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SPATIAL INTERACTION MODELS USING AGGREGATED INFORMATION

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1. INTRODUCTION

Despite their principal use as predictors of response to changing conditions, most travel demand models are estimated from data collected in single time periods. The fact of high mutual correlation between explanatory variables is perhaps the single greatest problem for the analyst, and has led to increasing use of disaggregate approaches, and a widespread recognition of the need for the 'temporal validation' of the models. Ideally, of course, such models would be estimated from data sets spanning different time periods; in practice the only historic information that is usually available is highly aggregate and incomplete.

For example, the calibration of a conventional mode-split/distribution model requires the collection of a sufficient set of zone-to-zone flows by each mode to identify the model parameters; such information will not normally exist for other time periods. However, it may happen that for previous years ticketing records and automatic traffic counts do provide information about certain specific groups of these flows. Such information is potentially useful, and may even allow a time dimension to be incorporated in the model specification.

The second major use of conventional models is to 'infill' incomplete data to establish a current pattern of demand; here too estimates of aggregate quantities often exist, and could be used.

In addition to such directly observed groups as flows on roads or revenue collected, there may be independent estimates of other aggregates which can also be used in the model fitting process, or in forecasts. The most familiar of these would be trip-end estimates calculated on the basis of socio-economic characteristics of households in each zone. More recently it has been suggested that the average amounts of time and money allocated to travel, so called 'travel budgets', can also be estimated directly and used to inform more detailed forecasts of travel patterns.

The aims of this paper are as follows; firstly (in Section 2), it will present some recent evidence for the nature of 'travel budgets', and for the implications that the stability of these budgets has for the form of travel demand models. Secondly (in Section 3), it will outline an approach by which all of the forms of data described above may be used during the model fitting, together with a corresponding algorithm which applies to the most common form of spatial interaction model, the logit model.

TRAVEL BUDGETS

Various researchers have established the relative invariance of the average amounts of time and money spent on travel by people living in different types of location, e.g. urban and rural dwellers. (See for example in Gunn, 1979). Interpretations of this phenomenon differ, however. Zahavi (1974, 1979) has taken this stability as evidence for the existence of 'optimal' amounts of travel, for given network conditions and population characteristics. Golob et al (1980) have developed this idea further, leading to a 'utility maximising' description of individual travel behaviour in which travel is treated as a directly demanded commodity. In this line of theory, the stability of travel budgets is identified as being primarily a feature of the demand for travel. Goodwin (1973) has suggested an alternative mechanism, namely the association between population densities, trip lengths and speeds (via congestions), which would account for the phenomenon as a result of characteristics of the 'supply' side of travel.

Potentially, then, the apparent 'stability' of travel budgets raises questions for the basic structure of forecasts of travel demand.

2.1 Some recent evidence

The nature of individual travel budgets is one of high person-to-person variability, with systematic differences according to various socio-economic characteristics. Table 1 sets out the main effects from a cross-sectional model fitted to some 10,000 individual travel diaries collected for the UK County Surveyors' Trip Rate Data Bank in 1974. Historically, in Britain, the trend has been of a steady growth in both time and money outlay on travel (Tanner, 1979).

Table 1: Analysis of Individual Travel Times : Minutes per day

Sex: male	0	**Income:**		**Day-of-Week:**	
female	-9	low	0	Mon	0
		med-low	0	Tues	0
Age: 5-16	0	med-high	+8	Wed	+5
17-24	+16	high	+20	Thurs	+6
25-59	0			Fri	+14
60+	-5	**Location:**			
		urban central	0	**Constant:**	63
Occupation:		suburban	+5		
professional	+14	rural/small town	-5	**Car-Ownership:**	
non-prof.	0	rural	0	0	0
non-worker	-19			1	0
				2+	0

Some recent work (Gunn 1980) has been directed towards a more detailed examination of these patterns, by the device of separating the amount of travel for activities which are, in the main, fixed in location and frequency of participation (activities deemed 'mandatory'), from the rest. The latter are 'discretionary', in the sense that they may be (more easily) varied in either or both of location and frequency of visit. Thus the amount of travel performed in connection with 'discretionary' activities is to some extent under the control of the individual, whereas that connected with 'mandatory' activities is not, at least in the short run.

Table 2 presents the main patterns of variation in individual travel times by various background variables.

Table 2: Analysis of individual travel : M-times and D-times : Minutes per day

	M	D		M	D		M	D
Sex: male	0	0	**Income:**			**Day:**		
female	-14	0	low	0	-4	Mon	0	0
			med-low	0	0	Tues	0	0
Age: 5-16	+16	-39	med-high	+6	0	Wed	0	+7
17-24	+13	0	high	+10	0	Thurs	0	+6
25-59	0	0				Fri	0	+11
60+	-14	+4	**Location:**					
			urban	0	0	**Cars:**		
Occupation:			suburban	+2	0	0	0	0
professional	+10	0	rural/small			1	0	0
non-prof.	0	0	town	-5	0	2+	0	0
non-worker	-31	+26	rural	0	0			
						Constant:	48	22

Given this analysis, it is possible to identify groups of individuals within which there are as few as possible systematic differences in average levels of travel, and then to compare travel time spent in connection with discretionary activities (D-travel) with that spent in connection with mandatory activities (M-travel). Figures 1 and 2 set out the average levels of D-travel for different levels of M-travel for two such groups of individuals, distinguishing between those reporting any travel at all, and those reporting both M-travel and D-travel. Figures 3 and 4 display the proportions of those interviewed reporting M-travel who did not report any D-travel, by level of M-travel. Broadly speaking, it may be seen that neither the frequency of reporting D-travel nor the level of D-travel performed were affected by the level of M-travel undertaken. There is no indication of any connection between travel associated with 'discretionary' activities and travel associated with 'mandatory' activities.

Table 3 sets out corresponding models fitted at the level of the household, relating time, cost and 'generalised cost' travel expenditures to background variables. (The 'generalised cost' calculation involved a single approximate value-of-time estimate for each individual, and is on a very crude basis.)

The variables involved are: IN, the number of household members with occupations involving travel; WINC, defined as household income multiplied by IN; NPRES, the number of household members (over 5 years of age); NCAR, defined as the number of cars available multiplied by NPRES; and TUES-FRID, being defined each as zero or NPRES, according to the day-of-the-week of the interview.

Table 3: Household Models

	Variables									
Expenditure	IN	WINC	NCAR	NPRES	TUE	WED	THUR	FRI	CONS	R^2
M-time	32	3.0	–	X	X	X	X	X	10	.49
D-time	X	X	2.0	10.4	-4	–	–	7	40	.05
M-cost	2	1.6	0.8	X	X	X	X	X	6	.23
D-cost	X	X	1.0	–	–	1	1	2	7	.04
M-gen cost	18	3.1	1.0	X	X	X	X	X	6	.40
D-gen cost	X	X	2.1	6.4	–	2	3	4	13	.12

X denotes not included in regression: – denotes non-significant, set to zero.

Figures 5, 6 and 7 set out average D-travel expenditures corresponding to different levels of M-travel expenditure for household groups selected to have similar income, size and 'active' membership, in the sense of the IN variable. As with individual travel times, there is little or no indication of any connection between household M-travel and D-travel at the household level, although there is much more fluctuation (as might be expected with smaller sample sizes).

A: IN=1, NPRES=2; B: IN=2, NPRES=3; C: IN=3, NPRES=4

These results cast a rather different light on the nature of 'travel budgets' than did the earlier observation that overall travel times were relatively invariant as between residents of different area types, made from the same data. Far from suggesting stable regularities in the amounts of travel undertaken, they suggest at least two independent dimensions of travel expenditure, namely the categories that we have labelled M-travel and D-travel. This is, of course, exactly what might have been expected from the conventional view of travel as having a purely derived demand.

2.2 'Behavioural' frameworks

The key issue concerns the way in which the travel costs of alternative activity schedules and location decisions affect behaviour. Formally, Golob et al give:

> "A definition of commodity groups relevant to modelling travel decisions is
>
> $$u = u\,(x,c,t) \tag{1}$$
>
> where u is household utility, x is the amount or quantity of travel, c is consumption of non-travel goods and services.... and t is leisure time.... Specifying price indices for travel and general consumption as p_x and p_c respectively, the household faces the following money budget constraint when allocating expenditures:
>
> $$p_x\,x + p_c\,c \leq Y \tag{2}$$
>
> where Y is household disposable income. Similarly, the time budget constraint is
>
> $$t_x\,x + t \leq T \tag{3}$$
>
> where t_x is given time per unit distance travelled, and T is the total time available to all household members."

Travel undertaken is then the outcome of a rational decision to maximise u(x,c,t) with respect to x,c and t, subject to constraints (2) and (b). Within this framework, travel is deemed 'useful' in its own right. Given realistic restrictions on the form of the function u(.), Golob et al then show exactly how powerful this assumption is; it is shown that, under this hypothesis,

a) travel can never decrease as income increases;
b) travel can never decrease as available time increases;
c) travel decreases with increasing (travel) costs; and
d) travel increases with increasing speeds.

The more conventional view of travel as a derived demand does not lend itself so readily to simple descriptions of overall travel behaviour; it is necessary to consider travel decisions jointly with the non-travel commodities whose consumption in particular locations necessitates the travel expenditure. For the purpose of comparison, consider a situation in which work hours and income are both fixed, as in the representation given by Golob et al. Further, suppose that work place is also fixed, and consider the travel related decisions of an unattached individual. Once again for simplicity, suppose there are only two non-travel commodities of interest, home location (a set of specific houses in particular areas at given prices) and leisure activities (available in varying degrees of attractiveness and at varying prices in different locations).

A more conventional representation of behaviour might distinguish between two different sorts of decision; corresponding broadly to long-term and short-term options, although the long-term decisions may well be informed by expectations of short-term decisions, so that no rigid ordering is necessary. For convenience, consider the process as a joint decision between K possible home locations and B possible alternative 'bundles' of consumption activities. A 'bundle' of activities consists of a set of activities with associated costs in specific locations, visited in a particular order. Thus there is a unique cost of consumption activities and cost of travel associated with each bundle, given a choice k of home location.

If we denote the 'attractiveness' of home location k, excluding rent and travel costs, by H_k, and that of consumption bundle b, excluding costs and travel costs, as L_b, and further denote rent by R_k, consumption times and costs by TP_b and P_b, work related travel quantity from home location k by TW_k and finally travel associated with consumption bundle b together with home location k by TL_{kb}, we can represent the 'rational decision' which results in travel activity as

$$\max_{k,b} U = U(H_k, L_b, TW_k, TL_{kb})$$

$$\text{subject to} \quad P_x\,(TW_k + TL_{kb}) + P_b + R_k \leq Y$$

$$t_x\,(TW_k + TL_{kb}) + TP_b \leq T$$

Subsequent reappraisal of either or both decision will be in order when the circumstances, prices or individual tastes change.

This sort of framework can be extended to accommodate joint or sequential choices involving individual or household decisions as to home location, work place, school location and so on. Decisions as to car purchase and use, together with mode split choices for different trips, all add extra dimensions of complexity to the representation. The outcome of such a choice process will be a set of realised trips; unfortunately, the theory says nothing in general about the travel these involve, let alone produces conclusions of the power of the four outline by Golob et al relating travel to incomes, available time, network speeds and travel costs. Realised travel would depend on the particular circumstances, including the exact natures and locations of alternative home and work places considered.

It then follows that observed patterns in realised travel expenditures should be interpreted as much in terms of the 'supply' of activities with associated costs in particular locations, and the network that connects them, as to the nature of the demand for travel.

This interpretation directs us to look for 'compensating mechanisms' in aggregate behaviour to explain the stability in travel budgets. Goodwin (1973) has suggested network congestion as one such mechanism. Minimisation of journey times, maximisation of access to facilities and 'low-density' housing taken as simultaneous objectives would also tend to have off-setting effects on overall travel. This suggests that independent estimates of travel budgets could be incorporated in the estimation of conventional models, and in their use for forecasting, providing an implicit assumption of the continuation of 'status quo' in terms of land use and network were acceptable. These conclusions are at direct odds with those of Zahavi; the role that is outlined for the incorporation of estimates of travel budgets into forecasts is of a minor, almost incidental nature, in contrast ot the central part that they play within the UMOT approach. Further research is underway to establish whether or not the trends discovered in the data set described above persist in later years.

Finally, it is interesting to note that the historic trends in national average travel times and costs per person, as estimated by Tanner (1979), are broadly consistent with the cross-sectional models outlined in Table 3 taken together with historic trends in activity rates, household sizes and age structures, incomes and car-ownership levels. Figure 5 plots the corresponding model "predictions" for travel costs and times per person over a 25-year period; note that the absolute levels of the two series are not to be compared, for one reason because the model refers only to weekday travel.

Over the period activity rates have risen and then declined, mainly as a result of increasing proportions of the retired offsetting greater female participation in the workforce; these trends have been counter-balanced to some extent by the effects of increased wealth and car-ownership. In the context of steadily reducing household sizes, this implies an increasing trend of travel per person. Whilst the models are obviously cruede (for example the effects associated with incomes and car-ownership must also reflect complicated changes in land-use), the general similarity in the overall trends is encouraging. Figure 6 illustrates the model estimates of the component 'mandatory' travel times per 'active' household member, and 'discretionary' travel times per household member, over the same period, roughly 2/3 of the overall growth is attributed to increased travel in connection with discretionary activities.

At a highly aggregate level, then, average travel expenditures have remained fairly stable over a twenty five year period; there is also little evidence of variation between residents of different types of area. Such information could be useful in the estimation and prediction of general spatial interactive models.

The estimation of conventional interaction models from general weighted aggregates of zone-to-zone flows.

The problem of fitting travel demand models defined at one level of aggregation from data at a higher level of aggregation has been considered by various authors in a variety of contexts. Specifically, for conventional logit-type models of zone-to-zone flows, Robillard (1975), Wills (1977) and others have considered estimation from traffic count data, Gunn et al (1980) have outlined a procedure using a combination of individual flow data and estimates of simple unweighted groups of flows (trip-ends), and Gunn (1977) has presented a procedure and estimation algorithm to deal with data grouped in any arbitrary manner. Douthesier and Daganzo (1979) have dealt with the same problem in the context of individual choice models.

The following approach is an extension of that outlined in Gunn (1977). Consider a conventional simultaneous mode-split/distribution model of the form

$$T_{ijk} = \alpha_i \beta_j \gamma_k F_k (\lambda_k^1, \lambda^{m_k}, S^{1_{ijk}}, S^{n_{ijk}} \qquad (5)$$

where T_{ijk} denotes the modelled flow zone i and zone j by mode k,

s^1_{ijk}, s^2_{ijk}, .., s^n_{ijk}, are given measures of the separation of zone i

from zone j by mode k, α_i, β_j, γ_h and λ^1_k, λ^m_k are unknown constants, and $f_k(x)$ is some given function. (This particular form is chosen for illustration only; the approach holds for any general model). Such models are usually estimated from a complete set of observations of all zone-to-zone flows by each mode, although a subset of such flows would suffice if chosen to allow all the unknown parameters to be identified. Commonly, a survey provides a set of estimates $\{t_{ijk}\}$ where t_{ijk} is a Poisson variate with mean T_{ijk}/N_{ijk} on the assumption that the model fits. Here $1/N_{ijk}$ denotes a sampling fraction, which will in general vary from flow to flow. It is now fairly generally accepted that the method of Maximum Likelihood (M.L.) is the most efficient way to estimate the unknown parameters from such a data set, although it is seldom used in practice. (The equations which define the M.L. solution contain terms such as $(T_{ijk}/N_{ijk} - l_{ijk})$. These are normally replaced by computationally simpler terms such as $(T_{ijk} - N_{ijk}\ t_{ijk}*)$ which thus leads to a non-likelihood-maximising set of parameter values).

Consider any arbitrary, weighted sum of flows; this can be written as

$$\sum_{ijk} W_{l_{ijk}} T_{ijk} = W^l T$$

Suppose that we have direct independent estimates $\{G^l, l=1, L\}$ of groups $\{\mathbf{W}^l, \mathbf{T}, l=1, ,L\}$ where the matrices $\mathbf{W}^l$ are given. Suppose further that we know the forms of the distributions of every G^l around $\mathbf{W}^l\,\mathbf{T}$. Assuming the model is correct, the problem is then to estimate the unknown parameters $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$, and $\underline{\lambda}$ - for brevity, denote these by $\underline{p}$ - together with their variance covariance matrix. We can write the likelihood function

$$L(\underline{p}\,1\,\{G^{l}\}) = \prod_{l-1}^{L} Pr(G^{l}/f)$$

and choose values of p to maximise this, or equivalently l = log L. In practice, only two distributions are likely to be encountered, namely the Poisson (for single flow estimates from surveys - see Kirby and Leese 1978) and the Normal (when groups of flows are described only by mean and variance, it is convenient to suppose that they are Normally distributed.) However, data divided into even two sets with different sorts of distribution could lead to unwieldy solution algorithms. This problem might be avoided if we can find a transformation for either set of data such that, after transformation, it is distributed as the other set. Variance stabilising transformation for Poisson data are available - the square root transform is the most common - but these are less satisfactory for low expected frequencies than high, and in practice T_{ijk}/N_{ijk} may be very small for many cells. Consequently it seems better to look for a 'variance destabilisation' of the Normal data.

Redhouse (1977) has suggested the following approach:
if X is normally distributed, with mean μ and variance σ^2, then mX/σ^2 will also be normally distributed with mean $m/n/\sigma^2$ and variance m^2/σ^2 (m being a constant); hence mX/σ^2 may be described as 'approximately Poisson' if $m \approx \mu$.

In practice of course, μ will be initially unknown, although it may be estimated by X, and then iteratively updated during the model fitting if so desired.

This approximation involves replacing terms such as

$$\frac{(x-\mu)}{\sigma^2} \quad by \quad \frac{(X-\mu)}{\sigma^2}\ \frac{m}{\mu}\ \frac{d\mu}{dp}$$

in the first-order conditions for a maximum of the likelihood function. Thus if m is sufficiently close to μ, neither the solution point nor the usual estimate of the various covariance matrix will be much affected. Accepting this approximation, we need only consider the problem of maximising a function of the form

$$l(\underline{p}) = \sum_{l=1}^{L} [-(\boldsymbol{W}^{l}\ \boldsymbol{T}(\underline{p})) + G^{l} \log(\boldsymbol{W}^{l}\boldsymbol{T}(\underline{p}))] \tag{8}$$

where both G and each element of the W matrix for a group with estimate Normally distributed with variance σ^2 have been multiplied by a factor m/σ^2
(Note that if m is iteratively updated to **W.T** during the estimation, there is as yet no proof of convergence under all circumstances). As usual, to be practically useful, a maximisation strategy must be found which avoids the need for matrix inversion, given the number of zones (and h ence paramaters) involved.

At the maximum, we shall have

$$\sum_{l-1}^{L} \frac{d}{dp} (\boldsymbol{W}^l \boldsymbol{T}(\underline{p})) \left[\frac{G^l}{\boldsymbol{W}^l \boldsymbol{T}(\underline{p})} - 1 \right] = 0 \ for \ \ l = 1, \ \ , P \tag{9}$$

where P = dim $\underline{p}$

In the specific case of the (logit) model of equation (5), these first order conditions will fall into four subsets. Denoting

$$S^l(\underline{p}) = \boldsymbol{W}^l \ \boldsymbol{T}(\underline{p}) = \sum_{ijk} W^l_{ijk} \ T_{ijk} \ (\underline{p}), \ and$$

$$D^l(P_i) = \frac{d}{dp} S^l(\underline{p}), \ \ where \ \underline{p} \ = (\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \lambda)$$

we have

$$D^l(\alpha_i) = \frac{1}{\alpha_i} \sum_{jk} W^l_{ijk} \ T_{ijk}(\underline{p}) = E^l(\frac{\alpha_{i\underline{p})}}{\alpha_i} \ i{=}1, \ dim\underline{\alpha} \tag{10}$$

$$D^l(\beta_j) = \frac{1}{\beta_j} \sum_{ik} W_{ijk} \ T_{ijk} \ (\underline{p}) \ = \frac{E^l(\beta_j, \ \ Ep)}{\beta_j} \ j{=}1 \tag{11}$$

$$D^l(\gamma_k) = \frac{1}{\gamma_k} \sum_{ij} W^l_{ijk} \ T_{ijk}(\underline{p}) = E^l(\gamma_k,\underline{p}) = \frac{E^l(\gamma_k,\underline{p})}{\gamma_k} \ k{=}1. \ dim\underline{\gamma} \tag{12}$$

$$D^l(\lambda^{r_k}) = \sum_{ij} \frac{1}{f_k} \frac{df_k}{d\lambda_k} W^l_{ijk} \ T_{ijk}(\underline{p}) \ \ k{=}1, dim\underline{\gamma}, r{=}1, m \tag{13}$$

For a simple example, let

$$f_k \ (x) = \exp \ (x) \quad k{=}1, \ \ dim \ \underline{\gamma}$$

and suppose m = n = 1, and $\lambda_k = \lambda$, k=1, dim $\underline{\gamma}$

Equations (9) to (13) then give the first-order conditions for a maximum as

The following iterative strategy may be used to solve these equations for $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ and λ.

$$\frac{1}{\alpha_i}\sum_{l=1}^{L} E^{l}(\gamma_l, \underline{p}) \; [\frac{G^{l}}{S^{l}}(\underline{p}) - 1] = 0 \quad i=1, \; dim\, \underline{\gamma} \tag{14}$$

$$\frac{1}{\beta_j}\sum_{l=1}^{L} E^{l}(\beta_j, \underline{p}) \; [\frac{G^{l}}{S^{l}}(\underline{p}) - 1] = 0 \quad j=1, \; dim\underline{\beta} \tag{15}$$

$$\frac{1}{\gamma_k}\sum_{l=1}^{L} E^{l}(\gamma_{k,l}\underline{p}) \; [\frac{G^{l}}{S^{l}}(\underline{p}) - 1] = 0 \quad k=1, \; dim\underline{\gamma} \tag{16}$$

$$\sum_{ijk} S_{ijk} \, W^{l}_{ijk} \, T_{ijk} \; [\frac{G^{l}}{S^{l}}(\underline{p}) - 1] = 0 \tag{17}$$

1. Choose starting values for all the parameters:

   call these $\underline{\alpha}^\circ$, $\underline{\beta}^\circ$, $\underline{\gamma}^\circ$ and λ° (collectively, $\underline{p}^\circ$) Set n = 0

2. For each i=1,.., dim $\underline{\alpha}$ calculate $S^l(\underline{p}^n)$ and $E^l(\alpha_i^n, \underline{p}^n)$ for l = 1, .., L;

$$Set \;\; \alpha_i^{n+1} = \alpha_i^{n} \left\{ \frac{\sum_{l=1}^{L} E^{l}(\gamma_i^n \; \underline{p}^n) \dfrac{G^{l}}{S^{l}(\underline{p}^n)}}{\sum_{l=1}^{L} E^{l}(\alpha_l^n, \underline{p}^n)} \right\}$$

3. Denote ($\underline{\alpha}^{nh}$, $\underline{\beta}^n$, $\underline{\gamma}^n$ and λ° (collectively, $\underline{p}^\circ$) Set n = 0
   Calculate $S^l(\underline{p}^{n+1/3})$ and $E^l(\beta^n_j, \underline{p}^{n+1/3})$ for l = 1, L
   For each j, j = 1, dim $\underline{\beta}$, Set

$$\beta_j^{n+1} = \beta_j^{n} \left\{ \frac{\sum_{l=1}^{L} E^{l}\left(S_j^n, \underline{p}^{n+\frac{1}{3}}\right) \dfrac{G^{l}}{S^{l}}\left(\underline{p}^{n+\frac{1}{3}}\right)}{\sum_{l=1}^{L} E^{l}\left(\beta_j^n, \underline{p}^{n+\frac{1}{3}}\right)} \right\}$$

4. Denote ($\underline{\alpha}^{n+1}$, $\underline{\beta}^{n+1}$, $\underline{\gamma}^n$, λ°) by $\underline{p}^{n+2/3}$
   Calculate $S^l(\underline{p}^{n+2/3})$ and $E^l(\gamma_k^n, \underline{p}^{n+2/3})$ for l=1, L
   For each k, k=1, .., dim $\underline{\gamma}$, set

$$\gamma_k^{n+1} = \gamma_k^n \left\{ \frac{\sum_{l=1}^{L} E^l\left(\gamma_k^n, \underline{p}^{n+\frac{2}{3}}\right) \frac{G^l}{S^l}\left(\underline{p}^{n+\frac{2}{3}}\right)}{\sum_{l=1}^{L} E^l\left(\gamma_k^n, \underline{p}^{n+\frac{2}{3}}\right)} \right\}$$

5. Repeat stages 2, 3 and 4 until $\underline{p}$ converges : at this value of $\underline{p}$ calculate

$$Z(\lambda) = \sum_{ijk} S_{ijk} \, W_{ijk}^l \, T_{ijk} \left[\frac{G^l}{S^l}(\underline{p}) - 1 \right]$$

6. Repeat steps 2, 3, 4 and 5 using different values of λ (and the previous final solutions of $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ as the starting point) until a value λ^* is found such that $Z(\lambda^*) = 0$.

   As usual, various search strategies are available to identify λ^* including second-order algorithms such as Newton Raphson.

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