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Abstract. Military strategists face a difficult task when engaged in a battle against an adversarial force. They have to predict both what tactics their opponent will employ and the outcomes of any resultant conflicts in order to make the best decision about their actions. Game theory has been the dominant technique used by analysts to investigate the possible actions that an enemy will employ. Traditional game theory can be augmented by use of Lanchester equations, a set of differential equations used to determine the outcome of a conflict. This paper demonstrates a novel combination of game theory and Lanchester equations using Colonel Blotto games. Colonel Blotto games, which are one of the oldest applications of game theory to the military domain, look at the allocation of troops and resources when fighting across multiple areas of operation. This paper demonstrates that employing Lanchester equations within a game overcomes some of practical problems faced when applying game theory.

1.0 INTRODUCTION

Military strategists face a difficult task when engaged in battle against an unknown adversarial force. They must allocate their resources in the most efficient manner possible so as to maximize their chance of defeating their opponent, while being mindful of their finite supply of soldiers, weapons, ammunition, etc. This task is not straightforward, nor trivial. To assist in this effort, military strategists typically employ computational approaches to help them evaluate potential battle outcomes and positively influence the overall outcome of a particular engagement. Mathematical models provide the theoretical foundation for these computational approaches. These models, while they can be effective, fall into two camps: 1) those that have a basis in Cold War Era tactics which focus on large scale conflicts with massive weaponry, or 2) newer, agent-based models which are computationally expensive and, often, unable to be validated [1]. This paper introduces two of these mathematical models for battle prediction, namely: Lanchester equations and game theoretic models. The theoretical foundations of game theory and then Lanchester models are discussed. Then, a comprehensive example is provided which showcases the authors' novel approach for combing these techniques used together within the Colonel Blotto problem framework.

2.0 GAME THEORY

Game theory is the study of decision problems involving more than one intelligent agent, or player, and it is used to model how these sophisticated agents interact. The term 'game theory' comes from the application of 'games of strategy' to economic problems by John Von Neumann and Oskar Morgenstern in their seminal book *Theory of Games and Economic Behavior* [2]. Game theory is concerned with determining the *best* way to play a game for any given set of rules. Defining what is meant by best is a non-trivial undertaking because any strategy used by a player must take into account the other player's strategy as well. Game theory is unconcerned about what are good strategies for playing the game unless that good strategy happens to be the best strategy; this distinguishes game theory from everyday gaming.

Game theory was started when, in the nineteenth century, Antoine Cournot proposed an idea that economists should look at situations where there are only a few competitors [3]. Economists had, until that point, only looked at markets without competition, called "Crusoe on his island", or markets when there was infinite competition, called "Multeity of atoms" [4]. The work by Cournot was virtually ignored until John Von Neumann and Oskar Morgenstern wrote their ground-breaking book during the Second World War [2]. This book became the bedrock of modern Game Theory. Seven years later, John Nash developed his Nash Equilibrium concept [5] which allowed Game Theory to become the useful technique within the modern day modeling community.

Classic applications in the defense realm range from looking at international power struggles in Mesquita and Lalman's War and Reason [6] to whether a victor should be magnanimous after the conflict in Bram's Theory of Moves [7]. Game theory has also been applied to the nuclear arms control negotiations that occurred between the United States of America (USA) and Union of Soviet Socialist Republics (USSR) during the Cold War [8].

2.1 Game Representation

Game theory attempts to mathematically capture behavior in strategic situations, or games, in which an individual's success in making choices depends on the choices of others. There are two standard forms with which game theory attempts to display the overall game under consideration; they are: *normal* and *extensive*. Normal-form games use a payoff matrix, for example see Figure 1. Extensive-form games use a game tree, for example see Figure 2.

		RED	
		A	B
BLUE	A	5, 5	0, 9
	B	9, 0	1, 1

Figure 1: Normal-form of a game

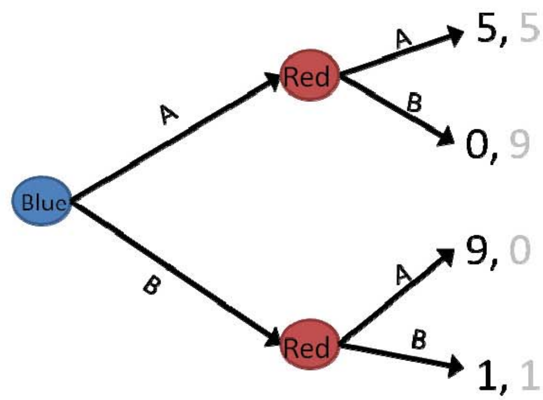

Figure 2: Extensive-form of a game

The rows of a payoff matrix represent the possible actions available to the blue player (BLUE) and the columns represent the possible actions available to the red player (RED). The two values in the cells of the matrix represent that reward or payoff that the players receive. The first number is the payoff to BLUE and the second number is the payoff to RED.

2.2 Nash Equilibrium

The main solution method for game theory is the Nash Equilibrium. It is concerned with the stability of the strategies chosen by the players. A player's strategy is how they choose their action for a game. If a player chooses to go with a single action then this is called a pure strategy. If a player chooses to randomly select an action from several different actions then this is called a mixed strategy. When mixed strategies are used by any of the players, we consider the player's payoff to be the expected payoff.

Given a particular set of strategies for the players it is a Nash Equilibrium if the following statement is true for all players: *Each player does not benefit from changing their current strategy, given the current strategy of the other players*. This does not mean that the players get the maximum payoff available to them within the game but that they gain the highest payoff available under the constraint of the other players' strategies. Some games have multiple Nash Equilibrium solutions.

2.3 Colonel Blotto

One of the oldest applications of game theory to the military domain are Colonel Blotto games. Colonel Blotto games look at the allocation of troops and resources when fighting across multiple areas of operation. The game is based around the simplest assumption that having more troops means that you are more likely to 'win" or control a particular area of operation. The actions that are available for the players are how they allocate their troops to the various areas of operations.

The original paper on Colonel Blotto games was written by Émile Borel, known for his contributions to measure theory, which acts as the foundation for all modern probability theory [9]. Though originally written in 1921, it was translated from French in the 1950's, which was the same time that game theory was becoming popular in the English speaking world due to Von Neumann and Morgenstern's work. It is no consequence that game theory's popularity coincided with the start of the Cold War.

2.4 Criticisms of Game Theory

There are several problems with the application of game theory and a brief summary is given here:

• Payoff determination: It can be difficult to determine what the payoffs for a game should be for each player's actions. It might seem that in some cases the payoffs are obvious, i.e. if the game is about winning money, then the money won or lost should be the payoff; however, this has been shown in examples like the Ultimatum game not to be the case [15]. As game theory's usefulness is about giving insight, and it is not an exact predictive solution, then the use of approximate payoffs is not a "show-stopper" for its application. In many games the outcome remains the same even with slight changes to the payoff. Given this limitation, the authors see utility in utilizing a Lanchester equation-based approach to determining more sophisticated payoffs.

• Determining the Nash Equilibrium: Games can be very easy to construct but can be difficult to solve. This problem can be seen in the game of chess. Chess is a relatively simple game that has been around for centuries, yet no solution to the game has been found, even though a solution has been shown to exist.

• Mixed strategies: When the solution to a game is a mixed strategy this implies that players should randomly choose between different actions. This might make some sense if the game is repeatedly played, but for one-off games this becomes problematic. Imagine that the mixed strategy says 'play action 'A' 99% of the time and action 'B' 1% of the time'. What would you do in this situation? It can be hard to explain this need to randomize to the player. It is also important that your opponent knows that you are going randomly choose your strategy, but how is this done? Should you phone your opponent and tell them you are rolling a die?

• Rational: Game theory assumes that the players are perfectly rational and infinitely intelligent. This assumption might be fine when highly skilled game players are being modeled with game theory, but is not necessarily the case for everyday people in their everyday lives.

3.0 LANCHESTER EQUATIONS

Lanchester analyzed World War I aircraft engagements and developed a theory to explain why concentration of military effort in these entanglements was advantageous [10]. This theory was encapsulated as laws, which are low-resolution aggregated models for determining a battle outcome between two opposing forces. For the purposes of this discussion, the two forces considered are denoted as the blue force (the force the analyst is planning for, aiding, etc.) and the red force (the force the analyst is trying to defeat). This model results in a system of differential equations as follows:

$$dB/dt = f_1(R,B\ldots) \quad (1)$$

$$dR/dt = f_2(R,B\ldots) \quad (2)$$

Where B denotes the blue force and R, the red force. This system of differential equations can be solved by substituting unique functions into Eqs. (1) and (2), depending on whether or not the analyst is determining the outcome of an ancient or modern conflict, resulting in Lanchester's Linear and Square Laws, respectively. The linear law was a formulation for ancient warfare, in which one-on-one combat was the only mechanism of engagement. Recognizing the limitation of his linear law to only describe one-on-one combat, Lanchester formulated his square law, which could be utilized to describe many-on-many combat.

3.1 Stochastic Lanchester Systems

The original formulation of Lanchester's models did not include stochastic behavior. That is, all parameters in the model are taken to be deterministic. This is a limiting assumption of the Lanchester models and does not represent reality, as uncertainty abounds, especially in combat situations. To that end, several extensions which amend Lanchester's model to include stochastic behavior have been developed.

Brown [11] developed an approximation to estimating the probability that *B* blue force members would neutralize *R* red force members (denoted as $P_{B,R}$).

$$P_{B,R} = \sum_{j=1}^{B} \frac{(-1)^{B-j} j^{B+R}}{[(B-j)!(R+j)!]} \quad (3)$$

Where *j* is a variable indice, and all other variables are as before. Equation 3 assumes that each force is depleted by one individual per round. That is, two blue and two red members can be reduced to two blue and one red member or one blue and two red members and not to one blue and one red members (in one battle step). This equation can be used to determine battle outcome probabilities to utilize as outcomes within a game theoretic construct.

3.2 Criticisms of Lanchester Equations

Critics of Lanchester models point out several deficiencies in the original formulation of Lanchester's laws that the authors would be remiss to not mention. Taylor succinctly summarizes them in [12]. Two criticisms were: "Tactical decision processes not considered" and "Battlefield intelligence not considered". Both these criticisms are addressed by coupling Lanchester equations with a game theoretic approach.

Several extensions to the original formulation help to alleviate many of the objections raised. For example, as discussed, the extension proposed by Hester and Tolk [13] allowed for a number of discrete "mini-battles", providing for the discretization of a battle landscape into smaller conflicts.

Despite these shortcomings, the authors believe that the novelty and usefulness of Lanchester models lie in part in their simplicity. The authors do not believe that a simpler, more efficient means of analyzing combat is available to today's modeler. Thus, many have adapted Lanchester's original equations to account for modern warfare scenarios and the authors believe this is a both a valid and useful approach to simple analytical combat modeling, especially when employed in conjunction with the game theoretic environment discussed in the following section.

4.0 MODEL

As mentioned previously, one major drawback in game theory is in the naïve generation of payoffs for games. In an effort to resolve this issue, the authors propose utilizing the battle prediction of Eq. 3 (derived from Lanchester's stochastic square model) to develop the payoffs of the game. A Colonel Blotto game is used as an example and the results are shown in Table 1.

Table 1: Battle Outcome Prediction as Payoffs in a Simple Game

# of BLUE soldiers \ # of RED soldiers	0	1	2	3	4
0	0, 0	0, 1	0, 1	0, 1	0, 1
1	1, 0	0.5, 0.5	0.167, 0.833	0.042, 0.958	0.008, 0.992
2	1, 0	0.833, 0.167	0.5, 0.5	0.225, 0.775	0.081, 0.919
3	1, 0	0.958, 0.042	0.775, 0.225	0.5, 0.5	0.26, 0.74
4	1, 0	0.992, 0.008	0.919, 0.081	0.74, 0.26	0.5, 0.5

Table 1 represents the normal-form of a game. This game is where the players have to allocate up to four soldiers to attack/defend a single guard post. The payoffs for this game are the players' chance of winning the guard post, as calculated using Eq. 3. Unsurprisingly, the Nash Equilibrium of this game is when both players allocate the maximum number of soldiers to the guard post.

Now let us consider the case when there are two guard posts to be attacked / defended. The blue player will allocate soldiers to the front guard post and the remainder will guard the rear post. Similarly, the red player will allocate soldiers to attack the front guard post and the remainder will attack the rear. Once force distributions are chosen via the Lanchester-assisted game theoretic model, an overall battle success probability can be calculated. Results for the analysis of this associated four person battle, and associated force division are shown in Table 2. In this case, the feasible scenarios for this battle are shown (with redundant scenarios eliminated), with the probability of the blue team winning in both a series and parallel (payoffs given in grey brackets) scenario also provided. Only one player's payoff needs to be presented in the table as the game is sum-zero (meaning that any red gain represents a blue loss). The actions represent the number of soldiers allocated to the front guard post with the remaining being allocated to the rear guard post.

Table 2: Battle Outcome Prediction as Payoffs in a Colonel Blotto Game.

# of BLUE soldiers at front post \ # of RED soldiers at front post	0	1	2	3	4
0	0.5 (0)	0.74 (0)	0.919 (0)	0.992 (0)	1 (0)
1	1 (0.26)	0.75 (0.25)	0.813 (0.129)	0.96 (0.04)	1 (0.08)
2	1 (0.081)	0.871 (0.188)	0.75 (0.25)	0.871 (0.188)	1 (0.081)
3	1 (0.008)	0.96 (0.04)	0.83 (0.129)	0.75 (0.25)	1 (0.26)
4	1 (0)	0.992 (0)	0.919 (0)	0.74 (0)	0.5 (0)

The Nash Equilibrium solution to this game is not obvious, unlike the single guard post version. The Gambit software package [14] was used to find the Nash Equilibrium of these games. In the game where only one guard post had to be defended by BLUE, a mixed strategy was found. Their percentage selections of the five actions for the blue player were: (27%, 15%, 16%, 15%, 27%) and for the red player (5%, 36%, 17%, 38%, 5%). This results in an expected payoff for the blue player of 0.86. To gain some understanding to what this payoff means, if the blue player had chosen just to randomly allocate all their soldiers to either of the guard posts, then his or her payoff would have been 0.75.

The mixed strategy for the game when BLUE must win both guard posts results in a payoff of 0.14. Again this game was solved by Gambit and the strategies are (0, 50%, 9%, 41%, 0) for the blue player and (45%, 0, 1%, 54%, 0) for the red player. The reason for these strategy allocations is not immediately obvious and further investigation would be required.

5.0 CONCLUSION

The authors believe this novel approach is useful when compared with the traditional Colonel Blotto approach, whereby payoffs are not analytically derived. This approach affords the battle planner the ability to understand battle results probabilistically, and it can be extended to more complicated battle scenarios (e.g. more combatants, more than two battles).

Military strategists are able to use Lanchester equations and game theory as tools to enable their decision making when planning for a battle against adversarial forces. Using Lanchester equations provides insights into the possible outcomes from a particular conflict, especially the expected attrition levels. Game theory gives insights to what strategies might be employed by an adversarial force and what is the best response to these strategies. A key strength of both of the analytical techniques is their simplicity, which helps any decision-maker gain clear insights into the problem being investigated by presenting the solution in a concise manner.

There is a temptation by modelers to increase the complexity of their models and simulations due to cheaply available computing power. As we saw in the example combining Lanchester equations and game theory given in the previous section, extra complexity within the model produces extra complexity within its solutions. This extra complexity of the solution is too often ignored by only presenting the simple statistics of the results to the decision-makers. The authors would argue that unless the richness of results from a complex model is going to be explored for further utility, then the modeler should utilize simple models to ensure they remain fully aware of what the results imply and gain the insight from that knowledge. Complexity can bring another problem during the modeling stage, as it is easy to construct games that cannot currently be solved – like chess.

As with all modeling methodologies, it is important to remember why you are using a particular technique. Both Lanchester models and game theory represent useful methodologies that can be imbedded into a larger simulation to solve the particular problems that they are designed for, i.e., attrition determination and strategy choice. It is in this embedded usage that the authors envision the true utility of this novel approach.

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