 | 1* | 7* |
| $7_{6}$ | $1_{24}$ | 3/5 | 4* | 2* | $8_{35}$ | $9_{33}$ | 5/3 | 6/5 |
| 5/6 | $9_{29}$ | 6/3 | $7_{5}$ | $1_{11}$ | 3/5 | $8_{34}$ | $4_{7}$ | $2_{12}$ |
| 6/5 | 7* | $9_{28}$ | 8* | $4_{10}$ | 1* | 5/3 | $2_{32}$ | 3/6 |
| $8_{20}$ | 5/6 | $2_{31}$ | 9* | $3_{14}$ | $7_{4}$ | 4* | 6/5 | $1_{25}$ |
| 3* | $4_{9}$ | $1_{23}$ | | $6_{17}$ | | $7_{3}$ | $9_{30}$ | $8_{21}$ |

Figure 125. Figure 4 for Puzzle 7

We can take

$3_{36}$r3: rcf-56(32)(33)
&49(36)(39)

$6_{37}$r4: rcf-35(44)(47)

and

$3_{38}$(28)g

in Figure 126.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9* | | $7_{1}$ | $1_{27}$ | $5_{18}$ | | 6* | 8* | |
| 4* | | $8_{19}$ | | 7* | | $1_{26}$ | $3_{38}$ | 5/3 |
| $1_{22}$ | 6/5 | 5/6 | $3_{36}$ | $8_{15}$ | 4/9 | 2* | $7_{2}$ | 9/4 |
| $2_{13}$ | 8* | $4_{8}$ | 5/3 | $9_{16}$ | $6_{37}$ | 3/5 | 1* | 7* |
| $7_{6}$ | $1_{24}$ | 3/5 | 4* | 2* | $8_{35}$ | $9_{33}$ | 5/3 | 6/5 |
| 5/6 | $9_{29}$ | 6/3 | $7_{5}$ | $1_{11}$ | 3/5 | $8_{34}$ | $4_{7}$ | $2_{12}$ |
| 6/5 | 7* | $9_{28}$ | 8* | $4_{10}$ | 1* | 5/3 | $2_{32}$ | 3/6 |
| $8_{20}$ | 5/6 | $2_{31}$ | 9* | $3_{14}$ | $7_{4}$ | 4* | 6/5 | $1_{25}$ |
| 3* | $4_{9}$ | $1_{23}$ | | $6_{17}$ | | $7_{3}$ | $9_{30}$ | $8_{21}$ |

Figure 126. Figure 5 for Puzzle 7

Finally, we can break up the flipflop chain in Figure 126 by taking all flips and complete the puzzle as shown in Figure 127.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9* | $3_{55}$ | $7_{1}$ | $1_{27}$ | $5_{18}$ | $2_{57}$ | 6* | 8* | $4_{58}$ |
| 4* | $2_{56}$ | $8_{19}$ | $6_{62}$ | 7* | $9_{63}$ | $2_{5}$ | $3_{38}$ | $5_{39}$ |
| $1_{22}$ | $6_{54}$ | $5_{51}$ | $3_{36}$ | $8_{15}$ | $4_{60}$ | 2* | $7_{2}$ | $9_{59}$ |
| $2_{13}$ | 8* | $4_{8}$ | $5_{46}$ | $9_{16}$ | $6_{37}$ | $3_{45}$ | 1* | 7* |
| $7_{6}$ | $1_{24}$ | $3_{49}$ | 4* | 2* | $8_{35}$ | $9_{33}$ | $5_{40}$ | $6_{48}$ |
| $5_{44}$ | $9_{29}$ | $6_{50}$ | $7_{5}$ | $1_{11}$ | $3_{47}$ | $8_{34}$ | $4_{7}$ | $2_{12}$ |
| $6_{43}$ | 7* | $9_{28}$ | 8* | $4_{10}$ | 1* | $5_{52}$ | $2_{32}$ | $3_{53}$ |
| $8_{20}$ | $5_{42}$ | $2_{31}$ | 9* | $3_{14}$ | $7_{4}$ | 4* | $6_{41}$ | $1_{25}$ |
| 3* | $4_{9}$ | $1_{23}$ | $2_{61}$ | $6_{17}$ | $5_{64}$ | $7_{3}$ | $9_{30}$ | $8_{21}$ |

Figure 127. Figure 6 for Puzzle 7

**Puzzle 8**

We can readily take the first thirtyfive steps of the puzzle in Figure 128 as shown in Figure 129.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 5* | | 6* | | | 2* | | |
| | | | | 8* | | | | |
| | | | | | | | | 9* |
| | | 7* | | 4* | | | 3* | |
| 5* | | | | | | 1* | | |
| 2* | | | | | | | | |
| | 6* | 4* | | | | | 7* | |
| | | | 2* | | 5* | | | |
| | | | 1* | | | | | |

Figure 128. Figure 1 for Puzzle 8

$2_{2}5_{3}1_{4}$r7 $1_{10}$r4

$4_{11}$b6

$5_{12}1_{13}$c5

$6_{22}$c6 $6_{23}$c3

$7_{32}$b6 $8_{33}$c6

$9_{34}$c5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $4_{27}$ | 5* | | 6* | $1_{13}$ | | 2* | $8_{35}$ | |
| $6_{26}$ | | $2_{9}$ | $5_{20}$ | 8* | | $4_{28}$ | $1_{16}$ | 9* |
| | | $1_{14}$ | $4_{21}$ | $2_{8}$ | | | $5_{19}$ | $6_{25}$ |
| | $1_{10}$ | 7* | | 4* | $2_{7}$ | $6_{24}$ | 3* | $5_{18}$ |
| 5* | | | | | $6_{22}$ | 1* | $2_{6}$ | |
| 2* | | $6_{23}$ | | $5_{12}$ | $1_{15}$ | | $9_{34}$ | |
| $1_{4}$ | 6* | 4* | | | $8_{33}$ | $5_{3}$ | 7* | $2_{2}$ |
| | | | 2* | $6_{31}$ | 5* | | $4_{29}$ | $1_{17}$ |
| | $2_{5}$ | $5_{1}$ | 1* | $7_{32}$ | $4_{11}$ | | $6_{30}$ | |

Figure 129. Figure 2 for Puzzle 8

We first come up with the chain in Figure 130. Note that 7/3(16) will be added to maintain the equilibrium as shown in Figure 131.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $4_{27}$ | 5* | | 6* | $1_{13}$ | | 2* | $8_{35}$ | 3/7 |
| $6_{26}$ | 7/3 | $2_{9}$ | $5_{20}$ | 8* | 3/7 | $4_{28}$ | $1_{16}$ | 9* |
| | | $1_{14}$ | $4_{21}$ | $2_{8}$ | | 7/3 | $5_{19}$ | $6_{25}$ |
| 9/8 | $1_{10}$ | 7* | 8/9 | 4* | $2_{7}$ | $6_{24}$ | 3* | $5_{18}$ |
| 5* | | | | 9/3 | $6_{22}$ | 1* | $2_{6}$ | |
| 2* | | $6_{23}$ | | $5_{12}$ | $1_{15}$ | | $9_{34}$ | |
| $1_{4}$ | 6* | 4* | 9/3 | 3/9 | $8_{33}$ | $5_{3}$ | 7* | $2_{2}$ |
| | | | 2* | $6_{31}$ | 5* | | $4_{29}$ | $1_{17}$ |
| | $2_{5}$ | $5_{1}$ | 1* | $7_{32}$ | $4_{11}$ | | $6_{30}$ | |

Figure 130. Figure 3 for Puzzle 8

We can take

$9_{36}$r1: rcf-37(16)(19)

and

$9_{37}$b2: rcf-37(16)(26)

in Figure 132,

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $4_{27}$ | 5* | | 6* | $1_{13}$ | 7/3 | 2* | $8_{35}$ | 3/7 |
| $6_{26}$ | 7/3 | $2_{9}$ | $5_{20}$ | 8* | 3/7 | $4_{28}$ | $1_{16}$ | 9* |
| | | $1_{14}$ | $4_{21}$ | $2_{8}$ | | 7/3 | $5_{19}$ | $6_{25}$ |
| 9/8 | $1_{10}$ | 7* | 8/9 | 4* | $2_{7}$ | $6_{24}$ | 3* | $5_{18}$ |
| 5* | | | | 9/3 | $6_{22}$ | 1* | $2_{6}$ | |
| 2* | | $6_{23}$ | | $5_{12}$ | $1_{15}$ | | $9_{34}$ | |
| $1_{4}$ | 6* | 4* | 9/3 | 3/9 | $8_{33}$ | $5_{3}$ | 7* | $2_{2}$ |
| | | | 2* | $6_{31}$ | 5* | | $4_{29}$ | $1_{17}$ |
| | $2_{5}$ | $5_{1}$ | 1* | $7_{32}$ | $4_{11}$ | | $6_{30}$ | |

Figure 131. Figure 4 for Puzzle 8

We shall expand the chain by further adding 7/3(54) and 3/7(64) in Figure 133.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $4_{27}$ | 5* | $9_{36}$ | 6* | $1_{13}$ | 7/3 | 2* | $8_{35}$ | 3/7 |
| $6_{26}$ | 7/3 | $2_{9}$ | $5_{20}$ | 8* | 3/7 | $4_{28}$ | $1_{16}$ | 9* |
| | | $1_{14}$ | $4_{21}$ | $2_{8}$ | $9_{37}$ | 7/3 | $5_{19}$ | $6_{25}$ |
| 9/8 | $1_{10}$ | 7* | 8/9 | 4* | $2_{7}$ | $6_{24}$ | 3* | $5_{18}$ |
| 5* | | | | 9/3 | $6_{22}$ | 1* | $2_{6}$ | |
| 2* | | $6_{23}$ | | $5_{12}$ | $1_{15}$ | | $9_{34}$ | |
| $1_{4}$ | 6* | 4* | 9/3 | 3/9 | $8_{33}$ | $5_{3}$ | 7* | $2_{2}$ |
| | | | 2* | $6_{31}$ | 5* | | $4_{29}$ | $1_{17}$ |
| | $2_{5}$ | $5_{1}$ | 1* | $7_{32}$ | $4_{11}$ | | $6_{30}$ | |

Figure 132. Figure 5 for Puzzle 8

Unlike the previous puzzles, we can right away take all the flips in column 4 of this chain as shown in Figure 134.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $4_{27}$ | 5* | $9_{36}$ | 6* | $1_{13}$ | 7/3 | 2* | $8_{35}$ | 3/7 |
| $6_{26}$ | 7/3 | $2_{9}$ | $5_{20}$ | 8* | 3/7 | $4_{28}$ | $1_{16}$ | 9* |
| | | $1_{14}$ | $4_{21}$ | $2_{8}$ | $9_{37}$ | 7/3 | $5_{19}$ | $6_{25}$ |
| 9/8 | $1_{10}$ | 7* | 8/9 | 4* | $2_{7}$ | $6_{24}$ | 3* | $5_{18}$ |
| 5* | | | 7/3 | 9/3 | $6_{22}$ | 1* | $2_{6}$ | |
| 2* | | $6_{23}$ | 3/7 | $5_{12}$ | $1_{15}$ | | $9_{34}$ | |
| $1_{4}$ | 6* | 4* | 9/3 | 3/9 | $8_{33}$ | $5_{3}$ | 7* | $2_{2}$ |
| | | | 2* | $6_{31}$ | 5* | | $4_{29}$ | $1_{17}$ |
| | $2_{5}$ | $5_{1}$ | 1* | $7_{32}$ | $4_{11}$ | | $6_{30}$ | |

Figure 133. Figure 6 for Puzzle 8

Then we take all flips for the chain related, followed by three block moves as shown in Figure 135.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $4_{27}$ | 5* | $9_{36}$ | 6* | $1_{13}$ | 7/3 | 2* | $8_{35}$ | 3/7 |
| $6_{26}$ | 7/3 | $2_{9}$ | $5_{20}$ | 8* | 3/7 | $4_{28}$ | $1_{16}$ | 9* |
| | | $1_{14}$ | $4_{21}$ | $2_{8}$ | $9_{37}$ | 7/3 | $5_{19}$ | $6_{25}$ |
| 9/8 | $1_{10}$ | 7* | $8_{38}$ | 4* | $2_{7}$ | $6_{24}$ | 3* | $5_{18}$ |
| 5* | | | $7_{39}$ | 9/3 | $6_{22}$ | 1* | $2_{6}$ | |
| 2* | | $6_{23}$ | $3_{40}$ | $5_{12}$ | $1_{15}$ | | $9_{34}$ | |
| $1_{4}$ | 6* | 4* | $9_{41}$ | 3/9 | $8_{33}$ | $5_{3}$ | 7* | $2_{2}$ |
| | | | 2* | $6_{31}$ | 5* | | $4_{29}$ | $1_{17}$ |
| | $2_{5}$ | $5_{1}$ | 1* | $7_{32}$ | $4_{11}$ | | $6_{30}$ | |

Figure 134. Figure 7 for Puzzle 8

Now by adding 3/7(22) to maintain the equilibrium for the remaining chain, we cantake

$8_{48}$b1: rcf-37(16)(19).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $4_{27}$ | 5* | $9_{36}$ | 6* | $1_{13}$ | 7/3 | 2* | $8_{35}$ | 3/7 |
| $6_{26}$ | 7/3 | $2_{9}$ | $5_{20}$ | 8* | 3/7 | $4_{28}$ | $1_{16}$ | 9* |
| $8_{48}$ | 3/7 | $1_{14}$ | $4_{21}$ | $2_{8}$ | $9_{37}$ | 7/3 | $5_{19}$ | $6_{25}$ |
| $9_{42}$ | $1_{10}$ | 7* | $8_{38}$ | 4* | $2_{7}$ | $6_{24}$ | 3* | $5_{18}$ |
| 5* | | | $7_{39}$ | $9_{44}$ | $6_{22}$ | 1* | $2_{6}$ | |
| 2* | | $6_{23}$ | $3_{40}$ | $5_{12}$ | $1_{15}$ | | $9_{34}$ | |
| $1_{4}$ | 6* | 4* | $9_{41}$ | $3_{43}$ | $8_{33}$ | $5_{3}$ | 7* | $2_{2}$ |
| $7_{46}$ | $9_{45}$ | | 2* | $6_{31}$ | 5* | | $4_{29}$ | $1_{17}$ |
| | $2_{5}$ | $5_{1}$ | 1* | $7_{32}$ | $4_{11}$ | $9_{47}$ | $6_{30}$ | |

Figure 135. Figure 8 for Puzzle 8

We can finally complete the puzzle as shown in Figure 136.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $4_{27}$ | 5* | $9_{36}$ | 6* | $1_{13}$ | $7_{58}$ | 2* | $8_{35}$ | $3_{59}$ |
| $6_{26}$ | $7_{56}$ | $2_{9}$ | $5_{20}$ | 8* | $3_{57}$ | $4_{28}$ | $1_{16}$ | 9* |
| $8_{48}$ | $3_{55}$ | $1_{14}$ | $4_{21}$ | $2_{8}$ | $9_{37}$ | $7_{54}$ | $5_{19}$ | $6_{25}$ |
| $9_{42}$ | $1_{10}$ | 7* | $8_{38}$ | 4* | $2_{7}$ | $6_{24}$ | 3* | $5_{18}$ |
| 5* | $8_{61}$ | $3_{53}$ | $7_{39}$ | $9_{44}$ | $6_{22}$ | 1* | $2_{6}$ | $4_{63}$ |
| 2* | $4_{62}$ | $6_{23}$ | $3_{40}$ | $5_{12}$ | $1_{15}$ | $8_{60}$ | $9_{34}$ | $7_{64}$ |
| $1_{4}$ | 6* | 4* | $9_{41}$ | $3_{43}$ | $8_{33}$ | $5_{3}$ | 7* | $2_{2}$ |
| $7_{46}$ | $9_{45}$ | $8_{52}$ | 2* | $6_{31}$ | 5* | $3_{51}$ | $4_{29}$ | $1_{17}$ |
| $3_{49}$ | $2_{5}$ | $5_{1}$ | 1* | $7_{32}$ | $4_{11}$ | $9_{47}$ | $6_{30}$ | $8_{50}$ |

Figure 136. Figure 9 for Puzzle 8

**Puzzle 9**

We can take to the fourteenth step in Figure 138 of the puzzle in Figure 137.

$2_2$c1 $7_9$b5 $9_{14}$b3

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | | | 5* | | | 6* | | |
| | | | | | 7* | | 2* | |
| | | | | | | | | |
| | 2* | | | | | | 7* | 4* |
| | | | 3* | 1* | | | | |
| | | | | | | | | |
| 4* | | 8* | | | | | | 7* |
| | | | | 2* | | 9* | | |
| 5* | | | 1* | | | | | |

Figure 137. Figure 1 for Puzzle 9

The 3 in box 5 restricts 3 of box 4 to be in column 5 as shown in Figure 138, which triggers the moves in row 7

$1_{15}3_{16}5_{17}$r7: 3c5b4.

.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | | $7_{12}$ | 5* | 3? | $2_4$ | 6* | | |
| | | | | 3? | 7* | | 2* | |
| $2_2$ | | | | 3? | $1_5$ | $7_1$ | | |
| | 2* | | | | | | 7* | 4* |
| $7_{13}$ | | | 3* | 1* | | | | $2_8$ |
| | | | $2_6$ | $7_9$ | | | | |
| 4* | $9_{14}$ | 8* | $6_{18}$ | $5_{17}$ | $3_{16}$ | $2_7$ | $1_{15}$ | 7* |
| | | | $7_{10}$ | 2* | | 9* | | |
| 5* | $7_{11}$ | $2_3$ | 1* | | | | | |

Figure 138. Figure 2 for Puzzle 9

Again, the 4 in box 8 restricts 4 of box 5 to be in column 6 as shown in Figure 139, which triggers the of move in box 6

$4_{19}$b6: 4c6b5.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | | $7_{12}$ | 5* | | $2_4$ | 6* | | |
| | | | | | 7* | | 2* | |
| $2_2$ | | | | | $1_5$ | $7_1$ | | |
| | 2* | | | | | | 7* | 4* |
| $7_{13}$ | | | 3* | 1* | 4? | | | $2_8$ |
| | | | $2_6$ | $7_9$ | 4? | | | |
| 4* | $9_{14}$ | 8* | $6_{18}$ | $5_{17}$ | $3_{16}$ | $2_7$ | $1_{15}$ | 7* |
| | | | $7_{10}$ | 2* | | 9* | | |
| 5* | $7_{11}$ | $2_3$ | 1* | $4_{19}$ | | | | |

Figure 139. Figure 3 for Puzzle 9

We can then take to the twentyeighth step in Figure 140 .

$4_{25}1_{26}$r1

$5_{28}$b7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $4_{25}$ | $7_{12}$ | 5* | | $2_4$ | 6* | | $1_{26}$ |
| | | | | | 7* | $4_{21}$ | 2* | |
| $2_2$ | | | $4_{27}$ | | $1_5$ | $7_1$ | $5_{28}$ | |
| | 2* | | | | | | 7* | 4* |
| $7_{13}$ | | | 3* | 1* | | | | $2_8$ |
| | | | $2_6$ | $7_9$ | | | | |
| 4* | $9_{14}$ | 8* | $6_{18}$ | $5_{17}$ | $3_{16}$ | $2_7$ | $1_{15}$ | 7* |
| | | | $7_{10}$ | 2* | $8_{24}$ | 9* | $4_{20}$ | $5_{22}$ |
| 5* | $7_{11}$ | $2_3$ | 1* | $4_{19}$ | $9_{23}$ | | | |

Figure 140. Figure 4 for Puzzle 9

Now, it’s time to look at the following chain of flipflops in Figure 141.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $4_{25}$ | $7_{12}$ | 5* | 8/9 | $2_4$ | 6* | 9/8 | $1_{26}$ |
| 8/9 | | | 9/8 | | 7* | $4_{21}$ | 2* | |
| $2_2$ | | | $4_{27}$ | | $1_5$ | $7_1$ | $5_{28}$ | 8/9 |
| 1/6 | 2* | | 8/9 | 9/8 | | | 7* | 4* |
| $7_{13}$ | | | 3* | 1* | | | | $2_8$ |
| 9/8 | | | $2_6$ | $7_9$ | | | | |
| 4* | $9_{14}$ | 8* | $6_{18}$ | $5_{17}$ | $3_{16}$ | $2_7$ | $1_{15}$ | 7* |
| 6/1 | | | $7_{10}$ | 2* | $8_{24}$ | 9* | $4_{20}$ | $5_{22}$ |
| 5* | $7_{11}$ | $2_3$ | 1* | $4_{19}$ | $9_{23}$ | | | |

Figure 141. Figure 5 for Puzzle 9

Look at the only grid left to be filled in box 7, where 3 is the residue of the chain in question! So, we take $3_{29}$b7: rcf-89(18)(39) in Figure 142.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $4_{25}$ | $7_{12}$ | 5* | 8/9 | $2_4$ | 6* | 9/8 | $1_{26}$ |
| 8/9 | | | 9/8 | | 7* | $4_{21}$ | 2* | $3_{29}$ |
| $2_2$ | | | $4_{27}$ | | $1_5$ | $7_1$ | $5_{28}$ | 8/9 |
| 1/6 | 2* | | 8/9 | 9/8 | | | 7* | 4* |
| $7_{13}$ | | | 3* | 1* | | | | $2_8$ |
| 9/8 | | | $2_6$ | $7_9$ | | | | |
| 4* | $9_{14}$ | 8* | $6_{18}$ | $5_{17}$ | $3_{16}$ | $2_7$ | $1_{15}$ | 7* |
| 6/1 | | | $7_{10}$ | 2* | $8_{24}$ | 9* | $4_{20}$ | $5_{22}$ |
| 5* | $7_{11}$ | $2_3$ | 1* | $4_{19}$ | $9_{23}$ | | | |

Figure 142. Figure 6 for Puzzle 9

We can continue to take seven more moves in Figure 143

$6_{32}$g

$3_{34}$c8

and break down as in Figure 144!

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $4_{25}$ | $7_{12}$ | 5* | 8/9 | $2_{4}$ | 6* | 9/8 | $1_{26}$ |
| 8/9 | | | 9/8 | $6_{31}$ | 7* | $4_{21}$ | 2* | $3_{29}$ |
| $2_{2}$ | | | $4_{27}$ | $3_{30}$ | $1_{5}$ | $7_{1}$ | $5_{28}$ | 8/9 |
| 1/6 | 2* | $3_{35}$ | 8/9 | 9/8 | | | 7* | 4* |
| $7_{13}$ | | | 3* | 1* | | | | $2_{8}$ |
| 9/8 | | | $2_{6}$ | $7_{9}$ | | | $3_{34}$ | $6_{32}$ |
| 4* | $9_{14}$ | 8* | $6_{18}$ | $5_{17}$ | $3_{16}$ | $2_{7}$ | $1_{15}$ | 7* |
| 6/1 | $3_{36}$ | | $7_{10}$ | 2* | $8_{24}$ | 9* | $4_{20}$ | $5_{22}$ |
| 5* | $7_{11}$ | $2_{3}$ | 1* | $4_{19}$ | $9_{23}$ | $3_{37}$ | $6_{33}$ | $8_{38}$ |

Figure 143. Figure 7 for Puzzle 9

Now, we can see 8 in box 9 will force us to take all the flops of the chain in question, excluding the flipflops 16(41)(81).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $4_{25}$ | $7_{12}$ | 5* | 8/9 | $2_{4}$ | 6* | 9/8 | $1_{26}$ |
| 8/9 | | | 9/8 | $6_{31}$ | 7* | $4_{21}$ | 2* | $3_{29}$ |
| $2_{2}$ | | | $4_{27}$ | $3_{30}$ | $1_{5}$ | $7_{1}$ | $5_{28}$ | 8/9 |
| 1/6 | 2* | $3_{35}$ | 8/9 | 9/8 | | | 7* | 4* |
| $7_{13}$ | | | 3* | 1* | | | | $2_{8}$ |
| 9/8 | | | $2_{6}$ | $7_{9}$ | | | $3_{34}$ | $6_{32}$ |
| 4* | $9_{14}$ | 8* | $6_{18}$ | $5_{17}$ | $3_{16}$ | $2_{7}$ | $1_{15}$ | 7* |
| 6/1 | $3_{36}$ | | $7_{10}$ | 2* | $8_{24}$ | 9* | $4_{20}$ | $5_{22}$ |
| 5* | $7_{11}$ | $2_{3}$ | 1* | $4_{19}$ | $9_{23}$ | $3_{37}$ | $6_{33}$ | $8_{38}$ |

Figure 144. Figure 8 for Puzzle 9

Thereafter, we can complete the puzzle rather easily as shown in Figure 145.

$6_{54}$c2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $4_{25}$ | $7_{12}$ | 5* | $9_{42}$ | $2_{4}$ | 6* | $8_{40}$ | $1_{26}$ |
| $9_{46}$ | $1_{61}$ | $5_{60}$ | $8_{43}$ | $6_{31}$ | 7* | $4_{21}$ | 2* | $3_{29}$ |
| $2_{2}$ | $8_{48}$ | $6_{49}$ | $4_{27}$ | $3_{30}$ | $1_{5}$ | $7_{1}$ | $5_{28}$ | $9_{39}$ |
| $1_{58}$ | 2* | $3_{35}$ | $9_{44}$ | $8_{45}$ | $6_{56}$ | $1_{63}$ | 7* | 4* |
| $7_{13}$ | $6_{54}$ | $4_{52}$ | 3* | 1* | $5_{55}$ | $8_{50}$ | $9_{41}$ | $2_{8}$ |
| $8_{47}$ | $5_{62}$ | $9_{51}$ | $2_{6}$ | $7_{9}$ | $4_{53}$ | $5_{64}$ | $3_{34}$ | $6_{32}$ |
| 4* | $9_{14}$ | 8* | $6_{18}$ | $5_{17}$ | $3_{16}$ | $2_{7}$ | $1_{15}$ | 7* |
| $6_{57}$ | $3_{36}$ | $1_{59}$ | $7_{10}$ | 2* | $8_{24}$ | 9* | $4_{20}$ | $5_{22}$ |
| 5* | $7_{11}$ | $2_{3}$ | 1* | $4_{19}$ | $9_{23}$ | $3_{37}$ | $6_{33}$ | $8_{38}$ |

Figure 145. Figure 9 for Puzzle 9

**Puzzle 10**

We can take to the sixth step in Figure 146 of the puzzle in Figure 147.

$2_2$b2 $4_3$b6 $4_6$r2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | | | | 4* | | | | |
| | | | | | | | 5* | 7* |
| | | | | 1* | | | | |
| 1* | 6* | | | | | 3* | | |
| | | | 2* | | | 8* | | |
| 4* | | | | 8* | | | | |
| | 2* | | 7* | | 6* | | | |
| | | | 5* | | | | | |
| | | | | | | 4* | | |

Figure 146. Figure 1 for Puzzle 10

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | | | | 4* | | | | |
| | $4_6$ | | | | | | 5* | 7* |
| | | | | 1* | | | | |
| 1* | 6* | $8_1$ | $4_5$ | | | 3* | | |
| | | | 2* | | | 8* | | |
| 4* | | $2_2$ | | 8* | | | | |
| | 2* | $4_4$ | 7* | | 6* | | | |
| | | | 5* | | $4_3$ | | | |
| | | | | | | 4* | | |

Figure 147. Figure 2 for Puzzle 10

In the setting of flipflops of Figure 148, 6 is the residue of column 5.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | | | | 4* | 5/7 | | | |
| | $4_6$ | | | 6 | | | 5* | 7* |
| | | | | 1* | 7/5 | | | |
| 1* | 6* | $8_1$ | $4_5$ | 5/7 | | 3* | | |
| | | | 2* | 7/5 | | 8* | | |
| 4* | | $2_2$ | | 8* | | | | |
| | 2* | $4_4$ | 7* | | 6* | | | |
| | | | 5* | | $4_3$ | | | |
| | | | | | | 4* | | |

Figure 148. Figure 3 for Puzzle 10

So, in Figure 149 we take the residue move $6_7$c5: rcf-57(45)(55), followed by the middle block move $6_8$MB.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | | | | 4* | | | | |
| | $4_6$ | | | $6_7$ | | | 5* | 7* |
| | | | | 1* | | | | |
| 1* | 6* | $8_1$ | $4_5$ | | | 3* | | |
| | | | 2* | | | 8* | | |
| 4* | | $2_2$ | $6_8$ | 8* | | | | |
| | 2* | $4_4$ | 7* | | 6* | | | |
| | | | 5* | | $4_3$ | | | |
| | | | | | | 4* | | |

Figure 149. Figure 4 for Puzzle 10

In the setting of flipflops of Figure 150, we can see 6 being the residue of column 5.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | | | | 4* | | | | |
| | $4_6$ | | | $6_7$ | | | 5* | 7* |
| | | | | 1* | | | | |
| 1* | 6* | $8_1$ | $4_5$ | | | 3* | | |
| | | | 2* | | 1 | 8* | 4/6 | 6/4 |
| 4* | | $2_2$ | $6_8$ | 8* | | | | |
| | 2* | $4_4$ | 7* | | 6* | | | |
| | | | 5* | | $4_3$ | | | |
| | | | | | | 4* | | |

Figure 150. Figure 5 for Puzzle 10

Therefore, we take the residue move $1_9$r5: rcf-46(58)(59) in Figure 151.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | | | | 4* | | | | |
| | $4_6$ | | | $6_7$ | | | 5* | 7* |
| | | | | 1* | | | | |
| 1* | 6* | $8_1$ | $4_5$ | | | 3* | | |
| | | | 2* | | $1_9$ | 8* | | |
| 4* | | $2_2$ | $6_8$ | 8* | | | | |
| | 2* | $4_4$ | 7* | | 6* | | | |
| | | | 5* | | $4_3$ | | | |
| | | | | | | 4* | | |

Figure 151. Figure 6 for Puzzle 10

In the setting of flipflops of Figure 152, we can see 3 and 9 being the residue of box 5.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | | | | 4* | | | | |
| | $4_6$ | | | $6_7$ | | | 5* | 7* |
| | | | | 1* | | | | |
| 1* | 6* | $8_1$ | $4_5$ | 5/7 | 9 | 3* | | |
| | | | 2* | 7/5 | $1_9$ | 8* | | |
| 4* | | $2_2$ | $6_8$ | 8* | 3 | | | |
| | 2* | $4_4$ | 7* | | 6* | | | |
| | | | 5* | | $4_3$ | | | |
| | | | | | | 4* | | |

Figure 152. Figure 7 for Puzzle 10

Therefore, we take

$3_{10}9_{11}$b5: rcf-57(45)(55),

followed by the next Eleven steps in Figure 153.

$3_{13}$r2 $8_{14}$b6

$1_{18}$c2 $2_{21}$r2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $1_{18}$ | | | 4* | | | | |
| $8_{15}$ | $4_6$ | $9_{22}$ | $3_{13}$ | $6_7$ | $2_{21}$ | $1_{20}$ | 5* | 7* |
| $2_{16}$ | | | | 1* | | | | |
| 1* | 6* | $8_1$ | $4_5$ | | $9_{11}$ | 3* | | |
| | | | 2* | | $1_9$ | 8* | | |
| 4* | | $2_2$ | $6_8$ | 8* | $3_{10}$ | | | |
| | 2* | $4_4$ | 7* | | 6* | | | |
| | $8_{17}$ | $1_{19}$ | 5* | | $4_3$ | | | |
| | | | $1_{12}$ | | $8_{14}$ | 4* | | |

Figure 153. Figure 8 for Puzzle 10

Now let us look at box 1 of Figure 154, we should take

$6_{23}$b1: u57c3b1c6b4

in Figure 155 to avoid possible multiple solutions.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $1_{18}$ | 7/5 | | 4* | 5/7 | | | |
| $8_{15}$ | $4_6$ | $9_{22}$ | $3_{13}$ | $6_7$ | $2_{21}$ | $1_{20}$ | 5* | 7* |
| $2_{16}$ | | 5/7 | | 1* | 7/5 | | | |
| 1* | 6* | $8_1$ | $4_5$ | | $9_{11}$ | 3* | | |
| | | | 2* | | $1_9$ | 8* | | |
| 4* | | $2_2$ | $6_8$ | 8* | $3_{10}$ | | | |
| | 2* | $4_4$ | 7* | | 6* | | | |
| | $8_{17}$ | $1_{19}$ | 5* | | $4_3$ | | | |
| | | | $1_{12}$ | | $8_{14}$ | 4* | | |

Figure 154. Figure 9 for Puzzle 10

We also add two flipflops 7/5(13) and 5/7(32) in Figure 155.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $1_{18}$ | 7/5 | | 4* | 5/7 | | | |
| $8_{15}$ | $4_{6}$ | $9_{22}$ | $3_{13}$ | $6_{7}$ | $2_{21}$ | $1_{20}$ | 5* | 7* |
| $2_{16}$ | 5/7 | $6_{23}$ | | 1* | 7/5 | | | |
| 1* | 6* | $8_{1}$ | $4_{5}$ | | $9_{11}$ | 3* | | |
| | | | 2* | | $1_{9}$ | 8* | | |
| 4* | | $2_{2}$ | $6_{8}$ | 8* | $3_{10}$ | | | |
| | 2* | $4_{4}$ | 7* | | 6* | | | |
| | $8_{17}$ | $1_{19}$ | 5* | | $4_{3}$ | | | |
| | | | $1_{12}$ | | $8_{14}$ | 4* | | |

Figure 155. Figure 10 for Puzzle 10

In order to get a better view of the situation, let us expland the chain of flipflops as displayed in Figure 156.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $1_{19}$ | 7/5 | | 4* | 5/7 | | | |
| $8_{16}$ | $4_{6}$ | $9_{22}$ | $3_{13}$ | $6_{8}$ | $2_{12}$ | $1_{21}$ | 5* | 7* |
| $2_{17}$ | 5/7 | $6_{23}$ | | 1* | 7/5 | | 3/4 | 4/3 |
| 1* | 6* | $8_{1}$ | $4_{5}$ | 5/7 | $9_{11}$ | 3* | | |
| | | | 2* | 7/5 | $1_{7}$ | 8* | 4/6 | 6/4 |
| 4* | | $2_{2}$ | $6_{9}$ | 8* | $3_{10}$ | | | |
| | 2* | $4_{4}$ | 7* | | 6* | | | |
| | $8_{18}$ | $1_{20}$ | 5* | | $4_{3}$ | | | |
| | | | $1_{15}$ | | $8_{14}$ | 4* | | |

Figure 156. Figure 11 for Puzzle 10

In order to speed up the solving process, let us further extend the chain of flipflops to much larger extent as displayed in Figure 157.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $1_{19}$ | 7/5 | | 4* | 5/7 | | | |
| $8_{16}$ | $4_{6}$ | $9_{22}$ | $3_{13}$ | $6_{8}$ | $2_{12}$ | $1_{21}$ | 5* | 7* |
| $2_{17}$ | 5/7 | $6_{23}$ | | 1* | 7/5 | | 3/4 | 4/3 |
| 1* | 6* | $8_{1}$ | $4_{5}$ | 5/7 | $9_{11}$ | 3* | | |
| 5/7 | | | 2* | 7/5 | $1_{7}$ | 8* | 4/6 | 6/4 |
| 4* | 7/5 | $2_{2}$ | $6_{9}$ | 8* | $3_{10}$ | 5/7 | | |
| | 2* | $4_{4}$ | 7* | | 6* | | | |
| | $8_{18}$ | $1_{20}$ | 5* | | $4_{3}$ | | | |
| 7/5 | | 5/7 | $1_{15}$ | | $8_{14}$ | 4* | | |

Figure 157. Figure 12 for Puzzle 10

Now, we can readily take to the thirtysecond step as in Figure 158.

$6_{29}2_{30}$c7

$7_{32}$r8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $1_{19}$ | 7/5 | | 4* | 5/7 | $6_{29}$ | | |
| $8_{16}$ | $4_{6}$ | $9_{22}$ | $3_{13}$ | $6_{8}$ | $2_{12}$ | $1_{21}$ | 5* | 7* |
| $2_{17}$ | 5/7 | $6_{23}$ | | 1* | 7/5 | | 3/4 | 4/3 |
| 1* | 6* | $8_{1}$ | $4_{5}$ | 5/7 | $9_{11}$ | 3* | | |
| 5/7 | $9_{25}$ | $3_{24}$ | 2* | 7/5 | $1_{7}$ | 8* | 4/6 | 6/4 |
| 4* | 7/5 | $2_{2}$ | $6_{9}$ | 8* | $3_{10}$ | 5/7 | | |
| $9_{28}$ | 2* | $4_{4}$ | 7* | | 6* | | | |
| $6_{27}$ | $8_{18}$ | $1_{20}$ | 5* | | $4_{3}$ | $2_{30}$ | $7_{32}$ | |
| 7/5 | $3_{26}$ | 5/7 | $1_{15}$ | $2_{31}$ | $8_{14}$ | 4* | | |

Figure 158. Figure 13 for Puzzle 10

We come to the point to break the chain almost all the way through as shown in Figure 159.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $1_{19}$ | $5_{36}$ | | 4* | $7_{42}$ | $6_{29}$ | | |
| $8_{16}$ | $4_{6}$ | $9_{22}$ | $3_{13}$ | $6_{8}$ | $2_{12}$ | $1_{21}$ | 5* | 7* |
| $2_{17}$ | $7_{35}$ | $6_{23}$ | | 1* | $5_{43}$ | | 3/4 | 4/3 |
| 1* | 6* | $8_{1}$ | $4_{5}$ | $7_{41}$ | $9_{11}$ | 3* | | |
| $7_{37}$ | $9_{25}$ | $3_{24}$ | 2* | $5_{40}$ | $1_{7}$ | 8* | 4/6 | 6/4 |
| 4* | $5_{34}$ | $2_{2}$ | $6_{9}$ | 8* | $3_{10}$ | $7_{33}$ | | |
| $9_{28}$ | 2* | $4_{4}$ | 7* | | 6* | | | |
| $6_{27}$ | $8_{18}$ | $1_{20}$ | 5* | | $4_{3}$ | $2_{30}$ | $7_{32}$ | |
| $5_{38}$ | $3_{26}$ | $7_{39}$ | $1_{15}$ | $2_{31}$ | $8_{14}$ | 4* | | |

Figure 159. Figure 14 for Puzzle 10

Finally, we can easily complete the puzzle in Figure 160.

$5_{44}$c7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $1_{19}$ | $5_{36}$ | $9_{47}$ | 4* | $7_{42}$ | $6_{29}$ | $8_{61}$ | $2_{62}$ |
| $8_{16}$ | $4_{6}$ | $9_{22}$ | $3_{13}$ | $6_{8}$ | $2_{12}$ | $1_{21}$ | 5* | 7* |
| $2_{17}$ | $7_{35}$ | $6_{23}$ | $8_{46}$ | 1* | $5_{43}$ | $9_{45}$ | $3_{51}$ | $4_{52}$ |
| 1* | 6* | $8_{1}$ | $4_{5}$ | $7_{41}$ | $9_{11}$ | 3* | $2_{63}$ | $5_{64}$ |
| $7_{37}$ | $9_{25}$ | $3_{24}$ | 2* | $5_{40}$ | $1_{7}$ | 8* | $4_{53}$ | $6_{54}$ |
| 4* | $5_{34}$ | $2_{2}$ | $6_{9}$ | 8* | $3_{10}$ | $7_{33}$ | $9_{57}$ | $1_{58}$ |
| $9_{28}$ | 2* | $4_{4}$ | 7* | $3_{49}$ | 6* | $5_{44}$ | $1_{59}$ | $8_{60}$ |
| $6_{27}$ | $8_{18}$ | $1_{20}$ | 5* | $9_{48}$ | $4_{3}$ | $2_{30}$ | $7_{32}$ | $3_{50}$ |
| $5_{38}$ | $3_{26}$ | $7_{39}$ | $1_{15}$ | $2_{31}$ | $8_{14}$ | 4* | $6_{55}$ | $9_{56}$ |

Figure 160. Figure 15 for Puzzle 10

**Puzzle 11**

Among 5000 Sudoku puzzles with 17 initial values in (2), I found the puzzle in Figure 161 is a rare unsolvable one!

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | 1* | | | 8* |
| 5* | | | | | | | 2* | |
| | 4* | | | 6* | | | | |
| | | | 2* | | | 4* | | |
| 8* | | | | | | 3* | | |
| | | 1* | | | | | | |
| | 2* | | | | | 7* | 4* | |
| | | | | 9* | 5* | | | |
| | | | 8* | | | | | |

Figure 161. Figure 1 for Puzzle 11

We can easily take to the fifteenth step as shown in Figure 162

$8_2$r7 $8_3$r3

$1_8 6_9$r2 $4_{11}$b4,

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | $2_{11}$ | 1* | | | 8* |
| 5* | $8_4$ | $6_9$ | | | | $1_8$ | 2* | $4_1$ |
| $1_{10}$ | 4* | $2_{12}$ | | 6* | $8_3$ | | | |
| | | | 2* | | | 4* | $8_6$ | |
| 8* | | | | | | 3* | | |
| $2_{13}$ | | 1* | | $8_5$ | | | | |
| | 2* | $8_2$ | | | | 7* | 4* | |
| | | | | 9* | 5* | $8_7$ | | $2_{15}$ |
| | | | 8* | | $2_{14}$ | | | |

Figure 162. Figure 2 for Puzzle 11

we found that no 2 could be filled in column 8 as shown in Figure 91.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | $2_{11}$ | 1* | No2 | | 8* |
| 5* | $8_4$ | $6_9$ | | | | $1_8$ | 2* | $4_1$ |
| $1_{10}$ | 4* | $2_{12}$ | | 6* | $8_3$ | No2 | | |
| | | | 2* | | | 4* | $8_6$ | |
| 8* | | | | | | 3* | | |
| $2_{13}$ | | 1* | | $8_5$ | | No2 | | |
| | 2* | $8_2$ | | | | 7* | 4* | |
| | | | | 9* | 5* | $8_7$ | | $2_{15}$ |
| | | | 8* | | $2_{14}$ | No2 | | |

Figure 163 Figure 3 for Puzzle 11

**Puzzle 12**

Unfortunately or fortunately, I further found the puzzles in Figures 164 and 167 are also rare unsolvable ones!

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 6* | | 7* | 8* | | |
| | 4* | 2* | | | | | | |
| | | | | | | | | |
| | | | 3* | 5* | | | | 2* |
| | 7* | | | | | | 1* | |
| 8* | | | | 4* | | | | |
| | | 5* | | 2* | | | | |
| 1* | | | | | | 7* | | |
| | | | 9* | | | | | |

Figure 164. Figure 1 for Puzzle 12

We can take the first eight steps in Figure 165 for the puzzle in Figure 164.

$2_1$r1

$7_6$b5

$7_7 8_8$r4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 6* | | 7* | 8* | $2_1$ | |
| | 4* | 2* | | | | | | |
| | | | | | | | | |
| | | | 3* | 5* | $8_8$ | | $7_7$ | 2* |
| $2_4$ | 7* | | | | | | 1* | |
| 8* | | | $7_5$ | 4* | $2_6$ | | | |
| | | 5* | | 2* | | | | |
| 1* | $2_3$ | | | | | 7* | | |
| | | | 9* | | | $2_2$ | | |

Figure 165. Figure 2 for Puzzle 12

We shall find that no 1 can be filled in box 5 as indicated in Figure 166.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 6* | | 7* | 8* | $2_1$ | |
| | 4* | 2* | | | | | | |
| | | | | | | | | |
| | | | 3* | 5* | $8_8$ | | $7_7$ | 2* |
| $2_4$ | 7* | | No1 | No1 | No1 | | 1* | |
| 8* | | | $7_5$ | 4* | $2_6$ | | | |
| | | 5* | | 2* | | | | |
| 1* | $2_3$ | | | | | 7* | | |
| | | | 9* | | | $2_2$ | | |

Figure 166. Figure 3 for Puzzle 12

**Puzzle 13**

For the puzzle of Figure 167, we can take the first thirty steps as shown in Figure 168.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | 6* | | | | 2* | |
| | | | | | | | 1* | |
| | 8* | | | | | | | |
| | | | | 3* | | 8* | | 5* |
| | | 6* | | | | 3* | | |
| 2* | | | | | | | | |
| 5* | 3* | | | | | 7* | | |
| | | | 2* | | 1* | | | |
| | | | 4* | | | | | |

Figure 167. Figure 1 for Puzzle 13

$2_2$b8 $1_8 4_9$r7

$3_{11}$b6 $3_{12}$r1

$4_{14}$b8 $6_{15}$r4

$8_{16}$b2 $8_{17} 5_{18}$c9

$4_{23}$c3 $6_{25}$c2

$6_{26}$b9 $7_{29}$b6.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | 6* | | | | 2* | $3_{12}$ |
| | $2_6$ | | | | | | 1* | $8_{22}$ |
| | 8* | | | $2_7$ | | | $5_{18}$ | $7_{28}$ |
| | | $4_{23}$ | | 3* | $2_3$ | 8* | $6_{15}$ | 5* |
| $8_{16}$ | | 6* | | | | 3* | | $2_2$ |
| 2* | | $3_1$ | | | | $1_{10}$ | | $4_{14}$ |
| 5* | 3* | $2_5$ | | | | 7* | $4_9$ | $1_8$ |
| $9_{30}$ | $4_{24}$ | $8_{21}$ | 2* | $7_{29}$ | 1* | $5_{19}$ | $3_{13}$ | $6_{26}$ |
| | $6_{25}$ | | 4* | $5_{20}$ | $3_{11}$ | $2_4$ | $8_{17}$ | $9_{27}$ |

Figure 168. Figure 2 for Puzzle 13

Next, we spot the grid (17), where only 9 fits. So we take

$9_{31}$g

in Figure 169 and we face a y-junction in column 5.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | 6* | 1? | | $9_{31}$ | 2* | $3_{12}$ |
| | $2_6$ | | | | | | 1* | $8_{22}$ |
| | 8* | | | $2_7$ | | | $5_{18}$ | $7_{28}$ |
| | | $4_{23}$ | | 3* | $2_3$ | 8* | $6_{15}$ | 5* |
| $8_{16}$ | | 6* | | 1? | | 3* | | $2_2$ |
| 2* | | $3_1$ | | | | $1_{10}$ | | $4_{14}$ |
| 5* | 3* | $2_5$ | | | | 7* | $4_9$ | $1_8$ |
| $9_{30}$ | $4_{24}$ | $8_{21}$ | 2* | $7_{29}$ | 1* | $5_{19}$ | $3_{13}$ | $6_{26}$ |
| | $6_{25}$ | | 4* | $5_{20}$ | $3_{11}$ | $2_4$ | $8_{17}$ | $9_{27}$ |

Figure 169. Figure 3 for Puzzle 13

Road 1. 1(15)

Case1. 1(31)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | No3 | No3 | 6* | $1_{1}$ | | $9_{31}$ | 2* | $3_{12}$ |
| $6_{3}$ | $2_{6}$ | No3 | | | | $4_{5}$ | 1* | $8_{22}$ |
| $1_{2}$ | 8* | $9_{8}$ | $3_{7}$ | $2_{7}$ | $4_{6}$ | $6_{4}$ | $5_{18}$ | $7_{28}$ |
| | | $4_{23}$ | | 3* | $2_{3}$ | 8* | $6_{15}$ | 5* |
| $8_{16}$ | | 6* | | | | 3* | | $2_{2}$ |
| 2* | | $3_{1}$ | | | | $1_{10}$ | | $4_{14}$ |
| 5* | 3* | $2_{5}$ | | | | 7* | $4_{9}$ | $1_{8}$ |
| $9_{30}$ | $4_{24}$ | $8_{21}$ | 2* | $7_{29}$ | 1* | $5_{19}$ | $3_{13}$ | $6_{26}$ |
| | $6_{25}$ | | 4* | $5_{20}$ | $3_{11}$ | $2_{4}$ | $8_{17}$ | $9_{27}$ |

Figure 170. Figure 4 for Puzzle 13

Case2. 1(33)

From Figures 170 and 171, we see that the first road took us to the dead end.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | 6* | $1_{1}$ | | $9_{31}$ | 2* | $3_{12}$ |
| $6_{4}$ | $2_{6}$ | $9_{7}$ | | No# | | $4_{6}$ | 1* | $8_{22}$ |
| $3_{3}$ | 8* | $1_{2}$ | | $2_{7}$ | | $6_{5}$ | $5_{18}$ | $7_{28}$ |
| | | $4_{23}$ | | 3* | $2_{3}$ | 8* | $6_{15}$ | 5* |
| $8_{16}$ | | 6* | | | | 3* | | $2_{2}$ |
| 2* | | $3_{1}$ | | | | $1_{10}$ | | $4_{14}$ |
| 5* | 3* | $2_{5}$ | | | | 7* | $4_{9}$ | $1_{8}$ |
| $9_{30}$ | $4_{24}$ | $8_{21}$ | 2* | $7_{29}$ | 1* | $5_{19}$ | $3_{13}$ | $6_{26}$ |
| | $6_{25}$ | | 4* | $5_{20}$ | $3_{11}$ | $2_{4}$ | $8_{17}$ | $9_{27}$ |

Figure 171. Figure 5 for Puzzle 13

Road 2. 1(55)

From Figure 172, the road not taken leads to nowhere either. Therefore, this puzzle is unsolvable. What a lesson to learn!

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | 6* | | | $9_{31}$ | 2* | $3_{12}$ |
| $6_{4}$ | $2_{6}$ | | | $4_{5}$ | | No# | 1* | $8_{22}$ |
| $3_{3}$ | 8* | | $1_{2}$ | $2_{7}$ | | | $5_{18}$ | $7_{28}$ |
| | | $4_{23}$ | | 3* | $2_{3}$ | 8* | $6_{15}$ | 5* |
| $8_{16}$ | | 6* | | $1_{1}$ | | 3* | | $2_{2}$ |
| 2* | | $3_{1}$ | | | | $1_{10}$ | | $4_{14}$ |
| 5* | 3* | $2_{5}$ | | | | 7* | $4_{9}$ | $1_{8}$ |
| $9_{30}$ | $4_{24}$ | $8_{21}$ | 2* | $7_{29}$ | 1* | $5_{19}$ | $3_{13}$ | $6_{26}$ |
| | $6_{25}$ | | 4* | $5_{20}$ | $3_{11}$ | $2_{4}$ | $8_{17}$ | $9_{27}$ |

Figure 172. Figure 6 for Puzzle 13

**Puzzle 14**

The puzzle in Figure 173 is a rare one with seventeen initial values, with two solutions as in Figures 174 and 175 !

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | 7* | | | | | 5* |
| | | | | | | | 7* | |
| | | | 2* | | | | | |
| 6* | 3* | | | | | 8* | | |
| 5* | | | | | | 2* | | |
| | | | | 1* | | | | |
| | | | | | 4* | 1* | | |
| | 7* | | | | | | | |
| | | 1* | 5* | | | | 9* | |

Figure 173. Figure 1 for Puzzle 14

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | 6 | 8 | 7* | 3 | 1 | 9 | 2 | 5* |
| 1 | 5 | 2 | 8 | 4 | 9 | 3 | 7* | 6 |
| 7 | 9 | 3 | 2* | 5 | 6 | 4 | 8 | 1 |
| 6* | 3* | 7 | 4 | 2 | 5 | 8* | 1 | 9 |
| 5* | 1 | 4 | 9 | 8 | 7 | 2* | 6 | 3 |
| 8 | 2 | 9 | 6 | 1* | 3 | 5 | 4 | 7 |
| 9 | 8 | 6 | 3 | 7 | 4* | 1* | 5 | 2 |
| 2 | 7* | 5 | 1 | 9 | 8 | 6 | 3 | 4 |
| 3 | 4 | 1* | 5* | 6 | 2 | 7 | 9* | 8 |

Figure 174. Figure 2 for Puzzle 14

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | 6 | 8 | 7* | 3 | 1 | 9 | 2 | 5* |
| 1 | 5 | 2 | 9 | 4 | 8 | 3 | 7* | 6 |
| 7 | 9 | 3 | 2* | 5 | 6 | 4 | 8 | 1 |
| 6* | 3* | 7 | 4 | 2 | 5 | 8* | 1 | 9 |
| 5* | 1 | 4 | 8 | 9 | 7 | 2* | 6 | 3 |
| 8 | 2 | 9 | 6 | 1* | 3 | 5 | 4 | 7 |
| 9 | 8 | 6 | 3 | 7 | 4* | 1* | 5 | 2 |
| 2 | 7* | 5 | 1 | 8 | 9 | 6 | 3 | 4 |
| 3 | 4 | 1* | 5* | 6 | 2 | 7 | 9* | 8 |

Figure 175. Figure 3 for Puzzle 14

**Puzzle 15**

Here is another one!

The puzzle in Figure 176 will lead to two solutions as shown in Figures 177 and 178.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | | | | | | | | 8* |
| | | | | 3* | | 7* | | |
| 6* | | | | | | | | |
| | 3* | | | | | 5* | 2* | |
| | | | 1* | | | | | 6* |
| | 7* | | | | | | | |
| | | | 2* | | 1* | | 4* | |
| | | | | | | 3* | 5* | |
| | | | 6* | | | | | |

Figure 176. Figure 1 for Puzzle 15

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | 5 | 3 | 7 | 9 | 2 | 4 | 6 | 8* |
| 2 | 9 | 4 | 8 | 3* | 6 | 7* | 1 | 5 |
| 6* | 8 | 7 | 5 | 1 | 4 | 2 | 9 | 3 |
| 4 | 3* | 1 | 9 | 6 | 8 | 5* | 2* | 7 |
| 5 | 2 | 8 | 1* | 4 | 7 | 9 | 3 | 6* |
| 9 | 7* | 6 | 3 | 2 | 5 | 1 | 8 | 4 |
| 3 | 8 | 5 | 2* | 7 | 1* | 6 | 4* | 9 |
| 7 | 6 | 2 | 4 | 8 | 9 | 3* | 5* | 1 |
| 4 | 1 | 9 | 6* | 5 | 3 | 8 | 7 | 2 |

Figure 177. Figure 2 for Puzzle 15

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | 5 | 3 | 7 | 9 | 4 | 2 | 6 | 8* |
| 2 | 9 | 4 | 8 | 3* | 6 | 7* | 1 | 5 |
| 6* | 8 | 7 | 5 | 1 | 2 | 4 | 9 | 3 |
| 4 | 3* | 1 | 9 | 6 | 8 | 5* | 2* | 7 |
| 5 | 2 | 8 | 1* | 4 | 7 | 9 | 3 | 6* |
| 9 | 7* | 6 | 3 | 2 | 5 | 1 | 8 | 4 |
| 3 | 8 | 5 | 2* | 7 | 1* | 6 | 4* | 9 |
| 7 | 6 | 2 | 4 | 8 | 9 | 3* | 5* | 1 |
| 4 | 1 | 9 | 6* | 5 | 3 | 8 | 7 | 2 |

Figure 178. Figure 3 for Puzzle 15

**Puzzle 16**

For the puzzle in Figure 179, if I told you that I could fill 1, 2, 3 in those grids in Figure 180, would you believe me?

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | | 8* | 2* | | | |
| 1* | | | | | | | 6* | |
| | | | | | | | | |
| | | 5* | | | | 7* | | 2* |
| | 6* | | 3* | | | | | |
| | 9* | | | | | | | |
| | | | 6* | 5* | | | 3* | |
| 3* | | | | | | 2* | | |
| | | | 1* | | | | | |

Figure 179. Figure 1 for Puzzle 16

How in the world can we see them? In fact, there are some hidden numbers as indicated in Figure 181, that might help you!

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | | 8* | 2* | | | |
| 1* | 2 | | | 3 | | | 6* | |
| | | | | | | | | |
| | | 5* | | | | 7* | 1 | 2* |
| | 6* | | 3* | | | | | |
| | 9* | | | | | | | |
| | | | 6* | 5* | | | 3* | |
| 3* | | | | | | 2* | | |
| | | | 1* | | | | | |

Figure 180. Figure 2 for Puzzle 16

In box 4, 1 and 6 can only be filled in (35) and (36) as shown. Since 6 can only be filled in either (45) or (46), where 1 should be avoided.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | | 8* | 2* | | | |
| 1* | | | | | | | 6* | |
| | | | | 1/6 | 6/1 | | | |
| | | 5* | | 6? | 6? | 7* | | 2* |
| | 6* | | 3* | | | | | |
| | 9* | | | | | | | |
| | | | 6* | 5* | | | 3* | |
| 3* | | | | | | 2* | | |
| | | | 1* | | | | | |

Figure 181. Figure 3 for Puzzle 16

Otherwise, we could flip flop 1 and 6 in Figure 182, if we were able to complete the puzzle so that multiple solutions would be obtained.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | | 8* | 2* | | | |
| 1* | | | | | | | 6* | |
| | | | | 1/6 | 6/1 | | | |
| | | 5* | | 6/1 | 1/6 | 7* | | 2* |
| | 6* | | 3* | | | | | |
| | 9* | | | | | | | |
| | | | 6* | 5* | | | 3* | |
| 3* | | | | | | 2* | | |
| | | | 1* | | | | | |

Figure 182. Figure 4 for Puzzle 16

Now, could we fill 1 in (42) as shown in Figure 183? The answer is no, since that is exactly the right spot for 3. So the right spot for 1 is (48).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | | 8* | 2* | | | |
| 1* | | | | | | | 6* | |
| | | | | | | | | |
| | 1? | 5* | | | | 7* | | 2* |
| | 6* | | 3* | | | | | |
| | 9* | | | | | | | |
| | | | 6* | 5* | | | 3* | |
| 3* | | | | | | 2* | | |
| | | | 1* | | | | | |

Figure 183. Figure 5 for puzzle 16

Next, let us look at column 5. Where can we fit 3 in Figure 184?

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | | 8* | 2* | | | |
| 1* | | | | | | | 6* | |
| | | | | 1/6 | 6/1 | | | |
| | 3 | 5* | | | | 7* | | 2* |
| | 6* | | 3* | | | | | |
| | 9* | | | | | | | |
| | | | 6* | 5* | | | 3* | |
| 3* | | | | | | 2* | | |
| | | | 1* | | | | | |

Figure 184. Figure 6 for puzzle 16

Now, could we fill 3 in (95) as shown in Figure185? The answer is no, since that is exactly the right spot for 2. So the right spot for 3 is (25).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | | 8* | 2* | | | |
| 1* | | | | | | | 6* | |
| | | | | 1/6 | 6/1 | | | |
| | 3 | 5* | | | | 7* | | 2* |
| | 6* | | 3* | | | | | |
| | | | | | | | | |
| | 9* | | | | | | | |
| | | | 6* | 5* | | | 3* | |
| 3* | | | | | | 2* | | |
| | | | 1* | 3? | | | | |

Figure 185. Figure 7 for puzzle 16

Could we fill 2 at (23) as in Figure 186? Since 6 needs to be at (13) in row 1 and in turn at (91), 3 and 5 are forced to be at (33) and (31), as shown in Figure 187, respectively.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | | 8* | 2* | | | |
| 1* | | 2? | | | | | 6* | |
| | | | | | | | | |
| | 3 | 5* | | | | 7* | | 2* |
| | 6* | | 3* | | | | | |
| | 9* | | | | | | | |
| | | | 6* | 5* | | | 3* | |
| 3* | | | | | | 2* | | |
| | | | 1* | | | | | |

Figure 186. Figure 8 for puzzle 1

As a result, 9 would not be able to fit in box 1 at all!

Therefore, the right spot for 2 is (22).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | $6_1$ | | 8* | 2* | | | |
| 1* | | 2? | | | | | 6* | |
| $5_4$ | | $3_3$ | | | | | | |
| | 3 | 5* | | | | 7* | | 2* |
| | 6* | | 3* | | | | | |
| | 9* | | | | | | | |
| | | | 6* | 5* | | | 3* | |
| 3* | | | | | | 2* | | |
| $6_2$ | | | 1* | | | | | |

Figure 187. Figure 9 for puzzle 16

Now, let us use the prescribed order to solve the puzzle in question restarting from Figure 188.

$2_2 3_3$b6

$3_5$r4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | | 8* | 2* | | | |
| 1* | | | | | | | 6* | |
| | | | | 1/6 | 6/1 | | $2_1$ | |
| | $3_5$ | 5* | | 6/1 | 1/6 | 7* | | 2* |
| | 6* | | 3* | | | | | |
| | 9* | | $2_4$ | | | | | |
| | | | 6* | 5* | | | 3* | |
| 3* | | | | | | 2* | | |
| | | | 1* | $2_2$ | $3_3$ | | | |

Figure 188. Figure 10 for puzzle 16

Next, to avoid the potential multiple solutions by flipflopping 1 and 6 as shown in Figure 188, we take in Figure 189

$1_6$ r4: u16r3b4r4b5.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | | 8* | 2* | | | |
| 1* | | | | | | | 6* | |
| | | | | | | | $2_1$ | |
| | $3_5$ | 5* | | | | 7* | $1_6$ | 2* |
| | 6* | | 3* | | | | | |
| | 9* | | $2_4$ | | | | | |
| | | | 6* | 5* | | | 3* | |
| 3* | | | | | | 2* | | |
| | | | 1* | $2_2$ | $3_3$ | | | |

Figure 189. Figure 11 for puzzle 16

The rest is easy as in Figure 190.

$3_7$c5: 16(35)(36) $6_8$r1

$3_{13}$b8 $3_{14}1_{15}$r1

$1_{17}$c2 $5_{18}$b3 $5_{21}$r1

$5_{22}$c6: 5r5b8 $9_{23}$b1

$4_{28}8_{29}$c2 $7_{32}$c1

$7_{34}$c8 $8_{47}$r5.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | $7_{30}$ | $6_8$ | $5_{21}$ | 8* | 2* | $3_{14}$ | $9_{31}$ | $1_{15}$ |
| 1* | $2_{24}$ | $9_{23}$ | $7_{58}$ | $3_7$ | $4_{59}$ | $8_{39}$ | 6* | $5_{40}$ |
| $5_{19}$ | $8_{18}$ | $3_9$ | $9_{60}$ | $6_{56}$ | $1_{55}$ | $4_{43}$ | $2_1$ | $7_{42}$ |
| $8_{33}$ | $3_{10}$ | 5* | $4_{61}$ | $9_{62}$ | $6_{57}$ | 7* | $1_6$ | 2* |
| $2_{26}$ | 6* | $1_{52}$ | 3* | $7_{53}$ | $8_{47}$ | $5_{41}$ | $4_{50}$ | $9_{46}$ |
| $7_{32}$ | 9* | $4_{51}$ | $2_4$ | $1_{54}$ | $5_{22}$ | $6_{12}$ | $8_{49}$ | $3_{13}$ |
| $9_{27}$ | $4_{28}$ | $2_{20}$ | 6* | 5* | $7_{37}$ | $1_{16}$ | 3* | $8_{38}$ |
| 3* | $1_{17}$ | $7_{35}$ | $8_{21}$ | $4_{63}$ | $9_{64}$ | 2* | $5_{21}$ | $6_7$ |
| $6_{10}$ | $5_{18}$ | $8_{36}$ | 1* | $2_2$ | $3_3$ | $9_{44}$ | $7_{34}$ | $4_{45}$ |

Figure 190. Figure 12 for puzzle 16

**Puzzle 17**

The first fifteen steps of the puzzle in Figure 191 is easy.

$1_{2}9_{3}$c$6_{4}$r7 $7_{10}$b2

$8_{11}$b6 $9_{12}$b8

$9_{14}$r1 $6_{15}$c4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 4* | | | | | 3* | | |
| 7* | | | 5* | | | | | |
| | | | | | | | | 1* |
| | 8* | 9* | | | | | | |
| | | | 8* | | | | 7* | |
| | | 1* | | | | 6* | | |
| 3* | | | | | | 2* | 8* | |
| | | | | 9* | 1* | | | |
| | | | | 6* | | | | |

Figure 191. Figure 1 for Puzzle 17

We take this rare opportunity to show you the "intersection move" 7(74) as shown in Figure 192.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $1_{5}$ | 4* | | $9_{14}$ | | | 3* | | |
| 7* | | | 5* | $1_{6}$ | | | | |
| | | | $6_{15}$ | | | | | 1* |
| | 8* | 9* | $1_{7}$ | | | | | |
| | | | 8* | | $9_{13}$ | $1_{8}$ | 7* | |
| | $7_{10}$ | 1* | | | | 6* | $9_{12}$ | $8_{1}$ |
| 3* | $1_{2}$ | $6_{4}$ | $7_{16}$ | | | 2* | 8* | $9_{3}$ |
| | | | | 9* | 1* | | | |
| | | | | 6* | $8_{11}$ | | $1_{9}$ | |

Figure 192. Figure 2 for Puzzle 17

Taking 7(75)/(76), we would run into the dilemma of having no 7 in column 4 as shown in Figure 193.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $1_{5}$ | 4* | | $9_{14}$ | | | 3* | | |
| 7* | | | 5* | $1_{6}$ | | | | |
| | | | $6_{15}$ | | | | | 1* |
| | 8* | 9* | $1_{7}$ | | | | | |
| | | | 8* | | $9_{13}$ | $1_{8}$ | 7* | |
| | $7_{10}$ | 1* | No7 | | | 6* | $9_{12}$ | $8_{1}$ |
| 3* | $1_{2}$ | $6_{4}$ | No7 | 7? | 7? | 2* | 8* | $9_{3}$ |
| | | | No7 | 9* | 1* | | | |
| | | | No7 | 6* | $8_{11}$ | | $1_{9}$ | |

Figure 193. Figure 3 for Puzzle 17

If we took 7(84)/(94), we would run into the dilemma of having no 7 in row 7 as shown in Figure 194.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $1_5$ | 4* | | $9_{14}$ | | | 3* | | |
| 7* | | | 5* | $1_6$ | | | | |
| | | | $6_{15}$ | | | | | 1* |
| | 8* | 9* | $1_7$ | | | | | |
| | | | 8* | | $9_{13}$ | $1_8$ | 7* | |
| | $7_{10}$ | 1* | | | | 6* | $9_{12}$ | $8_1$ |
| 3* | $1_2$ | $6_4$ | No7 | No7 | No7 | 2* | 8* | $9_3$ |
| | | | 7? | 9* | 1* | | | |
| | | | 7? | 6* | $8_{11}$ | | $1_9$ | |

Figure 194. Figure 4 for Puzzle 17

Thus we have

7(74): 7(75)/(76)
→No7c4
&7(84)/(94)
→No7r7

as shown in Figure 195.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $1_5$ | 4* | | $9_{14}$ | | 7? | 3* | | |
| 7* | $6_{18}$ | | 5* | $1_6$ | 7? | | | |
| | | | $6_{15}$ | | 7? | | | 1* |
| | 8* | 9* | $1_7$ | $7_{17}$ | $6_{16}$ | | | |
| $6_{19}$ | | | 8* | | $9_{13}$ | $1_8$ | 7* | |
| | $7_{10}$ | 1* | $4_{21}$ | | | 6* | $9_{12}$ | $8_1$ |
| 3* | $1_2$ | $6_4$ | $7_{20}$ | 4? | 4? | 2* | 8* | $9_3$ |
| | | | | 9* | 1* | | | |
| | | | | 6* | $8_{11}$ | | $1_9$ | |

Figure 195. Figure 5 for Puzzle 17

However, we shall follow the prescribed order to take four more steps as in Figure 196.

$6_{18}$b1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $1_5$ | 4* | $5_{29}$ | $9_{14}$ | $8_{30}$ | $2_{31}$ | 3* | $6_{26}$ | $7_{25}$ |
| 7* | $6_{18}$ | $8_{24}$ | 5* | $1_6$ | $3_{34}$ | $9_{22}$ | | |
| $9_{42}$ | | | $6_{15}$ | $4_{35}$ | $7_{28}$ | $8_{23}$ | $5_{32}$ | 1* |
| | 8* | 9* | $1_7$ | $7_{17}$ | $6_{16}$ | | | |
| $6_{19}$ | | | 8* | $2_{41}$ | $9_{13}$ | $1_8$ | 7* | |
| $2_{40}$ | $7_{10}$ | 1* | $4_{21}$ | $3_{39}$ | $5_{38}$ | 6* | $9_{12}$ | $8_1$ |
| 3* | $1_2$ | $6_4$ | $7_{20}$ | $5_{37}$ | $4_{36}$ | 2* | 8* | $9_3$ |
| $8_{33}$ | | | | 9* | 1* | | | $6_{27}$ |
| | $9_{43}$ | | | 6* | $8_{11}$ | | $1_9$ | |

Figure 196. Figure 6 for Puzzle 17

Finally, as in Figure 197, we can take

$4_{44}$g: 5(47)
→5(52)
→5(91)
→No5b9

as shown in Figure 198.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $1_{5}$ | 4* | $5_{29}$ | $9_{14}$ | $8_{30}$ | $2_{31}$ | 3* | $6_{26}$ | $7_{25}$ |
| 7* | $6_{18}$ | $8_{24}$ | 5* | $1_{6}$ | $3_{34}$ | $9_{22}$ | | |
| $9_{42}$ | | | $6_{15}$ | $4_{35}$ | $7_{28}$ | $8_{23}$ | $5_{32}$ | 1* |
| | 8* | 9* | $1_{7}$ | $7_{17}$ | $6_{16}$ | $5_{1}$ | | |
| $6_{19}$ | $5_{2}$ | | 8* | $2_{41}$ | $9_{13}$ | $1_{8}$ | 7* | |
| $2_{40}$ | $7_{10}$ | 1* | $4_{21}$ | $3_{39}$ | $5_{38}$ | 6* | $9_{12}$ | $8_{1}$ |
| 3* | $1_{2}$ | $6_{4}$ | $7_{20}$ | $5_{37}$ | $4_{36}$ | 2* | 8* | $9_{3}$ |
| $8_{33}$ | | | | 9* | 1* | No5 | No5 | $6_{27}$ |
| $5_{3}$ | $9_{43}$ | | | 6* | $8_{11}$ | No5 | $1_{9}$ | No5 |

Figure 197. Figure 7 for Puzzle 17

Now, according to the situation shown in Figure 198, we can take in Figure 200

$7_{20}4_{21}$r7:
7c6b4$4r7b6.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $1_{5}$ | 4* | $5_{29}$ | $9_{14}$ | $8_{30}$ | $2_{31}$ | 3* | $6_{26}$ | $7_{25}$ |
| 7* | $6_{18}$ | $8_{24}$ | 5* | $1_{6}$ | $3_{34}$ | $9_{22}$ | | |
| $9_{42}$ | | | $6_{15}$ | $4_{35}$ | $7_{28}$ | $8_{23}$ | $5_{32}$ | 1* |
| | 8* | 9* | $1_{7}$ | $7_{17}$ | $6_{16}$ | $4_{44}$ | | |
| $6_{19}$ | | | 8* | $2_{41}$ | $9_{13}$ | $1_{8}$ | 7* | |
| $2_{40}$ | $7_{10}$ | 1* | $4_{21}$ | $3_{39}$ | $5_{38}$ | 6* | $9_{12}$ | $8_{1}$ |
| 3* | $1_{2}$ | $6_{4}$ | $7_{20}$ | $5_{37}$ | $4_{36}$ | 2* | 8* | $9_{3}$ |
| $8_{33}$ | | | | 9* | 1* | | | $6_{27}$ |
| | $9_{43}$ | | | 6* | $8_{11}$ | | $1_{9}$ | |

Figure 198. Figure 8 for Puzzle 17

The rest is easy as in Figure 199.

$9_{22}$c7: 9r3b1

$7_{25}$b7 $5_{29}8_{30}$r1

$3_{34}$r2 $3_{39}$r6

$9_{42}$g

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $1_{5}$ | 4* | $5_{29}$ | $9_{14}$ | $8_{30}$ | $2_{31}$ | 3* | $6_{26}$ | $7_{25}$ |
| 7* | $6_{18}$ | $8_{24}$ | 5* | $1_{6}$ | $3_{34}$ | $9_{22}$ | $2_{62}$ | $4_{63}$ |
| $9_{42}$ | $2_{51}$ | $3_{50}$ | $6_{15}$ | $4_{35}$ | $7_{28}$ | $8_{23}$ | $5_{32}$ | 1* |
| $5_{47}$ | 8* | 9* | $1_{7}$ | $7_{17}$ | $6_{16}$ | $4_{44}$ | $3_{61}$ | $2_{64}$ |
| $6_{19}$ | $3_{48}$ | $4_{45}$ | 8* | $2_{41}$ | $9_{13}$ | $1_{8}$ | 7* | $5_{49}$ |
| $2_{40}$ | $7_{10}$ | 1* | $4_{21}$ | $3_{39}$ | $5_{38}$ | 6* | $9_{12}$ | $8_{1}$ |
| 3* | $1_{2}$ | $6_{4}$ | $7_{20}$ | $5_{37}$ | $4_{36}$ | 2* | 8* | $9_{3}$ |
| $8_{33}$ | $5_{52}$ | $2_{56}$ | $3_{58}$ | 9* | 1* | $7_{54}$ | $4_{59}$ | $6_{27}$ |
| $4_{46}$ | $9_{43}$ | $7_{55}$ | $2_{57}$ | 6* | $8_{11}$ | $5_{53}$ | $1_{9}$ | $3_{60}$ |

Figure 199. Figure 9 for Puzzle 17

**Puzzle 18**

This very puzzle as shown in Figure 200 has many virtues that will teach us what Sudoku is really about.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 4* | | 7* | | |
| | | | 5* | | | 3* | | |
| | 1* | 8* | | | | | | |
| 3* | | | | | | 5* | 4* | |
| | | 6* | 1* | | | | | |
| | | | 2* | | | | | |
| 5* | 2* | | | 3* | | | | |
| | | | | | | | | 1* |
| | | | | | | | 6* | |

Figure 200. Figure 1 for Puzzle 18

We can take to the eighth step

$1_1 2_2$r4

as in Figure 201. In Figure 202, 2 can only fit (13) or (16) in row 1.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 4* | | 7* | $1_6$ | |
| | | | 5* | $1_5$ | | 3* | | |
| | 1* | 8* | | | | | | |
| 3* | | $1_1$ | | | | 5* | 4* | $2_2$ |
| $2_8$ | | 6* | 1* | | | | | |
| | | | 2* | | | $1_7$ | | |
| 5* | 2* | | | 3* | $1_4$ | | | |
| | | | | | | | | 1* |
| $1_3$ | | | | | | | 6* | |

Figure 201. Figure 2 for Puzzle 18

If we took 2(13) in Figure 203, it would force no room for 5 to be in box 1.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 2? | | 4* | 2? | 7* | $1_6$ | |
| | | | 5* | $1_5$ | | 3* | | |
| | 1* | 8* | | | | | | |
| 3* | | $1_1$ | | | | 5* | 4* | $2_2$ |
| $2_8$ | | 6* | 1* | | | | | |
| | | | 2* | | | $1_7$ | | |
| 5* | 2* | | | 3* | $1_4$ | | | |
| | | | | | | | | 1* |
| $1_3$ | | | | | | | 6* | |

Figure 202. Figure 3 for Puzzle 18

So we take

$2_9$r1:2(13)
→3(12)
→No5b1

in Figure 204.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| No5 | 3 | 2? | | 4* | 2? | 7* | $1_6$ | |
| No5 | No5 | No5 | 5* | $1_5$ | | 3* | | |
| No5 | 1* | 8* | | | | | | |
| 3* | | $1_1$ | | | | 5* | 4* | $2_2$ |
| $2_8$ | | 6* | 1* | | | | | |
| | | | 2* | | | $1_7$ | | |
| 5* | 2* | | | 3* | $1_4$ | | | |
| | | | | | | | | 1* |
| $1_3$ | | | | | | | 6* | |

Figure 203. Figure 4 for Puzzle 18

Since 3 hidden in column 6 of box 5 as shown, we can take

$3_{11}$r3: 3c6b5

and the next nineteen steps in Figure 205.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 4* | $2_9$ | 7* | $1_6$ | |
| | | $2_{10}$ | 5* | $1_5$ | | 3* | | |
| | 1* | 8* | | | | | | |
| 3* | | $1_1$ | | | | 5* | 4* | $2_2$ |
| $2_8$ | | 6* | 1* | | 3? | | | |
| | | | 2* | | 3? | $1_7$ | | |
| 5* | 2* | | | 3* | $1_4$ | | | |
| | | | | | | | | 1* |
| $1_3$ | | | | | | | 6* | |

Figure 204. Figure 5 for Puzzle 18

$6_{12}$r7 $6_{13}$b8

$5_{16}4_{17}$r3 $3_{20}$b9

$5_{21}$c6:
34(65)(66)

$6_{22}8_{23}$b4 $6_{27}$g

We now face a stalemate.

Don’t panic!

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $6_{27}$ | $3_{30}$ | $5_{31}$ | $8_{23}$ | 4* | $2_9$ | 7* | $1_6$ | $9_{26}$ |
| | | $2_{10}$ | 5* | $1_5$ | $6_{22}$ | 3* | $8_{25}$ | $4_{18}$ |
| $4_{17}$ | 1* | 8* | $3_{11}$ | | | $6_{14}$ | $2_{15}$ | $5_{16}$ |
| 3* | | $1_1$ | | $6_{24}$ | 8? | 5* | 4* | $2_2$ |
| $2_8$ | | 6* | 1* | | 3/4 | 8? | | 8? |
| | | | 2* | 8? | 4/3 | $1_7$ | | $6_{13}$ |
| 5* | 2* | | $6_{12}$ | 3* | $1_4$ | | | |
| 8 | $6_{28}$ | $3_{29}$ | No8 | No8 | No8 | | $5_{19}$ | 1* |
| | | | | | | | | |
| $1_3$ | | | No8 | No8 | $5_{21}$ | | 6* | $3_{20}$ |

Figure 205. Figure 6 for Puzzle 18

There are only (47) and (65) for 8 to fit in box 5, but 8(65) would lead to the dead end in Figure 206.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $6_{27}$ | $3_{30}$ | $5_{31}$ | $8_{23}$ | 4* | $2_{9}$ | 7* | $1_{6}$ | $9_{26}$ |
| | | $2_{10}$ | 5* | $1_{5}$ | $6_{22}$ | 3* | $8_{25}$ | $4_{18}$ |
| $4_{17}$ | 1* | 8* | $3_{11}$ | | | $6_{14}$ | $2_{15}$ | $5_{16}$ |
| 3* | No4 | $1_{1}$ | | $6_{24}$ | $8_{32}$ | 5* | 4* | $2_{2}$ |
| $2_{8}$ | $5_{8}$ | 6* | 1* | | | | | |
| No4 | $8_{7}$ | No4 | 2* | | | $1_{7}$ | | $6_{13}$ |
| 5* | 2* | $4_{2}$ | $6_{12}$ | 3* | $1_{4}$ | | | |
| $8_{6}$ | $6_{28}$ | $3_{29}$ | | $2_{5}$ | | $4_{3}$ | $5_{19}$ | 1* |
| $1_{3}$ | | | $4_{1}$ | $8_{5}$ | $5_{21}$ | $2_{4}$ | 6* | $3_{20}$ |

Figure 206. Figure 7 for Puzzle 18

So, in Figure 207, we take

$8_{32}$b5: 34c6b5& 8(65)→8(81) →No8b6,

$4_{33}$c4:4(65)→4(73) →4(87)→2(97) →2(85)→8(81) →8(62)→2(52) →No4b2.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $6_{27}$ | $3_{30}$ | $5_{31}$ | $8_{23}$ | 4* | $2_{9}$ | 7* | $1_{6}$ | $9_{26}$ |
| 7/9 | 9/7 | $2_{10}$ | 5* | $1_{5}$ | $6_{22}$ | 3* | $8_{25}$ | $4_{18}$ |
| $4_{17}$ | 1* | 8* | $3_{11}$ | 9/7 | 7/9 | $6_{14}$ | $2_{15}$ | $5_{16}$ |
| 3* | 7/9 | $1_{1}$ | 9/7 | $6_{24}$ | $8_{32}$ | 5* | 4* | $2_{2}$ |
| $2_{8}$ | $5_{41}$ | 6* | 1* | 7/9 | $4_{38}$ | $8_{46}$ | $3_{36}$ | $7_{45}$ |
| 9/7 | $8_{40}$ | $4_{34}$ | 2* | $5_{39}$ | $3_{37}$ | $1_{7}$ | 7/9 | $6_{13}$ |
| 5* | 2* | 7/9 | $6_{12}$ | 3* | $1_{4}$ | $4_{43}$ | 9/7 | $8_{44}$ |
| $8_{35}$ | $6_{28}$ | $3_{29}$ | $4_{33}$ | | 9/7 | | $5_{19}$ | 1* |
| $1_{3}$ | $4_{42}$ | 9/7 | 7/9 | | $5_{21}$ | | 6* | $3_{20}$ |

Figure 207. Figure 8 for Puzzle 18

Finally, we can readily complete this puzzle in Figure 208.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $6_{27}$ | $3_{30}$ | $5_{31}$ | $8_{23}$ | 4* | $2_{9}$ | 7* | $1_{6}$ | $9_{26}$ |
| $9_{51}$ | $7_{50}$ | $2_{10}$ | 5* | $1_{5}$ | $6_{22}$ | 3* | $8_{25}$ | $4_{18}$ |
| $4_{17}$ | 1* | 8* | $3_{11}$ | $7_{60}$ | $9_{59}$ | $6_{14}$ | $2_{15}$ | $5_{16}$ |
| 3* | $9_{49}$ | $1_{1}$ | $7_{48}$ | $6_{24}$ | $8_{32}$ | 5* | 4* | $2_{2}$ |
| $2_{8}$ | $5_{41}$ | 6* | 1* | $9_{47}$ | $4_{38}$ | $8_{46}$ | $3_{36}$ | $7_{45}$ |
| $7_{52}$ | $8_{40}$ | $4_{34}$ | 2* | $5_{39}$ | $3_{37}$ | $1_{7}$ | $9_{53}$ | $6_{13}$ |
| 5* | 2* | $9_{55}$ | $6_{12}$ | 3* | $1_{4}$ | $4_{43}$ | $7_{54}$ | $8_{44}$ |
| $8_{35}$ | $6_{28}$ | $3_{29}$ | $4_{33}$ | $2_{62}$ | $7_{58}$ | $9_{61}$ | $5_{19}$ | 1* |
| $1_{3}$ | $4_{42}$ | $7_{56}$ | $9_{57}$ | $8_{63}$ | $5_{21}$ | $2_{64}$ | 6* | $3_{20}$ |

Figure 208. Figure 9 for Puzzle 18

**Puzzle 19**

The puzzle of Figure 209 can be solved rather easily as shown in Figure 210.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 1* | 7* | | | | | |
| | | | | | | | 5* | 3* |
| | | | | | | | | |
| 5* | 3* | | | 2* | | | | |
| | | | 1* | | | 7* | | 8* |
| 6* | | | | | | | | |
| 4* | 2* | | | | 5* | | 3* | |
| | | | 6* | | | 1* | | |
| | | | | | | | | |

Figure 209. Figure 1 for Puzzle 19

$1_3 3_4 9_5$r1 $6_{11}$b4

$5_{12}$ r2 $2_{13}$b4

$3_{14}$ $9_{15}$b2 $2_{17}$c4

$2_{18}$c3 $9_{23}$b5

$9_{24}3_{25}8_{26}6_{27}$r7

$4_{30}$c4 $8_{32}$r4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $2_{37}$ | $5_{9}$ | 1* | 7* | $3_{35}$ | $9_{53}$ | $6_{36}$ | $8_{27}$ | $4_{54}$ |
| $7_{46}$ | $6_{50}$ | $4_{40}$ | $2_{30}$ | $8_{48}$ | $1_{17}$ | $9_{55}$ | 5* | 3* |
| $8_{47}$ | $9_{51}$ | $3_{34}$ | $5_{8}$ | $6_{49}$ | $4_{52}$ | $2_{39}$ | $7_{15}$ | $1_{16}$ |
| 5* | 3* | $7_{20}$ | $8_{21}$ | 2* | $6_{13}$ | $4_{56}$ | $1_{18}$ | $9_{57}$ |
| $9_{43}$ | $4_{42}$ | $2_{38}$ | 1* | $5_{5}$ | $3_{6}$ | 7* | $6_{7}$ | 8* |
| 6* | $1_{2}$ | $8_{22}$ | $4_{31}$ | $9_{59}$ | $7_{60}$ | $3_{1}$ | $2_{19}$ | $5_{10}$ |
| 4* | 2* | $6_{24}$ | $9_{25}$ | $1_{4}$ | 5* | $8_{23}$ | 3* | $7_{14}$ |
| $3_{33}$ | $7_{62}$ | $5_{12}$ | 6* | $4_{58}$ | $8_{61}$ | 1* | $9_{44}$ | $2_{28}$ |
| $1_{3}$ | $8_{63}$ | $9_{41}$ | $3_{29}$ | $7_{64}$ | $2_{32}$ | $5_{11}$ | $4_{45}$ | $6_{26}$ |

Figure 210. Figure 2 for Puzzle 19

Would the flipflopping of 9(32) and 7(38) with 7(82) and 9(88) cause multiple solutions?

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $2_{37}$ | $5_{9}$ | 1* | 7* | $3_{35}$ | No8 | $6_{36}$ | $8_{27}$ | |
| $7_{1}$ | | | $2_{30}$ | No8 | $1_{17}$ | | 5* | 3* |
| $8_{2}$ | 9 | $3_{34}$ | $5_{8}$ | No8 | No8 | $2_{39}$ | $7_{15}$ | $1_{16}$ |
| 5* | 3* | $7_{20}$ | $8_{21}$ | 2* | $6_{13}$ | | $1_{18}$ | |
| | | $2_{38}$ | 1* | $5_{5}$ | $3_{6}$ | 7* | $6_{7}$ | 8* |
| 6* | $1_{2}$ | $8_{22}$ | $4_{31}$ | | | $3_{1}$ | $2_{19}$ | $5_{10}$ |
| 4* | 2* | $6_{24}$ | $9_{25}$ | $1_{4}$ | 5* | $8_{23}$ | 3* | $7_{14}$ |
| $3_{33}$ | | $5_{12}$ | 6* | | | 1* | 9 | $2_{28}$ |
| $1_{3}$ | 7 | | $3_{29}$ | $8_{3}$ | $2_{32}$ | $5_{11}$ | | $6_{26}$ |

Figure 211. Figure 3 for Puzzle 19

The answer is no, since the new 7(32) would conflict 7(21) in box1 as displayed in Figure 212. We need to look at the surroundings.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $2_{37}$ | $5_{9}$ | 1* | 7* | $3_{35}$ | | $6_{36}$ | $8_{27}$ | |
| 9? | | | $2_{30}$ | | $1_{17}$ | | 5* | 3* |
| 9? | | $3_{34}$ | $5_{8}$ | | | $2_{39}$ | $7_{15}$ | $1_{16}$ |
| 5* | 3* | $7_{20}$ | $8_{21}$ | 2* | $6_{13}$ | | $1_{18}$ | |
| No4 | $9_{1}$ | $2_{38}$ | 1* | $5_{5}$ | $3_{6}$ | 7* | $6_{7}$ | 8* |
| 6* | $1_{2}$ | $8_{22}$ | $4_{31}$ | | | $3_{1}$ | $2_{19}$ | $5_{10}$ |
| 4* | 2* | $6_{24}$ | $9_{25}$ | $1_{4}$ | 5* | $8_{23}$ | 3* | $7_{14}$ |
| $3_{33}$ | 7 | $5_{12}$ | 6* | | | 1* | 9 | $2_{28}$ |
| $1_{3}$ | | | $3_{29}$ | | $2_{32}$ | $5_{11}$ | | $6_{26}$ |

Figure 212. Figure 4 for Puzzle 19

For example, by moving 7(82) to 7(92), we would face the dilemma in Figure 213.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $2_{37}$ | $5_{9}$ | 1* | 7* | $3_{35}$ | | $6_{36}$ | $8_{27}$ | |
| | 9 | $4_{3}$ | $2_{30}$ | | $1_{17}$ | No# | 5* | 3* |
| | | $3_{34}$ | $5_{8}$ | | | $2_{39}$ | $7_{15}$ | $1_{16}$ |
| 5* | 3* | $7_{20}$ | $8_{21}$ | 2* | $6_{13}$ | | $1_{18}$ | |
| $9_{1}$ | $4_{2}$ | $2_{38}$ | 1* | $5_{5}$ | $3_{6}$ | 7* | $6_{7}$ | 8* |
| 6* | $1_{2}$ | $8_{22}$ | $4_{31}$ | | | $3_{1}$ | $2_{19}$ | $5_{10}$ |
| 4* | 2* | $6_{24}$ | $9_{25}$ | $1_{4}$ | 5* | $8_{23}$ | 3* | $7_{14}$ |
| $3_{33}$ | 7 | $5_{12}$ | 6* | | | 1* | 9 | $2_{28}$ |
| $1_{3}$ | | | $3_{29}$ | | $2_{32}$ | $5_{11}$ | | $6_{26}$ |

Figure 213. Figure 5 for Puzzle 19

In fact, the three different ways to move 9(32) as shown separately in Figures 212-214 each leading to a dilemma.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $2_{37}$ | $5_{9}$ | 1* | 7* | $3_{35}$ | | $6_{36}$ | $8_{27}$ | |
| | | 9 | $2_{30}$ | | $1_{17}$ | | 5* | 3* |
| | | $3_{34}$ | $5_{8}$ | | | $2_{39}$ | $7_{15}$ | $1_{16}$ |
| 5* | 3* | $7_{20}$ | $8_{21}$ | 2* | $6_{13}$ | | $1_{18}$ | |
| | | $2_{38}$ | 1* | $5_{5}$ | $3_{6}$ | 7* | $6_{7}$ | 8* |
| 6* | $1_{2}$ | $8_{22}$ | $4_{31}$ | | | $3_{1}$ | $2_{19}$ | $5_{10}$ |
| 4* | 2* | $6_{24}$ | $9_{25}$ | $1_{4}$ | 5* | $8_{23}$ | 3* | $7_{14}$ |
| $3_{33}$ | 7 | $5_{12}$ | 6* | | | 1* | 9 | $2_{28}$ |
| $1_{3}$ | | No# | $3_{29}$ | | $2_{32}$ | $5_{11}$ | | $6_{26}$ |

Figure 214. Figure 6 for Puzzle 19

**Puzzle 20**

We can readily fill with basic moves to the seventeenth step as shown in Figure 216 for the puzzle in Figure 215.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | | | | 7* | | 4* | | |
| | 3* | | 1* | | | | | |
| | | | | | | | | |
| | 5* | 1* | 3* | | | | | |
| | | | | 4* | | 8* | | |
| | | 6* | | | | 2* | | |
| 8* | | | | | 2* | | | |
| | | | 6* | | | | 5* | |
| | | | | | | | 1* | |

Figure 215. Figure 1 for Puzzle 20

$2_{1}8_{2}$r4 $6_{5}$b5

$1_{6}5_{7}$r5 $5_{13}$r7

$5_{14}$c7 $2_{19}$c8

$3_{20}6_{21}$r1: 3c8b8

$3_{22}$r7 $7_{23}$b5

$8_{25}$c8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | $1_{9}$ | | | 7* | $5_{16}$ | 4* | $6_{21}$ | $3_{20}$ |
| | 3* | | 1* | | | $5_{14}$ | $2_{19}$ | |
| $5_{15}$ | | | $2_{3}$ | | | $1_{8}$ | $8_{25}$ | |
| | 5* | 1* | 3* | $2_{1}$ | $8_{2}$ | | | |
| | | | $5_{7}$ | 4* | $6_{5}$ | 8* | | $1_{6}$ |
| | $8_{4}$ | 6* | $7_{23}$ | $9_{24}$ | $1_{12}$ | 2* | | $5_{18}$ |
| 8* | | $5_{13}$ | | $1_{11}$ | 2* | $3_{22}$ | | |
| $1_{10}$ | | | 6* | | | | 5* | |
| | | | | $5_{17}$ | | | 1* | |

Figure 216. Figure 2 for Puzzle 20

We can now complete the puzzle in Figure 217.

$4_{26}8_{27}$c4: 4(74)
→4(49)
→No4b9

$3_{34}$c5 $4_{39}$c9

$3_{45}$c1 $4_{46}$r2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | $1_{9}$ | $9_{28}$ | $8_{27}$ | 7* | $5_{16}$ | 4* | $6_{21}$ | $3_{20}$ |
| $7_{49}$ | 3* | $8_{30}$ | 1* | $6_{35}$ | $4_{46}$ | $5_{14}$ | $2_{19}$ | $7_{51}$ |
| $5_{15}$ | $6_{36}$ | $4_{48}$ | $2_{3}$ | $3_{34}$ | $9_{47}$ | $1_{8}$ | $8_{25}$ | $7_{52}$ |
| $9_{50}$ | 5* | 1* | 3* | $2_{1}$ | $8_{2}$ | $6_{44}$ | $7_{53}$ | $4_{39}$ |
| $3_{45}$ | $2_{57}$ | $7_{58}$ | $5_{7}$ | 4* | $6_{5}$ | 8* | $9_{34}$ | $1_{6}$ |
| $4_{40}$ | $8_{4}$ | 6* | $7_{23}$ | $9_{24}$ | $1_{12}$ | 2* | $3_{41}$ | $5_{18}$ |
| 8* | $7_{43}$ | $5_{13}$ | $9_{29}$ | $1_{11}$ | 2* | $3_{22}$ | $4_{42}$ | $6_{38}$ |
| $1_{10}$ | $4_{53}$ | $3_{60}$ | 6* | $8_{31}$ | $7_{62}$ | $9_{63}$ | 5* | $2_{33}$ |
| $6_{37}$ | $9_{56}$ | $2_{59}$ | $4_{26}$ | $5_{17}$ | $3_{61}$ | $7_{64}$ | 1* | $8_{32}$ |

Figure 217. Figure 3 for Puzzle 20

**Puzzle 21**

We can readily fill with basic moves to the thirty-nineth step as shown in Figure 219 for the puzzle in Figure 218.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | | | 1* | | | 8* | | |
| | | 2* | | | 6* | | | |
| | | | 4* | | | | | |
| | | | | 8* | | 1* | | |
| | | 5* | | | | | | 2* |
| | | | 7* | | | | | |
| 1* | 3* | | | | | | 4* | |
| | 8* | | | | 5* | | | |
| | | | | 2* | | | | |

Figure 218. Figure 1 for Puzzle 21

$2_1$b3 $2_3 5_4$c4

$5_7$r7 $8_{12} 2_{13}$b4

$1_{21}$c3 $4_{27}$r8

$4_{30}$b1 $6_{34} 3_{35}$b5

$6_{40}$c7: 6r9b3.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $5_{10}$ | $6_3$ | 1* | | $2_{13}$ | 8* | | $4_{31}$ |
| $8_{15}$ | $4_{30}$ | 2* | $5_4$ | | 6* | | | $1_{26}$ |
| | | $1_{21}$ | 4* | | $8_{12}$ | $5_9$ | $2_{14}$ | |
| $4_{33}$ | $6_2$ | $3_{17}$ | $2_3$ | 8* | $9_{36}$ | 1* | $5_8$ | |
| | $1_{22}$ | 5* | $6_{34}$ | $4_{29}$ | $3_{35}$ | | $8_{18}$ | 2* |
| | $2_5$ | $8_{16}$ | 7* | $5_6$ | $1_{23}$ | $4_{32}$ | $6_1$ | |
| 1* | 3* | $9_{39}$ | $8_{20}$ | $6_{38}$ | $7_{37}$ | $2_2$ | 4* | $5_7$ |
| $2_1$ | 8* | $4_{27}$ | | $1_{24}$ | 5* | $6_{40}$ | | |
| $5_{11}$ | No6 | No6 | | 2* | $4_{28}$ | | $1_{24}$ | $8_{19}$ |

Figure 219. Figure 2 for Puzzle 21

From Figure 220, we have

$6_{41}$(18)c8: 6(68)
→6(42)
→6(13)
→No6b3

$6_{42}$c3.

The rest is easy as shown.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $5_{10}$ | $7_{43}$ | 1* | $9_{45}$ | $2_{13}$ | 8* | $6_{41}$ | $4_{31}$ |
| $8_{15}$ | $4_{30}$ | 2* | $5_4$ | $3_{62}$ | 6* | $7_{61}$ | $9_{55}$ | $1_{26}$ |
| $6_{51}$ | $9_{50}$ | $1_{21}$ | 4* | $7_{63}$ | $8_{12}$ | $5_9$ | $2_{14}$ | $3_{64}$ |
| $4_{33}$ | $6_{49}$ | $3_{17}$ | $2_3$ | 8* | $9_{36}$ | 1* | $5_8$ | $7_{48}$ |
| $7_{46}$ | $1_{22}$ | 5* | $6_{34}$ | $4_{29}$ | $3_{35}$ | $9_{47}$ | $8_{18}$ | 2* |
| $9_{52}$ | $2_5$ | $8_{16}$ | 7* | $5_6$ | $1_{23}$ | $4_{32}$ | $3_{54}$ | $6_{53}$ |
| 1* | 3* | $9_{39}$ | $8_{20}$ | $6_{38}$ | $7_{37}$ | $2_2$ | 4* | $5_7$ |
| $2_1$ | 8* | $4_{27}$ | $3_{58}$ | $1_{24}$ | 5* | $6_{40}$ | $7_{56}$ | $9_{57}$ |
| $5_{11}$ | $7_{44}$ | $6_{42}$ | $9_{59}$ | 2* | $4_{28}$ | $3_{60}$ | $1_{24}$ | $8_{19}$ |

Figure 220. Figure 3 for Puzzle 21

**Puzzle 22**

We can readily fill with basic moves to the sixteenth step as shown in Figure 222 for the puzzle in Figure 221.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7* | | | | 3* | | 8* | | |
| | | | 5* | | | 2* | | |
| | | | | | | | | |
| | 2* | 5* | | 6* | 8* | | | |
| | | | | | | | 1* | |
| | 9* | | | | | | | |
| 3* | | | | | | 4* | 7* | |
| 1* | | | 9* | | | | | |
| | | | 2* | | | | | |

Figure 221. Figure 1 for Puzzle 22

$1_1$b2 $2_3 5_4 9_5$c1

$3_7 7_8 8_9 6_{10}$c4: 37r5b2

$4_{13}$g $4_{14}$g $6_{15}$g $7_{16}$c4.

As in Figure 222, we can take

$7_{17}$c6: 7r5b2&36(86)(96)

In Figure 223.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7* | | $4_{14}$ | $6_{10}$ | 3* | $2_6$ | 8* | | |
| $9_5$ | | | 5* | | | 2* | | $7_{12}$ |
| $2_3$ | | | $7_8$ | | | | | |
| $4_{13}$ | 2* | 5* | $1_2$ | 6* | 8* | $7_{16}$ | | |
| | 7? | 7? | $4_{11}$ | | | | 1* | |
| | 9* | $1_1$ | $3_7$ | | $7_{17}$ | | | |
| 3* | $6_{15}$ | | $8_9$ | | | 4* | 7* | |
| 1* | | | 9* | | 3/6 | | | |
| $5_4$ | | | 2* | | 6/3 | | | |

Figure 222. Figure 2 for Puzzle 22

We can complete the puzzle in Figure 223.

$7_{18}$c2: u47c2b3c5b6

$9_{20}$c7: 9r3b4&r5b5

$8_{29}6_{30}$r6

$1_{36}$c7

$1_{42}$c5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7* | $1_{40}$ | $4_{14}$ | $6_{10}$ | 3* | $2_6$ | 8* | $5_{61}$ | $9_{60}$ |
| $9_5$ | $3_{58}$ | $6_{57}$ | 5* | $8_{41}$ | $1_{46}$ | 2* | $4_{52}$ | $7_{12}$ |
| $2_3$ | $5_{59}$ | $8_{56}$ | $7_8$ | $9_{40}$ | $4_{47}$ | $1_{36}$ | $3_{55}$ | $6_{63}$ |
| $4_{13}$ | 2* | 5* | $1_2$ | 6* | 8* | $7_{16}$ | $9_{54}$ | $3_{53}$ |
| $6_{32}$ | $7_{18}$ | $3_{19}$ | $4_{11}$ | $2_{25}$ | $9_{33}$ | $5_{35}$ | 1* | $8_{34}$ |
| $8_{29}$ | 9* | $1_1$ | $3_7$ | $5_{31}$ | $7_{17}$ | $6_{30}$ | $2_{24}$ | $4_{26}$ |
| 3* | $6_{15}$ | $9_{21}$ | $8_9$ | $1_{42}$ | $5_{43}$ | 4* | 7* | $2_{23}$ |
| 1* | $8_{50}$ | $2_{22}$ | 9* | $7_{28}$ | $4_{45}$ | $3_{37}$ | $6_{64}$ | $5_{62}$ |
| $5_4$ | $4_{49}$ | $7_{27}$ | 2* | $6_{44}$ | $3_{39}$ | $9_{20}$ | $8_{51}$ | $1_{38}$ |

Figure 223. Figure 3 for Puzzle 22

**Puzzle 23**

We can take to the thirty-second step in Figure 225 of the puzzle in Figure 224.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | | | | | | | 5* | |
| | | 8* | | 3* | | | | |
| | | | | 1* | | | | |
| | | | 4* | | | 2* | 6* | |
| | 1* | | | | | | | |
| | | | 2* | | | | | |
| 6* | | | | | | 1* | | 3* |
| 4* | | | 5* | | 9* | | | |
| | | | | | | 7* | | |

Figure 224. Figure 1 for Puzzle 23

$1_2 3_3$c4 $3_{11}$b8

$5_{15}$c7

$6_{17} 4_{18} 2_{19} 8_{20}$r9

$4_{22}$r5 $6_{23} 7_{24}$b5

$7_{25}$r8 $2_{26} 4_{247}$c5

$8_{31}$b2: 46(62)(63)

$8_{32}$g.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | | $3_{13}$ | | $4_{27}$ | | | 5* | $1_6$ |
| $1_5$ | | 8* | | 3* | | | | |
| | | | | 1* | | | $3_{12}$ | |
| $3_{14}$ | $8_{31}$ | $5_1$ | 4* | $9_4$ | $1_8$ | 2* | 6* | No9 |
| | 1* | $2_1$ | $3_3$ | $8_3$ | $6_{23}$ | $5_{15}$ | No9 | $4_{22}$ |
| | | | 2* | $5_2$ | $7_{24}$ | $3_{11}$ | $1_7$ | $8_5$ |
| 6* | | | $8_{29}$ | $2_{26}$ | $4_{28}$ | 1* | $9_{30}$ | 3* |
| 4* | $3_{10}$ | $1_4$ | 5* | $7_{25}$ | 9* | | | |
| $8_{20}$ | $2_{19}$ | $9_{21}$ | $1_2$ | $6_{17}$ | $3_9$ | 7* | $4_{18}$ | $5_{16}$ |

Figure 225. Figure 2 for Puzzle 23

Due to the dilemma in Figure 225, we can complete the puzzle in Figure 226.

$5_{33} 9_{34}$r4: 5(43)
→5(65)→8(55)
→9(45)→No9b8

$5_{39}$c1 $9_{56}$r3

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | $6_{63}$ | $3_{13}$ | $7_{46}$ | $4_{27}$ | $8_{32}$ | $9_{64}$ | 5* | $1_6$ |
| $1_5$ | $9_{62}$ | 8* | $6_{58}$ | 3* | $5_{51}$ | $4_{61}$ | $2_{49}$ | $7_{45}$ |
| $7_{38}$ | $5_{55}$ | $4_{57}$ | $9_{56}$ | 1* | $2_{50}$ | $8_{47}$ | $3_{12}$ | $6_{53}$ |
| $3_{14}$ | $8_{31}$ | $7_{35}$ | 4* | $5_{33}$ | $1_8$ | 2* | 6* | $9_{34}$ |
| $9_{40}$ | 1* | $2_1$ | $3_3$ | $8_{42}$ | $6_{23}$ | $5_{15}$ | $7_{43}$ | $4_{22}$ |
| $5_{39}$ | $4_{59}$ | $6_{60}$ | 2* | $9_{41}$ | $7_{24}$ | $3_{11}$ | $1_7$ | $8_{44}$ |
| 6* | $7_{36}$ | $5_{37}$ | $8_{29}$ | $2_{26}$ | $4_{28}$ | 1* | $9_{30}$ | 3* |
| 4* | $3_{10}$ | $1_4$ | 5* | $7_{25}$ | 9* | $6_{54}$ | $8_{48}$ | $2_{52}$ |
| $8_{20}$ | $2_{19}$ | $9_{21}$ | $1_2$ | $6_{17}$ | $3_9$ | 7* | $4_{18}$ | $5_{16}$ |

Figure 226. Figure 3 for Puzzle 23

PRACTICE SET 1

**Puzzle 24**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 3* | 6* | | 5* | | |
| 4* | | | | | | | | 2* |
| 1* | | | | | | | | |
| | | | | | | 7* | 1* | |
| 2* | | | 8* | | | | | |
| | 8* | | 5* | | | | | |
| | 3* | | | | | 6* | | |
| | | | | 1* | 4* | | | |
| | | | | 7* | | | | |

**Puzzle 25**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | 7* | | | 9* | | | | |
| | | | | | | | 8* | 1* |
| | 2* | | | | | | | |
| | | 1* | 6* | | | | 4* | |
| 5* | | | | 7* | | 3* | | |
| | | | | | | | | |
| 2* | | | | | | 9* | | |
| | | | 1* | | 4* | | | |
| | | | 8* | | | | | |

**Puzzle 26**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1* | | 4* | | | | | |
| | | | | 8* | | 5* | | |
| | | | | | | 7* | | |
| 7* | | | 3* | | 4* | | | |
| | | | 1* | | | | 2* | |
| 8* | | | | | | | | |
| 2* | | | | 9* | | | | |
| | | | | | | | 4* | 3* |
| | | | | | | 6* | | 1* |

**Puzzle 27**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 6* | | | | 1* |
| | 2* | | 5* | | | | | |
| | | | | | | | | |
| | | | 3* | | | | 2* | |
| | | 7* | | | | 4* | | |
| 1* | | | | 5* | | | | |
| 7* | 3* | | | | 1* | | | |
| 6* | | | | | | | 5* | |
| | | | 2* | | | 8* | | |

**Puzzle 28**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | 5* | 1* |
| 2* | | | 8* | | | | | |
| 4* | | | | | | | | |
| | 1* | | | 5* | 7* | | | |
| 3* | | | | | | 2* | | |
| | | | | 6* | | 4* | | |
| | 5* | 7* | | | | | 6* | |
| | | | 2* | | | 3* | | |
| | | | | | | | | |

**Puzzle 29**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 7* | | 4* | | 6* | 5* | | |
| | | | 2* | | | 3* | 4* | |
| | | | | | | | | |
| | 8* | | | 1* | | | | |
| | | | | | | 4* | | |
| 1* | | | | | | | | |
| 3* | | | | | | | 1* | 7* |
| | | | 5* | | | | | 8* |
| 2* | | | | | | | | |

**Puzzle 30**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | | | | | | | 3* | 2* |
| | 5* | | 8* | | | | | |
| | | | | | | | | |
| | 6* | | 5* | | | 4* | | |
| 2* | | | 3* | | | | | |
| 7* | | | | | | | | |
| | | | | 2* | 1* | | | |
| | 4* | | | | | 6* | | |
| | | | | 7* | | 5* | | |

**Puzzle 24**

$1_{2}7_{3}4_{4}$c4 $3_{13}$b6

$4_{14}$c2 $6_{16}$c6

$7_{17}$b7 $2_{24}$r4

$2_{28}$b3 $6_{34}$r9

$9_{36}$b3 $8_{38}$b9

$9_{43}$r1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $8_{44}$ | $7_{23}$ | $9_{43}$ | 3* | 6* | $2_{30}$ | 5* | $4_{11}$ | $1_{5}$ |
| 4* | $5_{50}$ | $6_{48}$ | $1_{2}$ | $8_{57}$ | $9_{56}$ | $3_{58}$ | $7_{17}$ | 2* |
| 1* | $2_{29}$ | $3_{51}$ | $7_{3}$ | $4_{10}$ | $5_{53}$ | $8_{63}$ | $6_{41}$ | $9_{64}$ |
| $3_{33}$ | $9_{49}$ | $5_{52}$ | $4_{4}$ | $2_{24}$ | $6_{16}$ | 7* | 1* | $8_{1}$ |
| 2* | $1_{8}$ | $4_{21}$ | 8* | $3_{61}$ | $7_{12}$ | $9_{62}$ | $5_{40}$ | $6_{42}$ |
| $6_{32}$ | 8* | $7_{20}$ | 5* | $9_{60}$ | $1_{9}$ | $4_{22}$ | $2_{25}$ | $3_{59}$ |
| $7_{19}$ | 3* | $1_{7}$ | $2_{30}$ | $5_{54}$ | $8_{55}$ | 6* | $9_{37}$ | $4_{15}$ |
| $5_{45}$ | $6_{47}$ | $8_{46}$ | $9_{35}$ | 1* | 4* | $2_{26}$ | $3_{27}$ | $7_{18}$ |
| $9_{36}$ | $4_{14}$ | $2_{28}$ | $6_{34}$ | 7* | $3_{13}$ | $1_{6}$ | $8_{38}$ | $5_{39}$ |

**Puzzle 25**

$1_1$b1   $8_7$r1   $8_8$r7

$4_9$b9   $9_{11}$b6

$4_{12}8_{13}$c5: 4r2b1

$4_{15}$r1   $1_{14}$b9   $5_{16}$b3

$8_{16}4_{17}$r5   $2_{21}$g

$6_{22}$r3: 6r1b7   $6_{23}$r5

$3_{31}2_{32}6_{33}$b7   $4_{36}6_{37}$c1   $2_{52}$c4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | 7* | $8_{7}$ | $5_{34}$ | 9* | $1_{2}$ | $4_{15}$ | $6_{33}$ | $2_{32}$ |
| $9_{27}$ | $4_{39}$ | $5_{40}$ | $2_{52}$ | $3_{53}$ | $6_{50}$ | $7_{48}$ | 8* | 1* |
| $1_{1}$ | 2* | $6_{22}$ | $7_{47}$ | $4_{12}$ | $8_{14}$ | $5_{46}$ | $9_{30}$ | $3_{31}$ |
| $7_{38}$ | $3_{42}$ | 1* | 6* | $8_{13}$ | $5_{44}$ | $2_{29}$ | 4* | $9_{28}$ |
| 5* | $8_{16}$ | $9_{24}$ | $4_{17}$ | 7* | $2_{21}$ | 3* | $1_{4}$ | $6_{23}$ |
| $4_{36}$ | $6_{41}$ | $2_{25}$ | $9_{20}$ | $1_{3}$ | $3_{45}$ | $8_{19}$ | $7_{60}$ | $5_{59}$ |
| 2* | $1_{6}$ | $4_{10}$ | $3_{54}$ | $6_{51}$ | $7_{49}$ | 9* | $5_{55}$ | $8_{8}$ |
| $8_{18}$ | $9_{26}$ | $3_{62}$ | 1* | $5_{56}$ | 4* | $6_{35}$ | $2_{58}$ | $7_{61}$ |
| $6_{37}$ | $5_{43}$ | $7_{63}$ | 8* | $2_{57}$ | $9_{11}$ | $1_{5}$ | $3_{64}$ | $4_{9}$ |

**Puzzle 26**

$1_2$c1   $2_6 9_7$b9   $3_8$c5

$3_{10}$b7   $5_{12}$g   $8_{13}$b5

$8_{14}$g   $6_{16}$g: 57r7b9

$9_{17}$g   $1_{18}$g   $6_{20}$g

$8_{21}$g   $2_{23}$g

$9_{23}$c6: 26c6b4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $6_{20}$ | 1* | $7_{39}$ | 4* | $3_{8}$ | $5_{40}$ | $9_{17}$ | $8_{21}$ | $2_{22}$ |
| $9_{51}$ | $2_{63}$ | $3_{62}$ | $7_{37}$ | 8* | $6_{49}$ | 5* | $1_{5}$ | $4_{50}$ |
| $4_{48}$ | $8_{43}$ | $5_{44}$ | $9_{38}$ | $1_{4}$ | $2_{46}$ | 7* | $3_{11}$ | $6_{47}$ |
| 7* | $9_{58}$ | $2_{59}$ | 3* | $6_{35}$ | 4* | $1_{18}$ | $5_{26}$ | $8_{23}$ |
| $3_{52}$ | $5_{45}$ | $6_{55}$ | 1* | $7_{31}$ | $8_{13}$ | $4_{54}$ | 2* | $9_{25}$ |
| 8* | $4_{60}$ | $1_{19}$ | $5_{34}$ | $2_{36}$ | $9_{24}$ | $3_{53}$ | $6_{29}$ | $7_{30}$ |
| 2* | $3_{64}$ | $4_{61}$ | $6_{16}$ | 9* | $1_{3}$ | $8_{14}$ | $7_{28}$ | $5_{27}$ |
| $1_{2}$ | $6_{56}$ | $9_{57}$ | $8_{17}$ | $5_{33}$ | $7_{32}$ | $2_{6}$ | 4* | 3* |
| $5_{12}$ | $7_{41}$ | $8_{42}$ | $2_{10}$ | $4_{1}$ | $3_{9}$ | 6* | $9_{7}$ | 1* |

$5_{26}$b8: u57r6b8r7b9   $7_{32}$b6   $5_{34}6_{35}$b5   $2_{46}6_{47}$r3   $9_{51}$c1

**Puzzle 27**

$2_3 8_4$b3   $1_{10}$r8   $3_{16}$b2

$3_{18}$b9   $5_{19}$r5   $5_{20}3_{21}$r1

$6_{22}$b6   $4_{25}$r7   $6_{30}8_{31}$r5

$5_{35}$b7   $5_{36}$c7

$6_{39}7_{40}$r2   $8_{53}$r1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_{20}$ | $7_{54}$ | $4_{44}$ | $9_{45}$ | 6* | $3_{21}$ | $2_{8}$ | $8_{53}$ | 1* |
| $8_{57}$ | 2* | $1_{14}$ | 5* | $4_{33}$ | $7_{40}$ | $3_{36}$ | $6_{39}$ | $9_{56}$ |
| $3_{37}$ | $9_{63}$ | $6_{62}$ | $1_{13}$ | $2_{7}$ | $8_{51}$ | $5_{35}$ | $7_{55}$ | $4_{34}$ |
| $9_{58}$ | $6_{64}$ | $8_{61}$ | 3* | $1_{12}$ | $4_{42}$ | $1_{47}$ | 2* | $5_{38}$ |
| $2_{5}$ | $5_{19}$ | 7* | $8_{31}$ | $9_{32}$ | $6_{30}$ | 4* | $1_{11}$ | $3_{17}$ |
| 1* | $4_{43}$ | $3_{16}$ | $7_{41}$ | 5* | $2_{6}$ | $6_{42}$ | $9_{59}$ | $8_{60}$ |
| 7* | 3* | $5_{2}$ | $6_{22}$ | $8_{23}$ | 1* | $9_{26}$ | $4_{25}$ | $2_{9}$ |
| 6* | $8_{4}$ | $2_{3}$ | $4_{49}$ | $3_{29}$ | $9_{52}$ | $1_{10}$ | 5* | $7_{27}$ |
| $4_{45}$ | $1_{15}$ | $9_{46}$ | 2* | $7_{28}$ | $5_{1}$ | 8* | $3_{18}$ | $6_{24}$ |

**Puzzle 28**

$1_{1}5_{2}$b1 $1_{8}$c7 $2_{9}$r4

$3_{15}$b3 $4_{16}$r4 $3_{17}$b5

$1_{18}$r6 $4_{19}3_{20}$r1

$8_{23}$r4 $1_{25}$g $7_{26}$c1

$6_{27}$g $8_{30}1_{31}$c6

$6_{34}1_{35}$c1: 6r4b8

$3_{40}$r9 $7_{43}$b8 $8_{49}$r1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $7_{26}$ | $3_{20}$ | $9_{50}$ | $6_{27}$ | $2_{14}$ | $4_{19}$ | $8_{49}$ | 5* | 1* |
| 2* | $6_{52}$ | $5_{2}$ | 8* | $7_{62}$ | $1_{31}$ | $9_{63}$ | $4_{42}$ | $3_{60}$ |
| 4* | $8_{51}$ | $1_{1}$ | $5_{7}$ | $3_{61}$ | $9_{32}$ | $7_{64}$ | $2_{13}$ | $6_{59}$ |
| $9_{36}$ | 1* | $2_{9}$ | $4_{16}$ | 5* | 7* | $6_{58}$ | $3_{48}$ | $8_{57}$ |
| 3* | $4_{54}$ | $6_{53}$ | $1_{25}$ | $9_{33}$ | $8_{30}$ | 2* | $7_{43}$ | $5_{4}$ |
| $5_{3}$ | $7_{44}$ | $8_{45}$ | $3_{17}$ | 6* | $2_{11}$ | 4* | $1_{18}$ | $9_{39}$ |
| $8_{23}$ | 5* | 7* | $9_{24}$ | $4_{21}$ | $3_{22}$ | $1_{8}$ | 6* | $2_{12}$ |
| $6_{34}$ | $9_{55}$ | $4_{56}$ | 2* | $1_{37}$ | $5_{6}$ | 3* | $8_{47}$ | $7_{46}$ |
| $1_{35}$ | $2_{10}$ | $3_{15}$ | $7_{28}$ | $8_{38}$ | $6_{29}$ | $5_{5}$ | $9_{40}$ | $4_{41}$ |

**Puzzle 29**

$1_{4}$b4 $5_{8}3_{9}$b9

$7_{10}5_{11}$b7 $2_{15}3_{16}$r1

$7_{19}8_{20}$r9 $4_{24}$r3: 23r3b1

$8_{25}5_{26}$c1

$4_{38}9_{39}$c2: 23(32)(52)

$3_{46}2_{47}$c2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $9_{17}$ | 7* | $1_{5}$ | 4* | $3_{16}$ | 6* | 5* | $8_{11}$ | $2_{15}$ |
| $8_{25}$ | $6_{48}$ | $5_{50}$ | 2* | $7_{12}$ | $1_{4}$ | 3* | 4* | $9_{51}$ |
| $2_{1}$ | $2_{1}$ | $2_{49}$ | $8_{36}$ | $9_{37}$ | $5_{35}$ | $1_{2}$ | $7_{10}$ | $6_{52}$ |
| $5_{26}$ | 8* | $6_{60}$ | $9_{45}$ | 1* | $4_{42}$ | $7_{14}$ | $2_{59}$ | $3_{29}$ |
| $7_{21}$ | $2_{47}$ | $9_{62}$ | $3_{30}$ | $5_{34}$ | $8_{33}$ | 4* | $6_{63}$ | $1_{1}$ |
| 1* | $4_{38}$ | $3_{61}$ | $7_{23}$ | $6_{53}$ | $2_{54}$ | $8_{13}$ | $9_{64}$ | $5_{28}$ |
| 3* | $5_{32}$ | $8_{31}$ | $6_{44}$ | $4_{43}$ | $9_{55}$ | $2_{57}$ | 1* | 7* |
| $6_{27}$ | $1_{7}$ | $4_{40}$ | 5* | $2_{56}$ | $7_{22}$ | $9_{58}$ | $3_{9}$ | 8* |
| 2* | $9_{39}$ | $7_{19}$ | $1_{6}$ | $8_{20}$ | $3_{18}$ | $6_{41}$ | $5_{8}$ | $4_{3}$ |

**Puzzle 30**

$2_{1}3_{2}$c7

$1_{8}7_{9}9_{10}$c4: 4b3&6b9

$1_{12}$c2 $5_{13}$c1: 5r8b6

$7_{14}1_{15}$c7: 7r3b1

$3_{17}7_{18}$c2: 17r4b8→3r4b2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | $8_{35}$ | $6_{44}$ | $7_{9}$ | $5_{49}$ | $4_{50}$ | $9_{32}$ | 3* | 2* |
| $9_{33}$ | 5* | $2_{5}$ | 8* | $1_{16}$ | $3_{19}$ | $7_{14}$ | $6_{59}$ | $4_{60}$ |
| $4_{43}$ | $3_{17}$ | $7_{22}$ | $2_{4}$ | $6_{51}$ | $9_{36}$ | $1_{15}$ | $8_{62}$ | $5_{63}$ |
| $3_{39}$ | 6* | $8_{38}$ | 5* | $9_{37}$ | $2_{3}$ | 4* | $7_{24}$ | $1_{25}$ |
| 2* | $1_{12}$ | $5_{55}$ | 3* | $4_{57}$ | $7_{11}$ | $8_{31}$ | $9_{48}$ | $6_{53}$ |
| 7* | $9_{34}$ | $4_{56}$ | $1_{8}$ | $8_{58}$ | $6_{52}$ | $2_{1}$ | $5_{54}$ | $3_{29}$ |
| $5_{13}$ | $7_{18}$ | $9_{41}$ | $6_{45}$ | 2* | 1* | $3_{2}$ | $4_{61}$ | $8_{64}$ |
| $8_{28}$ | 4* | $1_{27}$ | $9_{10}$ | $3_{20}$ | $5_{21}$ | 6* | $2_{7}$ | $7_{23}$ |
| $6_{42}$ | $2_{6}$ | $3_{40}$ | $4_{46}$ | 7* | $8_{30}$ | 5* | $1_{26}$ | $9_{47}$ |

$8_{30}$b6 $8_{31}$c7: 8r1b1 $9_{33}$r2 $9_{34}$c2 $9_{36}$c6 $9_{37}8_{38}$r4 $5_{49}$r1

PRACTICE SET 2

**Puzzle 31**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | 7* | | | | | 4* | | |
| 8* | | | | 1* | | | | |
| | | | 2* | 5* | | | | |
| | | | 4* | | 3* | 7* | | |
| 1* | | | | | | | 5* | |
| | | | | | | | | |
| | 4* | 2* | 7* | | | | | |
| | | | | | | | 6* | 1* |
| | | | | | | | | |

**Puzzle 32**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 5* | 1* | | | |
| | 2* | | | | | 3* | | |
| | | | | | | | | |
| 7* | | | | | | | 1* | 5* |
| | | | 2* | 3* | | | | |
| | | | 4* | | | | | 8* |
| 5* | 7* | 8* | | | | | | |
| | | | | | | 6* | 4* | |
| 1* | | | | | | | | |

**Puzzle 33**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | | 4* | | | | | | |
| | | | | | | 8* | 2* | |
| | | | | | | 6* | | |
| | | | 3* | | 6* | 7* | | |
| | 2* | | | | | | | |
| | | | 7* | | | | | |
| 6* | | | | 5* | | | | 4* |
| | | | 4* | 2* | | | 5* | |
| | 8* | | | | | | | |

**Puzzle 34**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | | 4* | | 2* | | | | |
| 9* | | | 8* | | | 1* | | |
| 2* | | | | | | | | |
| | | | | 9* | | | 2* | |
| | 1* | | | | | 8* | | |
| | 3* | | | | | | | |
| | | | | | | | 4* | 9* |
| | | | 6* | | 1* | | | |
| | | | 3* | | | | | |

**Puzzle 35**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 6* | | | | | 1* |
| | 2* | | | 8* | | | | |
| | | | | | | | | |
| 1* | 9* | 4* | | | | | | |
| | | | | 3* | 2* | 8* | | |
| | | | | | | | | |
| 7* | | | 1* | | | | 5* | |
| 6* | | | 4* | | | | | |
| | | | | | | 2* | 3* | |

**Puzzle 36**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 1* | 5* | | | | | |
| | | | | | 1* | 3* | | |
| | | | | | | 8* | | |
| | | | | | | 6* | 9* | |
| | | | 7* | | | | 2* | |
| 3* | | | | | 8* | | | |
| | | | 2* | 1* | | | | |
| 8* | 4* | | | | | | | |
| | 3* | | | 6* | | | | |

**Puzzle 37**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 1* | 7* | | | | | |
| | | | | | | | 5* | 3* |
| | | | | | | | | |
| 5* | 3* | | | 2* | | | | |
| | | | 1* | | | 7* | | 8* |
| 6* | | | | | | | | |
| 4* | 2* | | | | 5* | | 3* | |
| | | | 6* | | | 1* | | |
| | | | | | | | | |

**Puzzle 31**

$1_{1}$r4 $2_{7}5_{8}$b1 $4_{11}$r2

$4_{12}$r8 $6_{13}$r7

$7_{15}6_{16}$b4 $6_{18}5_{19}$r4

$4_{22}$c1 $5_{23}$c4 $8_{25}$b3

$8_{26}$r3 $8_{28}2_{29}$r4

$3_{33}$r1 $7_{42}$r8

$3_{51}$c7: u38c2b2c7b8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | 7* | $1_{3}$ | $8_{32}$ | $3_{33}$ | $9_{34}$ | 4* | $2_{10}$ | $5_{9}$ |
| 8* | $2_{7}$ | $5_{8}$ | $6_{16}$ | 1* | $4_{11}$ | $3_{51}$ | $7_{55}$ | $9_{58}$ |
| $4_{22}$ | $9_{50}$ | $3_{49}$ | 2* | 5* | $7_{15}$ | $1_{2}$ | $8_{26}$ | $6_{17}$ |
| $5_{19}$ | $6_{18}$ | $9_{30}$ | 4* | $8_{28}$ | 3* | 7* | $1_{1}$ | $2_{29}$ |
| 1* | $3_{59}$ | $7_{47}$ | $9_{36}$ | $2_{39}$ | $6_{62}$ | $8_{63}$ | 5* | $4_{54}$ |
| $2_{20}$ | $8_{60}$ | $4_{48}$ | $1_{6}$ | $7_{40}$ | $5_{61}$ | $6_{64}$ | $9_{57}$ | $3_{58}$ |
| $9_{44}$ | 4* | 2* | 7* | $6_{13}$ | $1_{5}$ | $5_{24}$ | $3_{46}$ | $8_{27}$ |
| $7_{42}$ | $5_{21}$ | $8_{25}$ | $3_{35}$ | $4_{12}$ | $2_{38}$ | $9_{43}$ | 6* | 1* |
| $3_{45}$ | $1_{4}$ | $6_{14}$ | $5_{23}$ | $9_{37}$ | $8_{31}$ | $2_{41}$ | $4_{52}$ | $7_{53}$ |

**Puzzle 32**

$5_{9}8_{10}$c7 $7_{12}$b9

$2_{14}$b9: 2c7b8 $2_{16}$r1

$7_{17}$r8: 7c7b5 $8_{19}$r5

$5_{21}$b1 $5_{25}7_{26}$b3: 69r5b8

$7_{29}$c4 $4_{32}$c7 $4_{38}6_{39}$b3

$6_{45}$c8 $4_{48}$r1 $6_{53}$c4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $4_{48}$ | $6_{44}$ | $7_{24}$ | $3_{49}$ | 5* | 1* | $8_{10}$ | $9_{46}$ | $2_{16}$ |
| $9_{51}$ | 2* | $1_{7}$ | $7_{29}$ | $8_{30}$ | $4_{58}$ | 3* | $5_{23}$ | $7_{57}$ |
| $3_{50}$ | $8_{20}$ | $5_{21}$ | $6_{55}$ | $9_{59}$ | $2_{56}$ | $1_{6}$ | $7_{28}$ | $4_{60}$ |
| 7* | $3_{36}$ | $2_{34}$ | $8_{31}$ | $6_{62}$ | $9_{63}$ | $4_{32}$ | 1* | 5* |
| $8_{19}$ | $1_{8}$ | $4_{37}$ | 2* | 3* | $5_{25}$ | $7_{26}$ | $6_{45}$ | $9_{47}$ |
| $6_{40}$ | $5_{22}$ | $9_{41}$ | 4* | $1_{4}$ | $7_{27}$ | $2_{33}$ | $3_{1}$ | 8* |
| 5* | 7* | 8* | $1_{3}$ | $4_{61}$ | $6_{64}$ | $9_{15}$ | $2_{14}$ | $3_{13}$ |
| $2_{35}$ | $9_{43}$ | $3_{42}$ | $5_{2}$ | $7_{17}$ | $8_{18}$ | 6* | 4* | $1_{5}$ |
| 1* | $4_{38}$ | $6_{39}$ | $9_{54}$ | $2_{53}$ | $3_{52}$ | $5_{9}$ | $8_{11}$ | $7_{12}$ |

**Puzzle 33**

$2_{3}$b3 $2_{5}$b5 $4_{9}$b7

$6_{10}8_{11}$r8 $8_{13}$b1 $8_{14}$c4

$5_{18}$b5 $4_{19}$c6 $5_{20}$b8

$6_{22}5_{23}$r1 $7_{24}$b6: 7r9b9

$7_{25}$r1 $3_{29}$r1:
3r9b6→3c7b9

$9_{30}$c9 $9_{32}6_{33}$b6 $1_{36}$g

$1_{37}$g $9_{38}$g $9_{52}$c1 $7_{55}$c2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | $6_{22}$ | 4* | $2_{7}$ | $8_{17}$ | $3_{29}$ | $9_{28}$ | $7_{25}$ | $5_{23}$ |
| $5_{44}$ | $9_{53}$ | $7_{54}$ | $1_{46}$ | $6_{51}$ | $4_{19}$ | 8* | 2* | $3_{48}$ |
| $2_{4}$ | $3_{49}$ | $8_{13}$ | $5_{45}$ | $7_{50}$ | $9_{35}$ | 6* | $4_{9}$ | $1_{47}$ |
| $8_{16}$ | $5_{41}$ | $9_{38}$ | 3* | $4_{40}$ | 6* | 7* | $1_{37}$ | $2_{6}$ |
| $7_{63}$ | 2* | $6_{58}$ | $8_{14}$ | $1_{36}$ | $5_{18}$ | $4_{21}$ | $3_{64}$ | $9_{30}$ |
| $3_{62}$ | $4_{42}$ | $1_{43}$ | 7* | $9_{39}$ | $2_{5}$ | $5_{20}$ | $6_{61}$ | $8_{15}$ |
| 6* | $1_{56}$ | $2_{3}$ | $9_{32}$ | 5* | $7_{24}$ | $3_{59}$ | $8_{12}$ | 4* |
| $9_{52}$ | $7_{55}$ | $3_{57}$ | 4* | 2* | $8_{11}$ | $1_{60}$ | 5* | $6_{10}$ |
| $4_{1}$ | 8* | $5_{2}$ | $6_{33}$ | $3_{27}$ | $1_{34}$ | $2_{8}$ | $9_{31}$ | $7_{26}$ |

**Puzzle 34**

$1_{2}3_{3}$b1 $3_{11}$r4

$6_{13}$r7: 6c2b1

$9_{14}$b6 $9_{16}4_{17}$r8

$2_{18}4_{19}5_{20}$c2: 2r7b6

$4_{23}$r2 $5_{24}$b6 $5_{25}$r2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | $8_{56}$ | 4* | $1_{4}$ | 2* | $6_{40}$ | $9_{59}$ | $7_{60}$ | $3_{54}$ |
| 9* | $6_{39}$ | $3_{3}$ | 8* | $7_{38}$ | $4_{23}$ | 1* | $5_{25}$ | $2_{1}$ |
| 2* | $7_{57}$ | $1_{2}$ | $9_{15}$ | $3_{12}$ | $5_{26}$ | $4_{58}$ | $6_{47}$ | $8_{55}$ |
| $8_{30}$ | $4_{19}$ | $5_{33}$ | $7_{32}$ | 9* | $3_{11}$ | $6_{31}$ | 2* | $1_{6}$ |
| $7_{37}$ | 1* | $9_{46}$ | $5_{51}$ | $6_{41}$ | $2_{43}$ | 8* | $3_{53}$ | $4_{52}$ |
| $6_{36}$ | 3* | $2_{45}$ | $4_{50}$ | $1_{5}$ | $8_{29}$ | $5_{34}$ | $9_{48}$ | $7_{49}$ |
| $1_{8}$ | $5_{20}$ | $6_{13}$ | $2_{44}$ | $8_{28}$ | $7_{42}$ | $3_{10}$ | 4* | 9* |
| $3_{9}$ | $9_{16}$ | $7_{62}$ | 6* | $4_{17}$ | 1* | $2_{21}$ | $8_{61}$ | $5_{27}$ |
| $4_{22}$ | $2_{18}$ | $8_{63}$ | 3* | $5_{24}$ | $9_{14}$ | $7_{64}$ | $1_{7}$ | $6_{35}$ |

$8_{28}$c5 $8_{30}6_{31}7_{32}$r4: 78c3b3 $6_{36}$r6 $7_{38}$c5

**Puzzle 35**

$1_{1}4_{2}9_{3}$r5 $2_{5}$c1 $2_{6}3_{7}$c4

$3_{9}$r4 $3_{11}$c1 $4_{13}$b9

$5_{14}$g $8_{15}$b7 $8_{17}4_{18}$c1

$8_{27}5_{28}$c4:8r6b2&5r6b8

$7_{30}$r4 $6_{31}$c8

$5_{36}$b3 $9_{42}5_{43}$c9

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $3_{12}$ | $7_{60}$ | $8_{41}$ | 6* | $4_{26}$ | $5_{46}$ | $9_{63}$ | $2_{8}$ | 1* |
| $9_{19}$ | 2* | $6_{59}$ | $3_{7}$ | 8* | $1_{23}$ | $7_{64}$ | $4_{16}$ | $5_{43}$ |
| $4_{18}$ | $1_{22}$ | $5_{45}$ | $2_{6}$ | $9_{56}$ | $7_{55}$ | $6_{58}$ | $8_{15}$ | $3_{11}$ |
| 1* | 9* | 4* | $8_{27}$ | $5_{47}$ | $6_{48}$ | $3_{9}$ | $7_{30}$ | $2_{10}$ |
| $5_{14}$ | $6_{61}$ | $7_{62}$ | $9_{3}$ | 3* | 2* | 8* | $1_{1}$ | $4_{2}$ |
| $2_{5}$ | $8_{39}$ | $3_{38}$ | $7_{29}$ | $1_{24}$ | $4_{25}$ | $5_{44}$ | $6_{31}$ | $9_{42}$ |
| 7* | $3_{37}$ | $9_{33}$ | 1* | $2_{35}$ | $8_{51}$ | $4_{13}$ | 5* | $6_{50}$ |
| 6* | $5_{36}$ | $2_{34}$ | 4* | $7_{57}$ | $3_{40}$ | $1_{4}$ | $9_{32}$ | $8_{52}$ |
| $8_{17}$ | $4_{20}$ | $1_{21}$ | $5_{28}$ | $6_{49}$ | $9_{54}$ | 2* | 3* | $7_{53}$ |

**Puzzle 36**

$1_{2}$b3 $8_{3}$b6 $8_{4}3_{5}$c8

$4_{7}$b6 $3_{8}$c4 $1_{13}$r4

$1_{15}6_{16}2_{17}$r8

$6_{22}2_{23}9_{24}$r6

$7_{26}4_{27}$r4 $8_{32}$r1

$4_{35}$r5 $5_{54}$c8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $7_{57}$ | $8_{32}$ | 1* | 5* | $3_{10}$ | $2_{53}$ | $9_{60}$ | $6_{46}$ | $4_{40}$ |
| $5_{48}$ | $9_{56}$ | $4_{37}$ | $6_{42}$ | $8_{34}$ | 1* | 3* | $7_{55}$ | $2_{50}$ |
| $6_{47}$ | $2_{51}$ | $3_{1}$ | $4_{39}$ | $7_{44}$ | $9_{52}$ | 8* | $1_{18}$ | $5_{49}$ |
| $2_{25}$ | $1_{13}$ | $7_{26}$ | $3_{8}$ | $4_{27}$ | $5_{28}$ | 6* | 9* | $8_{12}$ |
| $4_{35}$ | $5_{36}$ | $8_{33}$ | 7* | $9_{31}$ | $6_{30}$ | $1_{14}$ | 2* | $3_{11}$ |
| 3* | $6_{22}$ | $9_{24}$ | $1_{9}$ | $2_{23}$ | 8* | $5_{61}$ | $4_{41}$ | $7_{62}$ |
| $9_{58}$ | $7_{59}$ | $5_{38}$ | 2* | 1* | $3_{6}$ | $4_{21}$ | $8_{4}$ | $6_{20}$ |
| 8* | 4* | $6_{16}$ | $9_{43}$ | $5_{29}$ | $7_{45}$ | $2_{17}$ | $3_{5}$ | $1_{15}$ |
| $1_{2}$ | 3* | $2_{19}$ | $8_{3}$ | 6* | $4_{7}$ | $7_{63}$ | $5_{54}$ | $9_{64}$ |

**Puzzle 37**

$1_{2}$b2 $5_{5}3_{6}6_{7}$r5

$7_{14}$r7: 7c3b2 $1_{16}$b7

$2_{19}$b8 $7_{20}8_{21}$r4

$8_{23}6_{24}$r7 $2_{28}$b9

$3_{29}2_{30}$c4 $6_{36}2_{37}$r1

$4_{40}$c3 $7_{46}$c1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $2_{37}$ | $5_{9}$ | 1* | 7* | $3_{35}$ | $9_{53}$ | $6_{36}$ | $8_{27}$ | $4_{54}$ |
| $7_{46}$ | $6_{50}$ | $4_{40}$ | $2_{30}$ | $8_{48}$ | $1_{17}$ | $9_{55}$ | 5* | 3* |
| $8_{47}$ | $9_{51}$ | $3_{34}$ | $5_{8}$ | $6_{49}$ | $4_{52}$ | $2_{39}$ | $7_{15}$ | $1_{16}$ |
| 5* | 3* | $7_{20}$ | $8_{21}$ | 2* | $6_{13}$ | $4_{56}$ | $1_{18}$ | $9_{57}$ |
| $9_{43}$ | $4_{42}$ | $2_{38}$ | 1* | $5_{5}$ | $3_{6}$ | 7* | $6_{7}$ | 8* |
| 6* | $1_{2}$ | $8_{22}$ | $4_{31}$ | $9_{59}$ | $7_{60}$ | $3_{1}$ | $2_{19}$ | $5_{10}$ |
| 4* | 2* | $6_{24}$ | $9_{25}$ | $1_{4}$ | 5* | $8_{23}$ | 3* | $7_{14}$ |
| $3_{33}$ | $7_{62}$ | $5_{12}$ | 6* | $4_{58}$ | $8_{61}$ | 1* | $9_{44}$ | $2_{28}$ |
| $1_{3}$ | $8_{63}$ | $9_{41}$ | $3_{29}$ | $7_{64}$ | $2_{32}$ | $5_{11}$ | $4_{45}$ | $6_{26}$ |

PRACTICE SET 3

**Puzzle 38**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 2* | 9* | 6* | | |
| | 1* | 4* | | | | | | |
| | | | | | | | | |
| | | | 7* | | | | | 1* |
| 6* | | | | 8* | | | | |
| | | 2* | | | | 3* | | |
| 2* | | | | | | 5* | 6* | |
| | 7* | | | | 1* | | | |
| | | | 3* | | | | | |

**Puzzle 39**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 8* | | 7* | 2* | |
| 5* | | 1* | | | | | | |
| | | | | | | | | |
| 6* | 2* | | | | | | | 1* |
| | | | 5* | | | | 4* | |
| 3* | | | | 7* | | | | |
| | 7* | | | | | 3* | | |
| | | | 4* | | 6* | | | |
| | | | 1* | | | | | |

**Puzzle 40**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 8* | 3* | | 6* | |
| | 5* | 4* | | | | | | |
| | | | | | | | | |
| | | | | 7* | | 2* | | |
| | 1* | | 4* | | | | | |
| 8* | | | | | | | 3* | |
| | | | 6* | | | 4* | | 1* |
| | | 9* | | | | 5* | | |
| 3* | | | | | | | | |

**Puzzle 41**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 7* | | 4* | | | 2* | | |
| | | 8* | | 5* | | | | |
| | | | | | | | | |
| | | 5* | | 8* | 3* | | | |
| 6* | | | | | | 7* | | |
| | | | | 9* | | | | |
| 7* | 4* | | 6* | | | | | |
| | | | | | | | 8* | 9* |
| | | | 1* | | | | | |

**Puzzle 42**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 3* | | | | | | | 6* |
| 2* | | | | | 4* | | | |
| | | | | | | | | |
| | 7* | 1* | | | | | 3* | |
| | | 6* | | 8* | | 2* | | |
| | | | | | | | | |
| 3* | | | | | | 4* | 7* | |
| 5* | | | 6* | 2* | | | | |
| | | | 1* | | | | | |

**Puzzle 43**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 5* | 2* | 6* | | 7* | | | |
| | | | | 1* | | 4* | | |
| | | | | | | | 9* | |
| | | | 5* | 8* | | | 2* | |
| 1* | | | | | | | | |
| | | | | | | | | |
| | 6* | | 2* | | | | | 3* |
| 4* | | | | | | 1* | | |
| | | | 9* | | | | | |

**Puzzle 44**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | 1* | | | 3* | | | | |
| | | 4* | | | | | 6* | 5* |
| | | | | | | | | |
| | 8* | | 2* | | | 3* | | |
| | | | 6* | | 4* | | | |
| | | | | | 5* | | | |
| | | | | 7* | | 1* | | |
| 6* | | 5* | | | | | | |
| 3* | | | | | | | | |

**Puzzle 38**

$1_{3}$r7 $2_{9}$c4 $6_{10}$b1

$2_{15}3_{16}$b5 $7_{17}$b2 $3_{20}$r7

$7_{21}2_{22}$b9 $7_{25}3_{26}$r1

$5_{28}$r8 $8_{29}$r6 $5_{31}$g

$4_{33}$b $9_{38}$r9 $5_{41}$c4

$4_{48}$c5 $4_{55}$c7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $7_{25}$ | $5_{31}$ | $3_{26}$ | $4_{50}$ | 2* | 9* | 6* | $1_{7}$ | $8_{54}$ |
| $8_{55}$ | 1* | 4* | $6_{14}$ | $7_{49}$ | $3_{46}$ | $9_{57}$ | $5_{44}$ | $2_{24}$ |
| $9_{56}$ | $2_{1}$ | $6_{10}$ | $5_{41}$ | $1_{6}$ | $8_{53}$ | $7_{58}$ | $3_{45}$ | $4_{59}$ |
| $5_{32}$ | $4_{33}$ | $9_{34}$ | 7* | $3_{16}$ | $6_{13}$ | $2_{23}$ | $8_{30}$ | 1* |
| 6* | $3_{18}$ | $7_{17}$ | $1_{5}$ | 8* | $2_{15}$ | $4_{62}$ | $9_{61}$ | $5_{47}$ |
| $1_{4}$ | $8_{29}$ | 2* | $9_{40}$ | $5_{42}$ | $4_{43}$ | 3* | $7_{19}$ | $6_{2}$ |
| 2* | $9_{35}$ | $1_{3}$ | $8_{51}$ | $4_{48}$ | $7_{52}$ | 5* | 6* | $3_{20}$ |
| $3_{27}$ | 7* | $5_{28}$ | $2_{9}$ | $6_{12}$ | 1* | $8_{63}$ | $4_{64}$ | $9_{60}$ |
| $4_{37}$ | $6_{11}$ | $8_{36}$ | 3* | $9_{38}$ | $5_{39}$ | $1_{8}$ | $2_{22}$ | $7_{21}$ |

**Puzzle 39**

$5_4$r1 $6_5$b5 $7_6$b6

$7_7 2_8 8_9$c4&$2_{10}$c5: 27r3b1

$4_{14}$c5 $5_{17}$b2 $7_{18} 3_{19}$r4

$9_{24}$r6 $8_{26}$g

$8_{27} 3_{28}$b1: 27(31)(33)

$4_{33}$b7 $6_{39} 5_{40}$c2 $5_{47}$r7 $5_{49}$c7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $9_{43}$ | $4_{38}$ | $6_{42}$ | $3_{31}$ | 8* | $1_1$ | 7* | 2* | $5_4$ |
| 5* | $3_{28}$ | 1* | $7_7$ | $2_{10}$ | $4_{16}$ | $8_{26}$ | $6_{45}$ | $9_{46}$ |
| $7_{57}$ | $8_{27}$ | $2_{58}$ | $9_{32}$ | $6_{11}$ | $5_{12}$ | $1_{30}$ | $3_{29}$ | $4_{33}$ |
| 6* | 2* | $5_{17}$ | $8_9$ | $4_{14}$ | $3_{19}$ | $9_{20}$ | $7_{18}$ | 1* |
| $8_{55}$ | $9_{41}$ | $7_{56}$ | 5* | $1_2$ | $2_{25}$ | $6_{22}$ | 4* | $3_{21}$ |
| 3* | $1_3$ | $4_{15}$ | $6_5$ | 7* | $9_{24}$ | $5_{49}$ | $8_{52}$ | $2_{51}$ |
| $4_{35}$ | 7* | $9_{48}$ | $2_8$ | $5_{47}$ | $8_{13}$ | 3* | $1_{37}$ | $6_{44}$ |
| $1_{36}$ | $5_{40}$ | $8_{54}$ | 4* | $3_{61}$ | 6* | $2_{50}$ | $9_{63}$ | $7_{23}$ |
| $2_{59}$ | $6_{39}$ | $3_{60}$ | 1* | $9_{62}$ | $7_6$ | $4_{34}$ | $5_{64}$ | $8_{53}$ |

**Puzzle 40**

$4_1 5_2$r1 $1_6$b3 $1_7$b8

$3_8$b9 $8_{15}$b1 $1_{18}$r1

$6_{19}$r8 $5_{17}$c1 $3_{18}$r7

$6_{22} 5_{23}$r4: 6c1b1

$9_{25}$r1: 9c1b1 $8_{27} 7_{28}$c7

$7_{38}$c4 $7_{40}$b3 $9_{51}$r2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $1_{18}$ | $7_{42}$ | $2_{43}$ | $5_2$ | 8* | 3* | $9_{25}$ | 6* | $4_1$ |
| $9_{51}$ | 5* | 4* | $1_{34}$ | $6_{53}$ | $7_{39}$ | $3_{14}$ | $8_{32}$ | $2_{36}$ |
| $6_{52}$ | $3_{13}$ | $8_{15}$ | $9_{60}$ | $2_{57}$ | $4_{59}$ | $7_{28}$ | $1_{33}$ | $5_{35}$ |
| $5_{23}$ | $9_{24}$ | $6_{22}$ | $3_{10}$ | 7* | $1_{11}$ | 2* | $4_3$ | $8_{17}$ |
| $2_{45}$ | 1* | $3_{12}$ | 4* | $9_{47}$ | $8_{16}$ | $6_{21}$ | $5_{37}$ | $7_{46}$ |
| 8* | $4_4$ | $7_{44}$ | $2_{56}$ | $5_{55}$ | $6_{54}$ | $1_7$ | 3* | $9_{48}$ |
| $7_{40}$ | $8_{29}$ | $5_{26}$ | 6* | $3_9$ | $9_{62}$ | 4* | $2_{63}$ | 1* |
| $4_5$ | $6_{19}$ | 9* | $8_{30}$ | $9_{31}$ | $2_{50}$ | 5* | $7_{49}$ | $3_8$ |
| 3* | $2_{41}$ | $1_6$ | $7_{38}$ | $4_{58}$ | $5_{61}$ | $8_{27}$ | $9_{64}$ | $6_{31}$ |

**Puzzle 41**

$8_3 9_4$r7 $3_{12}$c4 $5_{13}$b6

$5_{14} 3_{15}$b3 $7_{16} 4_{17}$r8

$1_{20}$g $6_{21}$b5 $1_{23}$g

$6_{24}$g $9_{25} 3_{26} 5_{27}$r5

$5_{33} 3_{34}$r1 $2_{39}$r8

$5_{43} 3_{44}$c7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $9_{31}$ | 7* | $6_{35}$ | 4* | $1_{34}$ | $1_{23}$ | 2* | $5_{33}$ | $8_6$ |
| $1_{48}$ | $2_{49}$ | 8* | $9_{11}$ | 5* | $7_{52}$ | $3_{44}$ | $6_{59}$ | $4_{58}$ |
| $5_{37}$ | $3_{46}$ | $4_{41}$ | $8_5$ | $6_{36}$ | $2_{51}$ | $9_{30}$ | $1_{55}$ | $7_{60}$ |
| $4_{42}$ | $9_{32}$ | 5* | $7_2$ | 8* | 3* | $1_{45}$ | $2_{62}$ | $6_{63}$ |
| 6* | $8_8$ | $2_{28}$ | $5_{27}$ | $1_{20}$ | $4_{22}$ | 7* | $9_{25}$ | $3_{26}$ |
| $3_{47}$ | $1_{50}$ | $7_1$ | $2_{29}$ | 9* | $6_{21}$ | $8_7$ | $4_{57}$ | $5_{56}$ |
| 7* | 4* | $9_4$ | 6* | $2_{19}$ | $8_3$ | $5_{43}$ | $3_{53}$ | $1_{54}$ |
| $2_{39}$ | $6_{38}$ | $1_{40}$ | $3_{12}$ | $7_{16}$ | $5_{13}$ | $4_{17}$ | 8* | 9* |
| $8_9$ | $5_{14}$ | $3_{15}$ | 1* | $4_{18}$ | $9_{10}$ | $6_{24}$ | $7_{61}$ | $2_{64}$ |

**Puzzle 42**

$2_{2}5_{3}$b2 $4_{4}$b6 $6_{5}$r7

$1_{8}$b3 $2_{12}$r7 $4_{13}$c2

$7_{15}$c1 $3_{17}$b6 $1_{20}$b4

$7_{26}4_{27}$r5 $2_{36}$r1 $6_{38}$r4

$5_{42}$r9 $7_{44}5_{45}$r2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $1_{9}$ | 3* | $5_{48}$ | $2_{36}$ | $7_{35}$ | $8_{47}$ | $9_{51}$ | $4_{14}$ | 6* |
| 2* | $9_{57}$ | $7_{44}$ | $5_{45}$ | $6_{7}$ | 4* | $1_{23}$ | $8_{61}$ | $3_{24}$ |
| $6_{6}$ | $4_{13}$ | $8_{58}$ | $9_{46}$ | $3_{19}$ | $1_{20}$ | $7_{52}$ | $2_{41}$ | $5_{62}$ |
| $8_{55}$ | 7* | 1* | $4_{30}$ | $5_{34}$ | $2_{31}$ | $6_{38}$ | 3* | $9_{56}$ |
| $4_{27}$ | $5_{3}$ | 6* | $3_{18}$ | 8* | $9_{28}$ | 2* | $1_{22}$ | $7_{26}$ |
| $9_{54}$ | $2_{2}$ | $3_{1}$ | $7_{29}$ | $1_{21}$ | $6_{33}$ | $8_{53}$ | $5_{43}$ | $4_{32}$ |
| 3* | $6_{5}$ | $2_{12}$ | $8_{49}$ | $9_{37}$ | $5_{50}$ | 4* | 7* | $1_{10}$ |
| 5* | $1_{8}$ | $4_{11}$ | 6* | 2* | $7_{16}$ | $3_{25}$ | $9_{63}$ | $8_{64}$ |
| $7_{15}$ | $8_{59}$ | $9_{60}$ | 1* | $4_{4}$ | $3_{17}$ | $5_{42}$ | $6_{39}$ | $2_{40}$ |

**Puzzle 43**

$1_{1}$c4 $4_{4}9_{5}3_{6}$r1

$2_{9}5_{10}$r2 $4_{14}7_{15}$c4

$2_{16}$c5: 79(55)(65)

$5_{19}$g $5_{20}$c1 $9_{21}$r8

$9_{22}3_{23}6_{24}$r8: 79(55)(65)
→36c6b5

$9_{28}4_{29}$r4 $8_{41}$b9 $3_{51}$c2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $9_{5}$ | 5* | 2* | 6* | $4_{4}$ | 7* | $3_{6}$ | $1_{3}$ | $8_{7}$ |
| $6_{59}$ | $7_{50}$ | $8_{57}$ | $3_{61}$ | 1* | $9_{8}$ | 4* | $5_{10}$ | $2_{9}$ |
| $3_{60}$ | $4_{30}$ | $1_{31}$ | $8_{62}$ | $2_{16}$ | $5_{18}$ | $7_{44}$ | 9* | $6_{43}$ |
| $7_{47}$ | $9_{28}$ | $4_{29}$ | 5* | 8* | $3_{53}$ | $6_{58}$ | 2* | $1_{2}$ |
| 1* | $8_{52}$ | $6_{55}$ | $4_{14}$ | $7_{40}$ | $2_{17}$ | $5_{63}$ | $3_{46}$ | $9_{38}$ |
| $2_{12}$ | $3_{51}$ | $5_{56}$ | $1_{1}$ | $9_{39}$ | $6_{54}$ | $8_{64}$ | $7_{45}$ | $4_{37}$ |
| $8_{49}$ | 6* | $7_{48}$ | 2* | $5_{19}$ | $1_{33}$ | $9_{27}$ | $4_{36}$ | 3* |
| 4* | $2_{12}$ | $9_{22}$ | $7_{15}$ | $3_{23}$ | $8_{25}$ | 1* | $6_{24}$ | $5_{21}$ |
| $5_{20}$ | $1_{32}$ | $3_{35}$ | 9* | $6_{26}$ | $4_{34}$ | $2_{11}$ | $8_{41}$ | $7_{42}$ |

**Puzzle 44**

$1_{1}$b3 $3_{2}$r2

$3_{3}$r5 $5_{5}$r1

$6_{11}4_{12}2_{13}$c5 $4_{20}$r4

$7_{23}$b5 $1_{25}$r4

$1_{27}$c4 $4_{29}$c7

$7_{32}9_{33}$c1 $7_{39}$c3

$3_{42}$r7 $7_{45}$c7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | 1* | $6_{15}$ | $5_{5}$ | 3* | $8_{37}$ | $4_{29}$ | $7_{53}$ | $9_{52}$ |
| $9_{33}$ | $3_{2}$ | 4* | $7_{24}$ | $2_{13}$ | $1_{28}$ | $8_{36}$ | 6* | 5* |
| $7_{32}$ | $5_{10}$ | $8_{34}$ | $4_{14}$ | $6_{11}$ | $9_{38}$ | $2_{55}$ | $3_{51}$ | $1_{54}$ |
| $5_{9}$ | 8* | $9_{26}$ | 2* | $1_{25}$ | $7_{23}$ | 3* | $4_{20}$ | $6_{17}$ |
| $1_{22}$ | $2_{40}$ | $3_{3}$ | 6* | $9_{58}$ | 4* | $5_{8}$ | $8_{62}$ | $7_{63}$ |
| $4_{21}$ | $6_{16}$ | $7_{39}$ | $3_{4}$ | $8_{59}$ | 5* | $4_{56}$ | $1_{57}$ | $2_{60}$ |
| $8_{35}$ | $4_{31}$ | $2_{41}$ | $9_{43}$ | 7* | $6_{19}$ | 1* | $5_{7}$ | $3_{42}$ |
| 6* | $9_{47}$ | 5* | $1_{27}$ | $4_{12}$ | $3_{48}$ | $7_{45}$ | $2_{61}$ | $8_{64}$ |
| 3* | $1_{46}$ | $1_{1}$ | $8_{44}$ | $5_{6}$ | $2_{49}$ | $6_{18}$ | $9_{50}$ | $4_{28}$ |

PRACTICE SET 4

**Puzzle 45**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 4* | | | 7* | |
| 6* | 5* | | | | | | | |
| | 2* | | | | | | | |
| | | | 3* | | | 6* | | 8* |
| 1* | | | | 7* | | | | |
| | | | | | | 2* | | |
| | | | | | 2* | 5* | 3* | |
| 7* | | | 8* | | | | | |
| 4* | | | | | | | | |

**Puzzle 46**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 3* | | | 7* | 9* | | | |
| | | | | | | | 1* | |
| | | | | | | | | |
| | 9* | | | | | 2* | | |
| 1* | | | 4* | | | | | |
| | | | 1* | 8* | | | | |
| 6* | | | 5* | | 3* | | | |
| | | | | 4* | | 9* | | 7* |
| | | | | | | | | |

**Puzzle 47**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 8* | 5* | | 4* | | | |
| | 3* | | | | | | | 1* |
| | | | 7* | | | | | |
| | | | 6* | | | | 5* | |
| 1* | 2* | | | | | | | |
| | | | | | | | | |
| 4* | 1* | | | 2* | | | | |
| 2* | | | | | | 3* | | |
| | | | | | | 8* | 6* | |

**Puzzle 48**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 7* | 4* | | | 3* | | | |
| | | | 6* | | | 1* | | 2* |
| | | | | | | | | |
| 5* | | | 1* | 2* | | | | |
| | 3* | | | | | | 4* | |
| | | | | 6* | | | | |
| 2* | | | 5* | | | | | |
| | | | | | | 8* | 7* | |
| 1* | | | | | | | | |

**Puzzle 49**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8* | | 9* | 6* | | | 4* | | |
| | | | 3* | | 5* | | | 7* |
| | | | | | | | | |
| | | | 7* | | 2* | | 5* | |
| | | 4* | | 9* | | | | |
| | | | | | | | | |
| 1* | 3* | | | | | | | |
| | | | | 2* | | 9* | | |
| | 5* | | | | | | | |

**Puzzle 50**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 8* | 6* | | 4* | | |
| | 2* | 3* | | | | | | |
| | 5* | | | | | 7* | | |
| | | | | 4* | 2* | | | |
| 1* | | | | | | 6* | | |
| | | | | | | | | |
| | | | | | 5* | | 3* | 2* |
| 6* | | | 1* | | | | | |
| | | | | | | | 8* | |

**Puzzle 51**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 1* | | 4* | | 5* |
| 6* | | | 9* | | | | | 8* |
| 3* | | | | | | 2* | | |
| 9* | | | | | | | 1* | |
| | | | | 2* | | | | |
| | | | | | 5* | | | |
| | | | 6* | | | | 3* | |
| | 2* | 8* | | | | | | |
| | | | 1* | | | | | |

**Puzzle 45**

$2_{1}5_{2}$c1 $7_{6}$r4:7c3b1

$4_{10}$c4 $4_{13}$c2

$1_{15}$b8 $3_{16}$r6:3c2b3

$8_{19}$b9 $6_{21}5_{22}$c8

$1_{25}$g $3_{26}$g $9_{31}$g

$9_{32}$g $9_{33}$c1

$1_{35}$r3: 47(23)(33) $3_{26}$g

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $3_{30}$ | $9_{56}$ | $1_{57}$ | $6_{50}$ | 4* | $5_{51}$ | $8_{37}$ | 7* | $2_{5}$ |
| 6* | 5* | $7_{39}$ | $1_{25}$ | $2_{4}$ | $8_{40}$ | $4_{36}$ | $9_{23}$ | $3_{26}$ |
| $8_{34}$ | 2* | $4_{38}$ | $9_{32}$ | $3_{42}$ | $7_{41}$ | $1_{35}$ | $6_{21}$ | $5_{24}$ |
| $2_{1}$ | $7_{6}$ | $9_{31}$ | 3* | $5_{54}$ | $1_{53}$ | 6* | $4_{14}$ | 8* |
| 1* | $2_{13}$ | $8_{46}$ | $2_{3}$ | 7* | $6_{48}$ | $3_{27}$ | $5_{22}$ | $9_{28}$ |
| $5_{2}$ | $6_{47}$ | $3_{16}$ | $4_{10}$ | $8_{45}$ | $9_{49}$ | 2* | $1_{15}$ | $7_{7}$ |
| $9_{33}$ | $8_{55}$ | $6_{58}$ | $7_{19}$ | $1_{60}$ | 2* | 5* | 3* | $4_{12}$ |
| 7* | $3_{44}$ | $5_{20}$ | 8* | $6_{62}$ | $4_{11}$ | $9_{29}$ | $2_{18}$ | $1_{61}$ |
| 4* | $1_{59}$ | $2_{19}$ | $5_{52}$ | $9_{63}$ | $3_{43}$ | $7_{8}$ | $8_{17}$ | $6_{64}$ |

**Puzzle 46**

$1_{3}$b4 $9_{5}$b5 $2_{10}$g

$7_{11}1_{12}$r7 $7_{13}$c4

$3_{17}$b5 $4_{18}$c2 $5_{24}$c5

$7_{30}3_{31}2_{32}$c1 $2_{34}$r2

$3_{35}$c4 $8_{38}$r1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $2_{32}$ | 3* | $1_{4}$ | $6_{36}$ | 7* | 9* | $4_{23}$ | $5_{39}$ | $8_{38}$ |
| 4* | $6_{48}$ | $7_{49}$ | $2_{34}$ | $5_{24}$ | $8_{29}$ | $3_{37}$ | 1* | $9_{9}$ |
| $9_{8}$ | $8_{43}$ | $5_{44}$ | $3_{35}$ | $1_{3}$ | $4_{2}$ | $7_{53}$ | $2_{55}$ | $6_{54}$ |
| $5_{33}$ | 9* | $8_{41}$ | $7_{13}$ | $3_{17}$ | $6_{42}$ | 2* | $4_{20}$ | $1_{1}$ |
| 1* | $2_{47}$ | $6_{61}$ | 4* | $9_{5}$ | $5_{51}$ | $8_{40}$ | $7_{52}$ | $3_{63}$ |
| $7_{30}$ | $4_{18}$ | $3_{62}$ | 1* | 8* | $2_{50}$ | $6_{60}$ | $9_{58}$ | $5_{64}$ |
| 6* | $7_{11}$ | $9_{7}$ | 5* | $2_{10}$ | 3* | $1_{12}$ | $8_{22}$ | $4_{21}$ |
| $3_{31}$ | $5_{45}$ | $2_{46}$ | $8_{26}$ | 4* | $1_{15}$ | 9* | $6_{27}$ | 7* |
| $8_{28}$ | $1_{16}$ | $4_{19}$ | $9_{6}$ | $6_{25}$ | $7_{14}$ | $5_{59}$ | $3_{57}$ | $2_{56}$ |

**Puzzle 47**

$1_{5}4_{6}$b9 $2_{7}1_{8}4_{9}3_{10}$c4

$3_{17}$r1 $4_{19}$b8 $5_{24}$c3

$5_{27}7_{28}$r2 $8_{29}$b3 $6_{32}$r7

$6_{33}$b2 $5_{34}$c2 $8_{35}$c4

$9_{37}$b3 $9_{40}7_{41}$b1

$8_{46}7_{47}$r5 $5_{50}$r6

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $7_{41}$ | $9_{40}$ | 8* | 5* | $1_{2}$ | 4* | $6_{44}$ | $2_{15}$ | $3_{17}$ |
| $5_{27}$ | 3* | $2_{3}$ | $8_{35}$ | $6_{58}$ | $9_{59}$ | $4_{26}$ | $7_{28}$ | 1* |
| $6_{43}$ | $4_{25}$ | $1_{1}$ | 7* | $3_{18}$ | $2_{11}$ | $9_{62}$ | $8_{63}$ | $5_{56}$ |
| $3_{22}$ | $7_{42}$ | $4_{24}$ | 6* | $9_{53}$ | $1_{14}$ | $2_{12}$ | 5* | $8_{54}$ |
| 1* | 2* | $9_{51}$ | $4_{9}$ | $8_{46}$ | $5_{52}$ | $7_{47}$ | $3_{20}$ | $6_{45}$ |
| $8_{30}$ | $6_{33}$ | $5_{50}$ | $2_{7}$ | $7_{48}$ | $3_{21}$ | $1_{13}$ | $4_{19}$ | $9_{55}$ |
| 4* | 1* | $6_{32}$ | $3_{10}$ | 2* | $1_{31}$ | $5_{61}$ | $9_{64}$ | $7_{49}$ |
| 2* | $8_{29}$ | $7_{38}$ | $9_{36}$ | $5_{57}$ | $6_{60}$ | 3* | $1_{5}$ | $4_{6}$ |
| $9_{37}$ | $5_{34}$ | $3_{23}$ | $1_{8}$ | $4_{16}$ | $7_{39}$ | 8* | 6* | $2_{4}$ |

**Puzzle 48**

$2_{1}1_{2}$r1 $2_{3}1_{4}$r8 $3_{14}$b5

$4_{15}7_{16}$r2: 4c6b5

$6_{17}$r5: 6c1b1 $7_{18}$c1: 7r5b5

$4_{19}$b2 $6_{24}$c2: 6r7b9

$5_{25}3_{26}$r8 $7_{29}$c4

$9_{31}$r5: u59r3b4r5b5

$9_{33}$b1 $9_{34}$c5 $5_{39}8_{40}9_{41}6_{42}$c8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $6_{45}$ | 7* | 4* | $2_{1}$ | $1_{2}$ | 3* | $9_{48}$ | $5_{39}$ | $8_{47}$ |
| $8_{44}$ | $5_{28}$ | $9_{33}$ | 6* | $4_{15}$ | $7_{16}$ | 1* | $3_{43}$ | 2* |
| $3_{46}$ | $1_{7}$ | $2_{9}$ | $8_{30}$ | $9_{34}$ | $5_{35}$ | $4_{62}$ | $6_{42}$ | $7_{61}$ |
| 5* | $4_{19}$ | $6_{23}$ | 1* | 2* | $9_{37}$ | $7_{60}$ | $8_{40}$ | $3_{59}$ |
| $9_{31}$ | 3* | $1_{6}$ | $1_{29}$ | $5_{36}$ | $8_{28}$ | $2_{11}$ | 4* | $6_{17}$ |
| $7_{18}$ | $2_{10}$ | $8_{32}$ | $3_{14}$ | 6* | $4_{21}$ | $5_{50}$ | $1_{5}$ | $9_{49}$ |
| 2* | $8_{53}$ | $3_{57}$ | 5* | $7_{55}$ | $1_{8}$ | $6_{63}$ | $9_{41}$ | $4_{64}$ |
| $4_{20}$ | $6_{24}$ | $5_{25}$ | $9_{27}$ | $3_{26}$ | $2_{3}$ | 8* | 7* | $1_{4}$ |
| 1* | $9_{52}$ | $7_{56}$ | $4_{22}$ | $8_{54}$ | $6_{15}$ | $3_{58}$ | $2_{12}$ | $5_{51}$ |

**Puzzle 49**

$5_{2}3_{3}2_{4}$r1 $7_{11}$b1

$9_{12}$b4 $8_{15}$r2

$4_{16}$b4 $4_{17}$r4

$9_{19}$b3 $2_{22}$c1

$6_{23}$g $4_{24}$g

$1_{29}$c3 $8_{31}$r4

$3_{33}$c5 $4_{40}$c4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8* | $2_{4}$ | 9* | 6* | $7_{34}$ | $1_{33}$ | 4* | $3_{3}$ | $5_{2}$ |
| $4_{24}$ | $6_{30}$ | $1_{29}$ | 3* | $8_{15}$ | 5* | $2_{14}$ | $9_{13}$ | 7* |
| $3_{10}$ | $7_{11}$ | $5_{9}$ | $2_{1}$ | $4_{16}$ | $9_{12}$ | $6_{50}$ | $8_{51}$ | $1_{45}$ |
| $6_{23}$ | $9_{20}$ | $3_{28}$ | 7* | $1_{32}$ | 2* | $8_{31}$ | 5* | $4_{17}$ |
| $5_{8}$ | $1_{43}$ | 4* | $8_{41}$ | 9* | $6_{48}$ | $3_{46}$ | $7_{58}$ | $2_{59}$ |
| $2_{22}$ | $8_{42}$ | $7_{27}$ | $4_{40}$ | $5_{7}$ | $3_{47}$ | $1_{44}$ | $6_{49}$ | $9_{18}$ |
| 1* | 3* | $2_{62}$ | $9_{21}$ | $6_{37}$ | $7_{56}$ | $5_{5}$ | $4_{63}$ | $8_{60}$ |
| $7_{25}$ | $4_{26}$ | $6_{54}$ | $5_{6}$ | 2* | $8_{55}$ | 9* | $1_{39}$ | $3_{38}$ |
| $9_{19}$ | 5* | $8_{61}$ | $1_{35}$ | $3_{36}$ | $4_{57}$ | $7_{52}$ | $2_{64}$ | $6_{53}$ |

**Puzzle 50**

$6_{1}$b1 $6_{2}$b9 $1_{7}$b5

$2_{8}3_{9}8_{10}$c7 $4_{17}$b1

$1_{20}5_{21}3_{22}$b4: 1r1b1

$3_{28}8_{29}$r8 $5_{37}$c1

$5_{38}$c5 $5_{43}$r5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $7_{61}$ | $1_{56}$ | $9_{62}$ | 8* | 6* | $3_{22}$ | 4* | $2_{15}$ | $5_{21}$ |
| $4_{17}$ | 2* | 3* | $7_{39}$ | $5_{38}$ | $1_{20}$ | $8_{10}$ | $6_{6}$ | $9_{26}$ |
| $8_{16}$ | 5* | $6_{1}$ | $2_{14}$ | $9_{25}$ | $4_{18}$ | 7* | $1_{24}$ | $3_{23}$ |
| $5_{37}$ | $6_{5}$ | $8_{35}$ | $9_{45}$ | 4* | 2* | $3_{9}$ | $7_{46}$ | $1_{27}$ |
| 1* | $9_{49}$ | $2_{11}$ | $3_{32}$ | $8_{33}$ | $7_{42}$ | 6* | $5_{43}$ | $4_{48}$ |
| $3_{30}$ | $4_{53}$ | $7_{54}$ | $5_{44}$ | $1_{7}$ | $6_{4}$ | $2_{8}$ | $9_{50}$ | $8_{34}$ |
| $9_{60}$ | $8_{36}$ | $4_{52}$ | $6_{3}$ | $7_{40}$ | 5* | $1_{59}$ | 3* | 2* |
| 6* | $3_{28}$ | $5_{63}$ | 1* | $2_{13}$ | $8_{29}$ | $9_{64}$ | $4_{51}$ | $7_{47}$ |
| $2_{12}$ | $7_{55}$ | $1_{57}$ | $5_{19}$ | $3_{31}$ | $9_{41}$ | $5_{58}$ | 8* | $6_{2}$ |

**Puzzle 51**

$2_{2}$c4 $3_{8}6_{9}$r1

$1_{11}$b7 $5_{14}4_{15}$r2

$1_{19}$b9: 58(77)(79)

$3_{21}6_{22}9_{23}7_{24}$r8

$5_{26}$c8 $5_{28}4_{29}$c1

$6_{44}$c7: 6r4b5

$7_{47}$c5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $8_{30}$ | $9_{54}$ | $7_{53}$ | $2_{2}$ | 1* | $3_{8}$ | 4* | $6_{9}$ | 5* |
| 6* | $1_{12}$ | $2_{3}$ | 9* | $5_{14}$ | $4_{15}$ | $3_{10}$ | $7_{16}$ | 8* |
| 3* | $5_{61}$ | $4_{62}$ | $8_{50}$ | $7_{47}$ | $6_{48}$ | 2* | $9_{17}$ | $1_{11}$ |
| 9* | $3_{59}$ | $5_{60}$ | $4_{32}$ | $6_{49}$ | $8_{51}$ | $7_{46}$ | 1* | $2_{5}$ |
| $4_{29}$ | $8_{55}$ | $6_{56}$ | $7_{52}$ | 2* | $1_{1}$ | $9_{42}$ | $5_{26}$ | $3_{40}$ |
| $2_{4}$ | $7_{45}$ | $1_{13}$ | $3_{41}$ | $9_{43}$ | 5* | $6_{44}$ | $8_{27}$ | $4_{31}$ |
| $1_{20}$ | $4_{63}$ | $9_{64}$ | 6* | $8_{35}$ | $2_{7}$ | $5_{33}$ | 3* | $7_{38}$ |
| $7_{24}$ | 2* | 8* | $5_{18}$ | $3_{21}$ | $9_{23}$ | $1_{19}$ | $4_{25}$ | $6_{22}$ |
| $5_{28}$ | $6_{59}$ | $3_{58}$ | 1* | $4_{36}$ | $7_{37}$ | $8_{34}$ | $2_{6}$ | $9_{39}$ |

PRACTICE SET 5

**Puzzle 52**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | | | | | | 3* | 4* | |
| 4* | | | | | 1* | | | |
| | | | 7* | | | 8* | | |
| | | 2* | | | | | | 1* |
| 6* | | | 3* | | | | | |
| | | | | | | | | |
| | | | | 3* | 2* | | | |
| | 7* | | | | | | 9* | |
| | 1* | | | 8* | | | | |

**Puzzle 53**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 1* | | 3* | | | 4* |
| 7* | | 6* | | | | | | |
| 8* | | | | | | | | |
| | 8* | | | | | | | 5* |
| | | | | 6* | | | 3* | |
| | | | 4* | | | | | |
| | | | | | | 8* | 1* | |
| | 1* | | 5* | | | | | |
| | | | 2* | | | 6* | | |

**Puzzle 54**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 4* | | 2* | | 5* | |
| 8* | | | | | | 7* | | |
| | | | | 6* | | | | |
| | 2* | | 7* | | | 4* | | |
| | 6* | | 3* | | | | | |
| | | | | | | 1* | | |
| | | | | | | | 3* | 8* |
| | | | | 2* | | | 6* | |
| 1* | | | | | | | | |

**Puzzle 55**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 5* | | | | | | 9* | |
| | | | 1* | 3* | | | | |
| | | | | | | | | |
| | | | | 5* | 8* | | 4* | |
| 1* | | 2* | | | | | | |
| | | | | | 9* | | | |
| 3* | | | 6* | | | 2* | | 1* |
| | | | | | | 3* | | |
| | 9* | | | | | | | |

**Puzzle 56**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 6* | 2* | | | | |
| | | 3* | | | | | | |
| 1* | | | | | | | | |
| 6* | 2* | | 3* | | | | | |
| | | | | | | | 5* | 1* |
| | | | | | | 8* | | |
| | | 4* | | 5* | 1* | | | |
| | 5* | | | | | 6* | 3* | |
| | | | | | | | | 7* |

**Puzzle 57**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | | | | | 1* | | | |
| | | | | | | 2* | 7* | |
| | | | | | | 3* | | |
| | | | 2* | 5* | | | | 8* |
| 6* | 3* | | | | | | | |
| | | 7* | | | | | | |
| 4* | | | | | | | 1* | 5* |
| | | | 7* | 6* | | | | |
| | | | 3* | | | | | |

**Puzzle 58**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 7* | | 5* | | 6* | |
| | 9* | 1* | | | | 4* | | |
| | | | | | | | | |
| 6* | | | | | | | 7* | 8* |
| 3* | | | 2* | | | | | |
| | | | | 9* | | | | |
| | | | | 1* | | 9* | | |
| 7* | | | 8* | | | | | |
| | 2* | | | | | | | |

**Puzzle 52**

$1_{3}$b7    $5_{7}$r3

$7_{8}$b71    $8_{12}5_{13}$r2

$7_{15}2_{16}9_{17}$c5

$2_{23}6_{24}$c2    $3_{27}8_{28}$b2

$8_{30}$g    $3_{35}2_{36}1_{37}$r8

$9_{40}$c6

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | $8_{14}$ | $1_{4}$ | $6_{41}$ | $2_{16}$ | $9_{40}$ | 3* | 4* | $7_{11}$ |
| 4* | $3_{9}$ | $7_{8}$ | $8_{12}$ | $5_{13}$ | 1* | $9_{46}$ | $2_{47}$ | $6_{45}$ |
| $9_{26}$ | $2_{23}$ | $6_{25}$ | 7* | $4_{2}$ | $3_{1}$ | 8* | $1_{3}$ | $5_{7}$ |
| $7_{10}$ | $4_{56}$ | 2* | $5_{55}$ | $9_{17}$ | $8_{34}$ | $6_{43}$ | $3_{29}$ | 1* |
| 6* | $9_{57}$ | $8_{28}$ | 3* | $1_{6}$ | $4_{58}$ | $7_{22}$ | $5_{48}$ | $2_{49}$ |
| $1_{5}$ | $5_{53}$ | $3_{27}$ | $2_{19}$ | $7_{15}$ | $6_{42}$ | $4_{51}$ | $8_{33}$ | $9_{52}$ |
| $8_{30}$ | $6_{24}$ | $5_{63}$ | $9_{64}$ | 3* | 2* | $1_{39}$ | $7_{21}$ | $4_{54}$ |
| $3_{35}$ | 7* | $4_{60}$ | $1_{37}$ | $6_{18}$ | $5_{59}$ | $2_{36}$ | 9* | $8_{31}$ |
| $1_{38}$ | 1* | $9_{62}$ | $4_{61}$ | 8* | $7_{20}$ | $5_{50}$ | $6_{44}$ | $3_{32}$ |

**Puzzle 53**

$1_1$b1 $6_2 8_3 7_4$r1

$5_{10}$b9 $8_{11}$c3

$1_{17}$c9 $7_{20}$b8

$7_{22} 3_{23}$c4 $3_{25}$r6:

3c2b1 $5_{26} 3_{37}$c7

$4_{32}$b1 $4_{33} 9_{34}$c8

$4_{36}$b6 $7_{42}$c5 $7_{48}$r8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_{62}$ | $9_{56}$ | $2_{63}$ | 1* | $8_3$ | 3* | $7_4$ | $6_2$ | 4* |
| 7* | $3_{31}$ | 6* | $9_{24}$ | $4_{41}$ | $5_{43}$ | $1_{20}$ | $2_{35}$ | $8_{16}$ |
| 8* | $4_{32}$ | $1_1$ | $6_9$ | $7_{42}$ | $2_{44}$ | $5_{26}$ | $9_{34}$ | $3_{29}$ |
| $6_6$ | 8* | $9_{46}$ | $3_{23}$ | $2_{45}$ | $1_{19}$ | $4_{40}$ | $7_{21}$ | 5* |
| $4_{39}$ | $5_{54}$ | $7_{53}$ | $8_{11}$ | 6* | $9_{57}$ | $2_{59}$ | 3* | $1_{17}$ |
| $1_{18}$ | $2_{55}$ | $3_{25}$ | 4* | $5_{47}$ | $7_{58}$ | $9_{60}$ | $8_{15}$ | $6_5$ |
| $9_{61}$ | $6_7$ | $5_{64}$ | $7_{22}$ | $3_{28}$ | $4_{36}$ | 8* | 1* | $2_{50}$ |
| $2_{49}$ | 1* | $8_{14}$ | 5* | $9_{37}$ | $6_8$ | $3_{27}$ | $4_{33}$ | $7_{48}$ |
| $3_{31}$ | $7_{52}$ | $4_{38}$ | 2* | $1_{13}$ | $8_{12}$ | 6* | $5_{10}$ | $9_{51}$ |

**Puzzle 54**

$1_2$b7 $6_3 3_4 8_5$c7

$1_{11}$r1: 1c3b2

$2_{12}$c1 $4_{14}$b7

$6_{18} 8_{19}$c4 $7_{20}$b4

$7_{22}$b9 $4_{25}$r7

$5_{30}$g $7_{34} 3_{35}$c2

$1_{41}$r4 $5_{49}$r8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $3_{47}$ | $1_{11}$ | $7_{46}$ | 4* | $8_{10}$ | 2* | $6_3$ | 5* | $9_{15}$ |
| 8* | $4_{16}$ | $6_6$ | $5_{57}$ | $9_{62}$ | $3_{61}$ | 7* | $1_9$ | $2_{13}$ |
| $2_{12}$ | $9_{53}$ | $5_{54}$ | $1_{21}$ | 6* | $7_{20}$ | $3_4$ | $8_8$ | $4_{14}$ |
| $5_{42}$ | 2* | $8_{40}$ | 7* | $1_{41}$ | $6_{38}$ | 4* | $9_{33}$ | $3_{36}$ |
| $7_{45}$ | 6* | $1_{43}$ | 3* | $4_{63}$ | $9_{64}$ | $8_5$ | $2_{31}$ | $5_{30}$ |
| $4_{51}$ | $3_{35}$ | $9_{55}$ | $2_1$ | $5_{44}$ | $8_{39}$ | 1* | $7_{32}$ | $6_{37}$ |
| $6_7$ | $5_{52}$ | $4_{25}$ | $9_{56}$ | $7_{23}$ | $1_{24}$ | $2_{27}$ | 3* | 8* |
| $9_{50}$ | $7_{34}$ | $3_{48}$ | $8_{19}$ | 2* | $4_{28}$ | $5_{49}$ | 6* | $1_2$ |
| 1* | $8_{29}$ | $2_{26}$ | $6_{18}$ | $3_{59}$ | $5_{58}$ | $9_{60}$ | $4_{17}$ | $7_{22}$ |

**Puzzle 55**

$1_1$b5 $3_9 2_{10}$c8

$2_{11} 3_{12}$c2 $5_{15}$b2 $6_{16}$b5

$6_{17} 7_{18}$c2: 48c2b2

$9_{20}$r2 $9_{22}$r7

$5_{26} 8_{27}$c4 $7_{35}$b6

$4_{38}$r8 $8_{42}$c1 $4_{44}$c5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $4_{43}$ | 5* | $1_4$ | $8_{27}$ | $7_{45}$ | $2_{31}$ | $6_{46}$ | 9* | $3_{14}$ |
| $9_{20}$ | $6_{17}$ | $7_{19}$ | 1* | 3* | $5_{48}$ | $4_{62}$ | $2_{10}$ | $8_{61}$ |
| $8_{42}$ | $2_{11}$ | $3_{13}$ | $9_{23}$ | $4_{44}$ | $6_{47}$ | $5_{49}$ | $1_3$ | $7_{50}$ |
| $6_{41}$ | $3_{12}$ | $9_{21}$ | $7_{55}$ | 5* | 8* | $1_2$ | 4* | $2_{52}$ |
| 1* | $8_{57}$ | 2* | $4_{54}$ | $6_{16}$ | $3_8$ | $9_{25}$ | $7_{56}$ | $5_{33}$ |
| 7* | $4_{58}$ | $5_{15}$ | $2_{53}$ | $1_1$ | 9* | $8_{59}$ | $3_9$ | $6_{31}$ |
| 3* | $7_{18}$ | $8_{37}$ | 6* | $9_{22}$ | $4_{36}$ | 2* | $5_{32}$ | 1* |
| $2_{29}$ | $1_5$ | $4_{38}$ | $5_{26}$ | $8_{34}$ | $7_{35}$ | 3* | $6_{40}$ | $9_{24}$ |
| $5_{28}$ | 9* | $6_{39}$ | $3_7$ | $2_{30}$ | $1_6$ | $7_{63}$ | $8_{60}$ | $4_{64}$ |

**Puzzle 56**

$1_{2}$r8 $6_{8}3_{9}7_{10}$r7

$2_{20}$b1 $4_{21}$r9

$6_{23}2_{24}$b8 $2_{27}$c4

$5_{29}$r2 $2_{34}8_{35}$c9

$8_{37}$c3 $7_{40}4_{41}$r5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_{32}$ | $4_{55}$ | $9_{48}$ | 6* | 2* | $3_{14}$ | $1_{7}$ | $7_{57}$ | $8_{35}$ |
| $1_{19}$ | $8_{44}$ | 3* | $1_{5}$ | $7_{58}$ | $5_{29}$ | $4_{62}$ | $9_{61}$ | $6_{26}$ |
| 1* | $7_{56}$ | $6_{16}$ | $8_{43}$ | $9_{59}$ | $4_{54}$ | $3_{13}$ | $2_{60}$ | $5_{33}$ |
| 6* | 2* | $5_{31}$ | 3* | $1_{4}$ | $8_{51}$ | $4_{63}$ | $4_{64}$ | $9_{36}$ |
| $4_{41}$ | $3_{11}$ | $8_{39}$ | $9_{42}$ | $6_{25}$ | $7_{40}$ | $2_{24}$ | 5* | 1* |
| $9_{46}$ | $1_{3}$ | $7_{47}$ | $5_{30}$ | $4_{50}$ | $2_{28}$ | 8* | $6_{23}$ | $3_{12}$ |
| $3_{9}$ | $6_{8}$ | 4* | $7_{10}$ | 5* | 1* | $9_{38}$ | $8_{37}$ | $2_{34}$ |
| $7_{18}$ | 5* | $1_{2}$ | $2_{27}$ | $8_{52}$ | $9_{53}$ | 6* | 3* | $4_{22}$ |
| $8_{45}$ | $9_{49}$ | $2_{20}$ | $4_{21}$ | $3_{15}$ | $6_{17}$ | $5_{1}$ | $1_{6}$ | 7* |

**Puzzle 57**

$1_{1}$b6 $3_{2}$r7

$5_{5}$b7 $7_{6}$r1

$6_{15}$r7 $5_{16}$r2:

5c6b6 $2_{17}$b4

$2_{20}$r5 $6_{22}$r4

$6_{26}$c9 $1_{29}$b8: 1c9b7

$8_{34}$r6 $8_{36}$r7: 8c4b4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | $7_{6}$ | $2_{18}$ | $4_{40}$ | $3_{4}$ | 1* | $9_{47}$ | $8_{46}$ | $2_{27}$ |
| $3_{3}$ | $8_{43}$ | $4_{56}$ | $5_{16}$ | $9_{38}$ | $6_{25}$ | 2* | 7* | $1_{55}$ |
| $9_{60}$ | $1_{61}$ | $6_{26}$ | $8_{39}$ | $7_{14}$ | $2_{17}$ | 3* | $5_{5}$ | $4_{57}$ |
| $9_{59}$ | $4_{58}$ | $9_{62}$ | 2* | 5* | $7_{10}$ | $6_{22}$ | $3_{12}$ | 8* |
| 6* | 3* | $8_{35}$ | $1_{24}$ | $4_{42}$ | $7_{41}$ | $5_{30}$ | $2_{20}$ | $7_{9}$ |
| $2_{32}$ | $5_{31}$ | 7* | $6_{23}$ | $8_{34}$ | $3_{11}$ | $1_{29}$ | $4_{50}$ | $9_{51}$ |
| 4* | $6_{15}$ | $3_{2}$ | $9_{37}$ | $2_{19}$ | $8_{36}$ | $7_{8}$ | 1* | 5* |
| $8_{44}$ | $2_{33}$ | $1_{54}$ | 7* | 6* | $1_{53}$ | $4_{48}$ | $9_{49}$ | $3_{13}$ |
| $7_{7}$ | $9_{63}$ | $5_{64}$ | 3* | $1_{1}$ | $5_{52}$ | $8_{45}$ | $6_{28}$ | $2_{21}$ |

**Puzzle 58**

$9_{1}1_{2}$r1 $1_{9}$b3

$2_{14}$r4 $2_{16}$b6

$7_{17}$r2 $8_{25}2_{26}$rb9

$8_{30}$c1 $4_{34}$b4

$4_{35}$c1 $4_{40}$c4:4r6b8

$6_{41}$b5 $4_{42}$r5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $2_{18}$ | $4_{45}$ | $3_{46}$ | 7* | $8_{32}$ | 5* | $1_{2}$ | 6* | $9_{1}$ |
| $8_{30}$ | 9* | 1* | $3_{54}$ | $6_{55}$ | $2_{17}$ | 4* | $5_{37}$ | $7_{19}$ |
| $5_{36}$ | $6_{39}$ | $7_{24}$ | $9_{7}$ | $4_{34}$ | $1_{8}$ | $8_{28}$ | $2_{29}$ | $3_{38}$ |
| 6* | $5_{43}$ | $9_{4}$ | $1_{11}$ | $3_{56}$ | $4_{57}$ | $2_{14}$ | 7* | 8* |
| 3* | $8_{31}$ | $4_{42}$ | 2* | $7_{22}$ | $6_{41}$ | $5_{44}$ | $9_{3}$ | $1_{12}$ |
| $1_{10}$ | $7_{23}$ | $2_{15}$ | $5_{48}$ | 9* | $8_{33}$ | $6_{63}$ | $3_{62}$ | $4_{60}$ |
| $4_{35}$ | $3_{47}$ | $5_{51}$ | $6_{53}$ | 1* | $7_{21}$ | 9* | $8_{25}$ | $2_{26}$ |
| 7* | $1_{9}$ | $6_{52}$ | 8* | $2_{16}$ | $9_{6}$ | $3_{64}$ | $4_{61}$ | $5_{50}$ |
| $9_{5}$ | 2* | $8_{27}$ | $4_{40}$ | $5_{49}$ | $3_{58}$ | $7_{20}$ | $1_{13}$ | $6_{59}$ |

PRACTICE SET 6

**Puzzle 59**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | | | | | 2* | | | |
| | | | 5* | | | | 7* | |
| | | | | | | | | |
| | | | 7* | 6* | | | 5* | |
| 2* | | | | | | 1* | | |
| | | | 3* | | | | | |
| | | | | 7* | | 8* | | 6* |
| 4* | 7* | 5* | | | | | | |
| | 3* | | | | | | | |

**Puzzle 60**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 8* | | | | | 7* | 4* | |
| | 3* | | | 2* | | | | |
| | | | | 1* | | | | |
| 2* | | | 6* | | | | 8* | |
| 1* | | | | | | | | |
| | | | | | | 5* | | |
| 9* | | | | | | | | 2* |
| | | | 8* | | | 3* | | |
| | | | 4* | | 3* | | | |

**Puzzle 61**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | | | | | | | | 8* |
| | | | 1* | | | | 6* | |
| 2* | | | | | | | | |
| | | | | 3* | 2* | 5* | 4* | |
| | 1* | | | | | | | |
| | | | | | | 2* | | |
| | 6* | | 7* | 1* | | | | |
| 4* | | | 6* | | | | | |
| | | | | | | 3* | | |

**Puzzle 62**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 5* | 7* | 3* | | | | | |
| 4* | | | | | | | | 2* |
| | | | | | | | | |
| 2* | | 4* | | 9* | | | | |
| | | | | 3* | | 6* | 5* | |
| 1* | | | | | | | | |
| | 8* | | | | | 3* | 7* | |
| | | | 1* | | | | | |
| | | | | | 4* | | | |

**Puzzle 63**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | 1* | | 3* | | | | |
| | | | | | | | 2* | |
| 8* | | | | | | | | |
| | 9* | | 2* | | 6* | | | |
| | | | | | | 1* | | 4* |
| | | | 7* | | | | | |
| | | | | 4* | | 3* | | |
| | 2* | | | | | | 6* | |
| 5* | | | 1* | | | | | |

**Puzzle 64**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 3* | 4* | | | | 1* | |
| | | | | | 8* | | 2* | |
| 8* | | | 6* | | | | | |
| 3* | 7* | | | | | 6* | | |
| | | | | 1* | | | | |
| 6* | | | | | | | | |
| | | | 3* | | | 8* | | |
| | 1* | | | | | 5* | | |
| | | 2* | | | | | | |

**Puzzle 65**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8* | 1* | | | | | | | |
| | | | 8* | | | 5* | | |
| | | | 3* | | | | | |
| | | 3* | | 9* | 4* | | | |
| 6* | | | | | | 2* | | |
| | | | | | 1* | | | |
| | | | | 3* | | | 1* | 9* |
| 2* | | | 7* | | | | | |
| | | | | | | | 4* | |

**Puzzle 59**

$2_{6}$b5 $5_{7}$b9 $3_{10}$r7

$3_{11}$r5 $3_{13}6_{14}8_{15}$c1

$3_{17}4_{18}$r7 $3_{20}$r5

$8_{21}$b3 $2_{22}1_{23}$r9

$2_{28}$c8 $3_{29}$c6

$8_{33}2_{34}$r2 $9_{39}$b5

$9_{45}$g $9_{46}$c8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | $6_{52}$ | $7_{2}$ | $8_{40}$ | $4_{56}$ | 2* | $5_{12}$ | $9_{46}$ | $3_{57}$ |
| $3_{13}$ | $8_{33}$ | $2_{34}$ | 5* | $9_{55}$ | $6_{32}$ | $4_{58}$ | 7* | $1_{26}$ |
| $5_{11}$ | $9_{51}$ | $4_{50}$ | $1_{25}$ | $3_{54}$ | $7_{1}$ | $6_{53}$ | $2_{28}$ | $8_{43}$ |
| $8_{15}$ | $1_{37}$ | $9_{45}$ | 7* | 6* | $4_{40}$ | $3_{59}$ | 5* | $2_{60}$ |
| 2* | $4_{48}$ | $3_{20}$ | $9_{39}$ | $5_{9}$ | $8_{41}$ | 1* | $6_{47}$ | $7_{4}$ |
| $7_{3}$ | $5_{10}$ | $6_{49}$ | 3* | $2_{6}$ | $1_{38}$ | $9_{62}$ | $8_{44}$ | $4_{63}$ |
| $9_{16}$ | $2_{35}$ | $1_{36}$ | $4_{18}$ | 7* | $5_{8}$ | 8* | $3_{17}$ | 6* |
| 4* | 7* | 5* | $6_{30}$ | $8_{31}$ | $3_{29}$ | $2_{61}$ | $1_{27}$ | $9_{64}$ |
| $6_{14}$ | 3* | $8_{21}$ | $2_{22}$ | $1_{23}$ | $9_{24}$ | $7_{5}$ | $4_{19}$ | $5_{7}$ |

**Puzzle 60**

$2_{3}$b6 $1_{9}$b1 $3_{11}$c8

$8_{15}$c1 $4_{19}$b9 $1_{23}$b8

$4_{25}8_{26}$c5 $4_{29}$r3 $5_{31}$b7

$9_{33}$c7: 9r3b1 $6_{35}$r2

$7_{37}$c1 $9_{40}$c4 $1_{44}$r5

$5_{45}$b5 $1_{57}$r7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_{38}$ | 8* | $2_{7}$ | $3_{14}$ | $6_{60}$ | $9_{59}$ | 7* | 4* | $1_{10}$ |
| $6_{35}$ | 3* | $1_{9}$ | $7_{36}$ | 2* | $4_{30}$ | $9_{33}$ | $5_{31}$ | $8_{17}$ |
| $4_{29}$ | $7_{53}$ | $9_{52}$ | $5_{41}$ | 1* | $8_{18}$ | $6_{34}$ | $2_{6}$ | $3_{13}$ |
| 2* | $9_{54}$ | $4_{27}$ | 6* | $3_{12}$ | $5_{45}$ | $2_{5}$ | 8* | $7_{56}$ |
| 1* | $5_{47}$ | $7_{51}$ | $9_{42}$ | $8_{26}$ | $1_{44}$ | $4_{22}$ | $3_{11}$ | $6_{50}$ |
| $3_{2}$ | $6_{48}$ | $8_{28}$ | $2_{4}$ | $4_{25}$ | $7_{46}$ | 5* | $1_{23}$ | $9_{49}$ |
| 9* | $4_{20}$ | $3_{1}$ | $1_{43}$ | $5_{56}$ | $6_{58}$ | $8_{16}$ | $7_{57}$ | 2* |
| $7_{37}$ | $1_{22}$ | $5_{39}$ | 8* | $9_{62}$ | $2_{3}$ | 3* | $6_{61}$ | $4_{19}$ |
| $8_{15}$ | $2_{8}$ | $6_{40}$ | 4* | $7_{63}$ | 3* | $1_{24}$ | $1_{64}$ | $5_{32}$ |

**Puzzle 61**

$1_{6}$c1 $3_{7}$c4 $2_{9}$b7

$4_{15}5_{16}7_{17}$c9: 7r5b5

$4_{21}$b9 $3_{27}$r8 $3_{31}$b5

$6_{32}$c1 $7_{33}$r4 $9_{35}$c1

$4_{38}$b2 $8_{44}$c4 $6_{55}$r3

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $9_{51}$ | $1_{56}$ | $4_{45}$ | $5_{43}$ | $6_{59}$ | $7_{58}$ | $2_{9}$ | 8* |
| $7_{34}$ | $4_{46}$ | $5_{47}$ | 1* | $2_{10}$ | $8_{52}$ | $9_{53}$ | 6* | $3_{8}$ |
| 2* | $8_{50}$ | $6_{55}$ | $3_{7}$ | $7_{60}$ | $9_{54}$ | $1_{57}$ | $5_{20}$ | $4_{15}$ |
| $6_{32}$ | $7_{33}$ | $8_{37}$ | $9_{36}$ | 3* | 2* | 5* | 4* | $1_{2}$ |
| $5_{39}$ | 1* | $2_{3}$ | $8_{44}$ | $4_{62}$ | $7_{61}$ | $6_{5}$ | $3_{14}$ | $9_{18}$ |
| $9_{35}$ | $3_{30}$ | $4_{38}$ | $5_{41}$ | $6_{31}$ | $1_{1}$ | 2* | $8_{19}$ | $7_{17}$ |
| $8_{40}$ | 6* | $3_{29}$ | 7* | 1* | $5_{42}$ | $4_{21}$ | $9_{25}$ | $2_{13}$ |
| 4* | $2_{12}$ | $7_{26}$ | 6* | $9_{28}$ | $3_{27}$ | $8_{22}$ | $1_{23}$ | $5_{16}$ |
| $1_{6}$ | $5_{48}$ | $9_{49}$ | $2_{11}$ | $8_{63}$ | $4_{64}$ | 3* | $7_{24}$ | $6_{4}$ |

**Puzzle 62**

$1_{4}$r7 $4_{5}2_{6}$r5

$5_{9}6_{10}$r4 $2_{12}$r1

$2_{16}$r7 $7_{20}$c2

$3_{29}$b1 $5_{35}$r7: 5c6b4

$6_{37}$r2 $7_{38}$r4

$2_{51}$c8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $6_{30}$ | 5* | 7* | 3* | $2_{12}$ | $9_{23}$ | $1_{54}$ | $4_{53}$ | $8_{55}$ |
| 4* | $9_{21}$ | $8_{28}$ | $6_{37}$ | $1_{14}$ | $7_{41}$ | $5_{42}$ | $3_{33}$ | 2* |
| $3_{29}$ | $1_{15}$ | $2_{16}$ | $8_{44}$ | $4_{13}$ | $5_{43}$ | $9_{49}$ | $6_{48}$ | $7_{40}$ |
| 2* | $6_{10}$ | 4* | $5_{9}$ | 9* | $1_{8}$ | $7_{38}$ | $8_{39}$ | $3_{34}$ |
| $8_{27}$ | $7_{20}$ | $9_{26}$ | $4_{5}$ | 3* | $2_{6}$ | 6* | 5* | $1_{7}$ |
| 1* | $3_{19}$ | $5_{11}$ | $7_{45}$ | $6_{47}$ | $8_{46}$ | $4_{52}$ | $2_{51}$ | $9_{22}$ |
| $9_{25}$ | 8* | $1_{4}$ | $2_{16}$ | $5_{35}$ | $6_{36}$ | 3* | 7* | $4_{3}$ |
| $7_{60}$ | $4_{2}$ | $6_{32}$ | 1* | $8_{62}$ | $3_{1}$ | $2_{63}$ | $9_{50}$ | $5_{58}$ |
| $1_{59}$ | $2_{17}$ | $3_{31}$ | $9_{24}$ | $7_{61}$ | 4* | $8_{64}$ | $1_{56}$ | $6_{57}$ |

**Puzzle 63**

$2_{1}$b1 $4_{7}$b5 $6_{13}$b6

$6_{17}$r5 $6_{21}$r1 $9_{23}$b1

$1_{24}$c8 $5_{25}$b9

$1_{27}$r8 $8_{30}$c2

$9_{31}$r7 $7_{33}$b2

$8_{37}$g $3_{38}$c8

$9_{44}$r1 $5_{49}$c4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | $7_{45}$ | 1* | $8_{43}$ | 3* | $2_{2}$ | $6_{21}$ | $5_{39}$ | $9_{44}$ |
| $6_{18}$ | $5_{51}$ | $9_{23}$ | $4_{12}$ | $7_{62}$ | $1_{61}$ | $8_{42}$ | 2* | $3_{53}$ |
| 8* | $3_{52}$ | $2_{1}$ | $5_{49}$ | $6_{22}$ | $9_{55}$ | $7_{47}$ | $4_{11}$ | $1_{54}$ |
| $7_{33}$ | 9* | $4_{8}$ | 2* | $1_{29}$ | 6* | $5_{40}$ | $3_{38}$ | $8_{41}$ |
| $2_{6}$ | $6_{17}$ | $5_{59}$ | $3_{16}$ | $9_{56}$ | $8_{60}$ | 1* | $9_{35}$ | 4* |
| $3_{34}$ | $1_{28}$ | $8_{58}$ | 7* | $5_{57}$ | $4_{7}$ | $2_{5}$ | $9_{36}$ | $6_{20}$ |
| $9_{31}$ | $8_{30}$ | $7_{32}$ | $6_{13}$ | 4* | $5_{26}$ | 3* | $1_{24}$ | $2_{4}$ |
| $1_{27}$ | 2* | $3_{28}$ | $9_{50}$ | $8_{63}$ | $7_{64}$ | $4_{10}$ | 6* | $5_{25}$ |
| 5* | $4_{9}$ | $6_{14}$ | 1* | $2_{3}$ | $3_{15}$ | $9_{46}$ | $8_{37}$ | $7_{48}$ |

**Puzzle 64**

$1_{4}$c1 $2_{10}$c7 $2_{13}$b2

$3_{18}$c6 $6_{22}$b7 $8_{24}$b3

$5_{32}$c4: 5r1b4 $7_{33}$r5

$7_{35}$r4 $5_{38}$b1 $7_{39}$r7

$7_{43}$b4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $2_{14}$ | $6_{23}$ | 3* | 4* | $9_{44}$ | $5_{42}$ | $7_{48}$ | 1* | $8_{3}$ |
| $1_{4}$ | $5_{38}$ | $9_{59}$ | $7_{43}$ | $3_{19}$ | 8* | $4_{58}$ | 2* | $6_{22}$ |
| 8* | $4_{55}$ | $7_{60}$ | 6* | $2_{15}$ | $1_{5}$ | $3_{21}$ | $5_{37}$ | $9_{61}$ |
| 3* | 7* | $1_{9}$ | $2_{17}$ | $4_{45}$ | $9_{46}$ | 6* | $8_{31}$ | $5_{35}$ |
| $9_{52}$ | $8_{26}$ | $4_{53}$ | $5_{32}$ | 1* | $6_{2}$ | $2_{10}$ | $7_{33}$ | $3_{20}$ |
| 6* | $2_{13}$ | $5_{36}$ | $8_{27}$ | $7_{34}$ | $3_{18}$ | $1_{8}$ | $9_{63}$ | $4_{62}$ |
| $7_{39}$ | $9_{54}$ | $6_{25}$ | 3* | $5_{41}$ | $2_{16}$ | 8* | $4_{64}$ | $1_{7}$ |
| $4_{51}$ | 1* | $8_{24}$ | $9_{47}$ | $6_{29}$ | $7_{50}$ | 5* | $3_{12}$ | $2_{11}$ |
| $5_{40}$ | $3_{1}$ | 2* | $1_{6}$ | $8_{28}$ | $4_{56}$ | $9_{57}$ | $6_{30}$ | $7_{49}$ |

**Puzzle 65**

$1_{2}$c4 $9_{8}4_{9}$c4 $2_{11}$r7

$4_{13}$r5 $9_{14}4_{15}$c7 $4_{17}$r2

$2_{21}$b4 $3_{22}9_{23}$c1 $2_{30}$r2

$6_{33}$r6: 6c5b4→6c5b6

$8_{37}$c6 $8_{38}$c7: 8r7b3

$8_{39}$r5 $7_{43}$c1

$6_{48}$b3 $7_{55}$r1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8* | 1* | $9_{55}$ | $9_{8}$ | $2_{21}$ | $5_{54}$ | $4_{15}$ | $6_{59}$ | $3_{25}$ |
| $4_{17}$ | $3_{24}$ | $2_{30}$ | 8* | $1_{7}$ | $6_{58}$ | 5* | $9_{29}$ | $7_{60}$ |
| $9_{23}$ | $5_{52}$ | $6_{56}$ | 3* | $4_{18}$ | $7_{57}$ | $1_{6}$ | $2_{31}$ | $8_{32}$ |
| $1_{4}$ | $7_{42}$ | 3* | $2_{34}$ | 9* | 4* | $8_{38}$ | $5_{62}$ | $6_{61}$ |
| 6* | $9_{16}$ | $4_{13}$ | $5_{35}$ | $8_{39}$ | $3_{1}$ | 2* | $7_{41}$ | $1_{5}$ |
| $5_{44}$ | $2_{36}$ | $8_{45}$ | $6_{33}$ | $7_{40}$ | 1* | $9_{14}$ | $3_{20}$ | $4_{19}$ |
| $7_{43}$ | $8_{51}$ | $5_{53}$ | $4_{9}$ | 3* | $2_{11}$ | $6_{47}$ | 1* | 9* |
| 2* | $4_{10}$ | $1_{3}$ | 7* | $6_{49}$ | $9_{28}$ | $3_{26}$ | $8_{63}$ | $5_{64}$ |
| $3_{22}$ | $6_{48}$ | $9_{27}$ | $1_{2}$ | $5_{50}$ | $8_{37}$ | $7_{46}$ | 4* | $2_{12}$ |

PRACTICE SET 7

**Puzzle 66**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | | | 1* | | | | 4* | |
| | | | 2* | | | 8* | | |
| 9* | | | | | | | | |
| | | | 5* | 4* | | | 6* | |
| | 8* | | | | | | | 1* |
| | 2* | | | | | | | |
| 4* | | | | 6* | | 3* | | |
| | 7* | | | | | 2* | | |
| | | | | | | | | |

**Puzzle 67**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 1* | 6* | 7* | | |
| | 4* | 8* | | | | | | |
| | | | | | | | | |
| 1* | | | | | 7* | 6* | | |
| | | 2* | 8* | | | | | |
| | | | 3* | | | | | |
| | | | | 4* | | | 8* | 2* |
| 7* | 5* | | | | | | | |
| | | | | | | | 3* | |

**Puzzle 68**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | | | | 5* | | | 3* | |
| | | | 2* | 1* | | | | |
| | | | | | | | | |
| | | | 1* | | 3* | | | 4* |
| 5* | | 2* | | | | | | |
| 8* | | | | | | | | |
| | | | | | 7* | 8* | 6* | |
| | 1* | | 4* | | | 5* | | |
| | | | | | | | | |

**Puzzle 69**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 4* | 5* | | | | |
| 1* | | | | | | 6* | | |
| | | | | 8* | | | | |
| 6* | | | 7* | | 1* | | | |
| | | | | | | 3* | | 8* |
| | | | | | | | 4* | |
| | 4* | 8* | 2* | | | | | |
| | 7* | | | | | 1* | | |
| | | 9* | | | | | | |

**Puzzle 70**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7* | | | | | | 6* | | 1* |
| | | | | 4* | 3* | | | |
| | | | | | | | | |
| 1* | | | 7* | | | | | |
| | | | | | 5* | | 4* | |
| | | 6* | | | | 8* | | |
| | 4* | 5* | | | | | 9* | |
| | 6* | | | 2* | | | | |
| | | | 1* | | | | | |

**Puzzle 71**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7* | | | | 1* | | | | |
| 4* | | | | | | 8* | | |
| | | | 2* | | | 3* | | |
| | | | 3* | | 8* | | | |
| 5* | | | | | | | 4* | |
| | | | 6* | | | | | |
| | 3* | | 5* | | | 2* | | |
| | | | | 4* | | | 7* | |
| | 1* | | | | | | | |

**Puzzle 72**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1* | | 2* | | | | | |
| 6* | | | | | | | 3* | |
| | | | | | | 5* | 7* | |
| 7* | | 3* | | | | | | |
| | | | 1* | | | | | |
| 5* | | | | | | | | |
| | 6* | | | | | 2* | | 1* |
| 4* | | | | 3* | | | | |
| | | | | 5* | | | | 8* |

**Puzzle 66**

$2_{1}$c1 $6_{7}$c2 $1_{11}$b7

$4_{13}$b9 $4_{14}$r2: 4c3b2

$8_{15}$c1 $1_{18}$r7

$8_{19}1_{20}3_{21}$r4: 3c3b3

$3_{23}$c1 $7_{24}$b1 $7_{27}$r4

$5_{29}9_{30}$b7: 37(19)(39)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | $5_{25}$ | $2_{6}$ | 1* | $8_{22}$ | $3_{53}$ | $9_{30}$ | 4* | $7_{52}$ |
| $3_{23}$ | $4_{14}$ | $7_{24}$ | 2* | $9_{31}$ | $6_{10}$ | 8* | $1_{11}$ | $5_{29}$ |
| 9* | $1_{17}$ | $8_{16}$ | $4_{62}$ | $7_{51}$ | $5_{63}$ | $6_{9}$ | $2_{5}$ | $3_{54}$ |
| $1_{20}$ | $3_{21}$ | $9_{28}$ | 5* | 4* | $8_{19}$ | $7_{27}$ | 6* | $2_{4}$ |
| $5_{48}$ | 8* | $6_{44}$ | $9_{42}$ | $2_{3}$ | $7_{50}$ | $4_{46}$ | $3_{36}$ | 1* |
| $7_{49}$ | 2* | $4_{45}$ | $6_{43}$ | $3_{55}$ | $1_{56}$ | $5_{47}$ | $8_{35}$ | $9_{32}$ |
| 4* | $9_{26}$ | $1_{18}$ | $7_{38}$ | 6* | $2_{2}$ | 3* | $5_{37}$ | $8_{33}$ |
| $8_{15}$ | 7* | $5_{60}$ | $3_{61}$ | $1_{57}$ | $4_{64}$ | 2* | $9_{40}$ | $6_{8}$ |
| $2_{1}$ | $6_{7}$ | $3_{59}$ | $8_{34}$ | $5_{58}$ | $9_{41}$ | $1_{12}$ | $7_{39}$ | $4_{13}$ |

$9_{32}8_{33}$c9 $3_{36}$c8 $5_{37}$r7 $1_{35}$r3: 47(23)(33) $3_{26}$g

**Puzzle 67**

$7_{1}$b9 $8_{3}$r1 $2_{7}$b3

$3_{8}$b6: 28(85)(86)

$4_{11}$b3 $5_{12}$r7

$6_{13}1_{14}$c4:6r8b9

$1_{15}$b9 $7_{16}$r2

$3_{20}2_{21}$c7: 3r5b2

$4_{23}$r4: 4c4b4 $4_{27}$c7 $1_{39}$r2 $5_{40}$r1: 5c3b2 $9_{42}$r4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $2_{31}$ | $3_{32}$ | $9_{41}$ | $4_{30}$ | 1* | 6* | 7* | $5_{40}$ | $8_{3}$ |
| $6_{56}$ | 4* | 8* | $2_{61}$ | $7_{10}$ | $5_{49}$ | $3_{21}$ | $1_{39}$ | $9_{62}$ |
| $5_{50}$ | $7_{53}$ | $1_{55}$ | $9_{60}$ | $3_{17}$ | $8_{18}$ | $2_{22}$ | $6_{59}$ | $4_{29}$ |
| 1* | $8_{5}$ | $5_{43}$ | $9_{42}$ | $2_{25}$ | 7* | 6* | $4_{23}$ | $3_{10}$ |
| $3_{33}$ | $6_{45}$ | 2* | 8* | $5_{44}$ | $4_{35}$ | $9_{28}$ | $7_{26}$ | $1_{37}$ |
| $4_{34}$ | $9_{51}$ | $7_{52}$ | 3* | $6_{46}$ | $1_{36}$ | $8_{4}$ | $2_{24}$ | $5_{38}$ |
| $9_{57}$ | $1_{54}$ | $6_{58}$ | $7_{2}$ | 4* | $3_{8}$ | $5_{12}$ | 8* | 2* |
| 7* | 5* | $3_{9}$ | $1_{14}$ | $8_{19}$ | $2_{20}$ | $4_{27}$ | $9_{63}$ | $6_{64}$ |
| $8_{6}$ | $2_{7}$ | $4_{11}$ | $6_{13}$ | $5_{47}$ | $9_{48}$ | $1_{15}$ | 3* | $7_{1}$ |

**Puzzle 68**

$1_{3}$b2 $5_{7}2_{8}$c6 $8_{10}2_{11}$r4

$4_{14}$r6 $3_{15}$b2 $8_{16}6_{17}$r8

$3_{20}$c4 $9_{21}$g:56(92)(93)
→7c1b3

$2_{22}$c1: 2r9b9 $4_{23}3_{24}$r7

$4_{29}8_{30}2_{31}$r1 $4_{45}$b5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | $8_{30}$ | $7_{36}$ | $9_{37}$ | 5* | $4_{29}$ | $1_{5}$ | 3* | $2_{31}$ |
| $4_{27}$ | $5_{41}$ | $3_{28}$ | 2* | 1* | $6_{49}$ | $9_{57}$ | $8_{47}$ | $7_{58}$ |
| $1_{4}$ | $2_{32}$ | $9_{42}$ | $3_{20}$ | $7_{35}$ | $8_{48}$ | $6_{51}$ | $4_{46}$ | $5_{50}$ |
| $9_{21}$ | $7_{38}$ | $6_{39}$ | 1* | $8_{10}$ | 3* | $2_{11}$ | $5_{9}$ | 4* |
| 5* | $3_{15}$ | 2* | $6_{53}$ | $4_{34}$ | $9_{44}$ | $7_{55}$ | $1_{6}$ | $8_{13}$ |
| 8* | $4_{14}$ | $1_{3}$ | $7_{54}$ | $2_{12}$ | $5_{7}$ | $3_{56}$ | $9_{59}$ | $6_{52}$ |
| $2_{22}$ | $9_{25}$ | $4_{23}$ | $5_{19}$ | $3_{24}$ | 7* | 8* | 6* | $1_{2}$ |
| $3_{62}$ | 1* | $8_{16}$ | 4* | $6_{17}$ | $2_{8}$ | 5* | $7_{60}$ | $9_{63}$ |
| $7_{61}$ | $6_{40}$ | $5_{43}$ | $8_{18}$ | $9_{26}$ | $1_{1}$ | $4_{45}$ | $2_{33}$ | $3_{64}$ |

**Puzzle 69**

$1_{1}$b3 $8_{4}$r4 $4_{8}$c7

$3_{12}$r4 $6_{13}$b3

$7_{14}$g: 259r4b8 $5_{20}$c3

$2_{23}5_{24}$c7 $7_{29}$b7

$3_{30}$r1: 3c2b2 $9_{33}$r3

$6_{36}9_{37}7_{38}$r7 $9_{41}$g

$3_{45}$g $2_{47}$r9: u23r3b7r9b9

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $8_{5}$ | $6_{31}$ | $2_{21}$ | 4* | 5* | $3_{30}$ | $9_{25}$ | $7_{29}$ | $1_{18}$ |
| 1* | $3_{34}$ | $4_{9}$ | $9_{41}$ | $7_{63}$ | $2_{62}$ | 6* | $8_{6}$ | $5_{22}$ |
| $7_{19}$ | $9_{33}$ | $5_{20}$ | $1_{3}$ | 8* | $6_{32}$ | $4_{8}$ | $2_{48}$ | $3_{51}$ |
| 6* | $8_{4}$ | $3_{12}$ | 7* | $4_{11}$ | 1* | $2_{23}$ | $5_{26}$ | $9_{27}$ |
| $4_{10}$ | $2_{60}$ | $7_{15}$ | $6_{43}$ | $9_{42}$ | $5_{59}$ | 3* | $1_{17}$ | 8* |
| $9_{35}$ | $5_{61}$ | $1_{16}$ | $3_{46}$ | $2_{64}$ | $8_{58}$ | $7_{14}$ | 4* | $6_{28}$ |
| $3_{39}$ | 4* | 8* | 2* | $1_{2}$ | $9_{37}$ | $5_{24}$ | $6_{36}$ | $7_{38}$ |
| $5_{48}$ | 7* | $6_{13}$ | $8_{55}$ | $3_{45}$ | $4_{54}$ | 1* | $9_{40}$ | $2_{49}$ |
| $2_{47}$ | $1_{1}$ | 9* | $5_{56}$ | $6_{44}$ | $7_{57}$ | $8_{7}$ | $3_{52}$ | $4_{53}$ |

**Puzzle 70**

$1_1$b3 $4_7$r1 $7_{11}$b4

$6_{12}8_{13}$b8: 6c4b4 $6_{16}$c6

$7_{17}6_{18}$r7 $7_{19}$b8 $5_{24}$b2

$9_{27}$r8 $8_{28}$r7 $2_{32}$c6

$3_{37}$g $3_{38}$b3 $9_{48}$c2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7* | $2_{49}$ | $4_{7}$ | $5_{52}$ | $9_{36}$ | $8_{31}$ | 6* | $3_{41}$ | 1* |
| $5_{63}$ | $1_{6}$ | $9_{51}$ | $6_{64}$ | 4* | 3* | $2_{58}$ | $7_{23}$ | $8_{43}$ |
| $6_{62}$ | $8_{42}$ | $3_{40}$ | $2_{61}$ | $7_{11}$ | $1_{5}$ | $9_{56}$ | $5_{55}$ | $4_{57}$ |
| 1* | $5_{24}$ | $8_{26}$ | 7* | $6_{12}$ | $4_{9}$ | $3_{47}$ | $2_{54}$ | $9_{53}$ |
| $3_{37}$ | $9_{48}$ | $2_{50}$ | $8_{13}$ | $1_{4}$ | 5* | $7_{19}$ | 4* | $6_{14}$ |
| $4_{8}$ | $7_{20}$ | 6* | $9_{33}$ | $3_{30}$ | $2_{32}$ | 8* | $1_{3}$ | $5_{25}$ |
| $2_{18}$ | 4* | 5* | $3_{29}$ | $8_{28}$ | $6_{16}$ | $1_{2}$ | 9* | $7_{17}$ |
| $9_{27}$ | 6* | $1_{1}$ | $4_{10}$ | 2* | $7_{22}$ | $5_{45}$ | $8_{44}$ | $3_{46}$ |
| $8_{39}$ | $3_{38}$ | $7_{21}$ | 1* | $5_{35}$ | $9_{34}$ | $4_{39}$ | $6_{15}$ | $2_{60}$ |

**Puzzle 71**

$3_1$c1 $4_7$b5 $7_{13}$b3

$5_{14}$r9 $7_{15}$b7

$7_{16}1_{17}8_{18}$c4

$2_{25}9_{26}7_{27}$c6

$1_{31}$r7: 1c8b7

$1_{32}5_{33}$c7 $2_{34}$c6

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7* | $8_{24}$ | $3_{6}$ | $4_{8}$ | 1* | $6_{35}$ | $5_{33}$ | $9_{62}$ | $2_{63}$ |
| 4* | $2_{49}$ | $6_{48}$ | $9_{19}$ | $3_{5}$ | $5_{34}$ | 8* | $1_{44}$ | $7_{15}$ |
| $1_{42}$ | $9_{52}$ | $5_{51}$ | 2* | $8_{23}$ | $7_{27}$ | 3* | $6_{47}$ | $4_{9}$ |
| $2_{41}$ | $4_{12}$ | $1_{43}$ | 3* | $9_{60}$ | 8* | $7_{22}$ | $5_{61}$ | $6_{54}$ |
| 5* | $6_{45}$ | $8_{39}$ | $7_{16}$ | $2_{57}$ | $1_{20}$ | $9_{56}$ | 4* | $3_{2}$ |
| $3_{1}$ | $7_{21}$ | $9_{46}$ | 6* | $5_{59}$ | $4_{7}$ | $1_{32}$ | $2_{58}$ | $8_{38}$ |
| $6_{36}$ | 3* | $4_{11}$ | 5* | $7_{28}$ | $9_{26}$ | 2* | $8_{37}$ | $1_{31}$ |
| $8_{40}$ | $5_{53}$ | $2_{50}$ | $1_{17}$ | 4* | $3_{4}$ | $6_{55}$ | 7* | $9_{64}$ |
| $9_{30}$ | 1* | $7_{13}$ | $8_{18}$ | $6_{29}$ | $2_{25}$ | $4_{10}$ | $3_{3}$ | $5_{14}$ |

$6_{36}$r7: 6c8b7 $2_{41}$c1 $6_{45}$b2 $6_{47}$b7

**Puzzle 72**

$1_16_2$b2 $1_5$b7 $2_8$b6

$3_9$r7 $5_{15}7_{16}$r1

$7_{23}$b5 $4_{26}$b9 $8_{27}$r8

$6_{29}$r9 $6_{31}$b4

$8_{32}$r3 $2_{33}$c1

$9_{35}$c3 $6_{46}$r4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $9_{34}$ | 1* | $5_{15}$ | 2* | $7_{16}$ | $3_{11}$ | $4_{55}$ | $8_{54}$ | $6_{50}$ |
| 6* | $2_{39}$ | $7_{37}$ | $5_{22}$ | $4_{62}$ | $8_{61}$ | $1_{5}$ | 3* | $9_{45}$ |
| $8_{32}$ | $3_{10}$ | $4_{38}$ | $9_{41}$ | $1_{6}$ | $6_{31}$ | 5* | 7* | $2_{40}$ |
| 7* | $8_{58}$ | 3* | $4_{42}$ | $2_{44}$ | $9_{59}$ | $6_{46}$ | $1_{7}$ | $5_{20}$ |
| $2_{33}$ | $4_{52}$ | $6_{2}$ | 1* | $8_{60}$ | $5_{21}$ | $7_{24}$ | $9_{53}$ | $3_{14}$ |
| 5* | $9_{57}$ | $1_{1}$ | $3_{12}$ | $6_{47}$ | $7_{23}$ | $8_{56}$ | $2_{43}$ | $4_{51}$ |
| $3_{9}$ | 6* | $8_{28}$ | $7_{30}$ | $9_{63}$ | $4_{64}$ | 2* | $5_{19}$ | 1* |
| 4* | $5_{17}$ | $2_{18}$ | $8_{27}$ | 3* | $1_{4}$ | $9_{49}$ | $6_{48}$ | $7_{25}$ |
| $1_{3}$ | $7_{36}$ | $9_{35}$ | $6_{29}$ | 5* | $2_{8}$ | $3_{13}$ | $4_{26}$ | 8* |

PRACTICE SET 8

**Puzzle 73**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | | | | 6* | | 2* | | 7* |
| | | | | 5* | 8* | | | |
| | | | | | 1* | | | |
| 4* | 3* | | 9* | | | | | |
| | | | | | | | 5* | 2* |
| | | | | | | | | |
| | | | 2* | | | 4* | | |
| | 5* | | | | | | 8* | |
| | | 1* | | | | | | |

**Puzzle 74**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | 3* | | | | 1* | | | |
| | | | 7* | | | 2* | | |
| | | | | | | | | |
| 4* | | 2* | | | | | | |
| | | | | 9* | | | | 3* |
| 7* | | | | | | | 1* | |
| | | 6* | 2* | | | 4* | | |
| | 8* | | | 3* | | | | |
| | | | | | | 7* | | |

**Puzzle 75**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9* | | | | | 4* | 2* | | |
| | 6* | | 1* | | | | | |
| | | | | | | | | |
| 2* | | | 3* | 9* | | | | |
| | | | 8* | | | | 1* | 4* |
| 6* | | | | | | | | |
| | | 1* | | | 5* | | | |
| | | | | 2* | | 9* | | |
| | 8* | | | | | | | |

**Puzzle 76**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | 1* | | | | | | 6* |
| | | | 7* | | | 2* | | |
| 5* | | | | | | | | |
| | 7* | | 3* | | 2* | | | |
| | | | | | | | 1* | 4* |
| | | | | | | | | |
| | | | 5* | 4* | 6* | | | |
| | | | | | | 8* | 3* | |
| | | | | 1* | | | | |

**Puzzle 77**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | | | 7* | 3* | | |
| | 5* | | 6* | | | | | |
| | | | | | | | | |
| | | 6* | 5* | 1* | | | | |
| | | | | | | | 8* | 2* |
| | | | | | | 4* | | |
| | | | 2* | | | | 7* | 6* |
| 3* | 4* | | | | | | | |
| 9* | | | | | | | | |

**Puzzle 78**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | | | 1* | | | | | 2* |
| | 7* | | | 3* | | | 8* | |
| | | | | | | | | |
| | 3* | | | | | 9* | 4* | |
| | 4* | | 5* | | | | | |
| | | | 2* | | | | | |
| 2* | | 1* | | | | | | |
| | | | | 9* | | 4* | | |
| 5* | | | | | | | | |

**Puzzle 79**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | | | 3* | 4* | | | | |
| | | | | | | 5* | 1* | |
| 2* | | | | | | | | |
| 4* | 3* | | | | | 7* | | |
| | | | | 1* | 8* | | | |
| | | | | | | | | |
| | 5* | | 2* | | | | | 4* |
| | | 8* | | | 7* | | | |
| | 1* | | | | | | | |

**Puzzle 73**

$2_{2}9_{3}$b4 $5_{4}8_{5}1_{6}$r1

$5_{8}2_{9}$r4 $1_{19}$r7

$8_{20}$b6

$8_{22}1_{23}6_{24}$r4

$4_{27}$c5: 4r6b8

$1_{30}$b5: 1r6b8

$3_{36}$g

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3* | $8_{5}$ | $5_{4}$ | $4_{7}$ | 6* | $9_{3}$ | 2* | $1_{6}$ | 7* |
| $1_{12}$ | $2_{11}$ | $4_{62}$ | $7_{59}$ | 5* | 8* | $6_{46}$ | $3_{61}$ | $9_{48}$ |
| $6_{54}$ | $7_{56}$ | $9_{63}$ | $3_{60}$ | $2_{2}$ | 1* | $5_{17}$ | $4_{64}$ | $8_{18}$ |
| 4* | 3* | $2_{7}$ | 9* | $7_{25}$ | $5_{8}$ | $8_{22}$ | $6_{24}$ | $1_{23}$ |
| $7_{55}$ | $9_{57}$ | $8_{58}$ | $1_{30}$ | $4_{27}$ | $6_{33}$ | $3_{29}$ | 5* | 2* |
| $5_{14}$ | $1_{31}$ | $6_{34}$ | $8_{32}$ | $3_{28}$ | $2_{13}$ | $9_{45}$ | $7_{43}$ | $4_{44}$ |
| $8_{52}$ | $6_{53}$ | $3_{37}$ | 2* | $1_{19}$ | $7_{41}$ | 4* | $9_{49}$ | $5_{16}$ |
| $2_{10}$ | 5* | $7_{40}$ | $6_{35}$ | $9_{21}$ | $4_{39}$ | $1_{26}$ | 8* | $3_{36}$ |
| $9_{51}$ | $4_{50}$ | 1* | $5_{15}$ | $8_{20}$ | $3_{38}$ | $7_{42}$ | $2_{1}$ | $6_{47}$ |

**Puzzle 74**

$2_1$r1 $3_2$b2 $3_3 1_4$c7

$7_7$r5: 24(58)(69)

$4_{16}$r5 $1_{18}$b5

$4_{20}3_{21}$b3 $1_{29}3_{30}$r7

$8_{34}$b2 $6_{37}$r1 $6_{38}$r8

$5_{47}$c7 $8_{48}$b8 $8_{55}$r

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5* | 3* | $8_{50}$ | $6_{37}$ | $2_1$ | 1* | $9_{49}$ | $4_{25}$ | $7_{27}$ |
| $6_{35}$ | $4_{22}$ | $9_{51}$ | 7* | $8_{61}$ | $3_5$ | 2* | $5_{63}$ | $1_{41}$ |
| $2_{13}$ | $7_{12}$ | $1_{42}$ | $9_{45}$ | $4_{24}$ | $5_{60}$ | $3_3$ | $8_{64}$ | $6_{40}$ |
| 4* | $6_{56}$ | 2* | $3_6$ | $1_{18}$ | $8_{55}$ | $5_{47}$ | $7_{28}$ | $9_{52}$ |
| $8_{34}$ | $1_{31}$ | $5_{26}$ | $4_{16}$ | 9* | $7_7$ | $6_{36}$ | $2_9$ | 3* |
| 7* | $9_{57}$ | $3_2$ | $5_{46}$ | $6_{58}$ | $2_8$ | $8_{48}$ | 1* | $4_{17}$ |
| $1_{29}$ | $5_{21}$ | 6* | 2* | $7_{10}$ | $9_{44}$ | 4* | $3_{30}$ | $8_{43}$ |
| $9_{33}$ | 8* | $7_{11}$ | $8_{39}$ | 3* | $4_{23}$ | $1_4$ | $6_{38}$ | $2_{15}$ |
| $3_{32}$ | $2_{14}$ | $4_{20}$ | $1_{19}$ | $5_{62}$ | $6_{59}$ | 7* | $9_{54}$ | $5_{53}$ |

**Puzzle 75**

$1_1$c1 $4_7$b5 $5_9$b5

$2_{10}6_{11}$r5 $8_{17}5_{18}$b2

$4_{20}$b1 $9_{21}$b3

$8_{24}3_{25}4_{26}6_{27}$b6: 6r8b3

$3_{35}$g $6_{36}$r1

$7_{37}$c2 $5_{44}4_{45}$c7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9* | $7_{37}$ | $3_{38}$ | $5_{29}$ | $6_{36}$ | 4* | 2* | $8_{32}$ | $1_2$ |
| $8_{19}$ | 6* | $2_{13}$ | 1* | $7_{54}$ | $9_{31}$ | $5_{44}$ | $4_{49}$ | $3_{56}$ |
| $1_1$ | $4_{20}$ | $5_{39}$ | $2_{12}$ | $3_{55}$ | $8_{30}$ | $7_{46}$ | $6_{59}$ | $9_{60}$ |
| 2* | $1_6$ | $4_7$ | 3* | 9* | $6_{15}$ | $8_{33}$ | $7_{51}$ | $5_{52}$ |
| $3_{41}$ | $9_{22}$ | $7_{42}$ | 8* | $5_8$ | $2_{10}$ | $6_{11}$ | 1* | 4* |
| 6* | $5_{18}$ | $8_{17}$ | $4_7$ | $1_5$ | $7_{16}$ | $3_{35}$ | $9_{61}$ | $2_{62}$ |
| $7_{51}$ | $2_{14}$ | 1* | $9_{23}$ | $8_{24}$ | 5* | $4_{45}$ | $3_{57}$ | $6_{58}$ |
| $4_{47}$ | $3_{40}$ | $6_{43}$ | $7_{28}$ | 2* | $1_4$ | 9* | $5_{48}$ | $8_{34}$ |
| $5_{50}$ | 8* | $9_{21}$ | $6_{27}$ | $4_{26}$ | $3_{25}$ | $1_3$ | $2_{63}$ | $7_{64}$ |

**Puzzle 76**

$1_1 4_2$r4 $3_3 8_4$b6 $4_5 1_6$r8

$2_{19}$c1: 2r6b8&r7b9

$5_{20}$r2 $6_{25}$c4 $5_{17}$c1

$3_{18}$r7 $6_{28}$r4 $7_{31}$b1

$2_{32}$r3 $7_{36}2_{37}$r9

$9_{39}$r1 $3_{45}$r3 $7_{52}$r6

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | $8_{40}$ | 1* | $2_{34}$ | $3_{27}$ | $9_{39}$ | $5_{24}$ | $7_{32}$ | 6* |
| $3_{55}$ | $6_{30}$ | $9_{54}$ | 7* | $5_{20}$ | $1_8$ | 2* | $4_{15}$ | $8_{47}$ |
| 5* | $2_{33}$ | $7_{31}$ | $4_{14}$ | $6_{26}$ | $8_{42}$ | $1_{10}$ | $9_{46}$ | $3_{45}$ |
| $1_1$ | 7* | $4_2$ | 3* | $8_{59}$ | 2* | $6_{28}$ | $5_{23}$ | $9_{61}$ |
| $2_{19}$ | $3_{41}$ | $8_{56}$ | $6_{25}$ | $9_{60}$ | $5_{21}$ | $7_{62}$ | 1* | 4* |
| $9_{53}$ | $5_{22}$ | $6_{29}$ | $1_{12}$ | $7_{52}$ | $4_{13}$ | $3_{48}$ | $8_{49}$ | $2_{51}$ |
| $8_{57}$ | $1_7$ | $3_{58}$ | 5* | 4* | 6* | $9_{63}$ | $2_{50}$ | $7_{64}$ |
| $6_8$ | $4_5$ | $5_7$ | $9_{35}$ | $2_{44}$ | $7_{43}$ | 8* | 3* | $1_6$ |
| $1_{36}$ | $9_{38}$ | $2_{37}$ | $8_4$ | 1* | $3_3$ | $4_{16}$ | $6_{17}$ | $5_{18}$ |

**Puzzle 77**

$4_{3}$r7 $3_{5}9_{6}$r7 $6_{7}$b3

$5_{11}1_{12}$r5 $5_{13}1_{14}$r7

$7_{16}$g $8_{17}2_{18}$r4 $7_{19}$g

$7_{21}3_{22}$c2 $2_{26}5_{27}$r1

$8_{34}$r1 $3_{37}9_{38}$c4 $5_{44}$c6

$8_{46}$c7: u58c5b6c7b9

$8_{47}$b9: 34(98)(99) $2_{51}$c7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | $9_{23}$ | $8_{34}$ | $1_{35}$ | $2_{26}$ | 7* | 3* | $6_{9}$ | $5_{27}$ |
| $2_{20}$ | 5* | $3_{55}$ | 6* | $9_{57}$ | $8_{45}$ | $1_{52}$ | $4_{58}$ | $7_{25}$ |
| $6_{8}$ | $7_{21}$ | $1_{54}$ | $4_{4}$ | $3_{56}$ | $5_{44}$ | $8_{46}$ | $2_{63}$ | $9_{62}$ |
| $8_{17}$ | $2_{18}$ | 6* | 5* | 1* | $4_{2}$ | $7_{16}$ | $9_{64}$ | $3_{61}$ |
| $5_{11}$ | $1_{12}$ | $4_{1}$ | $3_{37}$ | $7_{42}$ | $9_{43}$ | $6_{10}$ | 8* | 2* |
| $7_{19}$ | $3_{22}$ | $9_{24}$ | $8_{32}$ | $6_{31}$ | $2_{28}$ | 4* | $5_{29}$ | $1_{20}$ |
| $1_{14}$ | $8_{15}$ | $5_{13}$ | 2* | $4_{3}$ | $3_{5}$ | $9_{6}$ | 7* | 6* |
| 3* | 4* | $7_{40}$ | $9_{38}$ | $5_{49}$ | $6_{33}$ | $2_{51}$ | $1_{53}$ | $8_{47}$ |
| 9* | $6_{7}$ | $2_{41}$ | $7_{39}$ | $8_{48}$ | $1_{36}$ | $5_{50}$ | $3_{60}$ | $4_{59}$ |

**Puzzle 78**

$2_{1}1_{2}5_{3}$c2 $4_{9}$b3

$4_{10}$r1:
39(56)(66)→4c5b5

$3_{13}$c4: 3r8b3

$3_{14}$c1: 14(21)(31)

$7_{15}$b3 $9_{16}$g $6_{17}$g

$8_{18}$g $8_{19}$g $9_{20}$c4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | $5_{3}$ | $8_{26}$ | 1* | $7_{28}$ | $4_{10}$ | $3_{27}$ | $9_{22}$ | 2* |
| $1_{45}$ | 7* | $9_{16}$ | $6_{17}$ | 3* | $2_{4}$ | $5_{37}$ | 8* | $4_{47}$ |
| $1_{46}$ | $2_{1}$ | $3_{30}$ | $9_{20}$ | $5_{29}$ | $8_{25}$ | $1_{44}$ | $7_{57}$ | $6_{56}$ |
| $8_{23}$ | 3* | $2_{8}$ | $7_{21}$ | $1_{31}$ | $6_{34}$ | 9* | 4* | $5_{33}$ |
| $9_{59}$ | 4* | $6_{36}$ | 5* | $8_{24}$ | $3_{53}$ | $2_{7}$ | $1_{43}$ | $7_{58}$ |
| $7_{60}$ | $1_{2}$ | $5_{35}$ | 2* | $4_{11}$ | $9_{61}$ | $6_{62}$ | $3_{52}$ | $8_{54}$ |
| 2* | $9_{49}$ | 1* | $4_{12}$ | $6_{32}$ | $7_{42}$ | $8_{51}$ | $5_{38}$ | $3_{48}$ |
| $3_{14}$ | $6_{17}$ | $7_{15}$ | $8_{18}$ | 9* | $5_{39}$ | 4* | $2_{6}$ | $1_{40}$ |
| 5* | $8_{50}$ | $4_{9}$ | $3_{13}$ | $2_{5}$ | $1_{41}$ | $7_{63}$ | $6_{64}$ | $9_{55}$ |

$8_{23}$g $8_{24}$b5: 39(56)(66) $3_{27}$r1 $1_{31}$c5 $5_{33}$r4 $9_{44}$r7 $5_{56}$c9

**Puzzle 79**

$1_{1}5_{2}8_{3}$c1 $3_{7}$b1:
15(13)(33)

$4_{8}$r2 $1_{12}$r8 $2_{18}$b3

$3_{19}$b5 $6_{21}$c2 $8_{23}$b4

$7_{27}2_{28}$c5 $7_{27}2_{28}$c5

$5_{30}6_{31}$r4 $7_{40}3_{41}$r7

$5_{48}$b9 $6_{53}$c8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | $9_{36}$ | $1_{15}$ | 3* | 4* | $5_{17}$ | $2_{39}$ | $7_{38}$ | $8_{26}$ |
| $8_{3}$ | $4_{8}$ | $3_{7}$ | $6_{34}$ | $7_{27}$ | $1_{29}$ | 5* | 1* | $9_{35}$ |
| 2* | $7_{37}$ | $5_{16}$ | $2_{14}$ | $8_{23}$ | $9_{33}$ | $4_{55}$ | $6_{53}$ | $3_{56}$ |
| 4* | 3* | $9_{32}$ | $5_{30}$ | $2_{28}$ | $6_{31}$ | 7* | $8_{6}$ | $1_{4}$ |
| $5_{2}$ | $6_{21}$ | $7_{62}$ | $9_{63}$ | 1* | 8* | $3_{57}$ | $4_{54}$ | $2_{60}$ |
| $1_{1}$ | $8_{5}$ | $2_{61}$ | $7_{64}$ | $3_{19}$ | $4_{11}$ | $9_{58}$ | $5_{52}$ | $6_{59}$ |
| $7_{40}$ | 5* | $6_{22}$ | 2* | $9_{42}$ | $1_{13}$ | $8_{25}$ | $3_{41}$ | 4* |
| $3_{43}$ | $2_{18}$ | 8* | $4_{10}$ | $6_{50}$ | 7* | $1_{12}$ | $9_{45}$ | $5_{48}$ |
| $9_{44}$ | 1* | $4_{9}$ | $8_{24}$ | $5_{51}$ | $3_{20}$ | $6_{49}$ | $2_{47}$ | $7_{46}$ |

PRACTICE SET 9

**Puzzle 80**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 3* | | | 5* | 9* | |
| | 1* | | | | | | 7* | |
| 4* | | | | 6* | | | | |
| | | | 5* | | | 3* | | 4* |
| | | 1* | | | | | | |
| | | | 2* | | | | | |
| | 6* | | | 1* | 7* | | | |
| 3* | | | | | | 8* | | |
| | | | | | | | | |

**Puzzle 81**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | 1* | | | 3* | | | | |
| | | 4* | | | | | 6* | 5* |
| | | | | | | | | |
| | 8* | | 2* | | | 3* | | |
| | | | 6* | | 4* | | | |
| | | | | | 5* | | | |
| | | | | 7* | | 1* | | |
| 6* | | 5* | | | | | | |
| 3* | | | | | | | | |

**Puzzle 82**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 8* | | | | 1* | | | |
| | | | | | | 6* | 3* | |
| | | | | | | | | |
| | | | 3* | 4* | | 2* | | |
| | 7* | | | | | | 1* | 5* |
| | | | 6* | | | | | |
| | | | | 8* | | | | 1* |
| 3* | | 2* | | | | | | |
| 6* | | | | | | 4* | | |

**Puzzle 83**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 6* | | | | 1* |
| | 2* | | 5* | | | | | |
| | | | | | | | | |
| | | | 3* | | | | 2* | |
| | | 7* | | | | 4* | | |
| 1* | | | | 5* | | | | |
| 7* | 3* | | | | 1* | | | |
| 6* | | | | | | | 5* | |
| | | | 2* | | | 8* | | |

**Puzzle 84**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 7* | 1* | | 5* | | | |
| | 6* | 3* | | | | 7* | | |
| | | | | | | | | |
| | | | | 6* | | 3* | | 2* |
| 1* | | | | | | | | |
| | | | 5* | | | | | |
| 5* | 2* | | 4* | | | | 8* | |
| | | | | 3* | | | | |
| 9* | | | | | | | | |

**Puzzle 85**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 3* | | 7* | | |
| | 1* | | 6* | | | | | |
| | | | | | | 2* | | |
| | | | 1* | | | | 8* | 9* |
| 2* | | | | | | | | |
| 5* | | | | | | | | |
| | 9* | 6* | | | | | 1* | |
| | | | 4* | | 2* | 5* | | |
| | | | | 7* | | | | |

**Puzzle 86**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 4* | | 7* | | |
| | | | 5* | | | 3* | | |
| | 1* | 8* | | | | | | |
| 3* | | | | | | 5* | 4* | |
| | | 6* | 1* | | | | | |
| | | | 2* | | | | | |
| 5* | 2* | | | 3* | | | | |
| | | | | | | | | 1* |
| | | | | | | | 6* | |

**Puzzle 80**

$1_{2}$r1: 1c6b5     $4_{7}$b7

$2_{8}$g     $6_{9}$r7     $9_{10}$g

$7_{11}$b4     $7_{14}$b8     $3_{20}$c8

$3_{22}$r7     $5_{24}$g     $8_{26}$g

$6_{27}4_{28}$c4

$2_{31}$c5: u25c5b6c7b9

$3_{33}$g     $5_{37}$r7     $4_{45}$b2     $2_{47}$r5

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $6_{58}$ | $8_{62}$ | $2_{61}$ | 3* | $7_{11}$ | $4_{12}$ | 5* | 9* | $1_{2}$ |
| $5_{25}$ | 1* | $3_{23}$ | $9_{29}$ | $2_{31}$ | $8_{32}$ | $4_{7}$ | 7* | $6_{9}$ |
| 4* | $7_{55}$ | $9_{56}$ | $1_{6}$ | 6* | $5_{24}$ | $2_{8}$ | $3_{20}$ | $8_{21}$ |
| $7_{57}$ | $2_{60}$ | $6_{59}$ | 5* | $9_{52}$ | $1_{5}$ | 3* | $8_{49}$ | 4* |
| $8_{48}$ | $5_{39}$ | 1* | $7_{13}$ | $4_{46}$ | $3_{33}$ | $6_{16}$ | $2_{47}$ | $9_{19}$ |
| $9_{49}$ | $3_{34}$ | $4_{45}$ | 2* | $8_{50}$ | $6_{30}$ | $1_{4}$ | $5_{40}$ | $7_{14}$ |
| $2_{38}$ | 6* | $5_{37}$ | $8_{26}$ | 1* | 7* | $9_{10}$ | $4_{18}$ | $3_{22}$ |
| 3* | $4_{53}$ | $7_{54}$ | $6_{27}$ | $5_{36}$ | $9_{44}$ | 8* | $1_{3}$ | $2_{42}$ |
| $1_{1}$ | $9_{63}$ | $8_{64}$ | $4_{28}$ | $3_{35}$ | $2_{43}$ | $7_{15}$ | $6_{17}$ | $5_{41}$ |

**Puzzle 81**

$1_{1}$b3 $3_{2}$r2

$3_{3}$r5 $5_{5}$r1

$6_{11}4_{12}2_{13}$c5

$4_{20}$r4 $7_{23}$b5

$1_{25}$r4 $1_{27}$c4

$1_{28}$r2 $4_{29}$c7

$7_{32}9_{33}$c1 $3_{37}$c3 $3_{42}$r7 $1_{45}$c7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | 1* | $6_{15}$ | $5_{5}$ | 3* | $8_{37}$ | $4_{29}$ | $7_{53}$ | $9_{52}$ |
| $9_{33}$ | $3_{2}$ | 4* | $7_{24}$ | $2_{13}$ | $1_{28}$ | $8_{36}$ | 6* | 5* |
| $7_{32}$ | $5_{10}$ | $8_{34}$ | $4_{14}$ | $6_{11}$ | $9_{38}$ | $2_{55}$ | $3_{51}$ | $1_{54}$ |
| $5_{9}$ | 8* | $9_{26}$ | 2* | $1_{25}$ | $7_{23}$ | 3* | $4_{20}$ | $6_{17}$ |
| $1_{22}$ | $2_{40}$ | $3_{3}$ | 6* | $9_{58}$ | 4* | $5_{8}$ | $8_{62}$ | $7_{63}$ |
| $4_{21}$ | $6_{16}$ | $7_{39}$ | $3_{4}$ | $8_{59}$ | 5* | $9_{56}$ | $1_{57}$ | $2_{60}$ |
| $8_{35}$ | $4_{31}$ | $2_{41}$ | $9_{43}$ | 7* | $6_{19}$ | 1* | $5_{7}$ | $3_{42}$ |
| 6* | $9_{47}$ | 5* | $1_{27}$ | $4_{12}$ | $3_{48}$ | $7_{45}$ | $2_{61}$ | $8_{64}$ |
| 3* | $7_{46}$ | $1_{1}$ | $8_{44}$ | $5_{6}$ | $2_{49}$ | $6_{18}$ | $9_{50}$ | $4_{30}$ |

**Puzzle 82**

$1_{2}$b5 $6_{3}4_{4}3_{5}$r5

$2_{16}$b9 $2_{18}$c2

$2_{20}$b4 $4_{23}$b6

$4_{27}$b1 $8_{31}$b3

$8_{35}$r5: 8c4b4 $9_{37}$b4

$9_{39}$c2 $4_{47}$c3

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_{52}$ | 8* | $3_{11}$ | $7_{57}$ | $6_{9}$ | 1* | $9_{58}$ | $4_{29}$ | $2_{22}$ |
| $1_{34}$ | $2_{18}$ | $7_{48}$ | $5_{54}$ | $9_{37}$ | $4_{26}$ | 6* | 3* | $8_{55}$ |
| $9_{53}$ | $6_{8}$ | $4_{27}$ | $8_{56}$ | $3_{10}$ | $2_{20}$ | $1_{1}$ | $5_{51}$ | $7_{59}$ |
| $8_{32}$ | $9_{39}$ | $1_{33}$ | 3* | 4* | $5_{44}$ | 2* | $7_{46}$ | $6_{15}$ |
| $4_{4}$ | 7* | $6_{3}$ | $9_{36}$ | $2_{21}$ | $8_{35}$ | $3_{5}$ | 1* | 5* |
| $2_{19}$ | $3_{12}$ | $4_{40}$ | 6* | $1_{2}$ | $7_{45}$ | $8_{62}$ | $9_{63}$ | $4_{30}$ |
| $7_{49}$ | $4_{28}$ | $9_{47}$ | $2_{17}$ | 8* | $3_{7}$ | $5_{50}$ | $6_{14}$ | 1* |
| 3* | $1_{25}$ | 2* | $4_{23}$ | $5_{43}$ | $6_{13}$ | $7_{61}$ | $8_{64}$ | $9_{60}$ |
| 6* | $9_{41}$ | $8_{31}$ | $1_{24}$ | $7_{42}$ | $9_{38}$ | 4* | $2_{16}$ | $3_{6}$ |

**Puzzle 83**

$2_{3}8_{4}$b3 $1_{10}$r8

$3_{16}$b2 $3_{18}$b9

$3_{19}$r1$5_{22}$r1

$6_{23}$r9 $4_{26}$r7

$6_{30}8_{31}$r5 $5_{35}$c9

$6_{39}7_{40}$r2 $4_{50}8_{51}$r1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $5_{22}$ | $7_{52}$ | $4_{50}$ | $9_{45}$ | 6* | $3_{19}$ | $2_{8}$ | $8_{51}$ | 1* |
| $8_{57}$ | 2* | $1_{14}$ | 5* | $4_{33}$ | $7_{40}$ | $3_{37}$ | $6_{38}$ | $9_{56}$ |
| $3_{39}$ | $9_{63}$ | $6_{62}$ | $1_{13}$ | $2_{7}$ | $8_{46}$ | $5_{36}$ | $7_{55}$ | $4_{34}$ |
| $9_{58}$ | $6_{64}$ | $8_{61}$ | 3* | $1_{12}$ | $4_{42}$ | $7_{48}$ | 2* | $5_{35}$ |
| $2_{5}$ | $5_{21}$ | 7* | $8_{31}$ | $9_{32}$ | $6_{30}$ | 4* | $1_{11}$ | $3_{17}$ |
| 1* | $4_{43}$ | $3_{16}$ | $7_{41}$ | 5* | $2_{6}$ | $6_{49}$ | $9_{59}$ | $8_{60}$ |
| 7* | 3* | $5_{2}$ | $6_{24}$ | $8_{25}$ | 1* | $9_{27}$ | $4_{26}$ | $2_{9}$ |
| 6* | $8_{4}$ | $2_{3}$ | $4_{44}$ | $3_{20}$ | $9_{47}$ | $1_{10}$ | 5* | $7_{28}$ |
| $4_{53}$ | $1_{15}$ | $9_{54}$ | 2* | $7_{29}$ | $5_{1}$ | 8* | $3_{18}$ | $6_{23}$ |

**Puzzle 84**

$3_{2}6_{3}7_{4}$c1    $1_{8}$g    $1_{10}$c5

$3_{13}$r1    $5_{14}9_{15}$b1

$5_{24}$g: 48c3b3    $5_{18}4_{19}$r4

$9_{21}$g:
18(86)(96)→2c4b6

$2_{22}$g

$7_{23}$r3:
6(17)(19)→367r3b4    $2_{24}9_{25}$c5: 48(15)(25)    $7_{28}$r6    $6_{30}$r1    $7_{32}$r6    $4_{50}$c8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $2_{45}$ | $9_{15}$ | 7* | 1* | $8_{63}$ | 5* | $4_{64}$ | $3_{13}$ | $6_{30}$ |
| $8_{61}$ | 6* | 3* | $9_{21}$ | $4_{62}$ | $2_{43}$ | 7* | $5_{16}$ | $1_{12}$ |
| $4_{60}$ | $1_{9}$ | $5_{14}$ | $3_{36}$ | $7_{23}$ | $6_{42}$ | $9_{59}$ | $2_{44}$ | $8_{56}$ |
| $7_{4}$ | $5_{18}$ | $9_{17}$ | $8_{20}$ | 6* | $9_{19}$ | 3* | $1_{11}$ | 2* |
| 1* | $8_{54}$ | $6_{27}$ | $7_{33}$ | $2_{24}$ | $3_{34}$ | $5_{58}$ | $4_{50}$ | $8_{57}$ |
| $3_{2}$ | $4_{63}$ | $2_{26}$ | 5* | $1_{10}$ | $9_{35}$ | $8_{55}$ | $6_{31}$ | $7_{32}$ |
| 5* | 2* | $1_{8}$ | 4* | $9_{25}$ | $7_{28}$ | $6_{29}$ | 8* | $3_{6}$ |
| $6_{3}$ | $7_{7}$ | $4_{48}$ | $2_{22}$ | 3* | $8_{41}$ | $1_{39}$ | $9_{51}$ | $5_{52}$ |
| 9* | $3_{5}$ | $8_{47}$ | $6_{37}$ | $5_{1}$ | $1_{40}$ | $2_{38}$ | $7_{46}$ | $4_{49}$ |

**Puzzle 85**

$2_{1}$r4    $5_{9}$b3    $7_{10}4_{11}$r7

$1_{14}$r9    $1_{16}$r1    $6_{17}9_{18}$b6

$5_{23}4_{24}$b8    $7_{26}$r2    $8_{28}$c7

$9_{29}$b2    $8_{30}$b3: 8c2b2

$3_{32}$c7: 3r4b2    $3_{34}$g

$7_{36}$c3    $6_{41}$g    $9_{47}$c4

$5_{51}$b4    $9_{56}$c5    $8_{61}$r6

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $9_{48}$ | $5_{40}$ | $4_{37}$ | $2_{4}$ | 3* | $8_{50}$ | 7* | $6_{41}$ | $1_{16}$ |
| $8_{49}$ | 1* | $2_{5}$ | 6* | $5_{51}$ | $7_{26}$ | $9_{22}$ | $4_{53}$ | $3_{45}$ |
| $6_{42}$ | $7_{39}$ | $3_{35}$ | $9_{47}$ | $1_{15}$ | $4_{52}$ | 2* | $5_{54}$ | $8_{46}$ |
| $3_{34}$ | $4_{38}$ | $7_{36}$ | 1* | $2_{3}$ | $5_{25}$ | $6_{33}$ | 8* | 9* |
| 2* | $6_{62}$ | $9_{29}$ | $3_{59}$ | $4_{55}$ | $8_{63}$ | $1_{2}$ | $7_{21}$ | $5_{23}$ |
| 5* | $8_{61}$ | $1_{1}$ | $7_{27}$ | $9_{56}$ | $6_{64}$ | $3_{32}$ | $2_{8}$ | $4_{24}$ |
| $7_{10}$ | 9* | 6* | $5_{58}$ | $8_{57}$ | $3_{60}$ | $4_{11}$ | 1* | $2_{7}$ |
| $1_{13}$ | $3_{31}$ | $8_{30}$ | 4* | $6_{17}$ | 2* | 5* | $9_{19}$ | $7_{20}$ |
| $4_{12}$ | $2_{6}$ | $5_{9}$ | $9_{18}$ | 7* | $1_{14}$ | $8_{28}$ | $3_{44}$ | $6_{43}$ |

**Puzzle 86**

$5_{3}$r7    $4_{9}$c2    $6_{10}$b1

$7_{11}3_{12}$c4: 7r6b8

$1_{20}$b5    $2_{21}$b9    $3_{22}$c9

$3_{23}$c2    $4_{24}$c6    $4_{28}$b8

$1_{32}$r5    $1_{34}$r4: 1c2b1

$1_{37}$r7    $2_{40}$c2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $8_{51}$ | $3_{23}$ | $2_{52}$ | $5_{7}$ | 4* | 1* | 7* | 6* | $7_{17}$ |
| $6_{10}$ | $1_{39}$ | 7* | 5* | $2_{57}$ | $4_{24}$ | 3* | $5_{6}$ | $9_{44}$ |
| 4* | 1* | 8* | $7_{11}$ | $6_{14}$ | $3_{13}$ | $2_{27}$ | $1_{38}$ | $8_{43}$ |
| 3* | $2_{40}$ | $1_{34}$ | 4* | 5* | $8_{45}$ | 5* | 4* | $3_{22}$ |
| $3_{30}$ | 8* | 6* | 1* | $7_{19}$ | $9_{33}$ | $1_{32}$ | 2* | $5_{5}$ |
| $9_{42}$ | $5_{4}$ | $6_{35}$ | 2* | $1_{20}$ | $2_{46}$ | $8_{48}$ | $7_{18}$ | $4_{28}$ |
| 5* | 2* | $8_{50}$ | 2* | 3* | $7_{2}$ | $9_{49}$ | $4_{31}$ | $1_{37}$ |
| $1_{55}$ | $7_{1}$ | $3_{62}$ | $9_{59}$ | $4_{25}$ | $6_{15}$ | 5* | $8_{61}$ | 1* |
| $2_{54}$ | $4_{9}$ | $9_{63}$ | $1_{56}$ | $8_{60}$ | $5_{8}$ | 7* | 6* | $6_{16}$ |

PRACTICE SET 10

**Puzzle 87**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | 3* | | | | | 5* | | |
| | | | 4* | | | | | 8* |
| | | | | | | | | |
| | | | | 6* | 1* | | | |
| | | | | | | | 4* | 2* |
| | 8* | | | | | | | |
| 3* | | 4* | 2* | | | | | |
| | | | | 5* | | 8* | | |
| 7* | | | | | | | 1* | |

**Puzzle 88**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7* | | | 5* | | | 6* | 8* | |
| 2* | | | | | | | | |
| | | | | 1* | | | | |
| | 9* | | 2* | | | 5* | | |
| | | | 7* | | | 2* | | |
| | 1* | | | | | | | |
| | | | | 3* | | | 1* | 9* |
| | | | | | | | 3* | |
| 4* | | | | | | | | |

**Puzzle 89**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 4* | | | | 3* | 8* |
| | 5* | 1* | | | | | | |
| | | | | | | | | |
| | | | | 1* | | 5* | | 7* |
| 7* | | | 3* | | | | | |
| | 2* | | | | | 1* | | |
| 3* | | | 8* | | | | 6* | |
| 2* | | | | | | | | |
| | | | | | | 9* | | |

**Puzzle 90**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | 1* | | | 2* | | | | |
| | | | | | | | 3* | 9* |
| | | 7* | | | | | 4* | |
| 2* | | | | | | 8* | | 5* |
| | | | 4* | | | | | |
| | 9* | | | | | | | |
| | | | 1* | | 4* | | | |
| 5* | | | | | | 2* | | |
| | | | 3* | | | | | |

**Puzzle 91**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | | | 5* | | 6* | 3* | | |
| | | | 7* | | | | 4* | |
| | | | | | | | | |
| | 4* | | | | | | 2* | |
| | 3* | | | 8* | | | | |
| 5* | | | 1* | | | | | |
| | | | | 4* | | 9* | | |
| 7* | | | | | | | | 5* |
| | | | | 2* | | | | |

**Puzzle 92**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | | | | | 1* | 9* |
| | 6* | | 2* | | | | | |
| | | | | | | | 7* | |
| | | | 8* | 3* | | 2* | | |
| 7* | | | | | | 3* | | |
| | | 1* | | | | | | |
| | 8* | | | 6* | | 5* | | |
| | | | | | 7* | | 4* | |
| | | | | | | | | |

**Puzzle 93**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | 4* | | | | | | | |
| | | | 3* | | | 7* | | |
| | | | | | | | | |
| | | | | 4* | 5* | | 6* | |
| 7* | | | | | | 8* | | |
| | | 3* | | 2* | | | | |
| 1* | | | 7* | | | | | |
| | 6* | | | | | | 5* | |
| | | | | | 2* | | | 4* |

**Puzzle 87**

$4_2$r1 $2_6$b5

$2_7$b9 $5_8$b3

$5_{17}$b3 $3_{20}$b9

$5_{21}$c1 $7_{23}$r8

$1_{24}8_{25}$b6 $8_{26}1_{27}$r1

$6_{35}$c4 $7_{36}9_{37}$b5

$7_{50}$r1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | 3* | $1_{27}$ | $8_{26}$ | $2_{16}$ | $9_{51}$ | 5* | $7_{57}$ | $4_2$ |
| $2_{14}$ | $9_{45}$ | $5_{59}$ | 4* | $7_{61}$ | $6_{37}$ | $1_{29}$ | $3_{63}$ | 8* |
| $8_{10}$ | $4_{11}$ | $7_{60}$ | $1_{28}$ | $3_{62}$ | $5_{58}$ | $2_{15}$ | $9_{64}$ | $6_{55}$ |
| $4_{12}$ | $7_{42}$ | $2_{13}$ | $5_{22}$ | 6* | 1* | $3_{47}$ | $8_1$ | $9_{48}$ |
| $5_{21}$ | $1_{31}$ | $3_{39}$ | $9_{37}$ | $8_{34}$ | $7_{36}$ | $6_{40}$ | 4* | 2* |
| $9_{33}$ | 8* | $6_{41}$ | $3_{38}$ | $4_5$ | $2_6$ | $7_{43}$ | $5_{19}$ | $1_{30}$ |
| 3* | $5_{17}$ | 4* | 2* | $1_{24}$ | $8_{25}$ | $9_{49}$ | $6_{56}$ | $7_{54}$ |
| $1_{32}$ | $6_{44}$ | $9_{46}$ | $7_{23}$ | 5* | $4_4$ | 8* | $2_7$ | $3_{20}$ |
| 7* | $2_9$ | $8_8$ | $6_{35}$ | $9_{52}$ | $3_{53}$ | $4_3$ | 1* | $5_{18}$ |

**Puzzle 88**

$1_2 9_3$c1 $1_4$r1: 1c9b8

$1_5$c4 $1_6 9_7 3_8$c7

$7_{10}$b2 $9_{11}$c4

$2_{23}$r1 $5_{30}$c8

$7_{32} 4_{33}$r3 $4_{36}$r6: 4c6b6

$5_{40}$r5 $8_{50}$c4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7* | $4_{24}$ | $1_4$ | 5* | $9_{17}$ | $3_{15}$ | 6* | 8* | $2_{23}$ |
| 2* | $5_{31}$ | $3_{14}$ | $8_{50}$ | $6_{52}$ | $7_{53}$ | $1_6$ | $9_9$ | $4_{34}$ |
| $9_3$ | $6_{48}$ | $8_{47}$ | $4_{33}$ | 1* | $2_{25}$ | $3_8$ | $5_{30}$ | $7_{32}$ |
| $6_{43}$ | 9* | $7_{10}$ | 2* | $8_{42}$ | $1_{19}$ | 5* | $4_{37}$ | $3_{21}$ |
| $3_{22}$ | $8_{41}$ | $4_{29}$ | 7* | $5_{40}$ | $9_{18}$ | 2* | $6_{38}$ | $1_{20}$ |
| $5_{44}$ | 1* | $2_1$ | $3_{16}$ | $4_{36}$ | $6_{46}$ | $9_7$ | $7_{35}$ | $8_{39}$ |
| $8_{45}$ | $2_{28}$ | $5_{60}$ | $6_{51}$ | 3* | $4_{61}$ | $7_{55}$ | 1* | 9* |
| $1_2$ | $7_{49}$ | $6_{59}$ | $9_{11}$ | $2_{27}$ | $8_{58}$ | $4_{56}$ | 3* | $5_{63}$ |
| 4* | $3_{13}$ | $9_{12}$ | $1_5$ | $7_{54}$ | $5_{62}$ | $8_{57}$ | $2_{26}$ | $6_{64}$ |

**Puzzle 89**

$1_3 5_4$r1 $2_9$b1: 34(32)(33)

$3_{10}$b8 $5_{12}$c1 $6_{17}$g

$8_{19}$c7 $7_{20}$r8 $2_{23}$b9

$7_{26} 9_{27}$c1: 34r3b1→68(21)(31)

$4_{31}$g $3_{35}$b1 $7_{37}$c2

$4_{43}$r4 $9_{48}$r7 $7_{52}$c4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $7_{26}$ | $9_{28}$ | $2_9$ | 4* | $5_4$ | $1_3$ | $6_{17}$ | 3* | 8* |
| $8_{55}$ | 5* | 1* | $6_{53}$ | $3_{56}$ | $2_{50}$ | $4_{33}$ | $7_{20}$ | $9_{25}$ |
| $6_{54}$ | $3_{35}$ | $4_{32}$ | $9_{51}$ | $8_{63}$ | $7_{62}$ | $2_{34}$ | $1_7$ | $5_8$ |
| $9_{27}$ | $6_{38}$ | $3_{36}$ | $2_{44}$ | 1* | $8_{42}$ | 5* | $4_{43}$ | 7* |
| 4* | $1_1$ | $7_{41}$ | 3* | $9_{46}$ | $5_{13}$ | $8_{19}$ | $2_{45}$ | $6_{18}$ |
| $5_{12}$ | 2* | $8_{40}$ | $7_{52}$ | $6_{60}$ | $4_{59}$ | 1* | $9_{47}$ | $3_{10}$ |
| 3* | $4_{31}$ | $5_{16}$ | 8* | $2_{49}$ | $9_{48}$ | $7_{21}$ | 6* | $1_6$ |
| 2* | $8_{30}$ | $9_{29}$ | $1_5$ | $7_{64}$ | $6_{61}$ | $3_{11}$ | $5_{15}$ | $4_{24}$ |
| $1_2$ | $7_{37}$ | $6_{39}$ | $5_{14}$ | $4_{58}$ | $3_{57}$ | 9* | $8_{22}$ | $2_{23}$ |

**Puzzle 90**

$2_3$c4 $2_5$b7 $3_7 9_8$r1

$4_{11} 3_{12}$r4: 3r7b9→3c1b2

$5_{16} 8_{17}$c4: 58r2b1 $8_{18}$g

$7_{19}$g: 3r7b9 $5_{21} 2_{22} 8_{23}$c2

$6_{26}$g: 13c1b2 $6_{27}$c4

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6* | 1* | $4_2$ | $9_8$ | 2* | $3_7$ | $7_{54}$ | $5_{61}$ | $8_{62}$ |
| $8_{18}$ | $5_{21}$ | $2_{25}$ | $6_{27}$ | $4_1$ | $7_{30}$ | $1_{31}$ | 3* | 9* |
| $9_9$ | $3_{10}$ | 7* | $5_{16}$ | $1_{38}$ | $8_{39}$ | $6_{29}$ | 4* | $2_5$ |
| 2* | $4_{11}$ | $6_{26}$ | $7_{28}$ | $3_{12}$ | $1_{36}$ | 8* | $9_{37}$ | 5* |
| $1_{50}$ | $7_{20}$ | $8_{45}$ | 4* | $9_{43}$ | $5_{47}$ | $3_{52}$ | $2_6$ | $6_{49}$ |
| $4_{51}$ | 9* | $5_{46}$ | $2_3$ | $8_{44}$ | $6_{48}$ | $4_{15}$ | $7_{60}$ | $1_{59}$ |
| $7_{19}$ | $2_{22}$ | $9_{32}$ | 1* | $6_{57}$ | 4* | $5_{55}$ | $8_{58}$ | $3_{53}$ |
| 5* | $6_{24}$ | $3_{35}$ | $8_{17}$ | $7_{41}$ | $9_{42}$ | 2* | $1_{35}$ | $4_{14}$ |
| $4_{13}$ | $8_{23}$ | $1_{34}$ | 3* | $5_{56}$ | $2_4$ | $9_{40}$ | $6_{63}$ | $7_{64}$ |

$7_{30}$b4 $9_{32}$g: 3r7b9 $3_{33}$b3 $1_{36}$r4 $7_{41}$r8 $1_{50}$r5 $7_{54}$c7

**Puzzle 91**

$4_{1}$r1 $2_{4}$b9

$2_{5}$r1 $5_{14}7_{15}6_{16}$c5

$3_{18}$b5 $3_{22}9_{23}$r8

$8_{27}$r8 $8_{29}7_{30}$r1

$9_{32}5_{33}$c8 $7_{38}$c9

$1_{42}$r4 $8_{48}1_{49}$c2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | $2_{5}$ | $4_{1}$ | 5* | $9_{31}$ | 6* | 3* | $7_{30}$ | $8_{29}$ |
| $9_{37}$ | $5_{50}$ | $3_{62}$ | 7* | $1_{61}$ | $8_{26}$ | $2_{10}$ | 4* | $6_{59}$ |
| $6_{54}$ | $7_{40}$ | $8_{63}$ | $2_{9}$ | $3_{64}$ | $4_{11}$ | $5_{34}$ | $9_{32}$ | $1_{60}$ |
| $8_{43}$ | 4* | $7_{39}$ | $6_{17}$ | $5_{14}$ | $3_{18}$ | $1_{42}$ | 2* | $9_{35}$ |
| $2_{7}$ | 3* | $1_{44}$ | $4_{12}$ | 8* | $9_{19}$ | $6_{46}$ | $5_{33}$ | $7_{38}$ |
| 5* | $6_{45}$ | $9_{36}$ | 1* | $7_{15}$ | $2_{8}$ | $8_{47}$ | $3_{20}$ | $4_{13}$ |
| $3_{55}$ | $1_{49}$ | $5_{56}$ | $8_{25}$ | 4* | $7_{57}$ | 9* | $6_{52}$ | $2_{4}$ |
| 7* | $9_{23}$ | $2_{6}$ | $3_{22}$ | $6_{16}$ | $1_{28}$ | $4_{3}$ | $8_{27}$ | 5* |
| $4_{2}$ | $8_{48}$ | $6_{53}$ | $9_{24}$ | 2* | $5_{58}$ | $7_{41}$ | $1_{51}$ | $3_{21}$ |

**Puzzle 92**

$2_{1}$b7 $7_{2}1_{3}9_{4}$c7:

7r6b5 $1_{13}$r2 $3_{17}$g

$4_{19}$b8 $4_{20}$b3 $4_{22}$c5

$5_{26}$b7 $9_{27}$b5: 9c6b4

$6_{28}$c4 $3_{31}$r1

$2_{33}$g $3_{45}9_{46}$c3

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | $7_{6}$ | $2_{7}$ | $6_{28}$ | $5_{32}$ | $3_{31}$ | $8_{30}$ | 1* | 9* |
| $1_{13}$ | 6* | $8_{48}$ | 2* | $7_{8}$ | $9_{51}$ | $4_{24}$ | $3_{18}$ | $5_{26}$ |
| $5_{50}$ | $9_{49}$ | $3_{45}$ | $4_{23}$ | $1_{16}$ | $8_{52}$ | $6_{29}$ | 7* | $2_{1}$ |
| $6_{59}$ | $4_{25}$ | $9_{46}$ | 8* | 3* | $1_{12}$ | 2* | $5_{60}$ | $7_{10}$ |
| 7* | $2_{34}$ | $5_{47}$ | $9_{27}$ | $4_{22}$ | $6_{62}$ | 3* | $8_{63}$ | $1_{11}$ |
| $8_{58}$ | $3_{54}$ | 1* | $7_{9}$ | $2_{33}$ | $5_{61}$ | $9_{4}$ | $6_{64}$ | $4_{19}$ |
| $9_{38}$ | 8* | $7_{5}$ | $1_{15}$ | 6* | $4_{21}$ | 5* | $2_{37}$ | $3_{17}$ |
| $2_{36}$ | $5_{53}$ | $6_{44}$ | $3_{56}$ | $9_{40}$ | 7* | $1_{3}$ | 4* | $8_{42}$ |
| $3_{55}$ | $1_{14}$ | $4_{20}$ | $5_{57}$ | $8_{41}$ | $2_{35}$ | $7_{2}$ | $9_{39}$ | $6_{43}$ |

**Puzzle 93**

$7_{2}$b5 $3_{8}$b1 $3_{9}$r4

$2_{10}$r8:
2(28)(29)/(58)(59)

$4_{11}$c4 $2_{19}$b8

$6_{22}$c1 $8_{25}$b5

$5_{27}1_{28}$b2 $5_{32}$c1

$9_{36}$c4 $8_{39}$r9

$5_{48}$c7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2* | 4* | $8_{54}$ | $5_{35}$ | $7_{7}$ | $9_{53}$ | $1_{49}$ | $3_{44}$ | $6_{43}$ |
| $6_{22}$ | $9_{30}$ | $5_{62}$ | 3* | $1_{58}$ | $4_{18}$ | 7* | $2_{21}$ | $8_{63}$ |
| $3_{8}$ | $7_{6}$ | $1_{61}$ | $2_{1}$ | $6_{42}$ | $8_{60}$ | $4_{17}$ | $9_{51}$ | $5_{64}$ |
| $9_{29}$ | $1_{28}$ | $2_{20}$ | $8_{25}$ | 4* | 5* | $3_{9}$ | 6* | $7_{3}$ |
| 7* | $5_{27}$ | $6_{23}$ | $9_{36}$ | $3_{57}$ | $1_{59}$ | 8* | $4_{16}$ | $2_{19}$ |
| $4_{15}$ | $8_{26}$ | 3* | $6_{24}$ | 2* | $7_{2}$ | $5_{48}$ | $1_{50}$ | $9_{52}$ |
| 1* | $2_{13}$ | $4_{12}$ | 7* | $5_{34}$ | $6_{41}$ | $9_{47}$ | $8_{46}$ | $3_{45}$ |
| $8_{33}$ | 6* | $7_{5}$ | $4_{11}$ | $9_{55}$ | $3_{56}$ | $2_{10}$ | 5* | $1_{38}$ |
| $5_{32}$ | $3_{14}$ | $9_{31}$ | $1_{37}$ | $8_{39}$ | 2* | $6_{40}$ | $7_{4}$ | 4* |

**Puzzle 94**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | 1* | | 6* | |
| | | 7* | | | | 3* | | |
| 4* | | 5* | | | | | | |
| 7* | | | 4* | 5* | | | | |
| | 8* | | 6* | | | | 2* | |
| | | | | | | | | |
| | 6* | | 2* | 3* | | | | |
| | | | | | | 5* | | |
| | | | | | | 7* | | |

**Puzzle 95**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 3* | 5* | | | 4* | |
| | 7* | 6* | | | | | | |
| | | | | | | | | |
| | | 5* | | | | 7* | | 6* |
| 3* | | | | 2* | | | | |
| | | | | | | 1* | | |
| | 1* | | 7* | | 6* | | | |
| 8* | | | | | | | 2* | |
| | | | 5* | | | | | |

**Puzzle 96**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | | 6* | | 3* | | | | |
| | | | | | 5* | 4* | | |
| 7* | | | | | | | | |
| | 2* | | 8* | | 4* | | | |
| | | | | | | | 7* | 1* |
| | | | 2* | | | | | |
| 5* | | | | 1* | | | 6* | |
| | 4* | | | | | 8* | | |
| | | | | | | | | |

**Puzzle 97**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 6* | | | | 1* | | | |
| 2* | | | | | | | 7* | |
| | | | | | | | | |
| 3* | | | 2* | 6* | | | | |
| 8* | | | | | | | 1* | |
| | | | 3* | | | 4* | | |
| | 4* | | 5* | | | | | 8* |
| | 1* | | | 7* | | | | |
| | | | | | | 3* | | |

**Puzzle 98**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 5* | | | 2* | |
| 6* | | | | | | 4* | | |
| 3* | | | | | | | | |
| | 8* | 2* | 1* | | | | | |
| | | | | | | | 5* | |
| | | | | | | | | 6* |
| | 5* | 4* | | | | 2* | | |
| | | | 3* | | 6* | 1* | | |
| | | | 7* | | | | | |

**Puzzle 99**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 4* | | | | | | | 1* | 9* |
| | 6* | | 2* | | | | | |
| | | | | | | | 7* | |
| | | | 8* | 3* | | 2* | | |
| 7* | | | | | | 3* | | |
| | | 1* | | | | | | |
| | 8* | | | 6* | | 5* | | |
| | | | | | 7* | | 4* | |
| | | | | | | | | |

**Puzzle 100**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7* | | | | 8* | | | | |
| | | | | 4* | | | 1* | |
| | | | | | 3* | | 2* | |
| 3* | | 7* | | | | 5* | | |
| | | | 1* | | 6* | | | |
| 5* | | | 2* | | | | | |
| 6* | 1* | | | | | | | |
| | | | | | | 8* | | |
| | | | | | | | | 9* |

**Puzzle 94**

$5_{3}$r7 $4_{9}$c2

$6_{10}$b1 $7_{11}3_{12}$c4:

7r6b8 $1_{20}$b5

$2_{21}$b9 $3_{22}$c9

$3_{23}$c2 $4_{24}$c6

$4_{28}$b8 $1_{32}$r5

$1_{34}$r4: 1c2b1 $1_{37}$r7 $2_{40}$c2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $8_{51}$ | $3_{23}$ | $2_{52}$ | $5_{7}$ | $9_{53}$ | 1* | $4_{26}$ | 6* | $7_{17}$ |
| $6_{10}$ | $1_{39}$ | 7* | $8_{58}$ | $2_{57}$ | $4_{24}$ | 3* | $5_{6}$ | $9_{44}$ |
| 4* | $9_{41}$ | 5* | $7_{11}$ | $6_{14}$ | $3_{13}$ | $2_{27}$ | $1_{38}$ | $8_{43}$ |
| 7* | $2_{40}$ | $1_{34}$ | 4* | 5* | $8_{45}$ | $6_{36}$ | $9_{47}$ | $3_{22}$ |
| $3_{30}$ | 8* | $4_{29}$ | 6* | $7_{19}$ | $9_{33}$ | $1_{32}$ | 2* | $5_{5}$ |
| $9_{42}$ | $5_{4}$ | $6_{35}$ | $3_{12}$ | $1_{20}$ | $2_{46}$ | $8_{48}$ | $7_{18}$ | $4_{28}$ |
| $5_{3}$ | 6* | $8_{50}$ | 2* | 3* | $7_{2}$ | $9_{49}$ | $4_{31}$ | $1_{37}$ |
| $1_{55}$ | $7_{1}$ | $3_{62}$ | $9_{59}$ | $4_{25}$ | $6_{15}$ | 5* | $8_{61}$ | $2_{21}$ |
| $2_{54}$ | $4_{9}$ | $9_{63}$ | $1_{56}$ | $8_{60}$ | $5_{8}$ | 7* | $3_{64}$ | $6_{16}$ |

**Puzzle 95**

$2_{1}$b6 $2_{2}$b8

$6_{3}$r1 $7_{9}$c1

$1_{14}$b9 $5_{16}$b3

$3_{21}$b1 $3_{25}4_{26}$c9

$8_{27}$g: 49c3b3

$1_{30}$b2 $4_{45}$c2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $2_{37}$ | $8_{29}$ | $1_{31}$ | 3* | 5* | $7_{12}$ | $6_{3}$ | 4* | $9_{38}$ |
| $9_{57}$ | 7* | 6* | $2_{36}$ | $4_{56}$ | $8_{52}$ | $3_{23}$ | $1_{15}$ | $5_{20}$ |
| $4_{58}$ | $5_{18}$ | $3_{21}$ | $1_{33}$ | $6_{8}$ | $9_{55}$ | $2_{24}$ | $7_{11}$ | $8_{39}$ |
| $1_{30}$ | $2_{35}$ | 5* | $8_{53}$ | $3_{41}$ | $4_{54}$ | 7* | $9_{49}$ | 6* |
| 3* | $9_{46}$ | $7_{28}$ | $6_{7}$ | 2* | $1_{32}$ | $8_{48}$ | $5_{44}$ | $4_{26}$ |
| $6_{6}$ | $4_{45}$ | $8_{27}$ | $9_{47}$ | $7_{13}$ | $5_{43}$ | 1* | $3_{42}$ | $2_{2}$ |
| $5_{16}$ | 1* | $2_{17}$ | 7* | $9_{59}$ | 6* | $4_{63}$ | $8_{50}$ | $3_{25}$ |
| 8* | $6_{5}$ | $9_{61}$ | $4_{60}$ | $1_{34}$ | $3_{40}$ | $5_{19}$ | 2* | $7_{10}$ |
| $7_{9}$ | $3_{22}$ | $4_{62}$ | 5* | $8_{51}$ | $2_{1}$ | $9_{64}$ | $6_{4}$ | $1_{14}$ |

**Puzzle 96**

$1_{2}$b5 $4_{8}$b1 $7_{14}$r4

$7_{15}$b4 $6_{17}$r2

$6_{19}$r4: 6c1b3

$6_{20}7_{21}$c2: 6r5b5

$8_{25}$b8 $5_{26}$c7: 5r4b8

$3_{30}$c2 $5_{33}$b5

$8_{37}2_{38}$c5 $8_{42}$r1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1* | $9_{31}$ | 6* | $4_{9}$ | 3* | $7_{15}$ | $5_{26}$ | $8_{42}$ | $2_{43}$ |
| $2_{47}$ | $3_{30}$ | $8_{48}$ | $1_{7}$ | $6_{17}$ | 5* | 4* | $9_{44}$ | $7_{16}$ |
| 7* | $5_{27}$ | $4_{8}$ | $9_{41}$ | $8_{37}$ | $2_{40}$ | $1_{6}$ | $3_{32}$ | $6_{18}$ |
| $3_{62}$ | 2* | $1_{3}$ | 8* | $7_{14}$ | 4* | $6_{19}$ | $5_{36}$ | $9_{63}$ |
| $4_{13}$ | $8_{29}$ | $5_{28}$ | $6_{52}$ | $9_{39}$ | $3_{53}$ | $2_{1}$ | 7* | 1* |
| $9_{61}$ | $6_{20}$ | $7_{22}$ | 2* | $5_{33}$ | $1_{2}$ | $3_{60}$ | $4_{12}$ | $8_{25}$ |
| 5* | $7_{21}$ | $2_{46}$ | $3_{54}$ | 1* | $8_{55}$ | $9_{59}$ | 6* | $4_{11}$ |
| $6_{50}$ | 4* | $3_{57}$ | $7_{24}$ | $2_{38}$ | $9_{56}$ | 8* | $1_{5}$ | $5_{35}$ |
| $8_{49}$ | $1_{4}$ | $9_{58}$ | $5_{34}$ | $4_{10}$ | $6_{51}$ | $7_{23}$ | $2_{45}$ | $3_{64}$ |

**Puzzle 97**

$1_{2}$b5 $4_{8}$c1

$6_{12}2_{13}$r5 $7_{14}$c7

$7_{19}$c9 $6_{22}$c4

$8_{23}$b5 $8_{25}$b6

$3_{27}$b3 $6_{36}$c3 $5_{39}$r9

$5_{46}$c8 $5_{50}$r4 $7_{54}$c6

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $4_{8}$ | 6* | $7_{61}$ | $8_{34}$ | $9_{58}$ | 1* | $5_{63}$ | $3_{21}$ | $2_{49}$ |
| 2* | $3_{31}$ | $9_{56}$ | $6_{22}$ | $4_{18}$ | $5_{55}$ | $8_{35}$ | 7* | $1_{7}$ |
| $1_{4}$ | $8_{33}$ | $5_{62}$ | $7_{59}$ | $3_{30}$ | $2_{45}$ | $9_{64}$ | $6_{20}$ | $4_{17}$ |
| 3* | $9_{51}$ | $1_{3}$ | 2* | 6* | $4_{10}$ | $7_{14}$ | $8_{24}$ | $5_{50}$ |
| 8* | $2_{13}$ | $4_{9}$ | $9_{60}$ | $5_{59}$ | $7_{54}$ | $6_{12}$ | 1* | $3_{1}$ |
| $5_{41}$ | $7_{52}$ | $6_{36}$ | 3* | $1_{2}$ | $8_{23}$ | 4* | $2_{47}$ | $9_{53}$ |
| $7_{32}$ | 4* | $3_{27}$ | 5* | $2_{43}$ | $6_{38}$ | $1_{6}$ | $9_{44}$ | 8* |
| $9_{40}$ | 1* | $8_{26}$ | $4_{11}$ | 7* | $3_{29}$ | $2_{48}$ | $5_{46}$ | $6_{19}$ |
| $6_{37}$ | $5_{39}$ | $2_{28}$ | $1_{5}$ | $8_{25}$ | $9_{42}$ | 3* | $4_{16}$ | $7_{15}$ |

**Puzzle 98**

$1_2$b8    $5_3$b6

$2_9$r6: 2c5b6    $6_{10}$r4

$6_{11}3_{12}7_{13}$r7    $2_{17}$c4

$3_{19}$b7    $4_{21}$r3: 4c5b6

$4_{22}$c4    $4_{23}2_{24}8_{25}$r8

$9_{31}$c1    $8_{33}$c4

$9_{37}$g: 8c7b8    $4_{39}$c8    $1_{47}$b1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $8_{28}$ | $4_{29}$ | $1_{47}$ | $9_{34}$ | 5* | $3_{20}$ | $6_{14}$ | 2* | $7_{52}$ |
| 6* | $7_{48}$ | $5_6$ | $2_{17}$ | $8_{49}$ | $1_{46}$ | 4* | $3_{19}$ | $9_{36}$ |
| 3* | $2_{18}$ | $9_{35}$ | $6_{15}$ | $7_{50}$ | $4_{21}$ | $5_5$ | $8_{40}$ | $1_{53}$ |
| $5_8$ | 8* | 2* | 1* | $6_{10}$ | $7_{55}$ | $3_{56}$ | $9_{38}$ | $4_{42}$ |
| $9_{31}$ | $1_{57}$ | $6_{58}$ | $4_{22}$ | $3_{51}$ | $8_{54}$ | $7_{59}$ | 5* | $2_1$ |
| $4_{30}$ | $3_{62}$ | $7_{63}$ | $5_7$ | $9_{43}$ | $2_9$ | $8_{60}$ | $1_2$ | 6* |
| $7_{13}$ | 5* | 4* | $8_{33}$ | $1_{45}$ | $9_{44}$ | 2* | $6_{11}$ | $3_{12}$ |
| $2_{24}$ | $9_{26}$ | $8_{25}$ | 3* | $4_{23}$ | 6* | 1* | $7_{16}$ | $5_4$ |
| $1_{32}$ | $6_{61}$ | $3_{64}$ | 7* | $2_{27}$ | $5_3$ | $9_{37}$ | $4_{39}$ | $8_{41}$ |

**Puzzle 99**

$1_1 2_2 3_3$c1    $2_{11}$b4

$3_{15}$b6    $6_{18}$b3    $4_{19}$g

$4_{20}$c5    $7_{22}$r4    $5_{24}$c3

$8_{30}$r4    $6_{32}$b5    $7_{38}$b8

$5_{43}$c7    $3_{47}$r1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $2_2$ | $3_{47}$ | $4_{19}$ | $5_{35}$ | $7_{40}$ | $9_{41}$ | 8* | 1* | $6_{48}$ |
| 7* | $6_{49}$ | $8_{25}$ | $1_7$ | 3* | $4_{29}$ | $2_{14}$ | $5_{62}$ | $9_{63}$ |
| 5* | $9_{50}$ | $1_8$ | 6* | $2_{11}$ | $8_{28}$ | $3_{46}$ | $7_{59}$ | $4_{60}$ |
| 6* | $7_{22}$ | 9* | $3_{16}$ | $8_{30}$ | $5_{31}$ | 4* | $2_{10}$ | $1_4$ |
| $3_3$ | $4_{21}$ | $5_{24}$ | $9_{33}$ | 1* | 2* | $7_{53}$ | $6_{51}$ | $8_{52}$ |
| $1_1$ | $8_{26}$ | $2_9$ | 7* | $4_{20}$ | $6_{32}$ | $5_{43}$ | $9_{45}$ | $3_{44}$ |
| $8_{42}$ | $5_{27}$ | $7_{23}$ | 4* | $9_{39}$ | $1_6$ | 6* | $3_{17}$ | $2_{13}$ |
| $9_{56}$ | 2* | 3* | $8_{34}$ | $6_{36}$ | $7_{38}$ | $1_5$ | $4_{61}$ | $5_{64}$ |
| $4_{57}$ | 1* | $6_{18}$ | $2_{12}$ | $5_{37}$ | $3_{15}$ | $9_{54}$ | $8_{55}$ | $7_{58}$ |

**Puzzle 100**

$1_2$b2    $2_8 6_9$r4

$2_{10}$b4    $6_{13}$b9

$2_{14}6_{15}$c5: 2r7b9

$3_{20}7_{21}9_{22}$b5: 48c5b4

$3_{24}$c4: 3r7b9

$3_{25}$g:489r5b2&7c8b9

$8_{26}9_{27}$c8    $4_{30}3_{31}$r1    $5_{43}$r2    $9_{46}$c1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7* | $4_{30}$ | $2_{11}$ | $5_{32}$ | 8* | $1_4$ | $6_{19}$ | $9_{27}$ | $3_{31}$ |
| $8_{44}$ | $3_{33}$ | $6_{17}$ | $9_{42}$ | 4* | $2_{10}$ | $7_{40}$ | 1* | $5_{43}$ |
| $1_3$ | $9_{62}$ | $5_{63}$ | $7_{41}$ | $6_{15}$ | 3* | $4_{39}$ | 2* | $8_{45}$ |
| 3* | $2_8$ | 7* | $8_{52}$ | $9_{22}$ | $4_{53}$ | 5* | $6_9$ | $1_5$ |
| $9_{46}$ | $8_{49}$ | $4_{48}$ | 1* | $5_1$ | 6* | $2_{36}$ | $3_{25}$ | $7_{37}$ |
| 5* | $6_{12}$ | $1_2$ | 2* | $3_{20}$ | $7_{21}$ | $9_{28}$ | $8_{26}$ | $4_{29}$ |
| 6* | 1* | $8_{50}$ | $4_{54}$ | $7_{23}$ | $9_{59}$ | $3_{34}$ | $5_{58}$ | $2_{35}$ |
| $2_{16}$ | $7_{61}$ | $9_{64}$ | $3_{24}$ | $1_7$ | $5_{60}$ | 8* | $4_{55}$ | $6_{13}$ |
| $4_{47}$ | $5_{57}$ | $3_{38}$ | $6_{18}$ | $2_{14}$ | $8_{51}$ | $1_6$ | $7_{56}$ | 9* |

5. PADAGOGY EFFICIENCY

## 5.1. BIKINI AND OPEN TOP PROBLEMS

This is an excerpt from (21). The purpose to share some intuitive insights of typical optimization problems with those teaching as well as learning the standard method of using the differential calculus. Among other things, we shall explain intuitively the reason for the optimal solution to be attained at the critical point of the objective function in question. This kind of interpretation is often missing in existing textbooks.

In addition, we shall use the idea of the boundary being the marginal change of a well-rounded region (a region possessing an inscribed circle) with respect to the inradius (the radius of the inscribed circle) to solve optimization problems more efficiently and categorically.

We shall first explore the following three optimization problems.

- **Bikini problem.** Given a fixed material to form the total area of a bikini (two identical circles and one equilateral triangle), what is the maximum enclosure (combined perimeter)?
- **Minimum enclosure problem.** Given a fixed length, how can two well-rounded regions be formed with the minimum combined area?
- **Open top problem.** How can the largest open box be formed from a rectangular sheet of cardboard by first cutting off identical squares in all corners and then folding up the resulting flaps?

Take the last problem for example, we shall explain that to require the resulting box to be neither too shallow nor too narrow is the reason for the maximum volume to be attained at the critical point of the objective function in question.

In addition, we shall introduce a quick way of solving a wide spectrum of optimization problems in differential calculus based on the following three theorems. In other words, for a well-rounded region or a rectangle its boundary is the marginal change of its area.

**Theorem I.** For a polygon with an inscribed circle, its perimeter is the derivative of its area with respect to the inradius or apothem, the radius of the inscribed circle.

**Theorem II.** For a circle, its circumference is the derivative of its area with respect to the radius.

**Theorem III.** For a rectangle, its perimeter is the total differential derivative of its area.

## GENERALIZATION OF BIKINI PROBLEM

**Theorem 1.** Let *A* be the sum of the areas of *p* identical circles and *q* identical regular n-gons. Then the maximum combined perimeter is attained when the radius of the circles equals the inradius of the polygons. In this case, the maximum combined perimeter is

$$2\sqrt{\left(p\pi + qn\tan\frac{\pi}{n}\right)A}\ .$$

**Proof.** Let *x* be the radius of each circle and *y(x)* the inradius of each n-gon. Since

$$p\pi x^2 + qn\tan\frac{\pi}{n}\, y(x)^2 = A\ ,$$

It follows that

$$y'(x) = -\frac{p\pi x}{qn\pi\tan\frac{\pi}{n} y(x)}$$

and

$$y''(x) = -\frac{p\pi}{qn\tan\frac{\pi}{n}}\left[\frac{x}{y(x)}\right]' = -\frac{p\pi[qn\tan\frac{\pi}{n} y(x)^2 + p\pi x^2]}{(qn\tan\frac{\pi}{n})^2 y(x)^3} < 0\ .$$

Let P(x) denote the combined perimeter $2p\pi x + 2qn\tan\frac{\pi}{n} y(x)$. Then

$$P'(x) = 2p\pi + 2qn\tan\frac{\pi}{n} y'(x) = 2p\pi - \frac{2p\pi x}{y(x)}$$

and then

$$P''(x) = 2qn\tan\frac{\pi}{n} y''(x) < 0.$$

Hence the maximum of P(x) is attained when $P'(x) = 0$, i.e. $y(x) = x$. In this case,

$$P(x) = 2\left(p\pi + qn\tan\frac{\pi}{n}\right)x$$

$$= 2\left(p\pi + qn\tan\frac{\pi}{n}\right)\sqrt{\frac{A}{p\pi + qn\tan\frac{\pi}{n}}}$$

$$= 2\sqrt{\left(p\pi + qn\tan\frac{\pi}{n}\right)A}\ .$$

Similarly, we can prove the following theorems.

**Theorem 2.** Let ***A*** be the sum of the areas of $p$ identical regular $m$-gons and $q$ identical regular n-gons. Then the maximum combined perimeter is attained when all of the inradii are equal. In this case, the maximum combined perimeter is

$$2\sqrt{(pm\tan\pi/m + qn\tan\pi/n)A}\ .$$

**Theorem 3.** Let the sum of the volumes of $p$ identical equilateral tetrahedrons, $q$ identical spheres and $r$ identical cubes be fixed. Then the maximum combined surface area is attained when all the inradii and radii are identical.

**ENCLOSURE OF THE MINIMUM COMBINED AREA OF TWO REGIONS WITH A FIXED SUM OF BOUNDARIES**

Let ***P*** be a circle of radius x and ***Q*** a regular n-gon (or a triangle with fixed interior angles) of inradius $y$. We shall show that if the sum of ***P*** (the circumference of ***P)*** and ***Q*** (the perimeter of ***Q***) is fixed, then the minimum combined area enclosed is attained when $x = y$ . This is certainly not the case for an irregular n-gon ***Q*** with $n > 3$ . However, the same method will be applied to find the minimum combined area enclosed by ***P*** and ***Q*** with ***P*** + ***Q*** being fixed for various Q's.

**Theorem 4.** If the sum of ***P*** and ***Q*** is fixed, when $y = x$ the minimum combined area of ***P*** and ***Q*** (a regular n-gon) is attained as $(\pi + n\tan\pi/n)x^2$.

**Proof.** Write $y = y(x)$ . Since $2\pi x + 2ny(x)\tan\pi/n$ is fixed, we have

$$y'(x) = -\frac{\pi}{n\tan\pi/n}$$

Let $A(x)$ be the combined area of ***P*** and ***Q.*** Then $A(x) = \pi x^2 + ny(x)^2\tan\pi/n$ . It follows that

$$A'(x) = 2\pi x + 2ny(x)y'(x)\tan\pi/n \; = 2\pi[x - y(x)]$$

and

$$\text{A}''(\text{x}) = 2\pi\left[1 - y'^{(x)}\right] > 0 \; .$$

Therefore, the required minimum is attained when $y(x) = x$ .

Similarly, we can prove the following theorems.

**Theorem 5.** If the sum of $Q_m$ and $Q_n$ is fixed for regular m-gon $Q_m$ with inradius $x_m$ and n-gon $Q_n$ with inradius $x_n$ , then then the minimum combined area enclosed is attained as $(m\tan\pi/m \; + n\tan\pi/n){x_m}^2$.

**Theorem 6.** Let a circle ***P*** and an equilateral triangle (or a square) ***Q*** be enclosed by a fixed length. Then the minimum combined area is attained when ***P*** can be inscribed in ***Q.***

**Theorem 7.** Let a sphere ***P*** and an equilateral tetrahedron (or a cube) ***Q*** be enclosed by a fixed surface area. Then the minimum combined volume enclosed is attained when ***P*** can be inscribed in ***Q.***

## OPEN TOP PROBLEMS

Let $V(x)$ be the volume of the open box formed from the cardboard of length $a$ and width $b$ ($\leq a$) by cutting off identical squares of length $x$ in all corners. Then

$$V(x) = (a-2x)(b-2x)x,\ \ 0 \leq x \leq b/2\ .$$

Since

$$V'(x) = (a-2x)(b-2x) - 2x[(a-2x)+(b-2x)]$$

and since V(0)=0=V(b/2), it follows that the maximum volume is attained when the area of the bottom equals the lateral area of the open box so that the resulting box is neither too shallow nor too narrow; or when $\frac{2x}{a-2x} + \frac{2x}{b-2x} = 1$, namely when

$$x = \frac{a+b-\sqrt{a^2+b^2-ab}}{6}\ .$$

In the two dimensional case, the area of the open rectangle formed from a string of length $a$ by folding up both end segments of length $x$ *is* $V(x) = (a - 2x)x$, which attains the maximum when x=a/4.

## A QUICK WAY OF SOLVING OPTIMIZATION PROBLEMS

In some calculus textbooks such as [1], there are limited discussions along the line of Theorem II of section I which can be obtained by taking $n \to \infty$ in Theorem I, n being the number of sides of the polygon.

**Proof of Theorem I.** Let *x* be the inradius and *0* the center of the inscribed circle of the polygon *Q*. Let *A* and *B* be any two adjacent vertices of *Q*. Then the area of the triangle *AOAB is* the derivative of which is the length x(cot A +cot B) of the side *AB.* Since the sum of the areas of all such triangles and the sum of the lengths of all such sides are, respectively, the area and the perimeter of Q, the proof is completed.

The implication of Theorem I is that for a well-rounded region (with an inscribed circle), the marginal change of its area is its boundary.

**Proof of Theorem III.** Let x and $y$ be the $^1/_2$ of the length and width, respectively. Since $x$ and $y$ vary independently, the total differential derivative of the area 4xy is the perimeter 4(x + $y$) .

Theorems I, II and III along with their extensions Theorem IV, V and VI can be used to solve many optimization problems more efficiently and categorically as follows.

**Theorem IV.** For a circular right cylinder (or a polygonal right cylinder with an inscribed circular right cylinder), the area of its top (or bottom) is the derivative of its volume with respect to the height, and the area of its lateral surface is the derivative of its volume with respect to the radius (or inradius).

**Theorem V.** For a sphere (or a polygonal solid with an inscribed sphere), its surface area is the derivative of its volume with respect to the radius (or inradius).

**Theorem VI.** For a rectangular box, its surface area is the total differential derivative of its volume.

**Example 1.** Open Top Problem revisited.

**Solution.** Let $V(x)$ be the volume and $A(x)$ the area of the bottom of the open box. Then $V(x) = xA(x)$ . Thus the required maximum is attained when

$$xA'(x) + A(x) = V'(x) = 0 .$$

Since $A(x)$ varies negatively with x,

$$A'(x) = -2[(a-2x) + (b-2x)].$$

Therefore, the required x can be obtained by solving the following equation

$$2x[(a - 2x) + (b - 2x)] = (a - 2x)(b - 2x).$$

**Example 2.** Let $c$ be the sum of the areas of two well-rounded regions. Find the maximum sum P(x) of the boundaries.

**Solution.** Let x be the inradius of one region with the area $ax^2$ and $y(x)$ the inradius of the other region with the area $b[y(x)]^2$ . From $ax^2 + b[y(x)]^2 = c$, we can obtain

$y'(x) = -ax/by(x)$ . Since $P(x)= 2ax + 2by(x)$ , it follows that

$$P'(x) = 2a + 2by'(x) = 2a[1 - x/y(x)]$$

and

$$P''(x) = 2a\{-[y(x) - xy'(x)]/y(x)^2\} = -ac/y(x)^3 < 0 .$$

Therefore, the maximum of P(x) is $2\sqrt{c(a+b)}$ when $y(x) = x = \sqrt{c/(a+b)}$ .

**Example 3.** Given a fixed length $c$ to form two well-rounded regions, find the minimum combined area.

**Solution.** Let x and y(x) be the inradii of the given regions with the areas $ax^2$ and $b[y(x)]^2$ , respectively. Since the sum of the boundaries is $2ax + 2by(x) = c$, it follows that $y'(x) = -a/b$ . Hence the derivative of the combined area *A(x)* is

$$A'(x) = 2ax + 2by(x)y'(x) = 2a[x - y(x)]$$

so that

$$A''(x) = 2a[1 - y'(x)] = 2a(1 + a/b) > 0 .$$

Therefore, the minimum of *A(x)* is $c^2/4(a + b)$ when $y(x) = x = c/2(a + b)$ .

**Example 4.** Find the minimum surface area of a right circular cylindecr (or a right cylinder with an inscribed circular cylinder) of fixed volume c.

**Solution.** Let *x* be the radius (or inradius) of the top and y(x) the height of the cylinder. Then the area and the circumference (or the perimeter) of the top are $A(x) = ax^2$ and $A'(x) = 2ax$ , respectively. Since the volume y(x)A(x) is fixed, it follows that

$$y'(x) = - y(x)A'(x)/A(x) = - 2y(x)/x .$$

Let *S(x)* be the surface area of the cylinder. Then

$$S(x) = 2A(x) + 2axy(x).$$

Hence

$$S'(x) = 2A'(x)+2a[xy'(x)+ y(x)] = 2a[2x - y(x)] ,$$

and

$$S''(x) = 2a[2 - y'(x)] = 4a[1 + y(x)/x] > 0 .$$

Therefore, the minimum of *S(x)* is $3\sqrt[3]{2ac^2}$ when $y(x) = 2x = \sqrt[3]{c/2a}$ .

**Example 5.** Find the maximum volume of the above cylinder if the sum of the height and the girth (the circumference or the perimeter of the top) is fixed to be c.

**Solution.** We adopt the same notations and some of the results from Example 4. Since $y(x) + 2ax = c$ , the volume V(x) of the cylinder attains the maximum when

$$0 = V'(x) = y(x)A'(x) + A(x)y'(x) = y(x)(2ax) + ax^2(-2a) = 2ax[y(x) - ax] ,$$

namely when the height equals half of the girth. In this case, $x = c/3a$ so that the maximum volume is $c^3/27a$ , since

$$V''(x) = 2ax[y'(x) - a] + 2a[y(x) - ax] = -6a^2x < 0 .$$

## 5.3. SIMPLIFIED SIMPLEX METHOD

We shall simplify the simplex method of linear programming substantially by using cross-multiplication.

The simplex method in (4) was named in (3) among the top ten algorithms of the $20^{th}$ century. Although a number of variations in (7) and (14) have been introduced since then, the original algorithm has remained widely used for both reference and instruction in (5) and (10).

Over the years of teaching out of (10), I have encountered different types of "abnormal" problems that would give erroneous solutions had the algorithm stated in that book been used. For the remedies, whenever a pivot is found in either the same row or the same column as an old one (first type of abnormality), restart with the new pivot in the original system; otherwise, perform the "slope check" for each old pivot and when the "check number" is negative for a certain pivot (second type of abnormality), eliminate the entire inequality involving that pivot and restart with the new system. Furthermore, the new method of using cross-multiplication substantially simplified the process of finding the solutions of "normal" problems.

The proof of the maximization algorithm for the case of three variables will be given in theend via comparisons among values of the objective function at all feasible solutions of the variables. Short-hand notations for pertinent determinants of the coefficients of linear equations under the inequality constraints of a given problem enable us to efficiently express the relationship between function values at any pair of feasible solutions.

*Example 1*. Maximize $w = 3x + 4y$
subject to
$$x + 2y \le 8$$
$$2x + 3y \le 13$$
$$x \ge 0,\ y \ge 0.$$

*Solution*.

In stead of forming the first tableau $T_1$ of the coefficients of the original system $T_0$ as below

| | | |
|---|---|---|
| 3 | 4 | |
| 1 | 2 | 8 |
| 2 | 3 | 13 |

we shall directly find the pivot on $T_0$:

| | | |
|---|---|---|
| | v | |
| Maximize | $w = 3x + 4y$ | |
| subject to | $x + 2_1 y \le 8$ | 8/2 v |
| | $2x + 3y \le 13$ | 13/3 |
| | $x \ge 0,\ y \ge 0.$ | |

As illustrated above, find the greatest positive coefficient in the top row to yield the pivot column (as checked), find the least positive quotient of the last column over the pivot column to yield the pivot row (as checked) and find the pivot (as subscripted with 1).

Copy the pivot row from $T_0$, change the other coefficients of the pivot column to 0, Perform the cross-multiplication from the pivot $2_1$ to each of the remaining coefficients to yield the second tableau $T_2$ and find the pivot $1_2$ as illustrated below.

| | | | |
|---|---|---|---|
| v | | | |
| 2x3 - 4x1 = 2 | 0 | | |
| 1 | 2 | 8 | 8/1 |
| 2x2 - 3x1 = $1_2$ | 0 | 2 = 2x13 - 3x8 | 2/1 v |

Proceed as before to obtain the third tableau $T_3$:

| | | | |
|---|---|---|---|
| 0 | 0 | | |
| 0 | 2 | 6 | $y = 6/2 = 3$ |
| 1 | 0 | 2 | $x = 2/1 = 2$ |

No pivot can be found in $T_3$ and from which the variables involving pivots are solved.

This is a simplification of the method using the transformation [6],

$$\begin{matrix} p & q \\ r & s \end{matrix} \quad \rightarrow \quad \begin{matrix} 1/p & q/p \\ -r/p & s - r/p \end{matrix}$$

where p is the pivot, q is any other entry in the pivot row, r is any other entry in the pivot column and s is the entry in the row of r and the column of q as shown below.

| $T_1$: | $x$ | $y$ | 1 | |
|---|---|---|---|---|
| | $2_1$ | 1 | - 8 | = - $u$ |
| | 3 | 2 | - 13 | = - $v$ |
| | 4 | 3 | 0 | = $w$ |

| $T_2$: | $u$ | $y$ | 1 | |
|---|---|---|---|---|
| | 1/2 | 1/2 | - 4 | = - $x$ |
| | - 3/2 | $1/2_2$ | - 1 | = - $v$ |
| | -2 | 1 | 1 | = $w$ |

| $T_3$: | $u$ | $v$ | 1 | |
|---|---|---|---|---|
| | 2 | -1 | - 3 | = - $x$ |
| | - 3 | 2 | - 2 | = - $y$ |
| | 1 | -2 | 18 | = $w$ |

Since $w = 3(6.5) + 4(0) = 19.5$, we see that “the solution” obtained above was erroneous! This type of error occurs whenever the slope of $w$ is either greater than or less than the slopes of all non-trivial equations under the inequality constraints imposed on the problem.

Therefore, after finding each new pivot, the "slope check" is indispensable. To do that, in our case, perform the cross-multiplication from the top coefficient of the *y*-column of $T_0$ (the old pivot column) to the new pivot:

Slope check: (4)(2) – (3)(3) = - 1 < 0.

The negative check number eliminates the pivot in the *y*-column. Restart with the new pivot in

| | | | | |
|---|---|---|---|---|
| $T_{1:}$ | 3 | 4 | | |
| | 1 | 2 | 8 | |
| | $2_1$ | 3 | 13 | |
| | | | | |
| $T'_{2:}$ | 0 | -1 | | |
| | 0 | 1 | 3 | *y* = 0 (no pivot in *y* column) |
| | 2 | 3 | 13 | *x* = 13/2 = 6.5 |

Therefore, the maximum of *w* = 3(6.5) + 4(0) = 19.5.

v

*Example 2.* Maximize $w = 18\,x + 7\,y$

subject to

$3_1 x + \;\; y \;\leq 6$ 6/3 v

$3\,x + 2\,y \;\leq 15$ 15/3

$x \geq 0,\;\; y \geq 0.$

*Solution*.

| | | | | |
|---|---|---|---|---|
| $T_2$: | | v | | |
| | 0 | 3 | | |
| | 3 | $1_2$ | 6 | 6/1 v |
| | 0 | 3 | 27 | 27/3 |

The new pivot is in the same row as the one previously found. Restart with the new pivot.

| | | | | |
|---|---|---|---|---|
| $T_1$: | 18 | 7 | | |
| | 3 | $1_1$ | 6 | |
| | 3 | 2 | 15 | |
| | | | | |
| $T_2$': | - 3 | 0 | | |
| | 3 | 1 | 6 | *y* = 6/1 = 6 |
| | - 3 | 0 | 3 | *x* = 0 (no pivot in *x* column) |

Therefore, the maximum of *w* = 18(0) + 7(6) = 42.

***The algorithm for maximization with non-negative variables bounded from above:***

1) Use the original system of linear inequalities as the first tableau. If there is only one non-trivial inequality constraint, form the quotients to the top. The only pivot is on the variable of the column with the greatest quotient. Otherwise, find the greatest positive coefficient in the top row to yield the pivot column and then form the quotients of coefficients in the last column over the corresponding positive coefficients in the pivot column. Find the least positive quotient to determine the pivot row. Mark the pivot with an appropriate subscript. In the event of a tie when comparing quantities, all options need to be executed.

2) From the previous tableau, copy the pivot row and make the coefficients of the pivot column 0 except for the pivot. Starting with the pivot, cross-multiply the pivot column witheach of the other columns to yield the new tableau. Refer the pivot found in the new tableauback to the corresponding coefficient of the original system. If the new pivot is in the same row as an old one, restart with the new pivot in the original system.

3) In the original system, perform the slope check by cross-multiplying from each of the top coefficients of the columns involving pivot to the new pivot. If the check number of a certain column is negative, eliminate the inequality involving pivot in that column and restart with the new system.

4) Continue the same process until no more pivot could be found in the new tableau.

5) The solutions of the variables involving pivot can be obtained from the final tableau. The variables not involving pivot will yield the solution 0.

6) The maximum value of the objective function can be obtained by comparing the Function values among all options.

***The algorithm for minimization with non-negative variables bounded from below:***

1) Use the original system of linear inequalities as the first tableau. If there is only one non-trivial inequality constraint, form the quotients to the top. The only pivot is on the variable of the column with the least quotient. Otherwise, find the greatest positive coefficient in the last column to yield the pivot row and then form the quotients of coefficients in the top row over the corresponding positive coefficients in the pivot row. Find the least positive quotient to determine the pivot column. Mark the pivot with an appropriate subscript. In the event of a tie when comparing quantities, all options need to be executed.

2) From the previous tableau, copy the pivot column and make the coefficients of the pivot row 0 except for the pivot. Starting with the pivot, cross-multiply the pivot row with each of the other rows to yield the new tableau. Refer the pivot found in the new tableau back to the corresponding coefficient of the original system. If the new pivot is in the same column as an old one, restart with the new pivot in the original system.

3) Alongside each new tableau, display the table obtained by omitting the top row of the corresponding tableau constructed as though in the maximization case.

4) In the original system, perform the slope check by cross-multiplying from each of the top coefficients of the columns involving pivot to the new pivot. If the check number of a certain column is negative, eliminate the inequality involving the pivot in that column and restart with the new system.

5) Continue the same process until no more pivot could be found in the new tableau.

6) The solutions of the variables involving pivot can be obtained from the final table. The variables not involving pivot will yield the solution 0. If one of the solutions is negative, eliminate the inequality involving the first pivot and restart with the new system.

7) The minimum value of the objective function can be obtained by comparing the function values among all options.

v
5/2 6/3

*Example 3.* Minimize $w = 5x + 6y$

subject to

$x + y \geq 10$

$2x + 3_1 y \geq 12$ v

$x \geq 0, \; y \geq 0.$

*Solution*.

T2: v

| | | | | | | |
|---|---|---|---|---|---|---|
| 3/1 | 6/1 | | | | | |
| 3 | 6 | | | | | |
| $1_2$ | 1 | 18 v | | $1_2$ | 0 | 18 |
| 0 | 3 | 0 | | 2 | 3 | 12 |

Slope check: (6)(1) – (1)(5) = 1 > 0.

T3:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 3 | | | | | | |
| 1 | 0 | 0 | | 1 | 0 | 18 | $x = 18/1 = 18$ |
| 0 | 3 | 0 | | 0 | 3 | - 24 | $y = - 24/3 = -8$ |

Since $y = - 8$, eliminate the inequality involving the first pivot and restart with T':

| | | | |
|---|---|---|---|
| 5/1 | 6/1 | | |
| 5 | 6 | | $y = 0$ (no pivot in $y$ column) |
| $1_1$ | 1 | 10 | $x = 10/1 = 10$ |

Therefore, the maximum of $w = 5(10) + 6(0) = 50$.

*Example 4.* Minimize subject to

|  | 3/2 | 1/1 | 2/2 |  |  |  |
|---|---|---|---|---|---|---|
| Minimize | $w = 3x$ + | $y$ | + $2z$ |  |  |  |
| subject to | $x$ + | $y$ + | $z$ | $\geq$ | 42 |  |
|  | $3x$ + | $2y$ + | $z$ | $\geq$ | 49 |  |
|  | $2x$ + | $y$ + | $2z$ | $\geq$ | 56 | v |

$x \geq 0, y \geq 0, z \geq 0.$

*Solution*.

Option 1.

$T_1$:

|  | v |  |  |  |
|---|---|---|---|---|
| 3/2 | 1/1 | 2/2 |  |  |
| 3 | 1 | 2 |  |  |
| 1 | 1 | 1 | 42 |  |
| 3 | 2 | 1 | 49 |  |
| 2 | $1_1$ | 2 | 56 | v |

$T_2$:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 |  |  |  |  |  |  |
| - 1 | 1 | - 1 | - 14 | - 1 | 0 | - 1 | - 14 | $x = 0$ |
| - 1 | 2 | - 3 | - 63 | - 1 | 0 | - 3 | - 63 | $z = 0$ |
| 0 | 1 | 0 | 0 | 2 | 1 | 2 | 56 | $y = 56$ |

Hence $w = 3(0) + 1(56) + 2(0) = 56$.

Option 2.

$T_1$:

|  |  | v |  |  |
|---|---|---|---|---|
| 3/2 | 1/1 | 2/2 |  |  |
| 3 | 1 | 2 |  |  |
| 1 | 1 | 1 | 42 |  |
| 3 | 2 | 1 | 49 |  |
| 2 | 1 | $2_1$ | 56 | v |

$T_2$:

| v |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|
| 2/4 |  | 2/1 |  |  |  |  |  |  |
| 2 | 0 | 2 |  |  |  |  |  |  |
| 0 | 1 | 1 | 28 |  | 0 | 1 | 0 | 28 |
| $4_2$ | 3 | 1 | 42 | v | 4 | 3 | 0 | 42 |
| 0 | 0 | 2 | 0 |  | 2 | 1 | 2 | 56 |

Slope check: (2)(3) – (1)(3) = 3 > 0.

| | | | | | |
|---|---|---|---|---|---|
| $T_3$: | | | v | | |
| | | | 6/4 | | |
| | 2 | - 6 | 6 | | |
| | 0 | 4 | $4_3$ | 112 | v |
| | 4 | 0 | 0 | 0 | |
| | 0 | 0 | 8 | 0 | |

Since the new pivot is in the same column as the first one, restart from $T_1$with the new pivot:

| | | | | |
|---|---|---|---|---|
| $T_1$: | 3 | 1 | 2 | |
| | 1 | 1 | $1_1$ | 42 |
| | 3 | 2 | 1 | 49 |
| | 2 | 1 | 2 | 56 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $T'_2$: | v | | | | | | | | |
| | 1/2 | | 2/1 | | | | | | |
| | 1 | - 1 | 2 | | | | | | |
| | 0 | 0 | 1 | 0 | | 1 | 1 | 1 | 42 |
| | $2_2$ | 1 | 1 | 7 | v | $2_2$ | 1 | 0 | 7 |
| | 0 | - 1 | 2 | - 28 | | 0 | - 1 | 0 | - 28 |

Slope check: (2)(3) – (1)(3) = 3 > 0.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $T'_3$: | 1 | - 3 | 3 | | | | | | |
| | 0 | 0 | 2 | 0 | 0 | 1 | 2 | 77 | $z = 77/2$ |
| | 2 | 0 | 0 | 0 | 2 | 1 | 0 | 7 | $x = 7/2$ |
| | 0 | - 2 | 4 | - 56 | 0 | - 2 | 0 | - 56 | $y = 0$ |

Hence $w = 3(7/2) +1(0) +2(77/2) = 87.5$.

Therefore, Option 1 gives the minimum of $w = 56$.

**The proof of the maximization algorithm**

We can rearrange the variables and the given inequalities in such a manner that the pivots are to be found diagonally (in ascending order of both rows and columns). Since the method used here can be extended for more variables, we shall only consider the following problem with three positive variables in which all coefficients are positive.

| | | | |
|---|---|---|---|
| Maximize | $w = a_{01} x_1 + a_{02} x_2 + a_{03} x_3$ | | |
| subject to | $a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 \le a_{i4}$ | $(I_i)$ | $i = 1, 2, 3$ |
| | $a_{01} \ge a_{0j}$ | $(1_j)$ | $j = 2, 3$ |
| | $a_{i4}a_{11} - a_{14}a_{i1} \ge 0.$ | $(2_i)$ | $i = 2, 3$ |

We shall abbreviate $a_{ij}$ as [ij] and use $\{123\}_{123}$ to denote the determinant of the matrix $([ij])_{i, j \varepsilon 123}$, where i ε123 means that i takes on one element of {1, 2, 3} at a time in that order. Let c be a code consisting of some elements in {0, 1, 2, 3} and let d be a code consisting of some elements in {1, 2, 3, 4}. We shall further use $\{d\}_c$ to denote the determinant $([ij])_{i \varepsilon c, j \varepsilon d}$.

Thus, the inequality $(2_i)$ can be rewritten as $\{41\}_{i1} \ge 0$. Note that

$$\{rst\}_{ijk} = [ir]\{st\}_{jk} + [is]\{tr\}_{jk} + [it]\{rs\}_{jk}.$$

We shall first state and derive the following formulas concerning decompositions of determinants.

*Lemma 1.*

i) $\{rst\}_{ijk} = [kt]\{rs\}_{ij} + [jt]\{rs\}_{ki} + [it]\{rs\}_{jk}.$

ii) $[iu]\{rst\}_{ijk} = \{ru\}_{ij}\{st\}_{ik} + \{su\}_{ij}\{tr\}_{ik} + \{tu\}_{ij}\{rs\}_{ik}.$

iii) $[jt]\{rst\}_{ijk} = \{rt\}_{ij}\{st\}_{jk} - \{st\}_{ij}\{rt\}_{jk}.$

iv) $\{ru\}_{ij}\{st\}_{ij} + \{su\}_{ij}\{tr\}_{ij} + \{tu\}_{ij}\{rs\}_{ij} = 0.$

v) $\{ur\}_{ij}\{ust\}_{ijk} + \{us\}_{ij}\{utr\}_{ijk} + \{ut\}_{ij}\{urs\}_{ijk} = 0.$

*Proof.*

i) $\{rst\}_{ijk} = [ir]\{st\}_{jk} + [is]\{tr\}_{jk} + [it]\{rs\}_{jk}$

$= [ir][js][kt] - [ir][jt][ks] + [is][jt][kr] - [is][jr][[kt] + [it]\{rs\}_{jk}$

$= [kt]([ir][js] - [is][jr]) + [jt]([kr][is] - [ks][ir]) + [it]\{rs\}_{jk}$

$= [kt]\{rs\}_{ij} + [jt]\{rs\}_{ki} + [it]\{rs\}_{jk}$.

ii) $\{ru\}_{ij}\{st\}_{ik}+\{su\}_{ij}\{tr\}_{ik}+\{tu\}_{ij}\{rs\}_{ik}$

$= [ir][ju][is][kt] - [ir][[ju][it][ks] - [iu][jr]\{st\}_{ik} + [is][ju][it][kr] - [is][ju][ir][kt]$

$- [iu][js]\{tr\}_{ik} + [it][ju][ir][ks] - [it][[ju][is][kr] - [iu][jt]\{rs\}_{ik}$

$= - [iu]([jr]\{st\}_{ik} + [js]\{tr\}_{ik} + [jt]\{rs\}_{ik}) = - [iu]\{rst\}_{jik} = [iu]\{rst\}_{ijk}$.

iii) Taking k = j, j = i and u = t in ii), we have

$$[jt]\{rst\}_{jik} = \{rt\}_{ji}\{st\}_{jk} + \{st\}_{ji}\{tr\}_{jk} + \{tt\}_{ji}\{rs\}_{jk},$$

which implies

$$[jt]\{rst\}_{ijk} = \{rt\}_{ij}\{st\}_{jk} - \{st\}_{ij}\{rt\}_{jk.}$$

iv) Taking k = j in ii), we have

$$\{ru\}_{ij}\{st\}_{ij} + \{su\}_{ij}\{tr\}_{ij} + \{tu\}_{ij}\{rs\}_{ij} = [iu]\{rst\}_{ijj} = 0.$$

v) $\{ur\}_{ij}\{ust\}_{ijk} + \{us\}_{ij}\{utr\}_{ijk} + \{ut\}_{ij}\{urs\}_{ijk}$

$= \{ur\}_{ij}([iu]\{st\}_{jk} + [ju]\{st\}_{ki}+ [ku]\{st\}_{ij})$

$+ \{us\}_{ij}([iu]\{tr\}_{jk} + [ju]\{tr\}_{ki}+ [ku]\{tr\}_{ij})$

$+ \{ut\}_{ij}([iu]\{rs\}_{jk} + [ju]\{rs\}_{ki}+ [ku]\{rs\}_{ij})$

$= [iu](\{ur\}_{ij}\{st\}_{jk} + \{us\}_{ij}\{tr\}_{jk} + \{ut\}_{ij}\{rs\}_{jk})$

$+ [ju](\{ur\}_{ij}\{st\}_{ki} + \{us\}_{ij}\{tr\}_{ki} + \{ut\}_{ij}\{rs\}_{ki})$

$+ [ku](\{ur\}_{ij}\{st\}_{ij} + \{us\}_{ij}\{tr\}_{ij} + \{ut\}_{ij}\{rs\}_{ij})$

$= [iu][ju]\{rst\}_{jik} + [ju][iu]\{rst\}_{ijk} + [ku][iu]\{rst\}_{ijj} = 0.$

We shall introduce convenient notations for all possible corner solutions of the given problem. For i, j = 1, 2, 3, let $w_{ij}$ = [0j]([i4]/[ij]). For an ascending code ij, let $^{mn}w_{ij}$ = [0i]$^{mn}x_{ij}$ + [0j]$^{mn}x_{ji}$ , where ($^{mn}x_{ij}$ , $^{mn}x_{ji}$) is the possible non-degenerated solution of

$$[ui]x_i + [uj]x_j = [u4] , \quad u = m, n,$$

i.e. $^{mn}x_{ij}$ = {4j}$_{mn}$/{ij}$_{mn}$ and $^{mn}x_{ji}$ = {i4}$_{mn}$/{ij}$_{mn}$ . Note that m and n are interchangeable in the above notations. Furthermore, let $w^*$ = [01]$x_1^*$ + [02]$x_2^*$ + [03]$x_3^*$, where ($x_1^*$, $x_2^*$, $x_3^*$) is the possible non-degenerated solution of

$$[i1]x_1 + [i2]x_2 + [i3]x_3 = [i4], \quad i = 1, 2, 3,$$

i.e. $x_1^*$ ={423)$_{123}$/{123)$_{123}$ , $x_2^*$ ={143)$_{123}$/{123)$_{123}$ and $x_3^*$ ={124)$_{123}$/{123)$_{123}$.

In the tableau $T_1$:

| | | | |
|---|---|---|---|
| [01] | [02] | [03] | |
| [11] | [12] | [13] | [14] |
| [21] | [22] | [23] | [24] |
| [31] | [32] | [33] | [34] |

[11] is the pivot because of ($1_j$) and ($2_i$), j, i = 2, 3.

*Lemma 2.*

If $w_{i1} > w_{11}$, then ([i4]/[i1], 0, 0) is not a feasible solution.

*Proof.*

Since $w_{i1} > w_{11}$, {14}$_{1i}$ > 0. It follows that [11]([i4]/[i1]) + [12](0) + [13](0) > [14], i.e. ([i4]/[i1], 0, 0) does not satisfy ($I_1$). Perform the cross-multiplication at [11] to yield $T_2$:

| | | | |
|---|---|---|---|
| 0 | {12}$_{10}$ | {13}$_{10}$ | |
| [11] | [12] | [13] | [14] |
| 0 | {12}$_{12}$ | {13}$_{12}$ | {14}$_{12}$ |
| 0 | {12}$_{13}$ | {13}$_{13}$ | {14}$_{13}$ |

*Theorem 1.*

If

(3) $\{12\}_{01} \geq 0$

and

(4) $\{13\}_{01} \geq 0$,

then [11] is the only pivot of $T_0$ and $w_{11}$ is the maximum of $w$.

*Proof.*

Since there is no positive coefficient in the top row of $T_2$, [11] is the only pivot of $T_0$.

Due to Lemma 2, it suffices to show that $w_{11} \geq w$ in each of the following cases.

Case 1. $w = w_{kj}$ , where $w_{kj}$ is the least among $w_{ij}$ , i =1, 2, 3 and j = 2, 3.

From the inequality (j+1), we have

$$w_{11} - w \ \geq w_{11} - \mathrm{w}_{1j} = [01][14]/[11] - [0j][14]/[1j] = [14]\{1j\}_{01}/([11][1j]) \geq 0.$$

Case 2. $w = {}^{mn}w_{ij}$ , where ${}^{mn}x_{ij}$ and ${}^{mn}x_{ji}$ are positive and satisfying ($I_1$) with other variables 0.

From the inequalities (3) and (4), we have

$$\begin{aligned} w_{11} - w &= [01][14]/[11] - [0i]^{mn}x_{ij} - [0j]^{mn}x_{ji} \\ &= [01][14]/[11] - ([0i][11]/[11])^{mn}x_{ij} - ([0j][11]/[11])^{mn}x_{ji} \\ &\geq \ [01][14]/[11] - ([01][1i]/[11])^{mn}x_{ij} - ([01][1j]/[11])^{mn}x_{ji} \\ &= ([01]/[11])([14] - [1i]\,{}^{mn}x_{ij} - [1j]^{mn}x_{ji}) \geq 0. \end{aligned}$$

Case 3. $w = w^*$, where $x_j^* \geq 0$, j = 1, 2, 3.

From (3) and (4), we have

$$w_{11} - w^* = [01][14]/[11] - ([01]x_1^* + [02]x_2^* + [03]x_3^*)$$

$$\geq [01][14]/[11] - ([01][11]/[11])x_1^* - ([01][12]/[11])x_2^* - ([01][13]/[11])x_3^*$$

$$= ([01]/[11])([14] - [11]x_1^* - [12]x_2^* - [13]x_3^*) \geq 0.$$

If the pivot in $T_2$ exists, we can assume that

(3') $\{12\}_{01} < 0$.

If the pivot is [12], it replaces [11] as a pivot of $T_0$. In this case, we can rearrange $T_2$ into

$T_2$':

| | | | |
|---|---|---|---|
| $\{12\}_{10}$ | $\{13\}_{10}$ | 0 | |
| [12] | [13] | [11] | [14] |
| $\{12\}_{12}$ | $\{13\}_{12}$ | 0 | $\{14\}_{12}$ |
| $\{12\}_{13}$ | $\{13\}_{13}$ | 0 | $\{14\}_{13}$ |

and $T_0$ into $T_0$':

| | | | |
|---|---|---|---|
| [02] | [03] | [01] | |
| [12] | [13] | [11] | [14] |
| [22] | [23] | [21] | [24] |
| [32] | [33] | [31] | [34] |

If [12] is the only pivot of $T_0$', then similar to Theorem 1, we can prove that $w_{12}$ is the maximum of $w$. Otherwise, we can assume that [12] and [23] are the only pivots of $T_0$' and use the same method in the proof of the next theorem to prove that ${}^{12}w_{23}$ is the maximum of $w$.

Now, let us go back to $T_2$ and assume that $\{12\}_{12}$ is the pivot. Then we have

(5) $\{12\}_{10} \geq \{13\}_{10}$

(6) $\{12\}_{12} > 0$

(7) $[12]\{14\}_{12} - [14]\{12\}_{12} = [11]\{24\}_{12} \leq 0$

and

(8) $\{14\}_{12}\{12\}_{13} - \{12\}_{12}\{14\}_{13} = [11]\{124\}_{123} \geq 0.$

The slope check from [01] to [22] gives either

(9) $\{12\}_{02} \geq 0$

or

(9') $\{12\}_{02} < 0.$

*Theorem 2.*

If the inequalities (9) and

(10) $\{123\}_{012} \leq 0$

hold, then [11] and [22] are the only pivots of $T_0$ and ${}^{12}w_{12}$ is the maximum of $w$.

*Lemma 3.*

$w^* - {}^{mn}w_{jk} = x_i^*\{123\}_{0mn}/\{(jk)\}_{mn}$, where (21) = (12) = 12, (32) = (23) = 23, (13) = (31) = 31.

*Proof.*

Solving

$$[01]x_1 + [02]x_2 + [03]x_3 = w^*$$
$$[u1]x_1 + [u2]x_2 + [u3]x_3 = [u4] \qquad u = m, n$$

for $x_i$, we get $x_i^*\{123\}_{0mn} = w^*\{(jk)\}_{mn} + [0j]\{k4\}_{mn} + [0k]\{4j\}_{mn} = (w^* - {}^{mn}w_{jk})\{(jk)\}_{mn}$.

*Lemma 4.*

Let (01)' = 10, (02)' = 02, 1' = 2 and 2' = 1. Then, for k = 1, 2.

i) $^{12}w_{12} - w_{k3} = (\{12\}_{k0}\{34\}_{12} - [k4]\{123\}_{012})/([k3]\{12\}_{12})$.

ii) $^{12}w_{12} - {}^{12}w_{k3} = -{}^{12}x_{3k}\{123\}_{012}/\{12\}_{12}$.

iii) $^{12}w_{12} - {}^{k3}w_{12} = (\{12\}_{(0k)'}/\{12\}_{12})([k'4] - [k'1]^{k3}x_{12} - [k'2]^{k3}x_{21})$.

iv) $^{12}w_{12} - {}^{k3}w_{k'3} = (\{12\}_{(0k)'}/\{12\}_{12})([k'4]-[k'k']^{k3}x_{k'3}-[k'3]^{k3}x_{3k'}) - {}^{k3}x_{3k'}\{123\}_{012}/\{12\}_{12}$.

v) $^{12}w_{12} - {}^{k3}w_{k3} = (\{12\}_{(0k)'}/\{12\}_{12})([k'4]-[k'k]^{k3}x_{k3}-[k'3]^{k3}x_{3k}) - {}^{k3}x_{3k}\{123\}_{012}/\{12\}_{12}$ .

*Proof.*

We derive one formula in each category via Lemma 1 due to the similarity.

i) $^{12}w_{12} - w_{23}$

$= ([01]\{42\}_{12} + [02]\{14\}_{12})/\{12\}_{12} - [03]([24]/[23])$

$= [23]([01][14][22] - [01][12][24] + [02][11][24] - [02][14][21])/([23]\{12\}_{12})$

$- [24][03]\{12\}_{12}/([23]\{12\}_{12})$

$= \{[23][14]\{12\}_{02} + [24]([23]\{12\}_{10} + [03]\{12\}_{21})\}/([23]\{12\}_{12})$

$= \{[23][14]\{12\}_{02} + [24](\{312\}_{210} - [13]\{12\}_{02})\}/([23]\{12\}_{12})$

$= (\{12\}_{20}\{34\}_{12} - [24]\{123\}_{012})/([23]\{12\}_{12})\}$.

ii)

$^{12}w_{12} - {}^{12}w_{23}$

$= ([01]\{42\}_{12} + [02]\{14\}_{12})/\{12\}_{12} - ([02]\{43\}_{12} + [03]\{24\}_{12})/\{23\}_{12}$

$= \{[01]\{42\}_{12}\{23\}_{12} + [02](\{14\}_{12}\{23\}_{12} + \{34\}_{12}\{12\}_{12}) -$

$[03]\{24\}_{12}\{12\}_{12}\}/(\{12\}_{12}\{23\}_{12})$

$= ([01]\{24\}_{12}\{32\}_{12} + [02]\{24\}_{12}\{13\}_{12} + [03]\{24\}_{12}\{21\}_{12})/(\{12\}_{12}\{23\}_{12})$

$= (\{24\}_{12}/\{23\}_{12}\})/(\{132\}_{012}/\{12\}_{12}) = - {}^{12}x_{32}\{123\}_{012}/\{12\}_{12}$.

iii)

$$^{12}w_{12} - {}^{13}w_{12}$$

$$= ([01]\{42\}_{12} + [02]\{14\}_{12})/\{12\}_{12} - ([01]\{42\}_{13} + [02]\{14\}_{13})/\{12\}_{13}$$

$$= \{[01](\{12\}_{21}\{42\}_{13} - \{42\}_{21}\{12\}_{13}) + [02](\{21\}_{21}\{41\}_{13} -$$

$$\{41\}_{21}\{21\}_{13})\}/(\{12\}_{12}\{12\}_{13})$$

$$= ([01][12]\{142\}_{213} + [02][11]\{241\}_{213})/(\{12\}_{12}\{12\}_{13})$$

$$= \{12\}_{01}\{124\}_{123}/(\{12\}_{12}\{12\}_{13}) = (\{12\}_{10}/\{12\}_{12})([24] - [21]\,{}^{13}x_{12} - [22]\,{}^{13}x_{21}).$$

iv)

$$^{12}w_{12} - {}^{23}w_{13}$$

$$= ([01]\{42\}_{12} + [02]\{14\}_{12})/\{12\}_{12} - ([01]\{43\}_{23} + [03]\{14\}_{23})/\{13\}_{23}$$

$$= \{[01](\{42\}_{21}\{31\}_{23} + \{12\}_{21}\{43\}_{23}) + [02]\{14\}_{12}\{13\}_{23} -$$

$$[03]\{12\}_{12}\{14\}_{23}\}/(\{12\}_{12}\{13\}_{23})$$

$$= \{[01]([22]\{431\}_{213} - \{32\}_{21}\{14\}_{23}) + [02]\{41\}_{21}\{13\}_{23} +$$

$$[03]\{12\}_{21}\{14\}_{23}\}/(\{12\}_{12}\{13\}_{23})$$

$$= \{-[01][22]\{143\}_{123} + \{14\}_{23}([01]\{23\}_{21} + [03]\{12\}_{21}) +$$

$$[02]\{41\}_{21}\{13\}_{23}\}/(\{12\}_{12}\{13\}_{23})$$

$$= \{-[01][22]\{143\}_{123} + \{14\}_{23}(\{123\}_{021} - [02]\{31\}_{21}) + [02]\{41\}_{21}\{13\}_{23}\}/(\{12\}_{12}\{13\}_{23})$$

$$= \{-[01][22]\{143\}_{123} - \{14\}_{23}\{123\}_{012} + [02](\{41\}_{12}\{31\}_{23} -$$

$$\{31\}_{12}\{41\}_{23})\}/(\{12\}_{12}\{13\}_{23})$$

$$= (-[01][22]\{143\}_{123} - \{14\}_{23}\{123\}_{012} + [02][21]\{431\}_{123})/(\{12\}_{12}\{13\}_{23})$$

$$= \{12\}_{20}\{143\}_{123}/\{12\}_{12}\{13\}_{23} - (\{14\}_{23}/\{13\}_{23})(\{123\}_{012})/\{12\}_{12})$$

$$= (\{12\}_{02}/\{12\}_{12})([14] - [11]^{23}x_{13} - [13]^{23}x_{31}) - {}^{23}x_{31}\{123\}_{012}/\{12\}_{12}.$$

v)

$^{12}w_{12}$ - $^{13}w_{13}$

= ([01]{42}$_{12}$ + [02]{14}$_{12}$)/{12}$_{12}$ - ([01]{43}$_{23}$ + [03]{14}$_{13}$)/{13}$_{13}$

= {[01]({42}$_{12}${13}$_{13}$+{12}$_{12}${34}$_{13}$)+[02]{14}$_{12}${13}$_{13}$-

[03]{12}$_{12}${14}$_{13}$}/({12}$_{12}${13}$_{13}$)

= {[01]([12]{413}$_{123}$-{32}$_{12}${41}$_{13}$)+[02]{14}$_{12}${13}$_{13}$-[03]{12}$_{12}${14}$_{13}$}/({12}$_{12}${13}$_{13}$)

= {-[01][12]{143}$_{123}$ + {14}$_{13}$([01]{32}$_{12}$+[03]{21}$_{12}$) +

[02]{14}$_{12}${13}$_{13}$}/({12}$_{12}${13}$_{13}$)

= {-[01][12]{143}$_{123}$ + {14}$_{13}$({132}$_{012}$ - [02]{13}$_{12}$) + [02]{14}$_{12}${13}$_{13}$}/({12}$_{12}${13}$_{13}$)

= {-[01][12]{143}$_{123}$ - {14}$_{13}${123}$_{012}$ + [02]({41}$_{12}${31}$_{13}$ –

{31}$_{12}${41}$_{13}$)}/({12}$_{12}${13}$_{13}$)

= (-[01][12]{143}$_{123}$ - {14}$_{13}${123}$_{012}$ + [02][11]{431}$_{123}$)/({12}$_{12}${13}$_{13}$)

= {12}$_{10}${143}$_{123}$/{12}$_{12}${13}$_{13}$ - ({14}$_{13}$/{13}$_{13}$)({123}$_{012}$)/{12}$_{12}$)

= ({12}$_{10}$ /{12}$_{12}$)([24] - [21]$^{13}x_{13}$ - [23]$^{13}x_{31}$) - $^{13}x_{31}${123}$_{012}$/{12}$_{12}$.

Perform the cross-multiplication at {12}$_{12}$ in $T_2$ to yield $T_3$:

| | | | |
|---|---|---|---|
| 0 | 0 | [11]{123}$_{012}$ | |
| [11]{12}$_{12}$ | 0 | [11]{32}$_{12}$ | [11]{42}$_{12}$ |
| 0 | {12}$_{12}$ | {13}$_{12}$ | {14}$_{12}$ |
| 0 | 0 | [11]{123}$_{123}$ | [11] ]{124}$_{123}$ |

Because of (10), there is no pivot in $T_3$. Because of (6), it follows from (7) and ($2_2$) that $^{12}x_{12}$ and $^{12}x_{21}$ are non-negative. We shall assume the non-degenerated case so that {42}$_{12}$, {14}$_{12}$, $^{12}x_{12}$ and $^{12}x_{21}$ are all positive.

*Proof of Theorem 2.*

From Lemma 1.i) and (8), it follows that $({}^{12}x_{12}, {}^{12}x_{21}, 0)$ satisfies $(I_3)$:

$$[31]^{12}x_{12} + [32]^{12}x_{21} = ([31]\{42\}_{12} + [32]\{14\}_{12})/\{12\}_{12}$$

$$= (-\{124\}_{123} + [34]\{12\}_{12})/\{12\}_{12} \leq [34].$$

To prove that ${}^{12}w_{12}$ is the maximum of $w$, due to Lemma 2, we need only show that ${}^{12}w_{12}$ $\geq w$.

Case 1. $w = w^*$, where $x_3^*$ is non-negative.

Because of (10), it follows from lemma 3 that ${}^{12}w_{12} - w = - x_3^*\{123\}_{012}/\{12\}_{12} \geq 0$.

Case 2. $w = w_{k2}$, where $w_{k2}$ is the least among $w_{i2}$, i = 1, 2, 3.

From (9), we have

$${}^{12}w_{12} - w \geq [01]^{12}x_{12} + [02]^{12}x_{21} - w_{22}$$

$$= (\{12\}_{02}/[22])^{12}x_{12} + ([02]/[22])([21]^{12}x_{12} + [22]^{12}x_{21} - [24]) \geq 0.$$

Case 3. $w = w_{k3}$, where $w_{k3}$ is the least among $w_{i3}$, i = 1, 2, 3.

Because of (3'), (6), (9), (10) and Lemma 4.i), we have either ${}^{12}w_{12} \geq w_{13}$ or ${}^{12}w_{12} \geq w_{23}$.

Case 4. $w = {}^{12}w_{k3}$, where ${}^{12}x_{3k} \geq 0$, k = 1, 2.

Because of (6) and (10), it follows from Lemma 4.ii) that

$${}^{12}w_{12} - w = - {}^{12}x_{3k}\{123\}_{012}/\{12\}_{12} \geq 0.$$

Case 5. $w = {}^{k3}w_{12}$, where $({}^{k3}x_{12}, {}^{k3}x_{21}, 0)$ satisfies $(I_{k'})$, k= 1, 2.

Because of (3'), (6) and (9), it follows from Lemma 4.iii) that

$${}^{12}w_{12} - w = (\{12\}_{(0k)'}/\{12\}_{12})([k'4] - [k'1]^{k3}x_{12} - [k'2]^{k3}x_{21}) \geq 0.$$

Case 6. $w = {}^{k3}w_{k'3}$, where ${}^{k3}x_{3k'} \geq 0$, k = 1, 2, $(0, {}^{13}x_{23}, {}^{13}x_{32})$ satisfies $(I_2)$ and $({}^{23}x_{13}, 0, {}^{23}x_{31})$ satisfies $(I_1)$.

Because of (3'), (6), (9) and (10), it follows from Lemma 4.iv) that

$^{12}w_{12} - w = (\{12\}_{(0k)'}/\{12\}_{12})([k'4] - [k'k']^{k3}x_{k'3} - [k'3]^{k3}x_{3k'}) - {}^{k3}x_{3k'}\{123\}_{012}/\{12\}_{12} \geq 0.$

Case 7. $w = {}^{k3}w_{k3}$, where ${}^{k3}x_{3k} \geq 0$, k = 1, 2, (0, ${}^{13}x_{13}$ , ${}^{13}x_{31}$) satisfies ($I_2$) and (${}^{23}x_{23}$ , 0, ${}^{23}x_{32}$) satisfies ($I_1$).

Because of (3'), (6), (9) and (10), it follows from Lemma 4.v) that

$$^{12}w_{12} - w = (\{12\}_{(0k)'}/\{12\}_{12})([k'4] - [k'k]^{k3}x_{k3} - [k'3]^{k3}x_{3k}) - {}^{k3}x_{3k}\{123\}_{012}/\{12\}_{12} \geq 0.$$

Now, if the slope check from [01] to [22] in $T_0$ gives $\{12\}_{02} < 0$, the pivot [11] is eliminated.

Rearrange $T_0$ into $T_0$'':

| | | | |
|---|---|---|---|
| [02] | [03] | [01] | |
| [22] | [23] | [21] | [24] |
| [32] | [33] | [31] | [34] |
| [12] | [13] | [11] | [14] |

and $T_2$ into $T_2$'':

| | | | |
|---|---|---|---|
| $\{12\}_{10}$ | $\{13\}_{10}$ | 0 | |
| $\{12\}_{12}$ | $\{13\}_{12}$ | 0 | $\{14\}_{12}$ |
| $\{12\}_{13}$ | $\{13\}_{13}$ | 0 | $\{14\}_{13}$ |
| [12] | [13] | [11] | [14] |

If [22] is the only pivot in $T_0$'', then $w_{22}$ is the maximum of $w$ due to Theorem 1. If [22] and [33] are the only pivots in $T_0$'', then ${}^{23}w_{23}$ is the maximum of $w$ due to Theorem 2.

Finally, let's go back to $T_3$ . If

(10') $\{123\}_{012} > 0$

holds, due to Theorem 1 and Theorem 2, we need only consider the case that $[11]\{123\}_{123}$ is the pivot. In the three-pivot case, we require that

(11) $\{123\}_{123} > 0$

(12) $[11]^2(\{42\}_{12}\{123\}_{123} - \{32\}_{12}\{124\}_{123}) = [11]^2\{12\}_{12}\{423\}_{123} > 0$

(13) $[11](\{14\}_{12}\{123\}_{123} - \{13\}_{12}\{124\}_{123}) = [11]\{12\}_{12}\{143\}_{123} > 0$

(14) $\{124\}_{123} > 0$

(15) $\{13\}_{03} \geq 0$

and

(16) $\{23\}_{03} \geq 0$.

*Theorem 3.*

If the inequalities (9), (10’), (11), (12), (13), (14), (15) and (16) hold, then [11], [22] and [33] are the pivots of $T_0$ and $w^*$ is the maximum of $w$.

*Proof.*

Because of (15) and (16), there is no elimination of pivots as a result of the slope checks. Perform the cross-multiplication at $[11]\{123\}_{123}$ in $T_3$ to yield $T_4$:

$$\begin{array}{cccc} 0 & 0 & 0 & \\ [11]^2\{12\}_{12}\{123\}_{123} & 0 & 0 & [11]^2\{12\}_{12}\{423\}_{123} \\ 0 & [11]\{12\}_{12}\{123\}_{123} & 0 & [11]\{12\}_{12}\{143\}_{123} \\ 0 & 0 & [11]\{123\}_{123} & [11]\{124\}_{123} \end{array}$$

This is the final tableau, from which we see that $(x_1^*, x_2^*, x_3^*)$ is the positive solution due to (12), (13) and (14). Similar to (10’), we can rotate the pivots [11], [22] and [33] to obtain the inequalities

(17) $\{123\}_{031} > 0$

and

(18) $\{123\}_{023} > 0$.

To prove that $w^*$ is the maximum of $w$, due to Lemma 2, we need only show that $w^* \geq w$ in each of the following cases.

Case 1. $w = {}^{12}w_{12}$.

Because of (10’), it follows from lemma 3 that $w^* - w = x_3^*\{123\}_{012}/\{12\}_{12} > 0$.

Case 2. $w = w_{k2}$, where $w_{k2}$ is the least among $w_{i2}$, i = 1, 2, 3.

From Case 2 in the proof of Theorem 2 and Case 1, we have $w^* > {}^{12}w_{12} \geq w_{22} \geq w$.

Case 3. $w = w_{k3}$ , where $w_{k3}$ is the least among $w_{i3}$ , i = 1, 2, 3.

Because of (3'), (6), (9) and (10'), it follows from Lemma 4.i) and Case 1 that either

$w^* \geq {}^{12}w_{12} \geq w_{13}$ or $w^* \geq {}^{12}w_{12} \geq w_{23}$.

Case 4. $w = {}^{k3}w_{12}$ , where $({}^{k3}x_{12}, {}^{k3}x_{21}, 0)$ satisfies $(I_{k'})$, k= 1, 2.

Because of (3'), (6) and (9), it follows from Lemma 4.iii) and Case 1 that

$$w^* - w = (w^* - {}^{12}w_{12}) + ({}^{12}w_{12} - w)$$

$$\geq (\{12\}_{(0k)'}/\{12\}_{12})([k'4] - [k'1]^{k3}x_{12} - [k'2]^{k3}x_{21}) \geq 0.$$

Case 5. $w = {}^{12}w_{k3}$ , where ${}^{12}x_{3k} \geq 0$, k = 1, 2, where $({}^{12}x_{13}, 0, {}^{12}x_{31})$ and $(0, {}^{12}x_{23}, {}^{12}x_{32})$ satisfy $(I_3)$.

Since $\{14\}_{12}$ and $\{42\}_{12}$ are positive, it follows from

$${}^{12}x_{3k}(\{k43\}_{123}/\{k4\}_{12}) = \{k43\}_{312}/\{ k3\}_{12} = [3k]^{12}x_{k3} + [33]^{12}x_{3k} - [34] \leq 0.$$

that ${}^{12}x_{3k} = 0$ because of (12) and (13). Hence, by Case 3, we have $w^* \geq w_{k3} \geq w$.

Case 6. $w = {}^{k3}w_{k'3}$ , k = 1, 2, where $(0, {}^{13}x_{23}, {}^{13}x_{32})$ and $({}^{23}x_{13}, 0, {}^{23}x_{31})$ are feasible.

From (12), (13), (17), (18) and Lemma 3, it follows that

$$w^* - {}^{k3}w_{k'3} = x_k^*\{123\}_{0k3}/\{(k'3)\}_{k3}$$

$$= x_k^*\{123\}_{0k3}([k'4] - [k'k']^{k3}x_{k'3} - [k'3]^{k3}x_{3k'})/\{4(k'3)\}_{k'k3}\} \geq 0.$$

Case 7. $w = {}^{k3}w_{k3}$ , k = 1, 2, where $(0, {}^{23}x_{23}, {}^{23}x_{32})$ and $({}^{13}x_{13}, 0, {}^{13}x_{31})$ are feasible.

From (12), (13), (17), (18) and Lemma 3, it follows that

$$w^* - {}^{k3}w_{k3} = x_{k'}^*\{123\}_{0k3}/\{(k3)\}_{k3}$$

$$= x_{k'}^*\{123\}_{0k3}([k'4] - [k'k]^{k3}x_{k3} - [k'3]^{k3}x_{3k})\{4(k3)\}_{k'k3}\} \geq 0.$$

GLOSSARY

**Pythagorean Theorem:** The sum of the squares of the lengths of each of the right triangle's legs is the same as the square of the length of the triangle's hypotenuse.

**Combinatorics:** The branch of mathematics dealing with combinations of objects belonging to a finite set in accordance with certain constraints.

**Mathematical induction:** To prove a statement S(n) is true for any natural number n, it suffices first to establish the inductive basis [to prove S(1) is true] and then to provide the inductive step [to prove S(m+1) is true by assuming S(m) is true].

**Row move:** A move to place a number in a grid by observing a certain row.

**Column move:** A move to place a number in a grid by observing a certain column.

**Box move:** A move to place a number in a grid by observing a certain box.

**Grid move:** A move to place a number in a grid by observing a certain grid.

**Terminating move**: A move to place a number in a grid to fill up a row, column or box.

**Situational move:** A move after carefully studying the whole situation when stuck.

**Balducci assumption:** When $0 < t < 1$, the mortality rate ${}_tq_x$ can not be found in a life table. Under this assumption, the reciprocal interpolation is used.

**U-assumption:** When $0 < t < 1$, the mortality rate ${}_tq_x$ can not be found in a life table. Under this assumption, the linear interpolation is used.

**CSO:** The acronym for Commissioners Standard Ordinary.

**SOA:** The acronym for Society of actuaries.

**ARCH:** The acronym for Actuarial Research Clearing House, which is one of the two SOA publications of articles, the other is Transactions.

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APENDIX A: LIST OF TABLES

APENDIX B: LIST OF FIGURES

EDITORS PAGE

Editors of *"EVOLUTIONARY PROGRESS IN SCIENCE, TECHNOLOGY, ENGINEERING, ARTS AND MATHEMATICS (STEAM)"*

1. Dr. Lawrence K. Wang (王抗曝)

Lawrence K. Wang has over 30+ years of professional experience in facility design, environmental sustainability, natural resources, STEAM education, global pollution control, construction, plant operation, and management. He has expertise in water supply, air pollution control, solid waste disposal, drinking water treatment, waste treatment, and hazardous waste management. He was the Director/Acting President of the Lenox Institute of Water Technology, Engineering Director of Krofta Engineering Corporation and Zorex Corporation, and a Professor of RPI/SIT/UIUC, in the USA. He was also a Senior Advisor of the United Nations Industrial and Development Organization (UNIDO) in Austria. Dr. Wang is the author of over 700 technical papers and 45+ books, and is credited with 24 US patents and 5 foreign patents. He earned his two HS diplomas from the High School of National Taiwan Normal University and the State University of New York. He also earned his BS degree from National Cheng-Kung University, Taiwan, ROC, his two MS degrees from the University of Missouri and the University of Rhode Island, USA, and his PhD degree from Rutgers University, USA. Currently he is the Chief Series Editor of the Handbook of Environmental Engineering series (Springer); Chief Series Editor of the Advances in Industrial and Hazardous Wastes Treatment series, (CRC Press, Taylor & Francis); co-author of the Water and Wastewater Engineering series (John Wiley & Sons); and Co-Series Editor of the Handbook of Environment and Waste Management series (World Scientific). Dr. Wang is active in professional activities of AWWA, WEF, NEWWA, NEWEA, AIChE, ACS, OCEESA, etc.

2. Dr. Hung-ping Tsao（曹恆平）

Hung-ping Tsao has been a mathematician, a university professor, and an assistant actuary, serving private firms and universities in the United States and Taiwan for 30+ years. He used to be an Associate Member of the Society of Actuaries and a Member of the American Mathematical Society. His research have been in the areas of college mathematics, actuarial mathematics, management mathematics, classic number theory and Sudoku puzzle solving. In particular, bikini and open top problems are presented to share some intuitive insights and some type of optimization problems can be solved more efficiently and categorically by using the idea of the boundary being the marginal change of a well-rounded region with respect to its inradius; theory of interest, life contingency functions and pension funding are presented in more simplified and generalized fashions; the new way of the simplex method using cross-multiplication substantially simplified the process of finding the solutions of optimization problems; the generalization of triangular arrays of numbers from the natural sequence based to arithmetically progressive sequences based opens up the dimension of explorations; the introduction of step-by-step attempts to solve Sudoku puzzles makes everybody's life so much easier and other STEAM project development. Dr. Tsao is the author of 3 books and over 30 academic publications.  Among all of the above accomplishments, he is most proud of solving manually in the total of ten hours the hardest Sudoku posted online by Arto Inkala in early July of 2012. He earned his high school diploma from the High School of National Taiwan Normal University, his BS and MS degrees from National Taiwan Normal University, Taipei, Taiwan, his second MS degree from the UWM in USA, and a PhD degree from the University of Illinois, USA.

Editors of the eBOOK Series of the *"EVOLUTIONARY PROGRESS IN SCIENCE, TECHNOLOGY, ENGINEERING, ARTS AND MATHEMATICS (STEAM)"*

Dr. Lawrence K. Wang (王抗曝) - - left

Dr. Hung-ping Tsao (曹恆平) - - right

E-BOOK SERIES AND CHAPTER INTRODUCTON

Introduction to the eBOOK Series of the *"EVOLUTIONARY PROGRESS IN SCIENCE, TECHNOLOGY, ENGINEERING, ARTS AND MATHEMATICS (STEAM)"* and This Chapter *"*MATHEMATICS OF HUNG-PING TSAO*"*

The acronym STEM stands for "science, technology, engineering and mathematics". In accordance with the National Science Teachers Association (NSTA), "A common definition of STEM education is an interdisciplinary approach to learning where rigorous academic concepts are coupled with real-world lessons as students apply science, technology, engineering, and mathematics in contexts that make connections between school, community, work, and the global enterprise enabling the development of STEM literacy and with it the ability to compete in the new economy". The problem of this country has been pointed out by the US Department of Education that "All young people should be prepared to think deeply and to think well so that they have the chance to become the innovators, educators, researchers, and leaders who can solve the most pressing challenges facing our nation and our world, both today and tomorrow. But, right now, not enough of our youth have access to quality STEM learning opportunities and too few students see these disciplines as springboards for their careers." STEM learning and applications are very popular topics at present, and STEM related careers are in great demand. According to the US Department of Education reports that the number of STEM jobs in the United States will grow by 14% from 2010 to 2020, which is much faster than the national average of 5-8 % across all job sectors. Computer programming and IT jobs top the list of the hardest to fill jobs. Despite this, the most popular college majors are business, law, etc., not STEM related. For this reason, the US government has just extended a provision allowing foreign students that are earning degrees in STEM fields a seven month visa extension, now allowing them to stay for up to three years of "on the job training". So, at present STEM is a legal term.

The acronym STEAM stands for "science, technology, engineering, arts and mathematics". As one can see, STEAM (adds "arts") is simply a variation of STEM. The word of "arts" means application, creation, ingenuity, and integration, for enhancing STEM inside, or exploring of STEM outside. It may also mean that the word of "arts" connects all of the humanities through an idea that a person is looking for a solution to a very specific problem which comes out of the original inquiry process. STEAM is an academic term in the field of education. The University of San Diego and Concordia University offer a college degree with a STEAM focus. Basically STEAM is a framework for teaching or R&D, which is customizable and functional, thence the "fun" in functional. As a typical example, if STEM represents a normal cell phone communication tower looking like a steel truss or concrete column,STEAM will be an artificial green tree with all devices hided, but still with all cell phone communication functions. This ebook series presents the recent evolutionary progress in STEAM with many innovative chapters contributed by academic and professional experts.

This ebook chapter, "MATHEMATICS OF HUNG-PING TSAO" is Dr. Hung-ping Tsao's collection of thoughts, works and talks about various basic mathematical problems encountered through twenty years of learning plus twenty years of teaching. From time to time, he would share his innovative and artful ideas with all levels of audience by giving talks to college and high school students in U.S. as well as in Taiwan.