2009

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Ian Balitsky
Old Dominion University, ibalitsk@odu.edu

Giovanni A. Chirilli

Original Publication Citation
Balitsky, I. & Chirilli, G.A. (2009) Conformal kernel for the next-to-leading order BFKL equation in $\mathcal{N} = 4$ super Yang-Mills theory. *Physical Review D*, *79*(3), 1-5, Article 031502(R). https://doi.org/10.1103/PhysRevD.79.031502

PHYSICAL REVIEW D **79,** 031502(R) (2009)

Conformal kernel for the next-to-leading-order BFKL equation in $\mathcal{N} = 4$ super Yang-Mills theory

Ian Balitsky* and Giovanni A. Chirilli†

Physics Department, Old Dominion University, Norfolk, Virginia 23529, USA, and Theory Group, Jefferson Laboratory, 12000 Jefferson Ave, Newport News, Virginia 23606, USA

(Received 18 December 2008; published 25 February 2009)

Using the requirement of Möbius invariance of $\mathcal{N} = 4$ super Yang-Mills amplitudes in the Regge limit, we restore the explicit form of the conformal next-to-leading-order Balitsky-Fadin-Kuraev-Lipatov (BFKL) kernel out of the eigenvalues known from the forward next-to-leading-order BFKL result.

DOI: 10.1103/PhysRevD.79.031502 PACS numbers: 12.38.Bx, 12.38.Cy

The high-energy behavior of perturbative amplitudes is given by the Balitsky-Fadin-Kuraev-Lipatov (BFKL) Pomeron [1]. In the leading order, the BFKL equation is conformally invariant under the Möbius SL(2,C) group of transformations of the transverse plane. In the next-to-leading-order (NLO) the BFKL kernel in QCD is not invariant because of the running coupling, but the kernel in $\mathcal{N} = 4$ super Yang-Mills (SYM) is expected to be invariant. The eigenvalues of this conformal kernel are known from the calculation of forward NLO BFKL in the momentum space [2]. In a conformal theory it is possible to recover the amplitude of the nonforward scattering of two Reggeized gluons from the forward scattering amplitude. Using the NLO kernel for evolution of color dipoles in QCD [3], we guess the Möbius invariant kernel for $\mathcal{N} = 4$ SYM and check that it reproduces known eigenvalues [2].

At high energies the typical forward scattering amplitude has the form

$$A(s,0) = s\int \frac{d^2q}{q^2}\frac{d^2q'}{q'^2}F_A(q)F_B(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}f_+(\omega) \times \left(\frac{s}{qq'}\right)^{\omega}G_\omega(q,q'), \tag{1}$$

where $F_A(q)$, $F_B(q')$ are the impact factors, $f_+(\omega) = \frac{e^{i\pi\omega}-1}{\sin\pi\omega}$ is the signature factor, and $G_\omega(q,q')$ is the partial wave of the forward Reggeized gluon scattering amplitude, satisfying the BFKL equation

$$\omega G_\omega(q,q') = \delta^2(q-q') + \int d^2pK(q,p)G_\omega(p,q'). \tag{2}$$

In $\mathcal{N} = 4$ SYM the kernel $K(q,p)$ is known up to the next-to-leading order [2]

$$\int d^2pK(q,p)f(p) = \frac{\alpha_sN_c}{\pi^2}\int d^2p\left\{\frac{1}{(q-p)^2}\left(1-\frac{\alpha_sN_c\pi}{12}\right)\times\left[f(p)-\frac{q^2}{2p^2}f(q)\right]+\frac{\alpha_sN_c}{4\pi}\left[\Phi(q,p)-\frac{\ln^2q^2/p^2}{(q-p)^2}\right]\times f(p)\right\}+\frac{3\alpha_s^2N_c^2}{2\pi^2}\zeta(3)f(q), \tag{3}$$

where ζ is the Riemann zeta function and

$$\Phi(q,p) = \frac{(q^2-p^2)}{(q-p)^2(q+p)^2}\left[\ln\frac{q^2}{p^2}\ln\frac{q^2p^2(q-p)^4}{(q^2+p^2)^4} + 2\mathrm{Li}_2\left(-\frac{p^2}{q^2}\right)-2\mathrm{Li}_2\left(-\frac{q^2}{p^2}\right)\right] - \left[1-\frac{(q^2-p^2)^2}{(q-p)^2(q+p)^2}\right]\left[\int_0^1-\int_1^\infty\right]\times\frac{du}{(qu-p)^2}\ln\frac{u^2q^2}{p^2}. \tag{4}$$

Here Li_2 is the dilogarithm.

The eigenvalues of the kernel (3) are [2]

$$\int d^2p\left(\frac{p^2}{q^2}\right)^{-(1/2)+i\nu}e^{in\phi}K(q,p) = \omega(n,\nu),$$
$$\omega(n,\nu) = \frac{\alpha_s}{\pi}N_c\left[\chi\left(n,\frac{1}{2}+i\nu\right)+\frac{\alpha_sN_c}{4\pi}\delta\left(n,\frac{1}{2}+i\nu\right)\right],$$
$$\delta(n,\gamma) = 6\zeta(3)-\frac{\pi^2}{3}\chi(n,\gamma)-\chi''(n,\gamma)-2\Phi(n,\gamma) - 2\Phi(n,1-\gamma), \tag{5}$$

where $\chi(n,\gamma) = 2\psi(1)-\psi(\gamma+\frac{n}{2})-\psi(1-\gamma+\frac{n}{2})$ and

*balitsky@jlab.org
†chirilli@jlab.org

$$\Phi(n,\gamma)=\int_0^1\frac{dt}{1+t}t^{\gamma-1+(n/2)}\Big\{\frac{\pi^2}{12}-\frac{1}{2}\psi'\Big(\frac{n+1}{2}\Big)-\mathrm{Li}_2(t)-\mathrm{Li}_2(-t)-\Big(\psi(n+1)-\psi(1)+\ln(1+t)+\sum_{k=1}^{\infty}\frac{(-t)^k}{k+n}\Big)\ln t-\sum_{k=1}^{\infty}\frac{t^k}{(k+n)^2}\times[1-(-1)^k]\Big\}. \tag{6}$$

The Regge limit of the amplitude $A(x,y;x',y')$ in the coordinate space can be achieved as

$$x=\lambda x_*p_1+x_\perp,\qquad y=\lambda y_*p_2+y_\perp,$$
$$x'=\rho x'_\bullet p_2+x'_\perp,\qquad y'=\rho y'_\bullet p_2+y'_\perp, \tag{7}$$

with $\lambda,\rho\to\infty$ and $x_*>0>y_*$, $x'_\bullet>0>y'_\bullet$. Hereafter we use the notations $x_\bullet=p_1^\mu x_\mu$, $x_*=p_2^\mu x_\mu$ where p_1 and p_2 are lightlike vectors normalized by $2(p_1,p_2)=s$. These "Sudakov variables" are related to the usual light-cone coordinates $x^\pm=\frac{1}{\sqrt{2}}(x^0\pm x^3)$ by $x_*=x^+\sqrt{s/2}$, $x_\bullet=x^-\sqrt{s/2}$ so $x=\frac{2}{s}x_*p_1+\frac{2}{s}x_\bullet p_2+x_\perp$. We use the $(1,-1,-1,-1)$ metric so $x^2=\frac{4}{s}x_\bullet x_*-\vec{x}_\perp^2$.

In the Regge limit (7) the full conformal group reduces to Möbius subgroup SL(2,C) leaving the transverse plane $(0,0,z_\perp)$ invariant. In a conformal theory the four-point amplitude $A(x,y;x',y')$ depends on two conformal ratios which can be chosen as

$$R=\frac{(x-x')(y-y')^2}{(x-y)^2(x'-y')^2},$$
$$r=R\Big[1-\frac{(x-y')^2(y-x')^2}{(x-x')^2(y-y')^2}+\frac{1}{R}\Big]^2. \tag{8}$$

The conformal ratio R scales as $\lambda^2\rho^2$ while r does not depend on λ or ρ. Following Ref. [4] (see also Ref. [5]) it is convenient to introduce two SL(2,C)-invariant vectors

$$\kappa=\frac{\sqrt{s}}{2x_*}\Big(p_1-\frac{x^2}{s}p_2+x_\perp\Big)-\frac{\sqrt{s}}{2y_*}\Big(p_1-\frac{y^2}{s}p_2+y_\perp\Big),$$
$$\kappa'=\frac{\sqrt{s}}{2x'_\bullet}\Big(p_1-\frac{x'^2}{s}p_2+x'_\perp\Big)-\frac{\sqrt{s}}{2y'_\bullet}\Big(p_1-\frac{y'^2}{s}p_2+y'_\perp\Big), \tag{9}$$

such that

$$\kappa^2\kappa'^2=\frac{1}{R}\quad\text{and}\quad 4(\kappa\cdot\kappa')^2=\frac{r}{R} \tag{10}$$

(here $x^2=-x_\perp^2$, $x'^2=-x'^2_\perp$ and similarly for y). In the coordinate space the analog of Eq. (1) has the form

$$A(x,y;x',y')=\int d^2z_1d^2z_2d^2z'_1d^2z'_2I_A(x,y;z_1,z_2)\times\int\frac{d\omega}{2\pi}R^{\omega/2}\tilde{f}_+(\omega)G_\omega(z_1,z_2;z'_1,z'_2)\times I_B(x',y';z'_1,z'_2), \tag{11}$$

where $\tilde{f}_+(\omega)=(e^{i\pi\omega}-1)/\sin\pi\omega$ is the signature factor in the coordinate space. The partial wave of the scattering amplitude of two Reggeized gluons satisfies the equation

$$\omega G_\omega(z_1,z_2;z'_1,z'_2)=-\frac{1}{2}\ln^2\frac{(z_1-z'_1)^2(z_2-z'_2)^2}{(z_2-z'_1)^2(z_1-z'_2)^2}+\int d^2t_1d^2t_2K(z_1,z_2;t_1,t_2)\times G_\omega(t_1,t_2;z'_1,z'_2). \tag{12}$$

Here the first term in the right-hand side is the leading-order contribution coming from the two-gluon exchange.

The meaning of the Eq. (11) is that the amplitude is factorized into the product of three terms I_A, I_B, and G_ω corresponding to rapidities $\eta\sim\eta_A$, $\eta\sim\eta_B$, and $\eta_A>\eta>\eta_B$, respectively. With conformally invariant factorization of the amplitude into such a product, the impact factors and G_ω should be separately Möbius invariant leading to invariant kernel $K(z_1,z_2;t_1,t_2)$. The eigenfunctions of a conformal kernel are [6]

$$E_{\nu,n}(z_{10},z_{20})=\Big[\frac{\tilde{z}_{12}}{\tilde{z}_{10}\tilde{z}_{20}}\Big]^{(1/2)+i\nu+(n/2)}\times\Big[\frac{\bar{z}_{12}}{\bar{z}_{10}\bar{z}_{20}}\Big]^{(1/2)+i\nu-(n/2)}, \tag{13}$$

where $\tilde{z}=z_x+iz_y$, $\bar{z}=z_x-iz_y$ and $z_{10}\equiv z_1-z_0$, etc. Denoting the eigenvalues of the kernel K by $\omega(n,\nu)$

$$\int d^2t_1d^2t_2K(z_1,z_2;t_1,t_2)E_{\nu,n}(t_1-z_0,t_2-z_0)=\omega(n,\nu)E_{\nu,n}(z_{10},z_{20}) \tag{14}$$

and substituting the formal solution of Eq. (12) into Eq. (11), we obtain

$$A(x,y;x',y')=\sum_{n=-\infty}^{\infty}\int\frac{d\nu}{\pi^2}\frac{-2(\nu^2+\frac{n^2}{4})R^{(1/2)\omega(n,\nu)}}{[\nu^2+\frac{(n-1)^2}{4}][\nu^2+\frac{(n+1)^2}{4}]}\times\tilde{f}_+(\omega(n,\nu))\int d^2z_0d^2z_1d^2z_2I_A(x,y;z_1,z_2)\times E_{\nu,n}(z_{10},z_{20})\int d^2z'_1d^2z'_2I_B(x',y';z'_1,z'_2)\times E^*_{\nu,n}(z'_1-z_0,z'_2-z_0).$$

As demonstrated in Ref. [4] an impact factor depends on one conformal (Möbius invariant) ratio

$$I_A(x,y;z_1,z_2)=\frac{1}{z_{12}^4}I_A\Big(\frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big),$$
$$I_B(x',y';z'_1,z'_2)=\frac{1}{z'^4_{12}}I_B\Big(\frac{\kappa'^2(\zeta'_1\cdot\zeta'_2)}{2(\kappa'\cdot\zeta'_1)(\kappa'\cdot\zeta'_2)}\Big),$$

where $\zeta_1\equiv p_1+\frac{z_{1\perp}^2}{s}p_2+z_{1\perp}$ and similarly for other ζ's. This enables us to carry out the integrations over z_i and z'_i. The formulas are especially simple when we consider the correlator of four scalar currents such as $\mathrm{Tr}\{Z^2\}$ [$Z=\frac{1}{\sqrt{2}}\times(\phi_1+i\phi_2)$] so that only the term with $n=0$ contributes.

From conformal (Möbius) invariance we get [4]

$$\int \frac{d^2z_1 d^2z_2}{z_{12}^4} I_A\Big(\frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big)\Big(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\Big)^{(1/2)+i\nu} = \frac{1+4\nu^2}{8\pi}\frac{\Gamma^2(\frac12 - i\nu)}{\Gamma(1-2i\nu)}\Big(\frac{\kappa^2}{4(\kappa\cdot\zeta_0)^2}\Big)^{(1/2)+i\nu} I_A(\nu) \quad (15)$$

(here $\zeta_0 \equiv p_1 + \frac{z_{0\perp}^2}{s}p_2 + z_{0\perp}$) and therefore (cf. Ref. [4])

$$(x-y)^4(x'-y')^4\langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle = \frac{i}{2}\int d\nu \tilde f_+(\nu)\frac{\tanh\pi\nu}{\nu} I_A(\nu)I_B(-\nu)\Omega(r,\nu)R^{(1/2)\omega(\nu)}, \quad (16)$$

where $\mathcal{O} \equiv \frac{4\pi^2\sqrt{2}}{\sqrt{N_c^2-1}}\,\mathrm{Tr}\{Z^2\}$, $\omega(\nu)\equiv\omega(0,\nu)$, $\tilde f_+(\nu) \equiv \tilde f_+(\omega(\nu))$, and

$$\Omega(r,\nu) = \frac{\nu^2}{\pi^3}\int d^2z\Big(\frac{\kappa^2}{(\kappa\cdot\zeta)^2}\Big)^{(1/2)+i\nu}\Big(\frac{\kappa'^2}{(\kappa'\cdot\zeta)^2}\Big)^{(1/2)-i\nu}. \quad (17)$$

[Since the integral (17) does not scale with λ, ρ it can depend only on $\frac{(\kappa\cdot\kappa')^2}{\kappa^2\kappa'^2} = \frac{r}{4}$.] Equation (16), obtained in Ref. [7] from general consideration of the Regge limit in a conformal theory, proves the existence of the conformally invariant factorization (11). Note that all of the dependence on large energy (≡ large λ, ρ) is contained in $R^{(1/2)\omega(\nu)}$. For completeness, let us mention that in the leading order in perturbation theory $I(\nu) = \frac{2\pi^2\alpha_s}{\cosh\pi\nu}\frac{N_c}{\sqrt{N_c^2-1}}$.

To restore the NLO BFKL kernel in the coordinate representation (11) from the eigenvalues (5) in the momentum representation we must prove that Eq. (16) agrees with Eq. (1) with the same set of $\omega(\nu)$. (Strictly speaking, we need to demonstrate this property for arbitrary n but here we will do it only for $n = 0$.)

In order to perform Fourier transformation of the correlator (16) we need to relax the limit (7) by allowing small $x_\bullet \sim y_\bullet \sim 1/\lambda$ and $x'_* \sim y'_* \sim 1/\rho$. The conditions (10) for vectors (9) are now satisfied up to $\frac{1}{\lambda^2}$ and $\frac{1}{\rho^2}$ corrections. The correlator (16) takes the form

$$(x-y)^4(x'-y')^4\langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle = \frac{i}{2\pi^3}\int d\nu \tilde f_+(\nu)\nu\tanh\pi\nu\Big[\frac{16x_*y_*x'_\bullet y'_\bullet}{s^2(x-y)^2(x'-y')^2}\Big]^{(1/2)\omega(\nu)} \times\int d^2z_0\Big[\frac{\frac{(x-y)^2}{x_*y_*}}{(\frac{(x-z_0)_\perp^2}{x_*}-\frac{(y-z_0)_\perp^2}{y_*}-\frac4s(x-y)_\bullet)^2}\Big]^{(1/2)+i\nu} I_A(\nu) \times\Big[\frac{-\frac{(x'-y')^2}{x'_\bullet|y'_\bullet|}}{(\frac{(x'-z_0)_\perp^2}{x_\bullet}-\frac{(y'-z_0)_\perp^2}{y'_\bullet}-\frac4s(x'-y')_*)^2}\Big]^{(1/2)-i\nu} I_B(-\nu). \quad (18)$$

The forward scattering amplitude can be defined as (cf. Ref. [8])

$$A(s,0) = -i\int d^4z d^4x d^4y \langle \mathcal{O}(x_\bullet, x_* + z_*, x_\perp + z_\perp) \times \mathcal{O}^\dagger(0, z_*, z_\perp)\mathcal{O}(y_\bullet + z_\bullet, y_*, y_\perp) \times \mathcal{O}^\dagger(z_\bullet, 0, 0)\rangle e^{-ip_A\cdot x - ip_B\cdot y}, \quad (19)$$

where $p_A = p_1 + \frac{p_A^2}{s}p_2$ and $p_B = p_2 + \frac{p_B^2}{s}p_1$. Substituting Eq. (18) in Eq. (19) and performing the integrations over the coordinates we obtain

$$A(s,0) = \frac{\pi^2 s}{(p_A^2p_B^2)^{3/2}}\int d\nu I_A(\nu)I_B(-\nu) \times\Big(\frac{s}{\sqrt{p_A^2p_B^2}}\Big)^{\omega(\nu)} f_+(\nu)\Big(\frac{p_A^2}{p_B^2}\Big)^{i\nu} \times\Big|\frac{\Gamma^2(\frac32+\frac{\omega(\nu)}{2}+i\nu)}{\Gamma(3+\omega(\nu)+2i\nu)}\frac{\Gamma^2(\frac12-i\nu)}{\Gamma(1-2i\nu)}\Big|^2. \quad (20)$$

This should be compared with Eq. (1) which takes the form

$$A(s,0) = s\int\frac{d^2q}{q^2}\frac{d^2q'}{q'^2}F_A(q)F_B(q')\int\frac{d\nu}{2\pi^2}f_+(\nu) \times(q^2)^{-(1/2)+(i\nu/2)}(q'^2)^{-(1/2)-(i\nu/2)}\Big(\frac{s}{|q||q'|}\Big)^{\omega(\nu)} = \int\frac{d\nu}{2\pi^2}\frac{sf_+(\nu)}{(p_A^2p_B^2)^{3/2}}F_A(\nu)F_B(-\nu)\Big[\frac{s}{\sqrt{p_A^2p_B^2}}\Big]^{\omega(\nu)} \times\Big(\frac{p_A^2}{p_B^2}\Big)^{i\nu}. \quad (21)$$

It is clear that Eqs. (20) and (21) coincide after the redefinition of the impact factor

$$F_A(\nu) = \sqrt{2}\pi^2 I_A(\nu)\frac{\Gamma^2(\frac32 - i\nu + \frac{\omega(\nu)}{2})\Gamma^2(\frac12+i\nu)}{\Gamma(3-2i\nu+\omega(\nu))\Gamma(1+2i\nu)}$$

and similarly for F_B.

Now we are in a position to restore $K_{\mathrm{NLO}}(z_1,z_2;t_1,t_2)$ from the eigenvalues (5). Using the eigenvalues $\omega(n,\nu)$ and the requirement of conformal invariance it is possible to restore the conformal kernel for the BFKL equation [6]

$$K(z_1,z_2;z_3,z_4) = \frac{1}{z_{34}^4}\sum_{n=-\infty}^{\infty}\int\frac{d\nu}{\pi^4}\Big(\nu^2+\frac{n^2}{4}\Big)\omega(n,\nu) \times\int d^2z_0 E_{\nu,n}(z_{10},z_{20})E^*_{\nu,n}(z_{30},z_{40}). \quad (22)$$

At the leading-order level K is given by the BFKL kernel in the dipole form (the linear part of the BK equation [9,10])

$$K_{\mathrm{LO}}(z_1,z_2;z_3,z_4) = \frac{\alpha_sN_c}{2\pi^2}\Big[\frac{z_{12}^2\delta^2(z_{13})}{z_{14}^2z_{24}^2}+\frac{z_{12}^2\delta^2(z_{24})}{z_{13}^2z_{23}^2} - \delta^2(z_{13})\delta^2(z_{24}) \times\int d^2z\frac{z_{12}^2}{(z_1-z)^2(z_2-z)^2}\Big]. \quad (23)$$

At the NLO level, to perform explicitly the three summations and four integrations in Eq. (22) seems a formidable task. Instead, using the results of Ref. [3], we guess the NLO kernel in the form

$$K_{\rm NLO}(z_1, z_2; z_3, z_4) = -\frac{\alpha_s N_c}{4\pi}\frac{\pi^2}{3}K_{\rm LO}(z_1, z_2; z_3, z_4) + \frac{\alpha_s^2 N_c^2}{8\pi^4 z_{34}^4}\Big[\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \times \Big\{\Big(1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\Big)\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} + 2\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2}\Big\} + 12\pi^2\zeta(3) z_{34}^4 \delta(z_{13})\delta(z_{24})\Big] \tag{24}$$

and check that its eigenvalues coincide with $\omega_{\rm NLO}$ from Eq. (5). The explicit form of the NLO BFKL kernel (24) is the main result of the present paper. It is worth noting that the first term in braces on the right-hand side corresponds to the analytic term in the conformal part of NLO BK kernel in QCD [3].

Equation (12) with the kernel (24) is obviously conformally invariant. Let us prove that its eigenvalues are given by Eq. (5). The integral

$$\int d^2z_3 d^2z_4 K_{\rm NLO}(z_1, z_2; z_3, z_4)E_{n,\nu}(z_{30}, z_{40}) = \Big[c(n,\nu) + \frac{\alpha_s^2 N_c^2}{4\pi^2}\Big(6\zeta(3) - \frac{\pi^2}{3}\chi(n,\nu)\Big)\Big]E_{n,\nu}(z_{10}, z_{20}) \tag{25}$$

can be reduced to

$$\frac{\alpha_s^2 N_c^2}{8\pi^4}\int dz_3 dz_4 \frac{z_{12}^2}{z_{34}^2 z_{13}^2 z_{24}^2}\Big\{2\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \Big[\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} + 1\Big] \times \ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big\}\Big(\frac{\tilde{z}_{34}}{\tilde{z}_{12}}\Big)^{(1/2)+i\nu-(n/2)}\Big(\frac{\bar{z}_{34}}{\bar{z}_{12}}\Big)^{(1/2)+i\nu+(n/2)} = c(n,\nu)$$

by setting $z_0 = 0$ and making the inversion $x_i \to x_i/x^2$. Taking now $z_2 = 0$ we obtain

$$\frac{\alpha_s^2 N_c^2}{8\pi^4}\int d^2z \frac{z_1^2}{z^2}\Big(\frac{\tilde{z}}{\tilde{z}_1}\Big)^{(1/2)+i\nu-(n/2)}\Big(\frac{\bar{z}}{\bar{z}_1}\Big)^{(1/2)+i\nu+(n/2)} \times \int d^2z' \frac{1}{(z_1 - z - z')^2 z'^2}\Big\{2\ln\frac{z_1^2 z^2}{(z_1 - z')^2(z + z')^2} + \Big[1 + \frac{z_1^2 z^2}{(z_1 - z - z')^2 z'^2 - (z_1 - z')^2(z + z')^2}\Big] \times \ln\frac{(z_1 - z - z')^2 z'^2}{(z_1 - z')^2(z + z')^2}\Big\} = c(n,\nu). \tag{26}$$

Using now the integral

$$\int\frac{d^2z'}{\pi}\frac{\ln(z_1 - z')^2(z + z')^2/(z_1^2 z^2)}{(z_1 - z - z')^2 z'^2} = \frac{1}{(z_1 - z)^2}\ln^2\frac{z^2}{z_1^2}$$

and the integral J_{13} from Ref. [11]

$$\int\frac{d^2z'}{2\pi}\Big[1 + \frac{z_1^2 z^2}{(z_1 - z - z')^2 z'^2 - (z_1 - z')^2(z + z')^2}\Big] \times \frac{z'^{-2}}{(z_1 - z - z')^2}\ln\frac{(z_1 - z - z')^2 z'^2}{(z_1 - z')^2(z + z')^2} = \Phi(z_1, z),$$

[see Eq. (4) for the definition of Φ] we obtain

$$\frac{\alpha_s^2 N_c^2}{4\pi^3}\int d^2z (z^2/z_1^2)^{-(1/2)+i\nu+(n/2)} e^{-in\phi}\Big[-\frac{1}{(z_1 - z)^2} \times \ln^2\frac{z^2}{z_1^2} + \Phi(z_1, z)\Big] = c(n,\nu), \tag{27}$$

where ϕ is the angle between $\vec{z}$ and $\vec{z}_1$. The final step is to use integrals [2]

$$\int\frac{d^2z}{\pi}\frac{1}{(z_1 - z)^2}(z^2/z_1^2)^{\gamma} e^{in\phi}\ln^2\frac{z^2}{z_1^2} = \chi''(n,\gamma)$$

$$\int\frac{d^2z}{2\pi}\Big(\frac{z^2}{z_1^2}\Big)^{\gamma-1} e^{in\phi}\Phi(z_1, z) = -\Phi(n,\gamma) - \Phi(n, 1-\gamma).$$

Comparing to Eq. (3) we see that $c(n,\nu) = \frac{\alpha_s^2 N_c^2}{4\pi^2} \times [-\chi''(n,\nu) - 2\Phi(n, \frac{1}{2} + i\nu) - 2\Phi(n, \frac{1}{2} - i\nu)]$ which corresponds to $\omega_{\rm NLO}$ from Eq. (5).

Let us comment on the result in the literature that NLO BFKL in the coordinate space is not conformally invariant [12]. We think that the difference between our kernel and that of Ref. [12] is due to different cutoffs for longitudinal integrations. As we mentioned above, the conformal result for the NLO BFKL kernel (24) corresponds to the factorization in rapidity consistent with Möbius invariance. In other words, this kernel should describe the evolution of the color dipole with the conformally invariant rapidity cutoff. At present, there is no obvious way to impose such a cutoff, although we believe that it can be done by constructing a "composite operator" for a color dipole, order by order in the perturbation theory. We also think that the Fourier transform of Eq. (18) in the nonforward case would give the precise cutoff for the longitudinal integrations in the momentum space and the change in the cutoff will lead to the transformation $K_{\rm NLO} \to K_{\rm NLO} - [O, K_{\rm NLO}]$ [Eq. (1) of Ref. [12]] which shall cure the discrepancy with the results of Ref. [12].

One can also restore the NLO QCD kernel with the same rapidity cutoff implicitly defined above to satisfy the requirement of the conformal invariance of the $\mathcal{N} = 4$ kernel (24). Using the results of [3,13] one obtains

$$K_{\rm NLO}^{\rm QCD}(z_1, z_2; z_3, z_4) = K_{\rm NLO}(z_1, z_2; z_3, z_4) + \frac{\alpha_s}{4\pi}\Big(b \ln z_{12}^2\mu^2 + \frac{67}{9}N_c - \frac{10}{9}n_f\Big)K_{\rm LO}(z_1, z_2; z_3, z_4) + \frac{\alpha_s^2 N_c}{8\pi^3} b\Big[\delta^2(z_{13}) \times\Big(\frac{1}{z_{14}^2} - \frac{1}{z_{24}^2}\Big)\ln\frac{z_{14}^2}{z_{24}^2} + \delta^2(z_{24})\Big(\frac{1}{z_{13}^2} - \frac{1}{z_{23}^2}\Big)\ln\frac{z_{13}^2}{z_{23}^2} - \delta^2(z_{13})\delta^2(z_{24})\int d^2z_0\Big(\frac{1}{z_{10}^2} - \frac{1}{z_{20}^2}\Big)\ln\frac{z_{10}^2}{z_{20}^2}\Big] + \frac{\alpha_s^2 N_c^2}{8\pi^4 z_{34}^4}\Big[-3\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} + \Big(1 + \frac{n_f}{N_c^3}\Big)\Big(\frac{z_{13}^2 z_{24}^2 + z_{14}^2 z_{23}^2 - z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} - 2\Big)\Big], \quad (28)$$

where $b = 11N_c/3 - 2n_f/3$ and μ is the normalization point in the $\overline{\rm MS}$ scheme. This kernel has the QCD eigenvalues $\omega(n, \nu)$ from Ref. [14]. Note that Eq. (28) is different from the NLO BK kernel for the evolution of color dipoles in Ref. [3] since the "rigid cutoff" $\alpha < \sigma$ adopted in that paper is not conformally invariant.

The authors are grateful to L. N. Lipatov and J. Penedones for valuable discussions. This work was supported by Contract No. DE-AC05-06OR23177 under which the Jefferson Science Associates, LLC operate the Thomas Jefferson National Accelerator Facility. G. A. C.'s work was supported by DOE grant DE-FG02-97ER41028.

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