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Renormalons as Dilatation Modes in the Functional Space

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There are two possible sources of the factorial large-order behavior of a typical perturbative series. First, the number of different Feynman diagrams may be large; second, there may be abnormally large diagrams known as renormalons. It is well known that the large combinatorial number of diagrams is described by instanton-type solutions of the classical equations. We demonstrate that, from the functional-integral viewpoint, the renormalons do not correspond to a particular configuration but manifest themselves as dilatation modes in the functional space.

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It is well known that the perturbative series in a typical quantum field theory is at best asymptotic: the coefficients in front of a typical perturbative expansion grow like $n!$, where n is the order of the perturbation series. There are two sources of the $n!$ behavior which correspond to two different situations. In the first case all Feynman diagrams are ~ 1 but their number is large ($\sim n!$) [1] (for a review, see, e.g., Ref. [2]). In the second case, we have just one diagram but it is abnormally big, $\sim n!$ (the famous 't Hooft renormalon [3]). The first type of factorial behavior is not specific to a field theory; for example, it can be studied in quantum mechanics. However, renormalon singularities can occur only in field theories with a running coupling constant.

It is convenient to visualize the large-order behavior of perturbative series using the 't Hooft picture of singularities [3]. Consider Adler's function related to the polarization operator in (Euclidean) quantum chromodynamics (QCD),

$$D(q^2) = 4\pi^2 q^2 \frac{d}{dq^2}\,\frac{1}{3q^2}\,\Pi(q^2),$$
$$\Pi(q^2) = \int dx\, e^{iqx} \int D\bar{\psi} D\psi DA\, j_\mu(x) j_\mu(0) e^{-S_{\rm QCD}}.$$

Suppose we write down $D(q^2)$ as a Borel integral,

$$D[\alpha_s(q)] = \int_0^\infty dt D(t)\, e^{-[4\pi/\alpha_s(q)]t}. \tag{1}$$

The divergent behavior of the original series $D(\alpha_s(q))$ is encoded in the singularities of its Borel transform, as shown in Fig. 1.

The ultraviolet (UV) renormalons located at $t = -\frac{1}{b}, -\frac{2}{b}, -\frac{3}{b}, \dots$ ($b = 11 - \frac{2}{3}n_f$) come from the regions of hard momenta in Feynman diagrams; the infrared (IR) renormalons placed at $t = \frac{2}{b}, \frac{3}{b}, \dots,$ come from the regions of soft momenta (for a review of the renormalons, see Ref. [4], and references therein). The instanton singularities are located at $t = 1, 2, 3, \dots,$ and they correspond to the large number of graphs. (Actually, the first topologically trivial classical configuration, which contributes to the divergence of perturbation theory, is not an instanton itself but a weakly coupled instanton–anti-instanton pair [5].) The main result of this paper is to demonstrate that, unlike the instanton-type singularities, the renormalons do not correspond to a particular configuration but manifest themselves as dilatation modes in the functional space.

The interpretation of renormalons as dilatation modes is based upon the similarity of the functional integral, in the vicinity of a valley in the functional space [6], to the Borel representation (1). At first we will consider a quantum mechanical example without renormalons and then demonstrate that in field theory the same integral along the valley leads to renormalon singularities. With QCD in view, we consider the double-well anharmonic oscillator described by the functional integral

$$\int D\phi\, e^{-S(\phi)/g^2}, S(\phi) = \frac{1}{2}\int dt\,[\dot{\phi}^2 + \phi^2(1-\phi)^2].$$

The large-order behavior in this model is determined by the instanton–anti-instanton ($I\bar{I}$) configuration. The $I\bar{I}$ valley for the double-well system may be chosen as

$$f_\alpha(t-\tau) = \frac{1}{2}\tanh\frac{t-\tau+\alpha}{2} - (\alpha \leftrightarrow -\alpha). \tag{2}$$

It satisfies the valley equation [6,7]

$$\frac{12}{\xi^2}\, w_\alpha(t) f'_\alpha(t) = L_\alpha(t), \tag{3}$$

where $\xi \equiv e^\alpha$, $f'_\alpha \equiv \frac{\partial}{\partial\alpha} f_\alpha$, and $L_\alpha(t) = \frac{\delta S}{\delta\phi}|_{\phi=f_\alpha(t)}$. Here $w_\alpha(t) = \frac{\xi}{4}\sinh\alpha(\cosh\alpha\cosh t + 1)^{-1}$ is the measure in the functional space so that $(f,g) \equiv \int dt \times w_\alpha(t) f(t) g(t)$. The valley (2) connects two classical solutions: the perturbative vacuum at $\alpha = 0$ and the infinitely separated $I\bar{I}$ pair at $\alpha \to \infty$. The $I\bar{I}$

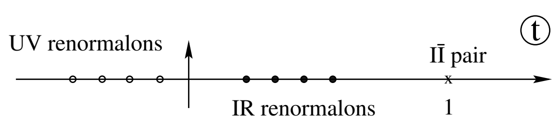

FIG. 1. The 't Hooft picture of singularities in the Borel plane for QCD.

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separation α and the position of the $I\bar{I}$ pair τ are the two collective coordinates of the valley. In order to integrate over the small fluctuations near (2) we insert two δ functions $\delta(\phi(t) - f_\alpha(t-\tau), \dot{f}_\alpha(t-\tau))$ and $\delta(\phi(t) - f_\alpha(t-\tau), f'_\alpha(t-\tau))$, restricting the integrations along the two collective coordinates, make a shift $\phi(t) \to \phi(t) + f_\alpha(t-\tau)$, expand in quantum deviations $\phi(t)$, and perform the (Gaussian) integration in the first nontrivial order in perturbation theory. After the shift, the linear term in the exponent $\int dt\, \phi(t) L_\alpha(t)$ disappears due to the valley equation (3), so our functional integral for the vacuum energy reduces to

$$T \int d\alpha \int D\phi\, (f_\alpha, L_\alpha)\, (\dot{f}_\alpha, \dot{f}_\alpha)\, \delta(\phi, f'_\alpha)\, \delta(\phi, \dot{f}_\alpha)\, e^{-(1/g^2)[S_\alpha + (1/2)\int dt\, \phi(t) \Box_\alpha \phi(t)]} + O(g^2). \quad (4)$$

Here, T (the total "volume" in one space-time dimension) is the result of the trivial integration over τ, $\Box_\alpha = -\partial^2 + 1 - 6f_\alpha(1 - f_\alpha)$ is the operator of the second derivative of the action, and

$$S_\alpha \equiv S(\xi) = \frac{6\xi^4 - 14\xi^2}{(\xi^2 - 1)^2} - \frac{17}{3} + \frac{12\xi^2 + 4}{(\xi^2 - 1)^3} \ln\xi \quad (5)$$

is the action of $I\bar{I}$ valley. By performing the Gaussian integrations, one obtains

$$T \frac{1}{g^2} \int_0^\infty d\alpha\, e^{-(1/g^2)S_\alpha} F(\alpha), \quad (6)$$

$$F(\alpha) = \frac{(\det \Box_\alpha)^{-1/2} (f'_\alpha, f'_\alpha)\, (\dot{f}_\alpha, \dot{f}_\alpha)}{(f'_\alpha, \Box_\alpha^{-1} w_\alpha f'_\alpha)^{1/2} (\dot{f}_\alpha, \Box_\alpha^{-1} w_\alpha \dot{f}_\alpha)^{1/2}}. \quad (7)$$

As $\alpha \to \infty$, we have the widely separated I and $\bar{I}$. In this case, the determinant of the $I\bar{I}$ configuration factorizes into a product of two one-instanton determinants (with zero modes excluded) so $F(\alpha) \to$ const at $\alpha \to \infty$. The divergent part of the integral (6) corresponds to the second iteration of the one-instanton contribution to the vacuum energy and therefore it must be subtracted from the $I\bar{I}$ contribution to E_{vac}:

$$g^2 E_{\mathrm{vac}}(g^2) = \int d\alpha\, [e^{-(1/g^2)S_\alpha} F(\alpha) - e^{-(1/g^2)2S_I} F(\infty)],$$

where S_I is the one-instanton action. Since S_α is a monotonic function of α, one can invert Eq. (5) and obtain

$$g^2 E_{\mathrm{vac}}(g^2) = \int_0^{2S_I} dS\, e^{-(1/g^2)S} \left(\frac{F(S)}{S - 2S_I} \right)_+, \quad (8)$$

which has the desired form of the Borel integral (1) for the vacuum energy. Thus the leading singularity for $E_{\mathrm{vac}}(S)$ is $F(2S_I)/(S - 2S_I)$. Note that our semiclassical calculation does not give the whole answer for the Borel transform of vacuum energy [$F(S) \neq E_{\mathrm{vac}}(S)$ in general]; since we threw away the higher quantum fluctuations around the $I\bar{I}$ pair in Eq. (4), it still determines the leading singularity in the Borel plane.

The situation with the instanton-induced asymptotics of perturbative series in a field theory such as QCD is pretty much similar, with one notable exception: in QCD there is an additional dimensional parameter ρ—the overall size of the $I\bar{I}$ configuration. The classical $I\bar{I}$ action does not depend on this parameter but the quantum determinant does, leading to the replacement

$$e^{-(1/g^2)S} \to e^{-[1/g^2(\rho)]S}. \quad (9)$$

Thus the Borel integrand has the following generic form:

$$F(S) \sim \int d\rho\, e^{-[1/g^2(\rho)]S} \mathcal{F}(\rho). \quad (10)$$

The divergence of this integral at either large or small ρ determines the position of the renormalon-type singularities of $F(S)$. We will demonstrate (by purely dimensional analysis) that $\mathcal{F}(\rho) \sim \rho^{-5}$ as $\rho \to \infty$ and $\mathcal{F}(\rho) \sim \rho$ as $\rho \to 0$, leading to the IR renormalon at $S = \frac{32\pi^2}{b}$ and the UV renormalon at $S = -\frac{16\pi^2}{b}$, respectively.

The $I\bar{I}$ valley in QCD [7] can be chosen as a conformal transformation of the spherical configuration

$$A_\mu^s(x) = -i(\sigma_\mu \bar{x} - x_\mu) x^{-2} f_\alpha(t), \quad (11)$$

with $t = \ln x^2/d^2$, where d is an arbitrary scale in which we use the notations $x \equiv x_\mu \sigma_\mu$, $\bar{x} \equiv x_\mu \bar{\sigma}_\mu$ with $\sigma_\mu = (1, -i\vec{\sigma})$, and $\bar{\sigma}_\mu = (1, i\vec{\sigma})$. To obtain the $I\bar{I}$ configuration with arbitrary sizes ρ_1, ρ_2 and separation R, one performs shift $x \to x - a$, inversion $x \to \frac{d^2}{x^2} x$, and second shift $x \to x - x_0$. The resulting valley $A_\mu^v(x - x_0)$ is the sum of the I with size ρ_1 and $\bar{I}$ with size ρ_2 in the singular gauge in the maximum attractive orientation, plus an additional term which is small at large $I\bar{I}$ separations (the explicit expressions can be found in [8]). The action of the $I\bar{I}$ valley is equal to the action of the spherical configuration (11) which is proportional to Eq. (5):

$$S^v(z) = 48\pi^2 S(\xi), \qquad \xi = z + \sqrt{z^2 - 1}, \quad (12)$$

where the "conformal parameter" z is given by

$$z = (\rho_1^2 + \rho_2^2 + R^2)/(2\rho_1\rho_2). \quad (13)$$

Let us now find the polarization operator in the valley background. The collective coordinates are the size of instantons ρ_i, separation R, overall position x_0, and the orientation in color space [the valley of a general color orientation has the form $\mathcal{O}_{ab} A_b^v$, where $\mathcal{O}$ is an arbitrary SU(3) matrix]. The structure of a Gaussian integral for the polarization operator is

$$\int_0^\infty \frac{d\rho_1\, d\rho_2}{\rho_1^5 \rho_2^5 g^{17}} d^4R\, d^4x_0\, d\mathcal{O}\, \Pi^v(q) e^{-S^v(z)/g^2} \Delta(\rho_i, R), \quad (14)$$

where the power 17 stands for the number of collective coordinates. Here $\Pi^v(q)$ is the Fourier transform of $\Pi^v(x) = (\sum e_q^2) \mathrm{Tr} \gamma_\mu G(x,0) \gamma_\mu G(0,x)$, where $G(x,y)$ is the Green function in the valley background. The factor $\Delta(\rho_i, R)$ in Eq. (14) is the quantum determinant—the result of Gaussian integrations near the $I\bar{I}$ valley [cf. Eq. (7)]. For our purposes, it is convenient to introduce the conformal parameter z and the average size $\rho = \sqrt{\rho_1\rho_2}$ as the collective coordinates in place of ρ_1 and ρ_2. We have

$$\int dz \frac{d\rho}{\rho^9} d^4 R\, d^4x_0 \frac{\Pi^v(q)}{g^{17}(\rho)} F(z, R^2/\rho^2) e^{-S^v(z)/g^2(\rho)}, \tag{15}$$

where $F(z, R^2/\rho^2)$ includes $\theta(z - 1 - \frac{R^2}{2\rho^2})$ [see Eq. (13)]. We have included in F the trivial integral over color orientation which gives the volume of the SU(3) group.

The main effect of the quantum determinant Δ is the replacement of the bare coupling constant g^2 in Eq. (14) by the effective coupling constant $g^2(\rho)$ in Eq. (15) so that the remaining function F is the (dimensionless) function of the ratio R^2/ρ^2 and the conformal parameter. This is almost evident from the renormalizability of the theory since the only dimensional parameters are ρ and R. Formally, one can prove that rescaling of the configuration by a factor λ (so that $\rho \to \lambda\rho$ and $R \to \lambda R$) leads to multiplication of the determinant by a factor $\lambda^{bS^v(z)/8\pi^2}$ due to the conformal anomaly (see, e.g., Ref. [9]).

We now consider the singularities of the integral (15). The function $F(z, R^2/\rho^2)$ is nonsingular since a singularity in Φ ($\equiv$ singularity in Δ) would mean a nonexisting zero mode in the quantum determinant. Moreover, the integration over R is finite due to $\theta(z - 1 - \frac{R^2}{2\rho^2})$ which means that the only source of singularity at finite z is the divergence of the ρ integral at either large or small ρ. [As $z \to \infty$, we can obtain the first instanton-type singularity located at $t = 1$ [10] in a way analogous to the derivation of Eq. (8).]

Let us demonstrate that the singularity of the integral (15) at large $\rho \gg \frac{1}{q}$ corresponds to the IR renormalon. The polarization operator $\Pi(x)$ in the background of the large-scale vacuum fluctuation reduces to [11]

$$\langle \Pi(x) \rangle_A \to -\frac{\sum e_q^2}{64\pi^4} \left(\frac{G^2(0)}{x^2} + c\alpha_s G^3(0) \ln x^2 + \dots \right), \tag{16}$$

where $G^2 \equiv 2\,\mathrm{Tr} G_{\xi\eta} G_{\xi\eta}$, $G^3 \equiv 2\,\mathrm{Tr} G_{\xi\eta} G_{\eta\sigma} G_{\sigma\xi}$, and c is an (unknown) constant. (The coefficient in front of G^3 vanishes at tree level [12]). Consider the leading term in this expansion. Since the field strength for the $I\bar{I}$ valley configuration depends only on $x - x_0$, we get $\int d^4x_0\, \mathrm{Tr} G^v_{\xi\eta}(0) G^v_{\xi\eta}(0) = 4S^v(z)$ so that the integral (15) reduces to

$$\frac{1}{q^2} \int_1^\infty dz \int_{1/q}^\infty \frac{d\rho}{\rho^9} d^4R\, g^{-17}(\rho) e^{-S^v(z)/g^2(\rho)} F\left(z, \frac{R^2}{\rho^2}\right),$$

where we have included the factor $\frac{1}{36} \sum e_q^2 S^v(z)$ in F. The (finite) integration over R can be performed, resulting in an additional dimensional factor ρ^4: $\int d^4R\, F(z, R^2/\rho^2) = \rho^4 \Phi(z)$, where the function Φ, defined by this formula, is dimensionless so that it can depend only on z. We get

$$\frac{1}{q^2} \int_1^\infty dz \int_{1/q}^\infty \frac{d\rho}{\rho^5} g^{-17}(\rho) e^{-[1/g^2(\rho)] S^v(z)} \Phi(z). \tag{17}$$

By inverting Eq. (5), we can write the corresponding contribution to Adler's function as an integral over the valley action ($t \equiv \frac{S}{16\pi^2}$ and $\alpha_s \equiv \frac{g^2}{4\pi}$):

$$\begin{aligned} D(q^2) &\simeq \frac{1}{3q^4} \int_0^1 dt \int_{1/q}^\infty \frac{d\rho}{\rho^5} g^{-17}(\rho) e^{-[4\pi/\alpha_s(\rho)]t} \Phi(t) \\ &= \frac{1}{q^4} \int_0^1 dt \int_{1/q}^\infty \frac{d\rho}{\rho^5} e^{-[4\pi/\alpha_s(\rho)]t} \Psi(t), \end{aligned} \tag{18}$$

where $\Psi(t) = \frac{1}{3\sqrt{\pi}(4\pi)^{17}} \int_0^t dt'\, (t - t')^{-1/2} \Phi^{(9)}(t')$ after nine integrations by parts and a Laplace transformation (we neglect terms $\sim e^{-4\pi/\alpha_s}$ corresponding to the $I\bar{I}$ singularity). At the one-loop level we obtain

$$D(t) \simeq \Psi(t) \int_{1/q}^\infty \frac{d\rho}{q^4\rho^5} (q^2\rho^2)^{bt} = \frac{1}{2 - bt}, \tag{19}$$

which is the first IR renormalon. Strictly speaking, we should integrate over ρ only up to $\rho < \Lambda_{\mathrm{QCD}}^{-1}$ since at $\rho \sim \Lambda_{\mathrm{QCD}}^{-1}$ our valley is washed out by the large-scale fluctuations populating the QCD vacuum (cf. Ref. [13]). At the one-loop level this leads to the replacement $(2 - bt)^{-1} \to (2 - bt)^{-1}[1 - (\Lambda^2/q^2)^{2-bt}]$, so we get

$$\begin{aligned} D(q^2) &= D^{\mathrm{pert}}(q^2) + D^{\mathrm{nonpert}}(q^2) \\ &\simeq \int_0^1 dt \frac{f(t)}{2 - bt} \left[e^{-[4\pi/\alpha_s(q)]t} - \frac{\Lambda^4}{q^4} \right]. \end{aligned} \tag{20}$$

Both D^{pert} and D^{nonpert} have the singularity $1/(2 - bt)$ which cancels in their sum. If we adopt the principal value prescription for the integration over t, the nonperturbative part is a real number (divided by q^4) which contributes to the phenomenological power correction $\frac{\pi}{3q^4} \alpha_s \langle G^2_{\mu\nu} \rangle$ (cf. Ref. [14]).

At the two-loop level we get

$$D^{\mathrm{pert}}(t) \simeq \Psi(t) \int_{1/q}^\infty \frac{d\rho}{q^4\rho^5} (q^2\rho^2)^{bt} [\alpha_s(\rho)/\alpha_s(q)]^{2b't/b}, \tag{21}$$

where $b' = 51 - \frac{19}{3} n_f$. By using integration by parts, it is easy to demonstrate that an extra $\alpha_s(\rho)$ does not change the singularity [cf. Eq. (18)] while an extra $\alpha_s(q)$ shifts it by one power of $2 - bt$. Thus, at the two-loop level the first IR renormalon is a branch point singularity $(t - \frac{2}{b})^{-1-4(b'/b^2)}$ [14]. The second term in the expansion (16) gives the second renormalon singularity located at $t = \frac{3}{b}$, and higher terms of the expansion of the polarization operator (16) will give the subsequent renormalons at $t = \frac{4}{b}, \frac{5}{b}$, etc.

Next we demonstrate that the divergence of the integral (15) at small ρ leads to the UV renormalon. In order to find the polarization operator in the valley background in this case we recall that a very small valley is an inversion of the very large spherical configuration (11). For a very large configuration we can use the formula (16) (in coordinate space) and obtain

$$\Pi^v(x) \to \frac{\sum e_q^2}{192\pi^4}\left[\frac{d^8 G_s^2(a)}{y^4 x_0^4 x^2}\left\{1 - \frac{4(x_0 y)^2}{x_0^2 y^2}\right\} + c\alpha_s \frac{d^{12} G_s^3(a)}{y^6 x_0^6}\left\{\left(3\ln\frac{x^2}{y^2 x_0^2} - 1\right)\frac{2(x_0 y)^2}{x_0^2 y^2} + 1\right\}\right], \tag{22}$$

where $y \equiv x - x_0$. Here $G^s_{\mu\nu}(a)$ is the field strength of the spherical configuration (11) calculated at $x = a$ ($\Leftrightarrow$ field strength at the origin of the configuration with center a). The integration over x and x_0 yields

$$\int dx\, dx_0\, e^{iqx}\Pi^v_{\mu\nu}(x) = c'\alpha_s(q) d^{12} G_s^3(a) q^4 \ln^2 q^2.$$

Note that the first $\sim G_s^2$ term in Eq. (22) vanishes. By rewriting d and a in terms of z, ρ, and R, we obtain $d^{12}G_s^3(a) = \rho^6 \mathcal{G}(z, R^2/\rho^2)$, where the dimensionless function $\mathcal{G}$ is nonsingular (the explicit expressions can be found in [8]). By performing the integration over R, we obtain the analog of Eq. (18) for the UV renormalon

$$D(q^2) \simeq q^2\alpha_s(q)\int_0^1 dt \int_0^{1/q} d\rho\,\rho \ln^2 q^2\rho^2 \times e^{-[4\pi/\alpha_s(\rho)]t}\tilde{\Psi}(t), \tag{23}$$

which corresponds to

$$D(t) \simeq \tilde{\Psi}(t) q^2 \int_0^{1/q} d\rho^2 (\ln q^2\rho^2)(q^2\rho^2)^{bt} \times \left(\frac{\alpha_s(\rho)}{\alpha_s(q)}\right)^{2b't/b} \tag{24}$$

[the extra $\alpha_s(q)$ in Eq. (23) is compensated by one power of $\ln q^2\rho^2$]. The integral over ρ diverges at $t = -\frac{1}{b}$ and gives the double pole at the one-loop level just as in the perturbative analysis. Subsequent terms in the expansion (16) correspond to UV renormalons located at $t = -\frac{2}{b}, -\frac{3}{b}, \ldots$. It should be mentioned that Eq. (24) does not reproduce the strength of the first UV renormalon at the two-loop level [15]. The reason is that in Eq. (16) we have neglected the anomalous dimensions of the operators $\sim[\alpha_s(q)/\alpha_s(\rho)]^{\gamma/b}$. Such factors can change the strength of the singularity. For the IR renormalon this does not matter since the operator G^2 is renorm invariant ($\gamma = 0$), and for the subsequent renormalons we can easily correct our results by the corresponding γ's. For the UV renormalons, we do not know how to use the conformal invariance with the anomalous dimensions included.

It is very important to note that we actually never use the explicit form of the valley configuration. What we have really used are the three features: (i) that the rescaling of the vacuum fluctuation with an action S by a factor λ multiplies the determinant by $\lambda^{bS/8\pi^2}$ leading to formula (9), (ii) the expansion of the polarization operator (16) in slow varying fields, and (iii) the conformal invariance of QCD at the tree level (for the UV renormalon we wrote down the small-size valley as an inversion of a large-scale spherical configuration). All of these properties hold true for an arbitrary vacuum fluctuation so we could take an arbitrary valley and arrive at the same results as Eqs. (21) and (24). It means that our result about the renormalon singularity coming from dilatation modes in the functional space is general. The reason why we chose to consider the $I\bar{I}$ valley is purely pedagogical.

Just as in the conventional approach, our method yields the position and strength of the IR renormalon singularity but not the coefficient in front of it (this coefficient determines the numerical value of asymptotics of perturbative series for $R_{e^+e^-\to\text{hadrons}}$). To go beyond that and find the coefficient would require integration over all possible valleys. This study is in progress.

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